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CONSTRUCTING MONOIDAL STRUCTURES ON FIBERED CATEGORIES VIA FACTORIZATIONS

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Abstract. Let $\mathscr{S}$ be a small category, and suppose that we are given two (non-full) subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ that generate all morphisms of $\mathscr{S}$ under composition in the same way as morphisms of quasi-projective algebraic varieties are generated by smooth morphisms and closed immersions. We show that a monoidal structure on a given $\mathscr{S}$-fibered category is completely determined by its restrictions to $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$; in fact, any such pair of monoidal structures satisfying a natural compatibility condition uniquely determines a monoidal structure over the whole of $\mathscr{S}$. The same principle applies to morphisms of $\mathscr{S}$-fibered categories and monoidality thereof. Under further assumptions on the two subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$, and with suitable restrictions on the $\mathscr{S}$-fibered categories and morphisms involved, we provide a variant of the above method in which inverse images under closed immersion are partially replaced by the corresponding direct images; the latter variant is more adapted to the setting of perverse sheaves.

Contents

Introduction 2289
Notation and conventions 2296
1 Adjointability of fibered categories and tensor structures 2296
2 The skeleton of a morphism 2313
3 The core of a localic morphism 2321
4 External tensor skeleta 2332
5 External tensor cores 2339
References 2343

Introduction

Motivation and goal of the paper. When studying a small category $\mathscr{S}$, it often proves useful to single out some distinguished classes of morphisms that one can control

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with relatively little effort; if these distinguished classes generate all morphisms in $\mathscr{S}$ under composition, one can hope to gain some control on the whole of $\mathscr{S}$ in this way. This principle is particularly effective in Algebraic Geometry, where results about the existence of distinguished factorizations for reasonable morphisms of schemes are among the most useful auxiliary tools in several situations. In the present paper, we are interested in the abstract setting of $\mathscr{S}$-fibered categories, regarded as contravariant pseudo-functors with domain $\mathscr{S}$. There is a general factorization principle to turn collections of categories parametrized by the objects of $\mathscr{S}$ into actual $\mathscr{S}$-fibered categories: roughly speaking, one constructs inverse image functors along arbitrary morphisms of $\mathscr{S}$ by first defining them for distinguished classes of morphisms and then pasting the partial constructions together. However, as the reader can guess, this idea does not work unless the partial constructions are compatible in a precise sense.

In order to illustrate the general principle with a concrete example, consider the category Var_k of quasi-projective algebraic varieties over a field k. Among all morphisms in Var_k there exist two distinguished classes with particularly nice properties: the class of smooth morphisms and the class of closed immersions. It is known that every morphism $f\colon T \to S$ in Var_k admits some factorization of the form

$$f\colon T \xrightarrow{t} P \xrightarrow{p} S \tag{1}$$

with t a closed immersion and p a smooth morphism. We want to use these morphisms as our basic building blocks. So suppose that we are given a collection of categories $\{\mathbb{H}(S)\}_{S\in\mathsf{Var}_k}$ that we want to turn into a Var_k-fibered category: this amounts to constructing inverse image functors along all morphisms of Var_k, compatibly with composition in the natural way. Suppose also that we already know how to do this separately for the classes of smooth morphisms and closed immersions. Given a general morphism $f\colon T \to S$ in Var_k, one can try to define the inverse image functor $f^*\colon \mathbb{H}(S) \to \mathbb{H}(T)$ as the composite

$$f^*\colon \mathbb{H}(S) \xrightarrow{p^*} \mathbb{H}(P) \xrightarrow{t^*} \mathbb{H}(T),$$

according to a factorization of the above form (1). The whole point is to show that the resulting functor is independent of the chosen factorization of f, and that the construction is compatible with composition. Assuming to have a positive answer to the former question, let us focus on the latter. Given a second morphism $g\colon S \to V$ in Var_k, with factorization

$$g\colon S \xrightarrow{s} Q \xrightarrow{q} V$$

into a closed immersion s followed by a smooth morphism q, one gets a factorization of the composite morphism $gf\colon T \to V$ as

$$gf\colon T \xrightarrow{t} P \xrightarrow{p} S \xrightarrow{s} Q \xrightarrow{q} V.$$

Since the latter does not quite have the correct form, one is led to choose a further factorization of the composite morphism $sp\colon P \to Q$ as

$$sp\colon P \xrightarrow{h} L \xrightarrow{l} Q$$

with h a closed immersion and l a smooth morphism. In this way, one can finally write gf as

$$gf\colon T \xrightarrow{ht} L \xrightarrow{lq} V$$

where now, by construction, ht is a closed immersion while lq is a smooth morphism. At this point, if one wants to compare the composite of inverse images

$$f^*g^*\colon \mathbb{H}(V) \xrightarrow{q^*} \mathbb{H}(Q) \xrightarrow{s^*} \mathbb{H}(S) \xrightarrow{p^*} \mathbb{H}(P) \xrightarrow{t^*} \mathbb{H}(T)$$

with the inverse image of the composite

$$(gf)^*\colon \mathbb{H}(V) \xrightarrow{q^*} \mathbb{H}(Q) \xrightarrow{l^*} \mathbb{H}(L) \xrightarrow{p^*} \mathbb{H}(P) \xrightarrow{t^*} \mathbb{H}(T),$$

one needs to compare the two possible paths of inverse image functors along the commutative square

$$\begin{array}{ccc} P & \xrightarrow{h} & L \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle l} \\ S & \xrightarrow{s} & Q. \end{array} \tag{2}$$

The theory of *exchange structures*, developed in [Ayo07a, § 1.2], provides a rigorous method for this: it is based on the datum of additional natural isomorphisms attached to squares of the form (2), compatibly with respect to vertical and horizontal concatenation of such squares.

The main goal of this paper is to adapt the factorization principle to other construction problems within the theory of $\mathscr{S}$-fibered categories: specifically, we are interested in morphisms of $\mathscr{S}$-fibered categories, monoidal structures on $\mathscr{S}$-fibered categories, and monoidal morphisms. Compared to the case of $\mathscr{S}$-fibered categories discussed above, these questions turn out to be somewhat more elementary, since the task becomes to construct natural transformations between already existing functors rather than to construct functors themselves: coherence of natural transformations is a property rather than an additional structure.

Our main motivation comes from the theory of perverse sheaves over quasi-projective algebraic varieties, where one wants to exploit the factorization of morphisms into closed immersions and smooth morphisms as displayed above. To fix notation, let k be a subfield of the complex numbers, and consider again the category Var_k of quasi-projective k-varieties. To every $S \in \mathsf{Var}_k$ one can attach the algebraically constructible $\mathbb{Q}$-linear derived category $D^b_c(S)$, a tensor-triangulated category. As S varies, these enjoy a full six-functor formalism and, in particular, assemble into a monoidal Var_k-fibered category. The theory of [BBD82] yields a perverse t-structure on $D^b_c(S)$, whose heart is the abelian category $\mathsf{Perv}(S)$ of *perverse sheaves*. Beilinson's classical result [Bei87, Thm. 1.3] asserts that the triangulated category $D^b_c(S)$ is canonically equivalent to $D^b(\mathsf{Perv}(S))$. This brings the question of how much of the six-functor formalism can be reconstructed from the abelian categories $\mathsf{Perv}(S)$. What makes this question non-trivial is the fact that most

functors belonging to the six-functor formalism are not t-exact with respect to the perverse t-structures. For instance, the usual tensor product functor

$$-\otimes-\colon D_c^b(S)\times D_c^b(S)\to D_c^b(S)$$

does not respect the perverse t-structures if $\dim(S)\geq 1$. Following [Shu08] and [MV20], in our paper [Ter24a] we explained how the entire structure of the constructible derived categories, regarded as a triangulated monoidal Var_k-fibered category, can be expressed in terms of the external tensor product functors

$$-\boxtimes-\colon D_c^b(S_1)\times D_c^b(S_2)\to D_c^b(S_1\times S_2),$$

which are always t-exact in each variable (see [BBD82, Prop. 4.2.8]); this includes associativity, commutativity, and unit constraints. However, the structure of a monoidal Var_k-fibered category is encoded in the monoidality isomorphisms for inverse image functors. While general morphisms in Var_k do not enjoy any t-exactness properties, it turns out that smooth morphisms and closed immersions are sufficiently well-behaved: for a smooth morphism $p\colon P\to S$, the inverse image functor p^* is t-exact up to shifting appropriately (see [BBD82, § 4.2.4]), while, for a closed immersion $t\colon T\to P$, the direct image functor t_* is t-exact on the nose (see [BBD82, Cor. 4.1.3]). This suggests that, when working with the external tensor product, it should be possible to reconstruct the monoidality isomorphism under general morphisms in Var_k in terms of natural isomorphisms of t-exact functors, by means of a factorization of the form (1). This turns out to be the case, but with a caveat: rephrasing monoidality of inverse images under closed immersions in terms of the corresponding direct images is not a purely formal step, since it crucially uses an important property of the six-functor formalism known as *localization*.

The above discussion in the setting of perverse sheaves holds, more generally, for systems of triangulated categories satisfying the six-functor formalism and endowed with reasonable perverse t-structures. The most prominent example is Saito's celebrated theory of mixed Hodge modules, developed in [Sai90]: in this case, the triangulated categories of coefficients are actually defined as the derived categories of the perverse hearts, and their functoriality is constructed starting from the level of abelian categories. Our work enlightens important category-theoretic features of Saito's construction, especially around the monoidal structure on mixed Hodge modules, which are not addressed systematically in [Sai90].

We ultimately intend to apply our abstract results to the emerging theory of perverse Nori motives, a further motivic refinement of Saito's theory introduced by Ivorra–Morel in [IM24]. As in the case of mixed Hodge modules, one has a system of abelian categories of enhanced perverse sheaves and one wants to equip their derived categories with a six-functor formalism. In this new setting, however, the perverse hearts are defined by a purely categorical procedure, which forces one to be very careful about the formal properties of functors and natural transformations relating them. In our paper [Ter26], exploiting the general techniques developed here, we construct a monoidal structure on derived perverse Nori motives by working directly with the external tensor product on

the perverse hearts. The only part of the monoidal structure that the techniques of the present paper do not reach is the unit constraint; in the companion paper [Ter24b], we develop further techniques to address this specific point.

MAIN RESULTS. Both for sake of clarity and in order not to restrict the application range of our results unnecessarily, we formulate everything in the abstract language of monoidal fibered categories. Throughout the paper, we work over a fixed base category $\mathscr{S}$; for our constructions involving the external tensor product, we need to assume that $\mathscr{S}$ admits binary products. Since our main results are motivated by geometric situations, we are led to introduce suitable assumptions on $\mathscr{S}$: we suppose to be given two (non-full) subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ mimicking the behavior of smooth morphisms and closed immersions of quasi-projective algebraic varieties; for part of our constructions, we also assume the existence of special morphisms in $\mathscr{S}^{sm}$ mimicking the behavior of open immersions.

Our final goal is to obtain a factorization method adapted to the theory of perverse sheaves. The arguments of [IM24, § 4.2] served both as an inspiration and as a model for our abstract results. We aim at expressing various structures on $\mathscr{S}$-fibered categories in terms of inverse images under $\mathscr{S}^{sm}$ and direct images under $\mathscr{S}^{cl}$. In order to make this idea work, we need to introduce suitable assumptions on the $\mathscr{S}$-fibered categories considered: we reproduce the setting of stable homotopy 2-functors from [Ayo07a, Ayo07b] for what concerns the localization axiom, but avoiding any use of the homotopy and stability axioms. This is based on the notion of *localic* triangulated $\mathscr{S}$-fibered category, which takes our setting close to that of [DG22, § 3].

Even if we are mainly interested in applying the factorization method to the construction of monoidal structures on $\mathscr{S}$-fibered categories and of monoidal morphisms, it is useful to start by applying it to plain morphisms of $\mathscr{S}$-fibered categories: other than interesting in its own, this case is implicitly needed in the study of monoidal morphisms, and the main arguments used for monoidal structures are just a notationally more intricate variant of those for plain morphisms. It seems natural to divide our construction into two main steps: firstly, we describe a general factorization method based on inverse images under $\mathscr{S}^{sm}$ and under $\mathscr{S}^{cl}$; secondly, we explain how to replace inverse images under $\mathscr{S}^{cl}$ with the corresponding direct images in the localic case.

We introduce these two constructions axiomatically via the notions of *$\mathscr{S}$-skeleton* and of *$\mathscr{S}$-core* of a morphism, respectively. While the first notion makes sense in general, the second one works only for certain morphisms between localic $\mathscr{S}$-fibered categories, which it seems natural to call *localic* morphisms; these include all morphisms of stable homotopy 2-functors in the sense of [Ayo10]. The skeleton of a morphism of $\mathscr{S}$-fibered categories consists of its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl}$-fibered categories, subject to a natural coherence condition with respect to squares of the form (2). For the core of a localic morphism, the main difference is that one regards the underlying morphism of $\mathscr{S}^{cl}$-fibered categories as a morphism of $\mathscr{S}^{cl,op}$-fibered categories, by replacing inverse images under $\mathscr{S}^{cl}$ with the corresponding direct images. In this case, the coherence condition between the two restrictions is slightly less obvious: the rough idea is to decompose a square of the form (2) into a Cartesian square and two triangles, and to analyze the

three pieces separately. After this preamble, we can state our first main result as follows:

0.1. THEOREM. [Proposition 2.10 and Theorem 3.12] *Let $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a morphism of $\mathscr{S}$-fibered categories. The following statements hold:*

1. *Suppose that $\mathscr{S}$ satisfies Assumption 2.2. Then, the morphism R is completely determined by its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl}$-fibered categories. Conversely, a pair of morphisms over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ gives rise to one over $\mathscr{S}$ if and only if it defines an $\mathscr{S}$-skeleton.*

2. *Suppose that $\mathscr{S}$ satisfies the stronger Assumption 3.2. In addition, suppose that the $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$ are localic, and that R is localic as well. Then, the morphism R is completely determined by its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl,op}$-fibered categories. Conversely, a pair of morphisms over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl,op}$ gives rise to (a localic) one over $\mathscr{S}$ if and only if it defines an $\mathscr{S}$-core.*

To treat the case of monoidal structures on $\mathscr{S}$-fibered categories, we apply the same method to monoidality isomorphisms: firstly, we observe that these are always determined by those under $\mathscr{S}^{sm}$ and under $\mathscr{S}^{cl}$; secondly, we explain how to rephrase monoidality isomorphisms under $\mathscr{S}^{cl}$ in terms of direct images in the localic case. We formulate this precisely via the notions of *external tensor skeleton* and of *external tensor core*, respectively; these involve suitable coherence conditions with respect to squares of the form (2), analogous to those for $\mathscr{S}$-skeleta and $\mathscr{S}$-cores.

As the adjective "external" suggests, we have chosen to work systematically with the external tensor product in place of the usual internal tensor product: this is possible thanks to the results of [Shu08] and [MV20], specialized to the triangulated setting in [Ter24a]. Compared to the internal tensor product, the external tensor product is better suited to factorization questions, for at least two reasons: first, because it often enjoys better adjointability properties (see Remark 1.14); second, because it allows us to interpret monoidal structures on $\mathscr{S}$-fibered categories as suitable 2-parameter families of morphisms of $\mathscr{S}$-fibered categories, so that we can deduce our results about monoidal structures from those about plain morphisms. Let us mention that the notion of external tensor core is adapted to external tensor structures satisfying suitable adjointability properties, analogous to those of localic $\mathscr{S}$-fibered categories and localic morphisms: we call them *localic* external tensor structures. This said, we can state our second main result as follows:

0.2. THEOREM. [Proposition 4.4 and Theorem 5.4] *Let $\mathbb{H}$ be an $\mathscr{S}$-fibered category endowed with an external tensor structure $(\boxtimes, m)$. The following statements hold:*

1. *Suppose that $\mathscr{S}$ satisfies Assumption 2.2. Then, the external tensor structure $(\boxtimes, m)$ is completely determined by its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl}$-fibered categories. Conversely, a pair of external tensor structures over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ gives rise to one over $\mathscr{S}$ if and only if it defines an external tensor skeleton.*

2. *Suppose that $\mathscr{S}$ satisfies the stronger Assumption 3.2. In addition, suppose that the $\mathscr{S}$-fibered category $\mathbb{H}$ is localic, and that $(\boxtimes, m)$ is localic as well. Then, the external tensor structure $(\boxtimes, m)$ is completely determined by its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl,op}$-fibered categories. Conversely, a pair of external tensor structures over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl,op}$ gives rise to (a localic) one over $\mathscr{S}$ if and only if it defines an external tensor core.*

Lastly, we introduce analogous notions and obtain analogous results about monoidal structures on morphisms between monoidal $\mathscr{S}$-fibered categories; again, we formulate everything in terms of external tensor structures. We can state our third main result as follows:

0.3. Theorem. [Proposition 4.14 and Theorem 5.12] *For $j = 1, 2$ let $\mathbb{H}_j$ be an $\mathscr{S}$-fibered category endowed with an external tensor structure $(\boxtimes_j, m_j)$. In addition, let $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a morphism of $\mathscr{S}$-fibered categories, and let ρ be an external tensor structure on R (with respect to $(\boxtimes_1, m_1)$ and $(\boxtimes_2, m_2)$). The following statements hold:*

1. *Suppose that $\mathscr{S}$ satisfies Assumption 2.2. Then, the external tensor structure ρ is completely determined by its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl}$-fibered categories. Conversely, a pair of external tensor structures over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ gives rise to one over $\mathscr{S}$ if and only if it defines an external tensor skeleton.*

2. *Suppose that $\mathscr{S}$ satisfies the stronger Assumption 3.2. In addition, suppose that the $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$ are localic, that $(\boxtimes_1, m_1)$ and $(\boxtimes_2, m_2)$ are localic, and that R is localic as well. Then, the external tensor structure ρ is completely determined by its restrictions to the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl,op}$-fibered categories. Conversely, a pair of external tensor structures over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl,op}$ gives rise to (a localic) one over $\mathscr{S}$ if and only if it defines an external tensor core.*

Along the way, in order to complete our main results about monoidal structures, we discuss how the natural coherence conditions of associativity and commutativity constraints for the external tensor product (as formulated in [Ter24a, §§ 2,5]) can be also checked over the subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$. On the other hand, we do not include unit constraints into the picture, as they are not adapted to the setting of external tensor cores (but only to that of external tensor skeleta).

Structure of the paper. Throughout the paper, we work over a fixed small category $\mathscr{S}$; at the beginning of each section, we specify the natural conditions that $\mathscr{S}$ has to satisfy in order to make our constructions possible.

Section 1 is mostly devoted to recalling the general conventions and notation employed in the paper: in particular, we review the notions of $\mathscr{S}$-fibered categories (and morphisms thereof) and external tensor structures (both on single $\mathscr{S}$-fibered categories and on morphisms of such). We also discuss the natural notions of left-adjointability and right-adjointability in each setting (Definitions 1.3, 1.8 and 1.12), and we prove some preliminary results about these (Lemmata 1.5, 1.9, 1.13 and 1.20).

Starting from Section 2, we assume to be given two distinguished subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ of $\mathscr{S}$ mimicking the usual properties of smooth morphisms and closed immersions of quasi-projective algebraic varieties (Assumption 2.2). In Section 2 we introduce the notion of *$\mathscr{S}$-skeleton* of a morphism of $\mathscr{S}$-fibered categories (Definition 2.8), and we show that giving a morphism of $\mathscr{S}$-fibered categories is equivalent to giving the underlying $\mathscr{S}$-skeleton (Proposition 2.10).

In Section 3, we work under a more specific set of assumptions on the base category $\mathscr{S}$ and on its subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ (Assumption 3.2). In the first place, we introduce the notion of *localic* $\mathscr{S}$-fibered category (Definition 3.5) and prove its basic properties (Lemma 3.7). We then introduce the notion of *$\mathscr{S}$-core* for localic morphisms between localic $\mathscr{S}$-fibered categories (Definition 3.11), and we show that giving such a morphism is equivalent to giving the underlying $\mathscr{S}$-core (Theorem 3.12).

In Section 4, we go back to the general setting considered in Section 2; in order to consider external tensor structures on $\mathscr{S}$-fibered categories, we assume that $\mathscr{S}$ admits binary products. Following the same route as in Section 2, we introduce the notion of *external tensor skeleton* on an $\mathscr{S}$-fibered category (Definition 4.2), and we show that giving an external tensor structure is equivalent to giving the underlying external tensor skeleton (Proposition 4.4). A similar result applies to external tensor structures on morphisms of $\mathscr{S}$-fibered categories (Proposition 4.14).

Finally, in Section 5 we reinforce again the assumptions on $\mathscr{S}$ as done in Section 3, and we only consider localic $\mathscr{S}$-fibered categories; as in Section 4, we assume that $\mathscr{S}$ admits binary products. Following the same route as in Section 3, we introduce the notion of *external tensor core* on a localic $\mathscr{S}$-fibered category $\mathbb{H}$ (Definition 5.3), and we show that giving a localic external tensor structure on $\mathbb{H}$ is equivalent to giving the underlying external tensor core (Theorem 5.4). Again, a similar result applies to external tensor structures on localic morphisms of localic $\mathscr{S}$-fibered categories (Theorem 5.12).

Notation and conventions

- Unless otherwise dictated, categories are assumed to be small with respect to some fixed universe.
- Given a category $\mathscr{C}$, the notation $A \in \mathscr{C}$ means that A is an object of $\mathscr{C}$.
- Given categories $\mathscr{C}_1, \dots, \mathscr{C}_n$, we let $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ denote their direct product category.

1. Adjointability of fibered categories and tensor structures

Throughout the paper, we work over a fixed small category $\mathscr{S}$; in this section $\mathscr{S}$ can be arbitrary, except for the fact that we will need to assume the existence of binary products when discussing monoidal structures.

This first section is devoted to recollections and preliminary results: after fixing our notation and conventions about $\mathscr{S}$-fibered categories and monoidal structures, we discuss

various adjointability properties in the setting of $\mathscr{S}$-fibered categories; these will be used extensively in the following sections. For the time being, we fix a (possibly non-full) subcategory $\mathscr{S}'$ of $\mathscr{S}$; in the following sections, we will consider two such subcategories at once and use more expressive notation.

1.1. Adjointable fibered categories. We keep the general notation and conventions about $\mathscr{S}$-fibered categories adopted in [Ter24a], following [Ayo07a, Ayo07b] and [CD19]. By an *$\mathscr{S}$-fibered category* $\mathbb{H}$ we mean a strictly unital pseudo-functor from $\mathscr{S}^{op}$ to the 2-category of small categories. In detail, this is the datum of

- for every $S \in \mathscr{S}$, a category $\mathbb{H}(S)$,
- for every morphism $f\colon T \to S$ in $\mathscr{S}$, a functor
$$f^*\colon \mathbb{H}(S) \to \mathbb{H}(T),$$
called the *inverse image functor along* f,
- for every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, a natural isomorphism of functors $\mathbb{H}(V) \to \mathbb{H}(T)$
$$\mathsf{conn} = \mathsf{conn}_{f,g}\colon (gf)^*A \xrightarrow{\sim} f^*g^*A,$$
called the *connection isomorphism* at (f, g)

satisfying the following conditions:

($\mathscr{S}$-fib-0) For every $S \in \mathscr{S}$, we have $\mathsf{id}_S^* = \mathsf{id}_{\mathbb{H}(S)}$.

($\mathscr{S}$-fib-1) For every triple of composable morphisms $f\colon T \to S$, $g\colon S \to V$ and $h\colon V \to W$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}(W) \to \mathbb{H}(T)$
$$\begin{array}{ccc} (hgf)^*A & \xrightarrow{\mathsf{conn}_{f,hg}} & f^*(hg)^*A \\ \downarrow{\scriptstyle \mathsf{conn}_{gf,h}} & & \downarrow{\scriptstyle \mathsf{conn}_{g,h}} \\ (gf)^*h^*A & \xrightarrow{\mathsf{conn}_{f,g}} & f^*g^*h^*A \end{array}$$
is commutative.

1.2. Remark.

1. The strict unitality of axiom ($\mathscr{S}$-fib-0) is mostly a matter of notational convenience; it is not a severe requirement (for example, see [Del01, Lemma 2.5]).
2. Axiom ($\mathscr{S}$-fib-1) is designed in such a way that all composites of connection isomorphisms (and inverses thereof) are coherent: in practice, this means that one can treat connection isomorphisms as equalities. For sake of clarity, we write connection isomorphisms explicitly in the rest of this section; starting from Section 2, we will systematically write them as equalities in order to lighten our notation.

Note that any $\mathscr{S}$-fibered category defines by restriction an $\mathscr{S}'$-fibered category. We are interested in the following adjointability property with respect to inverse image functors along morphisms of $\mathscr{S}'$:

1.3. Definition. *We say that an $\mathscr{S}$-fibered category $\mathbb{H}$ is* left-$\mathscr{S}'$-adjointable *(resp.* right-$\mathscr{S}'$-adjointable*) if, for every morphism $f\colon T \to S$ in $\mathscr{S}'$, the functor $f^*\colon \mathbb{H}(S) \to \mathbb{H}(T)$ admits a left-adjoint $f_\#$ (resp. a right-adjoint f_*). For sake of brevity, in the case when $\mathscr{S}' = \mathscr{S}$ we simply say* left-adjointable *(resp.* right-adjointable*) instead.*

1.4. Remark. Of course, in view of axiom ($\mathscr{S}$-fib-0), for every $S \in \mathscr{S}'$ the inverse image functor $\mathsf{id}_S^* = \mathsf{id}_{\mathbb{H}(S)}$ has a canonical two-sided adjoint given by $\mathsf{id}_{\mathbb{H}(S)}$ itself. We will always implicitly choose this as the left-adjoint and right-adjoint for id_S^*.

As we now explain, the adjoints of the inverse image functors naturally assemble into an $(\mathscr{S}')^{op}$-fibered category:

1.5. Lemma. *Let $\mathbb{H}$ be an $\mathscr{S}$-fibered category.*

1. *Suppose that $\mathbb{H}$ is left-$\mathscr{S}'$-adjointable. Then, associating*

   - *to every morphism $f\colon T \to S$ in $\mathscr{S}'$, the functor*
   $$f_\#\colon \mathbb{H}(T) \to \mathbb{H}(S),$$
   - *to every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}'$, the natural isomorphism of functors $\mathbb{H}(T) \to \mathbb{H}(V)$*
   $$\overline{\mathsf{conn}} = \overline{\mathsf{conn}}_{f,g}\colon (gf)_\# A \xrightarrow{\sim} g_\# f_\# A$$
   defined as the composite
   $$\begin{array}{ccccccc} (gf)_\# A & & & & & & g_\# f_\# A \\ \downarrow{\scriptstyle \eta} & & & & & & \uparrow{\scriptstyle \epsilon} \\ (gf)_\# f^* f_\# A & \xrightarrow{\eta} & (gf)_\# f^* g^* g_\# f_\# A & \xleftarrow{\mathsf{conn}_{f,g}} & (gf)_\# (gf)^* g_\# f_\# A & & \end{array}$$

   gives an $(\mathscr{S}')^{op}$-fibered category.

2. *Symmetrically, suppose that $\mathbb{H}$ is right-$\mathscr{S}'$-adjointable. Then, associating*

   - *to every morphism $f\colon T \to S$ in $\mathscr{S}'$, the functor*
   $$f_*\colon \mathbb{H}(T) \to \mathbb{H}(S),$$
   - *to every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}'$, the natural isomorphism of functors $\mathbb{H}(T) \to \mathbb{H}(V)$*
   $$\overline{\mathsf{conn}} = \overline{\mathsf{conn}}_{f,g}\colon (gf)_* A \xrightarrow{\sim} g_* f_* A$$
   defined as the composite
   $$\begin{array}{ccccccc} (gf)_* A & & & & & & g_* f_* A \\ \downarrow{\scriptstyle \eta} & & & & & & \uparrow{\scriptstyle \epsilon} \\ g_* g^* (gf)_* A & \xrightarrow{\eta} & g_* f_* f^* g^* (gf)_* A & \xleftarrow{\mathsf{conn}_{f,g}} & g_* f_* (gf)^* (gf)_* A & & \end{array}$$

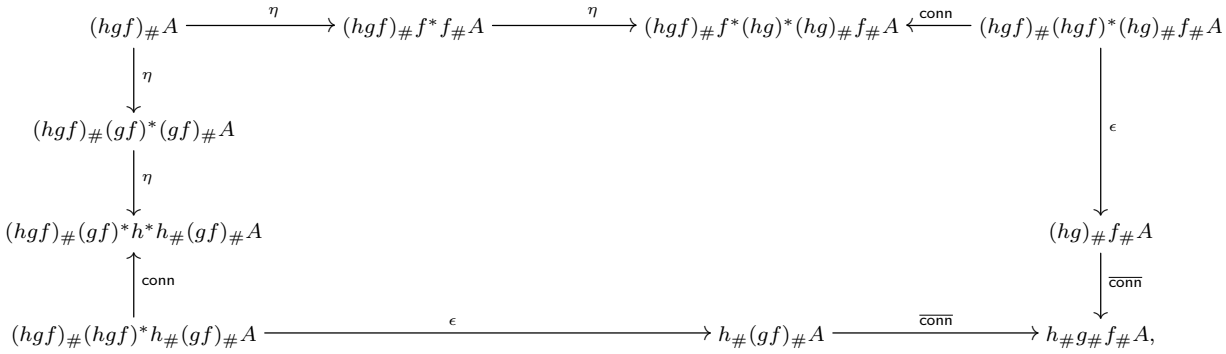
   gives an $(\mathscr{S}')^{op}$-fibered category.

PROOF. We only treat the case where $\mathbb{H}$ is left-$\mathscr{S}'$-adjointable; the other case is analogous.

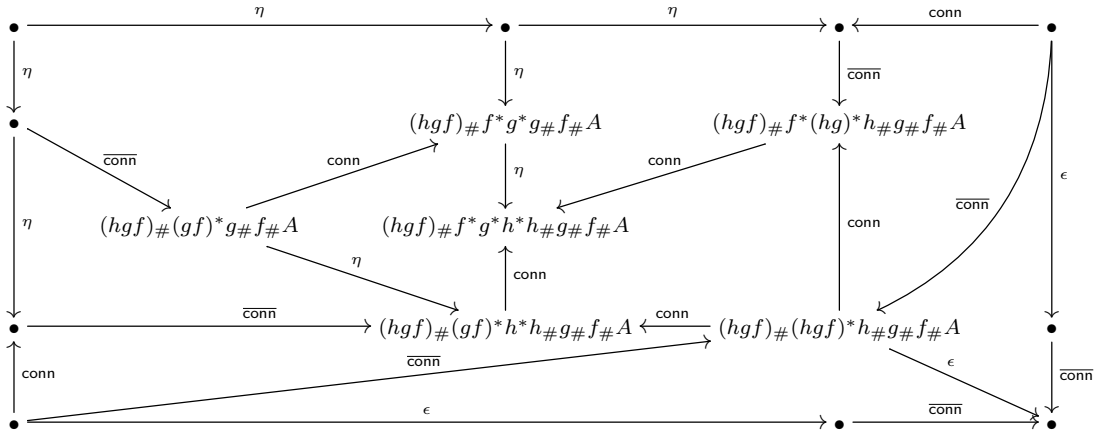
Since condition ($(\mathscr{S}')^{op}$-fib-0) holds by assumption (see Remark 1.4), we only need to check that the natural isomorphisms in the statement satisfy condition ($(\mathscr{S}')^{op}$-fib-1): given three composable morphisms $f\colon T \to S$, $g\colon S \to V$ and $h\colon V \to W$ in $\mathscr{S}'$, we have to show that the diagram of functors $\mathbb{H}(T) \to \mathbb{H}(W)$

$$\begin{array}{ccc} (hgf)_\# A & \xrightarrow{\overline{\mathsf{conn}}_{f,hg}} & (hg)_\# f_\# A \\ \downarrow{\scriptstyle \overline{\mathsf{conn}}_{gf,h}} & & \downarrow{\scriptstyle \overline{\mathsf{conn}}_{g,h}} \\ h_\#(gf)_\# A & \xrightarrow{\overline{\mathsf{conn}}_{g,f}} & h_\# g_\# f_\# A \end{array}$$

is commutative. Expanding the various definitions, we obtain the more explicit diagram



which we decompose as



Here, the central piece commutes by axiom ($\mathscr{S}'$-fib-1), while the other pieces commute by naturality and by construction. This proves the claim. ∎

1.6. ADJOINTABLE MORPHISMS OF FIBERED CATEGORIES. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two $\mathscr{S}$-fibered categories, in the sense of the previous subsection; for every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, we write the corresponding connection isomorphisms for $\mathbb{H}_1$ and $\mathbb{H}_2$ as $\mathsf{conn}^{(1)}_{f,g}$ and $\mathsf{conn}^{(2)}_{f,g}$, respectively.

Following the conventions of [Ayo10] and [CD19], by a *morphism* of $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ we mean a vertical natural transformation between the associated pseudo-functors. In detail, this is the datum of

- for every $S \in \mathscr{S}$, a functor
$$R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S),$$
called the *R-realization functor* over S,
- for every morphism $f\colon T \to S$ in $\mathscr{S}$, a natural isomorphism of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$
$$\theta = \theta_f\colon f^*R_S(A) \xrightarrow{\sim} R_T(f^*A),$$
called the *R-transition isomorphism* along f

satisfying the following condition:

(mor-$\mathscr{S}$-fib) For every pair of composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}$, the diagram of functors $\mathbb{H}_1(V) \to \mathbb{H}_2(T)$
$$\begin{array}{ccccc} (gf)^*R_V(A) & & \xrightarrow{\qquad\qquad\theta_{gf}\qquad\qquad} & & R_T((gf)^*A) \\ \Big\downarrow{\scriptstyle \mathsf{conn}^{(2)}_{f,g}} & & & & \Big\downarrow{\scriptstyle \mathsf{conn}^{(1)}_{f,g}} \\ f^*g^*R_V(A) & \xrightarrow{\theta_g} & f^*R_S(g^*A) & \xrightarrow{\theta_f} & R_T(f^*g^*A) \end{array}$$
is commutative.

For sake of clarity, it will be convenient to write a morphism $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ as a pair $(\mathscr{R}, \theta)$ consisting of

- a family of functors $\mathscr{R} := \{R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\}_{S\in\mathscr{S}}$,
- a system of natural isomorphisms $\theta = \left\{\theta_f\colon f^* \circ R_S \xrightarrow{\sim} R_T \circ f^*\right\}_{f\colon T\to S}$ satisfying condition (mor-$\mathscr{S}$-fib) above.

If we are already given a family $\mathscr{R}$ as above, we say that such a system of natural isomorphisms θ defines an *$\mathscr{S}$-structure* on $\mathscr{R}$.

1.7. Remark. Axiom (mor-$\mathscr{S}$-fib) is designed in such a way that all composites of connection isomorphisms and R-transition isomorphisms (and inverses thereof) are coherent. Together with axiom ($\mathscr{S}$-fib-0), it implies formally that $\theta_{\mathsf{id}_S} = \mathsf{id}_{R_S}$ for every $S \in \mathscr{S}$.

Note that any morphism of $\mathscr{S}$-fibered categories determines by restriction a morphism between the underlying $\mathscr{S}'$-fibered categories. In the case when the two $\mathscr{S}$-fibered categories involved are $\mathscr{S}'$-adjointable on the same side, it makes sense to consider the corresponding adjointability property for morphisms between them:

1.8. DEFINITION. *Suppose that both $\mathbb{H}_1$ and $\mathbb{H}_2$ are left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable). We say that a morphism of $\mathscr{S}$-fibered categories $R = (\mathscr{R}, \theta)\colon \mathbb{H}_1 \to \mathbb{H}_2$ is* left-$\mathscr{S}'$-adjointable *(resp.* right-$\mathscr{S}'$-adjointable*) if, for every morphism $f\colon T \to S$ in $\mathscr{S}'$, the natural transformation of functors $\mathbb{H}_1(T) \to \mathbb{H}_2(S)$*

$$f_\# R_T(A) \xrightarrow{\eta} f_\# R_T(f^* f_\# A) \xleftarrow{\theta_f} f_\# f^* R_S(f_\# A) \xrightarrow{\epsilon} R_S(f_\# A) \tag{3}$$

is invertible (resp. the natural transformation of functors $\mathbb{H}_1(T) \to \mathbb{H}_2(S)$

$$R_T(f_* A) \xrightarrow{\eta} f_* f^* R_T(f_* A) \xrightarrow{\theta_f} f_* R_S(f^* f_* A) \xrightarrow{\epsilon} f_* R_S(A) \tag{4}$$

is invertible). In this case, we let $\bar\theta = \bar\theta_f$ denote the natural transformation (3) (resp. the inverse of the natural transformation (4)). If we are already given a family of functors $\mathscr{R} := \{R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\}_{S \in \mathscr{S}}$, we say that an $\mathscr{S}$-structure θ on $\mathscr{R}$ is left-$\mathscr{S}'$-adjointable *(resp.* right-$\mathscr{S}'$-adjointable*) if so is the resulting morphism of $\mathscr{S}$-fibered categories $(\mathscr{R}, \theta)\colon \mathbb{H}_1 \to \mathbb{H}_2$.*

Note that the natural transformation $\bar\theta_f$ introduced in the definition is nothing but the mate of the natural isomorphism θ_f. In general, the mate of a natural isomorphism needs not be invertible; and even if f^* admits adjoints on both sides, it might happen that one between (3) and (4) is invertible while the other one is not. In any case, when the relevant mates are invertible, they define a morphism between the $(\mathscr{S}')^{op}$-fibered categories obtained via Lemma 1.5:

1.9. LEMMA. *Let $R = (\mathscr{R}, \theta)\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable) morphism between left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable) $\mathscr{S}$-fibered categories. Then, the natural isomorphisms $\bar\theta = \bar\theta_f$ define a morphism of $(\mathscr{S}')^{op}$-fibered categories.*

PROOF. We only treat the left-adjointable case; the other case is analogous.

We need to check that the natural isomorphisms in the statement satisfy condition (mor-$(\mathscr{S}')^{op}$-fib): given two composable morphisms $f\colon T \to S$ and $g\colon S \to V$ in $\mathscr{S}'$, we have to show that the diagram of functors $\mathbb{H}_1(T) \to \mathbb{H}_2(V)$

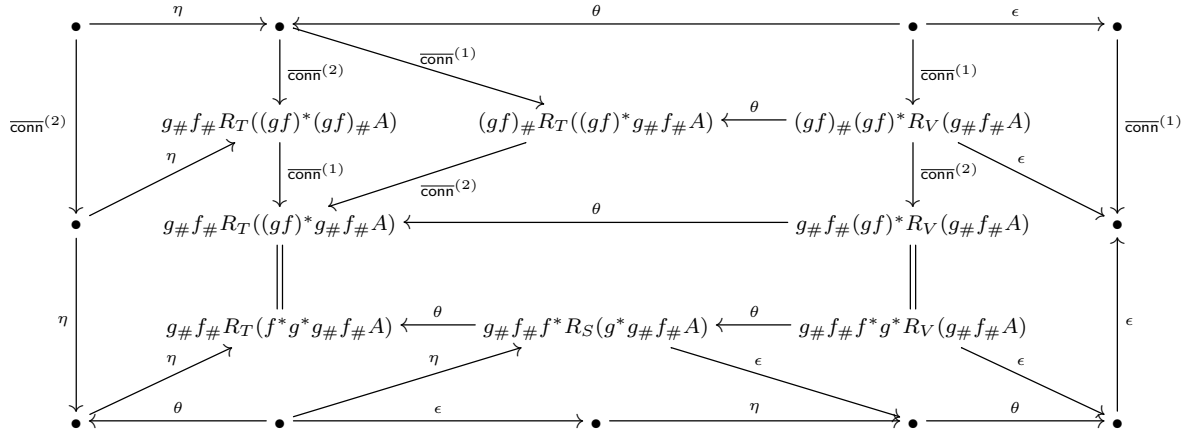
$$\begin{array}{ccccc} (gf)_\# R_T(A) & & \xrightarrow{\qquad\qquad\bar\theta_{gf}\qquad\qquad} & & R_V((gf)_\# A) \\ \| & & & & \| \\ g_\# f_\# R_T(A) & \xrightarrow{\bar\theta_f} & g_\# R_S(f_\# A) & \xrightarrow{\bar\theta_g} & R_V(g_\# f_\# A) \end{array}$$

is commutative. This claim is an instance of the calculus of mates, but we prefer to check it by hand. Expanding the definitions of the horizontal arrows, we obtain the more

explicit diagram

$$\begin{array}{ccccc}
(gf)_\# R_T((gf)^*(gf)_\# A) & \xleftarrow{\theta} & & & (gf)_\#(gf)^* R_V((gf)_\# A) \\
\uparrow{\scriptstyle \eta} & & & & \downarrow{\scriptstyle \epsilon} \\
(gf)_\# R_T(A) & & & & R_V((gf)_\# A) \\
\downarrow{\scriptstyle \overline{\mathsf{conn}}^{(2)}} & & & & \downarrow{\scriptstyle \overline{\mathsf{conn}}^{(1)}} \\
g_\# f_\# R_T(A) & & & & R_V(g_\# f_\# A) \\
\downarrow{\scriptstyle \eta} & & & & \uparrow{\scriptstyle \epsilon} \\
g_\# f_\# R_T(f^* f_\# A) & & & & g_\# g^* R_V(g_\# f_\# A) \\
\uparrow{\scriptstyle \theta} & & & & \uparrow{\scriptstyle \theta} \\
g_\# f_\# f^* R_S(f_\# A) & \xrightarrow{\epsilon} & g_\# R_S(f_\# A) & \xrightarrow{\eta} & g_\# R_S(g^* g_\# f_\# A),
\end{array}$$

which we decompose as



Here, the central rectangle commutes by axiom (mor-$\mathscr{S}'$-fib), while the other pieces commute by naturality and by construction. This proves the claim. ∎

1.10. Adjointable external tensor structures. As a preparation to the main results of the paper, we now discuss adjointability of monoidal structures on $\mathscr{S}$-fibered categories. Throughout, we assume that $\mathscr{S}$ admits binary products, and we let $\mathbb{H}$ be an $\mathscr{S}$-fibered category.

While monoidal structures are usually defined in terms of the internal tensor product, it is convenient for us to define them in terms of the external tensor product: this is possible thanks to Shulman's results [Shu08, Thm. 12.8, Prop. 13.30], expanded by Moeller–Vasilakopoulou in [MV20, Thm. 3.10]. For our purposes, it is convenient to follow the approach of [Ter24a, § 1], where an *external tensor structure* $(\boxtimes, m)$ on $\mathbb{H}$ is defined as the datum of

- for every $S_1, S_2 \in \mathscr{S}$, a functor
$$- \boxtimes - = - \boxtimes_{S_1,S_2} - \colon \mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2),$$
called the *external tensor product functor* over (S_1, S_2),
- for every choice of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2$, a natural isomorphism of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(T_1 \times T_2)$
$$m = m_{f_1,f_2} \colon f_1^* A_1 \boxtimes f_2^* A_2 \xrightarrow{\sim} (f_1 \times f_2)^*(A_1 \boxtimes A_2),$$
called the *external monoidality isomorphism* along (f_1, f_2)

satisfying the following condition:

(mETS) For every choice of composable morphisms $f_i \colon T_i \to S_i$ and $g_i \colon S_i \to V_i$, $i = 1, 2$, the diagram of functors $\mathbb{H}(V_1) \times \mathbb{H}(V_2) \to \mathbb{H}(T_1 \times T_2)$
$$\begin{array}{ccccc}
(g_1 f_1)^* A_1 \boxtimes (g_2 f_2)^* A_2 & & \xrightarrow{\qquad\qquad m \qquad\qquad} & & (g_1 f_1 \times g_2 f_2)^*(A_1 \boxtimes A_2) \\
\| & & & & \| \\
f_1^* g_1^* A_1 \boxtimes f_2^* g_2^* A_2 & \xrightarrow{m} & (f_1 \times f_2)^*(g_1^* A_1 \boxtimes g_2^* A_2) & \xrightarrow{m} & (f_1 \times f_2)^*(g_1 \times g_2)^*(A_1 \boxtimes A_2)
\end{array}$$
is commutative.

The usual associativity, commutativity and unit constraints can also be formulated in the external setting, as done explicitly in [Ter24a, § 2]; we will review part of these notions later on.

As explained in [Ter24a, Rmk. 1.14], giving an external monoidal structure on $\mathbb{H}$ is equivalent to giving an internal monoidal structure on the associated Grothendieck construction. Thus, our notion of external monoidal structure agrees with that of [Shu08] and [MV20].

1.11. Remark. Axiom (mETS) is designed in such a way that all composites of connection isomorphisms and external monoidality isomorphisms (and inverses thereof) are coherent. Together with axiom ($\mathscr{S}$-fib-0), it implies formally that $m_{\mathsf{id}_{S_1},\mathsf{id}_{S_2}} = \mathsf{id}_{-\boxtimes_{S_1,S_2}-}$ for every $S_1, S_2 \in \mathscr{S}$.

Note that any external tensor structure on $\mathbb{H}$ defines by restriction an external tensor structure on the underlying $\mathscr{S}'$-fibered category. In the case when $\mathbb{H}$ is $\mathscr{S}'$-adjointable, it makes sense to consider the corresponding property for external tensor structures on it:

1.12. Definition. *Suppose that $\mathbb{H}$ is left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable). We say that an external tensor structure $(\boxtimes, m)$ on $\mathbb{H}$ is* left-$\mathscr{S}'$-adjointable *(resp.* right-$\mathscr{S}'$-adjointable*) if, for every choice of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}'$, $i = 1, 2$, the natural transformation of functors $\mathbb{H}(T_1) \times \mathbb{H}(T_2) \to \mathbb{H}(S_1 \times S_2)$*
$$\begin{array}{ccc}
(f_1 \times f_2)_\#(A_1 \boxtimes A_2) & & f_{1,\#} A_1 \boxtimes f_{2,\#} A_2 \\
\downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \epsilon} \\
(f_1 \times f_2)_\#(f_1^* f_{1,\#} A_1 \boxtimes f_2^* f_{2,\#} A_2) & \xrightarrow{m} & (f_1 \times f_2)_\#(f_1 \times f_2)^*(f_{1,\#} A_1 \boxtimes f_{2,\#} A_2)
\end{array} \tag{5}$$

is invertible (resp. the natural transformation of functors $\mathbb{H}(T_1) \times \mathbb{H}(T_2) \to \mathbb{H}(S_1 \times S_2)$

$$\begin{array}{ccc} f_{1,*}A_1 \boxtimes f_{2,*}A_2 & & (f_1 \times f_2)_*(A_1 \boxtimes A_2) \\ \downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \epsilon} \\ (f_1 \times f_2)_*(f_1 \times f_2)^*(f_{1,*}A_1 \boxtimes f_{2,*}A_2) & \xleftarrow{\ m\ } & (f_1 \times f_2)_*(f_1^* f_{1,*}A_1 \boxtimes f_2^* f_{2,*}A_2) \end{array} \tag{6}$$

is invertible). In this case, we let $\bar{m} = \bar{m}_{f_1,f_2}$ *denote the inverse of the natural isomorphism* (5) *(resp. the natural isomorphism* (6)*).*

One can check that an external tensor structure $(\boxtimes, m)$ on $\mathbb{H}$ is left-$\mathscr{S}'$-adjointable if and only if, for every choice of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}'$, $i = 1, 2$, the natural transformation of functors $\mathbb{H}_1(T_1) \times \mathbb{H}_1(S_2) \to \mathbb{H}_2(S_1 \times S_2)$

$$\begin{array}{ccc} (f_1 \times \mathsf{id}_{S_2})_\#(A_1 \boxtimes B_2) & & f_{1,\#}A_1 \boxtimes B_2 \\ \downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \epsilon} \\ (f_1 \times \mathsf{id}_{S_2})_\#(f_1^* f_{1,\#}A_1 \boxtimes B_2) & \xrightarrow{\ m\ } & (f_1 \times \mathsf{id}_{S_2})_\#(f_1 \times \mathsf{id}_{S_2})^*(f_{1,\#}A_1 \boxtimes B_2) \end{array}$$

and the natural transformation of functors $\mathbb{H}_1(S_1) \times \mathbb{H}_1(T_2) \to \mathbb{H}(S_1 \times S_2)$

$$\begin{array}{ccc} (\mathsf{id}_{S_1} \times f_2)_\#(B_1 \boxtimes A_2) & & B_1 \boxtimes f_{2,\#}A_2 \\ \downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \epsilon} \\ (\mathsf{id}_{S_1} \times f_2)_\#(B_1 \boxtimes f_2^* f_{2,\#}A_2) & \xrightarrow{\ m\ } & (\mathsf{id}_{S_1} \times f_2)_\#(\mathsf{id}_{S_1} \times f_2)^*(B_1 \boxtimes f_{2,\#}A_2) \end{array}$$

are both invertible. Thus, the notion of left-adjointability just introduced recovers the formulation of the *projection formulae* in the language of external tensor structures, as stated in [Ter24a, § 1.3].

When the given external tensor structure is $\mathscr{S}'$-adjointable, the mates of the monoidality isomorphisms define an external tensor structure on the $(\mathscr{S}')^{op}$-fibered category obtained via Lemma 1.5:

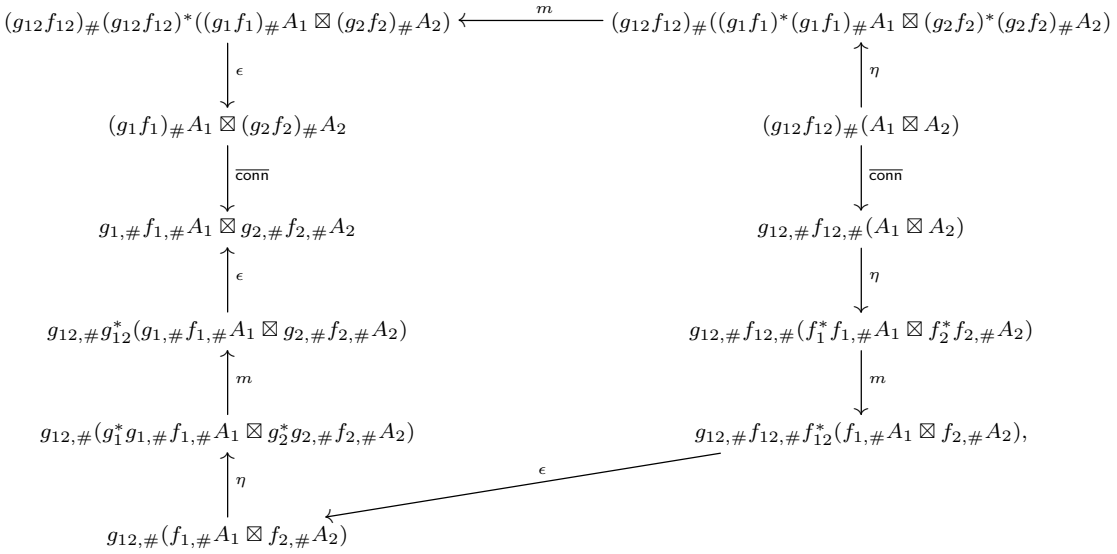
1.13. Lemma. *Suppose that* $\mathbb{H}$ *is left-*$\mathscr{S}'$*-adjointable (resp. right-*$\mathscr{S}'$*-adjointable), and let* $(\boxtimes, m)$ *be a left-*$\mathscr{S}'$*-adjointable (resp. right-*$\mathscr{S}'$*-adjointable) external tensor structure on it. Then, the natural isomorphisms* $\bar{m} = \bar{m}_f$ *define an external tensor structure over* $(\mathscr{S}')^{op}$*.*

Proof. We only treat the left-adjointable case; the other case is analogous.

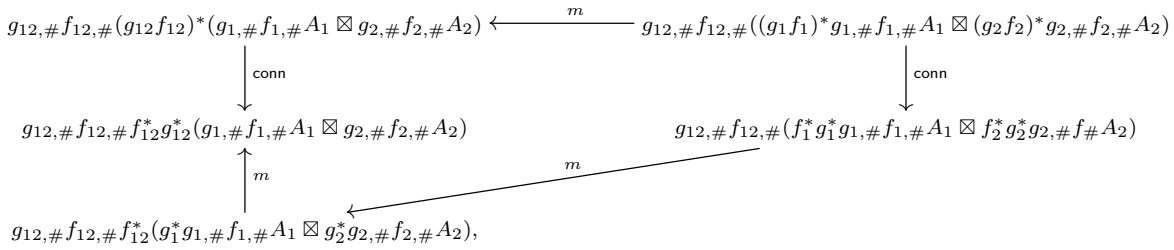
We need to check that the natural isomorphisms in the statement satisfy condition (mETS): given, for $i = 1, 2$, two composable morphisms $f_i \colon T_i \to S_i$ and $g_i \colon S_i \to V_i$ in $\mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}(T_1) \times \mathbb{H}(T_2) \to \mathbb{H}(S_1 \times S_2)$

$$\begin{array}{ccccc} (g_1 f_1)_\# A_1 \boxtimes (g_2 f_2)_\# A_2 & & \xrightarrow{\qquad\bar{m}\qquad} & & (g_1 f_1 \times g_2 f_2)_\#(A_1 \boxtimes A_2) \\ \| & & & & \| \\ g_{1,\#} f_{1,\#} A_1 \boxtimes g_{2,\#} f_{2,\#} A_2 & \xrightarrow{\bar{m}} & (g_1 \times g_2)_\#(f_{1,\#}A_1 \boxtimes f_{2,\#}A_2) & \xrightarrow{\bar{m}} & (g_1 \times g_2)_\#(f_1 \times f_2)_\#(A_1 \boxtimes A_2) \end{array}$$

is commutative. Unwinding the definitions of the monoidality isomorphisms (but not those of the connection isomorphisms), we obtain the diagram



where $f_1 \times f_2$ and $g_1 \times g_2$ are abbreviated as f_{12} and g_{12}, respectively. Note that this closely resembles the explicit diagram in the proof of Lemma 1.9, the only difference being the presence of external monoidality isomorphisms in place of R-transition isomorphisms. As the reader can easily check, it can be decomposed into a diagram of the same shape as the last diagram in the proof of Lemma 1.9, again with external monoidality isomorphisms in place of R-transition isomorphisms: in this case, the central piece is the diagram



which commutes by axiom (mETS) over $\mathscr{S}'$, while the other pieces commute by naturality and by construction. We leave the details to the interested reader. ∎

1.14. Remark. It does not seem interesting to study adjointability properties of internal tensor structures (that is, with respect to the internal tensor product): the main reason is that most internal tensor structures occurring in practice are not $\mathscr{S}'$-adjointable, even when the corresponding external tensor structures are so. To fix ideas, let $(\boxtimes, m)$ be a left-$\mathscr{S}'$-adjointable external tensor structure on a left-$\mathscr{S}'$-adjointable $\mathscr{S}$-fibered category $\mathbb{H}$, and write $(\otimes, m^i)$ for the internal tensor structure corresponding to it under the bijection of [Ter24a, Prop. 1.13]: as in [Ter24a, Lem. 1.12], the internal tensor product over S is defined by the formula

$$A \otimes B := \Delta_S^*(A \boxtimes B),$$

where $\Delta_S\colon S \to S \times S$ denotes the diagonal embedding, while the internal monoidality isomorphism m_f^i along $f\colon T \to S$ is defined as the composite

$$\begin{array}{ccccc} f^*A \otimes f^*B & & & & (f\times f)^*(A\otimes B). \\ \| & & & & \| \\ \Delta_T^*(f^*A \boxtimes f^*B) & \xrightarrow{m_f} & \Delta_T^*(f\times f)^*(A\boxtimes B) & = & f^*\Delta_S^*(A\boxtimes B) \end{array}$$

Studying left-$\mathscr{S}'$-adjointability of $(\otimes, m^i)$ amounts to studying whether, for a morphism $f\colon T \to S$ in $\mathscr{S}'$, the natural transformation of functors $\mathbb{H}(T)\times\mathbb{H}(T) \to \mathbb{H}(S)$

$$f_\#(A\otimes B) \xrightarrow{\eta} f_\#(f^*f_\#A \otimes f^*f_\#B) \xrightarrow{m_f^i} f_\#f^*(f_\#A\otimes f_\#B) \xrightarrow{\epsilon} f_\#A\otimes f_\#B$$

is invertible. Expanding the various definitions, one obtains the composite

$$\begin{array}{ccccc} f_\#\Delta_T^*(A\boxtimes B) & & & & \Delta_S^*(f_\#A\boxtimes f_\#B). \\ \downarrow{\scriptstyle\eta} & & & & \uparrow{\scriptstyle\epsilon} \\ f_\#\Delta_T^*(f^*f_\#A\boxtimes f^*f_\#B) & \xrightarrow{m_f} & f_\#\Delta_T^*(f\times f)^*(f_\#A\boxtimes f_\#B) & = & f_\#f^*\Delta_S^*(f_\#A\boxtimes f_\#B) \end{array}$$

As one can easily check, the latter coincides with the composite

$$\begin{array}{ccccc} f_\#\Delta_T^*(A\boxtimes B) & & & & \Delta_S^*(f_\#A\boxtimes f_\#B), \\ \downarrow{\scriptstyle\eta} & & & & \downarrow{\scriptstyle\overline{m}_f} \\ f_\#\Delta_T^*(f\times f)^*(f\times f)_\#(A\boxtimes B) & = & f_\#f^*\Delta_S^*(f\times f)_\#(A\boxtimes B) & \xrightarrow{\epsilon} & \Delta_S^*(f\times f)_\#(A\boxtimes B) \end{array}$$

which is invertible if and only if so is the Beck–Chevalley natural transformation

$$\begin{array}{ccc} f_\#\Delta_T^*(A\boxtimes B) & & \Delta_S^*(f\times f)_\#(A\boxtimes B) \\ \downarrow{\scriptstyle\eta} & & \uparrow{\scriptstyle\epsilon} \\ f_\#\Delta_T^*(f\times f)^*(f\times f)_\#(A\boxtimes B) & = & f_\#f^*\Delta_S^*(f\times f)_\#(A\boxtimes B) \end{array}$$

associated to the commutative diagram in $\mathscr{S}$

$$\begin{array}{ccc} T & \xrightarrow{\Delta_T} & T\times T \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times f} \\ S & \xrightarrow{\Delta_S} & S\times S. \end{array}$$

In typical geometric settings, one would try to conclude by applying some base-change result; however, the latter diagram is not Cartesian unless f is a monomorphism.

In the case when $\mathscr{S} = \mathsf{Var}_k$ is the category of algebraic varieties over a field k and $\mathbb{H}$ underlies a monoidal stable homotopy 2-functor in the sense of [Ayo07a, Ayo07b], the internal tensor structure on $\mathbb{H}$ is left-$\mathscr{S}'$-adjointable for $\mathscr{S}'$ the class of open immersions, and right-$\mathscr{S}'$-adjointable for $\mathscr{S}'$ the class of closed immersions; there does not seem to be much to say outside these two cases.

1.15. Associativity and commutativity constraints in the adjointable case. Next, we check that the notions of external associativity and commutativity constraints, as introduced in [Ter24a, § 2], are preserved in the adjointable case. Throughout, we fix an external tensor structure $(\boxtimes, m)$ on $\mathbb{H}$.

Following [Ter24a, Defn. 2.2], by an *external associativity constraint* a for $(\boxtimes, m)$ we mean the datum of

- for every $S_1, S_2, S_3 \in \mathscr{S}$, a natural isomorphism between functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \times \mathbb{H}(S_3) \to \mathbb{H}(S_1 \times S_2 \times S_3)$

$$a = a_{S_1,S_2,S_3} \colon (A_1 \boxtimes A_2) \boxtimes A_3 \xrightarrow{\sim} A_1 \boxtimes (A_2 \boxtimes A_3)$$

satisfying the following conditions:

(aETS-1) For every $S_1, S_2, S_3, S_4 \in \mathscr{S}$, the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \times \mathbb{H}(S_3) \times \mathbb{H}(S_4) \to \mathbb{H}(S_1 \times S_2 \times S_3 \times S_4)$

$$\begin{array}{ccccc}
((A_1 \boxtimes A_2) \boxtimes A_3) \boxtimes A_4 & & \xrightarrow{\quad a \quad} & & (A_1 \boxtimes A_2) \boxtimes (A_3 \boxtimes A_4) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
(A_1 \boxtimes (A_2 \boxtimes A_3)) \boxtimes A_4 & \xrightarrow{a} & A_1 \boxtimes ((A_2 \boxtimes A_3) \boxtimes A_4) & \xrightarrow{a} & A_1 \boxtimes (A_2 \boxtimes (A_3 \boxtimes A_4))
\end{array}$$

is commutative.

(aETS-2) For every choice of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2, 3$, the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \times \mathbb{H}(S_3) \to \mathbb{H}(T_1 \times T_2 \times T_3)$

$$\begin{array}{ccccc}
(f_1^* A_1 \boxtimes f_2^* A_2) \boxtimes f_3^* A_3 & \xrightarrow{m} & (f_1 \times f_2)^*(A_1 \boxtimes A_2) \boxtimes f_3^* A_3 & \xrightarrow{m} & (f_1 \times f_2 \times f_3)^*((A_1 \boxtimes A_2) \boxtimes A_3) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
f_1^* A_1 \boxtimes (f_2^* A_2 \boxtimes f_3^* A_3) & \xrightarrow{m} & f_1^* A_1 \boxtimes (f_2 \times f_3)^*(A_2 \boxtimes A_3) & \xrightarrow{m} & (f_1 \times f_2 \times f_3)^*(A_1 \boxtimes (A_2 \boxtimes A_3))
\end{array}$$

is commutative.

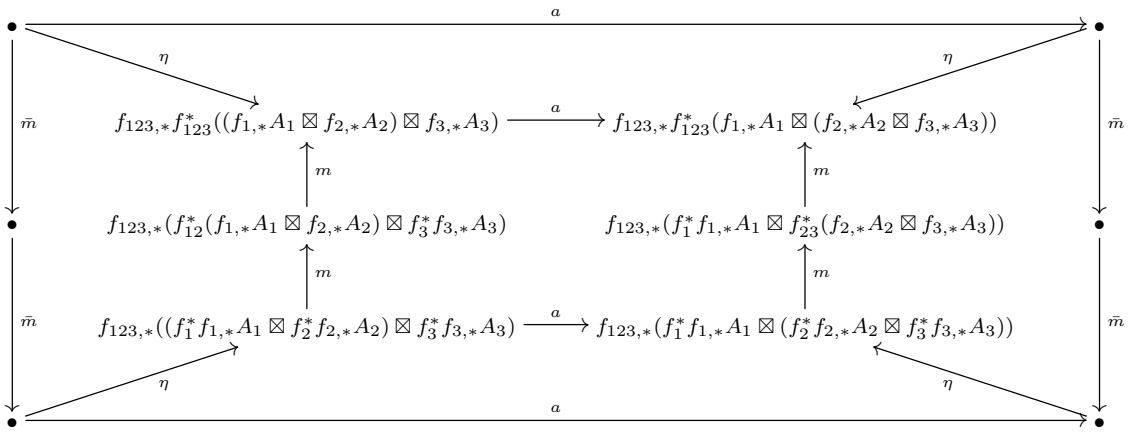
This notion behaves well under adjunction, as the following result shows:

1.16. Lemma. *Suppose that $\mathbb{H}$ is left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable), and that $(\boxtimes, m)$ is left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable) as well; let $(\boxtimes, \bar{m})$ denote the external tensor structure over $(\mathscr{S}')^{op}$ obtained from it via Lemma 1.13. Then, every external associativity constraint a for $(\boxtimes, m)$ also defines an associativity constraint for $(\boxtimes, \bar{m})$.*

PROOF. We only treat the right-adjointable case; the other case is analogous. Given an external associativity constraint a for $(\boxtimes, m)$, we need to check that it satisfies conditions (aETS-1) and (aETS-2) with respect to $(\boxtimes, \bar{m})$. Since condition (aETS-1) is clearly independent of whether we regard $\mathbb{H}$ as a fibered category over $\mathscr{S}'$ or over $(\mathscr{S}')^{op}$, we only need to check condition (aETS-2): given morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}'$, $i = 1, 2, 3$, we have to show that the diagram of functors $\mathbb{H}(T_1) \times \mathbb{H}(T_2) \times \mathbb{H}(T_3) \to \mathbb{H}(S_1 \times S_2 \times S_3)$

$$\begin{array}{ccc}
(f_{1,*}A_1 \boxtimes f_{2,*}A_2) \boxtimes f_{3,*}A_3 & \xrightarrow{a} & f_{1,*}A_1 \boxtimes (f_{2,*}A_2 \boxtimes f_{3,*}A_3) \\
\downarrow{\scriptstyle \bar{m}} & & \downarrow{\scriptstyle \bar{m}} \\
(f_1 \times f_2)_*(A_1 \boxtimes A_2) \boxtimes f_{3,*}A_3 & & f_{1,*}A_1 \boxtimes (f_2 \times f_3)_*(A_2 \boxtimes A_3) \\
\downarrow{\scriptstyle \bar{m}} & & \downarrow{\scriptstyle \bar{m}} \\
(f_1 \times f_2 \times f_3)_*((A_1 \boxtimes A_2) \boxtimes A_3) & \xrightarrow{a} & (f_1 \times f_2 \times f_3)_*(A_1 \boxtimes (A_2 \boxtimes A_3))
\end{array}$$

is commutative. To this end, we decompose it as



where $f_1 \times f_2 \times f_3$ is abbreviated as f_{123}, and similarly for the other product morphisms. Since the upper and lower pieces commute by naturality, while the central piece commutes by axiom (aETS-2) over $\mathscr{S}'$, it suffices to check that the two lateral pieces commute as well. This can be done by expanding the definition of the external monoidality isomorphisms $\bar{m}$ and filling the resulting diagrams appropriately; we leave the details to the interested reader. ∎

Following [Ter24a, Defn. 2.7], by an *external commutativity constraint* c for $(\boxtimes, m)$ we mean the datum of

- for every $S_1, S_2 \in \mathscr{S}$, a natural isomorphism between functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2)$
$$c = c_{S_1,S_2} \colon A_1 \boxtimes A_2 \xrightarrow{\sim} \tau^*(A_2 \boxtimes A_1),$$
where $\tau = \tau_{(12)} \colon S_1 \times S_2 \to S_2 \times S_1$ denotes the permutation isomorphism

satisfying the following conditions:

(*c*ETS-1) For every $S_1, S_2 \in \mathscr{S}$, the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2)$

$$\begin{array}{ccc} A_1 \boxtimes A_2 & \xrightarrow{c} & \tau^*(A_2 \boxtimes A_1) \\ & \searrow\!\!= & \downarrow c \\ & & \tau^*\tau^*(A_1 \boxtimes A_2) \end{array}$$

is commutative.

(*c*ETS-2) For every choice of of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2$, the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(T_1 \times T_2)$

$$\begin{array}{ccccc} f_1^*A_1 \boxtimes f_2^*A_2 & & \xrightarrow{\qquad\qquad c \qquad\qquad} & & \tau^*(f_2^*A_2 \boxtimes f_1^*A_1) \\ \downarrow m & & & & \downarrow m \\ (f_1 \times f_2)^*(A_1 \boxtimes A_2) & \xrightarrow{c} & (f_1 \times f_2)^*\tau^*(A_2 \boxtimes A_1) & = & \tau^*(f_2 \times f_1)^*(A_2 \boxtimes A_1) \end{array}$$

is commutative.

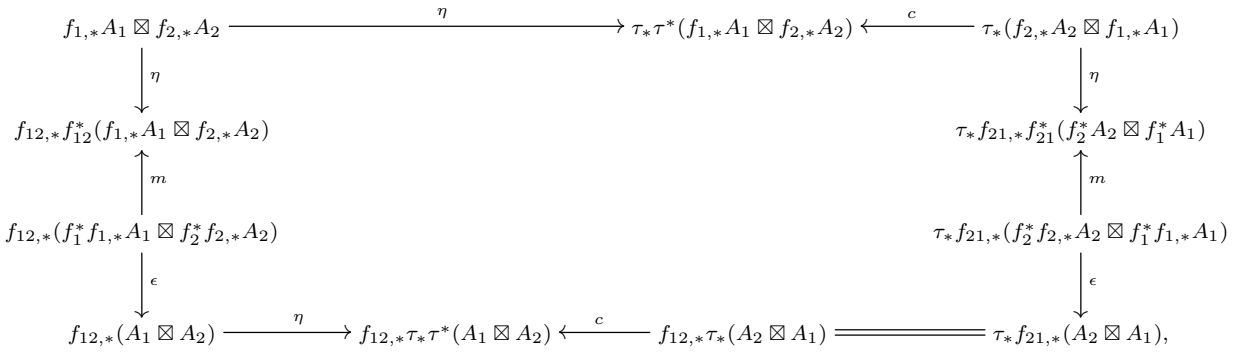
As in the case of associativity, this notion behaves well under adjunction:

1.17. Lemma. *Suppose that* $\mathbb{H}$ *is left-*$\mathscr{S}'$*-adjointable (resp. right-*$\mathscr{S}'$*-adjointable), and that* $(\boxtimes, m)$ *is left-*$\mathscr{S}'$*-adjointable (resp. right-*$\mathscr{S}'$*-adjointable) as well; let* $(\boxtimes, \bar{m})$ *denote the external tensor structure over* $(\mathscr{S}')^{op}$ *obtained from it via Lemma 1.13. Then, every external commutativity constraint* c *for* $(\boxtimes, m)$ *also defines a commutativity constraint for* $(\boxtimes, \bar{m})$.

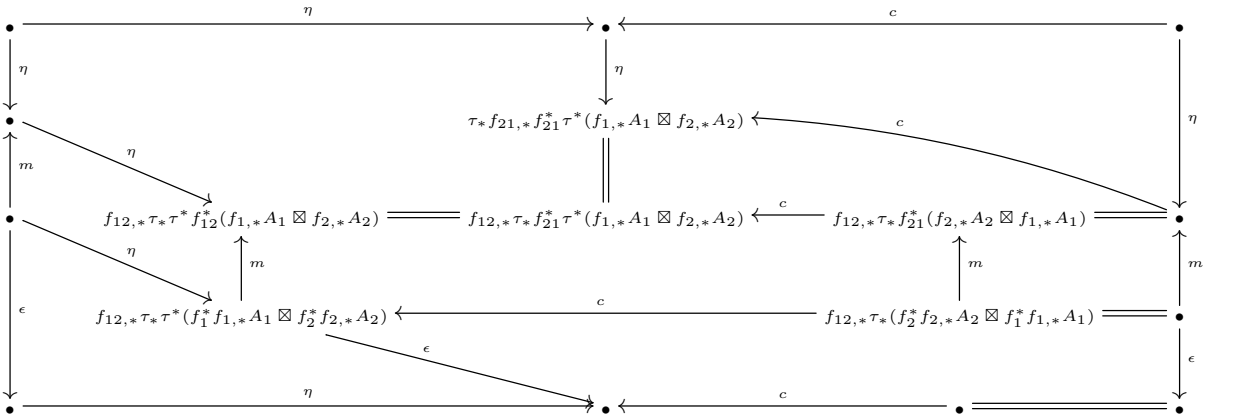
Proof. We only treat the right-adjointable case; the other case is analogous. Given an external commutativity constraint c for $(\boxtimes, m)$, we need to check that it satisfies conditions (*c*ETS-1) and (*c*ETS-2) with respect to $(\boxtimes, \bar{m})$. Since the only inverse image functors involved in condition (*c*ETS-1) are those over involutive permutation isomorphisms, one can easily check that they are independent of whether we regard $\mathbb{H}$ as a fibered category over $\mathscr{S}'$ or over $(\mathscr{S}')^{op}$. Therefore, we only need to check condition (*c*ETS-2): given morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}'$, $i = 1, 2$, we have to show that the diagram of functors $\mathbb{H}(T_1) \times \mathbb{H}(T_2) \to \mathbb{H}(S_1 \times S_2)$

$$\begin{array}{ccccc} f_{1,*}A_1 \boxtimes f_{2,*}A_2 & & \xrightarrow{\qquad\qquad c \qquad\qquad} & & \tau_*(f_{2,*}A_2 \boxtimes f_{1,*}A_1) \\ \downarrow \bar{m} & & & & \downarrow \bar{m} \\ (f_1 \times f_2)_*(A_1 \boxtimes A_2) & \xrightarrow{c} & (f_1 \times f_2)_*\tau_*(A_2 \boxtimes A_1) & = & \tau_*(f_2 \times f_1)_*(A_2 \boxtimes A_1) \end{array}$$

is commutative. Expanding the definitions of the monoidality isomorphisms and of the commutativity constraint, we obtain the more explicit diagram



where $f_1 \times f_2$ and $f_2 \times f_1$ are abbreviated as f_{12} and f_{21}, respectively. We decompose the latter as



Here, the central piece commutes by axiom (cETS-2) over $\mathscr{S}'$, while the other pieces commute by naturality and by construction. ∎

1.18. Remark. In [Ter24a, Defn. 2.17] we introduced the natural compatibility condition (acETS) between external associativity and commutativity constraints on a given external tensor structure; this does not involve inverse image functors, except those along permutation isomorphisms. As long as $\mathscr{S}'$ contains all (permutation) isomorphisms in $\mathscr{S}$, the validity of (acETS) does not depend on whether one regards $\mathbb{H}$ as an $\mathscr{S}'$-fibered or as an $(\mathscr{S}')^{op}$-fibered category.

1.19. Adjointable external tensor structures on morphisms. To conclude the present section, we combine the previous adjointability results for morphisms of $\mathscr{S}$-fibered categories and for monoidal structures in order to obtain a similar result for monoidal structures on morphisms.

For $j = 1, 2$, let $\mathbb{H}_j$ be an $\mathscr{S}$-fibered category endowed with an external tensor structure $(\boxtimes_j, m_j)$. In addition, let $R = (\mathscr{R}, \theta)\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a morphism of $\mathscr{S}$-fibered categories. Following [Ter24a, Defn. 3.7], by an *external tensor structure* ρ on R (with respect to $(\boxtimes_1, m_1)$ and $(\boxtimes_2, m_2)$) we mean the datum of

- for every $S_1, S_2 \in \mathscr{S}$, a natural isomorphism between functors $\mathbb{H}_1(S_1) \times \mathbb{H}_1(S_2) \to \mathbb{H}_2(S_1 \times S_2)$

$$\rho = \rho_{S_1,S_2} : R_{S_1}(A_1) \boxtimes_2 R_{S_2}(A_2) \xrightarrow{\sim} R_{S_1 \times S_2}(A_1 \boxtimes_1 A_2),$$

  called the *external R-monoidality isomorphism* over (S_1, S_2)

satisfying the following condition:

(mor-ETS) For every choice of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2$, the diagram of functors $\mathbb{H}_1(S_1) \times \mathbb{H}_1(S_2) \to \mathbb{H}_2(T_1 \times T_2)$

$$\begin{array}{ccc}
f_1^* R_{S_1}(A_1) \boxtimes_2 f_2^* R_{S_2}(A_2) & \xrightarrow{m_2} & (f_1 \times f_2)^*(R_{S_1}(A_1) \boxtimes_{"} R_{S_2}(A_2)) \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \rho} \\
R_{T_1}(f_1^* A_1) \boxtimes_2 R_{T_2}(f_2^* A_2) & & (f_1 \times f_2)^* R_{S_1 \times S_2}(A_1 \boxtimes_1 A_2) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \theta} \\
R_{T_1 \times T_2}(f_1^* A_1 \boxtimes_1 f_2^* A_2) & \xrightarrow{m_1} & R_{T_1 \times T_2}((f_1 \times f_2)^*(A_1 \boxtimes_1 A_2))
\end{array}$$

  is commutative.

As usual, note that any external tensor structure on R defines by restriction an external tensor structure on the underlying morphism of $\mathscr{S}'$-fibered categories.

We do not need to introduce any notion of adjointability for external tensor structures on morphisms, as the following result shows:

1.20. LEMMA. *For $j = 1, 2$ suppose that $\mathbb{H}_j$ is left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable) and that $(\boxtimes_j, m_j)$ is left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable), and let $(\boxtimes_j, \bar{m}_j)$ denote the external tensor structure over $(\mathscr{S}')^{op}$ obtained from it via Lemma 1.13. Suppose that, in addition, $R = (\mathscr{R}, \theta) \colon \mathbb{H}_1 \to \mathbb{H}_2$ is left-$\mathscr{S}'$-adjointable (resp. right-$\mathscr{S}'$-adjointable), and let $(\mathscr{R}, \bar{\theta}) \colon \mathbb{H}_1 \to \mathbb{H}_2$ denote the morphism of $(\mathscr{S}')^{op}$-fibered categories obtained from it via Lemma 1.9. Then, any external tensor structure ρ on $(\mathscr{R}, \theta)$ (with respect to $(\boxtimes_1, m_1)$ and $(\boxtimes_2, m_2)$) also defines an external tensor structure on $(\mathscr{R}, \bar{\theta})$ (with respect to $(\boxtimes_1, \bar{m}_1)$ and $(\boxtimes_2, \bar{m}_2)$).*

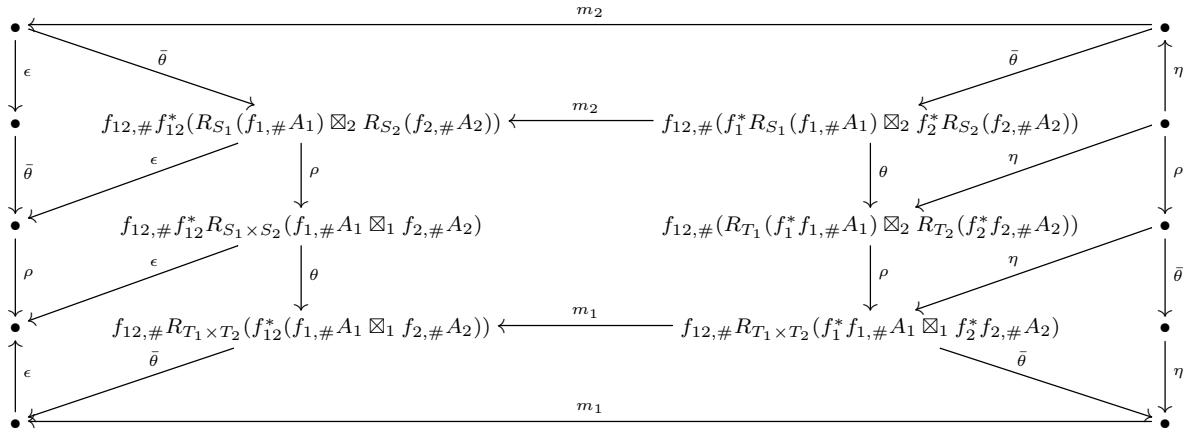
PROOF. We only prove the result in the left-adjointable case; the other case is analogous. We need to check that condition (mor-ETS) is satisfied with respect to the morphism $(\mathscr{R}, \bar{\theta})$: for every choice of morphisms $f_i \colon T_i \to S_i$ in $\mathscr{S}'$, $i = 1, 2$, we have to show that the diagram of functors $\mathbb{H}_1(T_1) \times \mathbb{H}_1(T_2) \to \mathbb{H}_2(S_1 \times S_2)$

$$\begin{array}{ccc}
f_{1,\#} R_{T_1}(A_1) \boxtimes_2 f_{2,\#} R_{T_2}(A_2) & \xrightarrow{\bar{m}_2} & (f_1 \times f_2)_\#(R_{T_1}(A_1) \boxtimes_2 R_{T_2}(A_2)) \\
\downarrow{\scriptstyle \bar{\theta}} & & \downarrow{\scriptstyle \rho} \\
R_{S_1}(f_{1,\#} A_1) \boxtimes_2 R_{S_2}(f_{2,\#} A_2) & & (f_1 \times f_2)_\# R_{T_1 \times T_2}(A_1 \boxtimes_1 A_2) \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \bar{\theta}} \\
R_{S_1 \times S_2}(f_{1,\#} A_1 \boxtimes_1 f_{2,\#} A_2) & \xrightarrow{\bar{m}_1} & R_{S_1 \times S_2}((f_1 \times f_2)_\#(A_1 \boxtimes_1 A_2))
\end{array}$$

is commutative. Unwinding the definitions of the external monoidality isomorphisms (but not those of the R-transition isomorphisms), we obtain the more explicit diagram

$$\begin{array}{ccc}
f_{12,\#}f_{12}^*(f_{1,\#}R_{T_1}(A_1)\boxtimes_2 f_{2,\#}R_{T_2}(A_2)) & \xleftarrow{m_2} & f_{12,\#}(f_1^*f_{1,\#}R_{T_1}(A_1)\boxtimes_2 f_2^*f_{2,\#}R_{T_2}(A_2)) \\
\downarrow{\scriptstyle\epsilon} & & \uparrow{\scriptstyle\eta} \\
f_{1,\#}R_{T_1}(A_1)\boxtimes_2 f_{2,\#}R_{T_2}(A_2) & & f_{12,\#}(R_{T_1}(A_1)\boxtimes_2 R_{T_2}(A_2)) \\
\downarrow{\scriptstyle\bar\theta} & & \downarrow{\scriptstyle\rho} \\
R_{S_1}(f_{1,\#}A_1)\boxtimes_2 R_{S_2}(f_{2,\#}A_2) & & f_{12,\#}R_{T_1\times T_2}(A_1\boxtimes_1 A_2) \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\bar\theta} \\
R_{S_1\times S_2}(f_{1,\#}A_1\boxtimes_1 f_{2,\#}A_2) & & R_{S_1\times S_2}(f_{12,\#}(A_1\boxtimes_1 A_2)) \\
\uparrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\eta} \\
R_{S_1\times S_2}(f_{12,\#}f_{12}^*(f_{1,\#}A_1\boxtimes_1 f_{2,\#}A_2)) & \xleftarrow{m_1} & R_{S_1\times S_2}(f_{12,\#}(f_1^*f_{1,\#}A_1\boxtimes_1 f_2^*f_{2,\#}A_2)),
\end{array}$$

where $f_1\times f_2$ is abbreviated as f_{12}. We decompose the latter as



Here, the central piece commutes by axiom (mor-ETS), while all the remaining pieces commute by naturality and by construction. ∎

1.21. Remark. In [Ter24a, § 3] we studied the natural compatibility conditions (mor-aETS) and (mor-cETS) between a given external tensor structure on R and two given external associativity or commutativity constraints on $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively: condition (mor-aETS) does not involve any inverse image functors, while condition (mor-cETS) only involves inverse images along permutation isomorphisms. As long as $\mathscr{S}'$ contains all (permutation) isomorphisms in $\mathscr{S}$, the validity of (mor-aETS) and (mor-cETS) does not depend on whether one regards $\mathbb{H}_1$ and $\mathbb{H}_2$ as $\mathscr{S}'$-fibered or as $(\mathscr{S}')^{op}$-fibered categories.

2. The skeleton of a morphism

In this section, we focus on morphisms of $\mathscr{S}$-fibered categories: the main goal is to show how a given morphism can be recovered from a smaller amount of data involving inverse image functors under two distinguished classes of morphisms of $\mathscr{S}$ only.

2.1. Axiomatics of factorization systems. In the first place, we have to spell out the precise conditions on the two classes of morphisms involved in order to get good factorization properties for morphisms of $\mathscr{S}$. In particular, we need to require that certain fibered products exist in $\mathscr{S}$. In detail:

2.2. Assumption. *We assume to be given two (possibly non-full) subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ of $\mathscr{S}$ satisfying the following conditions:*

(i) Both $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ have the same objects as $\mathscr{S}$ and contain all isomorphisms in $\mathscr{S}$.

(ii) Let $p\colon P \to S$ be a morphism in $\mathscr{S}^{sm}$. Then, for every morphism $f\colon T \to S$ in $\mathscr{S}$, the fibered product $P_T := P \times_S T$ exists in $\mathscr{S}$. Moreover, for every Cartesian square in $\mathscr{S}$ of the form

$$\begin{array}{ccc} P_T & \xrightarrow{f'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

with p in $\mathscr{S}^{sm}$, the base-changed morphism p' belongs to $\mathscr{S}^{sm}$ as well.

(iii) For every Cartesian square in $\mathscr{S}$ of the form

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p in $\mathscr{S}^{sm}$ and z in $\mathscr{S}^{cl}$, the base-changed morphism z' belongs to $\mathscr{S}^{cl}$ as well. Moreover, for every commutative diagram in $\mathscr{S}$ of the form

$$\begin{array}{ccc} Z & \xrightarrow{z} & S \\ & {\scriptstyle gz}\searrow & \downarrow{\scriptstyle g} \\ & & V \end{array}$$

with gz in $\mathscr{S}^{cl}$, the morphism z belongs to $\mathscr{S}^{cl}$ as well.

(iv) Every morphism $f\colon T \to S$ in $\mathscr{S}$ admits some factorization of the form

$$f\colon T \xrightarrow{t} P \xrightarrow{p} S$$

with t in $\mathscr{S}^{cl}$ and p in $\mathscr{S}^{sm}$.

We say that a morphism in $\mathscr{S}$ is a smooth morphism *if it belongs to $\mathscr{S}^{sm}$ and a* closed immersion *if it belongs to $\mathscr{S}^{cl}$.*

2.3. Example. The motivating example is when $\mathscr{S} = \mathsf{Var}_k$ is the category of quasi-projective algebraic varieties over a field k, with $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ the usual subcategories of smooth morphisms and closed immersions, respectively. In this case (i) is obvious, and both (ii) and (iii) are well-known. The quasi-projectivity assumption is only used in (iv) (see the discussion in [Ayo07a, § 1.3.5]). Of course, in this example there is no reason to require only fibered products with respect to a smooth morphism to exist in (ii). We have opted for this weak formulation not only because it suffices for all our constructions, but also in order to include the variant where $\mathscr{S} = \mathsf{Sm}_k$ is the category of smooth quasi-projective k-varieties: indeed, fibered products of smooth varieties need no be smooth in general, but they are so as soon as one of the two morphisms involved is smooth.

In the rest of the paper, we always work under Assumption 2.2; in Section 3 below we will need to strengthen it slightly for some constructions, but for the moment we prefer to work under the minimal necessary assumptions.

Our main results exploit systematically the factorization of morphisms in $\mathscr{S}$ into closed immersions and smooth morphisms, as described in Assumption 2.2(iv). It will be convenient to use the following notation:

2.4. Notation. For every morphism $f\colon T \to S$ in $\mathscr{S}$, we introduce the *factorization category* $\mathsf{Fact}_{\mathscr{S}}(f)$ where

- objects are triples $(P; t, p)$ where P is an object of $\mathscr{S}$, $t\colon T \to P$ is a closed immersion, and $p\colon P \to S$ is a smooth morphism such that $p \circ t = f$,
- a morphism $q\colon (P'; t', p') \to (P; t, p)$ is the datum of a smooth morphism $q\colon P' \to P$ such that $q \circ t' = t$ and $p \circ q = p'$.

The conditions listed in Assumption 2.2 have the following consequence on factorization categories:

2.5. Lemma. *For every morphism $f\colon T \to S$ in $\mathscr{S}$, the factorization category $\mathsf{Fact}_{\mathscr{S}}(f)$ is non-empty and connected.*

Proof. The fact that $\mathsf{Fact}_{\mathscr{S}}(f)$ is non-empty is Assumption 2.2(iv). To see that it is connected, we prove a more precise result, namely: every two objects $(P; t, p)$ and $(P'; t', p')$ in $\mathsf{Fact}_{\mathscr{S}}(f)$ are dominated by a third one $(P''; t'', p'')$. Indeed, it suffices to take $P'' = P \times_S P'$ (which exists in $\mathscr{S}$, by Assumption 2.2(ii)) together with the obvious morphisms $t''\colon T \to P''$ (which is a closed immersion, by Assumption 2.2(iii)) and $p''\colon P'' \to S$ (which is a smooth morphism, by Assumption 2.2(ii)). ∎

Even if the results of the present section do not require any adjointability notions, we need to keep track of certain adjointability properties in view of later applications. To this end, it is convenient to introduce some terminology:

2.6. DEFINITION.

1. *We say that an $\mathscr{S}$-fibered category $\mathbb{H}$ is* smooth-adjointable *(resp.* closed-adjointable*) if it is left-$\mathscr{S}^{sm}$-adjointable (resp. right-$\mathscr{S}^{cl}$-adjointable) in the sense of Definition 1.3.*

2. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be smooth-adjointable (resp. closed adjointable) $\mathscr{S}$-fibered categories. We say that a morphism of $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ is* smooth-adjointable *(resp.* closed-adjointable*) if it is left-$\mathscr{S}^{sm}$-adjointable (resp. right-$\mathscr{S}^{cl}$-adjointable) in the sense of Definition 1.8; similarly for $\mathscr{S}$-structures on a family of functors $\mathscr{R} = \{R_S\colon \mathbb{H}_1 \to \mathbb{H}_2\}_{S\in\mathscr{S}}$.*

3. *Let $\mathbb{H}$ be a smooth-adjointable (resp. closed-adjointable) $\mathscr{S}$-fibered category. We say that an external tensor structure $(\boxtimes, m)$ on $\mathbb{H}$ is* smooth-adjointable *(resp.* closed-adjointable*) if it is left-$\mathscr{S}^{sm}$-adjointable (resp. right-$\mathscr{S}^{cl}$-adjointable) in the sense of Definition 1.12.*

2.7. SKELETA OF MORPHISMS. For the rest of this section, we fix two $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$ as well as a family of functors $\mathscr{R} = \{R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\}_{S\in\mathscr{S}}$. Our goal is to turn $\mathscr{R}$ into a morphism of $\mathscr{S}$-fibered categories by constructing an $\mathscr{S}$-structure θ on it, starting from partial such structures over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$.

The following definition is inspired by the method of *exchange structures*, developed by Ayoub in [Ayo07a, § 1.2]; the main difference is that we apply it to morphisms rather than to single fibered categories, which considerably simplifies the discussion.

2.8. DEFINITION. *An $\mathscr{S}$-*skeleton *$(\theta^{sm}, \theta^{cl})$ on $\mathscr{R}$ is the datum of*

- *an $\mathscr{S}^{sm}$-structure θ^{sm} on $\mathscr{R}$,*
- *an $\mathscr{S}^{cl}$-structure θ^{cl} on $\mathscr{R}$*

satisfying the following exchange condition:

(ex-$\mathscr{S}$-fib) *For every commutative square in $\mathscr{S}$*

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p, q smooth morphisms and z, h closed immersions, the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(Q)$

$$\begin{array}{ccccc} q^*z^*R_S(A) & \xrightarrow{\theta^{cl}} & q^*R_Z(z^*A) & \xrightarrow{\theta^{sm}} & R_Q(q^*z^*A) \\ \| & & & & \| \\ h^*p^*R_S(A) & \xrightarrow{\theta^{sm}} & h^*R_P(p^*A) & \xrightarrow{\theta^{cl}} & R_Q(h^*p^*A) \end{array}$$

is commutative.

In case the $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$ are smooth-adjointable (in the sense of Definition 2.6(1)), we say that an $\mathscr{S}$-skeleton on $\mathscr{R}$ is smooth-adjointable *if the $\mathscr{S}^{sm}$-structure θ^{sm} is left-adjointable (in the sense of Definition 1.8).*

2.9. Remark. Let $f\colon T \to S$ be a morphism in $\mathscr{S}$ which is both a closed immersion and a smooth morphism. Applying condition (ex-$\mathscr{S}$-fib) to the commutative square

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathsf{id}_S} \\ S & \xrightarrow{\mathsf{id}_S} & S, \end{array}$$

we see that the two natural isomorphisms $\theta_f\colon f^* \circ R_S \xrightarrow{\sim} R_T \circ f^*$ obtained by regarding f either as a smooth morphism or as a closed immersion must in fact coincide; this follows from the equality $\theta_{\mathsf{id}_S} = \mathsf{id}_{R_S}$ (see Remark 1.7).

The following is the main result about $\mathscr{S}$-skeleta:

2.10. Proposition. *Every $\mathscr{S}$-structure on $\mathscr{R}$ determines by restriction an $\mathscr{S}$-skeleton on it. Conversely, every $\mathscr{S}$-skeleton on $\mathscr{R}$ uniquely extends to an $\mathscr{S}$-structure on it. Moreover, if $\mathbb{H}_1$ and $\mathbb{H}_2$ are smooth-adjointable, then a given $\mathscr{S}$-structure on $\mathscr{R}$ is smooth-adjointable if an only if the underlying $\mathscr{S}$-skeleton is smooth-adjointable.*

Proof. Suppose that we are given an $\mathscr{S}$-structure θ on $\mathscr{R}$, and let us check that its restrictions to $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ define indeed an $\mathscr{S}$-skeleton. Since axioms (mor-$\mathscr{S}^{sm}$-fib) and (mor-$\mathscr{S}^{cl}$-fib) are both special cases of axiom (mor-$\mathscr{S}$-fib), we only need to check that condition (ex-$\mathscr{S}$-fib) holds: given a commutative square in $\mathscr{S}$

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p, q smooth morphisms and z, h closed immersions, we have to show that the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(Q)$

$$\begin{array}{ccccc} q^*z^*R_S(A) & \xrightarrow{\theta} & q^*R_Z(z^*A) & \xrightarrow{\theta} & R_Q(q^*z^*A) \\ \| & & & & \| \\ h^*p^*R_S(A) & \xrightarrow{\theta} & h^*R_P(p^*A) & \xrightarrow{\theta} & R_Q(h^*p^*A) \end{array}$$

is commutative. We claim that, in fact, the conclusion is true with no assumption on the morphisms z, h, p, q. Indeed, if f denotes the composite morphism $p \circ h = z \circ q$, the above diagram coincides by definition with the outer rectangle of the diagram

$$\begin{array}{ccccc} q^*z^*R_S(A) & \xrightarrow{\theta} & q^*R_Z(z^*A) & \xrightarrow{\theta} & R_Q(q^*z^*A) \\ \| & & & & \| \\ f^*R_S(A) & & \xrightarrow{\qquad\qquad\theta\qquad\qquad} & & R_Q(f^*A) \\ \| & & & & \| \\ h^*p^*R_S(A) & \xrightarrow{\theta} & h^*R_P(p^*A) & \xrightarrow{\theta} & R_Q(h^*p^*A) \end{array}$$

where both pieces commute by axiom (mor-$\mathscr{S}$-fib).

Conversely, suppose that we are given an $\mathscr{S}$-skeleton $(\theta^{sm}, \theta^{cl})$ on $\mathscr{R}$, and let us show that it extends uniquely to an $\mathscr{S}$-structure θ: we have to construct R-transition isomorphisms along arbitrary morphisms of $\mathscr{S}$ and check that they satisfy condition (mor-$\mathscr{S}$-fib). To this end, we exploit the functoriality of factorization categories.

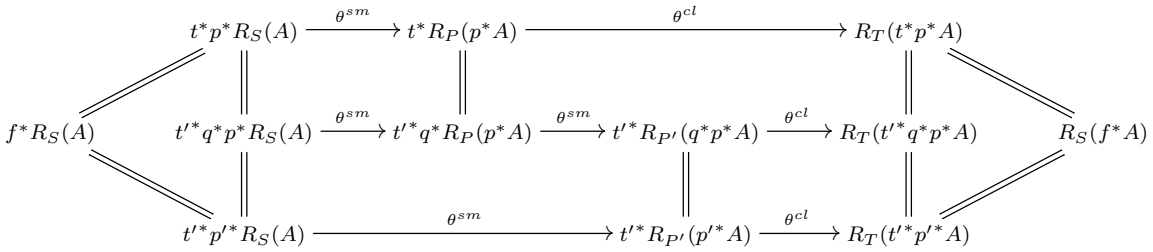
To begin with, fix a morphism $f\colon T \to S$ in $\mathscr{S}$, and let us construct the corresponding R-transition isomorphism. For every object $(P; t, p) \in \mathsf{Fact}_{\mathscr{S}}(f)$, consider the composite natural isomorphism of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\theta_f^{(t,p)}\colon f^*R_S(A) = t^*p^*R_S(A) \xrightarrow{\theta_p^{sm}} t^*R_P(p^*A) \xrightarrow{\theta_t^{cl}} R_T(t^*p^*A) = R_T(f^*A).$$

It follows from axiom (mor-$\mathscr{S}$-fib) that, if an $\mathscr{S}$-structure on $\mathscr{R}$ extending the data exists, then the R-transition isomorphism θ_f necessarily coincides with $\theta_f^{(t,p)}$; this also implies that the sought-after $\mathscr{S}$-structure θ on $\mathscr{R}$ is unique, provided it exists.

We claim that the natural isomorphism $\theta_f^{(t,p)}$ does not depend on the choice of $(P; t, p) \in \mathsf{Fact}_{\mathscr{S}}(f)$. To this end, using the connectedness of $\mathsf{Fact}_{\mathscr{S}}(f)$ (guaranteed by Lemma 2.5), it suffices to show that, for every arrow $q\colon (P'; t', p') \to (P; t, p)$ in $\mathsf{Fact}_{\mathscr{S}}(f)$, the equality $\theta_f^{(t',p')} = \theta_f^{(t,p)}$ holds. By definition, this amounts to showing that the outer part of the

diagram



is commutative. But the two lateral pieces commute by axiom ($\mathscr{S}$-fib-1), the two squares commute by naturality, the lower-left rectangle commutes by axiom (mor-$\mathscr{S}^{sm}$-fib), while the upper-right rectangle commutes by axiom (ex-$\mathscr{S}$-fib) applied to the commutative square in $\mathscr{S}$

$$\begin{array}{ccc} T & \xrightarrow{t'} & P' \\ \downarrow{\scriptstyle \mathsf{id}_T} & & \downarrow{\scriptstyle q} \\ T & \xrightarrow{t} & P, \end{array}$$

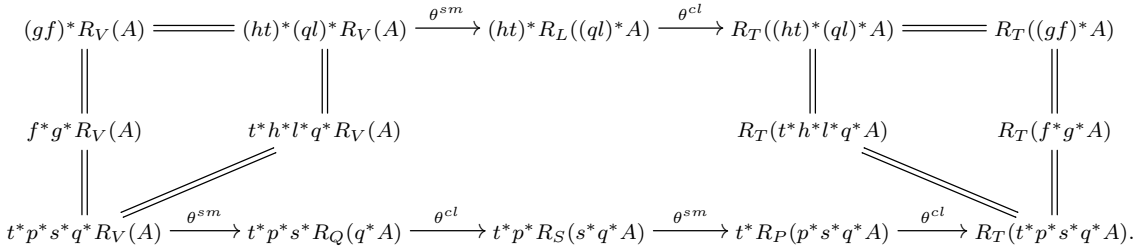
taking into account the equality $\theta_{\mathsf{id}_T} = \mathsf{id}_{R_T}$ (see Remark 1.7). This proves the claim. Thus, we have a well-defined natural isomorphism of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(T)$

$$\theta = \theta_f \colon f^* R_S(A) \xrightarrow{\sim} R_T(f^* A).$$

We now claim that, as f varies, these natural isomorphisms define an $\mathscr{S}$-structure on $\mathscr{R}$. To this end, we need to check that they satisfy axiom (mor-$\mathscr{S}$-fib): given two composable morphisms $f \colon T \to S$ and $g \colon S \to V$ in $\mathscr{S}$, we have to show that the diagram of functors $\mathbb{H}_1(V) \to \mathbb{H}_2(T)$

$$\begin{array}{ccccc} (gf)^* R_V(A) & & \xrightarrow{\quad\theta\quad} & & R_T((gf)^* A) \\ \| & & & & \| \\ f^* g^* R_V(A) & \xrightarrow{\theta} & f^* R_S(g^* A) & \xrightarrow{\theta} & R_T(f^* g^* A) \end{array}$$

is commutative. To this end, choose factorizations $(P; t, p) \in \mathsf{Fact}_{\mathscr{S}}(f)$ and $(Q; s, q) \in \mathsf{Fact}_{\mathscr{S}}(g)$, and then choose a further factorization $(L; h, l) \in \mathsf{Fact}_{\mathscr{S}}(s \circ p)$; in this way, we also obtain the factorization $(L; ht, ql) \in \mathsf{Fact}_{\mathscr{S}}(g \circ f)$. Expanding the definition of $\theta_f = \theta_f^{(t,p)}$ and $\theta_g = \theta_g^{(s,q)}$, we can express the above diagram more explicitly as the outer part of the diagram



Since the two lateral pieces commute by axiom ($\mathscr{S}$-fib-1), we are reduced to showing the commutativity of the central piece, which is the outer rectangle of the diagram

$$\begin{array}{ccccccc}
(ht)^*(ql)^*R_V(A) & & \xrightarrow{\theta^{sm}} & & (ht)^*R_L((ql)^*A) & & \xrightarrow{\theta^{cl}} & & R_T((ht)^*(ql)^*A) \\
\| & & & & \| & & & & \| \\
t^*h^*l^*q^*R_V(A) & \xrightarrow{\theta^{sm}} & t^*h^*l^*R_Q(q^*A) & \xrightarrow{\theta^{sm}} & t^*h^*R_L(l^*q^*A) & \xrightarrow{\theta^{cl}} & t^*R_P(h^*l^*q^*A) & \xrightarrow{\theta^{cl}} & R_T(t^*h^*l^*q^*A) \\
\| & & \| & & & & \| & & \| \\
t^*p^*s^*q^*R_V(A) & \xrightarrow{\theta^{sm}} & t^*p^*s^*R_Q(q^*A) & \xrightarrow{\theta^{cl}} & t^*p^*R_S(s^*q^*A) & \xrightarrow{\theta^{sm}} & t^*R_P(p^*s^*q^*A) & \xrightarrow{\theta^{cl}} & R_T(t^*p^*s^*q^*A).
\end{array}$$

Here, the two upper pieces commute by axioms (mor-$\mathscr{S}^{sm}$-fib) and (mor-$\mathscr{S}^{cl}$-fib), the two lateral lower pieces commute by naturality, while the central lower piece commutes by axiom (ex-$\mathscr{S}$-fib). This proves the claim, thereby showing that the given $\mathscr{S}$-skeleton on $\mathscr{R}$ uniquely extends to an $\mathscr{S}$-structure.

Lastly, the fact that a given $\mathscr{S}$-structure on $\mathscr{R}$ is smooth-adjointable if and only if so is the underlying $\mathscr{S}$-skeleton follows directly from Remark 2.9. ∎

In practice, it might be tricky to check the exchange condition of $\mathscr{S}$-skeleta for all commutative squares where the vertical (resp. horizontal) arrows are smooth morphisms (resp. closed immersions). However, the task usually simplifies in the case of Cartesian squares. This motivates the following result, which will be useful in the next section:

2.11. LEMMA. *Let θ^{sm} and θ^{cl} be an $\mathscr{S}^{sm}$-structure and an $\mathscr{S}^{cl}$-structure on $\mathscr{R}$, respectively. Then, the pair $(\theta^{sm}, \theta^{cl})$ defines an $\mathscr{S}$-skeleton on $\mathscr{R}$ if and only if it satisfies the following two conditions:*

(C-$\mathscr{S}$-fib) *For every Cartesian square in $\mathscr{S}$*

$$\begin{array}{ccc}
P_Z & \xrightarrow{z'} & P \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
Z & \xrightarrow{z} & S
\end{array}$$

with p a smooth morphism and z a closed immersion, the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(P_Z)$

$$\begin{array}{ccccc}
p'^*z^*R_S(A) & & \xrightarrow{\theta^{cl}} & & p'^*R_Z(z^*A) & & \xrightarrow{\theta^{sm}} & & R_{P_Z}(p'^*z^*A) \\
\| & & & & & & & & \| \\
z'^*p^*R_S(A) & & \xrightarrow{\theta^{sm}} & & z'^*R_P(p^*A) & & \xrightarrow{\theta^{sm}} & & R_{P_Z}(z'^*p^*A)
\end{array}$$

is commutative.

(T-$\mathscr{S}$-fib) *For every commutative triangle in* $\mathscr{S}$

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ & \searrow^{q} & \downarrow{\scriptstyle p} \\ & & S \end{array}$$

with p, q *smooth morphisms and* h *a closed immersion, the diagram of functors* $\mathbb{H}_1(S) \to \mathbb{H}_2(Q)$

$$\begin{array}{ccccc} q^*R_S(A) & & \xrightarrow{\quad\theta^{sm}\quad} & & R_Q(q^*A) \\ \| & & & & \| \\ h^*p^*R_S(A) & \xrightarrow{\theta^{sm}} & h^*R_P(p^*A) & \xrightarrow{\theta^{cl}} & R_S(h^*p^*A) \end{array}$$

is commutative.

Proof. By definition, the pair $(\theta^{sm}, \theta^{cl})$ defines an $\mathscr{S}$-skeleton on $\mathscr{R}$ if and only if it satisfies condition (ex-$\mathscr{S}$-fib) from Definition 2.8.

Note that condition (C-$\mathscr{S}$-fib) is the restriction of (ex-$\mathscr{S}$-fib) to the class of Cartesian squares, while condition (T-$\mathscr{S}$-fib) is equivalent to the restrictions of (ex-$\mathscr{S}$-fib) to the class of squares of the form

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{\mathsf{id}_S} & S, \end{array}$$

taking into account the equality $\theta_{\mathsf{id}_S} = \mathsf{id}_{R_S}$ (see Remark 1.7). Thus, condition (ex-$\mathscr{S}$-fib) trivially implies both (C-$\mathscr{S}$-fib) and (T-$\mathscr{S}$-fib).

Conversely, suppose that the latter two conditions are satisfied, and let us show that condition (ex-$\mathscr{S}$-fib) holds: given a commutative square

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

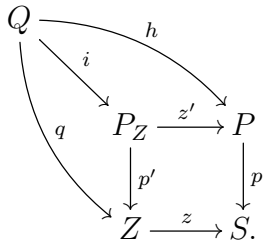
with z, h closed immersions and p, q smooth morphisms, we have to show that the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(Q)$

$$\begin{array}{ccccc} q^*z^*R_S(A) & \xrightarrow{\theta^{cl}} & q^*R_Z(z^*A) & \xrightarrow{\theta^{sm}} & R_Q(q^*z^*A) \\ \| & & & & \| \\ h^*p^*R_S(A) & \xrightarrow{\theta^{sm}} & h^*R_P(p^*A) & \xrightarrow{\theta^{cl}} & R_Q(h^*p^*A) \end{array}$$

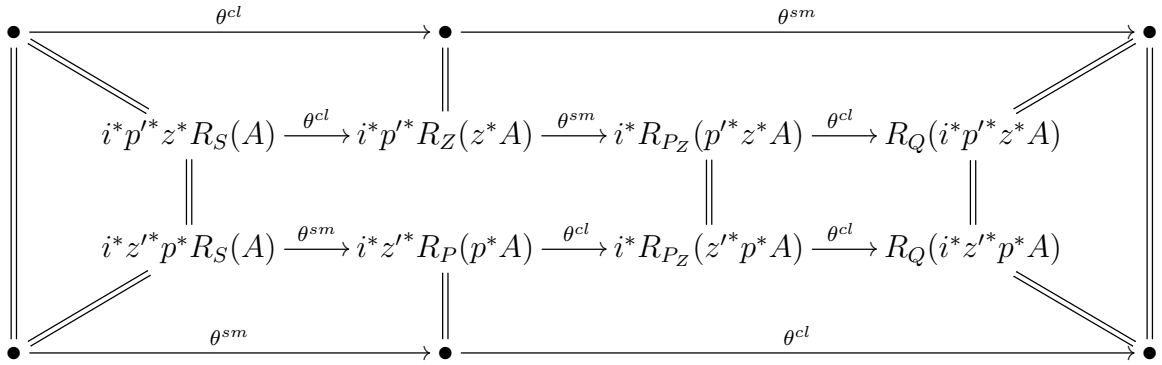
is commutative. To this end, form the Cartesian square

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

(which indeed exists, by Assumption 2.2(ii)) and let $i\colon Q \to P_Z$ be the morphism induced by q and h (which is a closed immersion, by Assumption 2.2(iii)): in this way, we obtain the commutative diagram



We can then decompose the diagram under consideration as



Here, the two lateral pieces commute by axiom ($\mathscr{S}$-fib-1), the upper-right piece commutes by axiom (T-$\mathscr{S}$-fib), the central-left rectangle commutes by axiom (C-$\mathscr{S}$-fib), the lower-right piece commutes by axiom ($\mathscr{S}^{cl}$-fib-1), while the remaining three pieces commute by naturality. This concludes the proof. ∎

3. The core of a localic morphism

In this section, we introduce a variant of the notion of $\mathscr{S}$-skeleton which is more adapted to the setting of perverse sheaves: we want to rephrase the axioms of $\mathscr{S}$-skeleta in such a way that inverse image functors under closed immersions are replaced by the corresponding direct image functors. It turns out that one still needs to use inverse images under closed immersions, but only in one specific situation; this does not cause any problems in our applications.

3.1. Localic fibered categories. First of all, we need to introduce some complements to the previous assumptions on the base category $\mathscr{S}$ and its subcategories $\mathscr{S}^{cl}$ and $\mathscr{S}^{sm}$. In detail:

3.2. Assumption. *We keep the assumptions and notation of Assumption 2.2. In addition, we assume the following:*

(i) The category $\mathscr{S}$ has an initial object $\emptyset$.

(ii) For every closed immersion $z\colon Z \to S$, there exists a distinguished smooth morphism $u\colon U \to S$ with the property that, for every $T \in \mathscr{S}$, the map

$$u \circ -\colon \mathrm{Hom}_{\mathscr{S}}(T, U) \to \mathrm{Hom}_{\mathscr{S}}(T, S)$$

identifies $\mathrm{Hom}_{\mathscr{S}}(T, U)$ with the subset of $\mathrm{Hom}_{\mathscr{S}}(T, S)$ consisting of all those morphisms $f\colon T \to S$ for which the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{z} & S \end{array}$$

is Cartesian. We call u the open immersion *complementary to z, and we call U the* open complement *of Z in S.*

3.3. Remark. Note that, for every closed immersion $z\colon Z \to S$, the complementary open immersion $u\colon U \to S$ is a monomorphism in $\mathscr{S}$ by definition.

3.4. Example. As for Assumption 2.2, the main examples of interest for us are when $\mathscr{S}$ is one among the category Var_k of quasi-projective k-varieties or its subcategory Sm_k of smooth k-varieties. In both cases, the notion of open immersion introduced above recovers the usual one.

Throughout this section, we work under Assumption 3.2 (which, by definition, also includes Assumption 2.2). While the results of Section 2 apply to general $\mathscr{S}$-fibered categories, the results presented below are specific to the setting of *triangulated $\mathscr{S}$-fibered categories*, as axiomatized in [Ter24a, §§ 4.1, 6.1]. In a triangulated $\mathscr{S}$-fibered category $\mathbb{H}$, the categories $\mathbb{H}(S)$ are triangulated, the inverse image functors $f^*\colon \mathbb{H}(S) \to \mathbb{H}(T)$ are exact, and the connection isomorphisms are required to be compatible with shifts. Similarly, in a *morphism of triangulated $\mathscr{S}$-fibered categories* $R\colon \mathbb{H}_1 \to \mathbb{H}_2$, the functors $R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)$ are triangulated, and the R-transition isomorphisms are required to be compatible with shifts.

In fact, our main applications concern the theory of stable homotopy 2-functors, as developed in [Ayo07a, Ayo07b] and [Ayo10]. In order not to discuss the specifics of this theory here, we have chosen to summarize the part of [Ayo07a, Defn. 1.4.1] revolving around open and closed immersions in a self-contained way:

3.5. Definition. *We say that a triangulated $\mathscr{S}$-fibered category $\mathbb{H}$ is* localic *if it satisfies the following conditions:*

(a) $\mathbb{H}(\emptyset)$ *is the zero triangulated category.*

(b) $\mathbb{H}$ *is smooth-adjointable in the sense of Definition 2.6(1). Moreover, for every Cartesian square in* $\mathscr{S}$

$$\begin{array}{ccc} P_T & \xrightarrow{f'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & S \end{array}$$

with p a smooth morphism, the Beck–Chevalley natural transformation of functors $\mathbb{H}(P) \to \mathbb{H}(T)$

$$p'_{\#} f'^{*} A \xrightarrow{\eta} p'_{\#} f'^{*} p^{*} p_{\#} A = p'_{\#} p'^{*} f^{*} p_{\#} A \xrightarrow{\epsilon} f^{*} p_{\#} A$$

is invertible.

(c) $\mathbb{H}$ *is closed-adjointable in the sense of Definition 2.6(1). Moreover, for every closed immersion $z\colon Z \to S$ in $\mathscr{S}$, the functor $z_*\colon \mathbb{H}(Z) \to \mathbb{H}(S)$ is fully faithful.*

(d) For every closed immersion $z\colon Z \to S$ in $\mathscr{S}$ with complementary open immersion $u\colon U \to S$, the pair of functors (z^*, u^*) is jointly conservative.

3.6. Example. In the case where $\mathscr{S} = \mathsf{Var}_k$ (with $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ the usual subcategories of smooth morphisms and closed immersions, respectively), the underlying triangulated $\mathscr{S}$-fibered category of any stable homotopy 2-functor is localic in virtue of the so-called *localization axiom* (see [Ayo07a, § 1.4.1]). Our axiomatic approach allows us to study localization triangles over Sm_k as well (compare with Example 2.3).

Let $\mathbb{H}$ be a localic triangulated $\mathscr{S}$-fibered category. Using Lemma 1.5, we can regard its underlying $\mathscr{S}^{sm}$-fibered category (resp. its underlying $\mathscr{S}^{cl}$-fibered category) as an $\mathscr{S}^{sm,op}$-fibered category (resp. as an $\mathscr{S}^{cl,op}$-fibered category). Since the left-adjoint (resp. the right-adjoint) of a triangulated functor between triangulated categories is also triangulated, compatibly with composition (for instance, see [Stacks, Tag 0A8C]), this is canonically a triangulated $\mathscr{S}^{sm,op}$-fibered category (resp. a triangulated $\mathscr{S}^{cl,op}$-fibered category). In the following, this fact will be systematically used without further comments.

The next lemma summarizes the basic properties of localic triangulated $\mathscr{S}$-fibered categories. In the setting of stable homotopy 2-functors, all these properties are established in [Ayo07a, § 1.4.4]; here, we just rearrange the argument in order to make sure that the same properties hold in our setting.

3.7. Lemma. *Fix a closed immersion $z\colon Z \to S$ in $\mathscr{S}$ with complementary open immersion $u\colon U \to S$. Then:*

1. *The functor $u_\#\colon \mathbb{H}(U) \to \mathbb{H}(S)$ is fully faithful.*
2. *The composite functors $z^* \circ u_\#$ and $u^* \circ z_*$ vanish.*
3. *There exists a canonical* localization distinguished triangle *of functors $\mathbb{H}(S) \to \mathbb{H}(S)$*

$$u_\# u^* A \xrightarrow{\epsilon} A \xrightarrow{\eta} z_* z^* A \to u_\# u^* A[1].$$

4. *For every Cartesian square in $\mathscr{S}$ of the form*

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p_Z} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p a smooth morphism and z a closed immersion, the Beck–Chevalley natural transformation of functors $\mathbb{H}(Z) \to \mathbb{H}(P)$

$$p^* z_* A \xrightarrow{\eta} z'_* z'^* p^* z_* A = z'_* p_Z^* z^* z_* A \xrightarrow{\epsilon} z'_* p_Z^* A$$

is invertible.

Proof.

1. Saying that $u_\#$ is fully faithful is equivalent to saying that the unit natural transformation $\eta\colon \mathsf{id}_{\mathbb{H}(S)} \to u^* u_\#$ is invertible. In order to show that this is the case, it suffices to apply Definition 3.5(b) to the square

$$\begin{array}{ccc} U & \xrightarrow{\mathsf{id}_U} & U \\ \downarrow{\scriptstyle \mathsf{id}_U} & & \downarrow{\scriptstyle u} \\ U & \xrightarrow{u} & S, \end{array}$$

which is Cartesian because u is a monomorphism (see Remark 3.3).

2. To prove the vanishing of $z^* \circ u_\#$, it suffices to apply Definition 3.5(a,b) to the Cartesian square

$$\begin{array}{ccc} \emptyset & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{z} & S. \end{array}$$

The vanishing of $u^* \circ z_*$ follows from this by adjunction.

3. For every object $A \in \mathbb{H}(S)$, complete the co-unit morphism $\epsilon\colon u_{\#}u^*A \to A$ to a distinguished triangle in $\mathbb{H}(S)$

$$u_{\#}u^*A \xrightarrow{\epsilon} A \to C(A) \to u_{\#}u^*A[1]. \tag{7}$$

Using the fully faithfulness of the functor $u_{\#}$, together with the triangular identity, one sees that the co-unit morphism $\epsilon\colon u^*u_{\#}u^*A \to u^*A$ is an isomorphism, inverse to $\eta\colon u^*A \to u^*u_{\#}u^*A$. Hence the object $u^*C(A)$ must vanish (being the cone of an isomorphism). For every morphism $\phi\colon A \to B$ in $\mathbb{H}(S)$, applying the criterion of [BBD82, Prop. 1.1.9] to the vanishing

$$\mathrm{Hom}_{\mathbb{H}(S)}(u_{\#}u^*A, C(B)[-1]) = \mathrm{Hom}_{\mathbb{H}(U)}(u^*A, u^*C(B)[-1]) = 0,$$

one deduces the existence of a unique morphism $C(\phi)\colon C(A) \to C(B)$ fitting into a morphism of triangles of the form

$$\begin{array}{ccccccc}
u_{\#}u^*A & \xrightarrow{\epsilon} & A & \longrightarrow & C(A) & \longrightarrow & u_{\#}u^*A[1] \\
\downarrow{\scriptstyle u_{\#}u^*\phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle C(\phi)} & & \downarrow{\scriptstyle u_{\#}u^*\phi[1]} \\
u_{\#}u^*B & \xrightarrow{\epsilon} & B & \longrightarrow & C(B) & \longrightarrow & u_{\#}u^*B[1].
\end{array}$$

As a consequence, the assignment $A \rightsquigarrow C(A)$ defines a functor $C\colon \mathbb{H}(S) \to \mathbb{H}(S)$ fitting into a canonical distinguished triangle of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$u_{\#}u^*A \xrightarrow{\epsilon} A \to C(A) \to u_{\#}u^*A[1].$$

Note that the composite natural isomorphism of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$u_{\#}u^*A \xrightarrow{\epsilon} A \xrightarrow{\eta} z_*z^*A$$

vanishes, because we have

$$\mathrm{Hom}_{\mathbb{H}(S)}(u_{\#}u^*A, z_*z^*A) = \mathrm{Hom}_{\mathbb{H}(Z)}(z^*u_{\#}u^*A, z^*A) = 0 \tag{8}$$

as $z^* \circ u_{\#} = 0$. Hence, there exists a morphism $\psi(A)\colon C(A) \to z_*z^*A$ fitting into a morphism of triangles of the form

$$\begin{array}{ccccccc}
u_{\#}u^*A & \xrightarrow{\epsilon} & A & \longrightarrow & C(A) & \longrightarrow & u_{\#}u^*A[1] \\
\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \psi(A)} & & \downarrow \\
0 & \longrightarrow & z_*z^*A & \longrightarrow & z_*z^*A & \longrightarrow & 0.
\end{array}$$

Applying the criterion of [BBD82, Prop. 1.1.9] to the vanishing (8), one deduces that such a morphism $\psi(A)$ is unique and, similarly, one shows that it is functorial in A. Thus, one obtains a canonical natural transformation of functors $\mathbb{H}(S) \to \mathbb{H}(S)$

$$\psi\colon C(A) \to z_*z^*A,$$

and it remains to check that this is object-wise invertible. To this end, since the pair (z^*, u^*) is conservative (by Definition 3.5(d)), and the image of both sides under u^* vanishes, it suffices to check that the morphism

$$z^*\psi\colon z^*C(A) \to z^*z_*z^*A$$

is invertible. But the arrow $z^*A \to z^*C(A)$ induced by the triangle (7) is invertible (since its cone vanishes, as $z^* \circ u_\# = 0$), and its composite with the above arrow coincides with the unit morphism $\eta\colon z^*A \to z^*z_*z^*A$. Since z_* is fully faithful (by Definition 3.5(c)), the latter is an isomorphism, inverse to $\epsilon\colon z^*z_*z^*A \to z^*A$. This concludes the proof.

4. Since z_* is fully faithful (by Definition 3.5(c)), the co-unit arrow $\epsilon\colon z'_*p_Z^*z^*z_*A \to z'_*p_Z^*A$ is invertible. Hence, it suffices to show that the unit arrow $\eta\colon p^*z_*A \to z'_*z'^*p^*z_*A$ is invertible as well. To this end, form the Cartesian diagram

$$\begin{array}{ccc} P_U & \xrightarrow{u'} & P \\ \downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{u} & S \end{array}$$

complementary to the one in the statement. As a consequence of the previous point, the invertibility of the above unit arrow is equivalent to the vanishing of the composite functor $u'_\#u'^* \circ p^* \circ z_*$. To show this, it suffices to use the existence of the connection isomorphism $u'^* \circ p^* = p_U^* \circ u^*$ and the vanishing of $u^* \circ z_*$ established in the second point.

■

3.8. Cores of localic morphisms. For the rest of this section, we fix two localic triangulated $\mathscr{S}$-fibered categories $\mathbb{H}_1$ and $\mathbb{H}_2$, as well as a family a triangulated functors $\mathscr{R} = \{R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\}_{S\in\mathscr{S}}$.

We say that an $\mathscr{S}$-structure θ on $\mathscr{R}$ is *triangulated* if it is compatible with shifts as in [Ter24a, Defn. 6.2]: in this case, the resulting morphism of $\mathscr{S}$-fibered categories is in fact a morphism of triangulated $\mathscr{S}$-fibered categories. Similarly, we say that an $\mathscr{S}$-skeleton on $\mathscr{R}$ is *triangulated* if so are the $\mathscr{S}^{sm}$-structure and $\mathscr{S}^{cl}$-structure defining it. As the reader can easily check, an $\mathscr{S}$-structure on $\mathscr{R}$ is triangulated if and only if so is the underlying $\mathscr{S}$-skeleton. All $\mathscr{S}$-structures and $\mathscr{S}$-skeleta considered in the present section are assumed to be triangulated.

We are going to introduce a variant of the notion of $\mathscr{S}$-skeleton on $\mathscr{R}$, in which inverse images under closed immersions are replaced by the corresponding direct images. To this end, we need to have a closer look at the adjointability properties of $\mathscr{S}^{sm}$-structures and $\mathscr{S}^{cl}$-structures in the localic setting. Here is the key observation:

3.9. LEMMA. *Suppose that we are given a triangulated $\mathscr{S}^{sm}$-structure on $\mathscr{R}$. Then, the following hold:*

1. *Any triangulated $\mathscr{S}^{cl}$-structure on $\mathscr{R}$ is necessarily right-adjointable.*
2. *If the given $\mathscr{S}^{sm}$-structure is left-adjointable, then any triangulated $\mathscr{S}^{cl,op}$-structure on $\mathscr{R}$ is necessarily left-adjointable.*

PROOF. It suffices to show that, given a closed immersion $z\colon Z \to S$ in $\mathscr{S}$, the following claims hold:

1. Given a natural isomorphism of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(Z)$
$$\theta_z\colon z^*R_S(A) \xrightarrow{\sim} R_Z(z^*A),$$
the natural transformation of functors $\mathbb{H}_1(Z) \to \mathbb{H}_2(S)$
$$R_S(z_*B) \xrightarrow{\eta} z_*z^*R_S(z_*B) \xrightarrow{\theta_z} z_*R_Z(z^*z_*B) \xrightarrow{\epsilon} z_*R_Z(B) \tag{9}$$
is always invertible.
2. Given a natural isomorphism of functors $\mathbb{H}_1(Z) \to \mathbb{H}_2(S)$
$$\bar{\theta}_z\colon z_*R_Z(B) \xrightarrow{\sim} R_S(z_*B),$$
the natural transformation of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(Z)$
$$z^*R_S(A) \xrightarrow{\eta} z^*R_S(z_*z^*A) \xleftarrow{\bar{\theta}_z} z^*z_*R_Z(z^*A) \xrightarrow{\epsilon} R_Z(z^*A) \tag{10}$$
is invertible, provided the given $\mathscr{S}^{sm}$-structure is left-adjointable.

Let $u\colon U \to S$ denote the open immersion complementary to z. We argue as follows:

1. Since $z_*\colon \mathbb{H}_1(Z) \to \mathbb{H}_1(S)$ is fully faithful (by Definition 3.5(c)), the natural transformation (9) is invertible as soon as the unit arrow $\eta\colon R_S \circ z_* \to z_*z^* \circ R_S \circ z_*$ is invertible. By the localization triangle, this is equivalent to the vanishing $u_\# u^* \circ R_S \circ z_* = 0$. To check the latter property, it suffices to combine the natural isomorphism $u^* \circ R_S = R_U \circ u^*$ (which exists by assumption) with the vanishing $u^* \circ z_* = 0$ (given by Lemma 3.7(2)).
2. Since $z_*\colon \mathbb{H}_2(Z) \to \mathbb{H}_2(S)$ is fully faithful (by Definition 3.5(c)), the natural transformation (10) is invertible as soon as the unit arrow $\eta\colon z^* \circ R_S \to z^* \circ R_S \circ z_*z^*$ is invertible. By the localization triangle, this is equivalent to the vanishing $z^* \circ R_S \circ u_\# u^* = 0$. To check the latter property, it suffices to combine the natural isomorphism $u_\# \circ R_U = R_S \circ u_\#$ (which exists by assumption) with the vanishing $z^* \circ u_\# = 0$ (given by Lemma 3.7(2)).

This proves the two claims. ∎

This leads to the following definition:

3.10. DEFINITION. *We say that a morphism between triangulated $\mathscr{S}$-fibered categories $R\colon \mathbb{H}_1 \to \mathbb{H}_2$ is* localic *if it is smooth-adjointable in the sense of Definition 2.6(2). If we are only given a collection of triangulated functors $\mathscr{R} = \{R_S\colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\}_{S\in\mathscr{S}}$, we say that a triangulated $\mathscr{S}$-structure θ on $\mathscr{R}$ is* localic *if so is the resulting morphism $(\mathscr{R},\theta)\colon \mathbb{H}_1 \to \mathbb{H}_2$.*

In the localic setting, we propose the following variant of the notion of $\mathscr{S}$-skeleton:

3.11. DEFINITION. *An* $\mathscr{S}$-core *on the family $\mathscr{R}$ is the datum of*

- *a left-adjointable triangulated $\mathscr{S}^{sm}$-structure θ^{sm} on $\mathscr{R}$,*
- *a triangulated $\mathscr{S}^{cl,op}$-structure $\bar{\theta}^{cl}$ on $\mathscr{R}$*

satisfying the following exchange conditions:

(C'-$\mathscr{S}$-fib) *For every Cartesian square in $\mathscr{S}$*

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

with p a smooth morphism and z a closed immersion, the diagram of functors $\mathbb{H}_1(Z) \to \mathbb{H}_2(P)$

$$\begin{array}{ccccc} p^*z_*R_Z(A) & \xrightarrow{\bar{\theta}^{cl}} & p^*R_S(z_*A) & \xrightarrow{\theta^{sm}} & R_{P_Z}(p^*z_*A) \\ \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\ z'_*p'^*R_S(A) & \xrightarrow{\theta^{sm}} & z'_*R_{P_Z}(p'^*A) & \xrightarrow{\bar{\theta}^{cl}} & R_P(z'_*p'^*A) \end{array}$$

is commutative.

(T-$\mathscr{S}$-fib) *For every commutative triangle in $\mathscr{S}$*

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle p} \\ & & S \end{array}$$

with p, q smooth morphisms and h a closed immersion, the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(Q)$

$$\begin{array}{ccccc} q^*R_S(A) & & \xrightarrow{\theta^{sm}} & & R_Q(q^*A) \\ \| & & & & \| \\ h^*p^*R_S(A) & \xrightarrow{\theta^{sm}} & h^*R_P(p^*A) & \xrightarrow{\theta^{cl}} & R_S(h^*p^*A) \end{array}$$

is commutative.

As explained in Remark 2.9, the exchange condition of $\mathscr{S}$-skeleta rules out any possible ambiguity on the natural isomorphism $\theta_f\colon f^*\circ R_S \xrightarrow{\sim} R_T\circ f^*$ in the case of a morphism $f\colon T\to S$ which is both a smooth morphism and a closed immersion. In the setting of $\mathscr{S}$-cores, it is natural to expect that the natural isomorphism $\theta_f^{sm}\colon f^*\circ R_S \xrightarrow{\sim} R_T\circ f^*$ obtained by regarding f as a smooth morphism is the mate of the natural isomorphism $\bar\theta_f^{cl}\colon f_*\circ R_T \xrightarrow{\sim} R_S\circ f_*$ obtained by regarding f as a closed immersion. While not obvious at first sight, this turns out to be true, as a consequence of the main result about $\mathscr{S}$-cores:

3.12. Theorem. *Every localic triangulated $\mathscr{S}$-structure on $\mathscr{R}$ determines by restriction and adjunction an $\mathscr{S}$-core on it. Conversely, every $\mathscr{S}$-core on $\mathscr{R}$ uniquely extends to a localic triangulated $\mathscr{S}$-structure on it.*

Proof. In view of Proposition 2.10, it suffices to show that localic $\mathscr{S}$-skeleta and $\mathscr{S}$-cores on $\mathscr{R}$ determine one another by adjunction. To this end, in view of Lemmata 2.11 and 3.9, it suffices to show that the exchange conditions (C-$\mathscr{S}$-fib) and (C'-$\mathscr{S}$-fib) are equivalent.

Assume that condition (C-$\mathscr{S}$-fib) holds, and let us prove that (C'-$\mathscr{S}$-fib) holds as well: given a Cartesian square

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

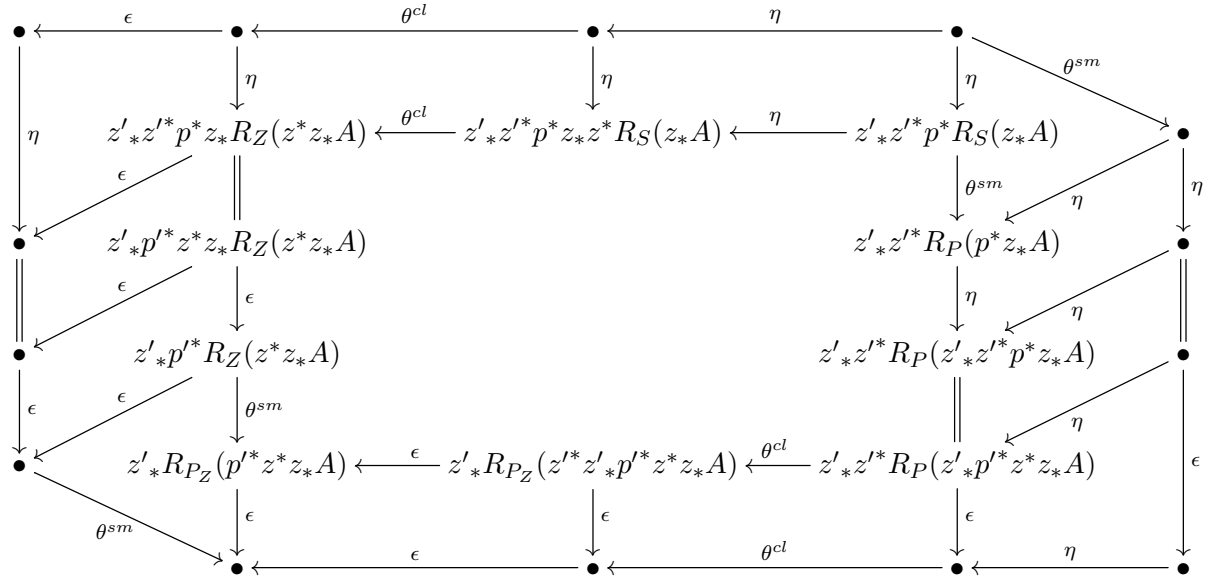
with z a closed immersion and p a smooth morphism, we have to show that the diagram of functors $\mathbb{H}_1(Z)\to\mathbb{H}_2(P)$

$$\begin{array}{ccccc} p^*z_*R_Z(A) & \xrightarrow{\bar\theta^{cl}} & p^*R_S(z_*A) & \xrightarrow{\theta^{sm}} & R_P(p^*z_*A) \\ \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\ z'_*p'^*R_Z(A) & \xrightarrow{\theta^{sm}} & z'_*R_{P_Z}(p'^*A) & \xrightarrow{\bar\theta^{cl}} & R_P(z'_*p'^*A) \end{array}$$

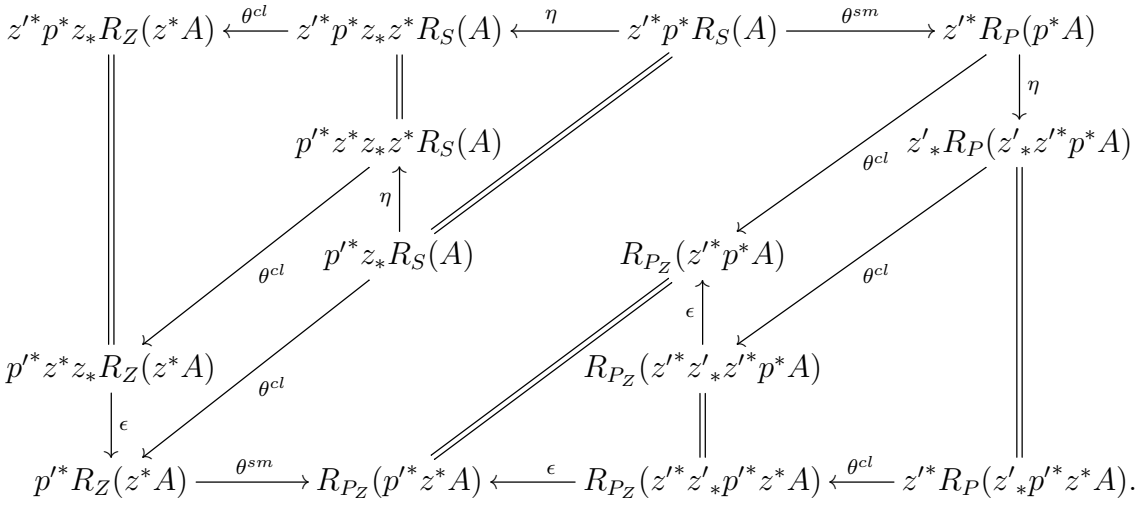
is commutative. Unwinding the various definitions, we obtain the more explicit diagram

$$\begin{array}{ccccccc} p^*z_*R_Z(A) & \xleftarrow{\epsilon} & p^*z_*R_Z(z^*z_*A) & \xleftarrow{\theta^{cl}} & p^*z_*z^*R_S(z_*A) & \xleftarrow{\eta} & p^*R_S(z_*A) \\ \downarrow{\scriptstyle\eta} & & & & & & \downarrow{\scriptstyle\theta^{sm}} \\ z'_*z'^*p^*z_*R_Z(A) & & & & & & R_P(p^*z_*A) \\ \| & & & & & & \downarrow{\scriptstyle\eta} \\ z'_*p'^*z^*z_*R_Z(A) & & & & & & R_P(z'_*z'^*p^*z_*A) \\ \downarrow{\scriptstyle\epsilon} & & & & & & \| \\ z'_*p'^*R_Z(A) & & & & & & R_P(z'_*p'^*z^*z_*A) \\ \downarrow{\scriptstyle\theta^{sm}} & & & & & & \downarrow{\scriptstyle\epsilon} \\ z'_*R_{P_Z}(p'^*A) & \xleftarrow{\epsilon} & z'_*R_{P_Z}(z'^*z'_*p'^*A) & \xleftarrow{\theta^{cl}} & z'_*z'^*R_{P_Z}(z'_*p'^*A) & \xleftarrow{\eta} & R_{P_Z}(z'_*p'^*A), \end{array}$$

which we decompose as



Since all boundary pieces in the latter diagram commute by naturality, we are reduced to considering the inner rectangle. The latter is obtained from the outer rectangle of the diagram of functors $\mathbb{H}_1(S) \to \mathbb{H}_2(P_Z)$



Here, the central piece commutes by condition (C-$\mathscr{S}$-fib), while all the other pieces commute by naturality and by construction.

Conversely, assume that condition (C'-$\mathscr{S}$-fib) holds, and let us prove that (C-$\mathscr{S}$-fib) holds as well: given a Cartesian square

$$\begin{array}{ccc} P_Z & \xrightarrow{z'} & P \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{z} & S \end{array}$$

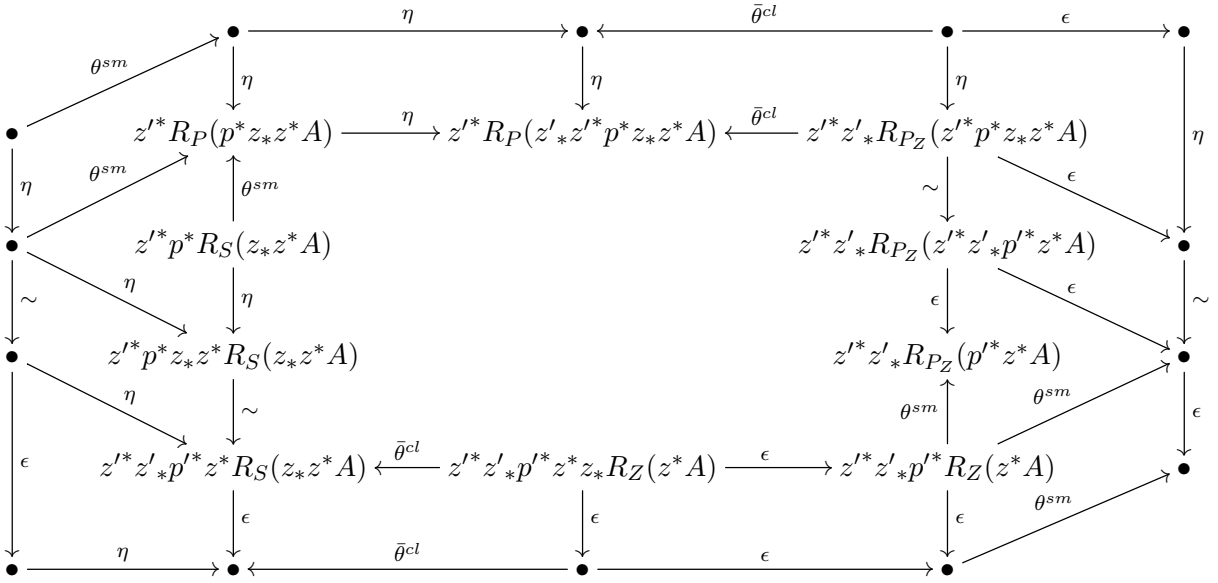
with z a closed immersion and p a smooth morphism, we have to show that the diagram of functors $\mathbb{H}_1(Z) \to \mathbb{H}_2(P)$

$$\begin{array}{ccccc} z'^*p^*R_S(A) & \xrightarrow{\theta^{sm}} & z'^*R_P(p^*A) & \xrightarrow{\theta^{cl}} & R_{P_Z}(z'^*p^*A) \\ \| & & & & \| \\ p'^*z^*R_S(A) & \xrightarrow{\theta^{cl}} & p'^*R_Z(z^*A) & \xrightarrow{\theta^{sm}} & R_{P_Z}(p'^*z^*A) \end{array}$$

is commutative. Unwinding the various definitions, we obtain the more explicit diagram

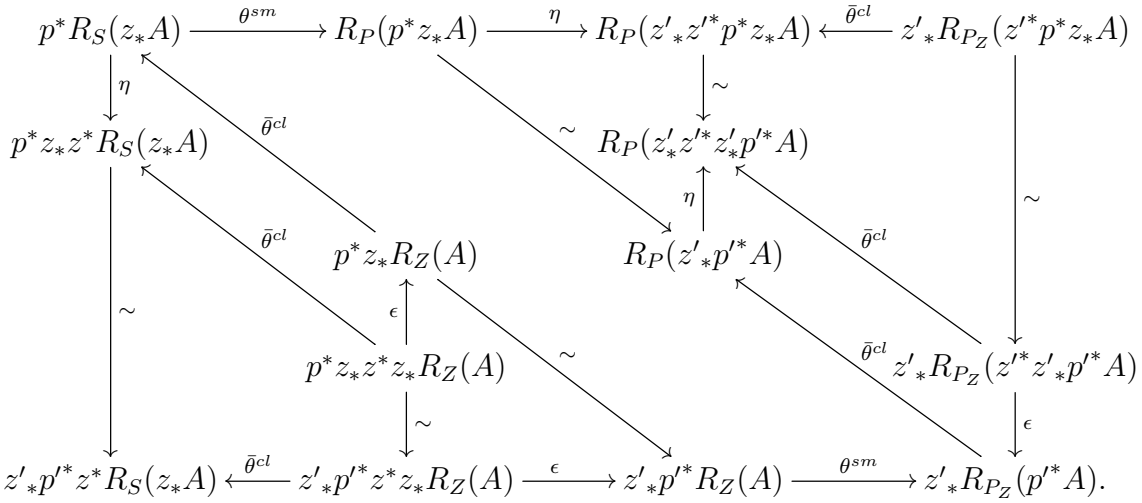
$$\begin{array}{ccccccc} z'^*R_P(p^*A) & \xrightarrow{\eta} & z'^*R_P(z'_*z'^*p^*A) & \xleftarrow{\bar\theta^{cl}} & z'^*z'_*R_{P_Z}(z'^*p^*A) & \xrightarrow{\epsilon} & R_{P_Z}(z'^*p^*A) \\ \uparrow{\scriptstyle\theta^{sm}} & & & & & & \downarrow{\scriptstyle\eta} \\ z'^*p^*R_S(A) & & & & & & R_{P_Z}(z'^*p^*z_*z^*A) \\ \downarrow{\scriptstyle\eta} & & & & & & \downarrow{\scriptstyle\sim} \\ z'^*p^*z_*z^*R_S(A) & & & & & & R_{P_Z}(z'^*z'_*p'^*z^*A) \\ \downarrow{\scriptstyle\sim} & & & & & & \downarrow{\scriptstyle\epsilon} \\ z'^*z'_*p'^*z^*R_S(A) & & & & & & R_{P_Z}(p'^*z^*A) \\ \downarrow{\scriptstyle\epsilon} & & & & & & \uparrow{\scriptstyle\theta^{sm}} \\ p'^*z^*R_S(A) & \xrightarrow{\eta} & p'^*z^*R_S(z_*z^*A) & \xleftarrow{\bar\theta^{cl}} & p'^*z^*z_*R_Z(z^*A) & \xrightarrow{\epsilon} & p'^*R_Z(z^*A), \end{array}$$

which we decompose as



Since all boundary pieces in the latter diagram commute by naturality, we are reduced to considering the inner diagram. The latter is obtained from the outer rectangle of the

diagram of functors $\mathbb{H}_1(Z) \to \mathbb{H}_2(P)$



Here, the central piece commutes by condition (C'-$\mathscr{S}$-fib), while all the other pieces commute by naturality and by construction. This concludes the proof. ∎

4. External tensor skeleta

In this section, we describe a variant of the notion of $\mathscr{S}$-skeleton from Section 2 in the setting of monoidal structures, both on single $\mathscr{S}$-fibered categories and on morphisms of such. We use the language of external tensor structures, following the conventions of [Ter24a] (as recalled in Subsection 1.10).

Throughout, we keep Assumption 2.2 (but not the stronger Assumption 3.2), and moreover we assume that the base category $\mathscr{S}$ admits binary products.

4.1. External tensor skeleta on fibered categories. Let $\mathbb{H}$ be an $\mathscr{S}$-fibered category; we let $\mathbb{H}^{sm}$ and $\mathbb{H}^{cl}$ denote the underlying $\mathscr{S}^{sm}$-fibered and $\mathscr{S}^{cl}$-fibered categories, respectively.

4.2. Definition. *An* external tensor skeleton $(\boxtimes, m^{sm}, m^{cl})$ *on* $\mathbb{H}$ *is the datum of*

- *an external tensor structure* $(\boxtimes^{sm}, m^{sm})$ *on* $\mathbb{H}^{sm}$,
- *an external tensor structure* $(\boxtimes^{cl}, m^{cl})$ *on* $\mathbb{H}^{cl}$

satisfying the following two conditions:

(ex-ETS-0) *For every* $S_1, S_2 \in \mathscr{S}$*, the two functors*

$$- \boxtimes^{sm} - = - \boxtimes^{sm}_{S_1,S_2} -, \; - \boxtimes^{cl} - = - \boxtimes^{cl}_{S_1,S_2} - \colon \mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2)$$

coincide, so that we can write both of them using the same symbol $- \boxtimes - = - \boxtimes_{S_1,S_2} -$.

(ex-ETS-1) *For every pair of commutative squares in* $\mathscr{S}$

$$\begin{array}{ccc} Q_i & \xrightarrow{h_i} & P_i \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle p_i} \\ Z_i & \xrightarrow{z_i} & S_i \end{array} \qquad (i = 1, 2)$$

with p_i, q_i *smooth morphisms and* z_i, h_i *closed immersions, the diagram of functors* $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(Q_1 \times Q_2)$

$$\begin{array}{ccccc} q_1^* z_1^* A_1 \boxtimes q_2^* z_2^* A_2 & \xrightarrow{m^{sm}} & (q_1 \times q_2)^*(z_1^* A_1 \boxtimes z_2^* A_2) & \xrightarrow{m^{cl}} & (q_1 \times q_2)^*(z_1 \times z_2)^*(A_1 \boxtimes A_2) \\ \| & & & & \| \\ h_1^* p_1^* A_1 \boxtimes h_2^* p_2^* A_2 & \xrightarrow{m^{cl}} & (h_1 \times h_2)^*(p_1^* A_1 \boxtimes p_2^* A_2) & \xrightarrow{m^{sm}} & (h_1 \times h_2)^*(p_1 \times p_2)^*(A_1 \boxtimes A_2) \end{array}$$

is commutative.

In case $\mathbb{H}$ *is smooth-adjointable (in the sense of Definition 2.6(1)), we say that an external tensor skeleton on* $\mathbb{H}$ *is* smooth-adjointable *if the external tensor structure* $(\boxtimes^{sm}, m^{sm})$ *is left-adjointable (in the sense of Definition 1.12).*

4.3. Remark. For $i = 1, 2$ let $f_i \colon T_i \to S_i$ be a morphism in $\mathscr{S}$ which is both a smooth morphism and a closed immersion. Specializing axiom (ex-ETS-1) to the commutative squares

$$\begin{array}{ccc} T_i & \xrightarrow{f_i} & S_i \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \mathsf{id}_{S_i}} \\ S_i & \xrightarrow{\mathsf{id}_{S_i}} & S_i \end{array} \qquad (i = 1, 2),$$

we see that the two natural isomorphisms of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(T_1 \times T_2)$

$$m^{sm} = m^{sm}_{f_1, f_2},\ m^{cl} = m^{cl}_{f_1, f_2} \colon f_1^* A_1 \boxtimes f_2^* A_2 \xrightarrow{\sim} (f_1 \times f_2)^*(A_1 \boxtimes A_2)$$

obtained by regarding f_1, f_2 as smooth morphisms or as closed immersions, respectively, must coincide; this follows from the equality $m_{\mathsf{id}_{S_1}, \mathsf{id}_{S_2}} = \mathsf{id}_{- \boxtimes_{S_1, S_2} -}$ (see Remark 1.11).

The first main result of the present section can be stated as follows:

4.4. Proposition. *Every external tensor structure on* $\mathbb{H}$ *determines by restriction an external tensor skeleton on it. Conversely, every external tensor skeleton on* $\mathbb{H}$ *uniquely extends to an external tensor structure on it. Moreover, if* $\mathbb{H}$ *is smooth-adjointable, then a given external tensor structure on it is smooth-adjointable if and only if the underlying external tensor skeleton is smooth-adjointable.*

PROOF. It suffices to adapt the proof of Proposition 2.10 in the obvious way, with condition (ex-ETS-1) playing here the same role as condition (ex-$\mathscr{S}$-fib) there. In particular, for every choice of arrows $f_i\colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2$, the external monoidality isomorphism of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(T_1 \times T_2)$

$$m = m_{f_1,f_2}\colon f_1^* A_1 \boxtimes f_2^* A_2 \xrightarrow{\sim} (f_1 \times f_2)^*(A_1 \boxtimes A_2)$$

can be defined as the composite

$$\begin{array}{ccccc} f_1^* A_1 \boxtimes f_2^* A_2 & & & & (f_1 \times f_2)^*(A_1 \boxtimes A_2) \\ \| & & & & \| \\ t_1^* p_1^* A_1 \boxtimes t_2^* p_2^* A_2 & \xrightarrow{m^{cl}} & (t_1 \times t_2)^*(p_1^* A_1 \boxtimes p_2^* A_2) & \xrightarrow{m^{sm}} & (t_1 \times t_2)^*(p_1 \times p_2)^*(A_1 \boxtimes A_2) \end{array}$$

associated to any choice of factorizations $(P_i; t_i, p_i) \in \mathsf{Fact}_{\mathscr{S}}(f_i)$, $i = 1, 2$. Arguing as for Proposition 2.10, one checks that the definition is independent of the chosen factorizations, and that condition (mETS) is satisfied over $\mathscr{S}$ as soon as it is satisfied over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$; we leave the details to the interested reader.

Lastly, the fact that a given external tensor structure on $\mathbb{H}$ is smooth-adjointable if and only if so is the underlying external tensor skeleton follows directly from Remark 4.3. ∎

Before proceeding further, let us quickly note the following analogue of Lemma 2.11, which will be useful in the next section:

4.5. LEMMA. *Let $(\boxtimes, m^{sm})$ and $(\boxtimes, m^{cl})$ be external tensor structures on $\mathbb{H}^{sm}$ and $\mathbb{H}^{cl}$, respectively, satisfying condition (ex-ETS-0). Then, the triple $(\boxtimes, m^{sm}, m^{cl})$ defines an external tensor skeleton on $\mathbb{H}$ if and only if it satisfies the following two conditions:*

(C-ETS-1) *For every pair of Cartesian squares in $\mathscr{S}$*

$$\begin{array}{ccc} P_{i,Z_i} & \xrightarrow{z_i'} & P_i \\ \downarrow{\scriptstyle p_i'} & & \downarrow{\scriptstyle p_i} \\ Z_i & \xrightarrow{z_i} & S_i \end{array} \qquad (i = 1, 2)$$

with p_1, p_2 smooth morphisms and z_1, z_2 closed immersions, the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(P_{1,Z_1} \times P_{2,Z_2})$

$$\begin{array}{ccccc} {p_1'}^* z_1^* A_1 \boxtimes {p_2'}^* z_2^* A_2 & \xrightarrow{m^{sm}} & (p_1' \times p_2')^*(z_1^* A_1 \boxtimes z_2^* A_2) & \xrightarrow{m^{cl}} & (p_1' \times p_2')^*(z_1 \times z_2)^*(A_1 \boxtimes A_2) \\ \| & & & & \| \\ {z_1'}^* p_1^* A_1 \boxtimes {z_2'}^* p_2^* A_2 & \xrightarrow{m^{cl}} & (z_1' \times z_2')^*(p_1^* A_1 \boxtimes p_2^* A_2) & \xrightarrow{m^{sm}} & (z_1' \times z_2')^*(p_1 \times p_2)^*(A_1 \boxtimes A_2) \end{array}$$

is commutative.

(T-ETS-1) *For every pair of commutative triangles in* $\mathscr{S}$

$$\begin{array}{ccc} Q_i & \xrightarrow{h_i} & P_i \\ & \searrow^{q_i} & \downarrow{\scriptstyle p_i} \\ & & S_i \end{array} \qquad (i = 1, 2)$$

with p_1, p_2, q_1, q_2 *smooth morphisms and* h_1, h_2 *closed immersions, the diagram of functors* $\mathbb{H}(S_1 \times S_2) \to \mathbb{H}(Q_1 \times Q_2)$

$$\begin{array}{ccccc} q_1^*A_1 \boxtimes q_2^*A_2 & & \xrightarrow{\quad m^{sm} \quad} & & (q_1 \times q_2)^*(A_1 \times A_2) \\ \| & & & & \| \\ h_1^*p_1^*A_1 \boxtimes h_2^*p_2^*A_2 & \xrightarrow{m^{cl}} & (h_1 \times h_2)^*(p_1^*A_1 \boxtimes p_2^*A_2) & \xrightarrow{m^{sm}} & (h_1 \times h_2)^*(p_1 \times p_2)^*(A_1 \times A_2) \end{array}$$

is commutative.

Proof. This can be proved along the lines of Lemma 2.11; we leave the details to the interested reader. ∎

4.6. Compatibility with associativity and commutativity constraints. In practice, one often wants to consider monoidal structures carrying (at least some among) compatible associativity, commutativity, and unit constraints. As explained in [Ter24a, §§ 2, 5], all these notions can be translated in the language of external tensor structures, and the translation is compatible with shifts in the triangulated setting. Our next task is to check that the axioms for associativity and commutativity constraints can be checked on the level of external tensor skeleta.

Let $(\boxtimes, m)$ be an external tensor structure on $\mathbb{H}$, and let $(\boxtimes, m^{sm}, m^{cl})$ denote the underlying external tensor skeleton.

4.7. Definition. *An* external associativity constraint *a for* $(\boxtimes, m^{sm}, m^{cl})$ *is the datum of*

- *an associativity constraint* a^{sm} *for* $(\boxtimes^{sm}, m^{sm})$,
- *an associativity constraint* a^{cl} *for* $(\boxtimes^{cl}, m^{cl})$

such that, for every choice of $S_1, S_2, S_3 \in \mathscr{S}$, *the two natural isomorphisms of functors* $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \times \mathbb{H}(S_3) \to \mathbb{H}(S_1 \times S_2 \times S_3)$

$$a^{sm} = a^{sm}_{S_1,S_2,S_3},\ a^{cl} = a^{cl}_{S_1,S_2,S_3} \colon (A_1 \boxtimes A_2) \boxtimes A_3 \xrightarrow{\sim} A_1 \boxtimes (A_2 \boxtimes A_3)$$

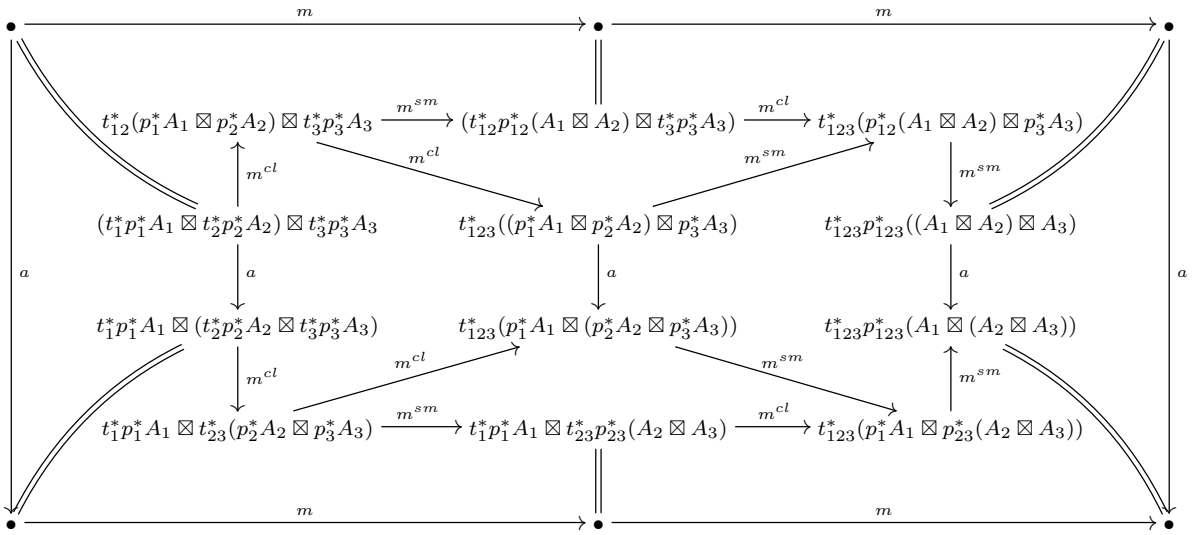
coincide, so that we can write both of them with the same symbol $a = a_{S_1,S_2,S_3}$.

4.8. Lemma. *Every external associativity constraint a for* $(\boxtimes, m^{sm}, m^{cl})$ *defines in fact an associativity constraint for* $(\boxtimes, m)$.

PROOF. Since condition (aETS-1) is trivially satisfied, we only need to check that condition (aETS-2) is satisfied as well: given three morphisms $f_i\colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2, 3$, we have to show that the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \times \mathbb{H}(S_3) \to \mathbb{H}(T_1 \times T_2 \times T_3)$

$$\begin{array}{ccccc}
(f_1^*A_1 \boxtimes f_2^*A_2) \boxtimes f_3^*A_3 & \xrightarrow{m} & (f_1 \times f_2)^*(A_1 \boxtimes A_2) \boxtimes f_3^*A_3 & \xrightarrow{m} & (f_1 \times f_2 \times f_3)^*((A_1 \boxtimes A_2) \boxtimes A_3) \\
\downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\
f_1^*A_1 \boxtimes (f_2^*A_2 \boxtimes f_3^*A_3) & \xrightarrow{m} & f_1^*A_1 \boxtimes (f_2 \times f_3)^*(A_2 \boxtimes A_3) & \xrightarrow{m} & (f_1 \times f_2 \times f_3)^*(A_1 \boxtimes (A_2 \boxtimes A_3))
\end{array}$$

is commutative. To this end, for $i = 1, 2, 3$ choose a factorization $(P_i; t_i, p_i) \in \mathsf{Fact}_{\mathscr{S}}(f_i)$. Then, we can decompose the diagram in question as



where $p_1 \times p_2$ and $t_1 \times t_2$ are abbreviated as p_{12} and t_{12}, respectively, and similarly for the other product morphisms. Here, the left-most and right-most central pieces commute by axioms (aETS-2) over $\mathscr{S}^{cl}$ and $\mathscr{S}^{sm}$, respectively, the upper and lower pieces commute by axiom (ETS-2), while the remaining four pieces commute by naturality. ∎

4.9. DEFINITION. *An* external commutativity constraint *c for $(\boxtimes, m^{sm}, m^{cl})$ is the datum of*

- *an external commutativity constraint c^{sm} for $(\boxtimes^{sm}, m^{sm})$,*
- *an external commutativity constraint c^{cl} for $(\boxtimes^{cl}, m^{cl})$*

such that, for every choice of $S_1, S_2 \in \mathscr{S}$, the two natural isomorphisms of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2)$

$$c^{sm} = c^{sm}_{S_1,S_2},\ c^{cl} = c^{cl}_{S_1,S_2}\colon A_1 \boxtimes A_2 \xrightarrow{\sim} \tau^*(A_2 \boxtimes A_1)$$

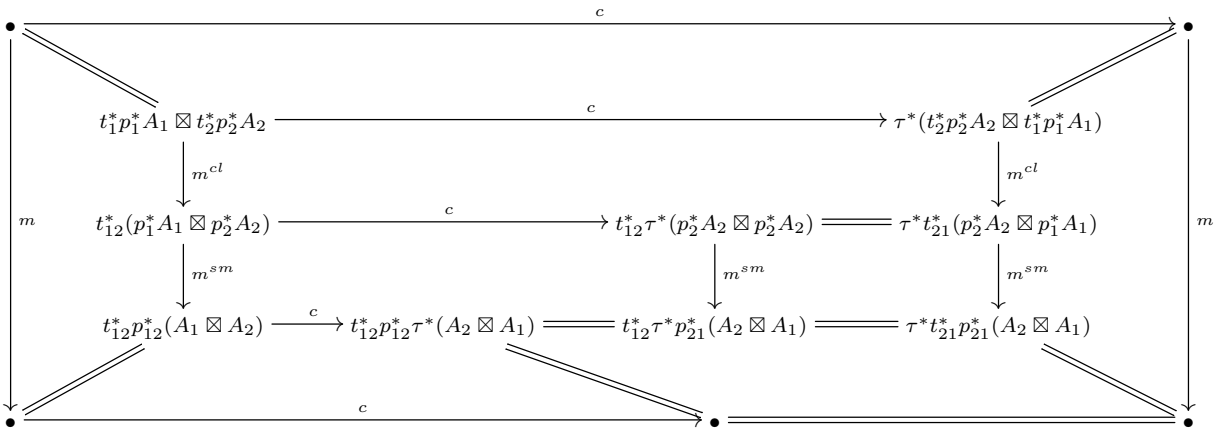
coincide, so that we can write both of them with the same symbol $c = c_{S_1,S_2}$.

4.10. LEMMA. *Every external commutativity constraint c for $(\boxtimes, m^{sm}, m^{cl})$ defines in fact a commutativity constraint for $(\boxtimes, m)$.*

PROOF. Since condition (cETS-1) is trivially satisfied, we only need to check that condition (cETS-2) is satisfied as well: given morphisms $f_i\colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2$, we have to show that the diagram

$$\begin{array}{ccc} f_1^*A_1 \boxtimes f_2^*A_2 & \xrightarrow{\quad c \quad} & \tau^*(f_2^*A_2 \boxtimes f_1^*A_1) \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ (f_1 \times f_2)^*(A_1 \boxtimes A_2) & \xrightarrow{c} (f_1 \times f_2)^*\tau^*(A_2 \boxtimes A_1) = & \tau^*(f_2 \times f_1)^*(A_2 \boxtimes A_1) \end{array}$$

is commutative. To this end, for $i = 1, 2$ choose a factorization $(P_i; t_i, p_i) \in \mathsf{Fact}_{\mathscr{S}}(f_i)$. Then, we can decompose the above diagram as



where $p_1 \times p_2$ and $t_1 \times t_2$ are abbreviated as p_{12} and t_{12}, respectively, as similarly for the other product morphisms. Here, the upper and lower-left central pieces commute by axiom (cETS-2) over $\mathscr{S}^{cl}$ and $\mathscr{S}^{sm}$, respectively, the two lateral pieces commute by axiom (ETS-2), the lower-right piece commutes by axiom ($\mathscr{S}$-fib-1), while the remaining three pieces commute by naturality. ∎

4.11. REMARK. The natural compatibility condition between external associativity and commutativity constraints (as spelled out in [Ter24a, Defn. 2.17]) does not involve inverse image functors, except those along permutation isomorphisms. Since the subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ contain all isomorphisms of $\mathscr{S}$ (by Assumption 2.2(i)), the same compatibility condition makes sense for external tensor skeleta.

4.12. EXTERNAL TENSOR SKELETA ON MORPHISMS. We now extend the notion of external tensor skeleta to morphisms of monoidal $\mathscr{S}$-fibered categories. For $j = 1, 2$, let $\mathbb{H}_j$ be an $\mathscr{S}$-fibered category with an external tensor structure $(\boxtimes_j, m_j)$, and let $(\boxtimes, m_j^{sm}, m_j^{cl})$ denote the underlying external tensor skeleton. Moreover, fix a morphism of $\mathscr{S}$-fibered categories $R = (\mathscr{R}, \theta)\colon \mathbb{H}_1 \to \mathbb{H}_2$, with external tensor skeleton $(\theta^{sm}, \theta^{cl})$.

4.13. DEFINITION. *An* external tensor skeleton (ρ^{sm}, ρ^{cl}) *on* R *(with respect to the external tensor skeleta* $(\boxtimes_1, m_1^{sm}, m_1^{cl})$ *and* $(\boxtimes_2, m_2^{sm}, m_2^{cl})$*) is the datum of*

- *an external tensor structure* ρ^{sm} *on* $R^{sm}\colon \mathbb{H}_1^{sm} \to \mathbb{H}_2^{sm}$ *(with respect to* $(\boxtimes_1, m_1^{sm})$ *and* $(\boxtimes_2, m_2^{sm})$*),*
- *an external tensor structure* ρ^{cl} *on* $R^{cl}\colon \mathbb{H}_1^{cl} \to \mathbb{H}_2^{cl}$ *(with respect to* $(\boxtimes_1, m_1^{cl})$ *and* $(\boxtimes_2, m_2^{cl})$*)*

such that, for every $S_1, S_2 \in \mathscr{S}$*, the two natural isomorphisms of functors* $\mathbb{H}_1(S_1) \times \mathbb{H}_1(S_2) \to \mathbb{H}_2(S_1 \times S_2)$

$$\rho^{sm} = \rho^{sm}_{S_1,S_2},\ \rho^{cl} = \rho^{cl}_{S_1,S_2}\colon R_{S_1}(A_1) \boxtimes_2 R_{S_2}(A_2) \xrightarrow{\sim} R_{S_2\times S_2}(A_1 \boxtimes_1 A_2)$$

coincide, so that we can write both of them using the same symbol $\rho = \rho_{S_1,S_2}$*.*

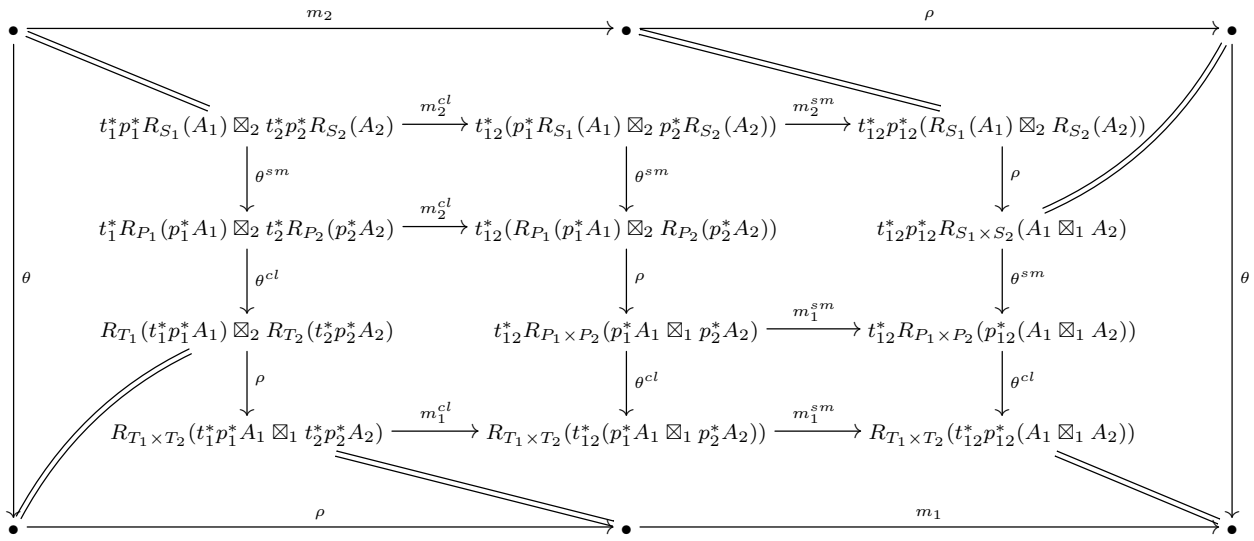
The second main result of the present section can be stated as follows:

4.14. PROPOSITION. *An external tensor skeleton on* R *(with respect to* $(\boxtimes_1, m_1^{sm}, m_1^{cl})$ *and* $(\boxtimes_2, m_2^{sm}, m_2^{cl})$*) is the same as an external tensor structure on* R *(with respect to* $(\boxtimes_1, m_1)$ *and* $(\boxtimes_2, m_2)$*).*

PROOF. Clearly, any external tensor structure ρ on R determines by restriction an external tensor skeleton on it. Conversely, suppose that we are given an external tensor skeleton (ρ^{sm}, ρ^{cl}) on R. In order to prove that it defines an external tensor structure, we need to check that it satisfies condition (mor-ETS): given morphisms $f_i\colon T_i \to S_i$ in $\mathscr{S}$, $i = 1, 2$, we have to show that the diagram of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(T_1 \times T_2)$

$$\begin{array}{ccccc}
f_1^* R_{S_1}(A_1) \boxtimes f_2^* R_{S_2}(A_2) & \xrightarrow{m_2} & (f_1 \times f_2)^*(R_{S_1}(A_1) \boxtimes R_{S_2}(A_2)) & \xrightarrow{\rho} & (f_1 \times f_2)^* R_{S_1\times S_2}(A_1 \boxtimes A_2) \\
\downarrow \theta & & & & \downarrow \theta \\
R_{T_1}(f_1^* A_1) \boxtimes R_{T_2}(f_2^* A_2) & \xrightarrow{\rho} & R_{T_1\times T_2}(f_1^* A_1 \boxtimes f_2^* A_2) & \xrightarrow{m_1} & R_{T_1\times T_2}((f_1 \times f_2)^*(A_1 \boxtimes A_2))
\end{array}$$

is commutative. To this end, for $i = 1, 2$ choose a factorization $(P_i; t_i, p_i) \in \mathsf{Fact}_{\mathscr{S}}(f_i)$. Then, we can decompose the above diagram as



where $t_1 \times t_2$ and $p_1 \times p_2$ are abbreviated as t_{12} and p_{12}, respectively. Here, the upper-right and lower-left central pieces commute by axiom (mor-ETS) over $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$, respectively, the upper-left and lower-right pieces commute by axiom (ETS-2), the two lateral pieces commute by axiom (mor-$\mathscr{S}$-fib), while the remaining pieces commute by naturality and by construction. ∎

4.15. Remark. The associativity and commutativity conditions for external tensor structures on morphisms (as spelled out in [Ter24a, § 3.3]) do not involve inverse image functors except those along permutation isomorphisms. Since the subcategories $\mathscr{S}^{sm}$ and $\mathscr{S}^{cl}$ contain all isomorphisms of $\mathscr{S}$ (by Assumption 2.2(i)), these conditions make sense also for external tensor skeleta.

5. External tensor cores

In this final section, we combine the approaches of Sections 3 and 4 in order to obtain an alternative description of external tensor structures which is more adapted to the setting of perverse sheaves.

As in the previous section, we assume that the base category $\mathscr{S}$ admits binary products. Moreover, throughout the present section, we work under Assumption 3.2 (which, by definition, includes Assumption 2.2).

5.1. External tensor cores on fibered categories. Let $\mathbb{H}$ be a localic triangulated $\mathscr{S}$-fibered category (in the sense of Definition 3.5).

5.2. Definition. *We say that a triangulated external tensor structure* $(\boxtimes, m)$ *on* $\mathbb{H}$ *is* localic *if it is smooth-adjointable (in the sense of Definition 2.6(3)).*

One can easily check that the analogue of Lemma 3.9 with respect to monoidality isomorphisms holds. This suggests the following variant of the notion of external tensor skeleton:

5.3. DEFINITION. *An* external tensor core $(\boxtimes, m^{sm}, \bar{m}^{cl})$ *on* $\mathbb{H}$ *is the datum of*

- *a triangulated external tensor structure* $(\boxtimes^{sm}, m^{sm})$ *on the* $\mathscr{S}^{sm}$*-fibered category* $\mathbb{H}^{sm}$,
- *a triangulated external tensor structure* $(\boxtimes^{cl}, \bar{m}^{cl})$ *on the* $\mathscr{S}^{cl,op}$*-fibered category* $\mathbb{H}^{cl}$

satisfying the following conditions:

(ex-ETS-0) *For every* $S_1, S_2 \in \mathscr{S}$*, the two functors*

$$- \boxtimes^{sm} - = - \boxtimes^{sm}_{S_1,S_2} -, \ - \boxtimes^{cl} - = - \boxtimes^{cl}_{S_1,S_2} - : \mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2)$$

coincide, so that we can write both of them using the same symbol $- \boxtimes - = - \boxtimes_{S_1,S_2} -$.

(C'-ETS-1) *For every pair Cartesian squares in* $\mathscr{S}$

$$\begin{array}{ccc} P_{i,Z_i} & \xrightarrow{z'_i} & P_i \\ \downarrow{\scriptstyle p'_i} & & \downarrow{\scriptstyle p_i} \\ Z_i & \xrightarrow{z_i} & S_i \end{array} \qquad (i = 1, 2)$$

with p_i *a smooth morphism and* z_i *a closed immersion, the diagram of functors* $\mathbb{H}(Z_1) \times \mathbb{H}(Z_2) \to \mathbb{H}(P_1 \times P_2)$

$$\begin{array}{ccccc} p_1^* z_{1,*} A_1 \boxtimes p_2^* z_{2,*} A_2 & \xrightarrow{m^{sm}} & (p_1 \times p_2)^*(z_{1,*} A_1 \boxtimes z_{2,*} A_2) & \xrightarrow{\bar{m}^{cl}} & (p_1 \times p_2)^*(z_1 \times z_2)_*(A_1 \boxtimes A_2) \\ \| & & & & \| \\ z'_{1,*} {p'_1}^* A_1 \boxtimes z'_{2,*} {p'_2}^* A_2 & \xrightarrow{\bar{m}^{cl}} & (z'_1 \times z'_2)_*({p'_1}^* A_1 \boxtimes {p'_2}^* A_2) & \xrightarrow{m^{sm}} & (z'_1 \times z'_2)_*(p'_1 \times p'_2)^*(A_1 \boxtimes A_2) \end{array}$$

is commutative.

(T-ETS-1) *For every pair of commutative triangles in* $\mathscr{S}$

$$\begin{array}{ccc} Q_i & \xrightarrow{h_i} & P_i \\ & {\scriptstyle q_i} \searrow & \downarrow{\scriptstyle p_i} \\ & & S_i \end{array} \qquad (i = 1, 2)$$

with p_i*,* q_i *smooth morphisms and* h_i *a closed immersion, the diagram of functors* $\mathbb{H}_1(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(Q_1 \times Q_2)$

$$\begin{array}{ccccc} q_1^* A_1 \boxtimes q_2^* A_2 & & \xrightarrow{m^{sm}} & & (q_1 \times q_2)^*(A_1 \boxtimes A_2) \\ \| & & & & \| \\ h_1^* p_1^* A_1 \boxtimes h_2^* p_2^* A_2 & \xrightarrow{m^{sm}} & (h_1 \times h_2)^*(p_1^* A_1 \boxtimes p_2^* A_2) & \xrightarrow{m^{cl}} & (h_1 \times h_2)^*(p_1 \times p_2)^*(A_1 \boxtimes A_2) \end{array}$$

is commutative.

This leads us to the first main result of this section:

5.4. THEOREM. *Every localic external tensor structure on $\mathbb{H}$ determines by restriction and adjunction an external tensor core. Conversely, every external tensor core on $\mathbb{H}$ uniquely extends to a localic external tensor structure.*

PROOF. This can be deduced from Proposition 4.14 in the same way as Theorem 3.12 can be deduced from Proposition 2.10; we leave the details to the interested reader. ∎

5.5. COMPATIBILITY WITH ASSOCIATIVITY AND COMMUTATIVITY CONSTRAINTS. As in the case of external tensor skeleta, the axioms for associativity and commutativity constraints on a localic external tensor structure can be checked on the underlying external tensor core.

Let again $\mathbb{H}$ be a localic triangulated $\mathscr{S}$-fibered category; let $(\boxtimes, m)$ be a localic triangulated external tensor structure on $\mathbb{H}$, and let $(\boxtimes, m^{sm}, m^{cl})$ denote the underlying external tensor core.

5.6. DEFINITION. *An* external associativity constraint *a for $(\boxtimes, m^{sm}, \bar{m}^{cl})$ is the datum of*

- *a triangulated external associativity constraint a^{sm} for $(\boxtimes, m^{sm})$,*
- *a triangulated external associativity constraint a^{cl} for $(\boxtimes, \bar{m}^{cl})$*

such that, for every $S_1, S_2, S_3 \in \mathscr{S}$, the two natural isomorphisms of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \times \mathbb{H}(S_3) \to \mathbb{H}(S_1 \times S_2 \times S_3)$

$$a^{sm} = a^{sm}_{S_1,S_2,S_3},\ a^{cl} = a^{cl}_{S_1,S_2,S_3} \colon (A_1 \boxtimes A_2) \boxtimes A_3 \xrightarrow{\sim} A_1 \boxtimes (A_2 \boxtimes A_3)$$

coincide, so that we can write both of them using the same symbol $a = a_{S_1,S_2,S_3}$.

5.7. LEMMA. *Every triangulated external associativity constraint for $(\boxtimes, m^{sm}, \bar{m}^{cl})$ is in fact an associativity constraint for $(\boxtimes, m)$.*

PROOF. In view of Lemma 4.8, it suffices to show that giving an external associativity constraint on $(\boxtimes, m^{cl})$ is equivalent to giving an external associativity constraint on $(\boxtimes, \bar{m}^{cl})$. This follows from Lemma 1.16. ∎

5.8. DEFINITION. *An* external commutativity constraint *c for $(\boxtimes, m^{sm}, \bar{m}^{cl})$ is the datum of*

- *a triangulated external commutativity constraint c^{sm} for $(\boxtimes, m^{sm})$*
- *a triangulated external commutativity constraint c^{cl} for $(\boxtimes, \bar{m}^{cl})$*

such that, for every $S_1, S_2 \in \mathscr{S}$, the two natural isomorphisms of functors $\mathbb{H}(S_1) \times \mathbb{H}(S_2) \to \mathbb{H}(S_1 \times S_2)$

$$c^{sm} = c^{sm}_{S_1,S_2},\ c^{cl} = c^{cl}_{S_1,S_2} \colon A_1 \boxtimes A_2 \xrightarrow{\sim} \tau^*(A_2 \boxtimes A_1)$$

coincide, so that we can write both of them using the same symbol $c = c_{S_1,S_2}$.

5.9. Lemma. *Every triangulated external commutativity constraint for* $(\boxtimes, m^{sm}, \bar{m}^{cl})$ *is in fact a commutativity constraint for* $(\boxtimes, m)$.

Proof. In view of Lemma 4.10, it suffices to show that giving an external commutativity constraint on $(\boxtimes, m^{cl})$ is equivalent to giving an external commutativity constraint on $(\boxtimes, \bar{m}^{cl})$. This follows from Lemma 1.17. ∎

5.10. External tensor cores on morphisms. We conclude by discussing the notion of external tensor core on morphisms of $\mathscr{S}$-fibered categories. For $j = 1, 2$, let $\mathbb{H}_j$ be a localic triangulated $\mathscr{S}$-fibered category, and let $(\boxtimes_j, m_j)$ be an external tensor structure on it; by Theorem 5.4, it is determined by the corresponding external tensor core $(\boxtimes_j, m_j^{sm}, \bar{m}_j^{cl})$. We also fix a localic morphism of triangulated $\mathscr{S}$-fibered categories $R = (\mathscr{R}, \theta) \colon \mathbb{H}_1 \to \mathbb{H}_2$; by Theorem 3.12, the $\mathscr{S}$-structure θ on the family $\mathscr{R} = \{R_S \colon \mathbb{H}_1(S) \to \mathbb{H}_2(S)\}_{S \in \mathscr{S}}$ is determined by the underlying $\mathscr{S}$-core $(\theta^{sm}, \bar{\theta}^{cl})$.

5.11. Definition. *An* external tensor core (ρ^{sm}, ρ^{cl}) *on* R *(with respect to* $(\boxtimes_1, m_1)$ *and* $(\boxtimes_2, m_2)$*) is the datum of*

- *a triangulated external tensor structure* ρ^{sm} *on* $R : \mathbb{H}_1^{sm} \to \mathbb{H}_2^{sm}$ *(with respect to* $(\boxtimes_1, m_1^{sm})$ *and* $(\boxtimes_2, m_2^{sm})$*),*
- *a triangulated external tensor structure* ρ^{cl} *on* $R : \mathbb{H}_1^{cl} \to \mathbb{H}_2^{cl}$ *(with respect to* $(\boxtimes, \bar{m}_1^{cl})$ *and* $(\boxtimes, \bar{m}_2^{cl})$*)*

such that, for every $S_1, S_2 \in \mathscr{S}$*, the two natural isomorphisms of functors* $\mathbb{H}_1(S_1) \times \mathbb{H}_1(S_2) \to \mathbb{H}_2(S_1 \times S_2)$

$$\rho^{sm} = \rho^{sm}_{S_1,S_2},\ \rho^{cl} = \rho^{cl}_{S_1,S_2} : R_{S_1}(A_1) \boxtimes_2 R_{S_2}(A_2) \xrightarrow{\sim} R_{S_1 \times S_2}(A_1 \boxtimes_1 A_2)$$

coincide, so that we can write both of them with the same symbol $\rho = \rho_{S_1,S_2}$.

We conclude by stating the second main result of this section:

5.12. Theorem. *Giving an external tensor core on the morphism* R *(with respect to* $(\boxtimes_1, m_1)$ *and* $(\boxtimes_2, m_2)$*) is the same as giving a triangulated external tensor structure on* R *(with respect to* $(\boxtimes_1, m_1)$ *and* $(\boxtimes_2, m_2)$*).*

Proof. In view of Proposition 4.14, it suffices to show that an external tensor core on R is the same as a localic triangulated external tensor skeleton on R. To this end, it suffices to check that the notion of external tensor structure on $R \colon \mathbb{H}_1^{cl} \to \mathbb{H}_2^{cl}$ is the same regardless of whether it is considered as a morphism of $\mathscr{S}^{cl}$-fibered categories or as a morphism of $\mathscr{S}^{cl,op}$-fibered categories. This follows easily from Lemma 1.20. ∎

5.13. Remark. Again, the associativity and commutativity conditions for external tensor structures on morphisms from [Ter24a, § 3.3] still make sense for external tensor cores.

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