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ON REPRESENTATION CATEGORIES OF A_∞-ALGEBRAS AND A_∞-COALGEBRAS

ABHISHEK BANERJEE, ANITA NAOLEKAR

Abstract. In this paper, we use the language of monads, comonads and Eilenberg-Moore categories to describe a categorical framework for A_∞-algebras and A_∞-coalgebras, as well as A_∞-modules and A_∞-comodules over them respectively. The resulting formalism leads us to investigate relations between representation categories of A_∞-algebras and A_∞-coalgebras. In particular, we relate A_∞-comodules and A_∞-modules by considering a rational pairing between an A_∞-coalgebra C and an A_∞-algebra A. The categorical framework also motivates us to introduce A_∞-contramodules over an A_∞-coalgebra C.

Contents

1. Introduction

A monad on a category $\mathcal{C}$ consists of an endofunctor $\mathbb{U} : \mathcal{C} \longrightarrow \mathcal{C}$ along with natural transformations $\mathbb{U} \circ \mathbb{U} \longrightarrow \mathbb{U}$ and $1 \longrightarrow \mathbb{U}$ satisfying conditions similar to an ordinary ring. As with a ring, one can study a monad by looking at its representation category, i.e., its category of modules. The role of modules over the monad $\mathbb{U}$ is played by the classical Eilenberg-Moore category $EM_{\mathbb{U}}$ (see [20], [21]). For instance, Borceux, Caenepeel and Janelidze [11] have shown that Eilenberg-Moore categories can be used to study a categorical form of Galois descent, extending from the usual notion of Galois extension of

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commutative rings. The dual notion of comonad resembles that of a coalgebra, and we have Eilenberg-Moore categories of comodules over them. For instance, Eilenberg-Moore categories of comodules can be used to study notions such as relative Hopf modules, or Doi-Hopf modules (see [14]).

In this paper, we consider A_∞-versions of monads, comonads and their representation categories. We develop Eilenberg-Moore categories for A_∞-monads and A_∞-comonads. We study a bar construction for A_∞-monads. We also study versions of distributive laws that enable us to lift A_∞-monads to the Eilenberg-Moore category, or extend A_∞-monads to the Kleisli category, of an ordinary monad. This gives us a categorical framework for A_∞-algebras and A_∞-coalgebras, as well as A_∞-modules and A_∞-comodules over them respectively, using the language of monads, comonads and Eilenberg-Moore categories. We are then motivated by this formalism to consider rational pairings between A_∞-coalgebras and A_∞-algebras, and also to introduce A_∞-contramodules.

We recall that an A_∞-algebra A consists of a graded vector space $A = \bigoplus\limits_{n\in\mathbb{Z}} A_n$ and a sequence $\{m_k : A^{\otimes k} \longrightarrow A\}_{k\geq 1}$ of homogeneous linear maps satisfying higher order associativity conditions. The A_∞-algebras were introduced by Stasheff [42], [43], and applied to the study of homotopy associative structures on H-spaces. In later years, A_∞-structures have found widespread applications in algebra, topology, geometry as well as in physics (see, for instance, [1], [9], [22], [23], [26], [27], [29], [30], [35], [41], [44]).

We investigate relations between three "module like" categories associated to A_∞-(co)-algebras: modules, comodules and contramodules. We define a rational pairing between an A_∞-coalgebra C and an A_∞-algebra A. This gives us an adjunction between the category of A_∞-modules over A and the A_∞-comodules over C. This extends the classical theory that embeds the comodules over a coalgebra as a coreflective subcategory of the modules over its dual algebra (see, for instance, [13, Chapter 1]).

On the other hand, if C' is an ordinary coassociative coalgebra, we know that there are two different representation categories of C'. One is the usual category of C'-comodules and the other is the category of C'-contramodules, introduced by Eilenberg and Moore in [21, IV.5]. At a categorical level, both are dual to the notion of modules over an algebra, which makes the study of contramodules as fundamental as that of comodules. Accordingly, our monadic approach motivates us to introduce A_∞-contramodules over an A_∞-coalgebra C. We also introduce a contratensor product that gives us an adjunction between categories of A_∞-comodules and A_∞-contramodules. For the contratensor product in the case of contramodules over a coassociative coalgebra, we refer the reader to [37].

We should mention that our approach to A_∞-structures in this paper using monads and comonads is inspired by Böhm, Brzeziński and Wisbauer, who provided a unifying framework for studying modules, comodules and contramodules over ordinary algebras and coalgebras in [10]. As such, we hope that this is the first step towards studying notions such as mixed distributive laws, entwining structures and Galois properties over A_∞-algebras and A_∞-coalgebras, as well as relative Hopf modules, Doi-Hopf modules, and Yetter-Drinfeld modules. The literature on the latter is vast and has been developed widely by several authors (see, for instance, [3], [4], [12], [14], [15], [16], [17], [18]).

We now describe the paper in more detail. We begin in Section 2 with an abelian category $\mathcal{C}$ that is K-linear (for a field K) and satisfies (AB5). An A_∞-monad $(\mathbb{U},\Theta)$ consists of a sequence $\mathbb{U}=\{\mathbb{U}_n\}_{n\in\mathbb{Z}}$ of endofunctors on $\mathcal{C}$ along with a collection Θ of natural transformations

$$\Theta=\{\theta_k(\bar{n}):\mathbb{U}_{\bar{n}}:=\mathbb{U}_{n_1}\circ...\circ\mathbb{U}_{n_k}\longrightarrow\mathbb{U}_{n_1+...+n_k+(2-k)}=\mathbb{U}_{\Sigma\bar{n}+(2-k)}\mid k\geq 1,\,\bar{n}=(n_1,...,n_k)\in\mathbb{Z}^k\}\tag{1.1}$$

satisfying certain relations. We say that $(\mathbb{U},\Theta)$ is even if $\theta_k(\bar{n})=0$ whenever k is odd. Dually, an A_∞-comonad $(\mathbb{V},\Delta)$ consists of a sequence $\mathbb{V}=\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ of endofunctors and a collection Δ of natural transformations

$$\Delta=\{\delta_k(\bar{n}):\mathbb{V}_{\Sigma\bar{n}+(2-k)}:=\mathbb{V}_{n_1+...+n_k+(2-k)}\longrightarrow\mathbb{V}_{n_1}\circ...\circ\mathbb{V}_{n_k}=\mathbb{V}_{\bar{n}}\mid k\geq 1,\,\bar{n}=(n_1,...,n_k)\in\mathbb{Z}^k\}\tag{1.2}$$

satisfying dual relations. Our first result is that if $\{(\mathbb{U}_n,\mathbb{V}_n)\}_{n\in\mathbb{Z}}$ is a sequence of adjoint functors, then $\mathbb{U}=\{\mathbb{U}_n\}_{n\in\mathbb{Z}}$ can be equipped with the structure of an even A_∞-monad (resp. an even A_∞-comonad) if and only if $\mathbb{V}=\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ can be equipped with the structure of an even A_∞-comonad (resp. an even A_∞-monad). Additionally, if $\mathbb{U}=\{\mathbb{U}_n\}_{n\in\mathbb{Z}}$ is an even A_∞-monad, we describe a process that gives a collection $\mathbb{V}^\oplus=\{\mathbb{V}^\oplus_n\}_{n\in\mathbb{Z}}$ of subfunctors of $\mathbb{V}=\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ carrying the structure of an A_∞-comonad $(\mathbb{V}^\oplus,\Delta^\oplus)$ that is locally finite. In other words, for each $k\geq 1$, $T\in\mathbb{Z}$, the induced morphism

$$\mathbb{V}^\oplus_{T+(2-k)}=\mathbb{V}^\oplus_{n_1+...+n_k+(2-k)}\longrightarrow\prod_{\bar{n}\in\mathbb{Z}^k,\Sigma\bar{n}=T}\mathbb{V}^\oplus_{n_1}\circ...\circ\mathbb{V}^\oplus_{n_k}\tag{1.3}$$

factors through the direct sum $\bigoplus\limits_{\bar{n}\in\mathbb{Z}^k,\Sigma\bar{n}=T}\mathbb{V}^\oplus_{n_1}\circ...\circ\mathbb{V}^\oplus_{n_k}$.

If $(\mathbb{U},\Theta)$ is an A_∞-monad, an A_∞-module (M,π) consists of a collection $M=\{M_n\}_{n\in\mathbb{Z}}$ of objects of $\mathcal{C}$ along with morphisms

$$\pi=\{\pi_k(\bar{n},l):\mathbb{U}_{\bar{n}}M_l\longrightarrow M_{\Sigma\bar{n}+l+(2-k)}\mid k\geq 1,\,\bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1},\,l\in\mathbb{Z}\,\}\tag{1.4}$$

satisfying certain conditions described in Definition 3.1. This gives the Eilenberg-Moore category $EM_{(\mathbb{U},\Theta)}$ of A_∞-modules over $(\mathbb{U},\Theta)$. We also describe a canonical functor $\mathcal{C}\longrightarrow EM_{(\mathbb{U},\Theta)}$. Further, if the functors $\{\mathbb{U}_n\}_{n\in\mathbb{Z}}$ preserve direct sums, this extends to a functor $\hat{\mathbb{U}}:\mathcal{C}^{\mathbb{Z}}\longrightarrow EM_{(\mathbb{U},\Theta)}$ from the category $\mathcal{C}^{\mathbb{Z}}$ of $\mathbb{Z}$-graded objects over $\mathcal{C}$. The A_∞-comodules over an A_∞-comonad $(\mathbb{V},\Delta)$ are defined in a dual manner. If $\{(\mathbb{U}_n,\mathbb{V}_n)\}_{n\in\mathbb{Z}}$ is a sequence of adjoint functors, we show (see Proposition 3.5) that the category of even A_∞-modules over $(\mathbb{U},\Theta)$ is isomorphic to the category of even A_∞-comodules over $(\mathbb{V},\Delta)$. We also define ∞-morphisms $\alpha=\{\,\alpha_k(\bar{n},l):\mathbb{U}_{\bar{n}}M^1_l\longrightarrow M^2_{\Sigma\bar{n}+l+(1-k)}\}$ between A_∞-modules (M^1,π^1), (M^2,π^2) (see Definition 3.6). Then, we show in Theorem 3.8 that even A_∞-modules over $(\mathbb{U},\Theta)$ with odd ∞-morphisms between them correspond to even A_∞-comodules over $(\mathbb{V},\Delta)$ with odd ∞-morphisms between them.

We study locally finite A_∞-comodules over A_∞-comonads in Section 4. Let $(\mathbb{V},\Delta)$ be an A_∞-comonad such that each $\mathbb{V}_n$ is exact. If $P=\{P_n\}_{n\in\mathbb{Z}}$ is an A_∞-comodule, we describe a process of restricting the structure maps $\rho_k(\bar{n},l) : P_{\Sigma\bar{n}+l+(2-k)} \longrightarrow \mathbb{V}_{\bar{n}}P_l$ of P to a subobject $P^{\oplus}$ such that for each $k \geq 1$, $l \in \mathbb{Z}$, $T \in \mathbb{Z}$, the induced morphism

$$\prod_{\bar{n}\in\mathbb{Z}^{k-1},\Sigma\bar{n}+l=T} \rho_k^{\oplus}(\bar{n},l) : P^{\oplus}_{T+(2-k)} \longrightarrow \prod_{\bar{n}\in\mathbb{Z}^{k-1},\Sigma\bar{n}+l=T} \mathbb{V}_{\bar{n}}P_l^{\oplus} \tag{1.5}$$

factors through the direct sum $\bigoplus\limits_{\bar{n}\in\mathbb{Z}^{k-1},\Sigma\bar{n}+l=T} \mathbb{V}_{\bar{n}}P_l^{\oplus}$. Further, we show (see Theorem 4.6) that this construction gives a right adjoint to the inclusion of locally finite A_∞-comodules into the Eilenberg-Moore category $EM^{(\mathbb{V},\Delta)}$ of A_∞-comodules over $(\mathbb{V},\Delta)$.

In Section 5, we develop the bar construction for an A_∞-monad $(\mathbb{U},\Theta)$. This gives a comonad $Bar(\mathbb{U},\Theta)$ which is graded and equipped with a coderivation. If $(\mathbb{W},\delta_1,\delta_2)$ is a conilpotent dg-comonad (see Definition 5.2), we show that morphisms from $(\mathbb{W},\delta_1,\delta_2)$ to $Bar(\mathbb{U},\Theta)$ correspond to families of natural transformations satisying conditions similar to the classical case of twisting morphisms. In Section 6, we study distributive laws between an A_∞-monad $(\mathbb{U},\Theta)$ and an ordinary monad $\mathbb{S}$. We show how these distributive laws can be used to lift $(\mathbb{U},\Theta)$ to an A_∞-monad $(\widetilde{\mathbb{U}},\widetilde{\Theta})$ on the Eilenberg-Moore category $EM_{\mathbb{S}}$ of $\mathbb{S}$. Similarly, we show how distributive laws can be used to extend $(\mathbb{U},\Theta)$ to an A_∞-monad $(\widehat{\mathbb{U}},\widehat{\Theta})$ on the Kleisli category $Kl(\mathbb{S})$ of $\mathbb{S}$.

In Section 7, we relate this formalism of monads and comonads to A_∞-algebras and A_∞-coalgebras. Motivated by a classical construction of Popescu [36] (see also Artin and Zhang [2]) on the noncommutative base change of an algebra, we construct examples of A_∞-monads and A_∞-comonads starting with a K-linear Grothendieck category $\mathcal{C}$. If A is an A_∞-algebra and $\mathbb{F}$ is a monad on $\mathcal{C}$ that preserves direct sums, the collection of functors $\mathbb{U}_{\mathcal{C}}^{A,\mathbb{F}} = \{\mathbb{U}_{\mathcal{C},n}^{A,\mathbb{F}} := A_n \otimes \mathbb{F}(\_\_) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ is an A_∞-monad on $\mathcal{C}$. Additionally, if A is even (i.e., the operations $m_k : A^{\otimes k} \longrightarrow A$ vanish whenever k is odd) and $\mathbb{F}$ has a right adjoint $\mathbb{G}$, the collection of functors $\mathbb{V}_{\mathcal{C}}^{A,\mathbb{G}} = \{\mathbb{V}_{\mathcal{C},n}^{A,\mathbb{G}} := \underline{Hom}(A_n, \mathbb{G}(\_\_)) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ is an A_∞-comonad on $\mathcal{C}$ (see Proposition 7.2). We also mention classes of A_∞-algebras that satisfy this evenness condition and have been studied extensively in the literature (see [19], [24], [28], [34]).

Henceforth, we fix $\mathcal{C} = Vect$, the category of K-vector spaces. If A is a $\mathbb{Z}$-graded vector space, we show that the collection of functors $\mathbb{U}^A = \{\mathbb{U}_n^A := A_n \otimes \_\_ : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}}$ can be equipped with the structure of an A_∞-monad if and only if A can be equipped with the structure of an A_∞-algebra. In that case, the Eilenberg-Moore category $EM_{\mathbb{U}_A}$ of A_∞-modules over $\mathbb{U}_A$ corresponds to A_∞-modules over the A_∞-algebra A. On the other hand, if C is an A_∞-coalgebra that is even, it follows from our earlier result in Theorem 2.3 that the right adjoints $\{Hom(C_{-n}, \_\_) : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}}$ can be equipped with the structure of an A_∞-monad. This is the key observation that motivates our definition of A_∞-contramodules in Section 9.

By a pairing of an A_∞-algebra A and an A_∞-coalgebra C, we will mean an ∞-morphism of A_∞-algebras $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ from A to the graded dual C^* of C. We show that such a pairing induces a functor $\iota(C,A) : \mathbf{M}^C \longrightarrow {}_A\mathbf{M}$ from right C-comodules to left A-modules. Additionally, when the pairing is rational in the sense of Definition 8.3, the functor $\iota(C,A)$ embeds $\mathbf{M}^C$ as a full subcategory of ${}_A\mathbf{M}$. Our main result in Section 8 is that a rational pairing $f=\{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ of an A_∞-algebra with an A_∞-coalgebra gives rise to an adjunction (see Theorem 8.6)

$$ {}_A\mathbf{M}(\iota(C,A)(M'),M) \cong \mathbf{M}^C(M',R_f(M)) \tag{1.6}$$

for any $M' \in \mathbf{M}^C$ and $M \in {}_A\mathbf{M}$.

We introduce A_∞-contramodules in Section 9. If A' is an ordinary algebra, we note that the structure map $A'\otimes M' \longrightarrow M'$ of a module M' may be expressed equivalently as a morphism $M' \longrightarrow [A',M'] := Vect(A',M')$. Accordingly, a contramodule over an ordinary coalgebra C', as defined by Eilenberg and Moore in [21, IV.5], consists of a space M'' along with a structure map $Vect(C',M'')=[C',M''] \longrightarrow M''$ satisfying certain coassociativity conditions. In particular, if V is any vector space, then $[C',V]$ is a C'-contramodule. In a categorical sense, it may be said therefore that both comodules and contramodules dualize the notion of module over an algebra. Even though the study of contramodules in the literature is not as developed as that of comodules, the topic has seen a lot of renewed interest in recent years (see, for instance, [5], [6], [7], [10], [37], [38], [39], [40], [46]).

Accordingly, our notion of an A_∞-contramodule over an A_∞-coalgebra C consists of a graded vector space $M \in Vect^{\mathbb{Z}}$ along with a collection of structure maps

$$ t_k^M : Vect^{\mathbb{Z}}(C^{\otimes k-1},M) = [C^{\otimes k-1},M] \longrightarrow M \qquad k\geq 1 \tag{1.7}$$

satisfying certain conditions set out in Definition 9.1. We show that there is a canonical functor $[C,\_\_] : Vect^{\mathbb{Z}} \longrightarrow \mathbf{M}_{[C,\_\_]}$ from graded vector spaces to the category $\mathbf{M}_{[C,\_\_]}$ of A_∞-contramodules over C. Further, we will describe a canonical functor that relates A_∞-contramodules to the Eilenberg-Moore category of A_∞-modules over the A_∞-monad $\{Hom(C_{-n},\_\_) : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}}$ described in Section 7. There is also a faithful functor κ^C that takes $\mathbf{M}_{[C,\_\_]}$ to A_∞-modules over C^*.

The final result of Section 9 relates A_∞-contramodules to A_∞-comodules, using contratensor products. For contramodules over coassociative coalgebras, the contratensor product was defined in [37], along with its right adjoint. We let C, D be A_∞-coalgebras. Let N be a space equipped with an A_∞-left C-coaction as well as an A_∞-right D-coaction (a (C,D)-bicomodule for instance) satisfying certain conditions. Using N, we define a contratensor product (see (9.22))

$$ \_\_ \boxtimes_C N : \mathbf{M}_{[C,\_\_]} \longrightarrow \mathbf{M}^D \tag{1.8}$$

We conclude with Theorem 9.8, which gives an adjunction of functors

$$ \mathbf{M}^D(M\boxtimes_C N, Q) \cong \mathbf{M}_{[C,\_\_]}(M,[N,Q]^D) \tag{1.9}$$

for $M\in \mathbf{M}_{[C,\_\_]}$ and $Q\in\mathbf{M}^D$. We also mention that for the convenience of the reader and in order to fix notation, we have collected the basic definitions of A_∞-algebras, A_∞-coalgebras, A_∞-modules and A_∞-comodules in an appendix in the final section of this paper (see, for instance, [27], [31]).

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Dedication: On Aug 20, 2026, Anita Naolekar, one of the authors of this paper, my research colleague and friend, passed away suddenly in an accident. This paper is dedicated to her memory.

2. A_∞-monads, A_∞-comonads and adjoints

Let K be a field. Throughout this section and the rest of this paper, we let $\mathcal{C}$ be a K-linear abelian category. We will suppose that $\mathcal{C}$ satisfies the (AB5) axiom. This means in particular that for any family $\{M_i\}_{i\in I}$ of objects in $\mathcal{C}$, we have an inclusion $\bigoplus\limits_{i\in I} M_i \hookrightarrow \prod\limits_{i\in I} M_i$.
For any tuple $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, we set $|\bar{n}| := k$ and $\Sigma\bar{n} := n_1 + ... + n_k$. We also write $\bar{n}^{op}$ for the opposite tuple $(n_k, ..., n_1) \in \mathbb{Z}^k$. On the other hand, for $T \in \mathbb{Z}$ and $k \geq 1$, we denote by $\mathbb{Z}(T, k)$ the set of all $\bar{n} \in \mathbb{Z}^k$ satisfying $\Sigma\bar{n} = T$.
For any collection $\mathbb{U} = \{\mathbb{U}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ of endofunctors on $\mathcal{C}$ indexed by $\mathbb{Z}$ and $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, we will denote by $\mathbb{U}_{\bar{n}}$ the composition $\mathbb{U}_{\bar{n}} := \mathbb{U}_{n_1} \circ ... \circ \mathbb{U}_{n_k}$. We start by considering an A_∞-monad on $\mathcal{C}$.

2.1. Definition. *An A_∞-monad $(\mathbb{U}, \Theta)$ on $\mathcal{C}$ consists of the following data:*
(a) A collection $\mathbb{U} = \{\mathbb{U}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ of endofunctors on $\mathcal{C}$.
(b) A collection Θ of natural transformations

$$\Theta = \{\theta_k(\bar{n}) : \mathbb{U}_{\bar{n}} = \mathbb{U}_{n_1} \circ ... \circ \mathbb{U}_{n_k} \longrightarrow \mathbb{U}_{n_1+...+n_k+(2-k)} = \mathbb{U}_{\Sigma\bar{n}+(2-k)} \mid k \geq 1,\ \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k\} \tag{2.1}$$

satisfying, for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 1$:

$$0 = \sum (-1)^{p+qr+q\Sigma\bar{n}} \theta_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'')(\mathbb{U}_{\bar{n}} * \theta_q(\bar{n}') * \mathbb{U}_{\bar{n}''}) : \mathbb{U}_{\bar{n}} \circ \mathbb{U}_{\bar{n}'} \circ \mathbb{U}_{\bar{n}''} \longrightarrow \mathbb{U}_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+(3-N)} \tag{2.2}$$

where the sum runs over partitions $\bar{z} = (\bar{n}, \bar{n}', \bar{n}'')$ with $|\bar{n}| = p$, $|\bar{n}'| = q$, $|\bar{n}''| = r$. We will say that $(\mathbb{U}, \Theta)$ is even if each $\theta_k(\bar{n}) = 0$ whenever k is odd.

2.2. Definition. *An A_∞-comonad $(\mathbb{V}, \Delta)$ on $\mathcal{C}$ consists of the following data:*
(a) A collection $\mathbb{V} = \{\mathbb{V}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ of endofunctors on $\mathcal{C}$.
(b) A collection Δ of natural transformations

$$\Delta = \{\delta_k(\bar{n}) : \mathbb{V}_{\Sigma\bar{n}+(2-k)} = \mathbb{V}_{n_1+...+n_k+(2-k)} \longrightarrow \mathbb{V}_{n_1} \circ ... \circ \mathbb{V}_{n_k} = \mathbb{V}_{\bar{n}} \mid k \geq 1,\ \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k\} \tag{2.3}$$

such that, for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 1$, we have

$$0 = \sum (-1)^{pq+r+q\Sigma\bar{n}} (\mathbb{V}_{\bar{n}} * \delta_q(\bar{n}') * \mathbb{V}_{\bar{n}''}) \delta_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'') : \mathbb{V}_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+(3-N)} \longrightarrow \mathbb{V}_{\bar{n}} \circ \mathbb{V}_{\bar{n}'} \circ \mathbb{V}_{\bar{n}''} \tag{2.4}$$

where the sum runs over partitions $\bar{z} = (\bar{n}, \bar{n}', \bar{n}'')$ with $|\bar{n}| = p$, $|\bar{n}'| = q$, $|\bar{n}''| = r$.

We will say that an A_∞-comonad $(\mathbb{V},\Delta)$ is locally finite if for each $k\geq 1$, $T\in\mathbb{Z}$, the induced morphism

$$\prod_{\bar{n}\in\mathbb{Z}(T,k)}\delta_k(\bar{n}):\mathbb{V}_{T+(2-k)}=\mathbb{V}_{n_1+...+n_k+(2-k)}\longrightarrow\prod_{\bar{n}\in\mathbb{Z}(T,k)}\mathbb{V}_{n_1}\circ...\circ\mathbb{V}_{n_k}=\prod_{\bar{n}\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}\quad(2.5)$$

factors through the direct sum $\bigoplus\limits_{\bar{n}\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}$.

We will say that $(\mathbb{V},\Delta)$ is even if each $\delta_k(\bar{n})=0$ whenever k is odd.

In any category $\mathcal{A}$ containing products and direct sums, we will say that a collection $\{\eta_i : X\longrightarrow X_i\}_{i\in I}$ of morphisms is locally finite if the induced morphism $\prod\limits_{i\in I}\eta_i : X\longrightarrow\prod\limits_{i\in I}X_i$ factors through the direct sum $\bigoplus\limits_{i\in I}X_i$. We now make two conventions that we will use for notation throughout the paper.

(1) Suppose (U,V) is a pair of adjoint functors between categories $\mathcal{A}$ and $\mathcal{B}$, so that we have natural isomorphisms

$$\mathcal{B}(UX,Y)\cong\mathcal{A}(X,VY)\quad(2.6)$$

for objects $X\in\mathcal{A}$, $Y\in\mathcal{B}$. For any morphism $f\in\mathcal{B}(UX,Y)$, we denote by f^R the corresponding morphism in $\mathcal{A}(X,VY)$. Conversely, for any $g\in\mathcal{A}(X,VY)$, we denote by g^L the corresponding morphism in $\mathcal{B}(UX,Y)$.

(2) Suppose (U_1,V_1) and (U_2,V_2) are pairs of adjoint functors between categories $\mathcal{A}$ and $\mathcal{B}$. Then, we know that there are isomorphisms

$$Nat(U_1,U_2)\cong Nat(V_2,V_1)\quad(2.7)$$

For any natural transformation $\eta\in Nat(U_1,U_2)$, we denote by η^R the corresponding natural transformation in $Nat(V_2,V_1)$. Conversely, for any $\zeta\in Nat(V_2,V_1)$, we denote by ζ^L the corresponding natural transformation in $Nat(U_1,U_2)$.

We will say that $(U=\{U_n:\mathcal{A}\longrightarrow\mathcal{B}\}_{n\in\mathbb{Z}}, V=\{V_n:\mathcal{B}\longrightarrow\mathcal{A}\}_{n\in\mathbb{Z}})$ is a $\mathbb{Z}$-system of adjoints between categories $\mathcal{A}$ and $\mathcal{B}$ if (U_n,V_n) is a pair of adjoint functors for each $n\in\mathbb{Z}$.

2.3. Theorem. *Let $(\mathbb{U}=\{\mathbb{U}_n:\mathcal{C}\longrightarrow\mathcal{C}\}_{n\in\mathbb{Z}},\mathbb{V}=\{\mathbb{V}_n:\mathcal{C}\longrightarrow\mathcal{C}\}_{n\in\mathbb{Z}})$ be a $\mathbb{Z}$-system of adjoints on $\mathcal{C}$. Then,*

(a) $\mathbb{U}$ can be equipped with the structure of an even A_∞-monad if and only if $\mathbb{V}$ can be equipped with the structure of an even A_∞-comonad.

(b) $\mathbb{U}$ can be equipped with the structure of an even A_∞-comonad if and only if $\mathbb{V}$ can be equipped with the structure of an even A_∞-monad.

Proof. We only prove (a) because (b) is similar. Suppose that $(\mathbb{U},\Theta=\{\theta_k(\bar{n})\})$ is an even A_∞-monad. Accordingly, we have transformations

$$(\theta_k(\bar{n}):\mathbb{U}_{\bar{n}}\longrightarrow\mathbb{U}_{\Sigma\bar{n}+(2-k)})\mapsto(\theta_k^R(\bar{n}):\mathbb{V}_{\Sigma\bar{n}+(2-k)}\longrightarrow\mathbb{V}_{\bar{n}^{op}})\quad(2.8)$$

Using the notation of (2.2), we see that the $\theta_k^R(\bar{n})$ satisfy, for any $\bar{z}\in\mathbb{Z}^N$, $N\geq 1$,

$$\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}(-1)^p(\mathbb{V}_{\bar{n}''^{op}}*\theta_q^R(\bar{n}')*\mathbb{V}_{\bar{n}^{op}})\theta^R_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')=0\quad(2.9)$$

as a transformation $\mathbb{V}_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+(3-N)} \longrightarrow \mathbb{V}_{\bar{n}''^{op}} \circ \mathbb{V}_{\bar{n}'^{op}} \circ \mathbb{V}_{\bar{n}^{op}}$ (note that q must be even). We now set $\delta_k(\bar{n}) := \theta^R_k(\bar{n}^{op})$. Accordingly, the condition in (2.9) now becomes

$$\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^p(\mathbb{V}_{\bar{n}''^{op}} * \delta_q(\bar{n}'^{op}) * \mathbb{V}_{\bar{n}^{op}})\delta_{p+1+r}(\bar{n}''^{op}, \Sigma\bar{n}'^{op} + (2-q), \bar{n}^{op}) = 0 \tag{2.10}$$

Comparing with (2.4), we see that $(\mathbb{V}, \Delta = \{\delta_k(\bar{n})\})$ satisfies the conditions for being an even A_∞-comonad. Further, these arguments can be reversed, showing that if $\mathbb{V}$ carries the structure of an even A_∞-comonad, then $\mathbb{U}$ carries structure of an even A_∞-monad. This proves the result. ∎

We will now describe the connection between A_∞-monads and locally finite A_∞-comonads. For this, we will first prove some simple lemmas on the collection $End(\mathcal{C})$ of endofunctors in $\mathcal{C}$. If $F \in End(\mathcal{C})$, a subfunctor $F' \hookrightarrow F$ is a natural transformation such that $F'(M) \longrightarrow F(M)$ is a monomorphism in $\mathcal{C}$ for each $M \in \mathcal{C}$. In order to avoid set theoretic complications, we will assume wherever necessary that the category $End(\mathcal{C})$ is well-powered, i.e., the collection of subfunctors of any $F \in End(\mathcal{C})$ forms a set.

2.4. Lemma. *Let F, $G \in End(\mathcal{C})$. Suppose that G preserves monomorphisms. Then, if $F'' \hookrightarrow F' \hookrightarrow F$ and $G'' \hookrightarrow G' \hookrightarrow G$ are subfunctors, the composition $G'' \circ F''$ is a subfunctor of $G' \circ F'$.*

Proof. We take some $M \in \mathcal{C}$ and consider the following commutative diagram

$$\begin{array}{ccccc}
 & & G \circ F''(M) & \xrightarrow{mono} & G \circ F'(M) & \xrightarrow{mono} & G \circ F(M) \\
 & & {\scriptstyle mono}\uparrow & & {\scriptstyle mono}\uparrow & & \\
G'' \circ F''(M) & \xrightarrow{mono} & G' \circ F''(M) & \xrightarrow{mono} & G' \circ F'(M) & &
\end{array} \tag{2.11}$$

Since G preserves monomorphisms, the morphisms in the top row of (2.11) are all monomorphisms. Further, the commutativity of the square in (2.11) shows that $G' \circ F''(M) \longrightarrow G' \circ F'(M)$ is a monomorphism. The composition in the bottom row now gives the desired result.

∎

2.5. Lemma. *Let $\{\eta_i : F \longrightarrow F_i\}_{i\in I}$ be a family of morphisms in $End(\mathcal{C})$. Fix an indexing set A. For each $i \in I$, let $\{F_i^\alpha\}_{\alpha\in A}$ be a family of subfunctors of F_i and let $\{F^\alpha\}_{\alpha\in A}$ be a family of subfunctors of F such that $\eta_i : F \longrightarrow F_i$ restricts to $\eta_i^\alpha : F^\alpha \longrightarrow F_i^\alpha$. For each $\alpha \in A$, suppose that $\{\eta_i^\alpha : F^\alpha \longrightarrow F_i^\alpha\}_{i\in I}$ is locally finite in $End(\mathcal{C})$. Then, the family*

$$\left\{ \eta_i^A = \sum_{\alpha\in A} \eta_i^\alpha : F^A = \sum_{\alpha\in A} F^\alpha \longrightarrow \sum_{\alpha\in A} F_i^\alpha = F_i^A \right\}_{i\in I} \tag{2.12}$$

is locally finite.

PROOF. Since $\{\eta_i^\alpha : F^\alpha \longrightarrow F_i^\alpha\}_{i\in I}$ is locally finite for each $\alpha \in A$, we have morphisms $F^\alpha \longrightarrow \bigoplus\limits_{i\in I} F_i^\alpha \longrightarrow \prod\limits_{i\in I} F_i^\alpha$. Summing over all $\alpha \in A$, we can consider for each $i_0 \in I$ the following composition

$$\sum_{\alpha\in A} F^\alpha \longrightarrow \sum_{\alpha\in A}\bigoplus_{i\in I} F_i^\alpha = \bigoplus_{i\in I}\sum_{\alpha\in A} F_i^\alpha \longrightarrow \sum_{\alpha\in A} F_{i_0}^\alpha \tag{2.13}$$

The second morphism in (2.13) is given by the canonical projection corresponding to $i_0 \in I$. The interchange in the middle term in (2.13) is justified by the fact that $\mathcal{C}$ satisfies (AB5). As we vary over all $i_0 \in I$, the compositions in (2.13) give us a factorization

$$\sum_{\alpha\in A} F^\alpha \longrightarrow \bigoplus_{i\in I}\sum_{\alpha\in A} F_i^\alpha \longrightarrow \prod_{i\in I}\sum_{\alpha\in A} F_i^\alpha \tag{2.14}$$

This proves the result. ∎

2.6. LEMMA. *Let F, $G \in End(\mathcal{C})$. Suppose that G preserves monomorphisms. Let $\{F^\alpha\}_{\alpha\in A}$ (resp. $\{G^\alpha\}_{\alpha\in A}$) be a family of subfunctors of F (resp. G). Then, $\sum_{\alpha\in A} G^\alpha F^\alpha$ is a subfunctor of $(\sum_{\alpha\in A} G^\alpha)(\sum_{\alpha\in A} F^\alpha)$.*

PROOF. For each fixed $\alpha_0 \in A$, we have subfunctors $F^{\alpha_0} \hookrightarrow (\sum_{\alpha\in A} F^\alpha) \hookrightarrow F$ and $G^{\alpha_0} \hookrightarrow (\sum_{\alpha\in A} G^\alpha) \hookrightarrow G$. Applying Lemma 2.4, it follows that each $G^{\alpha_0}F^{\alpha_0}$ is a subfunctor of $(\sum_{\alpha\in A} G^\alpha)(\sum_{\alpha\in A} F^\alpha)$. The result is now clear by summing over all $\alpha_0 \in A$. ∎

We now consider a $\mathbb{Z}$-system $(\mathbb{U} = \{\mathbb{U}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}, \mathbb{V} = \{\mathbb{V}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}})$ of adjoints on $\mathcal{C}$. Suppose that $(\mathbb{U}, \Theta)$ is an A_∞-monad, given by a collection of natural transformations

$$\Theta = \{\theta_k(\bar{n}) : \mathbb{U}_{\bar{n}} = \mathbb{U}_{n_1} \circ ... \circ \mathbb{U}_{n_k} \longrightarrow \mathbb{U}_{n_1+...+n_k+(2-k)} = \mathbb{U}_{\Sigma\bar{n}+(2-k)} \mid k \geq 1,\, \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k\} \tag{2.15}$$

satisfying the condition in (2.2). If $(\mathbb{U}, \Theta)$ is even, setting $\delta_k(\bar{n}) = \theta_k^R(\bar{n}^{op})$ as in the proof of Theorem 2.3, we know that $(\mathbb{V}, \Delta)$ is an A_∞-comonad, where

$$\Delta = \{\delta_k(\bar{n}) : \mathbb{V}_{\Sigma\bar{n}+(2-k)} = \mathbb{V}_{n_1+...+n_k+(2-k)} \longrightarrow \mathbb{V}_{n_1} \circ ... \circ \mathbb{V}_{n_k} = \mathbb{V}_{\bar{n}} \mid k \geq 1,\, \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k\} \tag{2.16}$$

We now consider a family $\mathbb{V}^\alpha = \{\mathbb{V}_n^\alpha \hookrightarrow \mathbb{V}_n\}_{n\in\mathbb{Z}}$ of subfunctors satisfying the following conditions

(a) For any $k \geq 1$ and $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, each of the transformations $\delta_k(\bar{n})$ in (2.16) restricts to the corresponding subfunctors

$$\Delta^\alpha = \{\delta_k^\alpha(\bar{n}) : \mathbb{V}^\alpha_{\Sigma\bar{n}+(2-k)} = \mathbb{V}^\alpha_{n_1+...+n_k+(2-k)} \longrightarrow \mathbb{V}^\alpha_{n_1} \circ ... \circ \mathbb{V}^\alpha_{n_k} = \mathbb{V}^\alpha_{\bar{n}} \mid k \geq 1,\, \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k\} \tag{2.17}$$

We note here that since each $\mathbb{V}_n$ is a right adjoint, it preserves monomorphisms. As such, it follows from Lemma 2.4 that each $\mathbb{V}^\alpha_{\bar{n}} = \mathbb{V}^\alpha_{n_1} \circ ... \circ \mathbb{V}^\alpha_{n_k}$ is a subfunctor of $\mathbb{V}_{\bar{n}} = \mathbb{V}_{n_1} \circ ... \circ \mathbb{V}_{n_k}$.

(b) For each $k \geq 1$ and $T \in \mathbb{Z}$, the collection

$$\left\{\delta_k^\alpha(\bar{n}) : \mathbb{V}^\alpha_{T+(2-k)} = \mathbb{V}^\alpha_{n_1+...+n_k+(2-k)} \longrightarrow \mathbb{V}^\alpha_{n_1} \circ ... \circ \mathbb{V}^\alpha_{n_k} = \mathbb{V}^\alpha_{\bar{n}}\right\}_{\bar{n}\in\mathbb{Z}(T,k)} \tag{2.18}$$

is locally finite.

2.7. Lemma. $(\mathbb{V}^\alpha, \Delta^\alpha)$ *is a locally finite* A_∞*-comonad.*

Proof. By assumption, the natural transformations $\delta_k(\bar{n})$ restrict to transformations $\delta_k^\alpha(\bar{n})$. Since each $\mathbb{V}_n$ preserves monomorphisms, we see that all the terms appearing in (2.4) restrict to corresponding subfunctors. As such, these transformations $\delta_k^\alpha(\bar{n})$ satisfy the condition in (2.4). Hence, $(\mathbb{V}^\alpha, \Delta^\alpha)$ is an A_∞-comonad. The fact that $(\mathbb{V}^\alpha, \Delta^\alpha)$ is locally finite follows directly from the assumption in (2.18). ∎

We now let $\{(\mathbb{V}^\alpha, \Delta^\alpha)\}_{\alpha \in A}$ denote the collection of such families of subfunctors. We note that this collection is non-empty since it contains the family of zero subfunctors. We now define

$$\mathbb{V}^\oplus = \left\{ \mathbb{V}_n^\oplus := \sum_{\alpha \in A} \mathbb{V}_n^\alpha \right\}_{n \in \mathbb{Z}} \tag{2.19}$$

2.8. Proposition. *The collection* $\mathbb{V}^\oplus = \{\mathbb{V}_n^\oplus\}_{n \in \mathbb{Z}}$ *is equipped with the structure of a locally finite* A_∞*-comonad.*

Proof. For any $k \geq 1$ and $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, we consider

$$\delta_k^\oplus(\bar{n}) = \sum_{\alpha \in A} \delta_k^\alpha(\bar{n}) : \sum_{\alpha \in A} \mathbb{V}^\alpha_{\Sigma \bar{n} + (2-k)} \longrightarrow \sum_{\alpha \in A} \mathbb{V}^\alpha_{n_1} \circ ... \circ \mathbb{V}^\alpha_{n_k} \hookrightarrow \left(\sum_{\alpha \in A} \mathbb{V}^\alpha_{n_1} \right) \circ ... \circ \left(\sum_{\alpha \in A} \mathbb{V}^\alpha_{n_k} \right) \tag{2.20}$$

where the inclusion in (2.20) follows from Lemma 2.6. Again since each $\mathbb{V}_n$ preserves monomorphisms, we see that all the transformations appearing in (2.4) restrict to corresponding subfunctors. It follows that the transformations $\delta_k^\oplus(\bar{n}) : \mathbb{V}^\oplus_{\Sigma \bar{n} + (2-k)} \longrightarrow \mathbb{V}^\oplus_{n_1} \circ ... \circ \mathbb{V}^\oplus_{n_k}$ satisfy the condition in (2.4), making $\left(\mathbb{V}^\oplus, \Delta^\oplus = \{\delta_k^\oplus(\bar{n}) \mid k \geq 1, \bar{n} \in \mathbb{Z}^k\}\right)$ an A_∞-comonad.
Additionally, using Lemma 2.5, we see that for each $k \geq 1$ and $T \in \mathbb{Z}$, we have a factorization

$$\begin{tikzcd}
\sum_{\alpha \in A} \mathbb{V}^\alpha_{T+(2-k)} \arrow[r] \arrow[dr] & \prod\limits_{\bar{n} \in \mathbb{Z}(T,k)} \sum_{\alpha \in A} \mathbb{V}^\alpha_{n_1} \circ ... \circ \mathbb{V}^\alpha_{n_k} \arrow[r] & \prod\limits_{\bar{n} \in \mathbb{Z}(T,k)} \left(\sum_{\alpha \in A} \mathbb{V}^\alpha_{n_1}\right) \circ ... \circ \left(\sum_{\alpha \in A} \mathbb{V}^\alpha_{n_k}\right) \\
& \bigoplus\limits_{\bar{n} \in \mathbb{Z}(T,k)} \sum_{\alpha \in A} \mathbb{V}^\alpha_{n_1} \circ ... \circ \mathbb{V}^\alpha_{n_k} \arrow[u] \arrow[r] & \bigoplus\limits_{\bar{n} \in \mathbb{Z}(T,k)} \left(\sum_{\alpha \in A} \mathbb{V}^\alpha_{n_1}\right) \circ ... \circ \left(\sum_{\alpha \in A} \mathbb{V}^\alpha_{n_k}\right) \arrow[u]
\end{tikzcd} \tag{2.21}$$

From (2.21), it is clear that $(\mathbb{V}^\oplus, \Delta^\oplus)$ is locally finite. ∎

2.9. Remark. *On the other hand, suppose that* $(\mathbb{U} = \{\mathbb{U}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n \in \mathbb{Z}}, \mathbb{V} = \{\mathbb{V}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n \in \mathbb{Z}})$ *is a* $\mathbb{Z}$*-system of adjoints such that* $\mathbb{V}$ *carries the structure of an even* A_∞*-monad. By Theorem 2.3, we know that* $\mathbb{U} = \{\mathbb{U}_n\}_{n \in \mathbb{Z}}$ *is canonically equipped with an* A_∞*-comonad structure. Additionally, if we suppose that each* $\mathbb{U}_n$ *preserves monomorphisms, we can construct by a process similar to (2.19) a family* $\mathbb{U}^\oplus = \{\mathbb{U}_n^\oplus \hookrightarrow \mathbb{U}_n\}_{n \in \mathbb{Z}}$ *of subfunctors equipped with a locally finite* A_∞*-comonad structure.*

3. Eilenberg-Moore categories for A_∞-monads and A_∞-comonads

We now consider modules over A_∞-monads and comodules over A_∞-comonads.

3.1. Definition. *Let $(\mathbb{U},\Theta)$ be an A_∞-monad over $\mathcal{C}$. A $(\mathbb{U},\Theta)$-module (M,π) consists of the following data:*
(a) A collection $M=\{M_n\}_{n\in\mathbb{Z}}$ of objects of $\mathcal{C}$.
(b) A collection of morphisms

$$\pi=\{\pi_k(\bar{n},l):\mathbb{U}_{\bar{n}}M_l\longrightarrow M_{\Sigma\bar{n}+l+(2-k)}\mid k\geq 1,\ \bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1},\ l\in\mathbb{Z}\ \}\tag{3.1}$$

satisfying, for each $\bar{z}\in\mathbb{Z}^N$, $N\geq 0$ and $l\in\mathbb{Z}$,

$$\begin{aligned}\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}&(-1)^{p+qr+q\Sigma\bar{n}}\pi_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'',l)\circ(\mathbb{U}_{\bar{n}}*\theta_q(\bar{n}'))(\mathbb{U}_{\bar{n}''}M_l)\\&=-\sum_{\bar{z}=(\bar{m},\bar{m}')}(-1)^{a+b\Sigma\bar{m}}\pi_{a+1}(\bar{m},\Sigma\bar{m}'+l+(2-b))\circ\mathbb{U}_{\bar{m}}(\pi_b(\bar{m}',l))\end{aligned}\tag{3.2}$$

On the left hand side of (3.2), we have $|\bar{n}|=p$, $|\bar{n}'|=q$ and $|\bar{n}''|=r-1$, while on the right hand side, we have $|\bar{m}|=a$ and $|\bar{m}'|=b-1$. Accordingly, both sides of (3.2) represent morphisms

$$\mathbb{U}_{\bar{n}}\mathbb{U}_{\bar{n}'}\mathbb{U}_{\bar{n}''}M_l=\mathbb{U}_{\bar{m}}\mathbb{U}_{\bar{m}'}M_l\longrightarrow M_{\Sigma\bar{m}+\Sigma\bar{m}'+l+(2-N)}=M_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+l+(2-N)}\tag{3.3}$$

A morphism $f:(M,\pi)\longrightarrow(M',\pi')$ of $\mathbb{U}$-modules consists of a collection $f=\{f_l:M_l\longrightarrow M'_l\}_{l\in\mathbb{Z}}$ such that $\pi'_k(\bar{n},l)\circ\mathbb{U}_{\bar{n}}(f_l)=f_{\Sigma\bar{n}+l+(2-k)}\circ\pi_k(\bar{n},l)$ for every $k\geq 1$, $\bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1}$, $l\in\mathbb{Z}$. We denote the category of $(\mathbb{U},\Theta)$-modules by $EM_{(\mathbb{U},\Theta)}$.
We will say that a $\mathbb{U}$-module (M,π) is even if $\pi_k(\bar{n},l)=0$ whenever k is odd. The full subcategory of even $\mathbb{U}$-modules will be denoted by $EM^e_{(\mathbb{U},\Theta)}$.

3.2. Definition. *Let $(\mathbb{V},\Delta)$ be an A_∞-comonad over $\mathcal{C}$. A $(\mathbb{V},\Delta)$-comodule (P,ρ) consists of the following data:*
(a) A collection $P=\{P_n\}_{n\in\mathbb{Z}}$ of objects of $\mathcal{C}$.
(b) A collection of morphisms

$$\rho=\{\rho_k(\bar{n},l):P_{\Sigma\bar{n}+l+(2-k)}\longrightarrow\mathbb{V}_{\bar{n}}P_l\mid k\geq 1,\ \bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1},\ l\in\mathbb{Z}\ \}\tag{3.4}$$

satisfying, for each $\bar{z}\in\mathbb{Z}^N$, $N\geq 0$ and $l\in\mathbb{Z}$,

$$\begin{aligned}\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}&(-1)^{pq+r+q\Sigma\bar{n}}(\mathbb{V}_{\bar{n}}*\delta_q(\bar{n}'))(\mathbb{V}_{\bar{n}''}P_l)\circ\rho_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'',l)\\&=-\sum_{\bar{z}=(\bar{m},\bar{m}')}(-1)^{ab+b\Sigma\bar{m}}\mathbb{V}_{\bar{m}}(\rho_b(\bar{m}',l))\circ\rho_{a+1}(\bar{m},\Sigma\bar{m}'+l+(2-b))\end{aligned}\tag{3.5}$$

On the left hand side of (3.5), we have $|\bar{n}|=p$, $|\bar{n}'|=q$ and $|\bar{n}''|=r-1$, while on the right hand side, we have $|\bar{m}|=a$ and $|\bar{m}'|=b-1$. Accordingly, both sides of (3.5) represent morphisms

$$P_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+l+(2-N)}=P_{\Sigma\bar{m}+\Sigma\bar{m}'+l+(2-N)}\longrightarrow\mathbb{V}_{\bar{m}}\mathbb{V}_{\bar{m}'}P_l=\mathbb{V}_{\bar{n}}\mathbb{V}_{\bar{n}'}\mathbb{V}_{\bar{n}''}P_l\tag{3.6}$$

A morphism $g : (P,\rho) \longrightarrow (P',\rho')$ *of* $\mathbb{V}$*-comodules consists of a collection* $g = \{g_l : P_l \longrightarrow P'_l\}_{l\in\mathbb{Z}}$ *such that* $\rho'_k(\bar{n},l)\circ g_{\Sigma\bar{n}+l+(2-k)} = \mathbb{V}_{\bar{n}}(g_l)\circ\rho_k(\bar{n},l)$ *for every* $k\geq 1$, $\bar{n} = (n_1,...,n_{k-1})\in\mathbb{Z}^{k-1}$, $l\in\mathbb{Z}$ *. We denote the category of* $(\mathbb{V},\Delta)$*-comodules by* $EM^{(\mathbb{V},\Delta)}$*.*
We will say that a $\mathbb{V}$*-comodule* (P,ρ) *is even if* $\rho_k(\bar{n},l) = 0$ *whenever* k *is odd. The full subcategory of even* $\mathbb{V}$*-comodules will be denoted by* $EM_e^{(\mathbb{V},\Delta)}$*.*

3.3. Proposition. *(a) Let* $(\mathbb{U},\Theta)$ *be an* A_∞*-monad over* $\mathcal{C}$*. Then, for any object* $M\in\mathcal{C}$*, the collection* $\{\mathbb{U}_nM\}_{n\in\mathbb{Z}}$ *can be equipped with a canonical* $(\mathbb{U},\Theta)$*-module structure.*
(b) Let $(\mathbb{V},\Delta)$ *be an* A_∞*-comonad over* $\mathcal{C}$*. Then, for any object* $M\in\mathcal{C}$*, the collection* $\{\mathbb{V}_nM\}_{n\in\mathbb{Z}}$ *can be equipped with a canonical* $(\mathbb{V},\Delta)$*-comodule structure.*

Proof. We prove only (a) because (b) is similar. For the sake of convenience, we set $Q_n := \mathbb{U}_nM$ for each $n\in\mathbb{Z}$. For $\bar{n} = (n_1,...,n_{k-1})\in\mathbb{Z}^{k-1}$, $k\geq 1$, $l\in\mathbb{Z}$, we set

$$\pi_k(\bar{n},l) : \mathbb{U}_{\bar{n}}Q_l = \mathbb{U}_{\bar{n}}\mathbb{U}_lM \xrightarrow{\theta_k(\bar{n},l)(M)} \mathbb{U}_{\Sigma\bar{n}+l+(2-k)}M = Q_{\Sigma\bar{n}+l+(2-k)} \tag{3.7}$$

We note that the condition in (2.2) for $(\mathbb{U},\Theta)$ to be an A_∞-monad implies in particular that for each $\bar{z}\in\mathbb{Z}^N$, $N\geq 0$ and $l\in\mathbb{Z}$, we must have

$$\begin{aligned}\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}&(-1)^{p+qr+q\Sigma\bar{n}}\theta_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'',l)(M)\circ(\mathbb{U}_{\bar{n}}*\theta_q(\bar{n}'))(\mathbb{U}_{\bar{n}''}\mathbb{U}_lM)\\ &= -\sum_{\bar{z}=(\bar{m},\bar{m}')}(-1)^{a+b\Sigma\bar{m}}\theta_{a+1}(\bar{m},\Sigma\bar{m}'+l+(2-b))(M)\circ\mathbb{U}_{\bar{m}}(\theta_b(\bar{m}',l)(M))\end{aligned} \tag{3.8}$$

On the left hand side of (3.8), we have $|\bar{n}| = p$, $|\bar{n}'| = q$ and $|\bar{n}''| = r-1$, while on the right hand side, we have $|\bar{m}| = a$ and $|\bar{m}'| = b-1$. Putting $\pi_k(\bar{n},l) = \theta_k(\bar{n},l)(M)$ and $Q_n = \mathbb{U}_nM$, it follows from (3.8) that

$$\begin{aligned}\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}&(-1)^{p+qr+q\Sigma\bar{n}}\pi_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'',l)\circ(\mathbb{U}_{\bar{n}}*\theta_q(\bar{n}'))(\mathbb{U}_{\bar{n}''}Q_l)\\ &= -\sum_{\bar{z}=(\bar{m},\bar{m}')}(-1)^{a+b\Sigma\bar{m}}\pi_{a+1}(\bar{m},\Sigma\bar{m}'+l+(2-b))\circ\mathbb{U}_{\bar{m}}(\pi_b(\bar{m}',l))\end{aligned} \tag{3.9}$$

This proves the result.

∎

Let $(\mathbb{U},\Theta)$ be an A_∞-monad over $\mathcal{C}$. We now let $\mathcal{C}^{\mathbb{Z}}$ denote the category of $\mathbb{Z}$-graded objects over $\mathcal{C}$. Then, for any $M = \{M_l\}_{l\in\mathbb{Z}}$ in $\mathcal{C}^{\mathbb{Z}}$, we set

$$\hat{\mathbb{U}}M = \left\{(\hat{\mathbb{U}}M)_l := \bigoplus_{x+y=l}\mathbb{U}_xM_y\right\}_{l\in\mathbb{Z}} \tag{3.10}$$

In particular, for $M\in\mathcal{C}$, we denote by $\hat{\mathbb{U}}M$ the object $\{\mathbb{U}_nM\}_{n\in\mathbb{Z}}\in EM_{(\mathbb{U},\Theta)}$ determined by Proposition 3.3.

3.4. Proposition. *Let* $(\mathbb{U},\Theta)$ *be an* A_∞*-monad over* $\mathcal{C}$ *such that each of the functors* $\{\mathbb{U}_n\}_{n\in\mathbb{Z}}$ *preserves direct sums. Then,* $\hat{\mathbb{U}}$ *determines a functor*

$$\hat{\mathbb{U}} : \mathcal{C}^{\mathbb{Z}} \longrightarrow EM_{(\mathbb{U},\Theta)} \tag{3.11}$$

Proof. For the sake of convenience, we set $Q_l := (\hat{\mathbb{U}}M)_l$ for each $l \in \mathbb{Z}$. For $\bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1}$, $k \geq 1$, $l \in \mathbb{Z}$, we define $\pi_k(\bar{n}, l) : \mathbb{U}_{\bar{n}}Q_l \longrightarrow Q_{\Sigma\bar{n}+l+(2-k)}$ by setting

$$\mathbb{U}_{\bar n}Q_l = \mathbb{U}_{\bar n}\left(\bigoplus_{x+y=l}\mathbb{U}_x M_y\right) = \left(\bigoplus_{x+y=l}\mathbb{U}_{\bar n}\mathbb{U}_x M_y\right) \xrightarrow[=\pi_k(\bar n,l)]{\bigoplus_{x+y=l}\theta_k(\bar n,x)(M_y)} \left(\bigoplus_{x+y=l}\mathbb{U}_{\Sigma\bar n+x+(2-k)}M_y\right) = Q_{\Sigma\bar n+l+(2-k)} \tag{3.12}$$

In (3.12) we have used the fact that each $\mathbb{U}_n$, and hence any $\mathbb{U}_{\bar{n}}$, preserves direct sums. As in (3.8), we note that the condition in (2.2) for $(\mathbb{U}, \Theta)$ to be an A_∞-monad implies in particular that for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 0$ and $x, y \in \mathbb{Z}$, we must have

$$\begin{aligned}\sum_{\bar z=(\bar n,\bar n',\bar n'')} &(-1)^{p+qr+q\Sigma\bar n}\theta_{p+1+r}(\bar n, \Sigma\bar n' + (2-q), \bar n'', x)(M_y) \circ (\mathbb{U}_{\bar n} * \theta_q(\bar n'))(\mathbb{U}_{\bar n''}\mathbb{U}_x M_y) \\ &= -\sum_{\bar z=(\bar m,\bar m')} (-1)^{a+b\Sigma\bar m}\theta_{a+1}(\bar m, \Sigma\bar m' + x + (2-b))(M_y) \circ \mathbb{U}_{\bar m}(\theta_b(\bar m', x)(M_y))\end{aligned} \tag{3.13}$$

Applying the definition of $\pi_k(\bar{n}, l)$ in (3.12) and taking the direct sum of the equalities in (3.13) over all x, $y \in \mathbb{Z}$ such that $l = x + y$, we obtain

$$\begin{aligned}\sum_{\bar z=(\bar n,\bar n',\bar n'')} &(-1)^{p+qr+q\Sigma\bar n}\pi_{p+1+r}(\bar n, \Sigma\bar n' + (2-q), \bar n'', l) \circ (\mathbb{U}_{\bar n} * \theta_q(\bar n'))(\mathbb{U}_{\bar n''}Q_l) \\ &= -\sum_{\bar z=(\bar m,\bar m')} (-1)^{a+b\Sigma\bar m}\pi_{a+1}(\bar m, \Sigma\bar m' + l + (2-b)) \circ \mathbb{U}_{\bar m}(\pi_b(\bar m', l))\end{aligned} \tag{3.14}$$

It is now clear that we have a functor $\hat{\mathbb{U}} : \mathcal{C}^{\mathbb{Z}} \longrightarrow EM_{(\mathbb{U},\Theta)}$. ∎

We mention here the following two simple facts on the formalism of adjoint functors, which we will use in the proof of the next result.

(1) Let (U_1, V_1) and (U_2, V_2) be pairs of adjoint endofunctors on a category $\mathcal{A}$. We consider a transformation $\theta : U_1 \longrightarrow U_2$ and the corresponding transformation $\theta^R : V_2 \longrightarrow V_1$ between right adjoints. Let X, $Y \in \mathcal{A}$ and let $f : U_2X \longrightarrow Y$ be a morphism. Then, we have commutative diagrams

$$\begin{tikzcd} & U_2X \arrow[dr, "f"] & \\ U_1X \arrow[ur, "\theta X"] \arrow[rr, "(f\circ\theta X)"] & & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & V_2Y \arrow[dr, "\theta^R Y"] & \\ X \arrow[ur, "f^R"] \arrow[rr, "(f\circ\theta X)^R"] & & V_1Y \end{tikzcd} \tag{3.15}$$

(2) Let (U_1, V_1) and (U_2, V_2) be pairs of adjoint endofunctors on a category $\mathcal{A}$. We consider objects X, Y, $Z \in \mathcal{A}$ as well as morphisms $f : U_1X \longrightarrow Y$ and $g : U_2Y \longrightarrow Z$. Then, we have commutative diagrams

$$\begin{tikzcd} & U_2Y \arrow[dr, "g"] & \\ U_2U_1X \arrow[ur, "U_2f"] \arrow[rr, "(g\circ U_2f)"] & & Z \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & V_1Y \arrow[dr, "V_1g^R"] & \\ X \arrow[ur, "f^R"] \arrow[rr, "(g\circ U_2f)^R"] & & V_1V_2Z \end{tikzcd} \tag{3.16}$$

3.5. PROPOSITION. *Let* $(\mathbb{U} = \{\mathbb{U}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\geq 0}, \mathbb{V} = \{\mathbb{V}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\geq 0})$ *be a* $\mathbb{Z}$*-system of adjoints on* $\mathcal{C}$*. Let* $(\mathbb{U}, \Theta)$ *be an even* A_∞*-monad and* $(\mathbb{V}, \Delta)$ *be the corresponding* A_∞*-comonad. Then, the categories* $EM^e_{(\mathbb{U},\Theta)}$ *and* $EM^{(\mathbb{V},\Delta)}_e$ *are isomorphic.*

PROOF. Let (M, π) be an even $(\mathbb{U}, \Theta)$-module as in Definition 3.1. Accordingly, for $\bar{n} \in \mathbb{Z}^{k-1}$, $k \geq 1$, $l \in \mathbb{Z}$, we have morphisms

$$(\pi_k(\bar{n}, l) : \mathbb{U}_{\bar{n}} M_l \longrightarrow M_{\Sigma\bar{n}+l+(2-k)}) \Rightarrow (\pi^R_k(\bar{n}, l) : M_l \longrightarrow \mathbb{V}_{\bar{n}^{op}} M_{\Sigma\bar{n}+l+(2-k)}) \tag{3.17}$$

We now define $P = \{P_l\}_{l\in\mathbb{Z}}$ by setting $P_l = M_{-l}$ for each $l \in \mathbb{Z}$ as well as

$$\rho_k(\bar{n}, l) : P_{\Sigma\bar{n}+l+(2-k)} = M_{-\Sigma\bar{n}-l-(2-k)} \xrightarrow{\pi^R_k(\bar{n}^{op}, -\Sigma\bar{n}-l-(2-k))} \mathbb{V}_{\bar{n}} M_{-l} = \mathbb{V}_{\bar{n}} P_l \tag{3.18}$$

for $l \in \mathbb{Z}$ and $\bar{n} \in \mathbb{Z}^{k-1}$. From the proof of Theorem 2.3, we know that the collection $\Delta = \{\delta_k(\bar{n})\}$ is given by setting $\delta_k(\bar{n}) = \theta^R_k(\bar{n}^{op})$. In the notation of (3.2), we now see that for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 0$, $l \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} &(-1)^{p+qr+q\Sigma\bar{n}} \pi_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'', l) \circ (\mathbb{U}_{\bar{n}} * \theta_q(\bar{n}'))(\mathbb{U}_{\bar{n}''} M_l) \\ &= -\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}} \pi_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \circ \mathbb{U}_{\bar{m}}(\pi_b(\bar{m}', l))\end{aligned} \tag{3.19}$$

Taking adjoints on both sides of (3.19), we obtain

$$\begin{aligned}\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} &(-1)^{p+qr+q\Sigma\bar{n}} (\pi_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'', l) \circ (\mathbb{U}_{\bar{n}} * \theta_q(\bar{n}'))(\mathbb{U}_{\bar{n}''} M_l))^R \\ &= -\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}} (\pi_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \circ \mathbb{U}_{\bar{m}}(\pi_b(\bar{m}', l)))^R\end{aligned} \tag{3.20}$$

We now consider one by one the terms in (3.20). For any term on the left hand side of (3.20), we apply (3.15) with

$$\begin{gathered} X = M_l \qquad Y = M_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+l+(2-N)} \\ \theta = \mathbb{U}_{\bar{n}} * \theta_q(\bar{n}') * \mathbb{U}_{\bar{n}''} : U_1 = \mathbb{U}_{\bar{n}}\mathbb{U}_{\bar{n}'}\mathbb{U}_{\bar{n}''} \longrightarrow \mathbb{U}_{\bar{n}}\mathbb{U}_{\Sigma\bar{n}'+(2-q)}\mathbb{U}_{\bar{n}''} = U_2 \\ f = \pi_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'', l) : U_2 X = \mathbb{U}_{\bar{n}}\mathbb{U}_{\Sigma\bar{n}'+(2-q)}\mathbb{U}_{\bar{n}''} M_l \longrightarrow M_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+l+(2-N)} = Y \end{gathered} \tag{3.21}$$

It follows therefore from (3.15) that the left hand side of (3.20) gives

$$\begin{aligned}&\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^{p+qr+q\Sigma\bar{n}} (\pi_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'', l) \circ (\mathbb{U}_{\bar{n}} * \theta_q(\bar{n}'))(\mathbb{U}_{\bar{n}''} M_l))^R \\ &= \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^{p+qr+q\Sigma\bar{n}} (\mathbb{V}_{\bar{n}''^{op}} * \theta^R_q(\bar{n}'))(\mathbb{V}_{\bar{n}^{op}}(M_{\bar{z}+l+(2-N)})) \circ \pi^R_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2-q), \bar{n}'', l)\end{aligned} \tag{3.22}$$

If we shift degrees by setting $l' = -\Sigma\bar{z} - l - (2-N)$ and use (3.18), then (3.22) becomes

$$-\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^{p+1} (\mathbb{V}_{\bar{n}''^{op}} * \delta_q(\bar{n}'^{op}))(\mathbb{V}_{\bar{n}^{op}}(P_{l'})) \circ \rho_{p+1+r}(\bar{n}''^{op}, \Sigma\bar{n}' + (2-q), \bar{n}^{op}, l') \tag{3.23}$$

where the sign is obtained from the fact that q and $p+1+r$ are both even. For any term on the right hand side of (3.20), we now apply (3.16) with

$$\begin{gathered} X = M_l \qquad Y = M_{\Sigma\bar{m}'+l+(2-b)} \qquad Z = M_{\Sigma\bar{m}+\Sigma\bar{m}'+l+(2-N)} \qquad U_1 = \mathbb{U}_{\bar{m}'} \qquad U_2 = \mathbb{U}_{\bar{m}} \\ f = \pi_b(\bar{m}', l) : U_1X = \mathbb{U}_{\bar{m}'}M_l \longrightarrow M_{\Sigma\bar{m}'+l+(2-b)} = Y \\ g = \pi_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) : U_2Y = \mathbb{U}_{\bar{m}}M_{\Sigma\bar{m}'+l+(2-b)} \longrightarrow M_{\Sigma\bar{m}+\Sigma\bar{m}'+l+(2-N)} = Z \end{gathered} \tag{3.24}$$

It follows therefore from (3.16) that the right hand side of (3.20) gives

$$\begin{aligned} &-\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}}(\pi_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \circ \mathbb{U}_{\bar{m}}(\pi_b(\bar{m}', l)))^R \\ &= -\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}}\mathbb{V}_{\bar{m}'^{op}}\pi^R_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \circ \pi^R_b(\bar{m}', l) \\ &= -\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}}\mathbb{V}_{\bar{m}'^{op}}\rho_{a+1}(\bar{m}^{op}, -\Sigma\bar{z} - l - (2-N)) \circ \rho_b(\bar{m}'^{op}, -\Sigma\bar{m}' - l - (2-b)) \end{aligned} \tag{3.25}$$

Again, if we shift degrees by setting $l' = -\Sigma\bar{z} - l - (2-N)$, then (3.25) becomes

$$\begin{aligned} &-\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}}\mathbb{V}_{\bar{m}'^{op}}\rho_{a+1}(\bar{m}^{op}, l') \circ \rho_b(\bar{m}'^{op}, \Sigma\bar{m} + l' + (1-a)) \\ &= \sum_{\bar{z}=(\bar{m},\bar{m}')} \mathbb{V}_{\bar{m}'^{op}}\rho_{a+1}(\bar{m}^{op}, l') \circ \rho_b(\bar{m}'^{op}, \Sigma\bar{m} + l' + (1-a)) \end{aligned} \tag{3.26}$$

where the last equality in (3.25) follows from the fact that $a+1$ and b must both be even. By inspecting (3.23) and (3.26), we see that the morphisms $\rho_k(\bar{n}, l)$ together make $P = \{P_n = M_{-n}\}_{n\in\mathbb{Z}}$ into a $(\mathbb{V}, \Delta)$-comodule. These arguments can be reversed, showing that even $(\mathbb{U}, \Theta)$-modules correspond to even $(\mathbb{V}, \Delta)$-comodules and vice-versa.

■

3.6. Definition. *Let $(\mathbb{U}, \Theta)$ be an A_∞-monad over $\mathcal{C}$ and let (M^1, π^1), (M^2, π^2) be $(\mathbb{U}, \Theta)$-modules. An ∞-morphism $\alpha : (M^1, \pi^1) \longrightarrow (M^2, \pi^2)$ consists of a collection*

$$\alpha = \{\ \alpha_k(\bar{n}, l) : \mathbb{U}_{\bar{n}}M^1_l \longrightarrow M^2_{\Sigma\bar{n}+l+(1-k)} \mid k \geq 1,\ \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1},\ l \in \mathbb{Z}\ \} \tag{3.27}$$

satisfying for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 0$, $l \in \mathbb{Z}$,

$$\begin{aligned} &\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}}\alpha_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \circ \mathbb{U}_{\bar{m}}(\pi^1_b(\bar{m}', l)) \\ &+ \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^{a+bc+b\Sigma\bar{n}}\alpha_{a+1+c}(\bar{n}, \Sigma\bar{n}' + (2-b), \bar{n}'', l) \circ (\mathbb{U}_{\bar{n}} * \theta_b(\bar{n}') * \mathbb{U}_{\bar{n}''})(M^1_l) \\ &\qquad = \sum_{\bar{z}=(\bar{p},\bar{p}')} \pi^2_{a+1}(\bar{p}, \Sigma\bar{p}' + l + (1-b)) \circ \mathbb{U}_{\bar{p}}(\alpha_b(\bar{p}', l)) \end{aligned} \tag{3.28}$$

where:
(1) In the first term on left hand side of (3.28), we have $|\bar{m}| = a$ and $|\bar{m}'| = b-1$.
(2) In the second term on left hand side of (3.28), we have $|\bar{n}| = a$, $|\bar{n}'| = b$ and $|\bar{n}''| = c-1$.
(3) On the right hand side of (3.28), we have $|\bar{p}| = a$ and $|\bar{p}'| = b-1$.

We observe that both sides of (3.28) represent morphisms $\mathbb{U}_{\bar{z}}M^1_l \longrightarrow M^2_{\Sigma\bar{z}+l+(1-N)}$. We will denote by $\widetilde{EM}_{(\mathbb{U},\Theta)}$ the category of $(\mathbb{U},\Theta)$-modules equipped with ∞-morphisms. For any (M,π) in $\widetilde{EM}_{(\mathbb{U},\Theta)}$, the identity $\iota : (M,\pi) \longrightarrow (M,\pi)$ is given by taking $\iota_1(l) = id : M_l \longrightarrow M_l$ for each $l \in \mathbb{Z}$ and $\iota_k(\bar{z},l) = 0$ otherwise. Given morphisms $\alpha : (M^1,\pi^1) \longrightarrow (M^2,\pi^2)$ and $\beta : (M^2,\pi^2) \longrightarrow (M^3,\pi^3)$, we define their composition $\gamma = \beta\alpha$ as follows

$$\gamma = \{\ \gamma_k(\bar{z},l) : \mathbb{U}_{\bar{z}}M^1_l \longrightarrow M^3_{\Sigma\bar{z}+l+(1-k)} \mid k \geq 1,\ \bar{z} = (z_1,...,z_{k-1}) \in \mathbb{Z}^{k-1},\ l \in \mathbb{Z}\ \}$$

$$\gamma_k(\bar{z},l) := \sum_{\bar{z}=(\bar{n},\bar{n}')} \beta_{a+1}(\bar{n}, \Sigma\bar{n}' + l + (1-b)) \circ \mathbb{U}_{\bar{n}}(\alpha_b(\bar{n}',l)) \tag{3.29}$$

where $|\bar{n}| = a$ and $|\bar{n}'| = b-1$ in (3.29). We will say that a morphism $\alpha : (M^1,\pi^1) \longrightarrow (M^2,\pi^2)$ is odd if $\alpha_k(\bar{n},l) = 0$ for any even k. In particular, any identity morphism is odd. From (3.29), we also notice that the composition of odd morphisms in $\widetilde{EM}_{(\mathbb{U},\Theta)}$ must be odd.

3.7. Definition. *Let $(\mathbb{V},\Delta)$ be an A_∞-comonad over $\mathcal{C}$ and let (P^1,ρ^1), (P^2,ρ^2) be $(\mathbb{V},\Delta)$-comodules. An ∞-morphism $\alpha : (P^1,\rho^1) \longrightarrow (P^2,\rho^2)$ consists of a collection*

$$\alpha = \{\ \alpha_k(\bar{n},l) : P^1_{\Sigma\bar{n}+l+(1-k)} \longrightarrow \mathbb{V}_{\bar{n}}P^2_l \mid k \geq 1,\ \bar{n} = (n_1,...,n_{k-1}) \in \mathbb{Z}^{k-1},\ l \in \mathbb{Z}\ \} \tag{3.30}$$

satisfying for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 0$, $l \in \mathbb{Z}$,

$$\begin{aligned}
&\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{ab+b\Sigma\bar{m}}\mathbb{V}_{\bar{m}}(\rho^2_b(\bar{m}',l)) \circ \alpha_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \\
&+ \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^{c+ab+b\Sigma\bar{n}}(\mathbb{V}_{\bar{n}} * \delta_b(\bar{n}') * \mathbb{V}_{\bar{n}''})(P^2_l) \circ \alpha_{a+1+c}(\bar{n}, \Sigma\bar{n}' + (2-b), \bar{n}'', l) \\
&\qquad = \sum_{\bar{z}=(\bar{p},\bar{p}')} \mathbb{V}_{\bar{p}}(\alpha_b(\bar{p}',l)) \circ \rho^1_{a+1}(\bar{p}, \Sigma\bar{p}' + l + (1-b))
\end{aligned} \tag{3.31}$$

where:
(1) In the first term on left hand side of (3.31), we have $|\bar{m}| = a$ and $|\bar{m}'| = b-1$.
(2) In the second term on left hand side of (3.31), we have $|\bar{n}| = a$, $|\bar{n}'| = b$ and $|\bar{n}''| = c-1$.
(3) On the right hand side of (3.31), we have $|\bar{p}| = a$ and $|\bar{p}'| = b-1$.

We observe that both sides of (3.31) represent morphisms $P^1_{\Sigma\bar{z}+l+(1-N)} \longrightarrow \mathbb{V}_{\bar{z}}P^2_l$. We will denote by $\widetilde{EM}^{(\mathbb{V},\Delta)}$ the category of $(\mathbb{V},\Delta)$-comodules equipped with ∞-morphisms. The composition of morphisms in $\widetilde{EM}^{(\mathbb{V},\Delta)}$ is defined in a manner dual to (3.29). We will say that a morphism α in $EM^{(\mathbb{V},\Delta)}$ is odd if $\alpha_k(\bar{n},l) = 0$ whenever k is even. We may also verify that the collection of odd morphisms in $\widetilde{EM}^{(\mathbb{V},\Delta)}$ is closed under composition.

For an A_∞-monad $(\mathbb{U},\Theta)$ we denote by $\widetilde{EM}^{eo}_{(\mathbb{U},\Theta)}$ the subcategory of $\widetilde{EM}_{(\mathbb{U},\Theta)}$ whose objects are even $(\mathbb{U},\Theta)$-modules with odd ∞-morphisms between them. Similarly, for an A_∞-comonad $(\mathbb{V},\Delta)$, we denote by $\widetilde{EM}^{(\mathbb{V},\Delta)}_{eo}$ the subcategory of $\widetilde{EM}^{(\mathbb{V},\Delta)}$ whose objects are even $(\mathbb{V},\Delta)$-comodules with odd ∞-morphisms between them. We now have the final result of this section.

3.8. Theorem. *Let* $(\mathbb{U} = \{\mathbb{U}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\geq 0}, \mathbb{V} = \{\mathbb{V}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\geq 0})$ *be a* $\mathbb{Z}$*-system of adjoints on* $\mathcal{C}$. *Let* $(\mathbb{U}, \Theta)$ *be an even* A_∞*-monad and* $(\mathbb{V}, \Delta)$ *be the corresponding* A_∞*-comonad. Then, the categories* $\widetilde{EM}^{eo}_{(\mathbb{U},\Theta)}$ *and* $\widetilde{EM}^{(\mathbb{V},\Delta)}_{eo}$ *are isomorphic.*

Proof. From Proposition 3.5, we already know that there is a one-one correspondence between the objects of $EM^{eo}_{(\mathbb{U},\Theta)}$ and $EM^{(\mathbb{V},\Delta)}_{eo}$. We consider therefore a morphism $\alpha : (M^1, \pi^1) \longrightarrow (M^2, \pi^2)$ in $EM^{eo}_{(\mathbb{U},\Theta)}$. Accordingly for any $k \geq 1$, $\bar{n} \in \mathbb{Z}^{k-1}$, $l \in \mathbb{Z}$ we have morphisms

$$(\alpha_k(\bar{n}, l) : \mathbb{U}_{\bar{n}} M^1_l \longrightarrow M^2_{\Sigma\bar{n}+l+(1-k)}) \Rightarrow (\alpha^R_k(\bar{n}, l) : M^1_l \longrightarrow \mathbb{V}_{\bar{n}^{op}} M^2_{\Sigma\bar{n}+l+(1-k)}) \tag{3.32}$$

From the proof of Proposition 3.5, we know that the object (P^1, ρ^1) (resp. (P^2, ρ^2)) in $EM^{(\mathbb{V},\Delta)}_{eo}$ corresponding to (M^1, π^1) (resp. (M^2, π^2)) is given by setting

$$\begin{aligned} &P^1_l := M^1_{-l} \qquad \rho^1_k(\bar{n}, l) : P^1_{\Sigma\bar{n}+l+(2-k)} = M^1_{-\Sigma\bar{n}-l-(2-k)} \xrightarrow{\pi^{1R}_k(\bar{n}^{op}, -\Sigma\bar{n}-l-(2-k))} \mathbb{V}_{\bar{n}} M^1_{-l} = \mathbb{V}_{\bar{n}} P^1_l \\ &P^2_l := M^2_{-l} \qquad \rho^2_k(\bar{n}, l) : P^2_{\Sigma\bar{n}+l+(2-k)} = M^2_{-\Sigma\bar{n}-l-(2-k)} \xrightarrow{\pi^{2R}_k(\bar{n}^{op}, -\Sigma\bar{n}-l-(2-k))} \mathbb{V}_{\bar{n}} M^2_{-l} = \mathbb{V}_{\bar{n}} P^2_l \end{aligned} \tag{3.33}$$

For $k \geq 1$, $\bar{n} \in \mathbb{Z}^{k-1}$, $l \in \mathbb{Z}$, we now define

$$\beta_k(\bar{n}, l) : P^1_{\Sigma\bar{n}+l+(1-k)} = M^1_{-\Sigma\bar{n}-l-(1-k)} \xrightarrow{\alpha^R_k(\bar{n}^{op}, -\Sigma\bar{n}-l-(1-k))} \mathbb{V}_{\bar{n}} M^2_{-l} = \mathbb{V}_{\bar{n}} P^2_l \tag{3.34}$$

We claim that (3.34) determines a morphism $(P^1, \rho^1) \longrightarrow (P^2, \rho^2)$ in $\widetilde{EM}^{(\mathbb{V},\Delta)}_{eo}$. From the proof of Theorem 2.3, we know that the collection $\Delta = \{\delta_k(\bar{n})\}$ is given by setting $\delta_k(\bar{n}) = \theta^R_k(\bar{n}^{op})$. In the notation of (3.28), we now see that for each $\bar{z} \in \mathbb{Z}^N$, $N \geq 0$, $l \in \mathbb{Z}$, we have

$$\begin{aligned} &\sum_{\bar{z}=(\bar{m},\bar{m}')} (-1)^{a+b\Sigma\bar{m}} \alpha_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) \circ \mathbb{U}_{\bar{m}}(\pi^1_b(\bar{m}', l)) \\ &+ \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} (-1)^{a+bc+b\Sigma\bar{n}} \alpha_{a+1+c}(\bar{n}, \Sigma\bar{n}' + (2-b), \bar{n}'', l) \circ (\mathbb{U}_{\bar{n}} * \theta_b(\bar{n}') * \mathbb{U}_{\bar{n}''})(M^1_l) \\ &\qquad = \sum_{\bar{z}=(\bar{p},\bar{p}')} \pi^2_{a+1}(\bar{p}, \Sigma\bar{p}' + l + (1-b)) \circ \mathbb{U}_{\bar{p}}(\alpha_b(\bar{p}', l)) \end{aligned} \tag{3.35}$$

We now consider one by one the terms in (3.35). For those in the first term on the left hand side of (3.35), we apply (3.16) with

$$\begin{aligned} &X = M^1_l \qquad Y = M^1_{\Sigma\bar{m}'+l+(2-b)} \qquad Z = M^2_{\Sigma\bar{z}+l+(1-N)} \qquad U_1 = \mathbb{U}_{\bar{m}'} \qquad U_2 = \mathbb{U}_{\bar{m}} \\ &f = \pi^1_b(\bar{m}', l) : U_1 X = \mathbb{U}_{\bar{m}'} M^1_l \longrightarrow M^1_{\Sigma\bar{m}'+l+(2-b)} = Y \\ &g = \alpha_{a+1}(\bar{m}, \Sigma\bar{m}' + l + (2-b)) : U_2 Y = \mathbb{U}_{\bar{m}} M^1_{\Sigma\bar{m}'+l+(2-b)} \longrightarrow M^2_{\Sigma\bar{z}+l+(1-N)} = Z \end{aligned} \tag{3.36}$$

It follows from (3.16) that the first term on the left hand side of (3.35) gives

$$
\begin{aligned}
&\sum_{\bar z=(\bar m,\bar m')}(-1)^{a+b\Sigma\bar m}(\alpha_{a+1}(\bar m,\Sigma\bar m'+l+(2-b))\circ\mathbb{U}_{\bar m}(\pi_b^1(\bar m',l)))^R\\
&=\sum_{\bar z=(\bar m,\bar m')}(-1)^{a+b\Sigma\bar m}\mathbb{V}_{\bar m'^{op}}(\alpha_{a+1}^R(\bar m,\Sigma\bar m'+l+(2-b)))\circ\pi_b^{1R}(\bar m',l)\\
&=\sum_{\bar z=(\bar m,\bar m')}(-1)^{a+b\Sigma\bar m}\mathbb{V}_{\bar m'^{op}}(\beta_{a+1}(\bar m^{op},-\Sigma\bar z-l-(1-N)))\circ\rho_b^1(\bar m'^{op},-\Sigma\bar m'-l-(2-b))\\
&=\sum_{\bar z=(\bar m,\bar m')}\mathbb{V}_{\bar m'^{op}}(\beta_{a+1}(\bar m^{op},l'))\circ\rho_b^1(\bar m'^{op},\Sigma\bar m+l'-a)
\end{aligned}
\tag{3.37}
$$

where we have set $l'=-\Sigma\bar z-l-(1-N)$, used the replacements in (3.33), (3.34) as well as the fact that a, b are both even. For those in the second term on the left hand side of (3.35), we apply (3.15) with

$$
\begin{gathered}
X=M_l^1\qquad Y=M^2_{\bar z+l+(1-N)}\\
\theta=\mathbb{U}_{\bar n}*\theta_b(\bar n')*\mathbb{U}_{\bar n''}:U_1=\mathbb{U}_{\bar n}\mathbb{U}_{\bar n'}\mathbb{U}_{\bar n''}\longrightarrow\mathbb{U}_{\bar n}\mathbb{U}_{\Sigma\bar n'+(2-b)}\mathbb{U}_{\bar n''}=U_2\\
f=\alpha_{a+1+c}(\bar n,\Sigma\bar n'+(2-b),\bar n'',l):U_2X=\mathbb{U}_{\bar n}\mathbb{U}_{\Sigma\bar n'+(2-b)}\mathbb{U}_{\bar n''}(M_l^1)\longrightarrow M^2_{\bar z+l+(1-N)}=Y
\end{gathered}
\tag{3.38}
$$

It follows from (3.15) that the second term on the left hand side of (3.35) gives

$$
\begin{aligned}
&\sum_{\bar z=(\bar n,\bar n',\bar n'')}(-1)^{a+bc+b\Sigma\bar n}(\alpha_{a+1+c}(\bar n,\Sigma\bar n'+(2-b),\bar n'',l)\circ(\mathbb{U}_{\bar n}*\theta_b(\bar n')*\mathbb{U}_{\bar n''})(M_l^1))^R\\
&=\sum_{\bar z=(\bar n,\bar n',\bar n'')}(-1)^{a+bc+b\Sigma\bar n}(\mathbb{V}_{\bar n''^{op}}*\theta_b^R(\bar n')*\mathbb{V}_{\bar n^{op}})(M^2_{\bar z+l+(1-N)})\circ\alpha^R_{a+1+c}(\bar n,\Sigma\bar n'+(2-b),\bar n'',l)\\
&=\sum_{\bar z=(\bar n,\bar n',\bar n'')}(-1)^{a}(\mathbb{V}_{\bar n''^{op}}*\delta_b(\bar n'^{op})*\mathbb{V}_{\bar n^{op}})(P^2_{l'})\circ\beta_{a+1+c}(\bar n''^{op},\Sigma\bar n'+(2-b),\bar n^{op},l')
\end{aligned}
\tag{3.39}
$$

where we have set $l'=-\Sigma\bar z-l-(1-N)$, used the replacements in (3.33), (3.34) as well as the fact that b, $a+c$ are both even. For those on the right hand side of (3.35), we apply (3.16) with

$$
\begin{gathered}
X=M_l^1\qquad Y=M^2_{\Sigma\bar p'+l+(1-b)}\qquad Z=M^2_{\Sigma\bar z+l+(1-N)}\qquad U_1=\mathbb{U}_{\bar p'}\qquad U_2=\mathbb{U}_{\bar p}\\
f=\alpha_b(\bar p',l):U_1X=\mathbb{U}_{\bar p'}M_l^1\longrightarrow M^2_{\Sigma\bar p'+l+(1-b)}=Y\\
g=\pi^2_{a+1}(\bar p,\Sigma\bar p'+l+(1-b)):U_2Y=\mathbb{U}_{\bar p}M^2_{\Sigma\bar p'+l+(1-b)}\longrightarrow M^2_{\Sigma\bar z+l+(1-N)}=Z
\end{gathered}
\tag{3.40}
$$

It follows from (3.16) that the the right hand side of (3.35) gives

$$
\begin{aligned}
&\sum_{\bar z=(\bar p,\bar p')}(\pi^2_{a+1}(\bar p,\Sigma\bar p'+l+(1-b))\circ\mathbb{U}_{\bar p}(\alpha_b(\bar p',l)))^R\\
&=\sum_{\bar z=(\bar p,\bar p')}\mathbb{V}_{\bar p'^{op}}(\pi^{2R}_{a+1}(\bar p,\Sigma\bar p'+l+(1-b)))\circ\alpha_b^R(\bar p',l)\\
&=\sum_{\bar z=(\bar p,\bar p')}\mathbb{V}_{\bar p'^{op}}(\rho^2_{a+1}(\bar p^{op},l'))\circ\beta_b(\bar p'^{op},\Sigma\bar p+l'+(1-a))
\end{aligned}
\tag{3.41}
$$

where we have set $l'=-\Sigma\bar z-l-(1-N)$ and used the replacements in (3.33), (3.34). By inspecting (3.37), (3.39) and (3.41) and comparing with (3.31), we conclude that

$\{\beta_k(\bar{n}, l)\}$ together determine a morphism $(P^1, \rho^1) \longrightarrow (P^2, \rho^2)$ in $\widetilde{EM}_{eo}^{(\mathbb{V},\Delta)}$. Finally, these arguments can be reversed, which shows that there is an isomorphism between categories $\widetilde{EM}^{eo}_{(\mathbb{U},\Theta)}$ and $\widetilde{EM}_{eo}^{(\mathbb{V},\Delta)}$. ∎

4. Locally finite comodules over A_∞-comonads

We suppose throughout that $(\mathbb{V}, \Delta)$ is an A_∞-comonad. We now define what it means for a $(\mathbb{V}, \Delta)$-comodule to be locally finite.

4.1. Definition. *Let (P, ρ) be a $(\mathbb{V}, \Delta)$-comodule. In the notation of Definition 3.2, we will say that (P, ρ) is locally finite if for any $T \in \mathbb{Z}$ and $k \geq 1$, the family of morphisms*

$$\left\{\rho_k(\bar{n}, l) : P_{T+(2-k)} \longrightarrow \mathbb{V}_{\bar{n}} P_l\right\}_{(\bar{n},l)\in\mathbb{Z}(T,k)} \tag{4.1}$$

in $\mathcal{C}$ is locally finite.

In the rest of this section, we suppose that each $\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ preserves monomorphisms. Given a $(\mathbb{V}, \Delta)$-comodule (P, ρ), we now consider, in the notation of Definition 3.2, a family of subobjects $\{P_n^\alpha \hookrightarrow P_n\}_{n\in\mathbb{Z}}$ satisfying the following conditions
(a) For any $\bar{n} \in \mathbb{Z}^{k-1}$, $l \in \mathbb{Z}$, the morphisms in (3.4) restrict to the corresponding subobjects

$$\rho^\alpha = \{\rho_k^\alpha(\bar{n}, l) : P^\alpha_{\Sigma\bar{n}+l+(2-k)} \longrightarrow \mathbb{V}_{\bar{n}} P_l^\alpha \mid k \geq 1,\, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1},\, l \in \mathbb{Z}\ \} \tag{4.2}$$

(b) For any $T \in \mathbb{Z}$ and $k \geq 1$, the family of morphisms

$$\left\{\rho_k^\alpha(\bar{n}, l) : P^\alpha_{T+(2-k)} \longrightarrow \mathbb{V}_{\bar{n}} P_l^\alpha\right\}_{(\bar{n},l)\in\mathbb{Z}(T,k)} \tag{4.3}$$

is locally finite in $\mathcal{C}$.

4.2. Lemma. *(P^α, ρ^α) is a locally finite $(\mathbb{V}, \Delta)$-comodule.*

Proof. Because the morphisms $\rho = \{\rho_k(\bar{n}, l) \mid k \geq 1,\, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1},\, l \in \mathbb{Z}\ \}$ in (3.4) restrict to

$$\rho^\alpha = \{\rho_k^\alpha(\bar{n}, l) \mid k \geq 1,\, \bar{n} = (n_1, ..., n_{k-1}) \in \mathbb{Z}^{k-1},\, l \in \mathbb{Z}\ \} \tag{4.4}$$

the morphisms $\rho^\alpha(\bar{n}, l)$ also satisfy the relations in (3.5). The local finiteness condition is clear from the assumption in (4.3). ∎

We now let $\{(P^\alpha,\rho^\alpha)\}_{\alpha\in A}$ denote the collection of such families of subobjects. We note that this collection is non-empty since it contains the zero family. We now define

$$P^\oplus = \left\{ P_n^\oplus := \sum_{\alpha\in A} P_n^\alpha \right\}_{n\in\mathbb{Z}} \tag{4.5}$$

4.3. Proposition. *The collection $P^\oplus = \{P_n^\oplus\}_{n\in\mathbb{Z}}$ is a locally finite A_∞-comodule over $(\mathbb{V},\Delta)$.*

Proof. Since each $\mathbb{V}_n$ preserves monomorphisms, for $\bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1}$ and $l\in\mathbb{Z}$, we have induced maps

$$\rho_k^\oplus(\bar{n},l) : P^\oplus_{\Sigma\bar{n}+l+(2-k)} = \sum_{\alpha\in A} P^\alpha_{\Sigma\bar{n}+l+(2-k)} \xrightarrow{\sum\limits_{\alpha\in A}\rho_k^\alpha(\bar{n},l)} \sum_{\alpha\in A}\mathbb{V}_{\bar{n}}P_l^\alpha \hookrightarrow \mathbb{V}_{\bar{n}}\left(\sum_{\alpha\in A}P_l^\alpha\right) = \mathbb{V}_{\bar{n}}(P_l^\oplus) \tag{4.6}$$

Again since the morphisms in (4.6) are restrictions of $\rho_k(\bar{n},l)$ to subobjects, it follows that they also satisfy the relations in (3.5). This makes $(P^\oplus,\rho^\oplus)$ a $(\mathbb{V},\Delta)$-comodule. Since each (P^α,ρ^α) is locally finite, we have for each $\alpha\in A$, $T\in\mathbb{Z}$ and $k\geq 1$, the following factorisation

$$\begin{tikzcd}
P^\alpha_{T+(2-k)} \arrow[r] \arrow[dr] & \left(\prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}}P_l^\alpha\right) \arrow[r] & \left(\prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}}P_l^\oplus\right) \\
 & \left(\bigoplus\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}}P_l^\alpha\right) \arrow[u] \arrow[r] & \left(\bigoplus\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}}P_l^\oplus\right) \arrow[u]
\end{tikzcd} \tag{4.7}$$

From (4.7), it follows that each of the compositions

$$\{P^\alpha_{T+(2-k)} \longrightarrow P^\oplus_{T+(2-k)} \longrightarrow \prod_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}}P_l^\oplus\}_{\alpha\in A} \tag{4.8}$$

factors through the subobject $\bigoplus\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}}P_l^\oplus$. Since $\mathcal{C}$ is abelian and $P^\oplus_{T+(2-k)} = \sum_{\alpha\in A}P^\alpha_{T+(2-k)}$, the result follows. ∎

Let (P,ρ^P) and (Q,ρ^Q) be $(\mathbb{V},\Delta)$-comodules. By Definition 3.2, a morphism $g:(P,\rho^P)\longrightarrow(Q,\rho^Q)$ in $EM^{(\mathbb{V},\Delta)}$ consists of a collection of morphisms $g=\{g_l:P_l\longrightarrow Q_l\}_{l\in\mathbb{Z}}$ such that

$$\begin{tikzcd}
P_{\Sigma\bar{n}+l+(2-k)} \arrow[r, "g_{\Sigma\bar{n}+l+(2-k)}"] \arrow[d, "\rho_k^P(\bar{n},l)"'] & Q_{\Sigma\bar{n}+l+(2-k)} \arrow[d, "\rho_k^Q(\bar{n},l)"] \\
\mathbb{V}_{\bar{n}}P_l \arrow[r, "\mathbb{V}_{\bar{n}}(g_l)"] & \mathbb{V}_{\bar{n}}Q_l
\end{tikzcd} \tag{4.9}$$

is commutative for any $\bar{n}\in\mathbb{Z}^{k-1}$, $k\geq 1$, $l\in\mathbb{Z}$.

4.4. Lemma. *Let* $(\mathbb{V},\Delta)$ *be an* A_∞*-comonad such that each* $\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ *is exact. Suppose that* $g:(P,\rho^P)\longrightarrow(Q,\rho^Q)$ *is a morphism in* $EM^{(\mathbb{V},\Delta)}$*. Then,*

$$R=\{R_l:=Ker(g_l:P_l\longrightarrow Q_l)\}_{l\in\mathbb{Z}}\qquad S=\{S_l:=Cok(g_l:P_l\longrightarrow Q_l)\}_{l\in\mathbb{Z}}\tag{4.10}$$

are canonically equipped with the structure of $(\mathbb{V},\Delta)$*-comodules.*

Proof. Since the functors $\mathbb{V}_{\bar{n}}$ are all exact, it is clear from (4.9) that we have induced morphisms

$$\rho_k^R(\bar{n},l):R_{\Sigma\bar{n}+l+(2-k)}\longrightarrow\mathbb{V}_{\bar{n}}R_l\qquad\rho_k^S(\bar{n},l):S_{\Sigma\bar{n}+l+(2-k)}\longrightarrow\mathbb{V}_{\bar{n}}S_l\tag{4.11}$$

Again since the functors $\mathbb{V}_{\bar{n}}$ are all exact, it follows that the morphisms in (4.11) satisfy the conditions in (3.5) for R and S to be $(\mathbb{V},\Delta)$-comodules.

■

4.5. Lemma. *Let* $(\mathbb{V},\Delta)$ *be an* A_∞*-comonad such that each* $\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ *is exact. Suppose that* $g:(P,\rho^P)\longrightarrow(Q,\rho^Q)$ *is a morphism in* $EM^{(\mathbb{V},\Delta)}$ *such that each* $g_l:P_l\longrightarrow Q_l$ *is an epimorphism. Then if* P *is locally finite, so is* Q*.*

Proof. For any $T\in\mathbb{Z}$ and $k\geq 1$, we consider the families

$$\{\rho_k^P(\bar{n},l):P_{T+(2-k)}\longrightarrow\mathbb{V}_{\bar{n}}P_l\}_{(\bar{n},l)\in\mathbb{Z}(T,k)}\qquad\{\rho_k^Q(\bar{n},l):Q_{T+(2-k)}\longrightarrow\mathbb{V}_{\bar{n}}Q_l\}_{(\bar{n},l)\in\mathbb{Z}(T,k)}\tag{4.12}$$

It is immediate that $\left(\prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\rho_k^Q(\bar{n},l)\right)\circ g_{T+(2-k)}=\left(\prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}(g_l)\right)\circ\left(\prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\rho_k^P(\bar{n},l)\right)$.

■

We now consider the commutative diagram

$$\begin{tikzcd}
R_{T+(2-k)} \arrow[r, "\upsilon_{T+(2-k)}"] & P_{T+(2-k)} \arrow[r, "\bigoplus\rho_k^P(\bar{n},l)"] \arrow[d, "g_{T+(2-k)}"'] & \bigoplus\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}P_l \arrow[r, "\iota_{T,k}^P"] \arrow[d, "\bigoplus\mathbb{V}_{\bar{n}}(g_l)"] & \prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}P_l \arrow[d, "\prod\mathbb{V}_{\bar{n}}(g_l)"] \\
 & Q_{T+(2-k)} & \bigoplus\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}Q_l \arrow[r, "\iota_{T,k}^Q"] & \prod\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)}\mathbb{V}_{\bar{n}}Q_l
\end{tikzcd}\tag{4.13}$$

where $R_{T+(2-k)}$ is the kernel of $g_{T+(2-k)}:P_{T+(2-k)}\longrightarrow Q_{T+(2-k)}$ and the factorization $\iota_{T,k}^P\circ\left(\bigoplus\rho_k^P(\bar{n},l)\right)=\prod\rho_k^P(\bar{n},l)$ is due to the fact that (P,ρ^P) is locally finite. From (4.13) we see that

$$\begin{aligned}\iota_{T,k}^Q\circ(\bigoplus\mathbb{V}_{\bar{n}}(g_l))\circ\left(\bigoplus\rho_k^P(\bar{n},l)\right)\circ\upsilon_{T+(2-k)}&=(\prod\mathbb{V}_{\bar{n}}(g_l))\circ\iota_{T,k}^P\circ\left(\bigoplus\rho_k^P(\bar{n},l)\right)\circ\upsilon_{T+(2-k)}\\&=(\prod\mathbb{V}_{\bar{n}}(g_l))\circ\left(\prod\rho_k^P(\bar{n},l)\right)\circ\upsilon_{T+(2-k)}\\&=\left(\prod\rho_k^Q(\bar{n},l)\right)\circ g_{T+(2-k)}\circ\upsilon_{T+(2-k)}=0\end{aligned}\tag{4.14}$$

Since $\iota^Q_{T,k}$ is a monomorphism, it follows from (4.14) that $(\bigoplus \mathbb{V}_{\bar{n}}(g_l)) \circ \left(\bigoplus \rho^P_k(\bar{n}, l)\right) \circ \upsilon_{T+(2-k)} = 0$. Since $R_{T+(2-k)}$ is the kernel of the epimorphism $g_{T+(2-k)} : P_{T+(2-k)} \longrightarrow Q_{T+(2-k)}$, we have an induced map $Q_{T+(2-k)} \longrightarrow \bigoplus\limits_{(\bar{n},l)\in\mathbb{Z}(T,k)} \mathbb{V}_{\bar{n}} Q_l$ which fits into the commutative diagram (4.13). Again since $g_{T+(2-k)}$ is an epimorphism, it is easily seen that this gives a factorization of $\prod \rho^Q_k(\bar{n}, l)$.

4.6. Theorem. *Let $(\mathbb{V}, \Delta)$ be an A_∞-comonad such that each $\{\mathbb{V}_n\}_{n\in\mathbb{Z}}$ is exact. Then, the association $Q \mapsto Q^\oplus$ determines a functor from $EM^{(\mathbb{V},\Delta)}$ to the full subcategory $EM^{(\mathbb{V},\Delta)}_{<\infty}$ of comodules in $EM^{(\mathbb{V},\Delta)}$ that are locally finite. Further, this functor is right adjoint to the inclusion $EM^{(\mathbb{V},\Delta)}_{<\infty} \hookrightarrow EM^{(\mathbb{V},\Delta)}$, i.e., there are natural isomorphisms*

$$EM^{(\mathbb{V},\Delta)}(P, Q) \cong EM^{(\mathbb{V},\Delta)}_{<\infty}(P, Q^\oplus) \tag{4.15}$$

for $P \in EM^{(\mathbb{V},\Delta)}_{<\infty}$ and $Q \in EM^{(\mathbb{V},\Delta)}$.

Proof. We consider some $P \in EM^{(\mathbb{V},\Delta)}_{<\infty}$, $Q \in EM^{(\mathbb{V},\Delta)}$ and a morphism $g : P \longrightarrow Q$ in $EM^{(\mathbb{V},\Delta)}$. Applying Lemma 4.4, we can consider the $(\mathbb{V}, \Delta)$-comodule R determined by setting

$$P_n \twoheadrightarrow R_n := Im(g_n : P_n \longrightarrow Q_n) \hookrightarrow Q_n \qquad n \in \mathbb{Z} \tag{4.16}$$

Since each $P_n \twoheadrightarrow R_n$ is an epimorphism and P is locally finite, it follows from Lemma 4.5 that R is also locally finite. As such, the family $\{R_n \hookrightarrow Q_n\}_{n\in\mathbb{Z}}$ of subobjects determines a locally finite $(\mathbb{V}, \Delta)$-comodule. From the definition in (4.5), it is now clear that each $R_n \subseteq Q^\oplus_n$.

It remains to show that the association $Q \mapsto Q^\oplus$ is a functor. For this, we consider a morphism $Q' \longrightarrow Q$ in $EM^{(\mathbb{V},\Delta)}$. Then $Q'^\oplus \in EM^{(\mathbb{V},\Delta)}_{<\infty}$ and if we apply the above reasoning to the composition $Q'^\oplus \longrightarrow Q' \longrightarrow Q$ in $EM^{(\mathbb{V},\Delta)}$, we see that there is an induced morphism $Q'^\oplus \longrightarrow Q^\oplus$. This proves the result. ∎

5. Bar construction for A_∞-monads

In this section, we will obtain the bar construction $Bar(\mathbb{U}, \Theta)$ of an A_∞-monad $(\mathbb{U}, \Theta)$, which gives a differential graded comonad on $\mathcal{C}$. Further, we show that morphisms to $Bar(\mathbb{U}, \Theta)$ from a conilpotent dg-comonad correspond to families of natural transformations that behave in a manner similar to the classical case of twisting morphisms (see, for instance, [27, § 3], [28, § 4], [32, § 2]). We mention here that the cobar construction of an A_∞-comonad can be obtained in an analogous manner. We begin with the following definition.

5.1. Definition. *A graded comonad $(\mathbb{W}, \delta_2)$ on $\mathcal{C}$ is given by the following*
(a) A collection $\{\mathbb{W}_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ of endofunctors on $\mathcal{C}$.
(b) A collection of natural transformations

$$\delta_2 = \{\delta_2(n_1, n_2) : \mathbb{W}_{n_1+n_2} \longrightarrow \mathbb{W}_{n_1} \circ \mathbb{W}_{n_2} \mid (n_1, n_2) \in \mathbb{Z}^2\} \tag{5.1}$$

such that for each $T \in \mathbb{Z}$, the induced morphism

$$\prod_{n_1+n_2=T} \delta_2(n_1,n_2) : \mathbb{W}_T \longrightarrow \prod_{n_1+n_2=T} \mathbb{W}_{n_1} \circ \mathbb{W}_{n_2} \tag{5.2}$$

factors through the direct sum $\bigoplus_{n_1+n_2=T} \mathbb{W}_{n_1} \circ \mathbb{W}_{n_2}$ *and for every n_1, n_2, $n_3 \in \mathbb{Z}$, we have*

$$(\mathbb{W}_{n_1} * \delta_2(n_2,n_3))\delta_2(n_1,n_2+n_3) = (\delta_2(n_1,n_2) * \mathbb{W}_{n_3})\delta_2(n_1+n_2,n_3) : \mathbb{W}_{n_1+n_2+n_3} \longrightarrow \mathbb{W}_{n_1} \circ \mathbb{W}_{n_2} \circ \mathbb{W}_{n_3} \tag{5.3}$$

A morphism $\beta : (\mathbb{W}, \delta_2) \longrightarrow (\mathbb{W}', \delta_2')$ of graded comonads on $\mathcal{C}$ is a collection of natural transformations $\beta = \{\beta_n : \mathbb{W}_n \longrightarrow \mathbb{W}'_n\}_{n \in \mathbb{Z}}$ such that $(\beta_{n_1} * \beta_{n_2})\delta_2(n_1,n_2) = \delta_2'(n_1,n_2)\beta_{n_1+n_2}$ for n_1, $n_2 \in \mathbb{Z}$. We will denote by $gr - Comon(\mathcal{C})$ the category of graded comonads over $\mathcal{C}$.

A differential graded comonad (or dg-comonad) $(\mathbb{W}, \delta_1, \delta_2)$ on $\mathcal{C}$ consists of a graded comonad $(\mathbb{W}, \delta_2)$ as well as a collection of natural transformations

$$\delta_1 = \{\delta_1(n) : \mathbb{W}_n \longrightarrow \mathbb{W}_{n+1} \mid n \in \mathbb{Z}\}$$

satisfying the following conditions (for n, n_1, $n_2 \in \mathbb{Z}$)

$$\begin{gathered} \delta_1(n+1)\delta_1(n) = 0 \\ \delta_2(n_1,n_2)\delta_1(n_1+n_2-1) = (\delta_1(n_1-1) * \mathbb{W}_{n_2})\delta_2(n_1-1,n_2) + (-1)^{n_1}(\mathbb{W}_{n_1} * \delta_1(n_2-1))\delta_2(n_1,n_2-1) \end{gathered} \tag{5.4}$$

as maps $\mathbb{W}_{n_1+n_2-1} \longrightarrow \mathbb{W}_{n_1} \circ \mathbb{W}_{n_2}$.

A morphism $\beta : (\mathbb{W}, \delta_1, \delta_2) \longrightarrow (\mathbb{W}', \delta_1', \delta_2')$ of dg-comonads on $\mathcal{C}$ is a collection of natural transformations $\beta = \{\beta_n : \mathbb{W}_n \longrightarrow \mathbb{W}'_n\}_{n \in \mathbb{Z}}$ such that

$$\delta_1'(n)\beta_n = \beta_{n+1}\delta_1(n) \qquad (\beta_{n_1} * \beta_{n_2})\delta_2(n_1,n_2) = \delta_2'(n_1,n_2)\beta_{n_1+n_2} \tag{5.5}$$

for n, n_1, $n_2 \in \mathbb{Z}$. We will denote by $dg - Comon(\mathcal{C})$ the category of dg-comonads over $\mathcal{C}$.

If $(\mathbb{W}, \delta_2)$ is a graded comonad on $\mathcal{C}$, we set for any $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, $k \geq 1$

$$\delta_2(\bar{n}) := (\mathbb{W}_{n_1} * ... * \mathbb{W}_{n_{k-2}} * \delta_2(n_{k-1},n_k))...(\mathbb{W}_{n_1} * \delta_2(n_3 + ... + n_k))\delta_2(n_1, n_2 + n_3 + ... + n_k) : \mathbb{W}_{\Sigma\bar{n}} \longrightarrow \mathbb{W}_{n_1} \circ ... \circ \mathbb{W}_{n_k} = \mathbb{W}_{\bar{n}} \tag{5.6}$$

We note that due to the coassociativity condition in (5.3), the natural transformation $\delta_2(\bar{n})$ appearing in (5.6) can be expressed in a number of equivalent ways, one for each way the sum $(n_1 + ... + n_k)$ can be "broken up" into the partition $(n_1, ..., n_k)$.

From the conditions in Definition 5.1, we know that for each $T \in \mathbb{Z}$ and $k \geq 1$, the morphism $\prod_{\bar{n} \in \mathbb{Z}(T,k)} \delta_2(\bar{n}) : \mathbb{W}_T \longrightarrow \prod_{\bar{n} \in \mathbb{Z}(T,k)} \mathbb{W}_{\bar{n}}$ factors through the direct sum $\bigoplus_{\bar{n} \in \mathbb{Z}(T,k)} \mathbb{W}_{\bar{n}}$.

We now have the following definition.

5.2. Definition. *Let $(\mathbb{W},\delta_2)$ be a graded comonad over $\mathcal{C}$. We will say that $(\mathbb{W},\delta_2)$ is conilpotent if for each object $M\in\mathcal{C}$ and $L\in\mathbb{Z}$, we have*

$$\mathbb{W}_L(M)=\bigcup_{k\geq 1}Ker\left(\mathbb{W}_L(M)\xrightarrow{\bigoplus\limits_{\bar{n}\in\mathbb{Z}(T,k)}\delta_2(\bar{n})(M)}\bigoplus_{\bar{n}\in\mathbb{Z}(L,k)}\mathbb{W}_{\bar{n}}(M)\right)\tag{5.7}$$

We now let $(\mathbb{U},\Theta)$ be an A_∞-monad, and let $(\mathbb{W},\delta_1,\delta_2)$ be a conilpotent dg-comonad over $\mathcal{C}$. For each $n\in\mathbb{Z}$, we set

$$A_n:=\prod_{k\in\mathbb{Z}}Nat(\mathbb{W}_k,\mathbb{U}_{k+n})=\{\zeta=(\zeta^{(k)})_{k\in\mathbb{Z}}\mid\zeta^{(k)}\in Nat(\mathbb{W}_k,\mathbb{U}_{k+n})\}\tag{5.8}$$

where $Nat(\mathbb{W}_k,\mathbb{U}_{k+n})$ is the collection of natural transformations from $\mathbb{W}_k$ to $\mathbb{U}_{k+n}$. We set $A:=\bigoplus\limits_{n\in\mathbb{Z}}A_n$. We now have maps $\{m_k:A^{\otimes k}\longrightarrow A\}_{k\geq 1}$ with m_k of degree $(2-k)$, given by components

$$\begin{gathered}m_k(\bar{n}):A_{n_1}\otimes A_{n_2}\otimes\ldots\otimes A_{n_k}\longrightarrow A_{\Sigma\bar{n}+(2-k)}\\(\zeta_{n_1}\otimes\ldots\otimes\zeta_{n_k})\mapsto\left(\sum_{\bar{t}\in\mathbb{Z}(l,k)}\theta_k(\bar{t}+\bar{n})(\zeta_{n_1}^{(t_1)}*\ldots*\zeta_{n_k}^{(t_k)})\delta_2(\bar{t})\right)_{l\in\mathbb{Z}}\end{gathered}\tag{5.9}$$

for $\bar{n}=(n_1,n_2,...,n_k)\in\mathbb{Z}^k$. In (5.9), we have suppressed the sign of the summand $\theta_k(\bar{t}+\bar{n})(\zeta_{n_1}^{(t_1)}*\ldots*\zeta_{n_k}^{(t_k)})\delta_2(\bar{t})$, which is $(-1)^{t_1(n_2+\ldots+n_k)+t_2(n_3+\ldots+n_k)+\ldots.+t_{k-1}n_k}$ as given by Koszul sign rule. For ease of notation, we set $A_{\bar{n}}:=A_{n_1}\otimes A_{n_2}\otimes\ldots\otimes A_{n_k}$. Accordingly, an element $(\zeta_{n_1}\otimes\ldots\otimes\zeta_{n_k})\in A_{n_1}\otimes A_{n_2}\otimes\ldots\otimes A_{n_k}=A_{\bar{n}}$ will often be denoted by $\zeta_{\bar{n}}$. Similarly, we write $\zeta_{\bar{n}}^{(\bar{t})}=\zeta_{n_1}^{(t_1)}*\ldots*\zeta_{n_k}^{(t_k)}$ for $\bar{n}$, $\bar{t}\in\mathbb{Z}^k$. We now have the following result.

5.3. Proposition. *The collection $(A,\{m_k:A^{\otimes k}\longrightarrow A\}_{k\geq 1})$ is an A_∞-algebra.*

Proof. We choose $\bar{z}\in\mathbb{Z}^N$ for some $N\geq 1$ and consider $\zeta_{\bar{z}}\in A_{\bar{z}}$. In the following, we have suppressed the signs for sake of convenience. With the sum running over partitions $\bar{z}=(\bar{n},\bar{n}',\bar{n}'')$ with $|\bar{n}|=p$, $|\bar{n}'|=q$, $|\bar{n}''|=r$, we see that we have for any $l\in\mathbb{Z}$

$$\begin{aligned}&\left(\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}m_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')(id\otimes m_q(\bar{n}')\otimes id)(\zeta_{\bar{z}})\right)^{(l)}\\&=\left(\sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}m_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')(\zeta_{\bar{n}}\otimes m_q(\bar{n}')(\zeta_{\bar{n}'})\otimes\zeta_{\bar{n}''})\right)^{(l)}\\&=\sum_{\bar{t}\in\mathbb{Z}(l,p+1+r)}\ \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}\theta_{p+1+r}(\bar{t}+(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}''))(\zeta_{n_1}^{(t_1)}*\ldots*\zeta_{n_p}^{(t_p)}*m_q(\bar{n}')(\zeta_{\bar{n}'})^{(t_{p+1})}*\zeta_{\bar{n}''}^{(t_{p+1+r})})\delta_2(\bar{t})\\&=\sum_{\bar{t}\in\mathbb{Z}(l,N)}\ \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}\theta_{p+1+r}((t_1,...,t_p,\textstyle\sum_{y=1}^{q}t_{p+y},t_{p+q+1},...,t_{p+q+r})\\&\quad+(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}''))(\mathbb{U}_{\bar{n}}*\theta_q((t_{p+1},...,t_{p+q})+\bar{n}')*\mathbb{U}_{\bar{n}''})(\zeta_{\bar{z}}^{(\bar{t})})\delta_2(\bar{t})\\&=\sum_{\bar{t}\in\mathbb{Z}(l,N)}\ \sum_{\bar{z}+\bar{t}=(\bar{n},\bar{n}',\bar{n}'')}\theta_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')(\mathbb{U}_{\bar{n}}*\theta_q(\bar{n}')*\mathbb{U}_{\bar{n}''})(\zeta_{\bar{z}}^{(\bar{t})})\delta_2(\bar{t})=0\end{aligned}$$

∎

As in [27, § 3.6], we recall here that any A_∞-algebra $(A, \{m_k : A^{\otimes k} \longrightarrow A\}_{k\geq 1})$ can be equivalently described using structure maps $(sA, \{b_k : (sA)^{\otimes k} \longrightarrow (sA)\}_{k\geq 1})$ given by

$$b_k(\bar{n}) := (-1)^{n_1(k-1)+...+n_{k-1}} m_k(n_1+1, ..., n_k+1) : (sA)_{n_1} \otimes ... \otimes (sA)_{n_k} \longrightarrow (sA)_{\Sigma\bar{n}+1} \tag{5.10}$$

for $\bar{n} = (n_1, n_2, ..., n_k) \in \mathbb{Z}^k$, where $sA = \{(sA)_j := A_{j+1}\}_{j\in\mathbb{Z}}$ is the suspension of A.
We will need the following basic example of a conilpotent graded comonad for describing the bar construction. Let $F = \{F_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ be a family of endofunctors. As always, for any $k \geq 1$ and $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, we set $F_{\bar{n}} := F_{n_1} \circ ... \circ F_{n_k}$. We now consider the graded comonad $\mathbb{T}^c(F)$ given by setting

$$\mathbb{T}^c(F)_L := \bigoplus_{\bar{n}\in\mathbb{Z}(L,k), k\geq 1} F_{\bar{n}} \qquad L \in \mathbb{Z} \tag{5.11}$$

The "comultiplication" $\delta_2^F : \mathbb{T}^c(F) \longrightarrow \mathbb{T}^c(F) \circ \mathbb{T}^c(F)$ is given on components by means of identity maps $F_{\bar{n}} \longrightarrow F_{\bar{n}_1} \circ F_{\bar{n}_2}$ for each partition $\bar{n} = (\bar{n_1}, \bar{n_2})$ of $\bar{n}$. If we consider some $M \in \mathcal{C}$, we see that $\delta_2^F(\bar{t})(M)|_{F_{\bar{n}}(M)} = 0$ for any $\bar{t}$ such that $|\bar{t}| > |\bar{n}|$. From the condition in (5.7), it is now clear that $(\mathbb{T}^c(F), \delta_2^F)$ is a conilpotent graded comonad.

5.4. PROPOSITION. *Let $(\mathbb{W}, \delta_2)$ be a conilpotent graded comonad over $\mathcal{C}$ and $F = \{F_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ be a family of endofunctors. Then, there is a one-one correspondence*

$$\prod_{L\in\mathbb{Z}} Nat(\mathbb{W}_L, F_L) \underset{h}{\overset{g}{\longleftrightarrow}} gr - Comon(\mathcal{C})((\mathbb{W}, \delta_2), (\mathbb{T}^c(F), \delta_2^F)) \tag{5.12}$$

PROOF. For $\zeta = \{\zeta^{(L)}\}_{L\in\mathbb{Z}} \in \prod_{L\in\mathbb{Z}} Nat(\mathbb{W}_L, F_L)$, we get $\chi := h(\zeta)$ by setting for each $M \in \mathcal{C}$

$$\chi_L(M) = \bigoplus_{\bar{n}\in\mathbb{Z}(L,k),k\geq 1} \chi_{\bar{n}}(M) = \bigoplus_{\bar{n}\in\mathbb{Z}(L,k),k\geq 1} \zeta^{(\bar{n})}\delta_2(\bar{n})(M) : \mathbb{W}_L(M) \longrightarrow \mathbb{T}^c(F)_L(M) = \bigoplus_{\bar{n}\in\mathbb{Z}(L,k),k\geq 1} F_{\bar{n}}(M) \tag{5.13}$$

where $\zeta^{(\bar{n})} = \zeta^{(n_1)} * ... * \zeta^{(n_k)}$ in (5.13) for $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$. It is clear from the conilpotence condition in (5.7) that the morphism in (5.13) is well defined. On the other hand, for $\chi \in gr - Comon(\mathcal{C})((\mathbb{W}, \delta_2), (\mathbb{T}^c(F), \delta_2^F))$, we get $\zeta := g(\chi)$ by setting $\zeta^{(L)}$ to be the composition of $\chi_L(M) : \mathbb{W}_L(M) \longrightarrow \mathbb{T}^c(F)_L(M)$ with the projection to $F_L(M)$ for each $M \in \mathcal{C}$. It may be verified that these two associations are inverse to each other. ■

5.5. LEMMA. *Let $F = \{F_n : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ be a family of endofunctors. Then, there is a one-one correspondence between the following*
(a) Families of natural transformations

$$\Gamma = \{\gamma_k(\bar{n}) : F_{\bar{n}} = F_{n_1} \circ ... \circ F_{n_k} \longrightarrow F_{\Sigma\bar{n}+1} \mid k \geq 1 \ \bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k\} \tag{5.14}$$

(b) Families of natural transformations $\delta_1^F = \{\delta_1^F(L) : \mathbb{T}^c(F)_L \longrightarrow \mathbb{T}^c(F)_{L+1} \mid L \in \mathbb{Z}\}$ which satisfy

$$\begin{aligned} \delta_2^F(L_1, L_2)\delta_1^F(L_1 + L_2 - 1) &= (\delta_1^F(L_1 - 1) * \mathbb{T}^c(F)_{L_2})\delta_2^F(L_1 - 1, L_2) \\ &\quad + (-1)^{L_1}(\mathbb{T}^c(F)_{L_1} * \delta_1^F(L_2 - 1))\delta_2^F(L_1, L_2 - 1) \end{aligned} \tag{5.15}$$

for $L_1,\ L_2 \in \mathbb{Z}$.

PROOF. Given a family Γ as in (a), we set for each $\bar{z}\in\mathbb{Z}^N$, $N\geq 1$:

$$\delta_1^F(\Sigma\bar{z})|_{F_{\bar{z}}} := \sum_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')}(-1)^{\Sigma\bar{n}}(F_{\bar{n}}*\gamma_{|\bar{n}'|}(\bar{n}')*F_{\bar{n}''}) : F_{\bar{z}}\longrightarrow \bigoplus_{\bar{z}=(\bar{n},\bar{n}',\bar{n}'')} F_{\bar{n}}\circ F_{\Sigma\bar{n}'+1}\circ F_{\bar{n}''}\hookrightarrow \mathbb{T}^c(\mathbb{F})_{\Sigma\bar{z}+1} \tag{5.16}$$

where the sum in (5.16) is taken over all partitions $\bar{z}=(\bar{n},\bar{n}',\bar{n}'')$. It may be verified that the family δ_1^F satisfies the condition in (5.15). Conversely, given a family δ_1^F as in (b), we set for each $\bar{z}\in\mathbb{Z}^N$, $N\geq 1$

$$\gamma_{|\bar{z}|}(\bar{z}) : F_{\bar{z}} \xrightarrow{\delta_1^F(\Sigma\bar{z})|_{F_{\bar{z}}}} \mathbb{T}^c(\mathbb{F})_{\Sigma\bar{z}+1}\longrightarrow F_{\Sigma\bar{z}+1} \tag{5.17}$$

where the second arrow in (5.17) is the canonical projection. It may be checked that these two associations are inverse to each other. ∎

Given a collection $F=\{F_n:\mathcal{C}\longrightarrow\mathcal{C}\}_{n\in\mathbb{Z}}$ of endofunctors, we now set $sF=\{(sF)_n := F_{n+1}\}_{n\in\mathbb{Z}}$. If $(\mathbb{U},\Theta)$ is an A_∞-monad, we can now consider the family $\Gamma^{s\mathbb{U}}$ of natural transformations

$$\begin{gathered}\gamma_k^{s\mathbb{U}}(\bar{n}) : (s\mathbb{U})_{\bar{n}} = (s\mathbb{U})_{n_1}\circ ...\circ(s\mathbb{U})_{n_k}\longrightarrow (s\mathbb{U})_{\Sigma\bar{n}+1}=\mathbb{U}_{\Sigma\bar{n}+2}\\ \gamma_k^{s\mathbb{U}}(\bar{n}) := (-1)^{n_1(k-1)+...+n_{k-1}}\theta_k(n_1+1,...,n_k+1)\end{gathered} \tag{5.18}$$

for $\bar{n}=(n_1,...,n_k)$, $k\geq 1$. Applying Lemma 5.5, the family of natural transformations in (5.18) corresponds to $\delta_1^{s\mathbb{U}}=\{\delta_1^{s\mathbb{U}}(L):\mathbb{T}^c(s\mathbb{U})_L\longrightarrow\mathbb{T}^c(s\mathbb{U})_{L+1}\mid L\in\mathbb{Z}\}$. Finally, using the fact that $(\mathbb{U},\Theta)$ is an A_∞-monad, it can be seen from the relations in (2.2) that each $\delta_1^{s\mathbb{U}}(L+1)\delta_1^{s\mathbb{U}}(L)=0$. We will now say that the dg-cononad given by

$$Bar(\mathbb{U},\Theta) := (\mathbb{T}^c(s\mathbb{U}),\delta_1^{s\mathbb{U}},\delta_2^{s\mathbb{U}}) \tag{5.19}$$

is the bar construction on the A_∞-monad $(\mathbb{U},\Theta)$.

5.6. THEOREM. *Let $(\mathbb{W},\delta_1,\delta_2)$ be a dg-comonad over $\mathcal{C}$ that is conilpotent. Let $(\mathbb{U},\Theta)$ be an A_∞-monad over $\mathcal{C}$ and let $(A,\{m_l:A^{\otimes l}\longrightarrow A\}_{l\geq 1})$ be the A_∞-algebra given by setting $A_l := \prod_{l'\in\mathbb{Z}} Nat(\mathbb{W}_{l'},\mathbb{U}_{l+l'})$. Then, there is a one-one correspondence between the following:*
(a) Morphisms $\chi:(\mathbb{W},\delta_1,\delta_2)\longrightarrow Bar(\mathbb{U},\Theta)=(\mathbb{T}^c(s\mathbb{U}),\delta_1^{s\mathbb{U}},\delta_2^{s\mathbb{U}})$ of dg-comonads over $\mathcal{C}$
(b) Elements $\zeta=\{\zeta^{(L)}\}_{L\in\mathbb{Z}}\in\prod_{L\in\mathbb{Z}} Nat(\mathbb{W}_L,(s\mathbb{U})_L)=A_1$ satisfying

$$\begin{aligned}&(\zeta^{(n_1-1)}*...*\zeta^{(n_k-1)})\delta_2(n_1-1,...,n_k-1)\delta_1(M)\\ &=\sum_{i=0}^{k-1}\sum_{j=1}^{\infty}(-1)^{n_1+...+n_i-i}(\zeta^{(n_1-1)}*...*\zeta^{(n_i-1)}*b_j(\zeta^{\otimes j})^{(n_{i+1}-2)}*\zeta^{(n_{i+2}-1)}*...*\zeta^{(n_k-1)})\delta_2(n_1-1,...,n_i-1,n_{i+1}-2,n_{i+2}-1,...,n_k-1)(M)\end{aligned} \tag{5.20}$$

for each fixed $k\geq 1$, $(n_1,...,n_k)\in\mathbb{Z}^k$ and $M\in\mathcal{C}$.

PROOF. We consider a morphism $\chi:(\mathbb{W},\delta_1,\delta_2)\longrightarrow(\mathbb{T}^c(s\mathbb{U}),\delta_1^{s\mathbb{U}},\delta_2^{s\mathbb{U}})$ of dg-comonads over $\mathcal{C}$. In particular, we see that $\chi\in gr-Comon(\mathcal{C})((\mathbb{W},\delta_2),(\mathbb{T}^c(s\mathbb{U}),\delta_2^{s\mathbb{U}}))$. Since $(\mathbb{W},\delta_2)$ is conilpotent, it follows from Proposition 5.4 that the morphism χ corresponds to an element $\zeta = \{\zeta^{(L)}\}_{L\in\mathbb{Z}} \in \prod\limits_{L\in\mathbb{Z}} Nat(\mathbb{W}_L, (s\mathbb{U})_L) = A_1$. Additionally, the morphism χ of graded comonads is well behaved with respect to the coderivations δ_1 and $\delta_1^{s\mathbb{U}}$. We now fix $k \geq 1$, $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$ and $M \in \mathcal{C}$. Composing $\delta_1^{s\mathbb{U}}(M)$ with the morphisms induced by χ, we have the sum of the following compositions over $0 \leq i \leq k-1$, $j \geq 1$

$$\begin{array}{c}
\mathbb{W}_{\Sigma\bar{n}-k-1}(M) \\
\Big\downarrow \scriptstyle \bigoplus\limits_{\bar{n}'\in\mathbb{Z}(n_{i+1}-2,j)} \zeta^{(n_1-1,\dots,n_i-1,\bar{n}',n_{i+2}-1,\dots,n_k-1)}\delta_2(n_1-1,\dots,n_i-1,\bar{n}',n_{i+2}-1,\dots,n_k-1)(M) \\
\bigoplus\limits_{\bar{n}'\in\mathbb{Z}(n_{i+1}-2,j)} \mathbb{U}_{n_1}\dots\mathbb{U}_{n_i}\mathbb{U}_{n'_1+1}\dots.\mathbb{U}_{n'_j+1}\mathbb{U}_{n_{i+2}}\dots\mathbb{U}_{n_k}(M) \\
\Big\downarrow \scriptstyle (-1)^{n_1+\dots+n_i-i}(-1)^{n'_1(j-1)+\dots+n'_{j-1}}(\mathbb{U}_{n_1}*\dots*\mathbb{U}_{n_i}*\theta_j(n'_1+1,\dots,n'_j+1)*\mathbb{U}_{n_{i+2}}*\dots*\mathbb{U}_{n_k})(M) \\
\mathbb{U}_{\bar{n}} = \mathbb{U}_{n_1}\dots\mathbb{U}_{n_k}(M)
\end{array} \tag{5.21}$$

Using the coassociativity condition on δ_2 in (5.3) and the description of the structure maps of the A_∞-algebra $\{b_l : (sA)^{\otimes l} \longrightarrow sA\}_{l\geq 1}$ in (5.10), we see that the sum of the compositions in (5.21) equates to

$$\sum_{i=0}^{k-1}\sum_{j=1}^{\infty}(-1)^{n_1+\dots+n_i-i}(\zeta^{(n_1-1)} * \dots * \zeta^{(n_i-1)} * b_j(\zeta^{\otimes j})^{(n_{i+1}-2)} * \zeta^{(n_{i+2}-1)} * \dots * \zeta^{(n_k-1)})\delta_2(n_1-1, \dots, n_i-1, n_{i+1}-2, n_{i+2}-1, \dots, n_k-1)(M) \tag{5.22}$$

On the other hand, composing δ_1 with the morphisms induced by χ, we have the composition

$$(\zeta^{(n_1-1)} * \dots * \zeta^{(n_k-1)})\delta_2(n_1-1, \dots, n_k-1)\delta_1(M) \tag{5.23}$$

The result is now clear from (5.22) and (5.23). ∎

6. Distributive laws and lifting of A_∞-monads to Eilenberg-Moore categories

Let $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ be a monad on $\mathcal{C}$, with "multiplication" $\theta_{\mathbb{S}} : \mathbb{S}\circ\mathbb{S} \longrightarrow \mathbb{S}$ and "unit" $\iota_{\mathbb{S}} : id \longrightarrow \mathbb{S}$. The standard Eilenberg-Moore category $EM_{\mathbb{S}}$ for the monad $\mathbb{S}$ consists of pairs $(M, \pi_M : \mathbb{S}(M) \longrightarrow M)$ satisfying $\pi_M \circ \mathbb{S}(\pi_M) = \pi_M \circ \theta_{\mathbb{S}}(M)$ and $id_M = \pi_M \circ \iota_{\mathbb{S}}(M)$ (see, for instance, [10]). The category $EM_{\mathbb{S}}$ is equipped with a pair $(L_{\mathbb{S}}, R_{\mathbb{S}})$ of adjoint functors, where $R_{\mathbb{S}} : EM_{\mathbb{S}} \longrightarrow \mathcal{C}$ is the forgetful functor and its left adjoint $L_{\mathbb{S}}$ is given by

$$L_{\mathbb{S}} : \mathcal{C} \longrightarrow EM_{\mathbb{S}} \qquad M \mapsto (\mathbb{S}(M), \theta_{\mathbb{S}}(M)) \tag{6.1}$$

If $\mathbb{T}$ is another monad on $\mathcal{C}$, it is a classical fact that liftings of $\mathbb{T}$ to a monad $\widetilde{\mathbb{T}}$ on the Eilenberg-Moore category of $\mathbb{S}$ correspond to monad distributive laws of $\mathbb{S}$ over $\mathbb{T}$ (see

Beck [8]). In this section, we will prove similar results, showing how an A_∞-monad on $\mathcal{C}$ can be lifted to an A_∞-monad on $EM_{\mathbb{S}}$. We note that dual results may be proved for lifting of A_∞-comonads to Eilenberg-Moore categories of comonads on $\mathcal{C}$.

More explicitly, let $\mathbb{F} : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor. A lifting of $\mathbb{F}$ to the Eilenberg-Moore category $EM_{\mathbb{S}}$ is given by a functor $\widetilde{\mathbb{F}} : EM_{\mathbb{S}} \longrightarrow EM_{\mathbb{S}}$ that fits into the following commutative diagram

$$\begin{tikzcd} EM_{\mathbb{S}} \arrow[r, "\widetilde{\mathbb{F}}"] \arrow[d, "R_{\mathbb{S}}"'] & EM_{\mathbb{S}} \arrow[d, "R_{\mathbb{S}}"] \\ \mathcal{C} \arrow[r, "\mathbb{F}"] & \mathcal{C} \end{tikzcd} \tag{6.2}$$

We know (see [8]) that such liftings are in one-one correspondence with distributive laws of the monad $\mathbb{S}$ over the endofunctor $\mathbb{F}$. Such a distributive law consists of a natural transformation $\lambda : \mathbb{S}\mathbb{F} \longrightarrow \mathbb{F}\mathbb{S}$ such that the following two diagrams commute

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\iota_{\mathbb{S}}\mathbb{F}"'] \arrow[dr, "\mathbb{F}\iota_{\mathbb{S}}"] & \\ \mathbb{S}\mathbb{F} \arrow[rr, "\lambda"] & & \mathbb{F}\mathbb{S} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^2\mathbb{F} \arrow[r, "\mathbb{S}\lambda"] \arrow[d, "\theta_{\mathbb{S}}\mathbb{F}"'] & \mathbb{S}\mathbb{F}\mathbb{S} \arrow[r, "\lambda\mathbb{S}"] & \mathbb{F}\mathbb{S}^2 \arrow[d, "\mathbb{F}\theta_{\mathbb{S}}"] \\ \mathbb{S}\mathbb{F} \arrow[rr, "\lambda"] & & \mathbb{F}\mathbb{S} \end{tikzcd} \tag{6.3}$$

Given a distributive law as in (6.3), the lift of $\mathbb{F}$ to $EM_{\mathbb{S}}$ is defined by setting $\widetilde{\mathbb{F}}(M, \pi_M) := (\mathbb{F}(M), \mathbb{F}(\pi_M) \circ \lambda(M))$. Conversely, given a lifting $\widetilde{\mathbb{F}}$ of $\mathbb{F}$ as in (6.2), the corresponding distributive law is defined by setting

$$\lambda : \mathbb{S}\mathbb{F} \xrightarrow{\quad \mathbb{S}\mathbb{F}\iota_{\mathbb{S}} \quad} \mathbb{S}\mathbb{F}\mathbb{S} = \mathbb{S}\mathbb{F}R_{\mathbb{S}}L_{\mathbb{S}} = R_{\mathbb{S}}L_{\mathbb{S}}R_{\mathbb{S}}\widetilde{\mathbb{F}}L_{\mathbb{S}} \longrightarrow R_{\mathbb{S}}\widetilde{\mathbb{F}}L_{\mathbb{S}} = \mathbb{F}R_{\mathbb{S}}L_{\mathbb{S}} = \mathbb{F}\mathbb{S} \tag{6.4}$$

where the morphism $R_{\mathbb{S}}L_{\mathbb{S}}R_{\mathbb{S}}\widetilde{\mathbb{F}}L_{\mathbb{S}} \longrightarrow R_{\mathbb{S}}\widetilde{\mathbb{F}}L_{\mathbb{S}}$ in (6.4) is induced by the counit $L_{\mathbb{S}}R_{\mathbb{S}} \longrightarrow id$ of the adjunction $(L_{\mathbb{S}}, R_{\mathbb{S}})$.

Now suppose that $\tau : \mathbb{F} \longrightarrow \mathbb{F}'$ is a natural transformation between endofunctors on $\mathcal{C}$. Suppose that $\mathbb{F}$ (resp. $\mathbb{F}'$) lifts to an endofunctor $\widetilde{\mathbb{F}}$ (resp. $\widetilde{\mathbb{F}}'$) on $EM_{\mathbb{S}}$, corresponding to a distributive law $\lambda : \mathbb{S}\mathbb{F} \longrightarrow \mathbb{F}\mathbb{S}$ (resp. $\lambda' : \mathbb{S}\mathbb{F}' \longrightarrow \mathbb{F}'\mathbb{S}$). The natural transformation τ is said to lift to $EM_{\mathbb{S}}$ (see [45, Definition 3.7]) if for each $(M, \pi_M) \in EM_{\mathbb{S}}$, the morphism $\tau(M) : \mathbb{F}(M) \longrightarrow \mathbb{F}'(M)$ gives a morphism $\widetilde{\tau}(M, \pi_M) : \widetilde{\mathbb{F}}(M, \pi_M) \longrightarrow \widetilde{\mathbb{F}}'(M, \pi_M)$ in $EM_{\mathbb{S}}$. In other words, the following diagram must commute

$$\begin{tikzcd} \mathbb{S}\mathbb{F}(M) \arrow[r, "\lambda(M)"] \arrow[d, "\mathbb{S}(\tau(M))"'] & \mathbb{F}\mathbb{S}(M) \arrow[r, "\mathbb{F}(\pi_M)"] & \mathbb{F}(M) \arrow[d, "\tau(M)"] \\ \mathbb{S}\mathbb{F}'(M) \arrow[r, "\lambda'(M)"] & \mathbb{F}'\mathbb{S}(M) \arrow[r, "\mathbb{F}'(\pi_M)"] & \mathbb{F}'(M) \end{tikzcd} \tag{6.5}$$

Additionally, we know from [45, Proposition 3.13] that such a lifting of τ exists if and only if the following diagram commutes

$$\begin{array}{ccc} \mathbb{S}\mathbb{F} & \xrightarrow{\lambda} & \mathbb{F}\mathbb{S} \\ {\scriptstyle \mathbb{S}\tau}\downarrow & & \downarrow{\scriptstyle \tau\mathbb{S}} \\ \mathbb{S}\mathbb{F}' & \xrightarrow{\lambda'} & \mathbb{F}'\mathbb{S} \end{array} \tag{6.6}$$

We need one more convention. If $\mathbb{F}_1$, ..., $\mathbb{F}_k$ is a family of endofunctors, along with distributive laws $\{\lambda_i : \mathbb{S}\mathbb{F}_i \longrightarrow \mathbb{F}_i\mathbb{S}\}_{1\leq i\leq k}$, we write

$$(\lambda_1, ..., \lambda_k) : \mathbb{S}(\mathbb{F}_1...\mathbb{F}_k) \xrightarrow{\lambda_1\mathbb{F}_2...\mathbb{F}_k} \mathbb{F}_1\mathbb{S}\mathbb{F}_2....\mathbb{F}_k \xrightarrow{\mathbb{F}_1\lambda_2\mathbb{F}_3...\mathbb{F}_k} \dots \xrightarrow{\mathbb{F}_1...\mathbb{F}_{k-1}\lambda_k} (\mathbb{F}_1...\mathbb{F}_k)\mathbb{S} \tag{6.7}$$

for the composition of the distributive laws λ_1, ... ,λ_k.

6.1. Definition. *Let $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ be a monad and let $(\mathbb{U}, \Theta)$ be an A_∞-monad on $\mathcal{C}$. A distributive law of the monad $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ over the A_∞-monad $(\mathbb{U}, \Theta)$ is given by a family $\Lambda = \{\lambda_n : \mathbb{S}\mathbb{U}_n \longrightarrow \mathbb{U}_n\mathbb{S}\}_{n\in\mathbb{Z}}$ such that*
(a) Each $\lambda_n : \mathbb{S}\mathbb{U}_n \longrightarrow \mathbb{U}_n\mathbb{S}$ is a distributive law for the monad $\mathbb{S}$ over the endofunctor $\mathbb{U}_n$.
(b) For each $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, $k \geq 1$, the following diagram commutes

$$\begin{array}{ccc} \mathbb{S}\mathbb{U}_{\bar{n}} = \mathbb{S}\mathbb{U}_{n_1}...\mathbb{U}_{n_k} & \xrightarrow{\lambda_{\bar{n}}=(\lambda_{n_1},...,\lambda_{n_k})} & \mathbb{U}_{n_1}...\mathbb{U}_{n_k}\mathbb{S} = \mathbb{U}_{\bar{n}}\mathbb{S} \\ {\scriptstyle \mathbb{S}\theta_k(\bar{n})}\downarrow & & \downarrow{\scriptstyle \theta_k(\bar{n})\mathbb{S}} \\ \mathbb{S}\mathbb{U}_{\Sigma\bar{n}+(2-k)} & \xrightarrow{\lambda_{\Sigma\bar{n}+(2-k)}} & \mathbb{U}_{\Sigma\bar{n}+(2-k)}\mathbb{S} \end{array} \tag{6.8}$$

6.2. Lemma. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a distributive law of the monad $\mathbb{S}$ over the A_∞-monad $(\mathbb{U}, \Theta)$. Then, $(\mathbb{U}, \Theta)$ lifts to an A_∞-monad $(\widetilde{\mathbb{U}}, \widetilde{\Theta})$ on the category $EM_{\mathbb{S}}$.*

Proof. Since each $\lambda_n : \mathbb{S}\mathbb{U}_n \longrightarrow \mathbb{U}_n\mathbb{S}$ is a distributive law, it follows from (6.2) and (6.3) that $\mathbb{U}_n$ lifts to $\widetilde{\mathbb{U}}_n : EM_{\mathbb{S}} \longrightarrow EM_{\mathbb{S}}$. Further, since (6.8) commutes, it follows from the criterion described in (6.6) that the natural transformations $\theta_k(\bar{n})$ lift to $\widetilde{\theta}_k(\bar{n}) : \widetilde{\mathbb{U}}_{\bar{n}} = \widetilde{\mathbb{U}}_{n_1}...\widetilde{\mathbb{U}}_{n_k} \longrightarrow \widetilde{\mathbb{U}}_{\Sigma\bar{n}+(2-k)}$. Accordingly, for any $(M, \pi_M) \in EM_{\mathbb{S}}$ and $\bar{z} \in \mathbb{Z}^N$, $N \geq 1$, it follows from (2.2) that the morphism in $\mathcal{C}$ underlying the sum

$$\sum (-1)^{p+qr+q\Sigma\bar{n}}\widetilde{\theta}_{p+1+r}(\bar{n}, \Sigma\bar{n}' + (2 - q), \bar{n}'')(\widetilde{\mathbb{U}}_{\bar{n}} * \widetilde{\theta}_q(\bar{n}') * \widetilde{\mathbb{U}}_{\bar{n}''}) \tag{6.9}$$

is zero, where the sum runs over partitions $\bar{z} = (\bar{n}, \bar{n}', \bar{n}'')$ with $|\bar{n}| = p$, $|\bar{n}'| = q$, $|\bar{n}''| = r$. ■

6.3. Theorem. *Let $(\mathbb{U}, \Theta)$ be an A_∞-monad and let $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ be a monad on $\mathcal{C}$. Then, there is a one one correspondence between the following*
(a) Liftings of the A_∞-monad $(\mathbb{U}, \Theta)$ to an A_∞-monad $(\widetilde{\mathbb{U}}, \widetilde{\Theta})$ on the category $EM_{\mathbb{S}}$.
(b) Distributive laws of the monad $\mathbb{S}$ over the A_∞-monad $(\mathbb{U}, \Theta)$.

PROOF. We know from Lemma 6.2 that a distributive law of the monad $\mathbb{S}$ over the A_∞-monad $(\mathbb{U}, \Theta)$ gives rise to a lifting of the A_∞-monad $(\mathbb{U}, \Theta)$ to $EM_{\mathbb{S}}$. On the other hand, let $(\widetilde{\mathbb{U}}, \widetilde{\Theta})$ be a lifting of the A_∞-monad $(\mathbb{U}, \Theta)$ to $EM_{\mathbb{S}}$. In particular, the lifting $\widetilde{\mathbb{U}}_n$ of the endofunctor $\mathbb{U}_n$ leads to a distributive law $\lambda_n : \mathbb{S}\mathbb{U}_n \longrightarrow \mathbb{U}_n\mathbb{S}$. Also, for any $k \geq 1$ and $\bar{n} \in \mathbb{Z}^k$, we know that the natural transformation $\theta_k(n) : \mathbb{U}_{\bar{n}} \longrightarrow \mathbb{U}_{\Sigma\bar{n}+(2-k)}$ lifts to $EM_{\mathbb{S}}$. It now follows from (6.6) that we must have $(\theta_k(\bar{n})\mathbb{S}) \circ (\lambda_{n_1}, ..., \lambda_{n_k}) = \lambda_{\Sigma\bar{n}+(2-k)} \circ (\mathbb{S}\theta_k(\bar{n}))$. Hence, the collection $\Lambda = \{\lambda_n : \mathbb{S}\mathbb{U}_n \longrightarrow \mathbb{U}_n\mathbb{S}\}_{n\in\mathbb{Z}}$ gives a distributive law of the monad $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ over the A_∞-monad $(\mathbb{U}, \Theta)$ in the sense of Definition 6.1. Since there is a one one correspondence between distributive laws and liftings of an endofunctor to $EM_{\mathbb{S}}$, it follows that these two associations are inverse to each other. ∎

We recall (see, for instance, [45, Proposition 3.30]) that a distributive law of $\mathbb{S}$ over a monad $\mathbb{T}$ gives rise to a new monad $\mathbb{T}\mathbb{S}$. We will now study something similar for A_∞-monads.

6.4. PROPOSITION. *Let $\Lambda = \{\lambda_n : \mathbb{S}\mathbb{U}_n \longrightarrow \mathbb{U}_n\mathbb{S}\}_{n\in\mathbb{Z}}$ be a distributive law of the monad $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ over the A_∞-monad $(\mathbb{U}, \Theta)$. Then, the pair $(\mathbb{U}^{(\mathbb{S})}, \Theta^{(\mathbb{S})})$ defined by setting $\mathbb{U}_n^{(\mathbb{S})} := \mathbb{U}_n\mathbb{S}$ and $\theta_k^{(\mathbb{S})}(\bar{n}) : \mathbb{U}_{\bar{n}}^{(\mathbb{S})} \longrightarrow \mathbb{U}^{(\mathbb{S})}_{\Sigma\bar{n}+(2-k)}$ as follows*

$$\begin{array}{c} \mathbb{U}_{\bar{n}}^{(\mathbb{S})} = (\mathbb{U}_{n_1}\mathbb{S})...(\mathbb{U}_{n_k}\mathbb{S}) \\ \Big\downarrow{\scriptstyle (\mathbb{U}_{n_1}\mathbb{S})...(\mathbb{U}_{n_{k-2}}\mathbb{S})\mathbb{U}_{n_{k-1}}\lambda_{n_k}\mathbb{S}} \\ \vdots \\ \Big\downarrow{\scriptstyle \mathbb{U}_{n_1}(\lambda_{n_2},...,\lambda_{n_k})\mathbb{S}^{k-1}} \\ \mathbb{U}_{\bar{n}}\mathbb{S}^k = \mathbb{U}_{n_1}...\mathbb{U}_{n_k}\mathbb{S}^k \\ \Big\downarrow{\scriptstyle \theta_k(\bar{n})\theta_{\mathbb{S}}^k} \\ \mathbb{U}_{\Sigma\bar{n}+(2-k)}\mathbb{S} = \mathbb{U}^{(\mathbb{S})}_{\Sigma\bar{n}+(2-k)} \end{array} \tag{6.10}$$

gives an A_∞-monad on $\mathcal{C}$, where $\theta_{\mathbb{S}}^k : \mathbb{S}^k \longrightarrow \mathbb{S}$ is obtained by iterating the multiplication $\theta_{\mathbb{S}} : \mathbb{S}^2 \longrightarrow \mathbb{S}$ on the monad S.

PROOF. We fix any $\bar{z} \in \mathbb{Z}^N$ and any partition $\bar{z} = (\bar{n}, \bar{n}', \bar{n}'')$ with $|\bar{n}| = p$, $|\bar{n}'| = q$, $|\bar{n}''| = r$. Since the distributive laws are well behaved both with respect to the multiplication $\theta_{\mathbb{S}}$ on the monad $\mathbb{S}$ as in (6.3) and the structure maps $\theta_k(\bar{n})$ of the A_∞-monad $(\mathbb{U}, \Theta)$ as in (6.8), we note that the following diagram commutes

$$\begin{tikzcd} \mathbb{U}_{\bar{n}}^{(\mathbb{S})} \circ \mathbb{U}_{\bar{n}'}^{(\mathbb{S})} \circ \mathbb{U}_{\bar{n}''}^{(\mathbb{S})} \arrow[r] \arrow[d, "\theta^{(\mathbb{S})}_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')(\mathbb{U}_{\bar{n}}^{(\mathbb{S})}*\theta_q^{(\mathbb{S})}(\bar{n}')*\mathbb{U}_{\bar{n}''}^{(\mathbb{S})})"'] & \mathbb{U}_{\bar{n}} \circ \mathbb{U}_{\bar{n}'} \circ \mathbb{U}_{\bar{n}''} \circ \mathbb{S}^N \arrow[d, "\mathbb{U}_{\bar{n}}\mathbb{U}_{\bar{n}'}\mathbb{U}_{\bar{n}''}\theta_{\mathbb{S}}^N"] \\ \mathbb{U}^{(\mathbb{S})}_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+(3-N)} = \mathbb{U}_{\Sigma\bar{n}+\Sigma\bar{n}'+\Sigma\bar{n}''+(3-N)}\mathbb{S} & \mathbb{U}_{\bar{n}} \circ \mathbb{U}_{\bar{n}'} \circ \mathbb{U}_{\bar{n}''} \circ \mathbb{S} \arrow[l, "\theta_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')(\mathbb{U}_{\bar{n}}*\theta_q(\bar{n}')*\mathbb{U}_{\bar{n}''})\mathbb{S}"'] \end{tikzcd} \tag{6.11}$$

Here the top horizontal arrow in (6.11) is obtained by repeatedly applying the distributive laws. It now follows from (2.2) that

$$\sum(-1)^{p+qr+q\Sigma\bar{n}}\theta^{(\mathbb{S})}_{p+1+r}(\bar{n},\Sigma\bar{n}'+(2-q),\bar{n}'')(\mathbb{U}^{(\mathbb{S})}_{\bar{n}}*\theta^{(\mathbb{S})}_{q}(\bar{n}')*\mathbb{U}^{(\mathbb{S})}_{\bar{n}''})=0 \tag{6.12}$$

where the sum runs over partitions $\bar{z}=(\bar{n},\bar{n}',\bar{n}'')$ with $|\bar{n}|=p$, $|\bar{n}'|=q$, $|\bar{n}''|=r$. ∎

Let $\Lambda=\{\lambda_n:\mathbb{S}\mathbb{U}_n\longrightarrow\mathbb{U}_n\mathbb{S}\}_{n\in\mathbb{Z}}$ be a distributive law of the monad $(\mathbb{S},\theta_{\mathbb{S}},\iota_{\mathbb{S}})$ over the A_∞-monad $(\mathbb{U},\Theta)$. By Theorem 6.3, $(\mathbb{U},\Theta)$ lifts to an A_∞-monad $(\widetilde{\mathbb{U}},\widetilde{\Theta})$ on the category $EM_{\mathbb{S}}$. Accordingly, we have the category $EM_{(\widetilde{\mathbb{U}},\widetilde{\Theta})}$ of $(\widetilde{\mathbb{U}},\widetilde{\Theta})$-modules in the sense of Definition 3.1. On the other hand, we have $(\mathbb{U}^{(\mathbb{S})},\Theta^{(\mathbb{S})})$ from Proposition 6.4, which is an A_∞-monad on $\mathcal{C}$. The category of $(\mathbb{U}^{(\mathbb{S})},\Theta^{(\mathbb{S})})$-modules is denoted by $EM_{(\mathbb{U}^{(\mathbb{S})},\Theta^{(\mathbb{S})})}$.

6.5. Proposition. *Let $\Lambda=\{\lambda_n:\mathbb{S}\mathbb{U}_n\longrightarrow\mathbb{U}_n\mathbb{S}\}_{n\in\mathbb{Z}}$ be a distributive law of the monad $(\mathbb{S},\theta_{\mathbb{S}},\iota_{\mathbb{S}})$ over the A_∞-monad $(\mathbb{U},\Theta)$. Then, there is a canonical functor $EM_{(\widetilde{\mathbb{U}},\widetilde{\Theta})}\longrightarrow EM_{(\mathbb{U}^{(\mathbb{S})},\Theta^{(\mathbb{S})})}$.*

Proof. By definition, a $(\widetilde{\mathbb{U}},\widetilde{\Theta})$-module consists of a collection $\{(M_n,\pi_n:\mathbb{S}M_n\longrightarrow M_n)\}_{n\in\mathbb{Z}}$ of objects of $EM_{\mathbb{S}}$ and a collection of morphisms in $EM_{\mathbb{S}}$:

$$\{\widetilde{\pi}_k(\bar{n},l):\widetilde{\mathbb{U}}_{\bar{n}}(M_l,\pi_l)\longrightarrow(M_{\Sigma\bar{n}+l+(2-k)},\pi_{\Sigma\bar{n}+l+(2-k)})\mid k\geq 1,\ \bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1},\ l\in\mathbb{Z}\ \} \tag{6.13}$$

Since $\widetilde{\mathbb{U}}_{\bar{n}}$ is a lifting of $\mathbb{U}_{\bar{n}}$ to $EM_{\mathbb{S}}$, there is a morphism $\mathbb{U}_{\bar{n}}M_l\longrightarrow M_{\Sigma\bar{n}+l+(2-k)}$ in $\mathcal{C}$ underlying $\widetilde{\pi}_k(\bar{n},l)$. We continue to denote this morphism by $\widetilde{\pi}_k(\bar{n},l):\mathbb{U}_{\bar{n}}M_l\longrightarrow M_{\Sigma\bar{n}+l+(2-k)}$. We also note the morphism

$$\pi_l\circ...\circ\mathbb{S}^{k-3}(\pi_l)\circ\mathbb{S}^{k-2}(\pi_l):\mathbb{S}^{k-1}(M_l)\xrightarrow{\mathbb{S}^{k-2}(\pi_l)}\mathbb{S}^{k-2}(M_l)\xrightarrow{\mathbb{S}^{k-3}(\pi_l)}\mathbb{S}^{k-3}(M_l)\xrightarrow{\mathbb{S}^{k-4}(\pi_l)}...\xrightarrow{\pi_l}M_l \tag{6.14}$$

For $k\geq 1$, $\bar{n}=(n_1,...,n_{k-1})\in\mathbb{Z}^{k-1}$, $l\in\mathbb{Z}$, we now consider the composition

$$\pi^{(\mathbb{S})}_k(\bar{n},l):\mathbb{U}^{(\mathbb{S})}_{\bar{n}}M_l\longrightarrow\mathbb{U}_{\bar{n}}\mathbb{S}^{k-1}M_l\xrightarrow{\pi_l\circ...\circ\mathbb{S}^{k-3}(\pi_l)\circ\mathbb{S}^{k-2}(\pi_l)}\mathbb{U}_{\bar{n}}M_l\xrightarrow{\widetilde{\pi}_k(\bar{n},l)}M_{\Sigma\bar{n}+l+(2-k)} \tag{6.15}$$

where the first morphism in (6.15) is obtained by repeatedly applying the distributive law $\Lambda=\{\lambda_n:\mathbb{S}\mathbb{U}_n\longrightarrow\mathbb{U}_n\mathbb{S}\}_{n\in\mathbb{Z}}$ as in top horizontal arrow in (6.11). As in the proof of Proposition 6.4, it may now be verified that the collection $\{M_n\}_{n\in\mathbb{Z}}$ along with the morphisms $\pi^{(\mathbb{S})}_k(\bar{n},l)$ satisfies the relations for being a $(\mathbb{U}^{(\mathbb{S})},\Theta^{(\mathbb{S})})$-module in the sense of Definition 3.1.

∎

We conclude this section with the following result. By a distributive law of the A_∞-monad $(\mathbb{U},\Theta)$ over the monad $(\mathbb{S},\theta_{\mathbb{S}},\iota_{\mathbb{S}})$, we mean a collection of natural transformations $\Lambda'=\{\lambda'_n:\mathbb{U}_n\mathbb{S}\longrightarrow\mathbb{S}\mathbb{U}_n\}_{n\in\mathbb{Z}}$ such that

(a) The following two diagrams commute for each $n \in \mathbb{Z}$

$$\begin{tikzcd} & \mathbb{U}_n \arrow[dl, "\mathbb{U}_n\iota_{\mathbb{S}}"'] \arrow[dr, "\iota_{\mathbb{S}}\mathbb{U}_n"] & \\ \mathbb{U}_n\mathbb{S} \arrow[rr, "\lambda'_n"'] & & \mathbb{S}\mathbb{U}_n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{U}_n\mathbb{S}^2 \arrow[r, "\lambda'_n\mathbb{S}"] \arrow[d, "\mathbb{U}_n\theta_{\mathbb{S}}"'] & \mathbb{S}\mathbb{U}_n\mathbb{S} \arrow[r, "\mathbb{S}\lambda'_n"] & \mathbb{S}^2\mathbb{U}_n \arrow[d, "\theta_{\mathbb{S}}\mathbb{U}_n"] \\ \mathbb{U}_n\mathbb{S} \arrow[rr, "\lambda'_n"'] & & \mathbb{S}\mathbb{U}_n \end{tikzcd} \tag{6.16}$$

(b) For each $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, $k \geq 1$, the following diagram commutes

$$\begin{tikzcd} \mathbb{U}_{\bar{n}}\mathbb{S} = \mathbb{U}_{n_1}...\mathbb{U}_{n_k}\mathbb{S} \arrow[r, "\mathbb{U}_{n_1}...\mathbb{U}_{n_{k-1}}\lambda'_{n_k}"] \arrow[d, "\theta_k(\bar{n})\mathbb{S}"'] & \mathbb{U}_{n_1}...\mathbb{U}_{n_{k-1}}\mathbb{S}\mathbb{U}_{n_k} \arrow[r, "\mathbb{U}_{n_1}...\mathbb{U}_{n_{k-2}}\lambda'_{n_{k-1}}\mathbb{U}_{n_k}"] & \dots \arrow[r, "\lambda_{n_1}\mathbb{U}_{n_2}...\mathbb{U}_{n_k}"] & \mathbb{S}\mathbb{U}_{n_1}...\mathbb{U}_{n_k} = \mathbb{S}\mathbb{U}_{\bar{n}} \arrow[d, "\mathbb{S}\theta_k(\bar{n})"] \\ \mathbb{U}_{\Sigma\bar{n}+(2-k)}\mathbb{S} \arrow[rrr, "\lambda'_{\Sigma\bar{n}+(2-k)}"] & & & \mathbb{S}\mathbb{U}_{\Sigma\bar{n}+(2-k)} \end{tikzcd} \tag{6.17}$$

We now recall (see, for instance, [25, § 5.2]) that the Kleisli category $Kl(\mathbb{S})$ of the monad $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ is the category whose objects are the same as those of $\mathcal{C}$, but a morphism $M \longrightarrow M'$ in $Kl(\mathbb{S})$ corresponds to a morphism $M \longrightarrow \mathbb{S}(M')$ in $\mathcal{C}$. The composition of $\alpha : M \longrightarrow \mathbb{S}(M')$ with $\beta : M' \longrightarrow \mathbb{S}(M'')$ in $Kl(\mathbb{S})$ is given by

$$M \xrightarrow{\alpha} \mathbb{S}(M') \xrightarrow{\mathbb{S}(\beta)} \mathbb{S}^2(M'') \xrightarrow{\theta_{\mathbb{S}}(M'')} \mathbb{S}(M'') \tag{6.18}$$

There is a canonical functor $J_{\mathbb{S}} : \mathcal{C} \longrightarrow Kl(\mathbb{S})$ that is identity on objects and which takes the morphism $\alpha : M \longrightarrow M'$ in $\mathcal{C}$ to $M \xrightarrow{\alpha} M' \xrightarrow{\iota_{\mathbb{S}}(M')} \mathbb{S}(M')$ in $Kl(\mathbb{S})$. From [25, Proposition 5.2.5], we know that a natural transformation $\lambda'_n : \mathbb{U}_n\mathbb{S} \longrightarrow \mathbb{S}\mathbb{U}_n$ satisfying the conditions in (6.16) corresponds to an extension $\widehat{\mathbb{U}}_n$ of the endofunctor $\mathbb{U}_n$ to the Kleisli category $Kl(\mathbb{S})$, i.e., $\widehat{\mathbb{U}}_n \circ J_{\mathbb{S}} = J_{\mathbb{S}} \circ \mathbb{U}_n$. From [45, Theorem 4.16], we know that the condition in (6.17) ensures that the natural transformations $\theta_k(\bar{n}) : \mathbb{U}_{\bar{n}} \longrightarrow \mathbb{U}_{\Sigma\bar{n}+(2-k)}$ extend to natural transformations $\widehat{\theta}_k(\bar{n}) : \widehat{\mathbb{U}}_{\bar{n}} \longrightarrow \widehat{\mathbb{U}}_{\Sigma\bar{n}+(2-k)}$. Proceeding as in the proof of Theorem 6.3, we can now prove the following result.

6.6. THEOREM. *Let $(\mathbb{U}, \Theta)$ be an A_∞-monad and let $(\mathbb{S}, \theta_{\mathbb{S}}, \iota_{\mathbb{S}})$ be a monad on $\mathcal{C}$. Then, there is a one one correspondence between the following*
(a) Extensions of the A_∞-monad $(\mathbb{U}, \Theta)$ to an A_∞-monad $(\widehat{\mathbb{U}}, \widehat{\Theta})$ on the category $Kl(\mathbb{S})$.
(b) Distributive laws of the A_∞-monad $(\mathbb{U}, \Theta)$ over the monad $\mathbb{S}$.

7. A_∞-(co)algebras and A_∞-(co)monads

In this section, we will show how to give examples of A_∞-monads and A_∞-comonads on any K-linear Grothendieck category. For basic definitions such as A_∞-algebras, A_∞-coalgebras, A_∞-modules over A_∞-algebras and A_∞-comodules over A_∞-coalgebras, we refer the reader to the appendix in Section 10.

Let $\mathcal{C}$ be a K-linear Grothendieck category. If R is a K-algebra, an R-object in $\mathcal{C}$ consists of $M \in \mathcal{C}$ along with a morphism $\xi_M : R \longrightarrow \mathcal{C}(M, M)$ of K-algebras. The category $\mathcal{C}_R$ of R-objects in $\mathcal{C}$ is a classical construction of Popescu [36, p 108]. This abstract module theory has also been developed extensively by Artin and Zhang [2], who studied Hilbert Basis Theorem, completions, localizations and Grothendieck's theory of flat descent in this context. In particular, if $\mathcal{C} = Mod\text{-}S$, the category of right modules over a K-algebra S, then $\mathcal{C}_R = Mod\text{-}(S \otimes_K R)$. Accordingly, the abstract module category $\mathcal{C}_R$ is often understood as a noncommutative base change of R by means of the category $\mathcal{C}$ (see Lowen and Van den Bergh [33]). If V is a right R-module, then there is a pair of adjoint functors (see [2, § B3, B4])

$$\_\_ \otimes V : \mathcal{C} \longrightarrow \mathcal{C}_R \qquad \underline{Hom}_R(V, \_\_) : \mathcal{C}_R \longrightarrow \mathcal{C} \tag{7.1}$$

Therefore, for $\mathcal{C} = \mathcal{C}_K$, an R-module object in $\mathcal{C}$ is determined by $M \in \mathcal{C}$ and a structure map $M \otimes R \longrightarrow M$ in $\mathcal{C}$ satisfying the usual associativity and unit conditions. We observe that $\_\_ \otimes R : \mathcal{C} \longrightarrow \mathcal{C}$ is an ordinary monad on $\mathcal{C}$. Similarly, if C' is a K-coalgebra, a C'-comodule object in $\mathcal{C}$ is determined by $P \in \mathcal{C}$ and a structure map $P \longrightarrow P \otimes C'$ satisfying coassociativity and counit conditions (see Brzeziński and Wisbauer [13, § 39.1]). Again, $\_\_ \otimes C'$ is an ordinary comonad on $\mathcal{C}$. This suggests how we can create A_∞-monads and A_∞-comonads on $\mathcal{C}$.

We let $Vect^{\mathbb{Z}}$ be the category of $\mathbb{Z}$-graded vector spaces. We know that $Vect^{\mathbb{Z}}$ is a closed symmetric monoidal category, with internal hom $[W', W]$ of two objects given by

$$[W', W]_p := \prod_{k \in \mathbb{Z}} Vect(W'_k, W_{k+p}) \qquad p \in \mathbb{Z} \tag{7.2}$$

Additionally, we consider the endofunctor $T : Vect^{\mathbb{Z}} \longrightarrow Vect^{\mathbb{Z}}$ that inverts the grading, i.e., $(TW)_k := W_{-k}$ for any $k \in \mathbb{Z}$ and $W \in Vect^{\mathbb{Z}}$.

We let $(\mathbb{F}, \theta_{\mathbb{F}})$ be a non-unital monad on $\mathcal{C}$. In other words, we have an endofunctor $\mathbb{F} \in End(\mathcal{C})$ along with a "multiplication" $\theta_{\mathbb{F}} : \mathbb{F} \circ \mathbb{F} \longrightarrow \mathbb{F}$ satisfying the usual associativity conditions. We note that any monad on $\mathcal{C}$ is equipped in particular with the structure of a non-unital monad. Similarly, a non-counital comonad $(\mathbb{G}, \Delta_{\mathbb{G}})$ consists of an endofunctor $\mathbb{G} \in End(\mathcal{C})$ along with a "comultiplication" $\Delta_{\mathbb{G}} : \mathbb{G} \longrightarrow \mathbb{G} \circ \mathbb{G}$ satisfying the usual coassociativity conditions. The following result can be used to construct several examples of A_∞-monads on $\mathcal{C}$.

7.1. Proposition. *Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be an A_∞-algebra and let $(\mathbb{F}, \theta_{\mathbb{F}})$ be a non-unital monad on $\mathcal{C}$. Suppose that $\mathbb{F}$ preserves direct sums. Then, the collection of functors*

$$\mathbb{U}_{\mathcal{C}}^{A,\mathbb{F}} = \{\mathbb{U}_{\mathcal{C},n}^{A,\mathbb{F}} := A_n \otimes \mathbb{F}(\_\_) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n \in \mathbb{Z}} \tag{7.3}$$

is an A_∞-monad on $\mathcal{C}$. In particular, the collection of functors $\mathbb{U}_{\mathcal{C}}^{A} = \{\mathbb{U}_{\mathcal{C},n}^{A} := A_n \otimes \_\_ : \mathcal{C} \longrightarrow \mathcal{C}\}_{n \in \mathbb{Z}}$ is an A_∞-monad on $\mathcal{C}$.

PROOF. Let $m_k : A^{\otimes k} \longrightarrow A$ of degree $(2-k)$ for $k \geq 1$ be the structure maps for A as an A_∞-algebra as in Definition 10.1. Since $\mathbb{F}$ preserves direct sums, we note that for $k \geq 1$ and $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, we have

$$\mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},n_1} \circ ... \circ \mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},n_k} = A_{n_1} \otimes ... \otimes A_{n_k} \otimes \mathbb{F}^k(\_\_) \tag{7.4}$$

We can now define natural transformations as in (2.1)

$$\Theta^{A,\mathbb{F}}_{\mathcal{C}} = \{\theta^{A,\mathbb{F}}_{\mathcal{C},k}(\bar{n}) : \mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},\bar{n}} = \mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},n_1} \circ ... \circ \mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},n_k} \longrightarrow \mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},n_1+...+n_k+(2-k)} = \mathbb{U}^{A,\mathbb{F}}_{\mathcal{C},\Sigma\bar{n}+(2-k)}\} \tag{7.5}$$

induced by the structure maps $A_{n_1} \otimes ... \otimes A_{n_k} \longrightarrow A_{n_1+...+n_k+(2-k)}$ of the A_∞-algebra A and the k-fold multiplication $\theta^{k-1}_{\mathbb{F}} : \mathbb{F}^k \longrightarrow \mathbb{F}$ on the monad $\mathbb{F}$. The relations in (2.2) now follow directly from the relations satisfied by the structure maps of A (see (10.1)).

■

We will say that an A_∞-algebra A is even if the structure maps $m_k : A^{\otimes k} \longrightarrow A$ vanish whenever k is odd. For instance, this condition is satisfied by $A_{2,p}$-algebras (for p even), which have been extensively studied in the literature (see, for instance, [19], [24], [28], [34]). An $A_{2,p}$-algebra is an A_∞-algebra with all the operations m_k vanishing except for $k = 2$ and $k = p$. The $A_{2,p}$-algebras were introduced by He and Lu in [24] and studied further by Dotsenko and Vallette in [19]. If A is a p-Koszul algebra, then the Ext-algebra $Ext^*(K_A, K_A)$ carries the structure of an $A_{2,p}$-algebra (see [24]). Here, K_A refers to the A-module structure on K obtained from the fact that $A_0 = K$. The authors in [24] also describe a general procedure for constructing an $A_{2,p}$-algebra starting from any positively graded associative algebra. Other motivating examples for $A_{2,p}$-algebras, such as the A_∞-Koszul dual of $K[t]/(t^p)$ for $p \geq 3$, appear in Lu, Palmieri, Wu, and Zhang [34] and in Keller [28, § 2.6].

7.2. PROPOSITION. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be an A_∞-algebra that is even and let $(\mathbb{F}, \theta_{\mathbb{F}})$ be a non-unital monad on $\mathcal{C}$. Suppose that $\mathbb{F}$ has a right adjoint $\mathbb{G}$. Then, the collection of functors*

$$\mathbb{V}^{A,\mathbb{G}}_{\mathcal{C}} = \{\mathbb{V}^{A,\mathbb{G}}_{\mathcal{C},n} := \underline{Hom}(A_n, \mathbb{G}(\_\_)) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}} \tag{7.6}$$

is an A_∞-comonad on $\mathcal{C}$. In particular, the collection of functors

$$\mathbb{V}^{A}_{\mathcal{C}} = \{\mathbb{V}^{A}_{\mathcal{C},n} := \underline{Hom}(A_n, \_\_) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}} \tag{7.7}$$

is an A_∞-comonad on $\mathcal{C}$. Further, we have isomorphisms of categories $EM^e_{\mathbb{U}^{A,\mathbb{F}}_{\mathcal{C}}} \cong EM_e^{\mathbb{V}^{A,\mathbb{G}}_{\mathcal{C}}}$ and $\widetilde{EM}^{eo}_{\mathbb{U}^{A,\mathbb{F}}_{\mathcal{C}}} \cong \widetilde{EM}_{eo}^{\mathbb{V}^{A,\mathbb{G}}_{\mathcal{C}}}$.

PROOF. Since $\mathbb{F}$ is a left adjoint (and hence it preserves direct sums), it follows by (7.1) that the functors $\underline{Hom}(A_n, \mathbb{G}(\_\_)) : \mathcal{C} \longrightarrow \mathcal{C}$ are right adjoint to the functors $A_n \otimes \mathbb{F}(\_\_) = \mathbb{F}(A_n \otimes \_\_) : \mathcal{C} \longrightarrow \mathcal{C}$. The result now follows from Theorem 2.3, Proposition 3.5 and Theorem 3.8. ■

The following result can be used to give several examples of A_∞-comonads.

7.3. Proposition. *Let $C = \bigoplus_{n\in\mathbb{Z}} C_n$ be an A_∞-coalgebra with structure maps $\{w_k : C \longrightarrow C^{\otimes k}\}_{k\geq 1}$ and let $(\mathbb{G}, \Delta_{\mathbb{G}})$ be a non-counital comonad on $\mathcal{C}$. Suppose that $\mathbb{G}$ preserves direct sums. Then, the collection of functors*

$$\mathbb{U}_{\mathcal{C}}^{TC,\mathbb{G}} = \{\mathbb{U}_{\mathcal{C},n}^{TC,\mathbb{G}} := C_{-n} \otimes \mathbb{G}(\_\_) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}} \tag{7.8}$$

can be equipped with the structure of an A_∞-comonad on $\mathcal{C}$. In particular, $\mathbb{U}_{\mathcal{C}}^{TC} = \{\mathbb{U}_{\mathcal{C},n}^{TC} := C_{-n} \otimes \_\_ : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ is an A_∞-comonad on $\mathcal{C}$.
If C is even, i.e., $w_k = 0$ whenever k is odd and $\mathbb{G}$ has a right adjoint $\mathbb{J}$, the collection of functors

$$\mathbb{V}_{\mathcal{C}}^{TC,\mathbb{J}} = \{\mathbb{V}_{\mathcal{C},n}^{TC,\mathbb{J}} := \underline{Hom}(C_{-n}, \mathbb{J}(\_\_)) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}} \tag{7.9}$$

is an A_∞-monad on $\mathcal{C}$. In particular, if C is even, $\mathbb{V}_{\mathcal{C}}^{TC} = \{\mathbb{V}_{\mathcal{C},n}^{TC} := \underline{Hom}(C_{-n}, \_\_) : \mathcal{C} \longrightarrow \mathcal{C}\}_{n\in\mathbb{Z}}$ is an A_∞-monad on $\mathcal{C}$.

Proof. In a manner similar to the proof of Proposition 7.1, it is clear from the definition of an A_∞-coalgebra in Definition 10.5 and the fact that $\mathbb{G}$ preserves direct sums that $\mathbb{U}_{\mathcal{C}}^{TC,\mathbb{G}}$ is an A_∞-comonad on $\mathcal{C}$. If C is even, so is $\mathbb{U}_{\mathcal{C}}^{TC,\mathbb{G}}$. By (7.1), the functors $\{\underline{Hom}(C_{-n}, \mathbb{J}(\_\_)) = \underline{Hom}((TC)_n, \mathbb{J}(\_\_))\}_{n\in\mathbb{Z}}$ are right adjoint to the functors $\{\mathbb{G}((TC)_n \otimes \_\_) = C_{-n} \otimes \mathbb{G}(\_\_)\}_{n\in\mathbb{Z}}$ and the result now follows directly from Theorem 2.3. ∎

For the rest of this section, we suppose that $\mathcal{C} = Vect$, the category of vector spaces over the field K. In this paper, for an A_∞-algebra A, the category of left A_∞-modules will always be denoted by ${}_A\mathbf{M}$ (see Definition 10.3). For an A_∞-coalgebra C, the category of left A_∞-comodules will be denoted by ${}^C\mathbf{M}$ (see Definition 10.6).

7.4. Proposition. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded vector space. Consider the collection of functors*

$$\mathbb{U}^A = \{\mathbb{U}_n^A := A_n \otimes \_\_ : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}} \tag{7.10}$$

Then, the family $\{\mathbb{U}_n^A\}$ can be equipped with the structure of an A_∞-monad if and only if $A = \bigoplus_{n\in\mathbb{Z}} A_n$ can be equipped with the structure of an A_∞-algebra.

Proof. If A is an A_∞-algebra, it follows from Proposition 7.1 that $\{\mathbb{U}_n^A\}$ can be equipped with the structure of an A_∞-monad. Conversely, suppose that the collection $\mathbb{U}^A$ as defined in (7.10) is an A_∞-monad. Then, by applying the natural transformations $\theta_k^A(\bar{n}) : \mathbb{U}_{\bar{n}}^A = \mathbb{U}_{n_1}^A \circ \ldots \circ \mathbb{U}_{n_k}^A \longrightarrow \mathbb{U}_{n_1+\ldots+n_k+(2-k)}^A = \mathbb{U}_{\Sigma\bar{n}+(2-k)}^A$ to the vector space K, we obtain morphisms $A_{n_1} \otimes \ldots \otimes A_{n_k} \longrightarrow A_{n_1+\ldots+n_k+(2-k)}$. The relations in (2.2) now ensure that these satisfy the conditions for being an A_∞-algebra. ∎

7.5. COROLLARY. *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be an A_∞-algebra. Then, we have*
(a) The category $EM_{\mathbb{U}_A}$ of modules over the A_∞-monad $\mathbb{U}^A = \{\mathbb{U}^A_n := A_n \otimes \_\_ : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}}$ is isomorphic to the category ${}_A\mathbf{M}$ of A_∞-modules over A.
(b) Suppose that $\mathbb{U}^A$ is even. Then, the collection of functors

$$\mathbb{V}^A = \{\mathbb{V}^A_n := Hom(A_n, \_\_) : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}} \tag{7.11}$$

is an A_∞-comonad.
(c) Suppose that $\mathbb{U}^A$ is even. Then, we have isomorphisms of categories $EM^e_{\mathbb{U}_A} \cong EM^{\mathbb{V}_A}_e$ and $\widetilde{EM}^{eo}_{\mathbb{U}_A} \cong \widetilde{EM}^{\mathbb{V}_A}_{eo}$.

PROOF. By inspecting the conditions in Definition 3.1, we can see directly that $\mathbb{U}^A$-modules correspond to A_∞-modules over A in the sense of Definition 10.3. This proves (a). Since the functors $\{Hom(A_n, \_\_)\}_{n\in\mathbb{Z}}$ are right adjoint to the functors $\{A_n \otimes \_\_\}_{n\in\mathbb{Z}}$, the result of (b) follows from Theorem 2.3. The result of (c) is also clear from Proposition 3.5 and Theorem 3.8. ■

7.6. PROPOSITION. *Let $C = \bigoplus_{n\in\mathbb{Z}} C_n$ be an A_∞-coalgebra. Then, there is a canonical functor*

$$\mathscr{I}^C : {}^C\mathbf{M} \longrightarrow EM^{\mathbb{U}^{TC}} \tag{7.12}$$

that embeds A_∞-comodules as a full subcategory of the Eilenberg-Moore category of comodules over the A_∞-comonad $\mathbb{U}^{TC}$.

PROOF. Let P be an A_∞-comodule over C in the sense of Definition 10.6, equipped with structure maps $\{w^P_k : P \longrightarrow C^{\otimes k-1} \otimes P\}_{k\geq 1}$. We set $Q = TP$, i.e., $Q_n := P_{-n}$ for $n \in \mathbb{Z}$. For any $k \geq 1$, $\bar{n} \in \mathbb{Z}^{k-1}$, $l \in \mathbb{Z}$, we now have

$$\begin{tikzcd}
\rho_k(\bar{n}, l) : Q_{\Sigma\bar{n}+l+(2-k)} = P_{-\Sigma\bar{n}-l-(2-k)} \arrow[r, "w^P_{k,-\Sigma\bar{n}-l-(2-k)}"] & (C^{\otimes k-1} \otimes P)_{-\Sigma\bar{n}-l} \arrow[d] \\
& (TC)_{n_1} \otimes ... \otimes (TC)_{n_{k-1}} \otimes (TP)_l = \mathbb{U}^{TC}_{\bar{n}} Q_l
\end{tikzcd} \tag{7.13}$$

where the second map in (7.13) is the canonical projection. Due to the relations in Definition 10.6 satisfied by the structure maps of P, it is clear that the collection $\rho_k(\bar{n}, l)$ satisfies the relations in (3.5) for Q to be a $\mathbb{U}^{TC}$-comodule. As such, setting $\mathscr{I}^C(P) = Q = TP$ defines a functor $\mathscr{I}^C : {}^C\mathbf{M} \longrightarrow EM^{\mathbb{U}^{TC}}$.
It remains to show that $\mathscr{I}^C$ determines a full embedding. For this, we consider $P' \in {}^C\mathbf{M}$ along with structure maps $\{w^{P'}_k : P' \longrightarrow C^{\otimes k-1} \otimes P'\}_{k\geq 1}$ as well as $Q' = TP' = \mathscr{I}^C(P') \in EM^{\mathbb{U}^{TC}}$ with structure maps $\rho'_k(\bar{n}, l) : Q'_{\Sigma\bar{n}+l+(2-k)} \longrightarrow \mathbb{U}^{TC}_{\bar{n}} Q'_l$ determined as in (7.13). Let $h = \{h_n\}_{n\in\mathbb{Z}} \in EM^{\mathbb{U}^{TC}}(\mathscr{I}^C(P), \mathscr{I}^C(P'))$. For each fixed $T \in \mathbb{Z}$ and $k \geq 1$, we now observe that the outer square as well as the lower square in the following diagram

commute

$$\begin{array}{ccc}
Q_{T+(2-k)} = P_{-T-(2-k)} & \xrightarrow{h_{T+(2-k)}} & Q'_{T+(2-k)} = P'_{-T-(2-k)} \\
\downarrow w^P_{k,-T-(2-k)} & & \downarrow w^{P'}_{k,-T-(2-k)} \\
\bigoplus\limits_{(\bar n,l)\in\mathbb{Z}(T,k)} (C_{-n_1}\otimes \dots \otimes C_{-n_{k-1}}\otimes P_{-l}) & \xrightarrow{\bigoplus\limits_{(\bar n,l)\in\mathbb{Z}(T,k)} (C_{-n_1}\otimes\dots\otimes C_{-n_{k-1}}\otimes h_l)} & \bigoplus\limits_{(\bar n,l)\in\mathbb{Z}(T,k)} (C_{-n_1}\otimes \dots \otimes C_{-n_{k-1}}\otimes P'_{-l}) \\
\downarrow & & \downarrow \\
\prod\limits_{(\bar n,l)\in\mathbb{Z}(T,k)} (C_{-n_1}\otimes \dots \otimes C_{-n_{k-1}}\otimes P_{-l}) & \xrightarrow{\prod\limits_{(\bar n,l)\in\mathbb{Z}(T,k)} (C_{-n_1}\otimes\dots\otimes C_{-n_{k-1}}\otimes h_l)} & \prod\limits_{(\bar n,l)\in\mathbb{Z}(T,k)} (C_{-n_1}\otimes \dots \otimes C_{-n_{k-1}}\otimes P'_{-l})
\end{array} \tag{7.14}$$

Since the direct sums in (7.14) embed into the direct products, it follows that the upper square in (7.14) also commutes. We now see that $h = \{h_n\}_{n\in\mathbb{Z}}$ also determines a morphism in ${}^C\mathbf{M}$. ∎

8. Rational pairings between A_∞-algebras and A_∞-coalgebras

If $W \in Vect^{\mathbb{Z}}$ is a graded vector space, we note that by (7.2), its graded dual $W^* = [W, K]$ is given by $W^*_n = Vect(W_{-n}, K)$ for $n \in \mathbb{Z}$. If C is an A_∞-coalgebra, equipped with structure maps $\{w_k : C \longrightarrow C^{\otimes k}\}_{k\geq 1}$, we know that its graded dual C^* is an A_∞-algebra with $m_k : (C^*)^{\otimes k} \longrightarrow C^*$ determined by

$$m_k(\phi_1 \otimes \cdots \otimes \phi_k) = (-1)^{k(|\phi_1|+\ldots+|\phi_k|+1)}(\mu \circ (\phi_1 \otimes \cdots \otimes \phi_k) \circ w_k) \tag{8.1}$$

where μ denotes the multiplication on K. If $M \in \mathbf{M}^C$ is a right C-comodule, we know that M may be treated as a left C^*-module with structure maps given by

$$(C^*)^{\otimes k-1} \otimes M \xrightarrow{(C^*)^{\otimes k-1}\otimes w^M_k} (C^*)^{\otimes k-1} \otimes M \otimes C^{\otimes k-1} \longrightarrow M \tag{8.2}$$

where $\{w^M_k : M \longrightarrow M \otimes C^{\otimes k-1}\}_{k\geq 1}$ comes from the right C-comodule structure of M.

8.1. Definition. *Let C be an A_∞-coalgebra and let A be an A_∞-algebra. A pairing between C and A is an A_∞-algebra morphism $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ from A to the graded dual C^* of C.*

8.2. Theorem. *Let C be an A_∞-coalgebra and let A be an A_∞-algebra. Then, a pairing $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ between C and A induces a functor $\iota(C, A) : \mathbf{M}^C \longrightarrow {}_A\mathbf{M}$.*

Proof. From (8.2), it follows that we have a functor $\mathbf{M}^C \longrightarrow {}_{C^*}\mathbf{M}$. Considering the morphism $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ of A_∞-algebras, we also have a functor ${}_{C^*}\mathbf{M} \longrightarrow {}_A\mathbf{M}$ given by restriction of scalars (see, for instance, [27, § 6.2]). The result is now clear. ∎

We now want to define a rational pairing between an A_∞-coalgebra and an A_∞-algebra. For this, we will need to refine the result of Theorem 8.2. First, we consider a morphism $f = \{f_k : A^{\otimes k} \longrightarrow B\}_{k\geq 1}$ of A_∞-algebras as in Definition 10.2. Let M be a left B-module, equipped with structure maps $\{m_k^{B,M} : B^{\otimes k-1} \otimes M \longrightarrow M\}_{k\geq 1}$. Then, the restriction of scalars makes M into an A-module, determined by setting (see, for instance, [27, § 6.2])

$$m_{k+1}^{A,M} : A^{\otimes k} \otimes M \longrightarrow M \qquad m_{k+1}^{A,M} = \sum (-1)^s m_{l+1}^{B,M}(f_{i_1} \otimes ... \otimes f_{i_l} \otimes M) \tag{8.3}$$

for $k \geq 0$, where the sum is taken over all decompositions $k = i_1 + ... + i_l$ and $s = (l-1)(i_1 - 1) + (l-2)(i_2 - 1) + \cdots + 2(i_{l-2} - 1) + (i_{l-1} - 1)$. We also note that $f = \{f_k : A^{\otimes k} \longrightarrow B\}_{k\geq 1}$ determines a morphism

$$\tilde{f} : \left(\bigoplus_{k\geq 0} A^{\otimes k}\right) \longrightarrow \left(\bigoplus_{l\geq 0} B^{\otimes l}\right) \tag{8.4}$$

of graded vector spaces, where the action of $\tilde{f}$ on the component $A^{\otimes k}$ is given by $\sum(-1)^s(f_{i_1} \otimes ... \otimes f_{i_l})$, where the signs are as in (8.3). Accordingly, for any $V \in Vect^{\mathbb{Z}}$, we have an induced morphism

$$[\tilde{f}, V] : \prod_{l\geq 0}[B^{\otimes l}, V] = \left[\left(\bigoplus_{l\geq 0} B^{\otimes l}\right), V\right] \longrightarrow \left[\left(\bigoplus_{k\geq 0} A^{\otimes k}\right), V\right] = \prod_{k\geq 0}[A^{\otimes k}, V] \tag{8.5}$$

In particular, we consider $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ determining the pairing between the A_∞-coalgebra C and the A_∞-algebra A. For any $V \in Vect^{\mathbb{Z}}$, we note that the closed symmetric structure on $Vect^{\mathbb{Z}}$ induces a canonical morphism

$$\alpha(C, A, V) : \prod_{l\geq 0} V \otimes C^{\otimes l} \longrightarrow \prod_{l\geq 0}[(C^*)^{\otimes l}, V] \xrightarrow{[\tilde{f}, V]} \prod_{k\geq 0}[A^{\otimes k}, V] \tag{8.6}$$

8.3. Definition. *Let $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ be a pairing of an A_∞-coalgebra C and an A_∞-algebra A. We will say that the pairing is rational if the canonical morphism $\alpha(C, A, V)$ as defined in (8.6) is a monomorphism in $Vect^{\mathbb{Z}}$ for any $V \in Vect^{\mathbb{Z}}$.*

8.4. Proposition. *Let $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ be a rational pairing of an A_∞-coalgebra C and an A_∞-algebra A. Then, the functor $\iota(C, A) : \mathbf{M}^C \longrightarrow {}_A\mathbf{M}$ embeds $\mathbf{M}^C$ as a full subcategory of ${}_A\mathbf{M}$.*

Proof. We consider M, $N \in \mathbf{M}^C$ with structure maps $\{w_k^M : M \longrightarrow M \otimes C^{\otimes k-1}\}_{k\geq 1}$ and $\{w_k^N : N \longrightarrow N \otimes C^{\otimes k-1}\}_{k\geq 1}$ respectively. If we treat M, N as left A-modules via the functor $\iota(C, A)$, we have to show that any A-module morphism $h : M \longrightarrow N$ is also a morphism in $\mathbf{M}^C$. We consider therefore the following diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & N \\ {\scriptstyle \prod w_{l+1}^M}\downarrow & & \downarrow{\scriptstyle \prod w_{l+1}^N} \\ \prod M \otimes C^{\otimes l} & \xrightarrow{\prod h \otimes C^{\otimes l}} & \prod N \otimes C^{\otimes l} \\ {\scriptstyle \alpha(C,A,M)}\downarrow & & \downarrow{\scriptstyle \alpha(C,A,N)} \\ \prod[A^{\otimes k}, M] & \xrightarrow{\prod[A^{\otimes k},h]} & \prod[A^{\otimes k}, N] \end{array} \tag{8.7}$$

Considering the explicit definition of restriction of scalars in (8.3) and the construction of the functor $\iota(C, A)$ in Theorem 8.2, we realize that the outer square in (8.7) commutes because h is an A-module morphism. The lower square in (8.7) commutes because h is a morphism of graded vector spaces. Accordingly, we have

$$\begin{aligned} &\alpha(C, A, N) \circ \left(\prod w_{l+1}^N\right) \circ h \\ &= \left(\prod[A^{\otimes k}, h]\right) \circ \alpha(C, A, M) \circ \left(\prod w_{l+1}^M\right) = \alpha(C, A, N) \circ \left(\prod h \otimes C^{\otimes l}\right) \circ \left(\prod w_{l+1}^M\right) \end{aligned} \tag{8.8}$$

Since the pairing is rational by assumption, we know that $\alpha(C, A, N)$ is a monomorphism. Accordingly, it follows from (8.8) that the upper square in (8.7) also commutes. This proves that h is a morphism of right C-comodules. ∎

8.5. Proposition. *Let $f = \{f_k : A^{\otimes k} \longrightarrow C^*\}_{k\geq 1}$ be a rational pairing of an A_∞-coalgebra C and an A_∞-algebra A. Then, the category $\mathbf{M}^C$, treated as a full subcategory of ${}_A\mathbf{M}$, is closed under direct sums, subobjects and quotients.*

Proof. It is clear that $\mathbf{M}^C$, treated as a full subcategory of ${}_A\mathbf{M}$, is closed under direct sums. Suppose we have a short exact sequence

$$0 \longrightarrow M' \xrightarrow{g} M \xrightarrow{h} M'' \longrightarrow 0 \tag{8.9}$$

in ${}_A\mathbf{M}$ such that $M \in \mathbf{M}^C$ with structure maps $\{w_{l+1}^M : M \longrightarrow M \otimes C^{\otimes l}\}_{l\geq 0}$. Since products of short exact sequences in $Vect^{\mathbb{Z}}$ are exact, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{g} & M & \xrightarrow{h} & M'' & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \prod w_{\bullet+1}^M} & & & & \\ 0 & \longrightarrow & \prod(M' \otimes C^\bullet) & \xrightarrow{\prod(g\otimes C^\bullet)} & \prod(M \otimes C^\bullet) & \xrightarrow{\prod(h\otimes C^\bullet)} & \prod(M'' \otimes C^\bullet) & \longrightarrow & 0 \\ & & {\scriptstyle \alpha(C,A,M')}\downarrow & & {\scriptstyle \alpha(C,A,M)}\downarrow & & {\scriptstyle \alpha(C,A,M'')}\downarrow & & \\ 0 & \longrightarrow & \prod[A^\bullet, M'] & \xrightarrow{\prod[A^\bullet,g]} & \prod[A^\bullet, M] & \xrightarrow{\prod[A^\bullet,h]} & \prod[A^\bullet, M''] & \longrightarrow & 0 \end{array} \tag{8.10}$$

Since g is an A-module morphism, we note that the composition $\alpha(C,A,M)\circ\left(\prod w^M_{\bullet+1}\right)\circ g$ factors through $\prod[A^\bullet,g]$. Accordingly, we have $(\prod[A^\bullet,h])\circ\alpha(C,A,M)\circ\left(\prod w^M_{\bullet+1}\right)\circ g=0$. It follows that

$$\alpha(C,A,M'')\circ\left(\prod(h\otimes C^\bullet)\right)\circ\left(\prod w^M_{\bullet+1}\right)\circ g=\left(\prod[A^\bullet,h]\right)\circ\alpha(C,A,M)\circ\left(\prod w^M_{\bullet+1}\right)\circ g=0 \tag{8.11}$$

Since f is a rational pairing, we know that $\alpha(C,A,M'')$ is a monomorphism, whence it follows from (8.11) that $(\prod(h\otimes C^\bullet))\circ\left(\prod w^M_{\bullet+1}\right)\circ g=0$. Since the middle row in (8.10) is short exact, there are now induced maps $M'\longrightarrow\prod(M'\otimes C^\bullet)$ and $M''\longrightarrow\prod(M''\otimes C^\bullet)$ making the diagram commutative and giving M', M'' the structure of C-comodules. This proves the result. ∎

Given the rational pairing $f=\{f_k:A^{\otimes k}\longrightarrow C^*\}_{k\geq 1}$, we now set for any $M\in {}_A\mathbf{M}$

$$R_f(M):=\sum_{g\in {}_A\mathbf{M}(\iota(C,A)(N),M),N\in\mathbf{M}^C} Im(g)\subseteq M \tag{8.12}$$

We now have the main result on rational pairings of A_∞-coalgebras and A_∞-algebras.

8.6. Theorem. *Let $f=\{f_k:A^{\otimes k}\longrightarrow C^*\}_{k\geq 1}$ be a rational pairing of an A_∞-coalgebra C and an A_∞-algebra A. Then, we have a functor $R_f:{}_A\mathbf{M}\longrightarrow\mathbf{M}^C$ that is right adjoint to $\iota(C,A):\mathbf{M}^C\longrightarrow {}_A\mathbf{M}$, i.e., we have natural isomorphisms*

$$ {}_A\mathbf{M}(\iota(C,A)(M'),M)\cong\mathbf{M}^C(M',R_f(M)) \tag{8.13}$$

for any $M'\in\mathbf{M}^C$ and $M\in {}_A\mathbf{M}$.

Proof. We consider some $g\in {}_A\mathbf{M}(\iota(C,A)(N),M)$, with $N\in\mathbf{M}^C$. Since $\mathbf{M}^C$ as a full subcategory of ${}_A\mathbf{M}$ is closed under quotients, we see that $Im(g)\in\mathbf{M}^C$. Applying Proposition 8.5 again, since $\mathbf{M}^C$ is closed under direct sums and quotients in ${}_A\mathbf{M}$, we see that the sum $R_f(M)=\sum Im(g)$ defined as in (8.12) must in $\mathbf{M}^C$.
Further, from the definition in (8.12), the image of any morphism $\iota(C,A)(M')\longrightarrow M$ in ${}_A\mathbf{M}$ must lie in $R_f(M)$. From this it is clear that we have an isomorphism

$$ {}_A\mathbf{M}(\iota(C,A)(M'),M)\cong\mathbf{M}^C(M',R_f(M)) \tag{8.14}$$

It remains to show that R_f is a functor. For this, we consider a morphism $h:M_1\longrightarrow M_2$ in ${}_A\mathbf{M}$. Since $R_f(M_1)$ lies in $\mathbf{M}^C$, it follows from the definition in (8.12) that the image of the composition $R_f(M_1)\hookrightarrow M_1\xrightarrow{h}M_2$ lies in $R_f(M_2)$. This proves the result.

∎

9. A_∞-contramodules

Let $C = \bigoplus\limits_{n\in\mathbb{Z}} C_n$ be an A_∞-coalgebra, equipped with structure maps $\{w_k : C \longrightarrow C^{\otimes k}\}_{k\geq 1}$, where w_k has degree $2-k$ (see Definition 10.5). For any $k \geq 1$ and $\bar{n} = (n_1, ..., n_k) \in \mathbb{Z}^k$, let $w_k(\bar{n}) : C_{-\Sigma\bar{n}-(2-k)} \longrightarrow C_{-n_1} \otimes ... \otimes C_{-n_k}$ be a component of the structure map $w_k : C \longrightarrow C^{\otimes k}$ of C. We define $w'_k : C \longrightarrow C^{\otimes k}$ whose components are given by $w'_k(\bar{n}) := (-1)^{k(\Sigma\bar{n}+1)} w_k(\bar{n}) : C_{-\Sigma\bar{n}-(2-k)} \longrightarrow C_{-n_1} \otimes ... \otimes C_{-n_k}$. We are now ready to introduce A_∞-contramodules over C.

9.1. Definition. *Let C be an A_∞-coalgebra. An A_∞-contramodule is a $\mathbb{Z}$-graded vector space M equipped with structure maps*

$$t_k^M : [C^{\otimes k-1}, M] \longrightarrow M \qquad k \geq 1 \tag{9.1}$$

with t_k^M of degree $(2-k)$ satisfying, for each $n \geq 1$,

$$\sum (-1)^{p+qr} t^M_{p+1+r} \circ [C^{\otimes p} \otimes w'_q \otimes C^{\otimes r-1}, M] + \sum (-1)^a t^M_{a+1} \circ [C^{\otimes a}, t^M_b] = 0 \tag{9.2}$$

where the first sum in (9.2) is taken over all $p \geq 0$, $q, r \geq 1$ such that $p+q+r = n$ and the second sum in (9.2) is taken over all $a \geq 0$, $b \geq 1$ with $a + b = n$. In (9.2), we choose the isomorphism $[C^{\otimes a}, [C^{\otimes b-1}, M]] \cong [C^{\otimes a+b-1}, M]$ making M a right A_∞-contramodule. A morphism $h : M \longrightarrow M'$ of A_∞-contramodules is a morphism of graded vector spaces that is compatible with the respective structure maps. We will denote by $\mathbf{M}_{[C,--]}$ the category of A_∞-contramodules over C.

We say that $M \in \mathbf{M}_{[C,--]}$ is even if $t_k^M = 0$ whenever k is odd. We denote by $\mathbf{M}^e_{[C,--]}$ the full subcategory of $\mathbf{M}_{[C,--]}$ consisting of even objects. By Proposition 7.3, we know that when C is even, the collection of functors

$$\mathbb{V}^{TC} = \{\mathbb{V}_n^{TC} := Hom(C_{-n}, --) : Vect \longrightarrow Vect\}_{n\in\mathbb{Z}} \tag{9.3}$$

is an A_∞-monad. The following result is now the A_∞-contramodule counterpart of Proposition 7.6.

9.2. Proposition. *Let C be an even A_∞-coalgebra. Then, there is a canonical functor*

$$\mathscr{I}_{[C,--]} : \mathbf{M}^e_{[C,--]} \longrightarrow EM^e_{\mathbb{V}^{TC}} \tag{9.4}$$

that embeds $\mathbf{M}^e_{[C,--]}$ as a subcategory of even modules over the A_∞-monad $\mathbb{V}^{TC}$. If C is such that $C_n \neq 0$ for only finitely many $n \in \mathbb{Z}$, then the categories $\mathbf{M}_{[C,--]}$ and $EM_{\mathbb{V}^{TC}}$ are isomorphic.

PROOF. Let $M \in \mathbf{M}^e_{[C,\text{--}]}$. For $k \geq 1$, $\bar{n} \in \mathbb{Z}^{k-1}$, $l \in \mathbb{Z}$, we define

$$\pi_k(\bar{n}, l) : \mathbb{V}^{TC}_{\bar{n}} M_l = [C_{-n_1} \otimes ... \otimes C_{-n_{k-1}}, M_l] \hookrightarrow [C^{\otimes k-1}, M]_{\Sigma\bar{n}+l} \xrightarrow{t^M_{k,\Sigma\bar{n}+l}} M_{\Sigma\bar{n}+l+(2-k)} \tag{9.5}$$

We denote by $\{\theta_k(\bar{n})\}$ the structure maps of the A_∞-monad $\mathbb{V}^{TC}$. For $N \geq 0$, $\bar{z} = (\bar{n}, \bar{n}', \bar{n}'') \in \mathbb{Z}^N$ with $|\bar{n}| = p$, $|\bar{n}'| = q$, $|\bar{n}''| = r-1$, we observe that the following diagram is commutative

$$\begin{array}{ccccc}
\mathbb{V}^{TC}_{\bar{n}}\mathbb{V}^{TC}_{\bar{n}'}\mathbb{V}^{TC}_{\bar{n}''}M_l & \xrightarrow{(\mathbb{V}^{TC}_{\bar{n}} * \theta_q(\bar{n}') * \mathbb{V}^{TC}_{\bar{n}''})M_l} & \mathbb{V}^{TC}_{\bar{n}}\mathbb{V}^{TC}_{\Sigma\bar{n}'+(2-q)}\mathbb{V}^{TC}_{\bar{n}''}M_l & \xrightarrow{\pi_{p+1+r}(\bar{n}, \Sigma\bar{n}'+(2-q), \bar{n}'', l)} & M_{\Sigma\bar{z}+l+(2-N)} \\
\downarrow & & \downarrow & & \downarrow \\
[C^{\otimes N}, M]_{\Sigma\bar{z}+l} & \xrightarrow{[C^{\otimes p} \otimes w'_q \otimes C^{\otimes r-1}, M]_{\Sigma\bar{z}+l}} & [C^{\otimes(p+r)}, M]_{\Sigma\bar{z}+l+(2-q)} & \xrightarrow{t^M_{p+1+r, \Sigma\bar{z}+l+(2-q)}} & M_{\Sigma\bar{z}+l+(2-N)}
\end{array} \tag{9.6}$$

Similarly, for $\bar{z} = (\bar{m}, \bar{m}') \in \mathbb{Z}^N$ with $|\bar{m}| = a$, $|\bar{m}'| = b-1$, we observe that the following diagram is commutative

$$\begin{array}{ccccc}
\mathbb{V}^{TC}_{\bar{m}}\mathbb{V}^{TC}_{\bar{m}'}M_l & \xrightarrow{\mathbb{V}^{TC}_{\bar{m}}(\pi_b(\bar{m}', l))} & \mathbb{V}^{TC}_{\bar{m}}M_{\Sigma\bar{m}'+l+(2-b)} & \xrightarrow{\pi_{a+1}(\bar{m}, \Sigma\bar{m}'+l+(2-b))} & M_{\Sigma\bar{z}+l+(2-N)} \\
\downarrow & & \downarrow & & \downarrow \\
[C^{\otimes a}, [C^{\otimes b-1}, M]]_{\Sigma\bar{z}+l} & \xrightarrow{[C^{\otimes a}, t^M_b]_{\Sigma\bar{z}+l}} & [C^{\otimes a}, M]_{\Sigma\bar{z}+l+(2-b)} & \xrightarrow{t^M_{a+1, \Sigma\bar{z}+l+(2-b)}} & M_{\Sigma\bar{z}+l+(2-N)}
\end{array} \tag{9.7}$$

Using the fact that the expressions in 9.2 are zero, it follows from (9.6) and (9.7) that the $\pi_k(\bar{n}, l)$ defined in (9.5) make M into a module over the A_∞-monad $\mathbb{V}^{TC}$. We therefore have the functor $\mathbf{M}^e_{[C,\text{--}]} \longrightarrow EM^e_{\mathbb{V}^{TC}}$.

Now suppose that $C_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. Then, it follows that for any $T \in \mathbb{Z}$, we have

$$[C^{\otimes k-1}, M]_T = \prod_{n_1+...+n_{k-1}+l=T} [C_{-n_1} \otimes ... \otimes C_{-n_{k-1}}, M_l] = \bigoplus_{n_1+...+n_{k-1}+l=T} \mathbb{V}^{TC}_{n_1}...\mathbb{V}^{TC}_{n_{k-1}}(M_l) \tag{9.8}$$

In other words, the structure maps $t^M_k : [C^{\otimes k-1}, M] \longrightarrow M$ of $M \in \mathbf{M}^e_{[C,\text{--}]}$ are completely determined by the induced morphisms $\mathbb{V}^{TC}_{n_1}...\mathbb{V}^{TC}_{n_{k-1}}(M_l) \longrightarrow M_{n_1+...+n_{k-1}+l+(2-k)}$ on components. It follows that the categories $\mathbf{M}^e_{[C,\text{--}]}$ and $EM^e_{\mathbb{V}^{TC}}$ are isomorphic. ∎

9.3. PROPOSITION. *Let C be an A_∞-coalgebra. Then, there exists a faithful functor $\kappa^C : \mathbf{M}_{[C,\text{--}]} \longrightarrow \mathbf{M}_{C^*}$. If the A_∞-coalgebra C is such that the canonical map $W \otimes C^* \longrightarrow [C, W]$ is an epimorphism for each $W \in Vect^{\mathbb{Z}}$, then κ^C is an isomorphism.*

PROOF. Let $M \in \mathbf{M}_{[C,\text{--}]}$, with structure maps as in (9.1). Then, for any $k \geq 1$, we consider the compositions

$$m^M_k : M \otimes (C^*)^{\otimes k-1} \longrightarrow [C^{\otimes k-1}, M] \xrightarrow{t^M_k} M \tag{9.9}$$

where the morphism $M \otimes (C^*)^{\otimes k-1} \longrightarrow [C^{\otimes k-1}, M]$ in (9.9) is induced by the canonical morphism $M \otimes (C^*)^{\otimes k-1} \otimes C^{\otimes k-1} \longrightarrow M$. This makes M into a C^*-module. It is also clear that the functor κ^C is faithful.

In particular, suppose that the A_∞-coalgebra C is such that the canonical map $W \otimes C^* \longrightarrow [C, W]$ is an epimorphism for each $W \in Vect^{\mathbb{Z}}$. We note that for each $n \in \mathbb{Z}$, the canonical map

$$(W \otimes C^*)_n = \bigoplus_{k\in\mathbb{Z}} (W_{k+n} \otimes (C^*)_{-k}) \longrightarrow \bigoplus_{k\in\mathbb{Z}} [C_k, W_{k+n}] \hookrightarrow \prod_{k\in\mathbb{Z}} [C_k, W_{k+n}] = [C, W]_n \quad (9.10)$$

is always a monomorphism, i.e. $W \otimes C^* \longrightarrow [C, W]$ is a monomorphism for each $W \in Vect^{\mathbb{Z}}$. By assumption, the morphisms in (9.10) are also epimorphisms. Then, the maps $M \otimes (C^*)^{\otimes k-1} \longrightarrow [C^{\otimes k-1}, M]$ which appear in (9.9) are all isomorphisms and it is clear that the functor κ^C is an isomorphism. ∎

9.4. Proposition. *Let C be an A_∞-coalgebra. Then, there is a functor $[C, \_\_] : Vect^{\mathbb{Z}} \longrightarrow \mathbf{M}_{[C,\_\_]}$.*

Proof. We consider $W \in Vect^{\mathbb{Z}}$ and put $N = [C, W]$. By setting $t_k^N := [w_k', W] : [C^{\otimes k-1}, N] = [C^{\otimes k-1}, [C, W]] = [C^{\otimes k}, W] \longrightarrow [C, W] = N$ where $w_k' : C \longrightarrow C^{\otimes k}$ is as at the beginning of this section, we obtain $N = [C, W] \in \mathbf{M}_{[C,\_\_]}$. ∎

We will now construct functors between categories of A_∞-comodules and A_∞-contramodules. Let $(C, w_\bullet^C)$, $(D, w_\bullet^D)$ be A_∞-coalgebras. By a (C, D)-space $(N, w_\bullet^L, w_\bullet^R)$, we will mean a graded vector space N equipped with families of morphisms

$$w_\bullet^L = \{w_p^L : N \longrightarrow C^{p-1} \otimes N\}_{p\geq 1} \qquad w_\bullet^R = \{w_q^R : N \longrightarrow N \otimes D^{\otimes q-1}\}_{q\geq 1} \qquad (9.11)$$

such that $(N, w_\bullet^L)$ is a left C-comodule and $(N, w_\bullet^R)$ is a right D-comodule. In particular, any (C, D)-bicomodule becomes a (C, D)-space in an obvious manner. Additionally, for Q, $Q' \in \mathbf{M}^D$, we set $[Q, Q']_p^D = \mathbf{M}^D(Q[-p], Q')$ for $p \in \mathbb{Z}$.

9.5. Proposition. *Let $(C, w_\bullet^C)$, $(D, w_\bullet^D)$ be A_∞-coalgebras and let $(N, w_\bullet^L, w_\bullet^R)$ be a (C, D)-space. Then, for each $Q \in \mathbf{M}^D$ and each $k \geq 1$, there is an isomorphism*

$$[C^{\otimes k-1}, [N, Q]^D] \cong [C^{\otimes k-1} \otimes N, Q]^D \qquad (9.12)$$

Proof. We consider $f \in [C^{\otimes k-1}, [N, Q)]^D]$. Then, f corresponds to

$$f' : C^{\otimes k-1} \otimes N \longrightarrow Q \qquad (\tilde{c} \otimes n) \mapsto f(\tilde{c})(n) \qquad (9.13)$$

In order to prove that $f' \in [C^{\otimes k-1} \otimes N, Q]^D$, we have to show that the following diagram commutes for each $l \geq 1$

$$\begin{array}{ccc} C^{\otimes k-1} \otimes N & \xrightarrow{C^{\otimes k-1} \otimes w_l^R} & C^{\otimes k-1} \otimes N \otimes D^{\otimes l-1} \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle (f' \otimes D^{\otimes l-1})} \\ Q & \xrightarrow{w_l^Q} & Q \otimes D^{\otimes l-1} \end{array} \quad \Rightarrow \quad \begin{array}{l} f(\tilde{c})(w_l^R(n)_0) \otimes w_l^R(n)_1 \\ = w_l^Q(f(\tilde{c})(n))_0 \otimes w_l^Q(f(\tilde{c})(n))_1 \end{array} \qquad (9.14)$$

where we have adapted the Sweedler notation in (9.14). But since $f(\tilde{c})$ is a D-comodule morphism, we already have

$$w_l^Q(f(\tilde{c})(n))_0 \otimes w_l^Q(f(\tilde{c})(n))_1 = f(\tilde{c})(w_l^R(n)_0) \otimes w_l^R(n)_1 \tag{9.15}$$

From (9.14) and (9.15), it follows that $f' \in \mathbf{M}^D(C^{\otimes k-1} \otimes N, Q)$. Conversely, we consider $g \in [C^{\otimes k-1} \otimes N, Q]^D$. Then, g corresponds to

$$g' : C^{\otimes k-1} \longrightarrow [N, Q] \qquad g'(\tilde{c})(n) := g(\tilde{c} \otimes n) \tag{9.16}$$

In order to show that g' takes values in $[N, Q]^D$, we have to show that for each $l \geq 1$ we must have

$$\begin{aligned} w_l^Q(g(\tilde{c} \otimes n))_0 \otimes w_l^Q(g(\tilde{c} \otimes n))_1 \ &= w_l^Q(g'(\tilde{c})(n))_0 \otimes w_l^Q(g'(\tilde{c})(n))_1 \\ &= g'(\tilde{c})(w_l^R(n)_0) \otimes w_l^R(n)_1 = g(\tilde{c} \otimes w_l^R(n)_0) \otimes w_l^R(n)_1 \end{aligned} \tag{9.17}$$

However since $g \in [C^{\otimes k-1} \otimes N, Q]^D$, we already know that

$$w_l^Q(g(\tilde{c} \otimes n))_0 \otimes w_l^Q(g(\tilde{c} \otimes n))_1 = g(\tilde{c} \otimes w_l^R(n)_0) \otimes w_l^R(n)_1 \tag{9.18}$$

From (9.17) and (9.18) it is clear that $g' \in [C^{\otimes k-1}, [N, Q]^D]$. This proves the result. ∎

We will say that $(N, w_\bullet^L, w_\bullet^R)$ is a commuting (C, D)-space if the C and D-coactions commute with each other, i.e., for each $k \geq 1$, $l \geq 1$, we have a commutative diagram

$$\begin{tikzcd} N \arrow[r, "w_l^R"] \arrow[d, "w_k^L"'] & N \otimes D^{\otimes l-1} \arrow[d, "w_k^L \otimes D^{\otimes l-1}"] \\ C^{\otimes k-1} \otimes N \arrow[r, "C^{\otimes k-1} \otimes w_l^R"] & C^{\otimes k-1} \otimes N \otimes D^{\otimes l-1} \end{tikzcd} \tag{9.19}$$

We note in particular that this means that each $w_l^R : N \longrightarrow N \otimes D^{\otimes l-1}$ is a morphism of left C-comodules, while each $w_k^L : N \longrightarrow C^{\otimes k-1} \otimes N$ is a morphism of right D-comodules. For the rest of this section, we will assume that both the C-comodule and D-comodule structures on N are even.

9.6. Proposition. *Let $(C, w_\bullet^C)$, $(D, w_\bullet^D)$ be even A_∞-coalgebras. Then, $(N, w_\bullet^L, w_\bullet^R)$ induces a functor $[N, \_\_]^D : \mathbf{M}^D \longrightarrow \mathbf{M}_{[C, \_\_]}$.*

Proof. We consider $Q \in \mathbf{M}^D$. We claim that $[N, Q]^D \in \mathbf{M}_{[C, \_\_]}$. From Proposition 9.5, we already have that $[C^{\otimes k-1}, [N, Q]^D] \cong [C^{\otimes k-1} \otimes N, Q]^D$ for each $k \geq 1$. We now have structure maps

$$t_k : [C^{\otimes k-1}, [N, Q]^D] \cong [C^{\otimes k-1} \otimes N, Q]^D \xrightarrow{[w_k^L, Q]^D} [N, Q]^D \tag{9.20}$$

where we have used the fact that each morphism $w_k^L : N \longrightarrow C^{\otimes k-1} \otimes N$ is a morphism of right D-comodules. It may be verified that the morphisms in (9.20) make $[N, Q]^D$ into an A_∞-contramodule over C. ∎

Let $(M, t_\bullet^M)$ be an A_∞-contramodule over C. Then, for each $k \geq 1$, we have a pair of canonical maps

$$[C^{\otimes k-1}, M] \otimes N \underset{t_k^M \otimes N}{\overset{(ev_{[C^{\otimes k-1},M]} \otimes N)\circ([C^{\otimes k-1},M]\otimes w_k^L)}{\rightrightarrows}} M \otimes N \tag{9.21}$$

where $ev_{[C^{\otimes k-1},M]} : [C^{\otimes k-1}, M] \otimes C^{\otimes k-1} \longrightarrow M$ is the canonical evaluation map. We now define the contratensor product $M \boxtimes_C N$ to be the coequalizer

$$M \boxtimes_C N := Coeq\left(\bigoplus_{k\geq 1} [C^{\otimes k-1}, M] \otimes N \underset{t_k^M \otimes N}{\overset{\bigoplus_{k\geq 1} (ev_{[C^{\otimes k-1},M]} \otimes N)\circ([C^{\otimes k-1},M]\otimes w_k^L)}{\rightrightarrows}} M \otimes N \right) \tag{9.22}$$

We note that (9.22) extends the contratensor product on contramodules over coassociative coalgebras defined in [37]. We now have the following result.

9.7. Proposition. *Let $(C, w_\bullet^C)$, $(D, w_\bullet^D)$ be even A_∞-coalgebras. Then, $(N, w_\bullet^L, w_\bullet^R)$ induces a functor $\_\_ \boxtimes_C N : \mathbf{M}_{[C,\_\_]} \longrightarrow \mathbf{M}^D$.*

Proof. By (9.22), we know that for any $M \in \mathbf{M}_{[C,\_\_]}$, the contratensor product $M \boxtimes_C N$ is a quotient of $M \otimes N$. We claim that the maps $M \otimes w_l^R : M \otimes N \longrightarrow M \otimes N \otimes D^{\otimes l-1}$ descend to the quotient $M \boxtimes_C N$. For this, we note that for any $k, l \geq 1$, we have the following commutative diagram

$$\begin{tikzcd}
{[C^{\otimes k-1}, M] \otimes N} \arrow[r, "{([C^{\otimes k-1},M]\otimes w_k^L)}"] \arrow[d, "{[C^{\otimes k-1},M]\otimes w_l^R}"'] & {[C^{\otimes k-1},M] \otimes C^{\otimes k-1}\otimes N} \arrow[r, "{(ev_{[C^{\otimes k-1},M]}\otimes N)}"] \arrow[d, "{[C^{\otimes k-1},M]\otimes C^{\otimes k-1}\otimes w_l^R}"'] & M \otimes N \arrow[d, "M\otimes w_l^R"'] \\
{[C^{\otimes k-1}, M] \otimes N \otimes D^{\otimes l-1}} \arrow[r, "{([C^{\otimes k-1},M]\otimes w_k^L)\otimes D^{\otimes l-1}}"] & {[C^{\otimes k-1},M] \otimes C^{\otimes k-1}\otimes N\otimes D^{\otimes l-1}} \arrow[r, "{(ev_{[C^{\otimes k-1},M]}\otimes N)\otimes D^{\otimes l-1}}"] & M \otimes N \otimes D^{\otimes l-1}
\end{tikzcd} \tag{9.23}$$

We note that the left side square in (9.23) commutes because N is a commuting (C, D)-space. Also, for any $k, l \geq 1$, we have the commutative diagram

$$\begin{tikzcd}
{[C^{\otimes k-1}, M] \otimes N} \arrow[r, "{[C^{\otimes k-1},M]\otimes w_l^R}"] \arrow[d, "t_k^M\otimes N"'] & {[C^{\otimes k-1}, M] \otimes N \otimes D^{\otimes l-1}} \arrow[d, "t_k^M\otimes N\otimes D^{\otimes l-1}"] \\
M \otimes N \arrow[r, "M\otimes w_l^R"] & M \otimes N \otimes D^{\otimes l-1}
\end{tikzcd} \tag{9.24}$$

Considering (9.23), (9.24) and the definition in (9.22), we have an induced map on the coequalizers $M \boxtimes_C N \longrightarrow (M \boxtimes_C N) \otimes D^{\otimes l-1}$ for each $l \geq 1$, making $M \boxtimes_C N$ into an A_∞-comodule over D. ∎

The following simple observation will be used in the proof of the next theorem. If $(Q, w^Q_\bullet)$ and $(Q', w^{Q'}_\bullet)$ are D-comodules, then $\mathbf{M}^D(Q,Q')$ may be expressed as the equalizer

$$\mathbf{M}^D(Q,Q') = Eq\left([Q,Q'] \overset{f\mapsto \prod w^{Q'}_\bullet \circ f}{\underset{f\mapsto \prod (f\otimes D^{\otimes \bullet -1})\circ w^Q_\bullet}{\rightrightarrows}} \prod [Q, Q'\otimes D^{\otimes \bullet -1}] \right) \tag{9.25}$$

Similarly, if $(M, t^M_\bullet)$ and $(M', t^{M'}_\bullet)$ are C-contramodules, then $\mathbf{M}_{[C,--]}(M,M')$ may be expressed as the equalizer

$$\mathbf{M}_{[C,--]}(M,M') = Eq\left([M,M'] \overset{f\mapsto \prod t^{M'}_\bullet \circ [C^{\bullet -1}, f]}{\underset{f\mapsto \prod f\circ t^M_\bullet}{\rightrightarrows}} \prod [[C^{\otimes \bullet -1}, M], M'] \right) \tag{9.26}$$

We are now ready to prove the main result of this section.

9.8. Theorem. *Let $(C, w^C_\bullet)$, $(D, w^D_\bullet)$ be even A_∞-coalgebras. Then, any commuting (C,D)-space $(N, w^L_\bullet, w^R_\bullet)$ that is even leads to an adjunction of functors*

$$\mathbf{M}^D(M \boxtimes_C N, Q) \cong \mathbf{M}_{[C,--]}(M, [N,Q]^D) \tag{9.27}$$

for $M \in \mathbf{M}_{[C,--]}$ and $Q \in \mathbf{M}^D$.

Proof. Using the definition in (9.22) as well as the expressions in (9.25) and (9.26), we can now compute directly

$$\begin{aligned}
&\mathbf{M}^D(M\boxtimes_C N, Q)\\
&= Eq\left([M\boxtimes_C N, Q] \rightrightarrows \prod_{l\geq 1} [M\boxtimes_C N, Q\otimes D^{\otimes l-1}] \right)\\
&= Eq\left(\left[Coeq\left(\bigoplus_{k\geq 1} [C^{\otimes k-1}, M]\otimes N \rightrightarrows M\otimes N \right), Q \right] \rightrightarrows \prod_{l\geq 1} \left[Coeq\left(\bigoplus_{k\geq 1} [C^{\otimes k-1}, M] \rightrightarrows M\otimes N \right), Q\otimes D^{\otimes l-1} \right] \right)\\
&= Eq\left(Eq\left([M\otimes N, Q] \rightrightarrows \prod_{k\geq 1} [[C^{\otimes k-1}, M]\otimes N, Q] \right) \rightrightarrows \prod_{l\geq 1} Eq\left([M\otimes N, Q\otimes D^{\otimes l-1}] \rightrightarrows \prod_{k\geq 1} [[C^{\otimes k-1}, M]\otimes N, Q\otimes D^{\otimes l-1}] \right) \right)\\
&= Eq\left(Eq\left([M, [N,Q]] \rightrightarrows \prod_{k\geq 1} [[C^{\otimes k-1}, M], [N,Q]] \right) \rightrightarrows \prod_{l\geq 1} Eq\left([M, [N, Q\otimes D^{\otimes l-1}]] \rightrightarrows \prod_{k\geq 1} [[C^{\otimes k-1}, M], [N, Q\otimes D^{\otimes l-1}]] \right) \right)\\
&= Eq\left(Eq\left([M, [N,Q]] \rightrightarrows \prod_{k\geq 1} [[C^{\otimes k-1}, M], [N,Q]] \right) \rightrightarrows Eq\left(\prod_{l\geq 1} [M, [N, Q\otimes D^{\otimes l-1}]] \rightrightarrows \prod_{l\geq 1}\prod_{k\geq 1} [[C^{\otimes k-1}, M], [N, Q\otimes D^{\otimes l-1}]] \right) \right)\\
&= Eq\left(Eq\left([M, [N,Q]] \rightrightarrows \prod_{l\geq 1} [M, [N, Q\otimes D^{\otimes l-1}]] \right) \rightrightarrows Eq\left(\prod_{k\geq 1} [[C^{\otimes k-1}, M], [N,Q]] \rightrightarrows \prod_{k\geq 1}\prod_{l\geq 1} [[C^{\otimes k-1}, M], [N, Q\otimes D^{\otimes l-1}]] \right) \right)\\
&= Eq\left(\left[M, Eq\left([N,Q] \rightrightarrows \prod_{l\geq 1} [N, Q\otimes D^{\otimes l-1}] \right) \right] \rightrightarrows \prod_{k\geq 1} \left[[C^{\otimes k-1}, M], Eq\left([N,Q] \rightrightarrows \prod_{l\geq 1} [N, Q\otimes D^{\otimes l-1}] \right) \right] \right)\\
&= Eq\left(\left[M, [N,Q]^D \right] \rightrightarrows \prod_{k\geq 1} \left[[C^{\otimes k-1}, M], [N,Q]^D \right] \right)\\
&= \mathbf{M}_{[C,--]}(M, [N,Q]^D)
\end{aligned}$$

■

We conclude with the following result.

9.9. Corollary. *Let $(C, w^C_\bullet)$, $(D, w^D_\bullet)$ be even A_∞-coalgebras. Let N_1 be an even left A_∞-comodule over C and N_2 be an even right A_∞-comodule over D. Then, we have an induced adjunction of functors, i.e., natural isomorphisms*

$$\mathbf{M}^D(M \boxtimes_C (N_1 \otimes N_2), Q) \cong \mathbf{M}_{[C,--]}(M, [(N_1 \otimes N_2), Q]^D) \tag{9.28}$$

for $M \in \mathbf{M}_{[C,--]}$ and $Q \in \mathbf{M}^D$.

Proof. If N_1 is a left C-comodule and N_2 is a right D-comodule, then $N_1 \otimes N_2$ carries left C-coactions and right D-coactions in the obvious manner. It is clear that the left C-coactions commute with the right D-coactions, making $N_1 \otimes N_2$ a commuting (C, D)-space. The result now follows from Theorem 9.8. ■

10. Appendix : Basic Definitions

For the convenience of the reader, we recall in this section the well known definitions of A_∞-algebras, A_∞-coalgebras as well as modules and comodules over them respectively.

10.1. Definition. *An A_∞-algebra over K is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{n\in\mathbb{Z}} A_n$ along with a family of graded K-linear maps $\{m_k : A^{\otimes k} \longrightarrow A\}_{k\geq 1}$ with m_k of degree $2-k$, satisfying for each $n \geq 1$:*

$$\sum (-1)^{p+qr} m_{p+1+r}(1^{\otimes p} \otimes m_q \otimes 1^{\otimes r}) = 0 \tag{10.1}$$

where 1 denotes the identity map and the sum is over all triples (p, q, r) such that $n = p+q+r$, $p, r \geq 0$, $q \geq 1$.

We should mention that the terms in (10.1) will have additional signs when applied to elements of A, following the usual Koszul sign conventions.

10.2. Definition. *Let A and B be A_∞-algebras. A morphism of A_∞-algebras $f : A \longrightarrow B$ is a family of graded K-linear maps $\{f_k : A^{\otimes k} \longrightarrow B\}_{k\geq 1}$ with f_k of degree $1-k$, satisfying for each $n \geq 1$:*

$$\sum (-1)^{p+qr} f_{p+1+r}(1^{\otimes p} \otimes m_q \otimes 1^{\otimes r}) = \sum (-1)^s m_t(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_t}) \tag{10.2}$$

where the first sum is over all (p, q, r) such that $n = p+q+r$ and the second sum is over all $1 \leq t \leq n$ and all decompositions $n = i_1 + i_2 + \cdots + i_t$, and the sign s is given by $s = (t-1)(i_1-1) + (t-2)(i_2-1) + \cdots + 2(i_{t-2}-1) + (i_{t-1}-1)$.

10.3. DEFINITION. *Let A be an A_∞-algebra over K. A (left) A_∞-module over A is a $\mathbb{Z}$-graded space M with maps $\{m_k^M : A^{\otimes k-1} \otimes M \longrightarrow M\}_{k\geq 1}$ with m_k^M of degree $2-k$, satisfying for each $n \geq 1$:*

$$\sum(-1)^{p+qr} m^M_{p+1+r}(1_A^{\otimes p} \otimes m_q \otimes 1_A^{\otimes r-1} \otimes 1_M) + \sum(-1)^a m^M_{a+1}(1_A^{\otimes a} \otimes m_b^M) = 0,$$

where the first sum is over all p, q, r such that $p+q+r = n$, $p \geq 0$, $q, r \geq 1$ and the second sum is over all a, b such that $a + b = n$, $a \geq 0$ and $b \geq 1$.
A morphism of left A_∞-modules $f : M \longrightarrow N$ is a family of maps $f = \{f_l : M_l \longrightarrow N_l\}_{l\in\mathbb{Z}}$ that is well behaved with respect to the structure maps of M and N. The category of left A_∞-modules over A will be denoted by ${}_A\mathbf{M}$. Similarly, we may define the category $\mathbf{M}_A$ of right A-modules.

10.4. DEFINITION. *Let M, N be left A_∞-modules over the A_∞-algebra A, with respective structure maps $\{m_k^M : A^{\otimes k-1} \otimes M \longrightarrow M\}_{k\geq 1}$ and $\{m_k^N : A^{\otimes k-1} \otimes N \longrightarrow N\}_{k\geq 1}$. An A_∞-morphism of (left) A_∞-modules $f : M \longrightarrow N$ is a family of maps*

$$f = \{f_k : A^{\otimes k-1} \otimes M \longrightarrow N\}_{k\geq 1}$$

with f_k of degree $1-k$, satisfying for each $n \geq 1$:

$$\sum(-1)^a f_{a+1}(1_A^{\otimes a} \otimes m_b^M) + \sum(-1)^{a+bc} f_{a+1+c}(1_A^{\otimes a} \otimes m_b \otimes 1_A^{\otimes c-1} \otimes 1_M) = \sum m^N_{a+1}(1_A^{\otimes a} \otimes f_b),$$

where:
(1) the first sum on the left side is taken over all a, b such that $n = a + b$, $a \geq 0$, $b \geq 1$
(2) the second sum on the left side is taken over all a, b, c such that $n = a + b + c$, $a \geq 0$, $b, c \geq 1$
(3) the sum on the right side is taken over all a, b such that $n = a + b$, $a \geq 0$ and $b \geq 1$.

10.5. DEFINITION. *An A_∞-coalgebra over K is a $\mathbb{Z}$-graded vector space $C = \bigoplus\limits_{n\in\mathbb{Z}} C_n$ along with a family of graded K-linear maps $\{w_k : C \longrightarrow C^{\otimes k}\}_{k\geq 1}$ with w_k of degree $2-k$, satisfying the following two conditions:*
(a) For each $n \geq 1$, we have

$$\sum(-1)^{pq+r}(1^{\otimes p} \otimes w_q \otimes 1^{\otimes r})w_{p+1+r} = 0, \ \forall\ k \geq 1,$$

where 1 denotes the identity map, and the sum is over all p, q, r such that $n = p+q+r$, $p, r \geq 0$, $q \geq 1$.
(b) The induced morphism $\prod\limits_{k\geq 1} w_k : C \longrightarrow \prod\limits_{k\geq 1} C^{\otimes k}$ factors through the direct sum $\bigoplus\limits_{k\geq 1} C^{\otimes k}$.

10.6. DEFINITION. *Let C be an A_∞-coalgebra over K with structure maps $\{w_k : C \longrightarrow C^{\otimes k}\}_{k\geq 1}$. A (left) A_∞-comodule over C is a $\mathbb{Z}$-graded space P with maps $\{w_k^P : P \longrightarrow C^{\otimes k-1} \otimes P\}_{k\geq 1}$ with w_k^P of degree $2-k$, satisfying, for each $n \geq 1$, we have*

$$\sum(-1)^{pq+r}(1_C^{\otimes p} \otimes w_q \otimes 1_C^{\otimes r-1} \otimes 1_P)w^P_{p+1+r} + \sum(-1)^{ab}(1^{\otimes a} \otimes w_b^P)w^P_{a+1} = 0,$$

where the first sum is over all p, q, r such that $n = p + q + r$, $p \geq 0$, $q, r \geq 1$ and the second sum is over all a, b such that $a + b = n$, $a \geq 0$, $b \geq 1$.
A morphism of left A_∞-comodules $g : P \longrightarrow Q$ is a family of maps $g = \{g_l : P_l \longrightarrow Q_l\}_{l \in \mathbb{Z}}$ that is well behaved with respect to the structure maps of P and Q. The category of left A_∞-comodules over C will be denoted by ${}^C\mathbf{M}$. Additionally, we will say that $P \in {}^C\mathbf{M}$ is strongly finite if the induced morphism $\prod_{k\geq 1} w_k^P : P \longrightarrow \prod_{k\geq 1}(C^{\otimes k-1} \otimes P)$ factors through the direct sum $\bigoplus_{k\geq 1}(C^{\otimes k-1} \otimes P)$. Similarly, we may define the category $\mathbf{M}^C$ of right C-comodules.

10.7. Definition. *Let C be an A_∞-coalgebra. Let P, Q be left A_∞-comodules over C with respective structure maps $\{w_k^P : P \longrightarrow C^{\otimes k-1} \otimes P\}_{k\geq 1}$ and $\{w_k^Q : Q \longrightarrow C^{\otimes k-1} \otimes Q\}_{k\geq 1}$. An A_∞-morphism of comodules $g : P \longrightarrow Q$ is a sequence of graded morphisms $\{g_k : P \longrightarrow C^{\otimes k-1} \otimes Q\}_{k\geq 1}$, with g_k of degree $1 - k$, satisfying for each $n \geq 1$:*

$$\sum(-1)^{ab}(1_C^{\otimes a} \otimes w_b^Q)g_{a+1} + \sum(-1)^{ab+c}(1_C^{\otimes a} \otimes w_b \otimes 1_C^{\otimes c-1} \otimes 1_Q)g_{a+1+c} = \sum(1_C^{\otimes a} \otimes g_b)w_{a+1}^P,$$

where
(1) the first sum on the left is over all a, b such that $n = a + b$, $a \geq 0$, $b \geq 1$.
(2) the second sum on the left is over all a, b, c such that $n = a + b + c$, $a \geq 0$, $b, c \geq 1$
(3) the sum on the right side is over all a, b such that $n = a + b$, $a \geq 0$, $b \geq 1$.

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Department of Mathematics,
Indian Institute of Science, Bangalore, India.

Stat-Math Unit, Indian Statistical Institute,
Bangalore, India.

Email: abhishekbanerjee1313@gmail.com
anita@isibang.ac.in

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