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EXTENDING MONOIDAL STRUCTURES ON FIBERED CATEGORIES VIA EMBEDDINGS

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Abstract. Let $\mathscr{S}$ be a small category, and suppose that we are given a full subcategory $\mathscr{U}$ such that every object of $\mathscr{S}$ can be embedded into some object of $\mathscr{U}$ in the same way as every quasi-projective algebraic variety admits a closed embedding into a smooth one. We show that every monoidal structure on a triangulated $\mathscr{S}$-fibered category satisfying certain natural conditions is completely determined by its restriction to $\mathscr{U}$; in fact, any monoidal structure over $\mathscr{U}$ satisfying similar natural conditions admits an essentially unique extension to the whole of $\mathscr{S}$. In this way, one can recover the unit constraint on the classical constructible derived categories over quasi-projective algebraic varieties from the abelian categories of perverse sheaves. The same principle applies to morphisms of $\mathscr{S}$-fibered categories and monoidality thereof. For instance, this allows one to fully understand the unitarity properties of the forgetful functors from Saito's categories of mixed Hodge modules to the underlying categories of perverse sheaves.

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Introduction

Motivation and goal of the paper. In Algebraic Geometry, studying smooth varieties often proves much easier than studying singular varieties: this basically happens every time one tries to study singular varieties via algebraic differential forms, which are well-behaved only in the smooth case. For instance, it is unreasonable to define algebraic

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de Rham cohomology of singular varieties via the usual algebraic de Rham complex, since this would not produce cohomology groups of the expected dimension (see [AK11, Ex. 4.4] for an explicit counterexample). More generally, the behavior of algebraic differential equations over a singular complex variety S is quite subtle both from the perspective of $\mathscr{D}$-modules and from that of local systems: on the side of $\mathscr{D}$-modules, a typical pathology is that the ring $\mathscr{D}(S)$ of algebraic differential operators over S is not necessarily Noetherian, so it is unreasonable to define $\mathscr{D}$-modules over S naively as modules over $\mathscr{D}(S)$; a typical issue on the side of local systems is that an ordinary local system over S does not define a perverse sheaf, and so it does not arise from a regular holonomic $\mathscr{D}$-module.

When the singular variety S is quasi-projective, a useful workaround is to choose a closed embedding $s\colon S \hookrightarrow Y$ into a smooth (still quasi-projective) variety Y: in the $\mathscr{D}$-module setting, one is led to define the category of $\mathscr{D}$-modules over S as the subcategory of those $\mathscr{D}$-modules over Y which vanish on the open complement $Y \setminus S$; in the local system setting, one can study those local systems over S that extend to local systems over Y in this way. In both cases, the main point is to check that the sought-after categories or objects are essentially independent of the chosen closed embedding $s\colon S \hookrightarrow Y$, and that they enjoy the expected functoriality with respect to morphisms of possibly singular varieties $f\colon T \to S$.

A posteriori, the success of this approach is explained by Saito's six-functor formalism of mixed Hodge modules over quasi-projective complex varieties, developed in [Sai90]. In fact, for every coefficient system $\mathbb{H}$ over quasi-projective varieties satisfying the six-functor formalism (as formalized in Ayoub's thesis [Ayo07a, Ayo07b]), a closed embedding $s\colon S \hookrightarrow Y$ determines an adjoint pair of the form

$$s^*\colon \mathbb{H}(Y) \rightleftarrows \mathbb{H}(S)\colon s_*$$

in which the direct image functor s_* is fully faithful while the inverse image functor s^* is monoidal. This allows one to identify the objects of $\mathbb{H}(S)$ with the objects of $\mathbb{H}(Y)$ supported on S, and also to identify the unit object $\mathbf{I}_S \in \mathbb{H}(S)$ with the pull-back $s^*\mathbf{I}_Y$. Combining these two facts, one is able to reconstruct the unit constraint over S

$$u_S\colon \mathbf{I}_S \otimes A \xrightarrow{\sim} A$$

from the unit constraint over Y, by the formula

$$\mathbf{I}_S \otimes A = s^*\mathbf{I}_Y \otimes s^*s_*A \xrightarrow{m_s} s^*(\mathbf{I}_Y \otimes s_*A) \xrightarrow{u_Y} s^*s_*A = A. \tag{1}$$

A similar strategy works for a morphism of coefficient systems $R\colon \mathbb{H}_1 \to \mathbb{H}_2$. In particular, in order to make sure that a given morphism is compatible with unit constraints over all quasi-projective varieties, it suffices to check its unitarity over smooth varieties.

The main goal of the present paper is to turn the embedding principle just sketched into a rigorous extension method, applicable to each aspect of the theory of monoidal categories over quasi-projective algebraic varieties. In particular, we address the problem of extending monoidal structures, morphisms of fibered categories, and monoidal structures

on morphisms from the smooth to the general case: this amounts to extending suitable functors and natural transformations given over smooth varieties to general varieties, and checking that these extensions still satisfy the relevant axioms.

In fact, this is the third of a series of papers devoted to the abstract monoidality properties of coefficient systems modelled on perverse sheaves. To fix notation, let k be a subfield of the complex numbers, and consider the category Var_k of quasi-projective algebraic k-varieties. To every $S \in \mathsf{Var}_k$ one can attach its algebraically constructible $\mathbb{Q}$-linear derived category, denoted $D^b_c(S)$. As S varies in Var_k, these categories assemble into a monoidal Var_k-fibered category, and this structure actually extends to a complete six-functor formalism. The classical theory of [BBD82] yields a perverse t-structure on $D^b_c(S)$, whose heart is the abelian category $\mathsf{Perv}(S)$ of *perverse sheaves*. As shown by Beilinson in [Bei87], the triangulated category $D^b_c(S)$ can be recovered as the bounded derived category $D^b(\mathsf{Perv}(S))$. It is thus natural to ask whether the six-functor formalism can be reconstructed on the level of perverse sheaves. In our papers [Ter24a] and [Ter24b] we address this problem systematically, especially for what concerns the monoidal structure: this yields a satisfying description of the constructible derived categories as a monoidal Var_k-fibered category in terms of functors respecting the perverse t-structures. The only question that the aforementioned papers leave open is how to describe the unit constraint in terms of perverse sheaves: the main issue, as alluded to above, is that the unit object $\mathbf{I}_S \in D^b_c(S)$ is not a perverse sheaf (up to shiftings) unless S is smooth. The results of the present paper allow us to overcome this specific issue.

The same issue, although rather formal-looking in the case of perverse sheaves, becomes more structural for refined coefficient systems such as Saito's mixed Hodge modules. Indeed, Saito's triangulated categories are defined in [Sai90] as the derived categories of their perverse hearts, and the six operations on mixed Hodge modules are obtained by cleverly combining various exact functors defined on the level of abelian categories. The same happens in the recent theory of perverse Nori motives, initiated by Ivorra–Morel in [IM24]: this is based on a system of abelian categories of motivic perverse sheaves defined in terms of a purely categorical universal property. Since objects and morphisms in these categories do not admit a concrete description, the same can be said about functors and natural transformations relating their derived categories. We ultimately want to apply our abstract results to the construction of the monoidal structure on derived perverse Nori motives, which we achieve in [Ter26] by working directly on the perverse hearts. The present work is exploited mostly in the construction of the motivic unit constraint: by working over smooth varieties only, we circumvent the issue that the motivic unit object cannot be defined directly over singular varieties. We plan to apply the present results to a larger extent in order to study Ivorra's perverse motivic realization functors, defined in [Ivo16]. Without going into the details, let us just mention that these realization functors are only available over smooth varieties: once again, the main obstruction to defining them in the singular case lies in the lack of perversity of the unit object. Our results would allow us to extend Ivorra's realization to possibly singular varieties automatically, on the condition of establishing its compatibility with the six operations over smooth varieties.

We hope to be able to say more about this problem in future work.

Main results. The basic principle behind our constructions is that, given a possibly singular quasi-projective variety S endowed with a closed immersion $s\colon S \hookrightarrow Y$ into a smooth variety Y, one can define several structures over S from analogous structures over Y. We work in a general axiomatic framework, over a base category $\mathscr{S}$ endowed with distinguished classes of smooth morphisms and closed immersions, as formalized in the companion paper [Ter24b]. We let $\mathscr{U}$ denote the full subcategory of $\mathscr{S}$ spanned by the smooth objects; our axioms ensure that every object of $\mathscr{S}$ admits a closed immersion into some object of $\mathscr{U}$.

Our constructions only apply to triangulated $\mathscr{S}$-fibered categories satisfying part of the axiomatics of stable homotopy 2-functors, as developed by Ayoub in [Ayo07a, Ayo07b] and [Ayo10]: we call them *geometric* triangulated $\mathscr{S}$-fibered categories, and we only allow certain morphisms of triangulated $\mathscr{S}$-fibered categories called *localic* morphisms. Our first main result allows one to recover localic morphisms of geometric $\mathscr{S}$-fibered categories from their restriction to the underlying $\mathscr{U}$-fibered categories. It can be stated as follows:

0.1. Theorem. [Theorem 2.7] *Let $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a localic morphism between geometric triangulated $\mathscr{S}$-fibered categories. Then, the morphism R is completely determined by its restriction to $\mathscr{U}$: in fact, every localic morphism between the underlying geometric triangulated $\mathscr{U}$-fibered categories extends uniquely over the whole of $\mathscr{S}$.*

The proof is based on some auxiliary constructions, exploiting the functoriality properties of closed immersions into $\mathscr{U}$ in a canonical and explicit way.

Following the same route, we study monoidal structures. Again, our constructions only apply to certain monoidal structures on geometric $\mathscr{S}$-fibered categories, which as in [Ter24b, § 5] we call *localic* monoidal structures: they are characterized by the *projection formulae* with respect to smooth morphisms, as formulated in [Ayo07a, Defn. 2.3.1]. Our second main result can be stated as follows:

0.2. Theorem. [Theorem 3.2, Propositions 3.7, 3.9, 3.11, Lemmata 3.14, 3.15, 3.16] *Let $\mathbb{H}$ be a geometric triangulated $\mathscr{S}$-fibered category equipped with a localic (unitary, symmetric, associative) triangulated monoidal structure. Then, the monoidal structure on $\mathbb{H}$ is completely determined by its restriction to $\mathscr{U}$: in fact, any localic (unitary, symmetric, associative) triangulated monoidal structure on the underlying geometric triangulated $\mathscr{U}$-fibered category extends uniquely over the whole of $\mathscr{S}$.*

Lastly, we combine the two extension methods just discussed to study monoidality of localic morphisms between geometric $\mathscr{S}$-fibered categories. Our third main result can be stated as follows:

0.3. Theorem. [Theorem 4.2, Lemmata 4.5, 4.6, 4.7] *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two geometric triangulated $\mathscr{S}$-fibered categories equipped with localic monoidal structures, and let $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a localic morphism equipped with a triangulated monoidal structure (with respect to the given monoidal structures on $\mathbb{H}_1$ and $\mathbb{H}_2$). Then, the monoidal structure on R is completely determined by its restriction to $\mathscr{U}$: in fact, any monoidal structure on*

the underlying morphism of geometric triangulated $\mathscr{U}$-fibered categories extends uniquely over the whole of $\mathscr{S}$.

Combining the main results, one can recover the unit constraint on the constructible derived categories working on the level of perverse sheaves: indeed, the formula (1) defines the unit constraint over a possibly singular variety S in terms of the unit constraint over the smooth ambient variety Y, where the unit object is a perverse sheaf (up to shiftings). The same formula holds in the settings of mixed Hodge modules and of perverse Nori motives; the unitarity of the forgetful functors to the constructible derived category over S follows from the corresponding property over Y, which can be checked on the perverse hearts.

On a technical level, the proofs of our three main results all follow the same pattern. For the result about morphisms of $\mathscr{S}$-fibered categories, we spell out the details of the proofs. In the case of monoidal structures on $\mathscr{S}$-fibered categories, the proofs are just a notationally more intricate variant of the previous ones, so we omit them. We give more details in the case of associativity, commutativity and unit constraints, and also for the result about monoidal structures on morphisms of $\mathscr{S}$-fibered categories.

Structure of the paper. Throughout the paper, we work over a fixed small category $\mathscr{S}$; at the beginning of each section, we specify the natural conditions that $\mathscr{S}$ has to satisfy in order to make our constructions possible.

In Section 1, after fixing our conventions and notation about $\mathscr{S}$-fibered categories, we introduce the main working hypotheses on $\mathscr{S}$ (Assumption 1.2) and we define the full subcategory of smooth objects $\mathscr{U}$ accordingly (Notation 1.3). After recording the main properties of $\mathscr{U}$ (Lemma 1.4), we establish a first extension result for sections of $\mathscr{S}$-fibered categories (Proposition 1.12).

Starting from Section 2, we strengthen our assumptions on $\mathscr{S}$ (Assumption 2.2), and we restrict to the class of geometric $\mathscr{S}$-fibered categories (Definition 2.4) and localic morphisms of such. Our first main result (Theorem 2.7) asserts that every geometric morphisms between localic $\mathscr{S}$-fibered categories is canonically determined by its restriction to $\mathscr{U}$ and that, conversely, any such morphism over $\mathscr{U}$ uniquely extends over the whole of $\mathscr{S}$. We also discuss how this extension result applies to images of sections along morphisms of $\mathscr{S}$-fibered categories (Lemma 2.13).

In Section 3, we explain how to extend monoidal structures on geometric $\mathscr{S}$-fibered categories in a similar fashion; this only applies to the class of localic monoidal structures. We obtain the extension result (Theorem 3.2) by suitably adapting the constructions of the previous section. To complete the discussion, we explain how to extend associativity, commutativity, and unit constraints (Propositions 3.7, 3.9 and 3.11), respecting their natural compatibility conditions (Lemmata 3.14, 3.15 and 3.16).

In the final Section 4, we explain how to extend monoidal structures on localic morphisms between monoidal geometric $\mathscr{S}$-fibered categories. After proving the basic extension result (Theorem 4.2), we check its compatibility with associativity, commutativity, and unit constraints (Lemmata 4.5, 4.6 and 4.7).

Notation and conventions

- Unless otherwise dictated, categories are assumed to be small with respect to some fixed universe.
- Given a category $\mathscr{C}$, the notation $A \in \mathscr{C}$ means that A is an object of $\mathscr{C}$.
- Given categories $\mathscr{C}_1, \dots, \mathscr{C}_n$, we let $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ denote their direct product category.

1. Extending sections of fibered categories

Throughout this paper, we work over a fixed small category $\mathscr{S}$ admitting finite products; we let $*$ denote the final object of $\mathscr{S}$.

In the first part of the present section, we introduce a convenient set of axioms on $\mathscr{S}$ mimicking certain natural properties of quasi-projective algebraic varieties over a field; all of our extension results will be stated uniformly in terms of this axiomatic framework. In the second part, we describe a first easy extension result, for sections of $\mathscr{S}$-fibered categories.

1.1. Axiomatics of embedding systems. Following the approach of the companion paper [Ter24b], we introduce suitable notions of smooth morphisms and closed immersions in $\mathscr{S}$ mimicking the properties of the usual classes of smooth morphisms and closed immersions between quasi-projective algebraic varieties over a field; in particular, this will allow us to factor morphisms of $\mathscr{S}$ into closed immersions and smooth morphisms. To be precise, throughout the entire paper we work under the following Assumption, which is [Ter24b, Assum. 2.1]:

1.2. Assumption. *We assume to be given two (possibly non-full) subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ of $\mathscr{S}$ satisfying the following conditions:*

(i) Both $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ have the same objects as $\mathscr{S}$ and contain all isomorphisms of $\mathscr{S}$.

(ii) Let $p\colon P \to S$ be a morphism in $\mathscr{S}^{sm}$. Then, for every morphism $f\colon T \to S$ in $\mathscr{S}$, the fibered product $P_T := P \times_S T$ exists in $\mathscr{S}$. Moreover, for every Cartesian square of the form

$$\begin{array}{ccc} P_T & \xrightarrow{f'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

with p in $\mathscr{S}^{sm}$, the morphism p' belongs to $\mathscr{S}^{sm}$ as well.

(iii) For every Cartesian square in $\mathscr{S}$ of the form

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p in $\mathscr{S}^{sm}$ and z in $\mathscr{S}^{cl}$, the morphism z' belongs to $\mathscr{S}^{cl}$ as well. Moreover, for every commutative diagram in $\mathscr{S}$ of the form

$$\begin{array}{ccc} Z & \xrightarrow{z} & S \\ & {\scriptstyle gz}\searrow & \downarrow{\scriptstyle g} \\ & & V \end{array}$$

with gz in $\mathscr{S}^{cl}$, the morphism z belongs to $\mathscr{S}^{cl}$ as well.

(iv) Every morphism $f\colon T \to S$ in $\mathscr{S}$ admits a factorization of the form

$$f\colon T \xrightarrow{t} P \xrightarrow{p} S$$

with t in $\mathscr{S}^{cl}$ and p in $\mathscr{S}^{sm}$.

We say that a morphism in $\mathscr{S}$ is a smooth morphism *if it belongs to $\mathscr{S}^{sm}$ and a* closed immersion *if it belongs to $\mathscr{S}^{cl}$.*

Following [Ter24b, Not. 2.7], for every morphism $f\colon T \to S$ in $\mathscr{S}$ we consider the associated *factorization category* $\mathsf{Fact}_{\mathscr{S}}(f)$, defined as follows:

- objects are triples $(P; t, p)$ where P is an object of $\mathscr{S}$, $t\colon T \to P$ is a closed immersion and $p\colon P \to S$ is a smooth morphism such that $p \circ t = f$,
- a morphism $q\colon (P'; t', p') \to (P; t, p)$ is the datum of a smooth morphism $q : P' \to P$ such that $q \circ t' = t$ and $p \circ q = p'$.

By [Ter24b, Lem. 2.8], the category $\mathsf{Fact}_{\mathscr{S}}(f)$ is non-empty and connected. More precisely, every two objects $(P; t, p)$ and $(P'; t', p')$ in $\mathsf{Fact}_{\mathscr{S}}(f)$ are dominated by a third one $(P''; t'', p'')$: just take $P'' = P \times_S P'$ with the obvious morphisms $t''\colon T \to P''$ and $p''\colon P'' \to S$.

Throughout the paper, we systematically work with a special full subcategory of $\mathscr{S}$, which should be thought of as the subcategory of smooth objects:

1.3. Notation. We let $\mathscr{U}$ denote the full subcategory of $\mathscr{S}$ collecting all those objects $Y \in \mathscr{S}$ such that the morphism to the final object $Y \to *$ is a smooth morphism.

1.4. Lemma. *The following statements hold:*

1. *The subcategory $\mathscr{U}$ is closed under finite products in $\mathscr{S}$.*
2. *For every $Y \in \mathscr{U}$ and every smooth morphism $p\colon X \to Y$, we have $X \in \mathscr{U}$.*
3. *For every $S \in \mathscr{S}$, there exists a closed immersion $s\colon S \to Y$ with $Y \in \mathscr{U}$.*

Proof. To prove the first point, given two objects $Y, Y' \in \mathscr{U}$, form the Cartesian diagram in $\mathscr{S}$

$$\begin{array}{ccc} Y \times Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & *. \end{array}$$

Since $Y' \in \mathscr{U}$, the projection $Y \times Y' \to Y$ is a smooth morphism (by Assumption 1.2(ii)). Since moreover $Y \in \mathscr{U}$, the thesis follows from the second point, which in turn is obvious (by the very definition of $\mathscr{U}$). Lastly, the third point follows by applying Assumption 1.2(iv) to the morphism $S \to *$. ∎

A further list of hypotheses about $\mathscr{S}$ and $\mathscr{U}$, needed from Section 2 onward, will be spelled out later in Assumption 2.2.

1.5. Remark. Note that the small category $\mathscr{U}$, together with its natural subcategories $\mathscr{U}^{sm} := \mathscr{U} \cap \mathscr{S}^{sm}$ and $\mathscr{U}^{cl} := \mathscr{U} \cap \mathscr{S}^{cl}$, still satisfies the conditions of Assumption 1.2: indeed, (i) follows from the fullness of $\mathscr{U}$ in $\mathscr{S}$; (ii) and (iii) follow from from the fact that, given two smooth morphisms in $\mathscr{U}$ with the same target, their fibered product in $\mathscr{S}$ belongs in fact to $\mathscr{U}$ (by conditions (ii) and (iii) over $\mathscr{S}$ together with Lemma 1.4(2)); (iv) follows from the fact that, for every morphism $f\colon T \to S$ in $\mathscr{U}$ and every factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(f)$, the object $X \in \mathscr{S}$ belongs in fact to $\mathscr{U}$ (again by Lemma 1.4(2)).

1.6. Example. As in [Ter24b], Assumption 1.2 is tailored to the setting where $\mathscr{S} = \mathsf{Var}_k$ is the category of quasi-projective algebraic varieties over a field k, and $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ denote the usual subcategories of smooth morphisms and closed immersions, respectively. In this setting, $\mathscr{U} = \mathsf{Sm}_k$ is the category of smooth quasi-projective k-varieties.

Our main constructions exploit systematically the possibility to embed objects of $\mathscr{S}$ into objects of $\mathscr{U}$. It will be convenient for us to use the following notation:

1.7. Notation. For every $S \in \mathscr{S}$, we introduce the *embedding category* $\mathsf{Emb}_{\mathscr{U}}(S)$ defined as follows:

- objects are pairs $(Y; s)$ with $Y \in \mathscr{U}$ and $s\colon S \to Y$ a closed immersion,
- a morphism $p\colon (Y'; s') \to (Y; s)$ is the datum of a smooth morphism $p\colon Y' \to Y$ such that $p \circ s' = s$.

Moreover, we let $\mathscr{G}_{\mathscr{U}}(S)$ denote the associated trivial groupoid: that is, the category whose objects are the same as for $\mathsf{Emb}_{\mathscr{U}}(S)$ and where each homomorphisms set contains exactly one element.

As a sanity check, let us record the following:

1.8. Lemma. *For every $S \in \mathscr{S}$, the category $\mathsf{Emb}_{\mathscr{U}}(S)$ is non-empty and connected.*

Proof. The fact that $\mathsf{Emb}_{\mathscr{U}}(S)$ is non-empty follows from Lemma 1.4(3). To see that it is connected, we prove a more precise result, namely: every two objects $(Y; s)$ and $(Y'; s')$ are dominated by a third one $(Y''; s'')$. To this end, it suffices to take $Y'' = Y \times Y'$ (which belongs to $\mathscr{U}$ by Lemma 1.4(1)): indeed, the canonical morphism $S \to Y''$ is a closed immersion (by Assumption 1.2(iii)), while the coordinate projections $Y'' \to Y$ and $Y'' \to Y'$ are smooth morphisms (by Assumption 1.2(ii)). ∎

1.9. Remark. The definition of $\mathscr{U}$ given in Notation 1.3 is adapted to the above notion of morphism in the embedding categories, guaranteeing in particular that they are connected. If one is willing to relax the notion of morphism $p\colon (Y'; s') \to (Y, s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$ by allowing $p\colon Y' \to Y$ to be not necessarily smooth, then one can replace $\mathscr{U}$ by any other full subcategory of $\mathscr{S}$ satisfying the conclusions of Lemma 1.4.

1.10. Extension of sections. In the rest of this section, we work over an *$\mathscr{S}$-fibered category* $\mathbb{H}$: following the conventions of [Ter24a, § 1], by this we mean a strictly unital contravariant pseudo-functor from $\mathscr{S}$ to the 2-category of small categories. In detail, this is the datum of

- for every $S \in \mathscr{S}$, a category $\mathbb{H}(S)$,
- for every morphism $f\colon T \to S$ in $\mathscr{S}$, a functor
$$f^*\colon \mathbb{H}(S) \to \mathbb{H}(T),$$
called the *inverse image functor* along f,
- for every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, a natural isomorphism of functors $\mathbb{H}(V) \to \mathbb{H}(T)$
$$\mathsf{conn} = \mathsf{conn}_{f,g}\colon (gf)^*A \xrightarrow{\sim} f^*g^*A,$$
called the *connection isomorphism* at (f, g)

such that the following conditions are satisfied:

($\mathscr{S}$-fib-0) For every $S \in \mathscr{S}$, we have $\mathsf{id}_S^* = \mathsf{id}_{\mathbb{H}(S)}$.

($\mathscr{S}$-fib-1) For every triple of composable morphisms $f\colon T \to S$, $g\colon S \to V$ and $h\colon V \to W$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}(W) \to \mathbb{H}(T)$

$$\begin{array}{ccc} (hgf)^*A & \xrightarrow{\mathsf{conn}_{f,hg}} & f^*(hg)^*A \\ \downarrow{\scriptstyle \mathsf{conn}_{gf,h}} & & \downarrow{\scriptstyle \mathsf{conn}_{g,h}} \\ (gf)^*h^*A & \xrightarrow{\mathsf{conn}_{f,g}} & f^*g^*h^*A \end{array}$$

is commutative.

1.11. Remark. The meaning of axiom ($\mathscr{S}$-fib-1) is that all composites of connection isomorphisms (and inverses thereof) are coherent. For sake of simplicity, starting from the next subsection, we will systematically write all connection isomorphisms as equalities.

As in [Ter24a, § 1], by a *section* A of $\mathbb{H}$ over $\mathscr{S}$ we mean the datum of

- for every $S \in \mathscr{S}$, an object $A_S \in \mathbb{H}(S)$,
- for every morphism $f\colon T \to S$ in $\mathscr{S}$, an isomorphism in $\mathbb{H}(T)$

$$A^* = A^*_f\colon f^*A_S \xrightarrow{\sim} A_T$$

satisfying the following condition:

($\mathscr{S}$-sect) For every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccc} f^*g^*A_V & \xrightarrow{A^*_g} & f^*A_S \\ {\scriptstyle \mathsf{conn}_{f,g}}\uparrow & & \downarrow{\scriptstyle A^*_f} \\ (gf)^*A_V & \xrightarrow{A^*_{gf}} & A_T \end{array}$$

is commutative.

Similarly, by a *morphism of sections* $w\colon A \to B$ of $\mathbb{H}$ over $\mathscr{S}$ we mean the datum of

- for every $S \in \mathscr{S}$, a morphism in $\mathbb{H}(S)$

$$w_S\colon A_S \to B_S$$

satisfying the following condition:

(mor-$\mathscr{S}$-sect) For every morphism $f\colon T \to S$ in $\mathscr{S}$, the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccc} f^*A_S & \xrightarrow{w_S} & f^*B_S \\ \downarrow{\scriptstyle A^*_f} & & \downarrow{\scriptstyle B^*_f} \\ A_T & \xrightarrow{w_T} & B_T \end{array}$$

is commutative.

Note that the $\mathscr{S}$-fibered category $\mathbb{H}$ defines by restriction a $\mathscr{U}$-fibered category, which we still write simply as $\mathbb{H}$ by abuse of notation. Similarly, every section of $\mathbb{H}$ over $\mathscr{S}$ defines a section of $\mathbb{H}$ over $\mathscr{U}$ by restriction; the same applies to morphisms of sections.

As a warm-up, we now show that the restriction operation from $\mathscr{S}$ to $\mathscr{U}$ for sections of $\mathbb{H}$ is reversible in a canonical way. Here is the precise statement:

1.12. Proposition. *Let A be a section of $\mathbb{H}$ over $\mathscr{U}$. Then, there exists a unique (up to unique isomorphism) section $\tilde{A}$ of $\mathbb{H}$ over $\mathscr{S}$ whose restriction to $\mathscr{U}$ coincides with A. The assignment $A \mapsto \tilde{A}$ defines a functor from sections over $\mathscr{U}$ to sections over $\mathscr{S}$.*

The rest of the present section is devoted to the proof of this result. The general idea is that, if the conclusion of the proposition holds, then the value $\tilde{A}_S$ must be canonically isomorphic to s^*A_Y for every choice of an embedding $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. The proof strategy consists in reversing this observation, by exploiting the functoriality of the embedding categories.

To begin with, we explain how to define the single values $\tilde{A}_S$ for all $S \in \mathscr{S}$.

1.13. Lemma. *Fix $S \in \mathscr{S}$. For every embedding $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, set $A^{(s)}_S := s^*A_Y$. Moreover, for every morphism $p\colon (Y';s') \to (Y;s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, consider the composite isomorphism in $\mathbb{H}(S)$*

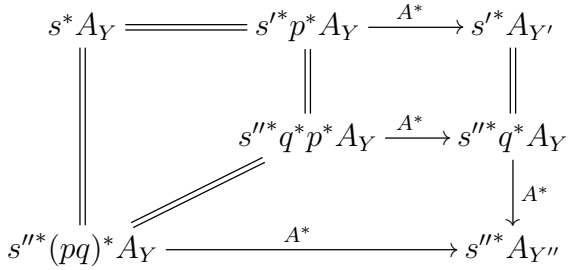
$$\alpha_p\colon A^{(s)}_S := s^*A_Y = s'^*p^*A_Y \xrightarrow{A^*_p} s'^*A_{Y'} =: A^{(s')}_S.$$

Then, the assignment $(Y;s) \mapsto A^{(s)}_S$ defines an $\mathsf{Emb}_{\mathscr{U}}(S)^{op}$-indexed system in $\mathbb{H}(S)$ with transition morphisms α_p.

Proof. We need to show that the assignment $p \mapsto \alpha_p$ is compatible with composition in the embedding category: given a pair of composable morphisms $q\colon (Y'';s'') \to (Y';s')$ and $p\colon (Y';s') \to (Y;s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, we have to check the equality between isomorphisms $A^{(s)}_S \xrightarrow{\sim} A^{(s'')}_S$

$$\alpha_{p\circ q} = \alpha_q \circ \alpha_p.$$

By construction, this amounts to checking that the outer part of the diagram in $\mathbb{H}(S)$



is commutative. But the upper-left piece commutes by axiom ($\mathscr{S}$-fib-1), the lower-right piece commutes by axiom ($\mathscr{U}$-sect), and the upper-right piece commutes by naturality. ∎

Given two embeddings $(Y^{(i)}; s^{(i)}) \in \mathsf{Emb}_{\mathscr{U}}(S)$, $i = 1, 2$, let $s^{(12)} \colon S \to Y^{(1)} \times Y^{(2)}$ denote the resulting morphism (which is a closed immersion, by Assumption 1.2(iii)), and let $p^{(i)} \colon Y^{(1)} \times Y^{(2)} \to Y^{(i)}$, $i = 1, 2$, denote the canonical projections (which are smooth morphisms, by Assumption 1.2(ii)). From Lemma 1.13, we obtain the isomorphism in $\mathbb{H}(S)$

$$\beta_{1,2} \colon A_S^{(s^{(1)})} \xrightarrow{\alpha_{p^{(1)}}} A_S^{(s^{(12)})} \xrightarrow{\alpha_{p^{(2)}}^{-1}} A_S^{(s^{(2)})}. \tag{2}$$

These satisfy the natural cocycle condition for triples of embeddings:

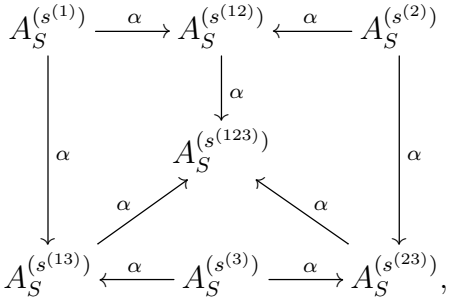
1.14. Corollary. *Suppose that we are given three embeddings* $(Y^{(i)}; s^{(i)}) \in \mathsf{Emb}_{\mathscr{U}}(S)$, $i = 1, 2, 3$, *and consider the isomorphisms in* $\mathbb{H}(S)$

$$\beta_{1,2} \colon A_S^{(s_1)} \xrightarrow{\sim} A_S^{(s_2)}, \quad \beta_{1,3} \colon A_S^{(s_1)} \xrightarrow{\sim} A_S^{(s_3)}, \quad \beta_{2,3} \colon A_S^{(s_2)} \xrightarrow{\sim} A_S^{(s_3)}$$

defined as in (2). *Then, we have the equality*

$$\beta_{1,3} = \beta_{2,3} \circ \beta_{1,2}.$$

Proof. This amounts to checking the commutativity of the outer part of the diagram in $\mathbb{H}(S)$



in which $s^{(123)} \colon S \to Y_1 \times Y_2 \times Y_3$ denotes the induced morphism (which is a closed immersion, by Assumption 1.2(iii)). This follows from the commutativity of the three inner pieces, which comes down to the compatibility of the assignment $p \mapsto \alpha_p$ with composition (given by Lemma 1.13). ∎

As a consequence, the assignment $(Y; s) \mapsto A_S^{(s)}$ defines a $\mathscr{G}_{\mathscr{U}}(S)$-indexed system in $\mathbb{H}(S)$ with invertible transition morphisms of the form (2). Thus, the colimit

$$\tilde{A}_S := \varinjlim_{(Y;s) \in \mathscr{G}_{\mathscr{U}}(S)} A_S^{(s)}$$

exists in $\mathbb{H}(S)$. We prefer to state the results below using this colimit instead of some specific $A_S^{(s)}$, mostly in order to avoid making choices. On the other hand, when studying the properties of $\tilde{A}_S$, we will often replace it with some representative $A_S^{(s)}$, via the canonical isomorphism $A_S^{(s)} \xrightarrow{\sim} \tilde{A}_S$ defined by the universal property of the colimit.

Our next task is to turn the collection of objects $\left\{\tilde{A}_S\right\}_{S\in\mathscr{S}}$ into a section of $\mathbb{H}$ over $\mathscr{S}$, by constructing the isomorphisms $\tilde{A}^*_f\colon f^*\tilde{A}_S \xrightarrow{\sim} \tilde{A}_T$ associated to morphisms $f\colon T\to S$ in $\mathscr{S}$. The idea is to compare some representatives $f^*A^{(s)}_S$ and $A^{(t)}_T$, defined in terms of embeddings $(Y;s)\in\mathsf{Emb}_{\mathscr{U}}(S)$ and $(X,t)\in\mathsf{Emb}_{\mathscr{U}}(T)$. However, this only works if f extends to a morphism $p\colon X\to Y$, which needs not be the case in general. The trick is to start from an object $(Y;s)\in\mathsf{Emb}_{\mathscr{U}}(S)$ and define $(X;t)$ in terms of some factorization $(X;t,p)\in\mathsf{Fact}_{\mathscr{S}}(s\circ f)$. Note that, since the morphism $p\colon X\to Y$ is smooth (by definition), the object X automatically belongs to $\mathscr{U}$ (by Lemma 1.4(2)), and so the pair $(X;t)$ defines indeed an object of $\mathsf{Emb}_{\mathscr{U}}(T)$. This allows us to construct the sought-after isomorphisms, as follows:

1.15. Lemma. *Fix a morphism $f\colon T\to S$ in $\mathscr{S}$. For every embedding $(Y;s)\in\mathsf{Emb}_{\mathscr{U}}(S)$ and every factorization $(X;t,p)\in\mathsf{Fact}_{\mathscr{S}}(s\circ f)$, define the composite isomorphism in $\mathbb{H}(T)$*

$$A^{*,(s,t,p)}_f\colon f^*A^{(s)}_S := f^*s^*A_Y = t^*p^*A_Y \xrightarrow{A^*_p} t^*A_X =: A^{(t)}_T.$$

Then, the induced isomorphism in $\mathbb{H}(T)$

$$\tilde{A}^* = \tilde{A}^*_f\colon f^*\tilde{A}_S = f^*A^{(s)}_S \xrightarrow{A^{*,(s,t,p)}_f} A^{(t)}_T = \tilde{A}_T$$

is independent of the choice of $(Y;s)\in\mathsf{Emb}_{\mathscr{U}}(S)$ and of $(X;t,p)\in\mathsf{Fact}_{\mathscr{S}}(s\circ f)$.

Proof. In view of the way the objects $\tilde{A}_S$ and $\tilde{A}_T$ are defined, as using the connectedness of the embedding categories via products (given by Lemma 1.8), we are reduced to proving the following claim: given a morphism $r\colon (Y';s')\to(Y;s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, a factorization $(X';t',p')\in\mathsf{Fact}_{\mathscr{S}}(s'\circ f)$, and a morphism $q\colon (X';t')\to(X;t)$ in $\mathsf{Emb}_{\mathscr{U}}(T)$ fitting into a commutative diagram of the form

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{r} & Y, \end{array}$$

the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccc} f^*A^{(s)}_S & \xrightarrow{A^{*,(s,t,p)}_f} & A^{(t)}_T \\ \downarrow{\scriptstyle \alpha_r} & & \downarrow{\scriptstyle \alpha_q} \\ f^*A^{(s')}_S & \xrightarrow{A^{*,(s',t',p')}_f} & A^{(t')}_T \end{array}$$

is commutative. Unwinding the various definitions, we obtain the outer part of the diagram

$$\begin{array}{ccccc}
f^*s^*A_Y & = & t^*p^*A_Y & \xrightarrow{A^*} & t^*A_X \\
\| & & \| & & \| \\
 & & t'^*q^*p^*A_Y & \xrightarrow{A^*} & t'^*q^*A_X \\
 & & \| & & \downarrow{\scriptstyle A^*} \\
f^*s'^*r^*A_Y & = & t'^*p'^*r^*A_Y & & \\
\downarrow{\scriptstyle A^*} & & \downarrow{\scriptstyle A^*} & & \\
f^*s'^*A_{Y'} & = & t'^*p^*A_{Y'} & \xrightarrow{A^*} & t'^*A_{X'}.
\end{array}$$

Here, the upper-left piece commutes by axiom ($\mathscr{S}$-fib-1), the lower-right piece commutes by axiom ($\mathscr{U}$-sect), while the other two pieces commute by naturality. This proves the claim. ∎

Proof of Proposition 1.12. In the first place, we claim that the natural isomorphisms $\tilde{A}^*_f$ defined in Lemma 1.15 turn the family $\left\{\tilde{A}_S\right\}_{S\in\mathscr{S}}$ into a section $\tilde{A}$ of the $\mathscr{S}$-fibered category $\mathbb{H}$. This amounts to showing that condition ($\mathscr{S}$-sect) is satisfied: given two composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, we have to check that the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccc}
f^*g^*\tilde{A}_V & \xrightarrow{\tilde{A}^*} & f^*\tilde{A}_S \\
\| & & \downarrow{\scriptstyle \tilde{A}^*} \\
(gf)^*\tilde{A}_V & \xrightarrow{\tilde{A}^*} & \tilde{A}_T
\end{array}$$

is commutative. To this end, choose an embedding $(W; v) \in \mathsf{Emb}_{\mathscr{U}}(V)$, then a factorization $(Y; s, q) \in \mathsf{Fact}_{\mathscr{S}}(v \circ g)$, and then a factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; note that the objects X and Y automatically belong to $\mathscr{U}$ (by Lemma 1.4(2)), so we obtain embeddings $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and $(X; t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. By construction, it now suffices to show that the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccc}
f^*g^*A_V^{(v)} & \xrightarrow{A_g^{*,(v,s,q)}} & f^*A_S^{(s)} \\
\| & & \downarrow{\scriptstyle A_f^{*,(s,t,p)}} \\
(gf)^*A_V^{(v)} & \xrightarrow{A_{gf}^{*,(v,t,qp)}} & A_T^{(t)}
\end{array}$$

is commutative. Unwinding the various definitions, we obtain the outer part of the diagram

$$\begin{array}{ccccc} f^*g^*v^*A_W & = & f^*s^*q^*A_W & \xrightarrow{A^*} & f^*s^*A_Y \\ \| & & \| & & \| \\ & & t^*p^*q^*A_W & \xrightarrow{A^*} & t^*p^*A_Y \\ & & \| & & \downarrow{\scriptstyle A^*} \\ (gf)^*v^*A_W & = & t^*(qp)^*A_W & \xrightarrow{A^*} & t^*A_X. \end{array}$$

Here, the left-most piece commutes by axiom ($\mathscr{S}$-fib-1), the lower-right square commutes by axiom ($\mathscr{U}$-sect), while the upper-right square commutes by naturality. This proves the claim.

Note that the restriction of the section $\tilde{A}$ to $\mathscr{U}$ coincides with the original section A (by construction). We now claim that this property characterizes $\tilde{A}$ up to unique isomorphism. To see this, let A' be another section of $\mathbb{H}$ over $\mathscr{S}$ extending A. For every $S \in \mathscr{S}$, consider the isomorphism in $\mathbb{H}(S)$

$$w_S \colon \tilde{A}_S := \varinjlim_{(Y;s)\in\mathscr{G}_{\mathscr{U}}(S)} \tilde{A}^{(s)}_S := \varinjlim_{(Y;s)\in\mathscr{G}_{\mathscr{U}}(S)} s^*A_Y = \varinjlim_{(Y,s)\in\mathscr{G}(S)} s^*A'_Y \xrightarrow{\varinjlim A'^*_s} A'_S.$$

Note that, for every embedding $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, the isomorphism w_S coincides with the composite

$$\tilde{A}_S = A^{(s)}_S := s^*A_Y = s^*A'_Y \xrightarrow{A'^*_s} A'_S.$$

Let us show that these isomorphisms satisfy condition (mor-$\mathscr{S}$-sect): for every morphism $f \colon T \to S$ in $\mathscr{S}$, we have to check that the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccc} f^*\tilde{A}_S & \xrightarrow{w_S} & f^*A'_S \\ \downarrow{\scriptstyle \tilde{A}^*} & & \downarrow{\scriptstyle A'^*} \\ \tilde{A}_T & \xrightarrow{w_T} & A'_T \end{array}$$

is commutative. To this end, choose an embedding $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and then a factorization $(X;t,p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$. By construction, it now suffices to show that the outer part of the diagram in $\mathbb{H}(T)$

$$\begin{array}{ccccc} f^*s^*A_Y & = & f^*s^*A'_Y & \xrightarrow{A'^*} & f^*A'_S \\ \| & & \| & & \downarrow{\scriptstyle A'^*} \\ t^*p^*A_Y & = & t^*p^*A'_Y & \xrightarrow{A'^*} t^*A'_X \xrightarrow{A'^*} & A'_T \end{array}$$

is commutative. But the left-most square clearly commutes, while the right-most rectangle commutes by axiom ($\mathscr{S}$-sect) applied to A'. Thus, we obtain an isomorphism of sections

$w\colon \tilde{A} \xrightarrow{\sim} A'$ of $\mathbb{H}$ over $\mathscr{S}$; by construction, this extends the identity automorphism of A and, in fact, it is uniquely determined by this property.

Lastly, the functoriality of the assignment $A \mapsto \tilde{A}$ follows from the fact that every passage in our construction via Lemmata 1.13 and 1.15 is visibly functorial. This concludes the proof. ∎

2. Extending morphisms of fibered categories

Starting from the present section, we need to introduce further hypotheses on the base category $\mathscr{S}$, and we restrict our attention to a certain class of triangulated $\mathscr{S}$-fibered categories. After introducing the new setting, we prove our first main extension result, for suitable morphisms between such $\mathscr{S}$-fibered categories. Moreover, we check its compatibility with the extension result for sections obtained previously.

2.1. Geometric fibered categories. In the first place, we need to strengthen the working hypotheses on the base category $\mathscr{S}$. From now on, we will work under the following assumption, which is an improved version of [Ter24b, Assum. 3.1]:

2.2. Assumption. *We keep Assumption 1.2, and we let $\mathscr{U}$ be as in Notation 1.3. In addition, we assume that:*

(i) The category $\mathscr{S}$ has an initial object $\emptyset$.

(ii) For every closed immersion $z\colon Z \to S$, there exists a distinguished smooth morphism $u\colon U \to S$ with the property that, for every $T \in \mathscr{S}$, the map

$$u \circ -\colon \mathrm{Hom}_{\mathscr{S}}(T, U) \to \mathrm{Hom}_{\mathscr{S}}(T, S)$$

identifies $\mathrm{Hom}_{\mathscr{S}}(T, U)$ with the subset of $\mathrm{Hom}_{\mathscr{S}}(T, S)$ consisting of all those morphisms $f\colon T \to S$ for which the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{z} & S \end{array}$$

is Cartesian. We call u the open immersion *complementary to z, and we call U the* open complement *of Z in S.*

(iii) For every $S \in \mathscr{S}$, there exists a finite sequence of closed immersions

$$\emptyset = Z_{-1} \to Z_0 \to Z_1 \to \cdots \to Z_n = S$$

such that, for each $r = 0, \ldots, n$, the open complement of Z_{r-1} in Z_r belongs to $\mathscr{U}$.

2.3. Example. As in Example 1.6, the assumption is tailored to the setting where $\mathscr{S} = \mathsf{Var}_k$ is the category of quasi-projective algebraic varieties over a field k, with $\mathscr{U} = \mathsf{Sm}_k$ the full subcategory of smooth quasi-projective k-varieties; in this setting, what we call open immersions in (ii) are just the usual open immersions between k-varieties. Note that, in order for (iii) to be satisfied, one needs to know that every $S \in \mathsf{Var}_k$ admits a smooth dense open subvariety: this is true if and only if k is perfect (see [Stacks, Tag 0B8X]).

In the rest of the paper we work in the setting of triangulated $\mathscr{S}$-fibered categories, as axiomatized in [Ter24a, §§ 4-6]. More precisely, under the new hypotheses, it is natural to work with the class of localic $\mathscr{S}$-fibered categories, introduced in [Ter24b, § 3]. A triangulated $\mathscr{S}$-fibered category is called *localic* if it satisfies the following conditions:

(a) $\mathbb{H}(\emptyset)$ is the zero triangulated category.

(b) For every smooth morphism $p\colon P \to S$, the functor $p^*\colon \mathbb{H}(S) \to \mathbb{H}(P)$ has a left-adjoint $p_\#\colon \mathbb{H}(P) \to \mathbb{H}(S)$. Moreover, for every Cartesian square in $\mathscr{S}$

$$\begin{array}{ccc} P_T & \xrightarrow{f'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

with p a smooth morphism, the Beck–Chevalley natural transformation of functors $\mathbb{H}(P) \to \mathbb{H}(T)$

$$p'_\# f'^* A \xrightarrow{\eta} p'_\# f'^* p^* p_\# A = p'_\# p'^* f^* p_\# A \xrightarrow{\epsilon} f^* p_\# A$$

is invertible.

(c) For every closed immersion $z\colon Z \to S$ in $\mathscr{S}$, the functor $z^*\colon \mathbb{H}(S) \to \mathbb{H}(Z)$ has a right-adjoint $z_*\colon \mathbb{H}(Z) \to \mathbb{H}(S)$, which moreover is fully faithful.

(d) For every closed immersion $z\colon Z \to S$ in $\mathscr{S}$ with complementary open immersion $u\colon U \to S$, the pair of functors (z^*, u^*) is jointly conservative: that is, a morphism is $\mathbb{H}(S)$ is an isomorphism if and only if so are its images under z^* and u^*.

As explained in [Ter24b, Lem. 3.5], in a localic triangulated $\mathscr{S}$-fibered category $\mathbb{H}$ one can form the so-called localization triangle associated to a closed immersion $z\colon Z \to S$ with complementary open immersion $u\colon U \to S$. More precisely, the following hold:

1. The functor $u_\#\colon \mathbb{H}(U) \to \mathbb{H}(S)$ is fully faithful.
2. The composite functors $z^* \circ u_\#$ and $u^* \circ z_*$ vanish.
3. There exists a canonical *localization distinguished triangle* of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$u_\# u^* A \xrightarrow{\epsilon} A \xrightarrow{\eta} z_* z^* A \to u_\# u^* A[1].$$

4. For every Cartesian square in $\mathscr{S}$ of the form

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p_Z} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p a smooth morphism and z a closed immersion, the Beck–Chevalley natural transformation of functors $\mathbb{H}(Z) \to \mathbb{H}(P)$

$$p^* z_* A \xrightarrow{\eta} z'_* z'^* p^* z_* A = z'_* p_Z^* z^* z_* A \xrightarrow{\epsilon} z'_* p_Z^* A$$

is invertible.

To prove these properties, the existence of the left-adjoint functor $p_\#$ is only used when the smooth morphism p is an open immersion; as the reader can check, the same applies to the results of the present paper. Note that any localic $\mathscr{S}$-fibered category defines by restriction a localic $\mathscr{U}$-fibered category (see Remark 1.5).

It turns out that, in order to make our constructions work, we need slightly stronger assumptions on the $\mathscr{S}$-fibered categories involved:

2.4. Definition. *We say that a triangulated $\mathscr{S}$-fibered category $\mathbb{H}$ is* geometric *if it is localic and, for every smooth morphism $p\colon X \to Y$ in $\mathscr{U}$, the functor $p^*\colon \mathbb{H}(Y) \to \mathbb{H}(X)$ admits a right-adjoint $p_*\colon \mathbb{H}(X) \to \mathbb{H}(Y)$.*

Note that the direct image functors p_* in the definition are not required to satisfy any base-change properties. Clearly, the localic $\mathscr{U}$-fibered category underlying a geometric $\mathscr{S}$-fibered category is still geometric (see again Remark 1.5).

2.5. Example. When $\mathscr{S} = \mathsf{Var}_k$ is the category of quasi-projective varieties over a perfect field k (as in Example 1.6 above), the main examples of geometric Var_k-fibered categories come from the theory of stable homotopy 2-functors, developed in [Ayo07a, Ayo07b], which axiomatizes the properties of six-functor formalisms: indeed, the underlying triangulated Var_k-fibered category of any stable homotopy 2-functor is geometric by definition; in this case, every inverse image functor (not only those along smooth morphisms in $\mathscr{U}$) admits a right-adjoint.

Let now $\mathbb{H}_1$ and $\mathbb{H}_2$ be two $\mathscr{S}$-fibered categories. Following [Ter24a, § 3], by a *morphism* of $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ we mean the datum of

- for every $S \in \mathscr{S}$, a functor

$$R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S),$$

called the *R-realization functor* over S,

- for every morphism $f\colon T \to S$ in $\mathscr{S}$, a natural isomorphism of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$
$$\theta = \theta_f\colon f^* R_S(A) \xrightarrow{\sim} R_T(f^* A),$$
called the *R-transition isomorphism* along f

such that the following condition is satisfied:

(mor-$\mathscr{S}$-fib) For every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}_1(V) \to \mathbb{H}_2(T)$
$$\begin{array}{ccccc} (gf)^* R_V(A) & & \xrightarrow{\qquad\qquad\theta_{gf}\qquad\qquad} & & R_T((gf)^* A) \\ \downarrow{\scriptstyle \mathsf{conn}^{(2)}_{f,g}} & & & & \downarrow{\scriptstyle \mathsf{conn}^{(1)}_{f,g}} \\ f^* g^* R_V(A) & \xrightarrow{\theta_g} & f^* R_S(g^* A) & \xrightarrow{\theta_f} & R_T(f^* g^* A) \end{array}$$
is commutative.

In the triangulated setting, it only makes sense to consider *morphisms of triangulated $\mathscr{S}$-fibered categories*, as axiomatized in [Ter24a, § 6]: in this case, the functors $R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)$ are triangulated, and the R-transition isomorphisms θ_f are required to be compatible with shifts. In the localic setting, it is natural to further restrict our attention to *localic* morphisms: namely, those morphisms of triangulated $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ such that, for every smooth morphism $p\colon P \to S$ in $\mathscr{S}$, the composite natural transformation of functors $\mathbb{H}_1(P) \to \mathbb{H}_2(S)$
$$p_\# R_P(A) \xrightarrow{\eta} p_\# R_P(p^* p_\# A) \xleftarrow{\theta_p} p_\# p^* R_S(p_\# A) \xrightarrow{\epsilon} R_S(p_\# A)$$
is invertible. No further conditions are needed in the case of geometric $\mathscr{S}$-fibered categories: in particular, we do not require the realization functors to commute with the direct image functors $p_*\colon \mathbb{H}(X) \to \mathbb{H}(Y)$ along smooth morphisms $p\colon X \to Y$ in $\mathscr{U}$ – in fact, this usually does not happen for morphisms of stable homotopy 2-functors (see Example 2.5). Note that any morphism of triangulated $\mathscr{S}$-fibered categories determines by restriction a morphism of triangulated $\mathscr{U}$-fibered categories, and the restriction of a localic morphism is localic as well (compare with Remark 1.5).

Given two morphisms of $\mathscr{S}$-fibered categories $R, R'\colon \mathbb{H}_1 \to \mathbb{H}_2$, there is a natural notion of 2-*morphism* $\gamma\colon R \to R'$. Namely, this is the datum of

- for every $S \in \mathscr{S}$, a natural transformation of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$
$$\gamma_S\colon R_S(A) \to R'_S(A)$$

satisfying the following condition:

(2-mor-$\mathscr{S}$-fib) For every morphism $f\colon T \to S$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\begin{array}{ccc} f^*R_S(C) & \xrightarrow{\gamma_S} & f^*R'_S(C) \\ \downarrow{\scriptstyle \theta_f^{(1)}} & & \downarrow{\scriptstyle \theta_f^{(2)}} \\ R_T(f^*C) & \xrightarrow{\gamma_T} & R'_T(f^*C) \end{array}$$

is commutative.

A 2-morphism of $\mathscr{S}$-fibered categories γ is called a 2-*isomorphism* if all natural transformations γ_S are invertible; in this case, their inverses define a 2-morphism as well. In the triangulated setting, it only makes sense to consider *triangulated* 2-*morphisms*: this amounts to requiring that the natural transformation γ_S be compatible with shifts in the obvious way.

In the present paper, we use the notion of 2-morphism only in order to state our extension result for morphisms of $\mathscr{S}$-fibered categories in a precise form; to this end, we only need to consider 2-isomorphisms. Note that any morphism between triangulated $\mathscr{S}$-fibered categories which is 2-isomorphic to a localic morphism is itself localic.

2.6. Extension of localic morphisms. From now until the end of the present section, we fix two triangulated $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$, which we assume to be geometric in the sense of Definition 2.4 (thus, in particular, localic).

For localic morphisms between geometric $\mathscr{S}$-fibered categories, the restriction from $\mathscr{S}$ to $\mathscr{U}$ is reversible in a canonical way. Here is the precise statement:

2.7. Theorem. *Let $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a localic morphism between the underlying geometric triangulated $\mathscr{U}$-fibered categories. Then, there exists a unique (up to unique 2-isomorphism) morphism of triangulated $\mathscr{S}$-fibered categories $\tilde{R}\colon \mathbb{H}_1 \to \mathbb{H}_2$ whose restriction to $\mathscr{U}$ coincides with R, and it is automatically localic.*

The result applies, in particular, to morphisms of stable homotopy 2-functors, as axiomatized in [Ayo10] (compare with Example 2.5).

Most of the present section is devoted to the proof of Theorem 2.7. In the course of the proof, we construct the extension $\tilde{R}$ canonically from R; this makes the assignment $R \mapsto \tilde{R}$ compatible with 2-morphisms. Similarly to the case of Proposition 1.12, the general idea is that, if the conclusion of the theorem holds, then the functor $\tilde{R}_S$ must be naturally isomorphic to the composite $s^* \circ R_Y \circ s_*$ for every choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Reversing this observation, we are able to define the single functors $\tilde{R}_S$ for all $S \in \mathscr{S}$.

2.8. Lemma. *Fix an object $S \in \mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the triangulated functor*

$$R_S^{(s)}\colon \mathbb{H}_1(S) \xrightarrow{s_*} \mathbb{H}_1(Y) \xrightarrow{R_Y} \mathbb{H}_2(Y) \xrightarrow{s^*} \mathbb{H}_2(S).$$

Moreover, for every morphism $p\colon (Y';s') \to (Y;s)$ *in* $\mathsf{Emb}_{\mathscr{U}}(S)$*, consider the composite natural transformation of functors* $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{lccccc} \lambda_p\colon & R_S^{(s)}(A) & & & & R_S^{(s')}(A). \\ & \| & & & & \| \\ & s^*R_Y(s_*A) & = & s'^*p^*R_Y(p_*s'_*A) & \xrightarrow{\theta_p} s'^*R_{Y'}(p^*p_*s'_*A) \xrightarrow{\epsilon} & s'^*R_Y(s'_*A) \end{array}$$

Then, each natural transformation λ_p *is invertible, and the assignment* $(Y;s) \mapsto R_S^{(s)}$ *defines an* $\mathsf{Emb}_{\mathscr{U}}(\mathscr{S})^{op}$*-indexed system of triangulated functors* $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$ *with triangulated transition* 2*-isomorphisms* λ_p*.*

Proof. To show that the natural transformation λ_p in the statement is invertible, it suffices to check that so is the co-unit arrow $\epsilon\colon s'^*R_{Y'}(p^*p_*s'_*A) \to s'^*R_Y(s'_*A)$. To this end, let $u\colon U \to Y'$ denote the open immersion complementary to s'. Using the associated localization triangle, it suffices to show that the two co-unit arrows

$$\epsilon\colon s'^*R_{Y'}(s'_*s'^*p^*p_*s'_*A) \to s'^*R_{Y'}(s'_*s'^*s'_*A)$$

and

$$\epsilon\colon s'^*R_{Y'}(u_\#u^*p^*p_*s'_*A) \to s'^*R_{Y'}(u_\#u^*s'_*A)$$

are both invertible. For the first one, this follows from the commutative diagram

$$\begin{array}{ccc} s'^*R_{Y'}(s'_*s'^*p^*p_*s'_*A) & \xrightarrow{\epsilon} & s'^*R_{Y'}(s'_*s'^*s'_*A) \\ \| & & \downarrow{\scriptstyle \epsilon} \\ s'^*R_{Y'}(s'_*s^*s_*A) & \xrightarrow{\epsilon} & s'^*R_{Y'}(s'_*A), \end{array}$$

in which the other two co-unit arrows are invertible because s_* and s'_* are both fully faithful (as $\mathbb{H}_1$ is localic). For the second one, this follows from the fact that both terms are zero: indeed, we have

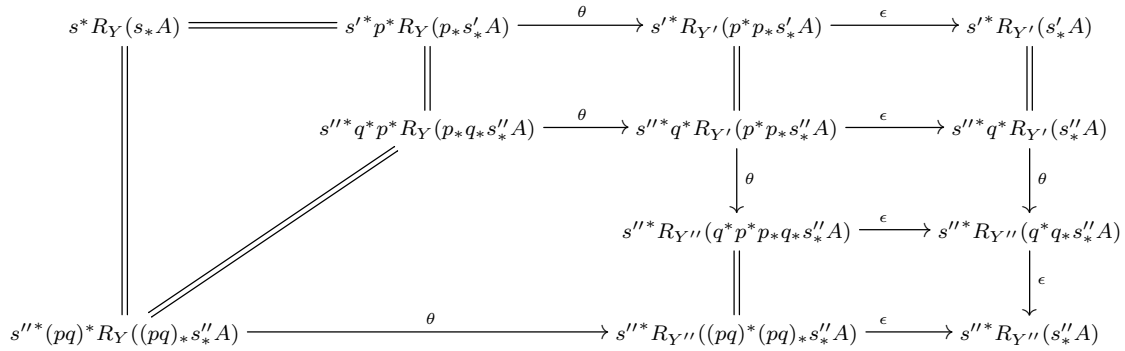
$$s'^* \circ R_{Y'} \circ u_\# = s'^*u_\# \circ R_U = 0,$$

where the first passage witnesses the isomorphism $u_\# \circ R_U \xrightarrow{\sim} R_{Y'} \circ u_\#$ (as R is a localic morphism) and the second passage follows from the vanishing $(s')^* \circ u_\# = 0$ (as $\mathbb{H}_2$ is localic).

To conclude, with have to show that the assignment $p \mapsto \lambda_p$ is compatible with composition in the embedding category: given a pair of composable morphisms $q\colon (Y'';s'') \to (Y';s')$ and $p\colon (Y';s') \to (Y;s)$ in $\mathsf{Emb}_{\mathscr{U}}(\mathscr{S})$, we have to check the equality between natural isomorphisms of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\lambda_{p\circ q} = \lambda_q \circ \lambda_p.$$

By construction, this amounts to showing that the outer part of the diagram



is commutative. But the left-most piece commutes by axiom ($\mathscr{S}$-fib-1), the lower-central piece commutes by axiom (mor-$\mathscr{U}$-fib), while all the other pieces commute by naturality or by construction. This proves the claim. ∎

2.9. Remark. Note that the construction of the natural isomorphism λ_p exploits the existence of the direct image functor $p_*\colon \mathbb{H}_1(Y') \to \mathbb{H}_1(Y)$: this is where we need to use the assumption that $\mathbb{H}_1$ is geometric. On the other hand, the assumption that $\mathbb{H}_2$ is geometric actually plays no role. The same applies to the other extension results collected below.

Given two embeddings $(Y^{(i)}; s^{(i)}) \in \mathsf{Emb}_{\mathscr{U}}(S)$, $i = 1, 2$, let $s^{(12)}\colon S \to Y^{(1)} \times Y^{(2)}$ denote the resulting morphism (which is a closed immersion, by Assumption 1.2(iii)), and let $p^{(i)}\colon Y^{(1)} \times Y^{(2)} \to Y^{(i)}$, $i = 1, 2$, denote the canonical projections (which are smooth morphisms, by Assumption 1.2(ii)). From Lemma 2.8, we obtain the natural isomorphism of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\mu_{1,2}\colon R_S^{(s^{(1)})}(A) \xrightarrow{\lambda_{p^{(1)}}} R_S^{(s^{(12)})}(A) \xrightarrow{\lambda^{-1}_{p^{(2)}}} R_S^{(s^{(2)})}(A). \tag{3}$$

These satisfy the natural cocycle condition for triples of embeddings:

2.10. Corollary. *Suppose that we are given three embeddings* $(Y^{(i)}; s^{(i)}) \in \mathsf{Emb}_{\mathscr{U}}(S)$, $i = 1, 2, 3$, *and consider the natural isomorphisms of functors* $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\mu_{1,2}\colon R_S^{(s^{(1)})} \xrightarrow{\sim} R_S^{(s^{(2)})}, \quad \mu_{1,3}\colon R_S^{(s^{(1)})} \xrightarrow{\sim} R_S^{(s^{(3)})}, \quad \mu_{2,3}\colon R_S^{(s^{(2)})} \xrightarrow{\sim} R_S^{(s^{(3)})}$$

defined as in (3). *Then, we have the equality*

$$\mu_{1,3} = \mu_{2,3} \circ \mu_{1,2}.$$

PROOF. This amounts to checking the commutativity of the outer part of the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccccc}
R_S^{(s^{(1)})}(A) & \xrightarrow{\lambda} & R_S^{(s^{(12)})}(A) & \xleftarrow{\lambda} & R_S^{(s^{(2)})}(A) \\
\downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\
 & \nearrow{\scriptstyle\lambda} & R_S^{(s^{(123)})}(A) & \nwarrow{\scriptstyle\lambda} & \\
R_S^{(s^{(13)})}(A) & \xleftarrow{\lambda} & R_S^{(s^{(3)})}(A) & \xrightarrow{\lambda} & R_S^{(s^{(23)})}(A),
\end{array}$$

in which $s^{(123)}\colon S \to Y_1 \times Y_2 \times Y_3$ denotes the induced morphism (which is a closed immersion, by Assumption 1.2(iii)). This follows from the commutativity of the three inner pieces, which comes down to the compatibility of the assignment $p \mapsto \lambda_p$ with composition (given by Lemma 2.8). ∎

As a consequence, the assignment $(Y; s) \mapsto R_S^{(s)}$ defines a $\mathscr{G}_{\mathscr{U}}(S)$-indexed system of triangulated functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$ with invertible triangulated transition natural transformations of the form (3). Thus, the formula

$$\tilde{R}_S(A) := \varinjlim_{(Y,s)\in\mathscr{G}(S)} R_S^{(s)}(A)$$

defines a triangulated functor $\tilde{R}_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)$, endowed with canonical triangulated natural isomorphisms $R_S^{(s)} \xrightarrow{\sim} \tilde{R}_S$; the natural isomorphism witnessing the compatiblity of $\tilde{R}_S$ with shifts is induced by the analogous natural isomorphisms for each $R_S^{(s)}$, and $\tilde{R}_S$ preserves distinguished triangles because so does each $R_S^{(s)}$.

We want to turn the collection of triangulated functors $\mathscr{R} = \left\{\tilde{R}_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\right\}_{S\in\mathscr{S}}$ just obtained into a morphism of triangulated $\mathscr{S}$-fibered categories $\tilde{R}\colon \mathbb{H}_1 \to \mathbb{H}_2$. To this end, as done for Lemma 1.15, we use the embedding categories in combination with the factorization categories.

2.11. LEMMA. *Fix a morphism $f\colon T \to S$ in $\mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and every factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$, consider the composite natural transformation of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$*

$$\begin{array}{lccccccc}
\theta_f^{(s,t,p)}\colon & f^*R_S^{(s)}(A) & & & & & & R_T^{(t)}(f^*A). \\
 & \| & & & & & & \| \\
 & f^*s^*R_Y(s_*A) & & & & & & t^*R_X(t_*f^*A) \\
 & \| & & & & & & \uparrow{\scriptstyle\epsilon} \\
 & t^*p^*R_Y(s_*A) & \xrightarrow{\theta_p} & t^*R_X(p^*s_*A) & \xrightarrow{\eta} & t^*R_X(t_*t^*p^*s_*A) & = & t^*R_X(t_*f^*s^*s_*A)
\end{array}$$

Then $\theta_f^{(s,t,p)}$ *is invertible, and the induced natural isomorphism of functors* $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\tilde{\theta} = \tilde{\theta}_f \colon f^* \tilde{R}_S(A) = f^* R_S^{(s)}(A) \xrightarrow{\theta_f^{(s,t,p)}} R_T^{(t)}(f^* A) = \tilde{R}_T(f^* A)$$

is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ *and of* $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$.

Proof. We start by showing the invertibility of $\theta_f^{(s,t,p)}$. By construction, this is equivalent to showing the invertibility of the unit arrow $\eta \colon t^* R_X(A) \to t^* R_X(t_* t^* A)$. To this end, let $u \colon U \to X$ denote the open immersion complementary to $t \colon T \to X$. Using the associated localization triangle, we see that the invertibility of the above unit arrow is equivalent to the vanishing of the composite functor $t^* \circ R_X \circ u_\#$. But we have

$$t^* \circ R_X \circ u_\# = t^* u_\# \circ R_U = 0,$$

where the first passage witnesses the natural isomorphism $u_\# \circ R_U \xrightarrow{\sim} R_X \circ u_\#$ (as R is a localic morphism) and the second one follows from the vanishing $t^* \circ u_\# = 0$ (as $\mathbb{H}_2$ is localic).

Next, we show that the resulting natural isomorphism $\tilde{\theta}_f$ in the statement is indeed independent of the choice of $(Y; s)$ and of $(X; t, p)$. In view of the way the functors $\tilde{R}_S$ and $\tilde{R}_T$ are defined, and using the connectedness of the embedding categories via products (given by Lemma 1.8), we are reduced to proving the following claim: given morphisms $r \colon (Y'; s') \to (Y; s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$ and $q \colon (X'; t') \to (X; t)$ in $\mathsf{Emb}_{\mathscr{U}}(T)$ fitting into a commutative diagram of the form

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{r} & Y, \end{array}$$

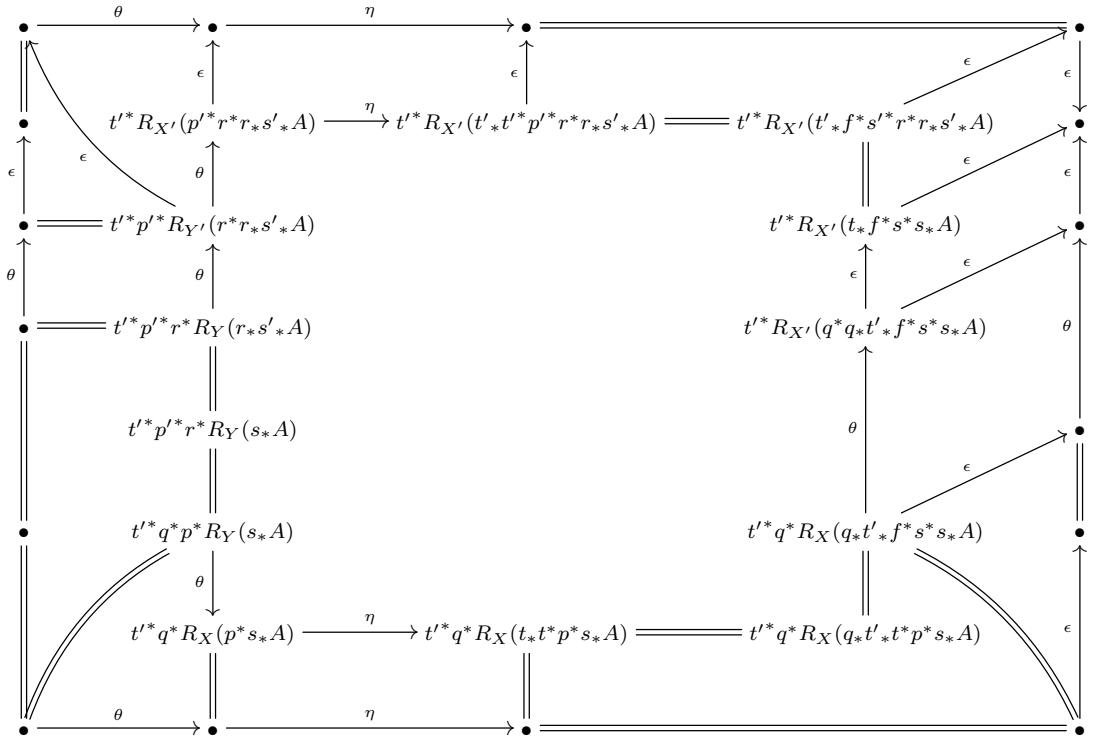
the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\begin{array}{ccc} f^* R_S^{(s')}(A) & \xrightarrow{\theta_f^{(s',t',p')}} & R_T^{(t')}(f^* A) \\ {\scriptstyle \lambda_r}\uparrow & & \uparrow{\scriptstyle \lambda_q} \\ f^* R_S^{(s)}(A) & \xrightarrow{\theta_f^{(s,t,p)}} & R_T^{(t)}(f^* A) \end{array}$$

is commutative. Expanding the various definitions, we obtain the more explicit diagram

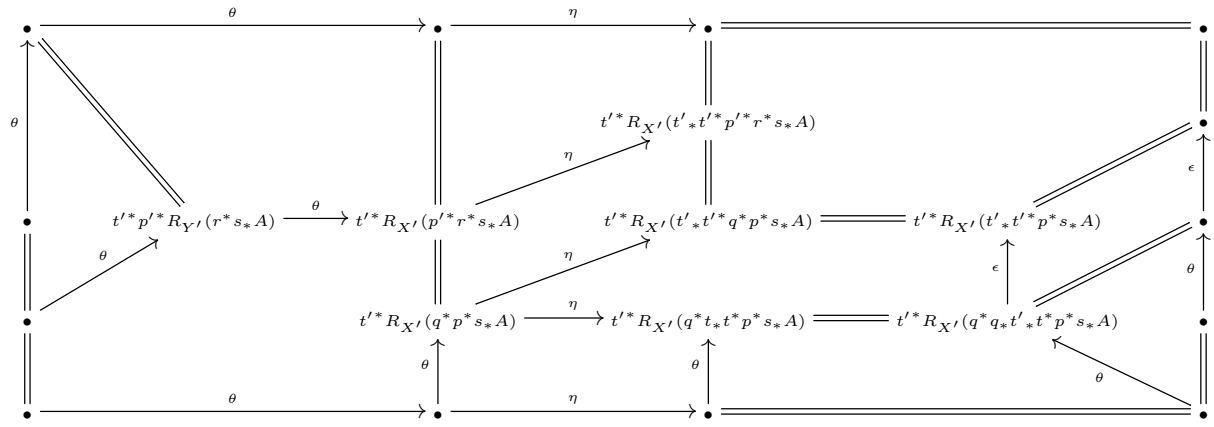
$$\begin{array}{ccccccc}
t'^*p'^*R_{Y'}(s'_*A) & \xrightarrow{\theta} & t'^*R_{X'}(p'^*s'_*A) & \xrightarrow{\eta} & t'^*R_{X'}(t'_*t'^*p'^*s'_*A) & = & t'^*R_{X'}(t'_*f^*s'^*s'_*A) \\
\| & & & & & & \downarrow{\scriptstyle\epsilon} \\
f^*s'^*R_{Y'}(s'_*A) & & & & & & t'^*R_{X'}(t'_*f^*A) \\
\uparrow{\scriptstyle\epsilon} & & & & & & \uparrow{\scriptstyle\epsilon} \\
f^*s'^*R_Y(r^*r_*s'_*A) & & & & & & t'^*R_{X'}(q^*q_*t'_*f^*A) \\
\uparrow{\scriptstyle\theta} & & & & & & \uparrow{\scriptstyle\theta} \\
f^*s'^*r^*R_Y(r_*s'_*A) & & & & & & t'^*q^*R_X(q_*t'_*f^*A) \\
\| & & & & & & \| \\
f^*s^*R_Y(s_*A) & & & & & & t^*R_X(t_*f^*A) \\
\| & & & & & & \uparrow{\scriptstyle\epsilon} \\
t^*p^*R_Y(s_*A) & \xrightarrow{\theta} & t^*R_X(p^*s_*A) & \xrightarrow{\eta} & t^*R_X(t_*t^*p^*s_*A) & = & t^*R_X(t_*f^*s^*s_*A),
\end{array}$$

which we decompose as



Here, the lower-left and lower-right boundary pieces commute by axiom ($\mathscr{S}$-fib-1), while all the other boundary pieces commute by naturality and by construction. Hence, we are

reduced to considering the central part of the diagram, which we further decompose as



Here, the upper-right piece commutes by axiom ($\mathscr{S}$-fib-1), the lower-left piece commutes by axiom (mor-$\mathscr{U}$-fib), while all the other pieces commute by naturality and by construction. This proves the claim. ∎

Proof of Theorem 2.7. In the first place, we show that the natural isomorphisms $\tilde{\theta}_f$ constructed in Lemma 2.11 turn the triangulated functors $\tilde{R}_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)$ into a morphism of $\mathscr{S}$-fibered categories $\tilde{R}\colon \mathbb{H}_1 \to \mathbb{H}_2$. For this, we only need to show that they satisfy condition (mor-$\mathscr{S}$-fib): given two composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, we have to check that the diagram of functors $\mathbb{H}_1(V) \to \mathbb{H}_2(T)$

$$\begin{array}{ccc} (gf)^*\tilde{R}_V(A) & \xrightarrow{\quad\tilde{\theta}\quad} & \tilde{R}_T((gf)^*A) \\ \| & & \| \\ f^*g^*\tilde{R}_V(A) & \xrightarrow{\tilde{\theta}} f^*\tilde{R}_T(g^*A) \xrightarrow{\tilde{\theta}} & \tilde{R}_S(f^*g^*A) \end{array}$$

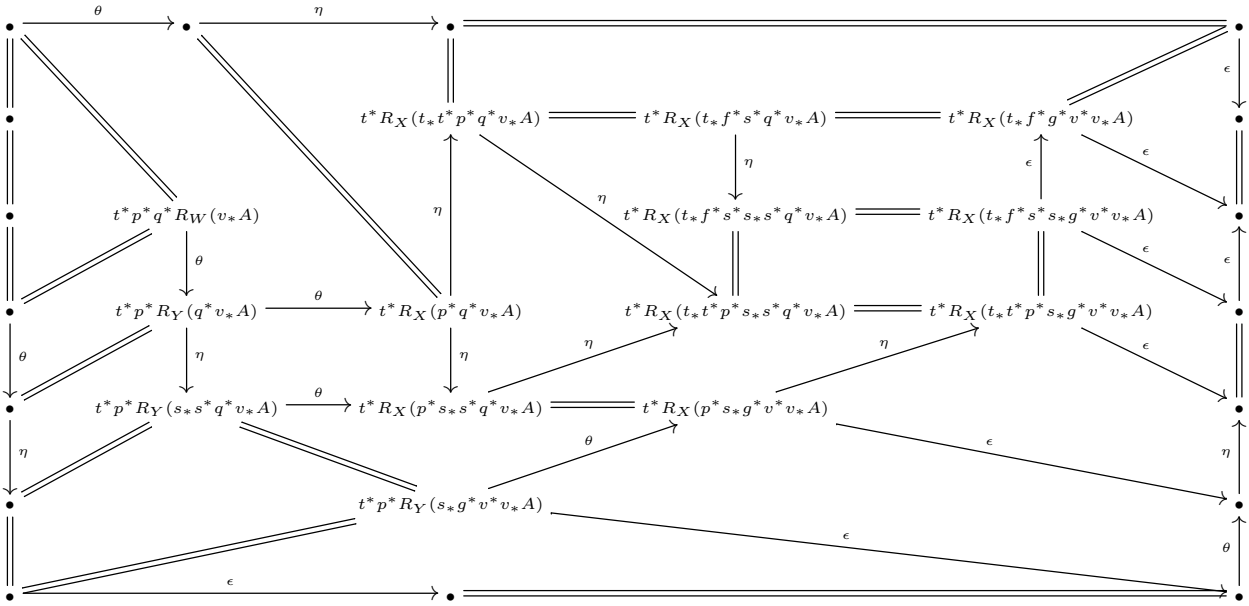
is commutative. To this end, choose an embedding $(W; v) \in \mathsf{Emb}_{\mathscr{U}}(V)$, then a factorization $(Y; s, q) \in \mathsf{Fact}_{\mathscr{S}}(v \circ g)$, and then a factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; note that the objects X and Y automatically belong to $\mathscr{U}$ (by Lemma 1.4(2)), so we obtain embeddings $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and $(X; t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. By construction, it now suffices to show that the diagram

$$\begin{array}{ccc} (gf)^*R_V(A) & \xrightarrow{\quad\theta_{gf}^{(v,t,qp)}\quad} & R_T((gf)^*A) \\ \| & & \| \\ f^*g^*R_V(A) & \xrightarrow{\theta_g^{(v,s,q)}} f^*R_T(g^*A) \xrightarrow{\theta_f^{(s,t,p)}} & R_S(f^*g^*A) \end{array}$$

is commutative. Unwinding the various definition, we obtain the more explicit diagram

$$
\begin{array}{ccccccc}
t^*(qp)^*R_W(v_*A) & \xrightarrow{\theta} & t^*R_X((qp)^*v_*A) & \xrightarrow{\eta} & t^*R_X(t_*t^*(qp)^*v_*A) & = & t^*R_X(t_*(gf)^*v^*v_*A) \\
\| & & & & & & \downarrow{\scriptstyle\epsilon} \\
(gf)^*v^*R_W(v_*A) & & & & & & t^*R_X(t_*(gf)^*A) \\
\| & & & & & & \| \\
f^*g^*v^*R_W(v_*A) & & & & & & t^*R_X(t_*f^*g^*A) \\
\| & & & & & & \uparrow{\scriptstyle\epsilon} \\
f^*s^*q^*R_W(v_*A) & & & & & & t^*R_X(t_*f^*s^*s_*g^*A) \\
\downarrow{\scriptstyle\theta} & & & & & & \| \\
f^*s^*R_Y(q^*v_*A) & & & & & & t^*R_X(t_*t^*p^*s_*g^*A) \\
\downarrow{\scriptstyle\eta} & & & & & & \uparrow{\scriptstyle\eta} \\
f^*s_*R_Y(s_*s^*q^*v_*A) & & & & & & t^*R_X(p^*s_*g^*A) \\
\| & & & & & & \uparrow{\scriptstyle\theta} \\
f^*s^*R_Y(s_*g^*v^*v_*A) & & \xrightarrow{\epsilon} & & f^*s^*R_Y(s_*g^*A) & = & t^*p^*R_Y(s_*g^*A),
\end{array}
$$

which we decompose as



Here, the upper-left piece commutes by axiom (mor-$\mathscr{U}$-fib), the upper-right piece commutes by axiom ($\mathscr{S}$-fib-1), while all the other pieces commute by naturality and by construction. This proves the claim.

Arguing in the same way, one checks that the natural isomorphisms witnessing the compatibility of the triangulated functors $\tilde{R}_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)$ with shifts are coherent with respect to inverse images (as in [Ter24a, Defn. 6.2]). In conclusion, we obtain a morphism of triangulated $\mathscr{S}$-fibered categories as in the statement. As a consequence of the explicit definition of $\tilde{R}$, it is clear that its restriction to $\mathscr{U}$ coincides with R.

We now check that the extended morphism $\tilde{R}\colon \mathbb{H}_1 \to \mathbb{H}_2$ is localic: given a smooth morphism $p\colon P \to S$ in $\mathscr{S}$, we have to show that the natural transformation of functors $\mathbb{H}_1(P) \to \mathbb{H}_2(S)$

$$p_\# \tilde{R}_P(A) \xrightarrow{\eta} p_\# \tilde{R}_P(p^* p_\# A) \xleftarrow{\theta_p} p_\# p^* \tilde{R}_S(p_\# A) \xrightarrow{\epsilon} \tilde{R}_S(p_\# A)$$

is invertible. We argue by induction on the minimal length $n \geq 0$ of a sequence of closed immersions of the form

$$\emptyset = Z_{-1} \to Z_0 \to Z_1 \to \cdots \to Z_n = S$$

as in Assumption 2.2(iii). The base step is when $n = 0$, in which case we have $S \in \mathscr{U}$. But then P belongs to $\mathscr{U}$ as well (by Lemma 1.4(2)), so the claim true by assumption (as R is a localic morphism). For the inductive step, assume that $n \geq 1$ and that the claim is known to hold for Z_{n-1}. For notational simplicity, write Z_{n-1} simply as Z, and write $z\colon Z \to S$ for the corresponding closed immersion; moreover, let $u\colon U \to S$ denote the complementary open immersion. Consider the commutative diagram of functors $\mathbb{H}_1(P) \to \mathbb{H}_2(S)$

$$\begin{array}{ccccccc}
u_\# u^* p_\# \tilde{R}_T(A) & \longrightarrow & p_\# \tilde{R}_T(A) & \longrightarrow & z_* z^* p_\# \tilde{R}_T(A) & \longrightarrow & u_\# u^* p_\# \tilde{R}_T(A)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
u_\# u^* \tilde{R}_S(p_\# A) & \longrightarrow & \tilde{R}_S(p_\# A) & \longrightarrow & z_* z^* \tilde{R}_S(p_\# A) & \longrightarrow & u_\# u^* \tilde{R}_S(p_\# A)[1],
\end{array}$$

in which both rows are induced by the associated localization triangle. In order to show that the second vertical arrow is invertible, it suffices to prove the invertibility of the first and third vertical arrows. To this end, form the diagram with Cartesian squares

$$\begin{array}{ccccc}
P_U & \xrightarrow{u'} & P & \xleftarrow{z'} & P_Z \\
\downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_Z} \\
U & \xrightarrow{u} & S & \xleftarrow{z} & Z,
\end{array}$$

and observe that P_U belongs to $\mathscr{U}$ (by Lemma 1.4(2)). By construction, the two lateral vertical arrows in the above morphism of triangles coincide with the marked arrows in the two diagrams of functors $\mathbb{H}_1(P) \to \mathbb{H}_2(S)$

$$\begin{array}{ccccc}
u_\# u^* p_\# \tilde{R}_P(A) & \xleftarrow{\sim} & u_\# p_{U,\#} u'^* \tilde{R}_P(A) & \xrightarrow{\theta} & u_\# p_{U,\#} \tilde{R}_{P_U}(u'^* A) \\
\downarrow{\scriptstyle \star} & & & & \downarrow{\scriptstyle \sim} \\
u_\# u^* \tilde{R}_S(p_\# A) & \xrightarrow{\theta} & u_\# \tilde{R}_U(u^* p_\# A) & \xleftarrow{\sim} & u_\# \tilde{R}_U(p_{U,\#} u'^* A)
\end{array}$$

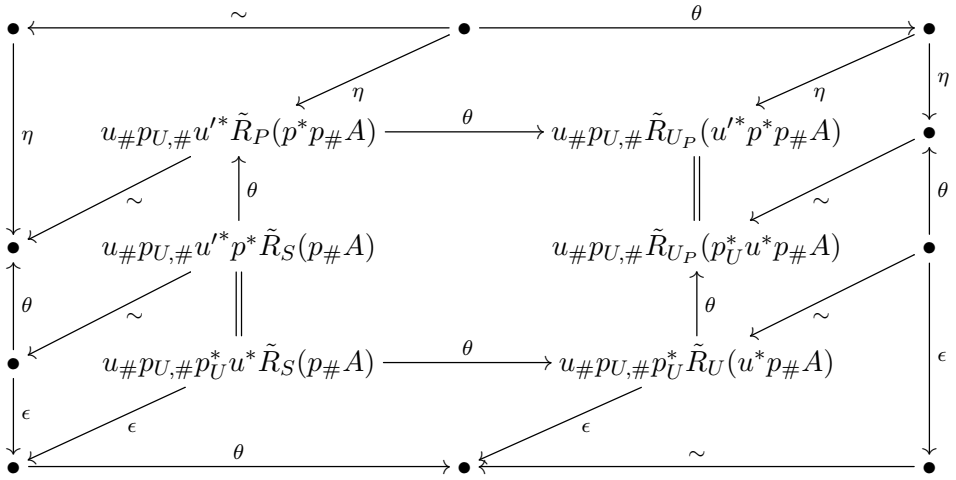
and

$$\begin{array}{ccccc} z_*z^*p_\#\tilde{R}_P(A) & \xleftarrow{\sim} & z_*p_{Z,\#}z'^*\tilde{R}_P(A) & \xrightarrow{\theta} & z_*p_{Z,\#}\tilde{R}_{P_Z}(z'^*A) \\ \downarrow{\scriptstyle \star} & & & & \downarrow{\scriptstyle \sim} \\ z_*z^*\tilde{R}_S(p_\#A) & \xrightarrow{\theta} & z_*\tilde{R}_Z(z^*p_\#A) & \xleftarrow{\sim} & z_*\tilde{R}_Z(p_{Z,\#}z'^*A). \end{array}$$

Note that all arrows in these two diagrams, except possibly the marked one, are already known to be invertible: indeed, all horizontal arrows are invertible (as $\mathbb{H}_1$ and $\mathbb{H}_2$ are localic), and the right-most vertical arrows are invertible as well (because R is a localic morphism of $\mathscr{U}$-fibered category in the first case, by inductive hypothesis in the second case). Therefore, in order to conclude, it suffices to show that these two diagrams are commutative. Let us show, for example, that the first diagram is commutative; a similar argument works for the second diagram. Unwinding the various definitions in the first diagram, we obtain the more explicit diagram

$$\begin{array}{ccccc} u_\#u^*p_\#\tilde{R}_P(A) & \xleftarrow{\sim} & u_\#p_{U,\#}u'^*\tilde{R}_P(A) & \xrightarrow{\theta} & u_\#p_{U,\#}\tilde{R}_{U_P}(u'^*A) \\ \downarrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle \eta} \\ u_\#u^*p_\#\tilde{R}_P(p^*p_\#A) & & & & u_\#p_{U,\#}\tilde{R}_{U_P}(p_U^*p_{U,\#}u'^*A) \\ \uparrow{\scriptstyle \theta} & & & & \uparrow{\scriptstyle \theta} \\ u_\#u^*p_\#p^*\tilde{R}_S(p_\#A) & & & & u_\#p_{U,\#}p_U^*\tilde{R}_U(p_{U,\#}u'^*A) \\ \downarrow{\scriptstyle \epsilon} & & & & \downarrow{\scriptstyle \epsilon} \\ u_\#u^*\tilde{R}_S(p_\#A) & \xrightarrow{\theta} & u_\#\tilde{R}_U(u^*p_\#A) & \xleftarrow{\sim} & u_\#\tilde{R}_U(p_{U,\#}u'^*A), \end{array}$$

which we decompose as



Here, the central rectangle commutes by axiom (mor-$\mathscr{U}$-fib), while all the other pieces commute by naturality and by construction. This proves the claim.

Lastly, we check that $\tilde{R}$ is characterized (up to unique 2-isomorphism) as the unique extension of R to a morphism of triangulated $\mathscr{S}$-fibered categories. To this end, let $R'\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a second morphism of triangulated $\mathscr{S}$-fibered categories extending R; for every morphism $f\colon T \to S$ in $\mathscr{S}$, let $\theta' = \theta'_f\colon f^* \circ R_S \xrightarrow{\sim} R_T \circ f^*$ denote the associated R'-transition isomorphism. In view of the way the $\tilde{R}$-transition isomorphisms $\tilde{\theta}_f$ are constructed in Lemma 2.11, it suffices to show that the same formula is satisfied by the R'-transition isomorphism: this amounts to checking that, given a commutative diagram in $\mathscr{S}$ of the form

$$\begin{array}{ccc} T & \xrightarrow{t} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{s} & Y \end{array}$$

with t and s closed immersions, the outer part of the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\begin{array}{ccccccc} f^*R'_S(A) & & & \xrightarrow{\theta'} & & & R'_T(f^*A) \\ \uparrow{\scriptstyle \epsilon} & & & \nearrow{\scriptstyle \epsilon} & & & \uparrow{\scriptstyle \epsilon} \\ f^*R'_S(s^*s_*A) & \xrightarrow{\theta'} & R'_T(f^*s^*s_*A) & \xleftarrow{\epsilon} & R'_T(t^*t_*f^*s^*s_*A) & \xrightarrow{\epsilon} & R'_T(t^*t_*f^*A) \\ \uparrow{\scriptstyle \theta'} & & \| & & \| & & \uparrow{\scriptstyle \theta'} \\ f^*s^*R'_Y(s_*A) & & R'_T(t^*p^*s_*A) & \xrightarrow{\eta} & R'_T(t^*t_*t^*p^*s_*A) & & t^*R'_X(t_*f^*A) \\ \| & & \uparrow{\scriptstyle \theta'} & & \uparrow{\scriptstyle \theta'} & & \uparrow{\scriptstyle \epsilon} \\ t^*p^*R'_Y(s_*A) & \xrightarrow{\theta'} & t^*R'_X(p^*s_*A) & \xrightarrow{\eta} & t^*R'_X(t_*t^*p^*s_*A) & = & t^*R'_X(t_*f^*s^*s_*A) \end{array}$$

is commutative. But the lower-left piece commutes by axiom (mor-$\mathscr{S}$-fib) applied to R', while all the other pieces commute by naturality and by construction. This proves the claim, thereby concluding the proof. ∎

2.12. Extending images of sections. To complete the discussion, we check that the extension results of Theorem 2.7 and Proposition 1.12 are compatible. This is based on the notion of *image* of a section A of $\mathbb{H}_1$ over $\mathscr{S}$ under a morphism of $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$. Following [Ter24a, § 1], this is the section $R(A)$ of $\mathbb{H}_2$ over $\mathscr{S}$ defined as follows:

- for every $S \in \mathscr{S}$, the object $R(A)_S \in \mathbb{H}_2(S)$ is just $R_S(A_S)$,
- for every morphism $f\colon T \to S$ in $\mathscr{S}$, the isomorphism in $\mathbb{H}_2(T)$

$$R(A)^*_f\colon f^*R(A)_S \xrightarrow{\sim} R(A)_T$$

is the composite

$$f^*R_S(A_S) \xrightarrow{\theta_f} R_T(f^*A_S) \xrightarrow{A^*_f} R_T(A_T).$$

This assignment defines indeed a section of $\mathbb{H}_2$ over $\mathscr{S}$.

As in the previous subsection, we let $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a localic morphism between geometric triangulated $\mathscr{S}$-fibered categories.

2.13. Lemma. *Let A be a section of $\mathbb{H}_1$ over $\mathscr{U}$. Then, there exists a unique isomorphism of sections of $\mathbb{H}_2$ over $\mathscr{S}$*

$$\tilde{R}(\tilde{A}) \xrightarrow{\sim} \widetilde{R(A)}$$

extending the identity of $R(A)$ over $\mathscr{U}$.

Proof. In the first place, we define an isomorphism between the values of the two sections at each fixed $S \in \mathscr{S}$. To this end, for every $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the isomorphism in $\mathbb{H}_2(S)$

$$R_S^{(s)}(A_S^{(s)}) \xrightarrow{\sim} R(A)_S^{(s)}$$

defined as the composite

$$R_S^{(s)}(A_S^{(s)}) := s^* R_Y(s_* s^* A_Y) \xleftarrow{\eta} s^* R_Y(A_Y) =: s^* R(A)_Y =: R(A)_S^{(s)}.$$

We claim that the resulting isomorphism in $\mathbb{H}_2(S)$

$$\tilde{R}_S(\tilde{A}_S) \xrightarrow{\sim} \widetilde{R(A)_S}$$

is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Using the connectedness of the embedding category via products (given by Lemma 1.8), it suffices to prove the following claim: for every morphism $p\colon (Y'; s') \to (Y; s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, the diagram in $\mathbb{H}_2(S)$

$$\begin{array}{ccc}
R_S^{(s)}(A_S^{(s)}) & \xrightarrow{\sim} & R(A)_S^{(s)} \\
\downarrow{\scriptstyle \lambda_p} & & \\
R_S^{(s')}(A_S^{(s)}) & & \downarrow{\scriptstyle \alpha_p} \\
\downarrow{\scriptstyle \alpha_p} & & \\
R_S^{(s')}(A_S^{(s')}) & \xrightarrow{\sim} & R(A)_S^{(s')}
\end{array}$$

is commutative. Unwinding the various definitions, we obtain the outer part of the diagram

$$\begin{array}{ccccc}
s'^* p^* R_Y(p_* s'_* s^* A_Y) & = & s^* R_Y(s_* s^* A_Y) & \xleftarrow[\eta]{} & s^* R_Y(A_Y) \\
\downarrow{\scriptstyle \theta} & & \| & & \| \\
s'^* R_{Y'}(p^* p_* s'_* s^* A_Y) & = & s'^* R_{Y'}(p^* s_* s^* A_Y) & \xleftarrow[\theta]{} & s'^* p^* R_Y(s_* s^* A_Y) \xleftarrow[\eta]{} s'^* p^* R_Y(A_Y) \\
\downarrow{\scriptstyle \epsilon} & \nearrow & \| & & \downarrow{\scriptstyle \theta} \\
s'^* R_{Y'}(s'_* s^* A_Y) & & s'^* R_{Y'}(p^* p_* s'_* s'^* p^* A_Y) & \xleftarrow[\eta]{} & s'^* R_{Y'}(p^* A_Y) \\
\| & \swarrow{\scriptstyle \epsilon} & & \swarrow{\scriptstyle \eta} & \downarrow{\scriptstyle A^*} \\
s'^* R_{Y'}(s'_* s'^* p^* A_Y) & & \xrightarrow[A^*]{} & & s'^* R_{Y'}(s'_* s'^* A_{Y'}) \xleftarrow[\eta]{} s'^* R_{Y'}(A_{Y'}),
\end{array}$$

where all pieces commute by naturality and by construction. This proves the claim.

In order to conclude the proof, it remains to show that the isomorphisms just obtained assemble into an isomorphism of sections of $\mathbb{H}_2$ over $\mathscr{S}$. This amounts to checking that they satisfy condition (mor-$\mathscr{S}$-sect): for every morphism $f\colon T \to S$ in $\mathscr{S}$, we have to check that the diagram in $\mathbb{H}_2(T)$

$$\begin{array}{ccc} f^*\tilde{R}(\tilde{A})_S & \xrightarrow{\sim} & f^*\widetilde{R(\tilde{A})}_S \\ \downarrow{\scriptstyle \tilde{R}(\tilde{A})^*} & & \downarrow{\scriptstyle R(\tilde{A})^*} \\ \tilde{R}(\tilde{A})_T & \xrightarrow{\sim} & \widetilde{R(\tilde{A})}_T \end{array}$$

is commutative. To this end, choose an embedding $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and a factorization $(X;t,p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; as usual, note that we obtain an object $(X;t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. By construction, it now suffices to check that the diagram

$$\begin{array}{ccc} f^*R_S^{(s)}(A_S^{(s)}) & \xrightarrow{\sim} & f^*R(A)_S^{(s)} \\ \downarrow{\scriptstyle \theta_f^{(s,t,p)}} & & \\ R_T^{(t)}(f^*A_S^{(s)}) & & \downarrow{\scriptstyle R(A)_f^{*,(s,t,p)}} \\ \downarrow{\scriptstyle A_f^{*,(s,t,p)}} & & \\ R_T^{(t)}(A_T^{(t)}) & \xrightarrow{\sim} & R(A)_T^{(t)} \end{array}$$

is commutative. Unwinding the definitions, we obtain the outer part of the diagram

$$\begin{array}{ccccc} t^*p^*R_Y(s_*s^*A) & = & f^*s^*R_Y(s_*s^*A_Y) & \xleftarrow{\eta} & f^*s^*R_Y(A_Y) \\ \downarrow{\scriptstyle \theta} & \swarrow{\scriptstyle \eta} & & & \| \\ t^*R_X(p^*s_*s^*A_Y) & & & & t^*p^*R_Y(A_Y) \\ \downarrow{\scriptstyle \sim} & & \swarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \theta} \\ t^*R_X(t_*f^*s^*A_Y) & & & & t^*R_X(p^*A_Y) \\ \| & & \swarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle A^*} \\ t^*R_X(t_*t^*p^*A_Y) & \xrightarrow[A^*]{} & t^*R_X(t_*t^*A_X) & \xleftarrow{\eta} & t^*R_X(A_X), \end{array}$$

where all pieces commute by naturality and by construction. This concludes the proof. ∎

3. Extending monoidal structures

In this long section, we prove an extension result for suitable triangulated monoidal structures on geometric $\mathscr{S}$-fibered categories. We start by showing how to extend the single tensor product functors together with the monoidality isomorphisms. In a second moment, we explain how to extend associativity, commutativity, and unit constraints accordingly.

3.1. Extending internal tensor structures. Throughout, we fix a triangulated $\mathscr{S}$-fibered category $\mathbb{H}$, which we assume to be geometric in the sense of Definition 2.4 (thus, in particular, localic).

In principle, the notion of monoidal structure on $\mathbb{H}$ can be axiomatized in several different ways, according to the presence of some among associativity, commutativity, or unit constraints. For our purposes, it is convenient to start by considering a minimal notion which only includes the single tensor product functors together with the monoidality isomorphisms. Following [Ter24a, Defn. 1.7], by an *internal tensor structure* $(\otimes, m)$ on the $\mathscr{S}$-fibered category $\mathbb{H}$ we mean the datum of

- for every $S \in \mathscr{S}$, a functor
$$- \otimes - = - \otimes_S - \colon \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S),$$
called the *internal tensor product functor* over S,
- for every morphism $f \colon T \to S$ in $\mathscr{S}$, a natural isomorphism between functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$
$$m = m_f \colon f^*A \otimes f^*B \xrightarrow{\sim} f^*(A \otimes B),$$
called the *internal monoidality isomorphism* along f

satisfying the following condition:

(mITS) For every pair of composable morphisms $f \colon T \to S$ and $g \colon S \to V$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}(V) \times \mathbb{H}(V) \to \mathbb{H}(T)$
$$\begin{array}{ccc} (gf)^*A \otimes (gf)^*B & \xrightarrow{\qquad m \qquad} & (gf)^*(A \otimes B) \\ \| & & \| \\ f^*g^*A \otimes f^*g^*B & \xrightarrow{m} f^*(g^*A \otimes g^*B) \xrightarrow{m} & f^*g^*(A \otimes B) \end{array}$$
is commutative.

In the triangulated setting, it only makes sense to consider *triangulated internal tensor structures* on $\mathbb{H}$, as axiomatized in [Ter24a, Defn. 4.5]: in this case, the internal tensor product functors $- \otimes_S -$ are triangulated in each variable (subject to a suitable coherence condition), and the internal monoidality isomorphisms m_f are required to be compatible with shifts. In the localic setting, it is natural to further restrict our attention to *localic* triangulated internal tensor structures: namely, those triangulated internal tensor structures $(\otimes, m)$ such that, for every smooth morphism $p \colon P \to S$ in $\mathscr{S}$, the composite natural transformation of functors $\mathbb{H}(P) \times \mathbb{H}(S) \to \mathbb{H}(S)$
$$p_\#(A \otimes p^*B) \xrightarrow{\eta} p_\#(p^*p_\#A \otimes p^*B) \xrightarrow{m} p_\#p^*(p_\#A \otimes B) \xrightarrow{\epsilon} p_\#A \otimes B$$

and the composite natural transformation of functors $\mathbb{H}(S) \times \mathbb{H}(P) \to \mathbb{H}(S)$

$$p_\#(p^*A \otimes B) \xrightarrow{\eta} p_\#(p^*A \otimes p^*p_\#B) \xrightarrow{m} p_\#p^*(A \otimes p_\#B) \xrightarrow{\epsilon} A \otimes p_\#B$$

are both invertible. In view of [Ter24a, Lem. 1.20, Lem. 1.21], these correspond under [Ter24a, Thm. 4.1] to the localic external tensor structures introduced in [Ter24b, Defn. 5.1]. No further conditions are needed in the case of geometric $\mathscr{S}$-fibered categories. As usual, note that any triangulated internal tensor structure on $\mathbb{H}$ over $\mathscr{S}$ defines by restriction a triangulated internal tensor structure on the underlying triangulated $\mathscr{U}$-fibered category, and the restriction of a localic internal tensor structure is localic as well (compare with Remark 1.5).

There is also a natural notion of *equivalence* between internal tensor structures on an $\mathscr{S}$-fibered category $\mathbb{H}$, as axiomatized in [Ter24a, Defn. 1.7(2)]; it admits a triangulated variant, given in [Ter24a, Defn. 4.5(2)]. This will allow us to state our extension result for internal tensor structures in a precise form. By construction, any triangulated internal tensor structure which is equivalent to a localic one is itself localic.

For localic internal tensor structures on geometric $\mathscr{S}$-fibered categories, the restriction from $\mathscr{S}$ to $\mathscr{U}$ is reversible in a canonical way. Here is the precise statement:

3.2. Theorem. *Let $(\otimes, m)$ be a localic internal tensor structure on the underlying geometric triangulated $\mathscr{U}$-fibered category of $\mathbb{H}$. Then, there exists a unique (up to unique equivalence) triangulated internal tensor structure $(\tilde{\otimes}, \tilde{m})$ on the $\mathscr{S}$-fibered category $\mathbb{H}$ whose restriction to $\mathscr{U}$ coincides with $(\otimes, m)$, and it is automatically localic.*

The result applies, in particular, to monoidal structures on stable homotopy 2-functors, as axiomatized in [Ayo07a, § 2.3] (compare with Example 2.5). The proof goes along the same lines as that of Theorem 2.7. For sake of brevity, we omit most of the details in the proofs and refer to the analogous arguments presented in Subsection 2.6.

To begin with, we construct the single internal tensor product functors over all $S \in \mathscr{S}$.

3.3. Lemma. *Fix $S \in \mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the functor*

$$- \otimes_s - \colon \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S), \quad A \otimes_s B := s^*(s_*A \otimes s_*B).$$

Moreover, for every morphism $p\colon (Y'; s') \to (Y; s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, consider the composite natural transformation of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccccc}
\phi_p\colon & A \otimes_s B & & A \otimes_{s'} B. & \\
& \| & & \| & \\
& s^*(s_*A \otimes s_*B) & & s'^*(s'_*A \otimes s'_*B) & \\
& \| & & \uparrow{\scriptstyle \epsilon} & \\
s'^*p^*(p_*s'_*A \otimes p_*s'_*B) & & \xleftarrow{m} & & s'^*(p^*p_*s'_*A \otimes p^*p_*s'_*B)
\end{array}$$

Then, each natural transformation ϕ_p is invertible, and the assignment $(Y; s) \mapsto - \otimes_s -$ defines an $\mathsf{Emb}_{\mathscr{U}}(S)^{op}$-indexed system of bi-triangulated functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$ with transition isomorphisms ϕ_p.

PROOF. Showing that ϕ_p is invertible is equivalent to showing the invertibility of the co-unit arrow $\epsilon\colon s'^*(p^*p_*s'_*A \otimes p^*p_*s'_*B) \to s'^*(s'_*A \otimes s'_*B)$, which follows from the invertibility of the two single co-unit arrows $\epsilon\colon s'^*(p^*p_*s'_*A \otimes p^*p_*s'_*B) \to s'^*(p^*p_*s'_*A \otimes s'_*B)$ and $\epsilon\colon s'^*(p^*p_*s'_*A \otimes s'_*B) \to s'^*(s'_*A \otimes s'_*B)$. This can be checked as for the first statement of Lemma 2.8. To prove the compatibility of the assignment $p \mapsto \phi_p$ with composition in the embedding category, it suffices to adapt the argument for the second statement of Lemma 2.8. ∎

Given two embeddings $(Y^{(i)}; s^{(i)}) \in \mathsf{Emb}_{\mathscr{U}}(S)$, $i = 1, 2$, write $s^{(12)}\colon S \to Y^{(1)} \times Y^{(2)}$ for the resulting morphism (which is a closed immersion, by Assumption 1.2(iii)) and $p^{(i)} : Y^{(1)} \times Y^{(2)} \to Y^{(i)}$, $i = 1, 2$, for the canonical projections (which are smooth morphisms, by Assumption 1.2(ii)). From Lemma 3.3, we obtain the natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\psi_{1,2}\colon A \otimes_{s^{(1)}} B \xrightarrow{\phi_{p^{(1)}}} A \otimes_{s^{(12)}} B \xrightarrow{\phi^{-1}_{p^{(2)}}} A \otimes_{s^{(2)}} B. \tag{4}$$

These satisfy the natural cocycle condition for triples of embeddings:

3.4. COROLLARY. *Suppose that we are given three embeddings* $(Y^{(i)}; s^{(i)}) \in \mathsf{Emb}_{\mathscr{U}}(S)$, $i = 1, 2, 3$, *and consider the natural isomorphisms of functors* $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\psi_{1,2}\colon A \otimes_{s^{(1)}} B \xrightarrow{\sim} A \otimes_{s^{(2)}} B, \quad \psi_{1,3}\colon A \otimes_{s^{(1)}} B \xrightarrow{\sim} A \otimes_{s^{(3)}} B, \quad \psi_{2,3}\colon A \otimes_{s^{(2)}} B \xrightarrow{\sim} A \otimes_{s^{(3)}} B$$

defined as in (4). *Then, we have the equality*

$$\psi_{1,3} = \psi_{2,3} \circ \psi_{1,2}.$$

PROOF. This follows from Lemma 3.3, in the same way as Corollary 2.10 follows from Lemma 2.8. ∎

As a consequence, the assignment $(Y; s) \mapsto - \otimes_s -$ defines a $\mathscr{G}_{\mathscr{U}}(S)$-indexed system of bi-triangulated functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$ with invertible bi-triangulated transition natural isomorphisms of the form (4). Thus, the formula

$$A \tilde{\otimes}_S B := \varinjlim_{(Y;s) \in \mathscr{G}(S)} A \otimes_s B$$

defines a bi-triangulated functor $- \tilde{\otimes} -\colon \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$, endowed with canonical bi-triangulated natural isomorphisms $- \otimes_s - \xrightarrow{\sim} - \tilde{\otimes} -$; the natural isomorphisms witnessing the compatibility of $- \tilde{\otimes} -$ with shifts are induced by the analogous natural isomorphisms for each $- \otimes_s -$, and $- \tilde{\otimes} -$ preserves distinguished triangles in each variable since so does each $- \otimes_s -$.

Next, we want to turn the collection of bi-triangulated functors just obtained into a triangulated internal tensor structure on the triangulated $\mathscr{S}$-fibered category $\mathbb{H}$. To this end, we need to construct the associated internal monoidality isomorphisms. This can be done as follows:

3.5. Lemma. *Fix a morphism $f\colon T \to S$ in $\mathscr{S}$. For every embedding $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and every factorization $(X;t,p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$, the composite natural transformation of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$*

$$\begin{array}{ccc} t^*(p^*s_*A \otimes p^*s_*B) & & t^*(t_*f^*A \otimes t_*f^*B) \\ \downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \epsilon} \\ t^*(t_*t^*p^*s_*A \otimes t_*t^*p^*s_*B) & = & t^*(t_*f^*s^*s_*A \otimes t_*f^*s^*s_*B) \end{array} \tag{5}$$

is invertible. Moreover, consider the natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$

$$m_f^{(s,t,p)}\colon \quad \begin{array}{ccc} f^*A \otimes_t f^*B & & f^*(A \otimes_s B), \\ \| & & \| \\ t^*(t_*f^*A \otimes t_*f^*B) & & f^*s^*(s_*A \otimes s_*B) \\ \sim\uparrow & & \| \\ t^*(p^*s_*A \otimes p^*s_*B) & \xrightarrow{\ m\ } & t^*p^*(s_*A \otimes s_*B) \end{array}$$

where the unnamed natural isomorphism is the composite (5). Then, the induced natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$

$$\tilde{m} = \tilde{m}_f\colon f^*A\tilde{\otimes}f^*B = f^*A \otimes_t f^*B \xrightarrow{m_f^{(s,t,p)}} f^*(A \otimes_s B) = f^*(A\tilde{\otimes}B)$$

is independent of the choice of $(Y;s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and of $(X;t,p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$.

Proof. Showing the invertibility of (5) is equivalent to showing the invertibility of the unit arrow $\eta\colon t^*(p^*s_*A \otimes p^*s_*B) \to t^*(t_*t^*p^*s_*A \otimes t_*t^*p^*s_*B)$, which follows from the invertibility of the two single unit arrows $\eta\colon t^*(p^*s_*A \otimes p^*s_*B) \to t^*(t_*t^*p^*s_*A \otimes p^*s_*B)$ and $\eta\colon t^*(t_*t^*p^*s_*A \otimes p^*s_*B) \to t^*(t_*t^*p^*s_*A \otimes t_*t^*p^*s_*B)$. This can be checked as for the first statement of Lemma 2.11. Once this is achieved, the independence of the resulting natural isomorphism $\tilde{m}_f$ from the choice of $(Y;s)$ and $(X;t,p)$ follows as for the second statement of Lemma 2.11. ∎

Proof of Theorem 3.2. In the first place, one has to show that the natural isomorphisms $\tilde{m}_f$ constructed in Lemma 3.5 turn the collection of bi-triangulated functors $-\tilde{\otimes}-$ into an internal tensor structure $(\tilde{\otimes}, \tilde{m})$ on $\mathbb{H}$: this amounts to checking that they satisfy condition (mITS), which can be done using the same argument as for Theorem 2.7.

Similarly, one checks that the natural isomorphisms witnessing the compatibility of the extended internal tensor product functors with shifts are coherent with respect to inverse images. In conclusion, one obtains a triangulated internal tensor structure as in the statement. As a consequence of the explicit definition of the extended internal tensor product functors as a colimit, it is clear that the restriction of $(\tilde{\otimes}, \tilde{m})$ to $\mathscr{U}$ is canonically equivalent to $(\otimes, m)$.

To check that the internal tensor structure $(\tilde{\otimes}, \tilde{m})$ is localic, and that it is the unique (up to canonical equivalence) extension of $(\otimes, m)$ to an internal tensor structure over $\mathscr{S}$, one proceeds as in the proof of Theorem 2.7; we leave the details to the interested reader. ∎

3.6. Extending associativity, commutativity, and unit constraints. As in the previous subsection, we work over a geometric triangulated $\mathscr{S}$-fibered category $\mathbb{H}$. Moreover, we fix a localic internal tensor structure $(\otimes, m)$ on the underlying geometric triangulated $\mathscr{U}$-fibered category, and we let $(\tilde{\otimes}, \tilde{m})$ denote the internal tensor structure over $\mathscr{S}$ obtained from it via Theorem 3.2. We want to check that the extended internal tensor structure $(\tilde{\otimes}, \tilde{m})$ inherits associativity, commutativity, and unit constraints defined on $(\otimes, m)$.

We start by studying *internal associativity constraints* a for $(\tilde{\otimes}, \tilde{m})$: as in [Ter24a, Defn. 2.1], by this we mean the datum of

- for every $S \in \mathscr{S}$, a natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$a = a_S \colon (A \tilde{\otimes} B) \tilde{\otimes} C \xrightarrow{\sim} A \tilde{\otimes} (B \tilde{\otimes} C)$$

satisfying the following conditions:

(aITS-1) For every $S \in \mathscr{S}$, the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccccc}
((A \tilde{\otimes} B) \tilde{\otimes} C) \tilde{\otimes} D & & \xrightarrow{\quad a \quad} & & (A \tilde{\otimes} B) \tilde{\otimes} (C \tilde{\otimes} D) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
(A \tilde{\otimes} (B \tilde{\otimes} C)) \tilde{\otimes} D & \xrightarrow{a} & A \tilde{\otimes} ((B \tilde{\otimes} C) \tilde{\otimes} D) & \xrightarrow{a} & A \tilde{\otimes} (B \tilde{\otimes} (C \tilde{\otimes} D))
\end{array}$$

is commutative.

(aITS-2) For every morphism $f \colon T \to S$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$

$$\begin{array}{ccccc}
(f^* A \tilde{\otimes} f^* B) \tilde{\otimes} f^* C & \xrightarrow{\tilde{m}} & f^*(A \tilde{\otimes} B) \tilde{\otimes} f^* C & \xrightarrow{\tilde{m}} & f^*((A \tilde{\otimes} B) \tilde{\otimes} C) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
f^* A \tilde{\otimes} (f^* B \tilde{\otimes} f^* C) & \xrightarrow{\tilde{m}} & f^* A \tilde{\otimes} f^*(B \tilde{\otimes} C) & \xrightarrow{\tilde{m}} & f^*(A \tilde{\otimes} (B \tilde{\otimes} C))
\end{array}$$

is commutative.

We only consider *triangulated internal associativity constraints*, as in [Ter24a, Defn. 5.1]: in this case, the natural isomorphisms a_S are required to be compatible with shifts in each variable in the obvious way.

3.7. Proposition. *Let a be a triangulated internal associativity constraint for $(\otimes, m)$. Then, there exists a unique triangulated internal associativity constraint $\tilde{a}$ for $(\tilde{\otimes}, \tilde{m})$ whose restriction to $\mathscr{U}$ coincides with a.*

As usual, we start by defining the single natural isomorphisms $\tilde{a}_S$. This can be done as follows:

3.8. LEMMA. *Fix $S \in \mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the composite triangulated natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$*

$$
\begin{array}{ccc}
a^{(s)} = a^{(s)}_S\colon & (A \otimes_s B) \otimes_s C & A \otimes_s (B \otimes_s C). \\
& \| & \| \\
& s^*(s_*s^*(s_*A \otimes s_*B) \otimes s_*C) & s^*(s_*A \otimes s_*s^*(s_*B \otimes s_*C)) \\
& \uparrow{\scriptstyle \eta} & \uparrow{\scriptstyle \eta} \\
& s^*((s_*A \otimes s_*B) \otimes s_*C) & \xrightarrow{\quad a \quad} s^*(s_*A \otimes (s_*B \otimes s_*C))
\end{array}
$$

Then, the induced triangulated natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\tilde{a} = \tilde{a}_S\colon (A\tilde{\otimes}B)\tilde{\otimes}C = (A \otimes_s B) \otimes_s C \xrightarrow{a^{(s)}} A \otimes_s (B \otimes_s C) = A\tilde{\otimes}(B\tilde{\otimes}C)$$

is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$.

PROOF. To make sure that the natural isomorphism $a^{(s)}_S$ is well-defined, one has to check the invertibility of the unit arrows $\eta\colon s^*((s_*A \otimes s_*B) \otimes s_*C) \to s^*(s_*s^*(s_*A \otimes s_*B) \otimes s_*C)$ and $\eta\colon s^*(s_*A \otimes (s_*B \otimes s_*C)) \to s^*(s_*A \otimes s_*s^*(s_*B \otimes s_*C))$. This can be done as usual, via the associated localization triangles.

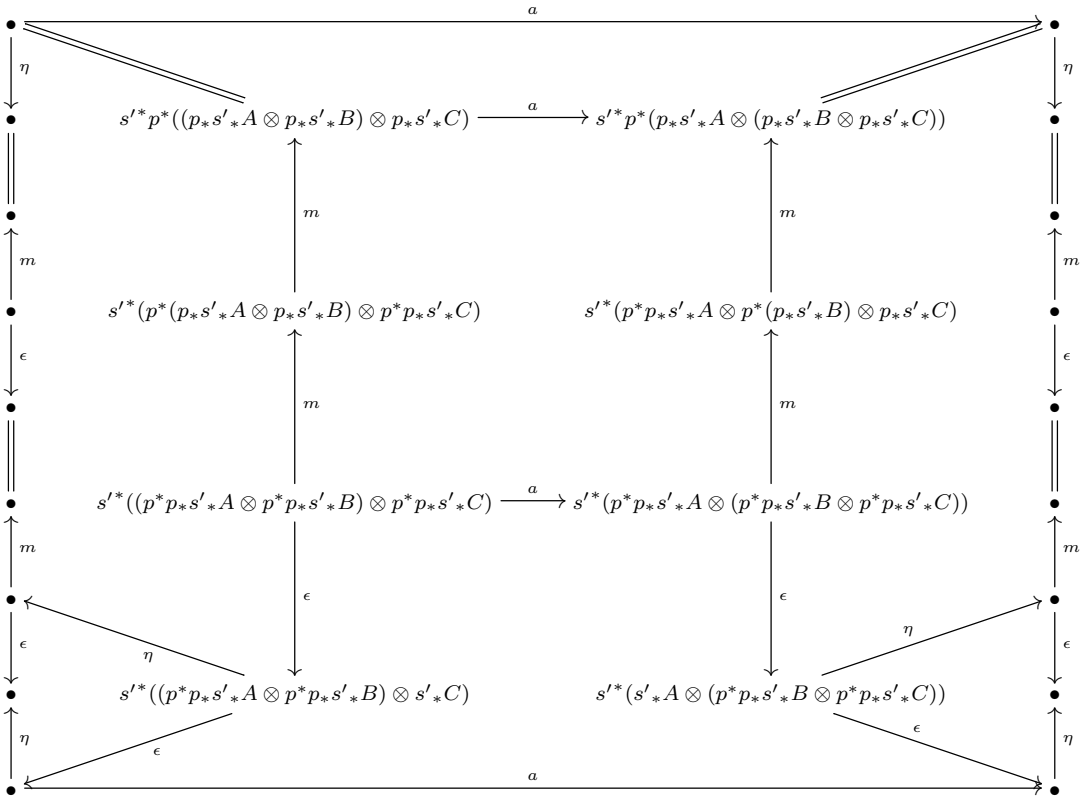
Let us show that the resulting natural isomorphism $\tilde{a}$ is independent of the choice of $(Y, s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Using the connectedness of the embedding category via products (given by Lemma 1.8), we are reduced to proving the following claim: given a morphism $p\colon (Y'; s') \to (Y; s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$
\begin{array}{ccc}
(A \otimes_s B) \otimes_s C & \xrightarrow{a^{(s)}} & A \otimes_s (B \otimes_s C) \\
\downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \phi_p} \\
(A \otimes_s B) \otimes_{s'} C & & A \otimes_{s'} (B \otimes_s C) \\
\downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \phi_p} \\
(A \otimes_{s'} B) \otimes_{s'} C & \xrightarrow{a^{(s')}} & A \otimes_{s'} (B \otimes_{s'} C)
\end{array}
$$

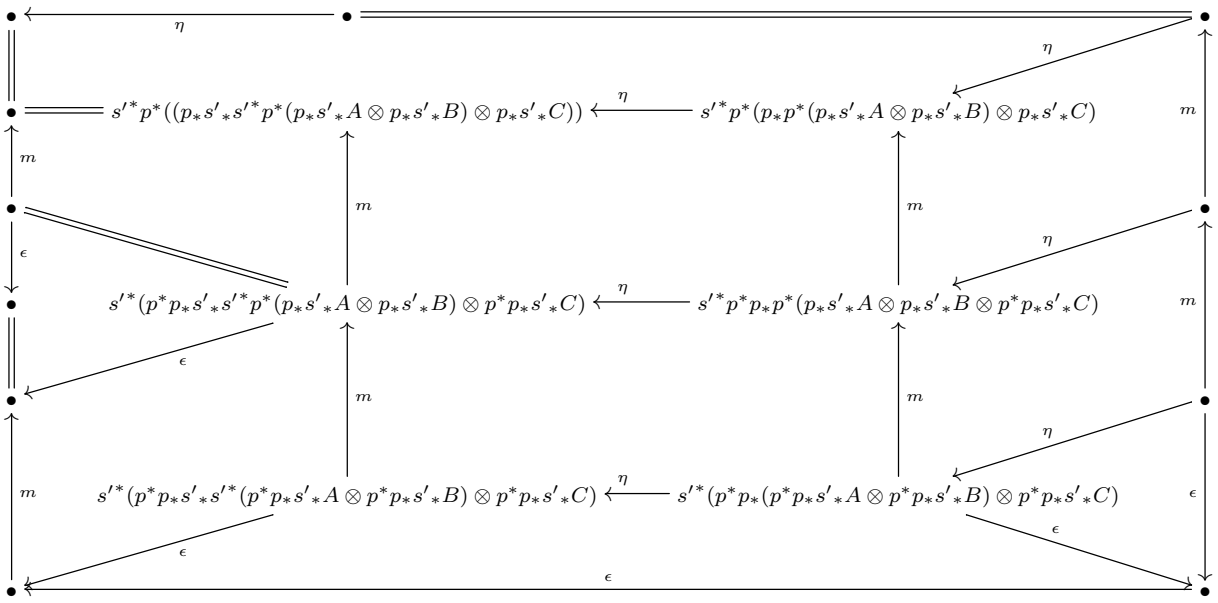
is commutative. Unwinding the various definitions, we obtain the more explicit diagram

$$
\begin{array}{ccc}
s^*((s_*A \otimes s_*B) \otimes s_*C) & \xrightarrow{a} & s^*(s_*A \otimes (s_*B \otimes s_*C)) \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
s^*(s_*s^*(s_*A \otimes s_*B) \otimes s_*C) & & s^*(s_*A \otimes s_*s^*(s_*B \otimes s_*C)) \\
\| & & \| \\
s'^*p^*(p_*s'_*s^*(s_*A \otimes s_*B) \otimes p_*s'_*C) & & s'^*p^*(p_*s'_*A \otimes p_*s'_*s^*(s_*B \otimes s_*C)) \\
\uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m} \\
s'^*(p^*p_*s'_*s^*(s_*A \otimes s_*B) \otimes p^*p_*s'_*C) & & s'^*(p^*p_*s'_*A \otimes p^*p_*s'_*s^*(s_*B \otimes s_*C)) \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\
s'^*(s'_*s^*(s_*A \otimes s_*B) \otimes s'_*C) & & s'^*(s'_*A \otimes s'_*s^*(s_*B \otimes s_*C)) \\
\| & & \| \\
s'^*(s'_*s'^*p^*(p_*s'_*A \otimes p_*s'_*B) \otimes s'_*C) & & s'^*(s'_*A \otimes s'_*s'^*p^*(p_*s'_*B \otimes p_*s'_*C)) \\
\uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m} \\
s'^*(s'_*s'^*(p^*p_*s'_*A \otimes p^*p_*s'_*B) \otimes s'_*C) & & s'^*(s'_*A \otimes s'_*s'^*(p^*p_*s'_*B \otimes p^*p_*s'_*C)) \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\
s'^*(s'_*s'^*(s'_*A \otimes s'_*B) \otimes s'_*C) & & s'^*(s'_*A \otimes s'_*s'^*(s'_*B \otimes s'_*C)) \\
\uparrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \eta} \\
s'^*((s'_*A \otimes s'_*B) \otimes s'_*C) & \xrightarrow{a} & s'^*(s'_*A \otimes (s'_*B \otimes s'_*C)),
\end{array}
$$

which we decompose as



Here, the central piece commutes by axiom (aITS-2) over $\mathscr{U}$, the upper piece and the two lower-lateral pieces commute by naturality, and the lower-central piece commutes by construction. Hence, it remains to show that the left-most and right-most parallelograms are commutative as well. But the left-most one can be decomposed as



where all pieces commute by naturality and construction; similarly for the right-most one. This proves the claim. ∎

Proof of Proposition 3.7. For sake of brevity, we only show that the natural isomorphisms $\tilde{a}$ constructed in Lemma 3.8 define an internal associativity constraint for $(\tilde{\otimes}, \tilde{m})$; the statements about its restriction to $\mathscr{U}$, its uniqueness, and its compatibility with shifts are left to the interested reader.

Let us show that the extended natural isomorphisms $\tilde{a}$ still satisfy conditions (aITS-1) and (aITS-2). We start from condition (aITS-1): given $S \in \mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccccc}
((A\tilde{\otimes}B)\tilde{\otimes}C)\tilde{\otimes}D & & \xrightarrow{\quad\tilde{a}\quad} & & (A\tilde{\otimes}B)\tilde{\otimes}(C\tilde{\otimes}D) \\
\downarrow{\scriptstyle \tilde{a}} & & & & \downarrow{\scriptstyle \tilde{a}} \\
(A\tilde{\otimes}(B\tilde{\otimes}C))\tilde{\otimes}D & \xrightarrow{\tilde{a}} & A\tilde{\otimes}((B\tilde{\otimes}C)\tilde{\otimes}D) & \xrightarrow{\tilde{a}} & A\tilde{\otimes}(B\tilde{\otimes}(C\tilde{\otimes}D))
\end{array}$$

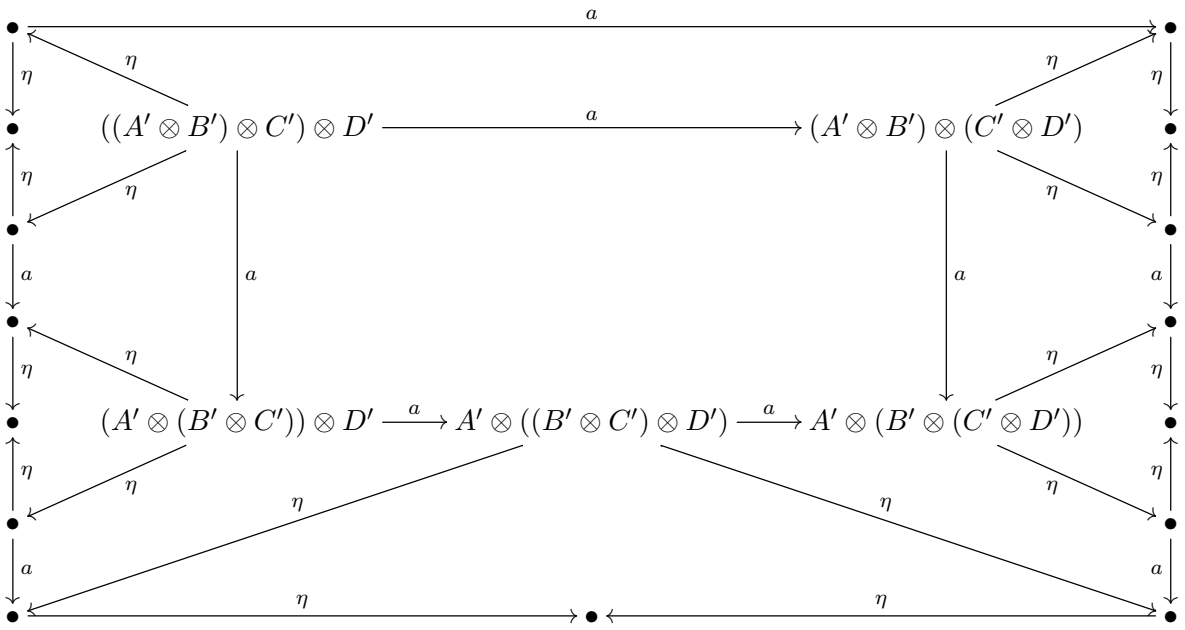
is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram

$$\begin{array}{ccccc}
((A \otimes_s B) \otimes_s C) \otimes_s D & & \xrightarrow{\quad a_S^{(s)}\quad} & & (A \otimes_s B) \otimes_s (C \otimes_s D) \\
\downarrow{\scriptstyle a_S^{(s)}} & & & & \downarrow{\scriptstyle a_S^{(s)}} \\
(A \otimes_s (B \otimes_s C)) \otimes_s D & \xrightarrow{a_S^{(s)}} & A \otimes_s ((B \otimes_s C) \otimes_s D) & \xrightarrow{a_S^{(s)}} & A \otimes_s (B \otimes_s (C \otimes_s D))
\end{array}$$

is commutative. Unwinding the various definitions, we are reduced to considering the diagram of functors $\mathbb{H}(Y) \times \mathbb{H}(Y) \times \mathbb{H}(Y) \to \mathbb{H}(Y)$

$$\begin{array}{ccccc}
(s_*s^*(A' \otimes B') \otimes C') \otimes D' & & \xrightarrow{\quad a\quad} & & s_*s^*(A' \otimes B') \otimes (C' \otimes D') \\
\downarrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle \eta} \\
s_*s^*(s_*s^*(A' \otimes B') \otimes C') \otimes D' & & & & s_*s^*(A' \otimes B') \otimes s_*s^*(C' \otimes D') \\
\uparrow{\scriptstyle \eta} & & & & \uparrow{\scriptstyle \eta} \\
s_*s^*((A' \otimes B') \otimes C') \otimes D' & & & & (A' \otimes B') \otimes s_*s^*(C' \otimes D') \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
s_*s^*(A' \otimes (B' \otimes C')) \otimes D' & & & & A' \otimes (B' \otimes s_*s^*(C' \otimes D')) \\
\downarrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle \eta} \\
s_*s^*(A' \otimes s_*s^*(B' \otimes C')) \otimes D' & & & & A' \otimes s_*s^*(B' \otimes s_*s^*(C' \otimes D')) \\
\uparrow{\scriptstyle \eta} & & & & \uparrow{\scriptstyle \eta} \\
(A' \otimes s_*s^*(B' \otimes C')) \otimes D' & & & & A' \otimes s_*s^*(B' \otimes (C' \otimes D')) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
A' \otimes (s_*s^*(B' \otimes C') \otimes D') & \xrightarrow{\eta} & A' \otimes s_*s^*(s_*s^*(B' \otimes C') \otimes D') & \xleftarrow{\eta} & A' \otimes s_*s^*((B' \otimes C') \otimes D'),
\end{array}$$

which we decompose as



Here, the central rectangle commutes by axiom (aITS-1) over $\mathscr{U}$, while all the other pieces commute by naturality. This proves the claim.

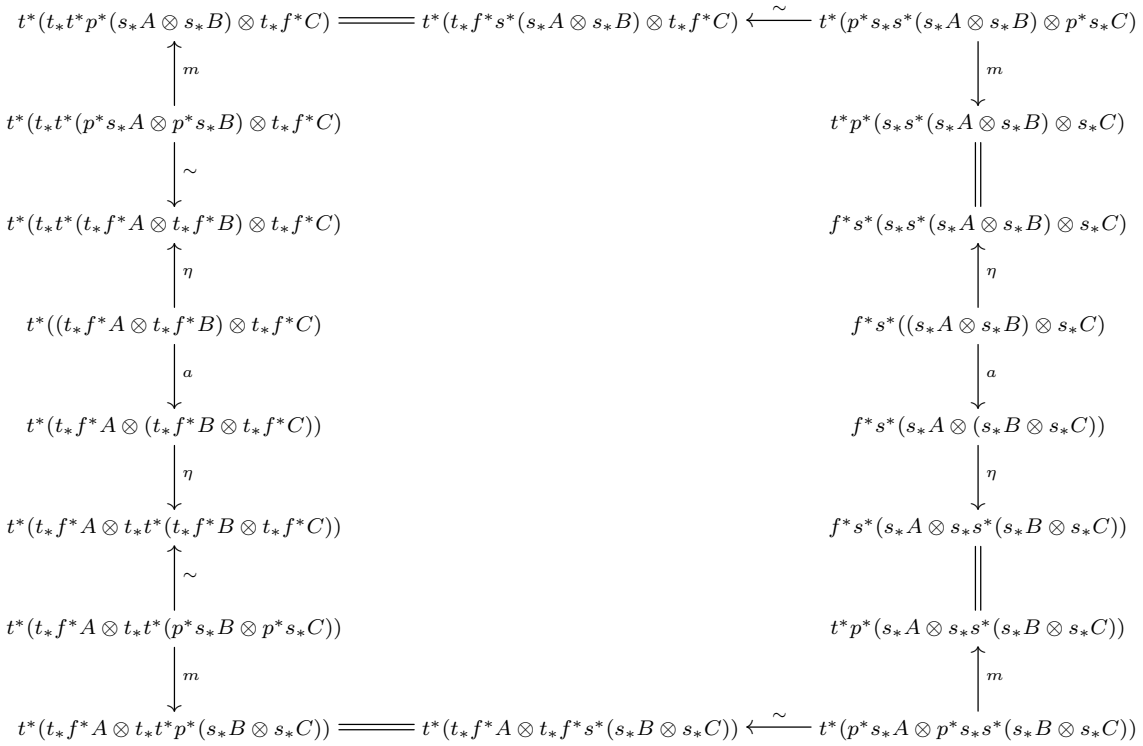
Next, we treat condition (aITS-2): given a morphism $f\colon T \to S$ in $\mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$

$$\begin{array}{ccccc} (f^*A\tilde{\otimes}f^*B)\tilde{\otimes}f^*C & \xrightarrow{\tilde{m}} & f^*(A\tilde{\otimes}B)\tilde{\otimes}f^*C & \xrightarrow{\tilde{m}} & f^*((A\tilde{\otimes}B)\tilde{\otimes}C) \\ \downarrow{\scriptstyle \tilde{a}} & & & & \downarrow{\scriptstyle \tilde{a}} \\ f^*A\tilde{\otimes}(f^*B\tilde{\otimes}f^*C) & \xrightarrow{\tilde{m}} & f^*A\tilde{\otimes}f^*(B\tilde{\otimes}C) & \xrightarrow{\tilde{m}} & f^*(A\tilde{\otimes}(B\tilde{\otimes}C)) \end{array}$$

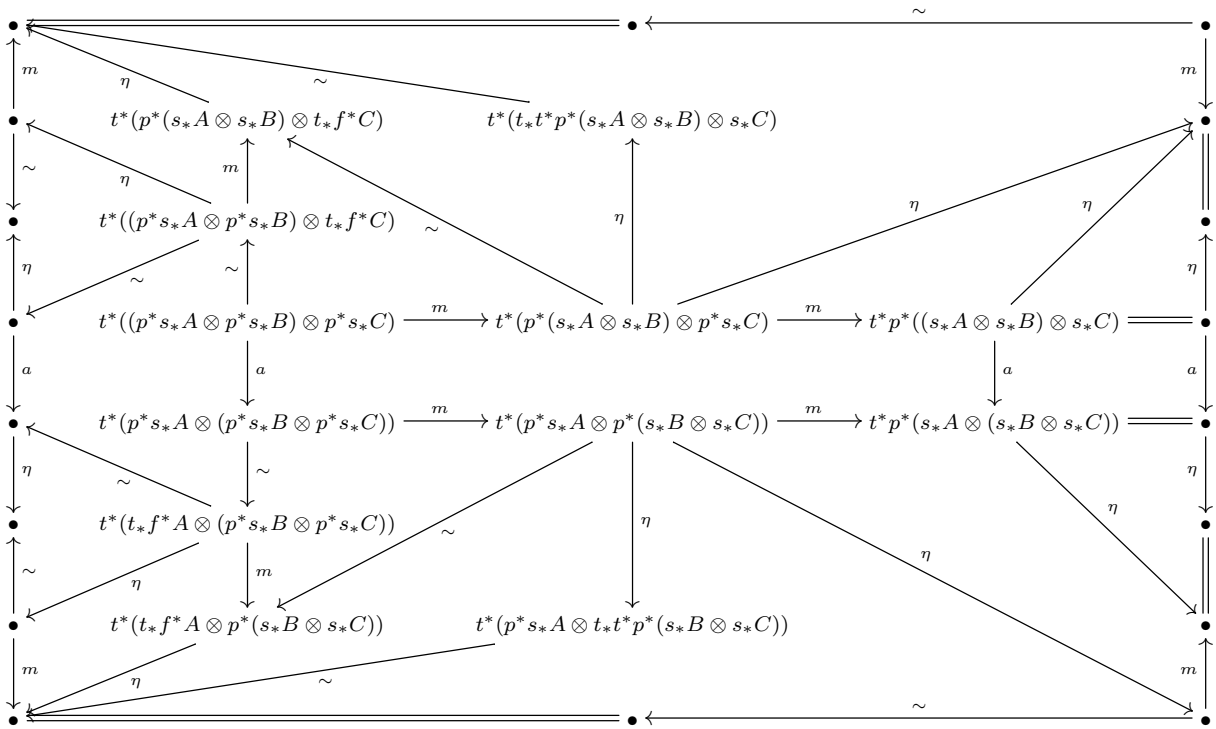
is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and then a factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; note that X automatically belongs to $\mathscr{U}$ (by Lemma 1.4(2)), so we obtain an embedding $(X; t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. By construction, it now suffices to show that the diagram

$$\begin{array}{ccccc} (f^*A \otimes_t f^*B) \otimes_t f^*C & \xrightarrow{m_f^{(s,t,p)}} & f^*(A \otimes_s B) \otimes_t f^*C & \xrightarrow{m_f^{(s,t,p)}} & f^*((A \otimes_s B) \otimes_s C) \\ \downarrow{\scriptstyle a_T^{(t)}} & & & & \downarrow{\scriptstyle a_S^{(s)}} \\ f^*A \otimes_t (f^*B \otimes_t f^*C) & \xrightarrow{m_f^{(s,t,p)}} & f^*A \otimes_t f^*(B \otimes_s C) & \xrightarrow{m_f^{(s,t,p)}} & f^*(A \otimes_s (B \otimes_s C)) \end{array}$$

is commutative. Unwinding the various definitions, we obtain the more explicit diagram



which we decompose as



Here, the central rectangle commutes by axiom (aITS-2) over $\mathscr{U}$, while all the other pieces commutes by naturality and by construction. This proves the claim, thereby concluding the proof. ∎

In the same spirit, we study *internal commutativity constraints* c for $(\otimes, m)$: as in [Ter24a, Defn. 2.6], by this we mean the datum of

- for every $S \in \mathscr{S}$, a natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$c = c_S \colon A \tilde{\otimes} B \xrightarrow{\sim} B \tilde{\otimes} A$$

satisfying the following condition

(cITS-1) For every $S \in \mathscr{S}$, the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} A\tilde{\otimes}B & \xrightarrow{c} & B\tilde{\otimes}A \\ & \searrow^{\mathsf{id}} & \downarrow{\scriptstyle c} \\ & & A\tilde{\otimes}B \end{array}$$

is commutative.

(cITS-2) For every morphism $f \colon T \to S$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$

$$\begin{array}{ccc} f^*A\tilde{\otimes}f^*B & \xrightarrow{c} & f^*B\tilde{\otimes}f^*A \\ \downarrow{\scriptstyle \tilde{m}} & & \downarrow{\scriptstyle \tilde{m}} \\ f^*(A\tilde{\otimes}B) & \xrightarrow{c} & f^*(B\tilde{\otimes}A) \end{array}$$

is commutative.

We only consider *triangulated internal commutativity constraints*, as in [Ter24a, Defn. 5.5]: in this case, the natural isomorphisms c_S are required to be compatible with shifts in each variable in the obvious way.

3.9. Proposition. *Let c be a triangulated internal commutativity constraint for $(\otimes, m)$. Then, there exists a unique triangulated internal commutativity constraint $\tilde{c}$ for $(\tilde{\otimes}, \tilde{m})$ whose restriction to $\mathscr{U}$ coincides with c.*

The single natural isomorphisms $\tilde{c}_S$ can be constructed as follows:

3.10. Lemma. *Fix $S \in \mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the composite triangulated natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$*

$$c^{(s)} = c_S^{(s)} \colon A \otimes_s B = s^*(s_*A \otimes s_*B) \xrightarrow{c} s^*(s_*B \otimes s_*A) = (B \otimes_s A).$$

Then, the induced triangulated natural isomorphism of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\tilde{c} = \tilde{c}_S \colon A\tilde{\otimes}B = A \otimes_s B \xrightarrow{c_S^{(s)}} (B \otimes_s A) = (B\tilde{\otimes}A)$$

is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$.

PROOF. Using the connectedness of the embedding category via products (given by Lemma 1.8), we are reduced to proving the following claim: given a morphism $p\colon (Y';s') \to (Y;s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, the diagram of functors $\mathbb{H}(S)\times\mathbb{H}(S)\to\mathbb{H}(S)$

$$\begin{array}{ccc} A\otimes_s B & \xrightarrow{c_S^{(s)}} & B\otimes_s A \\ \downarrow{\scriptstyle \phi_p} & & \downarrow{\scriptstyle \phi_p} \\ A\otimes_{s'} B & \xrightarrow{c_S^{(s')}} & B\otimes_{s'} A \end{array}$$

is commutative. Unwinding the various definitions, we obtain the outer part of the diagram

$$\begin{array}{ccc} s^*(s_*A\otimes s_*B) & \xrightarrow{c} & s^*(s_*B\otimes s_*A) \\ \| & & \| \\ s'^*p^*(p_*s'_*A\otimes p_*s'_*B) & \xrightarrow{c} & s'^*p^*(p_*s'_*B\otimes p_*s'_*A) \\ \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m} \\ s'^*(p^*p_*s'_*A\otimes p^*p_*s'_*B) & \xrightarrow{c} & s'^*(p^*p_*s'_*B\otimes p^*p_*s'_*A) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ s'^*(s'_*A\otimes s'_*B) & \xrightarrow{c} & s'^*(s'_*B\otimes s'_*A). \end{array}$$

Here, the central piece commutes by axiom (cITS-2) over $\mathscr{U}$, while the other two pieces commute by naturality. This proves the claim. ∎

PROOF OF PROPOSITION 3.9. For sake of brevity, we only show that the natural isomorphisms $\tilde{c}$ constructed in Lemma 3.10 define an internal commutativity constraint for $(\tilde{\otimes},\tilde{m})$; the other properties are left to the interested reader.

Let us show that the extended natural isomorphisms $\tilde{c}$ still satisfy conditions (cITS-1) and (cITS-2). We start from condition (cITS-1): given $S\in\mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}(S)\times\mathbb{H}(S)\to\mathbb{H}(S)$

$$\begin{array}{ccc} A\tilde{\otimes}B & \xrightarrow{\tilde{c}} & B\tilde{\otimes}A \\ & {\scriptstyle \mathsf{id}}\searrow & \downarrow{\scriptstyle \tilde{c}} \\ & & A\tilde{\otimes}B \end{array}$$

is commutative. To this end, choose an embedding $(Y;s)\in\mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram

$$\begin{array}{ccc} A\otimes_s B & \xrightarrow{c_S^{(s)}} & B\otimes_s A \\ & {\scriptstyle \mathsf{id}}\searrow & \downarrow{\scriptstyle c_S^{(s)}} \\ & & A\otimes_s B \end{array}$$

is commutative. Unwinding the definitions, we obtain the more explicit diagram

$$\begin{array}{ccc} s^*(s_*A \otimes s_*B) & \xrightarrow{c} & s^*(s_*B \otimes s_*A) \\ & \searrow^{\mathsf{id}} & \downarrow{\scriptstyle c} \\ & & s^*(s_*A \otimes s_*B), \end{array}$$

which is commutative by axiom (cITS-1) over $\mathscr{U}$. This proves the claim.

We now check condition (cITS-2): given a morphism $f\colon T \to S$ in $\mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(T)$

$$\begin{array}{ccc} f^*A \tilde{\otimes} f^*B & \xrightarrow{\tilde{c}} & f^*B \tilde{\otimes} f^*A \\ \downarrow{\scriptstyle \tilde{m}} & & \downarrow{\scriptstyle \tilde{m}} \\ f^*(A \tilde{\otimes} B) & \xrightarrow{\tilde{c}} & f^*(B \tilde{\otimes} A) \end{array}$$

is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and then a factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; as usual, note that X automatically belongs to $\mathscr{U}$, so we obtain an embedding $(X; t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. By construction, it now suffices to show that the diagram

$$\begin{array}{ccc} f^*A \otimes_t f^*B & \xrightarrow{c_T^{(t)}} & f^*B \otimes_t f^*A \\ \downarrow{\scriptstyle m_f^{(s,t,p)}} & & \downarrow{\scriptstyle m_f^{(s,t,p)}} \\ f^*(A \otimes_s B) & \xrightarrow{c_S^{(s)}} & f^*(B \otimes_s A) \end{array}$$

is commutative. Unwinding the various definitions, we obtain the outer part of the diagram

$$\begin{array}{ccc} t^*(t_*f^*A \otimes t_*f^*B) & \xrightarrow{c} & t^*(t_*f^*B \otimes t_*f^*A) \\ \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\ t^*(p^*s_*A \otimes p^*s_*B) & \xrightarrow{c} & t^*(p^*s_*B \otimes p^*s_*A) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ t^*p^*(s_*A \otimes s_*B) & \xrightarrow{c} & t^*p^*(s_*B \otimes s_*A) \\ \| & & \| \\ f^*s^*(s_*A \otimes s_*B) & \xrightarrow{c} & f^*s^*(s_*B \otimes s_*A) \end{array}$$

where the central piece commutes by axiom (cITS-2) over $\mathscr{U}$ while the other two pieces commute by naturality. This proves the claim, thereby concluding the proof. ∎

Lastly, we study *internal unit constraints* u for $(\tilde{\otimes}, \tilde{m})$: as in [Ter24a, Defn. 2.11], by this we mean the datum of

- a section $\mathbf{I}$ of the $\mathscr{S}$-fibered category $\mathbb{H}$, called the *unit section*,

- for every $S \in \mathscr{S}$, two natural isomorphisms of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$u_r = u_{r,S} \colon A \tilde{\otimes} \mathbf{I}_S \xrightarrow{\sim} A, \qquad u_l = u_{l,S} \colon \mathbf{I}_S \tilde{\otimes} B \xrightarrow{\sim} B$$

satisfying the following conditions:

(uITS-0) For every $S \in \mathscr{S}$, the diagram in $\mathbb{H}(S)$

$$\begin{array}{ccc} \mathbf{I}_S \otimes \mathbf{I}_S & = & \mathbf{I}_S \otimes \mathbf{I}_S \\ & {\scriptstyle u_r} \searrow \quad \swarrow {\scriptstyle u_l} & \\ & \mathbf{I}_S & \end{array}$$

is commutative.

(uITS-1) For every morphism $f \colon T \to S$ in $\mathscr{S}$, the two diagrams of functors $\mathbb{H}(S) \to \mathbb{H}(T)$

$$\begin{array}{ccc} f^*A \tilde{\otimes} f^*\mathbf{I}_S & \xrightarrow{\mathbf{I}^*} & f^*A \tilde{\otimes} \mathbf{I}_T \\ \downarrow{\scriptstyle \tilde{m}} & & \downarrow{\scriptstyle u_r} \\ f^*(A \tilde{\otimes} \mathbf{I}_S) & \xrightarrow{u_r} & f^*A \end{array} \qquad \begin{array}{ccc} f^*\mathbf{I}_S \tilde{\otimes} f^*B & \xrightarrow{\mathbf{I}^*} & \mathbf{I}_T \tilde{\otimes} f^*B \\ \downarrow{\scriptstyle \tilde{m}} & & \downarrow{\scriptstyle u_l} \\ f^*(\mathbf{I}_S \tilde{\otimes} B) & \xrightarrow{u_l} & f^*B \end{array}$$

are commutative.

We only consider *triangulated internal unit constraints*, as in [Ter24a, Defn. 5.9]: in this case, the natural isomorphisms $u_{r,S}$ and $u_{l,S}$ are required to be compatible with shifts in each variable in the obvious way.

3.11. Proposition. *Let u be a triangulated internal unit constraint on $(\otimes, m)$ with unit section $\mathbf{I}$; let $\tilde{\mathbf{I}}$ denote the section of $\mathbb{H}$ over $\mathscr{S}$ extending $\mathbf{I}$ as in Proposition 1.12. Then, there exists a unique triangulated internal unit constraint $\tilde{u}$ for $(\tilde{\otimes}, \tilde{m})$ with unit section $\tilde{\mathbf{I}}$ whose restriction to $\mathscr{U}$ coincides with u.*

The single natural isomorphisms $\tilde{u}_{r,S}$ and $\tilde{u}_{l,S}$ can be constructed as follows:

3.12. Lemma. *Fix $S \in \mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the composite triangulated natural isomorphisms of functors $\mathbb{H}(S) \to \mathbb{H}(S)$*

$$u_r^{(s)} = u_{r,S}^{(s)} \colon A \otimes_s \mathbf{I}_S^{(s)} := s^*(s_*A \otimes s_*s^*\mathbf{I}_Y) \xleftarrow{\eta} s^*(s_*A \otimes \mathbf{I}_Y) \xrightarrow{u_r} s^*s_*A \xrightarrow{\epsilon} A$$

and

$$u_l^{(s)} = u_{l,S}^{(s)} \colon \mathbf{I}_S^{(s)} \otimes_s A := s^*(s_*s^*\mathbf{I}_Y \otimes s_*A) \xleftarrow{\eta} s^*(\mathbf{I}_Y \otimes s_*A) \xrightarrow{u_r} s^*s_*A \xrightarrow{\epsilon} A.$$

Then, the induced triangulated natural isomorphisms of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$\tilde{u}_r = \tilde{u}_{r,S} \colon A \tilde{\otimes} \tilde{\mathbf{I}}_S = A \otimes_s \mathbf{I}_S^{(s)} \xrightarrow{u_r^{(s)}} A, \qquad \tilde{u}_l = \tilde{u}_{l,S} \colon \tilde{\mathbf{I}}_S \tilde{\otimes} B = \mathbf{I}_S^{(s)} \otimes_s B \xrightarrow{u_l^{(s)}} B$$

are independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$.

PROOF. The invertibility of the two unit arrows $\eta\colon s^*(s_*A \otimes \mathbf{I}_Y) \to s^*(s_*A \otimes s_*s^*\mathbf{I}_Y)$ and $\eta\colon s^*(\mathbf{I}_Y \otimes s_*B) \to s^*(s_*s^*\mathbf{I}_Y \otimes s_*B)$ can be checked as usual, via the associated localization triangles. As a consequence, the natural isomorphisms $u^{(s)}_{r,S}$ and $u^{(s)}_{l,S}$ are well-defined.

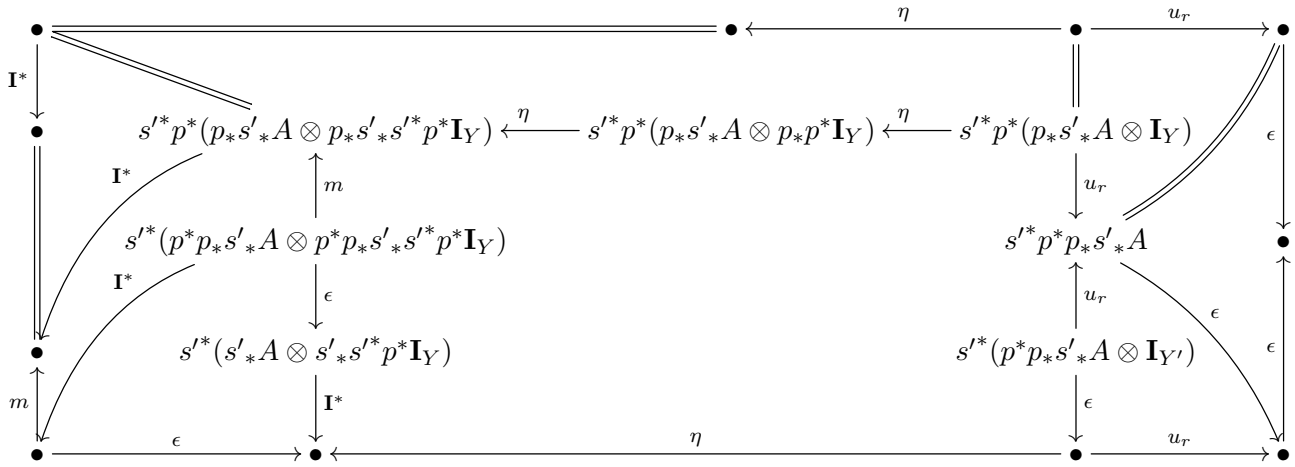
Let us show that the resulting natural isomorphism $\tilde{u}_{r,S}$ is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$; the same argument works for $\tilde{u}_{l,S}$. Using the connectedness of the embedding category via products (given by Lemma 1.8), we are reduced to proving the following claim: given a morphism $p\colon (Y'; s') \to (Y; s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, the diagram of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} A \otimes_s \mathbf{I}^{(s)}_S & & \\ \downarrow{\scriptstyle \alpha_p} & \searrow^{u^{(s)}_r} & \\ A \otimes_s \mathbf{I}^{(s')}_S & & A \\ \downarrow{\scriptstyle \phi_p} & \nearrow_{u^{(s')}_r} & \\ A \otimes_{s'} \mathbf{I}^{(s')}_S & & \end{array}$$

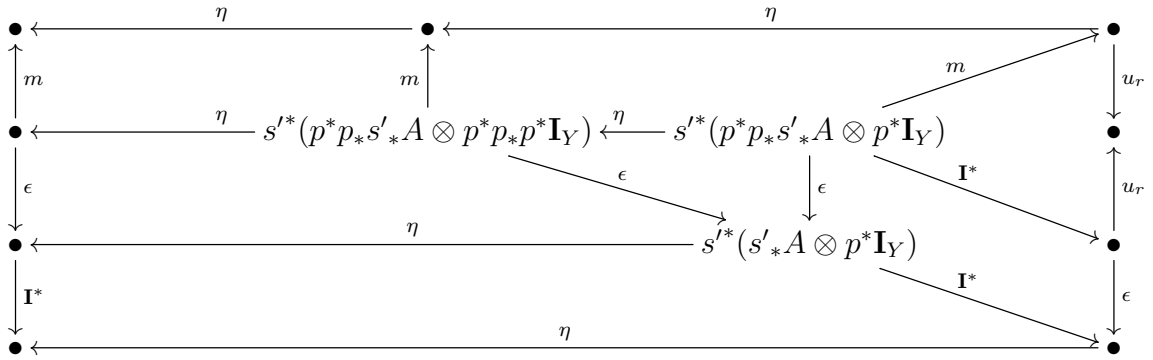
is commutative. Unwinding the various definitions, we obtain the more explicit diagram

$$\begin{array}{ccccccc} s^*(s_*A \otimes s_*s'^*p^*\mathbf{I}_Y) & = & s^*(s_*A \otimes s_*s^*\mathbf{I}_Y) & \xleftarrow{\eta} & s^*(s_*A \otimes \mathbf{I}_Y) & \xrightarrow{u_r} & s^*s_*A \\ \downarrow{\scriptstyle \mathbf{I}^*} & & & & & & \\ s^*(s_*A \otimes s_*s'^*\mathbf{I}_{Y'}) & & & & & & \downarrow{\scriptstyle \epsilon} \\ \| & & & & & & \\ & & & & & & A \\ & & & & & & \uparrow{\scriptstyle \epsilon} \\ s'^*p^*(p_*s'_*A \otimes p_*s'_*s'^*\mathbf{I}_{Y'}) & & & & & & \\ \uparrow{\scriptstyle m} & & & & & & \\ s'^*(p^*p_*s'_*A \otimes p^*p_*s'_*s'^*\mathbf{I}_{Y'}) & \xrightarrow{\epsilon} & s'^*(s'_*A \otimes s'_*s'^*\mathbf{I}_{Y'}) & \xleftarrow{\eta} & s'^*(s'_*A \otimes \mathbf{I}_{Y'}) & \xrightarrow{u_r} & s'^*s'_*A, \end{array}$$

which we decompose as



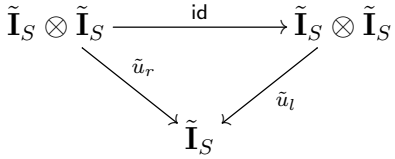
Here, the upper piece commutes by construction, while the six lateral pieces commute by naturality. We are thus reduced to considering the central rectangle, which we further decompose as



Here, the upper-right triangle commutes by axiom (uITS-2) over $\mathscr{U}$, while all the remaining pieces commute by naturality and by construction. This proves the claim. ∎

Proof of Proposition 3.11. For sake of brevity, we only show that the natural isomorphisms $\tilde{u}_{r,S}$ and $\tilde{u}_{l,S}$ constructed in Lemma 3.12 define an internal unit constraint for $(\tilde{\otimes}, \tilde{m})$; the other properties are left to the interested reader.

Let us show that the extended natural isomorphisms $\tilde{u}_r$ and $\tilde{u}_l$ satisfy conditions (uITS-1) and (uITS-2). We start from condition (uITS-1): given $S \in \mathscr{S}$, we have to show that the diagram in $\mathbb{H}(S)$



is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}(S)$. Then, by construction, it suffices to show that the diagram

$$\begin{array}{ccc} \mathbf{I}_S^{(s)} \otimes_s \mathbf{I}_S^{(s)} & = & \mathbf{I}_S^{(s)} \otimes_s \mathbf{I}_S^{(s)} \\ & {\scriptstyle u_{r,S}^{(s)}} \searrow \quad \swarrow {\scriptstyle u_{l,S}^{(s)}} & \\ & \mathbf{I}_S^{(s)} & \end{array}$$

is commutative. Unwinding the definitions, we obtain the outer part of the diagram

$$\begin{array}{ccccccc} s^*(s_*s^*\mathbf{I}_Y \otimes s_*s^*\mathbf{I}_Y) & = & & = & & = & s^*(s_*s^*\mathbf{I}_Y \otimes s_*s^*\mathbf{I}_Y) \\ \uparrow{\scriptstyle \eta} & & & & & & \uparrow{\scriptstyle \eta} \\ s^*(s_*s^*\mathbf{I}_Y \otimes \mathbf{I}_Y) & \xleftarrow{\eta} & s^*(\mathbf{I}_Y \otimes \mathbf{I}_Y) & = & s^*(\mathbf{I}_Y \otimes \mathbf{I}_Y) & \xrightarrow{\eta} & s^*(\mathbf{I}_Y \otimes s_*s^*\mathbf{I}_Y) \\ \downarrow{\scriptstyle u_r} & & {\scriptstyle u_r}\searrow & & \swarrow{\scriptstyle u_l} & & \downarrow{\scriptstyle u_l} \\ s^*s_*s^*\mathbf{I}_Y & & \xrightarrow{\epsilon} & & \xleftarrow{\epsilon} & & s^*s_*s^*\mathbf{I}_Y \end{array}$$

Here, the central triangular piece commutes by axiom (uITS-1) over $\mathscr{U}$, while all the other pieces commute by naturality. This proves the claim.

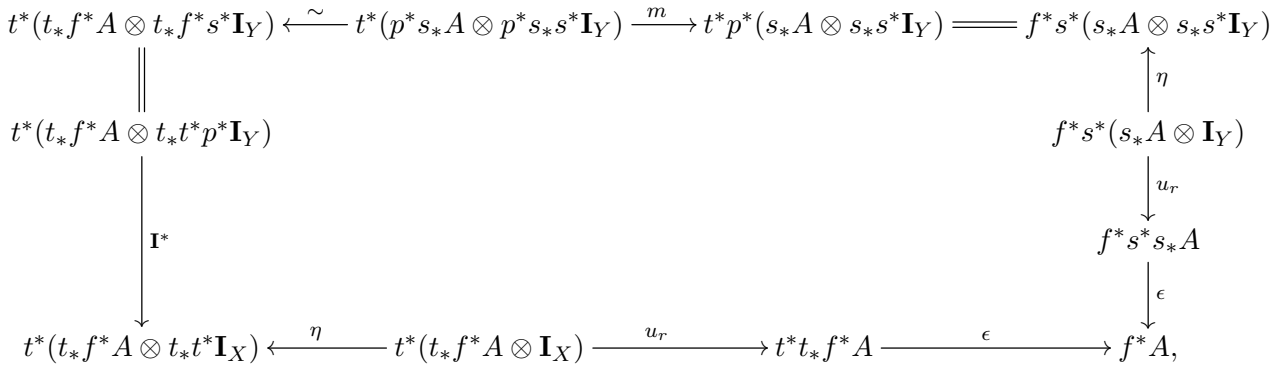
We now check condition (uITS-2): given a morphism $f\colon T \to S$ in $\mathscr{S}$, we have to show that the diagrams of functors $\mathbb{H}(S) \to \mathbb{H}(T)$

$$\begin{array}{ccc} f^*A\tilde{\otimes}f^*\tilde{\mathbf{I}}_S & \xrightarrow{\tilde{m}} & f^*(A\tilde{\otimes}\tilde{\mathbf{I}}_S) \\ \downarrow{\scriptstyle \tilde{\mathbf{I}}^*} & & \downarrow{\scriptstyle \tilde{u}_r} \\ f^*A\tilde{\otimes}\tilde{\mathbf{I}}_T & \xrightarrow{\tilde{u}_r} & f^*A \end{array} \qquad \begin{array}{ccc} f^*\tilde{\mathbf{I}}_S\tilde{\otimes}f^*B & \xrightarrow{\tilde{m}} & f^*(\tilde{\mathbf{I}}_S\tilde{\otimes}B) \\ \downarrow{\scriptstyle \tilde{\mathbf{I}}^*} & & \downarrow{\scriptstyle \tilde{u}_l} \\ \tilde{\mathbf{I}}_T\tilde{\otimes}f^*B & \xrightarrow{\tilde{u}_l} & f^*B \end{array}$$

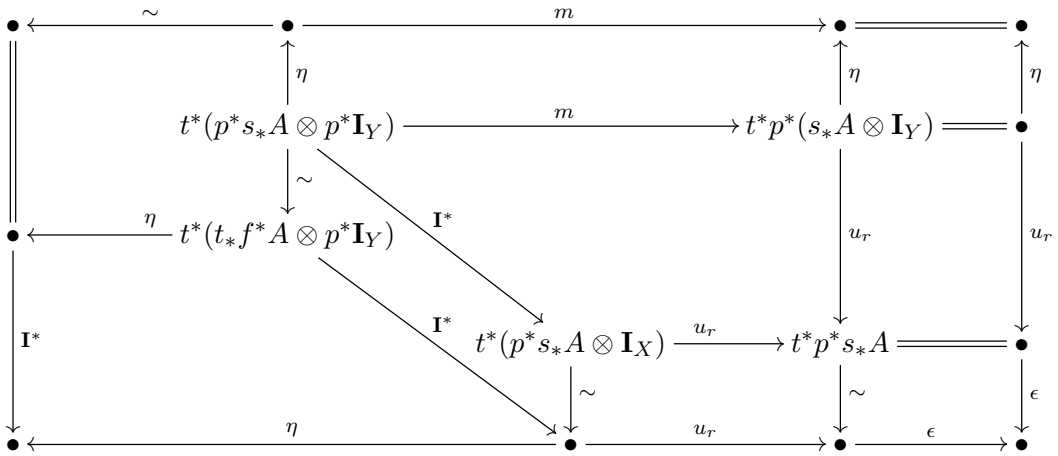
are commutative. For sake of brevity, we only treat the first diagram; the same argument works for the second diagram. To check the commutativity of the left-most diagram, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and then a factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; as usual, note that X automatically belongs to $\mathscr{U}$, so we obtain an embedding $(X; t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. By construction, it now suffices to show that the diagram

$$\begin{array}{ccc} f^*A \otimes_t f^*\mathbf{I}_S^{(s)} & \xrightarrow{m_f^{(s,t,p)}} & f^*(A \otimes_s \mathbf{I}_S^{(s)}) \\ \downarrow{\scriptstyle \mathbf{I}_f^{*,(s,t,p)}} & & \downarrow{\scriptstyle u_{r,S}^{(s)}} \\ f^*A \otimes_t \mathbf{I}_T^{(t)} & \xrightarrow{u_{r,T}^{(t)}} & f^*A \end{array}$$

is commutative. Unwinding the definitions, we obtain the more explicit diagram



which we decompose as



Here, the central inner piece commutes by axiom (uITS-2) over $\mathscr{U}$, while all the other pieces commute by naturality and by construction. This proves the claim, thereby concluding the proof. ∎

3.13. Extended compatibility conditions. As in the previous subsection, we work over a geometric triangulated $\mathscr{S}$-fibered category $\mathbb{H}$ endowed with a localic triangulated internal tensor structure $(\otimes, m)$ over $\mathscr{U}$, and we write $(\tilde{\otimes}, \tilde{m})$ for its extension over $\mathscr{S}$ obtained via Theorem 3.2. To complete the discussion of the present section, it remains to check that the natural mutual compatibility conditions between associativity, commutativity, and unit constraints on $(\otimes, m)$ pass automatically to $(\tilde{\otimes}, \tilde{m})$.

We start by considering associativity and commutativity, following [Ter24a, Defn. 2.16]. Two internal associativity and commutativity constraints a and c for $(\tilde{\otimes}, \tilde{m})$ are compatible if they satisfy the following additional condition:

(*ac*ITS) For every $S \in \mathscr{S}$, the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccccc}
(A\tilde{\otimes}B)\tilde{\otimes}C & \xrightarrow{c} & (B\tilde{\otimes}A)\tilde{\otimes}C & \xrightarrow{a} & B\tilde{\otimes}(A\tilde{\otimes}C) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle c} \\
A\tilde{\otimes}(B\tilde{\otimes}C) & \xrightarrow{c} & (B\tilde{\otimes}C)\tilde{\otimes}A & \xrightarrow{a} & B\tilde{\otimes}(C\tilde{\otimes}A)
\end{array}$$

is commutative.

The definition carries through without change to the triangulated setting.

3.14. Lemma. *Suppose that we are given compatible triangulated associativity and commutativity constraints a and c for $(\otimes, m)$. Then, the extended constraints $\tilde{a}$ and $\tilde{c}$ for $(\tilde{\otimes}, \tilde{m})$ are compatible as well.*

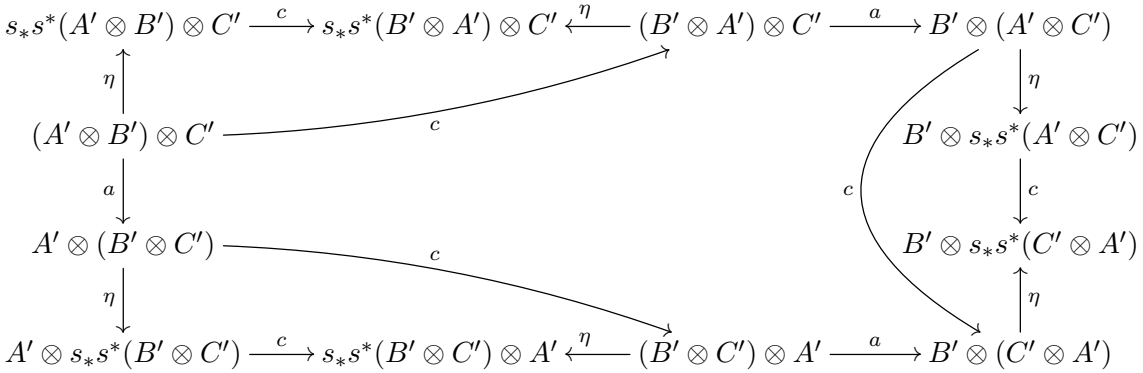
Proof. Let us show that the constraints $\tilde{a}$ and $\tilde{c}$ still satisfy condition (*ac*ITS): given $S \in \mathscr{S}$, we have to check that the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccccc}
(A\tilde{\otimes}B)\tilde{\otimes}C & \xrightarrow{\tilde{c}} & (B\tilde{\otimes}A)\tilde{\otimes}C & \xrightarrow{\tilde{a}} & B\tilde{\otimes}(A\tilde{\otimes}C) \\
\downarrow{\scriptstyle \tilde{a}} & & & & \downarrow{\scriptstyle \tilde{c}} \\
A\tilde{\otimes}(B\tilde{\otimes}C) & \xrightarrow{\tilde{a}} & (B\tilde{\otimes}C)\tilde{\otimes}A & \xrightarrow{\tilde{a}} & B\tilde{\otimes}(C\tilde{\otimes}A)
\end{array}$$

is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram

$$\begin{array}{ccccc}
(A \otimes_s B) \otimes_s C & \xrightarrow{c_S^{(s)}} & (B \otimes_s A) \otimes_S C & \xrightarrow{a_S^{(s)}} & B \otimes_s (A \otimes_s C) \\
\downarrow{\scriptstyle a_S^{(s)}} & & & & \downarrow{\scriptstyle c_S^{(s)}} \\
A \otimes_s (B \otimes_s C) & \xrightarrow{c_S^{(s)}} & (B \otimes_s C) \otimes_s A & \xrightarrow{a_S^{(s)}} & B \otimes_s (C \otimes_s A)
\end{array}$$

is commutative. Unwinding the definitions, we are reduced to considering the outer part of the diagram of functors $\mathbb{H}(Y) \times \mathbb{H}(Y) \times \mathbb{H}(Y) \to \mathbb{H}(Y)$



Here, the central piece commutes by axiom (*ac*ITS) over $\mathscr{U}$, while all the other pieces commute by naturality. ∎

Next, we consider associativity and unitarity, following [Ter24a, Defn. 2.20]. Two internal associaitivity and unit constraints a and u for $(\tilde{\otimes}, \tilde{m})$ are compatible if they satisfy the following additional condition:

(auITS) For every $S \in \mathscr{S}$, the three diagrams of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} (\mathbf{I}_S \tilde{\otimes} B)\tilde{\otimes} C & \xrightarrow{\quad a \quad} & \mathbf{I}_S \tilde{\otimes}(B \tilde{\otimes} C) \\ & {\scriptstyle u_l} \searrow \quad \swarrow {\scriptstyle u_l} & \\ & B \otimes C, & \end{array} \qquad \begin{array}{ccc} (A \tilde{\otimes} \mathbf{I}_S)\tilde{\otimes} C & \xrightarrow{\quad a \quad} & A \tilde{\otimes}(\mathbf{I}_S \tilde{\otimes} C) \\ & {\scriptstyle u_r} \searrow \quad \swarrow {\scriptstyle u_l} & \\ & A \tilde{\otimes} C & \end{array}$$

and

$$\begin{array}{ccc} (A \tilde{\otimes} B)\tilde{\otimes} \mathbf{I}_S & \xrightarrow{\quad a \quad} & A \tilde{\otimes}(B \tilde{\otimes} \mathbf{I}_S) \\ & {\scriptstyle u_r} \searrow \quad \swarrow {\scriptstyle u_r} & \\ & A \tilde{\otimes} B & \end{array}$$

are commutative.

The definition carries through without change to the triangulated setting.

3.15. Lemma. *Suppose that we are given compatible triangulated associativity and unit constraints a and u for $(\otimes, m)$. Then, the extended constraints $\tilde{a}$ and $\tilde{u}$ for $(\tilde{\otimes}, \tilde{m})$ are compatible as well.*

Proof. Let us show that the constraints $\tilde{c}$ and $\tilde{u}$ still satisfy condition (auITS): given $S \in \mathscr{S}$, we have to check that the three diagrams of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} (\tilde{\mathbf{I}}_S \tilde{\otimes} B)\tilde{\otimes} C & \xrightarrow{\quad \tilde{a} \quad} & \tilde{\mathbf{I}}_S \tilde{\otimes}(B \tilde{\otimes} C) \\ & {\scriptstyle \tilde{u}_l} \searrow \quad \swarrow {\scriptstyle \tilde{u}_l} & \\ & B \tilde{\otimes} C, & \end{array} \qquad \begin{array}{ccc} (A \tilde{\otimes} \tilde{\mathbf{I}}_S)\tilde{\otimes} C & \xrightarrow{\quad \tilde{a} \quad} & A \tilde{\otimes}(\tilde{\mathbf{I}}_S \tilde{\otimes} C) \\ & {\scriptstyle \tilde{u}_r} \searrow \quad \swarrow {\scriptstyle \tilde{u}_l} & \\ & A \tilde{\otimes} C & \end{array}$$

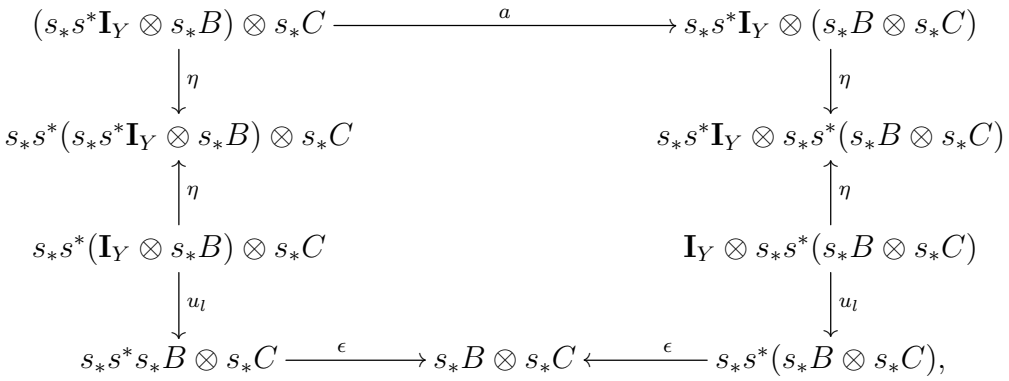
and

$$\begin{array}{ccc} (A \tilde{\otimes} B)\tilde{\otimes} \tilde{\mathbf{I}}_S & \xrightarrow{\quad \tilde{a} \quad} & A \tilde{\otimes}(B \tilde{\otimes} \tilde{\mathbf{I}}_S) \\ & {\scriptstyle \tilde{u}_r} \searrow \quad \swarrow {\scriptstyle \tilde{u}_r} & \\ & A \tilde{\otimes} B & \end{array}$$

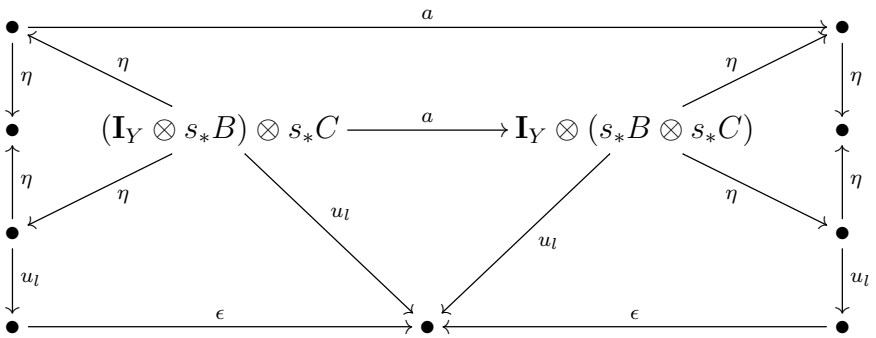
are commutative. For sake of brevity, we only treat the first diagram; the same argument works for the other two diagrams. To check the commutativity of the first diagram, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} (\mathbf{I}_S^{(s)} \otimes_s B) \otimes_s C & \xrightarrow{\quad a_S^{(s)} \quad} & \mathbf{I}_S^{(s)} \otimes_s (B \otimes_s C) \\ & {\scriptstyle u_{l,S}^{(s)}} \searrow \quad \swarrow {\scriptstyle u_{l,S}^{(s)}} & \\ & B \otimes_s C & \end{array}$$

is commutative. Unwinding the various definitions, we are reduced to considering the diagram of functors $\mathbb{H}(S) \times \mathbb{H}(S) \to \mathbb{H}(Y)$



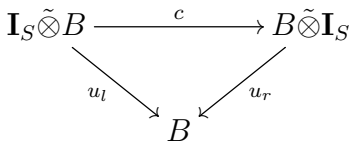
which we decompose as



Here, the central triangle commutes by axiom (auITS) over $\mathscr{U}$, while all the other pieces commute by naturality and by construction. ∎

Lastly, we consider commutativity and unitarity, following [Ter24a, Defn. 2.24]. Two internal commutativity and unit constraints c and u for $(\tilde{\otimes}, \tilde{m})$ are compatible if they satisfy the following additional condition:

(cuITS) For every $S \in \mathscr{S}$, the diagram of functors $\mathbb{H}(S) \to \mathbb{H}(S)$



is commutative.

The definition carries through without change to the triangulated setting.

3.16. Lemma. *Suppose that we are given compatible triangulated commutativity and unit constraints c and u for $(\otimes, m)$. Then, the extended constraints $\tilde{c}$ and $\tilde{u}$ for $(\tilde{\otimes}, \tilde{m})$ are compatible as well.*

PROOF. Let us show that the constraints $\tilde{c}$ and $\tilde{u}$ still satisfy condition (*cu*ITS): given $S \in \mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} \tilde{\mathbf{I}}_S \tilde{\otimes} B & \xrightarrow{\tilde{c}} & B \tilde{\otimes} \tilde{\mathbf{I}}_S \\ & {\scriptstyle \tilde{u}_l} \searrow \quad \swarrow {\scriptstyle \tilde{u}_r} & \\ & B & \end{array}$$

is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$\begin{array}{ccc} \mathbf{I}_S^{(s)} \otimes_s B & \xrightarrow{c_S^{(s)}} & B \otimes_s \mathbf{I}_S^{(s)} \\ & {\scriptstyle u_{l,S}^{(s)}} \searrow \quad \swarrow {\scriptstyle u_{r,S}^{(s)}} & \\ & B & \end{array}$$

is commutative. Unwinding the various definitions, we can write it more explicitly as the outer part of the diagram

$$\begin{array}{ccc} s^*(s_* s^* \mathbf{I}_Y \otimes s_* B) & \xrightarrow{c} & s^*(s_* B \otimes s_* s^* \mathbf{I}_Y) \\ {\scriptstyle \eta} \uparrow & & \uparrow {\scriptstyle \eta} \\ s^*(\mathbf{I}_Y \otimes s_* B) & \xrightarrow{c} & s^*(s_* B \otimes \mathbf{I}_Y) \\ & {\scriptstyle u_l} \searrow \quad \swarrow {\scriptstyle u_l} & \\ & s^* s_* B & \\ & {\scriptstyle \epsilon} \downarrow & \\ & B & \end{array}$$

Here, the lower triangle commutes by axiom (*cu*ITS) over $\mathscr{U}$, while the other piece commutes by naturality. ∎

4. Extending monoidal morphisms

Throughout this final section, we fix two geometric triangulated $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$ endowed with localic triangulated internal tensor structures $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively; we also fix a localic morphism of triangulated $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$. We are going to prove an extension result for monoidal structures on R (with respect to the given monoidal structures of $\mathbb{H}_1$ and $\mathbb{H}_2$), compatible with associativity, commutativity, and unit constraints.

As the reader can guess, one could also start from localic internal tensor structures on the underlying $\mathscr{U}$-fibered categories of $\mathbb{H}_1$ and $\mathbb{H}_2$, and from a localic morphism of $\mathscr{U}$-fibered categories, and extend everything at once: the single monoidal structures, the

morphism, and also any monoidal structure on the latter. For notational simplicity, we prefer to work directly with the extended monoidal structures on $\mathbb{H}_1$ and $\mathbb{H}_2$ and the extended morphism. However, in the proof of the results below, we will often need to use the constructions of Sections 2 and 3 implicitly.

4.1. Extending internal tensor structures on morphisms. We are interested in studying *internal tensor structures* ρ on R (with respect to $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$). Following [Ter24a, Defn. 3.6], by this we mean the datum of

- for every $S \in \mathscr{S}$, a natural isomorphism of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\rho = \rho_S : R_S(A) \otimes_2 R_S(B) \xrightarrow{\sim} R_S(A \otimes_1 B),$$

  called the *internal R-monoidality isomorphism* at S

satisfying the following condition:

(mor-ITS) For every morphism $f\colon T \to S$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\begin{array}{ccccc} f^*R_S(A) \otimes_2 f^*R_S(B) & \xrightarrow{m_2} & f^*(R_S(A) \otimes_2 R_S(B)) & \xrightarrow{\rho} & f^*R_S(A \otimes_1 B) \\ \downarrow{\scriptstyle \theta} & & & & \downarrow{\scriptstyle \theta} \\ R_T(f^*A) \otimes_2 R_T(f^*B) & \xrightarrow{\rho} & R_T(f^*A \otimes_1 f^*B) & \xrightarrow{m_1} & R_T(f^*(A \otimes_1 B)) \end{array}$$

  is commutative.

In the triangulated setting, it only makes sense to consider *triangulated internal tensor structures* on R, as defined in [Ter24a, Defn. 6.5]: in this case, the R-monoidality isomorphisms are required to be compatible with shifts in each variable. No additional condition is needed in the localic setting.

Note that any triangulated internal tensor structure on R defines by restriction a triangulated internal tensor structure on the underlying morphism of $\mathscr{U}$-fibered categories. In the reverse direction, we have the following result:

4.2. Theorem. *Let ρ be a triangulated internal tensor structure on the morphism of geometric $\mathscr{U}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$. Then, there exists a unique triangulated internal tensor structure $\tilde{\rho}$ on the morphism of $\mathscr{S}$-fibered categories R whose restriction to $\mathscr{U}$ coincides with ρ.*

The result applies, in particular, to monoidal morphisms between stable homotopy 2-functors, as axiomatized in [Ayo10].

As usual, the general idea is that, if the conclusion of the theorem holds, then the natural isomorphism $\tilde{\rho}_S$ must be determined from ρ_Y for every choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. With the notation in the statement of the theorem, the precise construction goes as follows:

4.3. Lemma. *Fix $S \in \mathscr{S}$. For every embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$, consider the composite natural isomorphism of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$*

$$\begin{array}{ccc}
\rho_S^{(s)}\colon & R_S^{(s)}(A) \otimes_{2,s} R_S^{(s)}(B) & R_S^{(s)}(A \otimes_{1,s} B). \\
& \| & \| \\
& s^*(s_*s^*R_Y(s_*A) \otimes_2 s_*s^*R_Y(s_*B)) & s^*R_Y(s_*s^*(s_*A \otimes s_*B)) \\
& \uparrow{\scriptstyle \eta} & \uparrow{\scriptstyle \eta} \\
& s^*(R_Y(s_*A) \otimes_2 R_Y(s_*B)) & \xrightarrow{\quad\rho\quad} s^*R_Y(s_*A \otimes_1 s_*B)
\end{array}$$

Then, the induced natural isomorphism of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\tilde{\rho} = \tilde{\rho}_S\colon R_S(A) \otimes_2 R_S(B) = R_S^{(s)}(A) \otimes_{2,s} R_S^{(s)}(B) \xrightarrow{\rho_S^{(s)}} R_S^{(s)}(A \otimes_{1,s} B) = R_S(A \otimes_1 B)$$

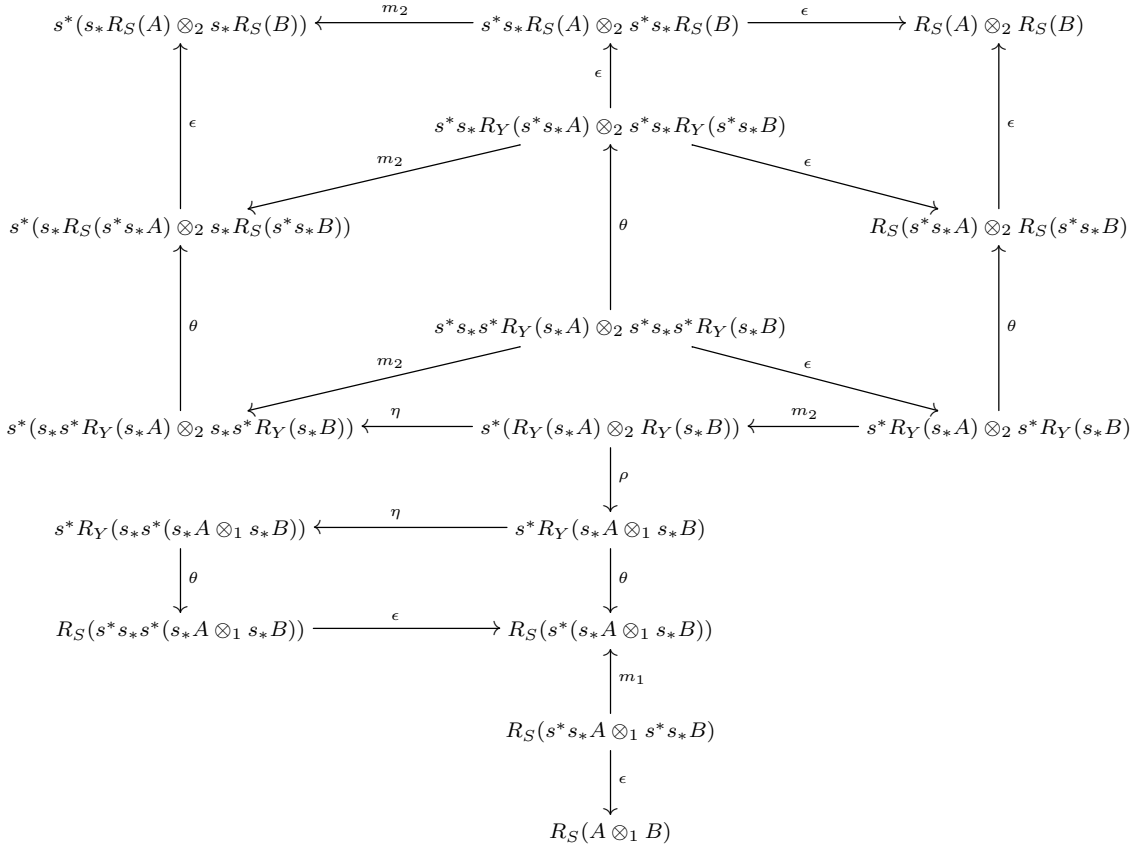
coincides with the composite natural isomorphism

$$\begin{array}{ccc}
R_S(A) \otimes_2 R_S(B) & & R_S(A \otimes_1 B) \\
\uparrow{\scriptstyle \epsilon} & & \uparrow{\scriptstyle \epsilon} \\
R_S(s^*s_*A) \otimes_2 R_S(s^*s_*B) & & R_S(s^*s_*A \otimes_1 s^*s_*B) \\
\uparrow{\scriptstyle \theta} & & \downarrow{\scriptstyle m_1} \\
s^*R_Y(s_*A) \otimes s^*R_Y(s_*B) & & R_S(s^*(s_*A \otimes_1 s_*B)) \\
\downarrow{\scriptstyle m_2} & & \uparrow{\scriptstyle \theta} \\
s^*(R_Y(s_*A) \otimes_2 R_Y(s_*B)) & \xrightarrow{\quad\rho\quad} & s^*R_Y(s_*A \otimes_1 s_*B)
\end{array}$$

and is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$.

Proof. In the first place, we fix an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and show that the two natural isomorphisms in the statement coincide. Unwinding the two lateral natural isomorphisms appearing in the definition of $\tilde{\rho}_S$ (as in the proofs of Theorems 2.7 and 3.2), this amounts to showing that the outer part of the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to$

$\mathbb{H}_2(S)$



is commutative. Since all pieces commute by naturality and by construction, this proves the claim.

Let us show that the result natural isomorphism $\tilde{\rho}$ is independent of the choice of $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Using the connectedness of the embedding category $\mathsf{Emb}_{\mathscr{U}}(S)$ via products (given by Lemma 1.8), we are reduced to showing the following claim: given a morphism $p\colon (Y'; s') \to (Y; s)$ in $\mathsf{Emb}_{\mathscr{U}}(S)$, the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

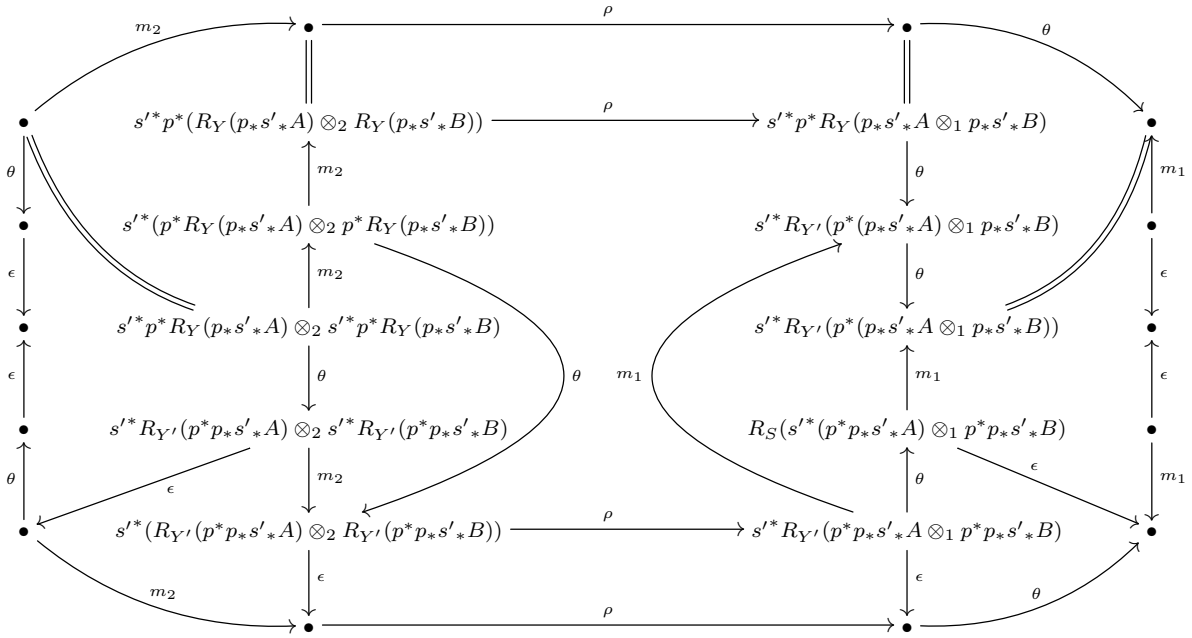
$$\begin{array}{ccc}
R_S^{(s)}(A) \otimes_{2,s} R_S^{(s)}(B) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}(A \otimes_{1,s} B) \\
\| & & \| \\
R_S(A) \otimes_2 R_S(B) & & R_S(A \otimes_1 B) \\
\| & & \| \\
R_S^{(s')}(A) \otimes_{2,s'} R_S^{(s')}(B) & \xrightarrow{\rho_S^{(s')}} & R_S^{(s')}(A \otimes_{1,s'} B)
\end{array}$$

is commutative. To this end, it is convenient to use the alternative description of the two composite natural transformations established above. Unwinding the definitions, we

obtain the more explicit diagram

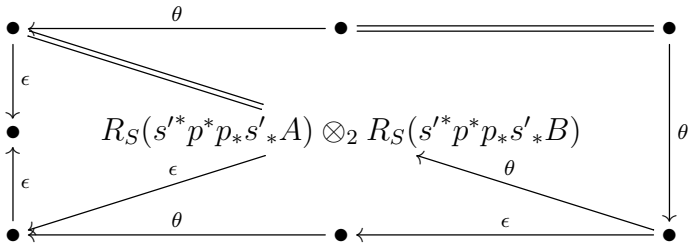
$$\begin{array}{ccc}
s^*(R_Y(s_*A) \otimes_1 R_Y(s_*B)) & \xrightarrow{\rho} & s^* R_Y(s_*A \otimes_1 s_*B) \\
\uparrow{\scriptstyle m_2} & & \downarrow{\scriptstyle \theta} \\
s^* R_Y(s_*A) \otimes_2 s^* R_Y(s_*B) & & R_S(s^*(s_*A \otimes_1 s_*B)) \\
\downarrow{\scriptstyle \theta} & & \uparrow{\scriptstyle m_1} \\
R_S(s^* s_* A) \otimes_2 R_S(s^* s_* B) & & R_S(s^* s_* A \otimes_1 s^* s_* B) \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\
R_S(A) \otimes_2 R_S(B) & & R_S(A \otimes_1 B) \\
\uparrow{\scriptstyle \epsilon} & & \uparrow{\scriptstyle \epsilon} \\
R_S(s'^* s'_* A) \otimes_2 R_S(s'^* s'_* B) & & R_S(s'^* s'_* A \otimes_1 s'^* s'_* B) \\
\uparrow{\scriptstyle \theta} & & \downarrow{\scriptstyle m_1} \\
s'^* R_Y(s'_* A) \otimes_2 s'^* R_Y(s'_* B) & & R_S(s'^*(s'_* A \otimes_1 s'_* B)) \\
\downarrow{\scriptstyle m_2} & & \uparrow{\scriptstyle \theta} \\
s'^*(R_Y(s'_* A) \otimes_2 R_Y(s'_* B)) & \xrightarrow{\rho} & s'^* R_Y(s'_* A \otimes_1 s'_* B),
\end{array}$$

which we decompose as

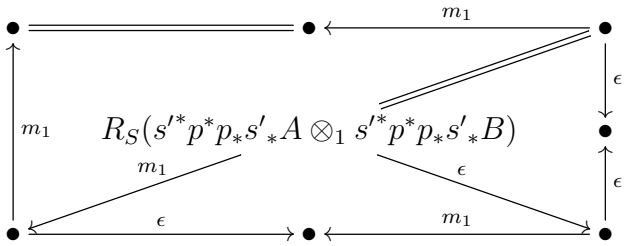


Here, the central piece commutes by axiom (mor-ITS) over $\mathscr{U}$, the upper-left piece commutes by axioms (mITS), and the upper-right piece commutes by axiom (mor-$\mathscr{S}$-fib).

Moreover, the central-upper and central-lower squares, the two lateral pieces in the central rectangle as well as the two lower-lateral pieces all commute by naturality. We are thus reduced to checking the commutativity of the two central lateral pieces, which we now study separately. The left-most central piece can be further decomposed as



where the upper-right triangle commutes by axiom (mor-$\mathscr{S}$-fib) while the other two pieces commute by naturality. Similarly, the right-most central piece can be further decomposed as



where the upper-left triangle commutes by axiom (mITS) while the other two pieces commute by naturality. This proves the claim, thereby concluding the proof. ∎

Proof of Theorem 4.2. For sake of brevity, we only show that the natural isomorphisms $\tilde{\rho}_S$ constructed in Lemma 4.3 define an internal tensor structure on R over $\mathscr{S}$; the other properties are left to the interested reader.

We have to check that the extended natural isomorphisms $\tilde{\rho}$ still satisfy condition (mor-ITS): given a morphism $f\colon T \to S$ in $\mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(T)$

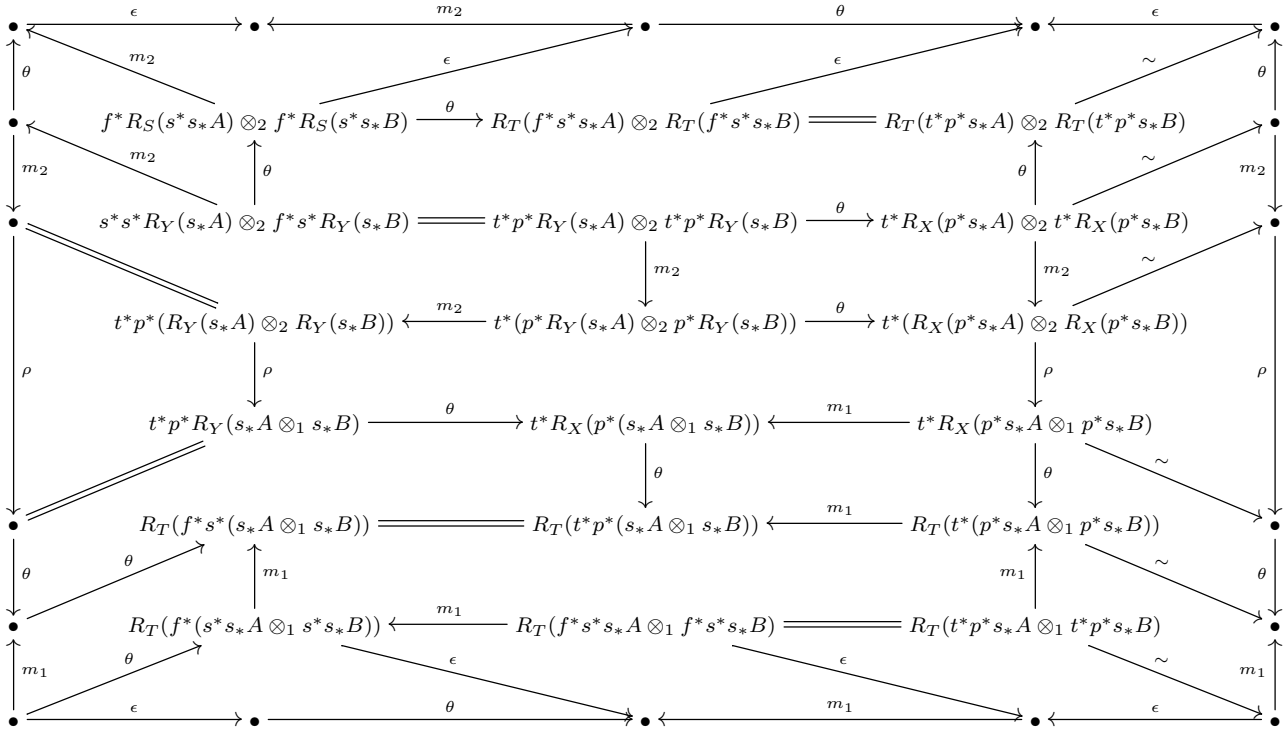
$$\begin{array}{ccc}
f^*R_S(A) \otimes_2 f^*R_S(B) & \xrightarrow{m_2} & f^*(R_S(A) \otimes_2 R_S(B)) \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \tilde{\rho}} \\
f^*R_S(A \otimes_1 B) & & R_T(f^*A) \otimes_2 R_T(f^*B) \\
\downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \theta} \\
R_T(f^*A \otimes_1 f^*B) & \xrightarrow{m_1} & R_T(f^*(A \otimes_1 B))
\end{array}$$

is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$ and then a factorization $(X; t, p) \in \mathsf{Fact}_{\mathscr{S}}(s \circ f)$; as usual, note that X automatically belongs to $\mathscr{U}$, so we obtain an embedding $(X; t) \in \mathsf{Emb}_{\mathscr{U}}(T)$. Unwinding the definitions of $\tilde{\rho}_S$ and $\tilde{\rho}_T$

according to the alternative formula provided in Lemma 4.3, we obtain the more explicit diagram

$$
\begin{array}{ccccc}
f^*(R_S(A)\otimes_2 R_S(B)) & \xleftarrow{m_2} & f^*R_S(A)\otimes_2 f^*R_S(B) & \xrightarrow{\theta} & R_T(f^*A)\otimes_2 R_T(f^*B) \\
\uparrow{\scriptstyle \epsilon} & & & & \uparrow{\scriptstyle \epsilon} \\
f^*(R_S(s^*s_*A)\otimes_2 R_S(s^*s_*B)) & & & & R_T(t^*t_*f^*A)\otimes_2 R_T(t^*t_*f^*B) \\
\uparrow{\scriptstyle \theta} & & & & \uparrow{\scriptstyle \theta} \\
f^*(s^*R_Y(s_*A)\otimes_2 s^*R_Y(s_*B)) & & & & t^*R_X(t_*f^*A)\otimes_2 t^*R_X(t_*f^*B) \\
\downarrow{\scriptstyle m_2} & & & & \downarrow{\scriptstyle m_2} \\
f^*s^*(R_Y(s_*A)\otimes_2 R_Y(s_*B)) & & & & t^*(R_X(t_*f^*A)\otimes_2 R_X(t_*f^*B)) \\
\downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle \rho} \\
f^*s^*R_Y(s_*A\otimes_1 s_*B) & & & & t^*R_X(t_*f^*A\otimes_1 t_*f^*B) \\
\downarrow{\scriptstyle \theta} & & & & \downarrow{\scriptstyle \theta} \\
f^*R_S(s^*(s_*A\otimes_1 s_*B)) & & & & R_T(t^*(t_*f^*A\otimes_1 t_*f^*B)) \\
\uparrow{\scriptstyle m_1} & & & & \uparrow{\scriptstyle m_1} \\
f^*R_S(s^*s_*A\otimes_1 s^*s_*B) & & & & R_T(t^*t_*f^*A\otimes_1 t^*t_*f^*B) \\
\downarrow{\scriptstyle \epsilon} & & & & \downarrow{\scriptstyle \epsilon} \\
f^*R_S(A\otimes_1 B) & \xrightarrow{\theta} & R_T(f^*(A\otimes_1 B)) & \xleftarrow{m_1} & R_T(f^*A\otimes_1 f^*B),
\end{array}
$$

which we decompose as



Here, the central rectangle commutes by axiom (mor-ITS) over $\mathscr{U}$, the other four six-term pieces commute by axioms (mor-$\mathscr{S}$-fib) and (mITS), and the remaining pieces commute by naturality and by construction. This proves the claim. ∎

4.4. Extending associative, commutative, and unitary morphisms. As in the previous subsection, we fix two geometric triangulated $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$ endowed with localic triangulated internal tensor structures $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively. Moreover, we fix a localic morphism of $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ as well as a triangulated internal tensor structure ρ on the underlying morphism of triangulated $\mathscr{U}$-fibered categories, and we write $\tilde{\rho}$ for its extension over $\mathscr{S}$ obtained via Theorem 4.2. To complete the discussion of the present section, it remains to check that $\tilde{\rho}$ inherits the associativity, commutativity, and unitarity properties enjoyed by ρ.

We start by studying associative morphisms, as in [Ter24a, Defn. 3.12]. Suppose we are given internal associativity constraints a_1 and a_2 for $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively. The internal tensor structure ρ is *associative* (with respect to a_1 and a_2) if the following condition holds:

(mor-aITS) For every $S \in \mathscr{S}$, the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccccc} (R_S(A) \otimes_2 R_S(B)) \otimes_2 R_S(C) & \xrightarrow{\rho} & R_S(A \otimes_1 B) \otimes_2 R_S(C) & \xrightarrow{\rho} & R_S((A \otimes_1 B) \otimes_1 C) \\ \downarrow{\scriptstyle a_2} & & & & \downarrow{\scriptstyle a_1} \\ R_S(A) \otimes_2 (R_S(B) \otimes_2 R_S(C)) & \xrightarrow{\rho} & R_S(A) \otimes_2 R_S(B \otimes_1 C) & \xrightarrow{\rho} & R_S(A \otimes_1 (B \otimes_1 C)) \end{array}$$

is commutative.

No additional conditions are required in the triangulated setting.

4.5. Lemma. *Let a_1 and a_2 be triangulated associativity constraints for $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively, and suppose that ρ is associative (with respect to a_1 and a_2). Then, the extended tensor structure $\tilde{\rho}$ is associative as well (with respect to the extended constraints $\tilde{a}_1$ and $\tilde{a}_2$).*

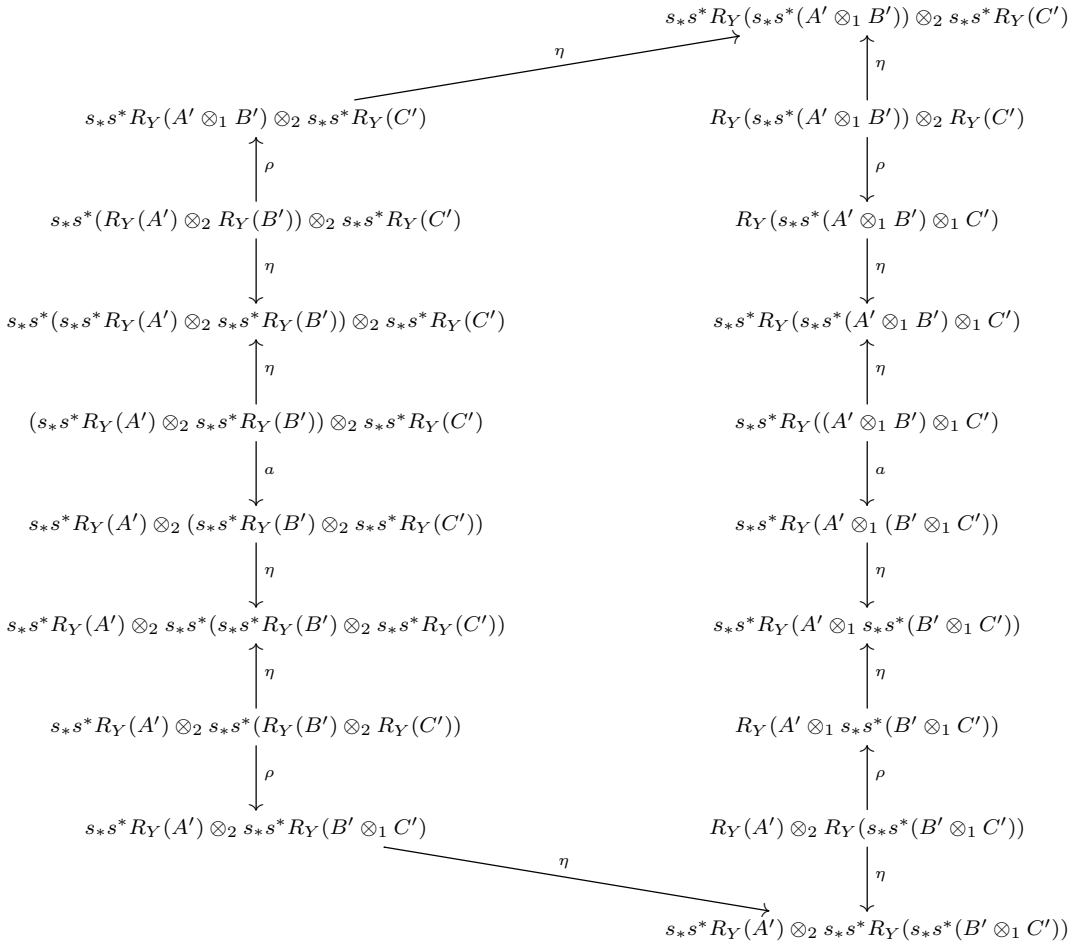
Proof. We need to show that condition (mor-aITS) is still satisfied after extension: given $S \in \mathscr{S}$, we have to check that the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccccc}
(R_S(A) \otimes_2 R_S(B)) \otimes_2 R_S(C) & \xrightarrow{\tilde{\rho}} & R_S(A \otimes_1 B) \otimes_2 R_S(C) & \xrightarrow{\tilde{\rho}} & R_S((A \otimes_1 B) \otimes_1 C) \\
\downarrow{\scriptstyle a_2} & & & & \downarrow{\scriptstyle a_1} \\
R_S(A) \otimes_2 (R_S(B) \otimes_2 R_S(C)) & \xrightarrow{\tilde{\rho}} & R_S(A) \otimes_2 R_S(B \otimes_1 C) & \xrightarrow{\tilde{\rho}} & R_S(A \otimes (B \otimes C))
\end{array}$$

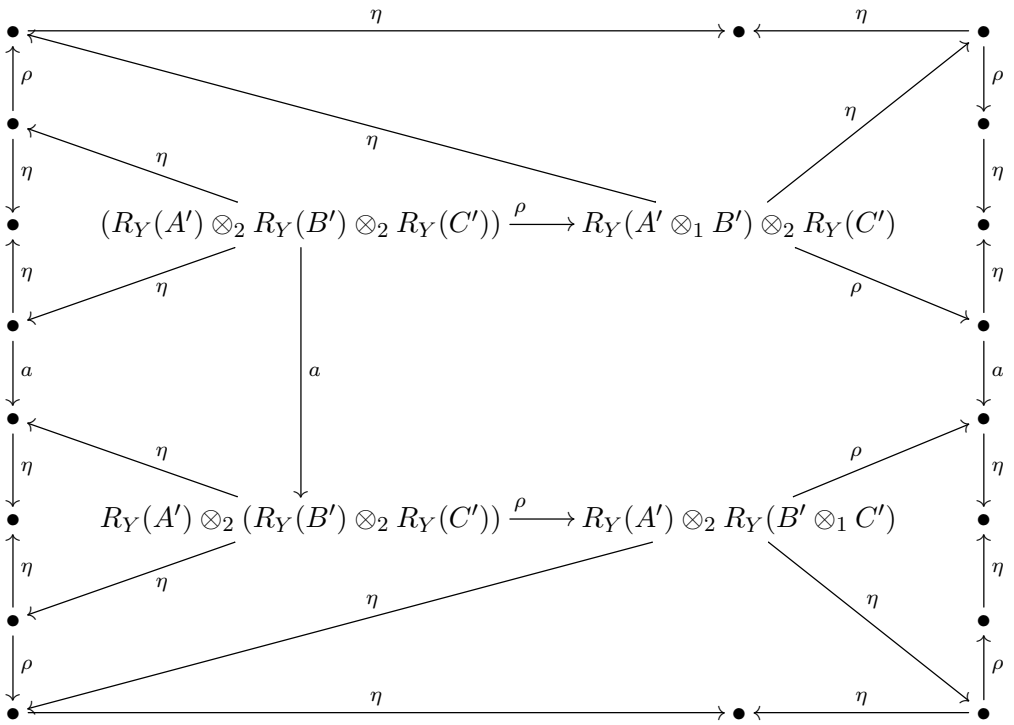
is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram

$$\begin{array}{ccccc}
(R_S^{(s)}(A) \otimes_{2,s} R_S^{(s)}(B)) \otimes_{2,s} R_S^{(s)}(C) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}(A \otimes_{1,s} B) \otimes_{2,s} R_S^{(s)}(C) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}((A \otimes_{1,s} B) \otimes_{1,s} C) \\
\downarrow{\scriptstyle a_{2,S}^{(s)}} & & & & \downarrow{\scriptstyle a_{1,S}^{(s)}} \\
R_S^{(s)}(A) \otimes_{2,s} (R_S^{(s)}(B) \otimes_{2,s} R_S^{(s)}(C)) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}(A) \otimes_{2,s} R_S^{(s)}(B \otimes_{1,s} C) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}(A \otimes_{1,s} (B \otimes_{1,s} C))
\end{array}$$

is commutative. Unwinding the various definitions, we are reduced to checking the commutativity of the diagram of functors $\mathbb{H}_1(Y) \times \mathbb{H}_1(Y) \times \mathbb{H}_1(Y) \to \mathbb{H}_2(Y)$



which we decompose as



Here, the central-right piece commutes by axiom (mor-aITS) over $\mathscr{U}$, while all the other pieces commute by naturality and by construction. ∎

Next, we study symmetric morphisms, as in [Ter24a, Defn. 3.16]. Suppose we are given internal commutativity constraints c_1 and c_2 for $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively. The internal tensor structure ρ is *symmetric* (with respect to c_1 and c_2) if the following condition holds:

(mor-cITS) For every $S \in \mathscr{S}$, the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccc} R_S(A) \otimes_2 R_S(B) & \xrightarrow{\rho} & R_S(A \otimes_1 B) \\ \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_1} \\ R_S(B) \otimes_2 R_S(A) & \xrightarrow{\rho} & R_S(B \otimes_1 A) \end{array}$$

is commutative.

No additional conditions are required in the triangulated setting.

4.6. Lemma. *Let c_1 and c_2 be triangulated commutativity constraints for $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively, and suppose that ρ is symmetric (with respect ot c_1 and c_2). Then, the extended tensor structure $\tilde{\rho}$ is symmetric as well (with respect to the extended constraints $\tilde{c}_1$ and $\tilde{c}_2$).*

PROOF. We need to check that condition (mor-cITS) is still satisfied after extension: given $S \in \mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}_1(S) \times \mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccc} R_S(A) \otimes_2 R_S(B) & \xrightarrow{\tilde{\rho}} & R_S(A \otimes_1 B) \\ \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_1} \\ R_S(B) \otimes_2 R_S(A) & \xrightarrow{\tilde{\rho}} & R_S(B \otimes_1 A) \end{array}$$

is commutative. To this end, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to show that the diagram

$$\begin{array}{ccc} R_S^{(s)}(A) \otimes_{2,s} R_S^{(s)}(B) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}(A \otimes_{1,s} B) \\ \downarrow{\scriptstyle c_{2,S}^{(s)}} & & \downarrow{\scriptstyle c_{1,S}^{(s)}} \\ R_S^{(s)}(B) \otimes_{2,s} R_S^{(s)}(A) & \xrightarrow{\rho_S^{(s)}} & R_S^{(s)}(B \otimes_{1,s} A) \end{array}$$

is commutative. Unwinding the various definitions, we are reduced to considering the diagram of functors $\mathbb{H}_1(Y) \times \mathbb{H}_1(Y) \to \mathbb{H}_2(Y)$

$$\begin{array}{ccccccc} s_*s^*R_Y(A') \otimes_2 s_*s^*R_Y(B') & \xleftarrow{\eta} & R_Y(A') \otimes_2 R_Y(B') & \xrightarrow{\rho} & R_Y(A' \otimes_1 B') & \xrightarrow{\eta} & s_*s^*R_Y(A' \otimes_1 B') \\ \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\ s_*s^*R_Y(B') \otimes_2 s_*s^*R_Y(A') & \xleftarrow{\eta} & R_Y(B') \otimes_2 R_Y(A') & \xrightarrow{\rho} & R_Y(B' \otimes_1 A') & \xrightarrow{\eta} & s_*s^*R_Y(B' \otimes_1 A'). \end{array}$$

Here, the central piece commutes by axiom (mor-cITS) over $\mathscr{U}$, while the other two pieces commute by naturality. ∎

Lastly, we study unitary morphisms, as in [Ter24a, Defn. 3.20]. Suppose we are given internal unit constraints u_1 and u_2 for $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, with unit sections $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$, respectively, as well as an isomorphism of sections $w\colon R(\mathbf{I}^{(1)}) \xrightarrow{\sim} \mathbf{I}^{(2)}$. The internal tensor structure ρ is *unitary via* w (with respect to u_1 and u_2) if the following condition holds:

(mor-uITS) For every $S \in \mathscr{S}$, the two diagrams of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccc} R_S(A) \otimes_2 R_S(\mathbf{I}_S^{(1)}) & \xrightarrow{w_S} & R_S(A) \otimes_2 \mathbf{I}_S^{(2)} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle u_{2,r}} \\ R_S(A \otimes_1 \mathbf{I}_S^{(1)}) & \xrightarrow{u_{1,r}} & R_S(A) \end{array} \qquad \begin{array}{ccc} R_S(\mathbf{I}_S^{(1)}) \otimes_2 R_S(B) & \xrightarrow{w_S} & \mathbf{I}_S^{(2)} \otimes_2 R_S(B) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle u_{2,l}} \\ R_S(\mathbf{I}_S^{(1)} \otimes_1 B) & \xrightarrow{u_{1,l}} & R_S(B) \end{array}$$

are commutative.

No additional conditions are required in the triangulated setting.

4.7. Lemma. *Let u_1 and u_2 be triangulated unit constraints with unit sections $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ for $(\otimes_1, m_1)$ and $(\otimes_2, m_2)$, respectively, and suppose that ρ is unitary via $w\colon R(\mathbf{I}^{(1)}) \xrightarrow{\sim} \mathbf{I}^{(2)}$ (with respect to to u_1 and u_2). Then, the extended tensor structure $\tilde{\rho}$ is unitary as well, via the extended isomorphism $\tilde{w}\colon \tilde{R}(\tilde{\mathbf{I}}^{(1)}) \xrightarrow{\sim} \tilde{\mathbf{I}}^{(2)}$ (with respect to the extended constraints $\tilde{u}_1$ and $\tilde{u}_2$).*

Proof. We need to show that condition (mor-uITS) is still satisfied after extension: given $S \in \mathscr{S}$, we have to check that the two diagrams of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(S)$

$$\begin{array}{ccc} R_S(A) \otimes_2 R_S(\mathbf{I}_S^{(1)}) & \xrightarrow{w_S} & R_S(A) \otimes_2 \mathbf{I}_S^{(2)} \\ \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle u_{2,r}} \\ R_S(A \otimes_1 \mathbf{I}_S^{(1)}) & \xrightarrow{u_{1,r}} & R_S(A) \end{array} \qquad \begin{array}{ccc} R_S(\mathbf{I}_S^{(1)}) \otimes_2 R_S(B) & \xrightarrow{w_S} & \mathbf{I}_S^{(2)} \otimes_2 R_S(B) \\ \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle u_{2,l}} \\ R_S(\mathbf{I}_S^{(1)} \otimes_1 B) & \xrightarrow{u_{1,l}} & R_S(B) \end{array}$$

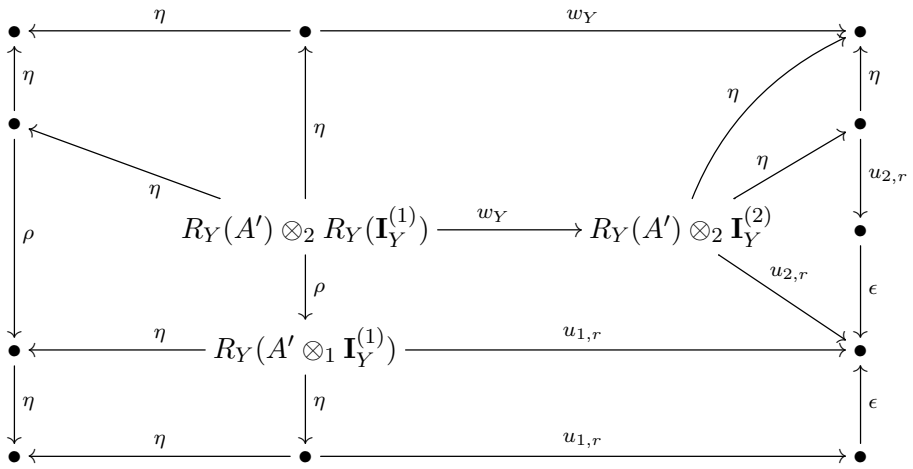
are commutative. For sake of brevity, we only treat the left-most diagram; the same argument works for the right-most diagram. To check the commutativity of the left-most diagram, choose an embedding $(Y; s) \in \mathsf{Emb}_{\mathscr{U}}(S)$. Then, by construction, it suffices to check that the diagram

$$\begin{array}{ccc} R_S^{(s)}(A) \otimes_2 R_S^{(s)}(\mathbf{I}_S^{(1,s)}) & \xrightarrow{w_S^{(s)}} & R_S^{(s)}(A) \otimes_2 \mathbf{I}_S^{(2,s)} \\ \downarrow{\scriptstyle \rho_S^{(s)}} & & \downarrow{\scriptstyle u_{2,r}^{(s)}} \\ R_S^{(s)}(A \otimes_1 \mathbf{I}_S^{(1,s)}) & \xrightarrow{u_{1,r}^{(s)}} & R_S^{(s)}(A) \end{array}$$

is commutative. Unwinding the various definitions, we are reduced to considering the diagram of functors $\mathbb{H}_1(Y) \to \mathbb{H}_2(Y)$

$$\begin{array}{ccccc} s_*s^*R_Y(A') \otimes_2 s_*s^*R_Y(s_*s^*\mathbf{I}_Y^{(1)}) & \xleftarrow{\eta} & s_*s^*R_Y(A') \otimes_2 s_*s^*R_Y(\mathbf{I}_Y^{(1)}) & \xrightarrow{w_Y} & s_*R_Y(A') \otimes_2 s_*s^*\mathbf{I}_Y^{(2)} \\ \uparrow{\scriptstyle \eta} & & & & \uparrow{\scriptstyle \eta} \\ R_Y(A') \otimes_2 R_Y(s_*s^*\mathbf{I}_Y^{(1)}) & & & & s_*s^*R_Y(A') \otimes_2 \mathbf{I}_Y^{(2)} \\ \downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle u_{2,r}} \\ R_Y(s_*A \otimes_1 s_*s^*\mathbf{I}_Y^{(1)}) & & & & s_*s^*R_Y(A') \\ \downarrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle \epsilon} \\ R_Y(s_*s^*(A' \otimes_1 s_*s^*\mathbf{I}_Y^{(1)})) & \xleftarrow{\eta} & R_Y(s_*s^*(A' \otimes_1 \mathbf{I}_Y^{(1)})) & \xrightarrow{u_{1,r}} & R_Y(s_*s^*A'), \end{array} \qquad \begin{array}{c} \\ \\ \\ \\ R_Y(A') \\ \uparrow{\scriptstyle \epsilon} \\ \\ \end{array}$$

which we decompose as



Here, the central piece commutes by axiom (mor-uITS) over $\mathscr{U}$, while all the other pieces commute by naturality and by construction. ∎

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