Theory and Applications of Categories, Vol. 45, No. 45, 2026, pp. 1904–1958.

REALISATIONS OF POSETS AND TAMENESS

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Abstract. We introduce a construction called realisation which transforms posets into posets. We show that realisations share several key features with upper semilattices. For example, we define local dimensions of points in a poset and show that these numbers for realisations behave in a similar way as they do for upper semilattices. Furthermore, similarly to upper semilattices, realisations have well behaved discrete approximations which are suitable for capturing homological properties of functors indexed by them. These discretisations are convenient and effective for describing tameness of functors. Homotopical and homological properties of tame functors, particularly those indexed by realisations, are discussed, with emphasis on the use of Koszul complexes to compute Betti diagrams of minimal free resolutions of tame functors indexed by upper semilattices and realisations.

1. Introduction

Motivation: If J is a finite poset, then the category of functors $\mathrm{Fun}(J, \mathcal{C})$ inherits key structural properties from $\mathcal{C}$. For example, if $\mathcal{C}$ is abelian (or a model category), then so is $\mathrm{Fun}(J, \mathcal{C})$ in which explicit constructions of minimal resolutions (or cofibrant replacements) can be obtained using J and resolutions (or cofibrant replacements) in $\mathcal{C}$. These are some of the reasons why functors indexed by finite posets are convenient to encode homological changes and have been extensively used in algebraic topology, algebraic geometry, homotopy theory, representation theory and, more recently, in topological data analysis (TDA). If J is infinite, then the category $\mathrm{Fun}(J, \mathcal{C})$ can be much more complex, for example minimal resolutions or cofibrant replacements may not exist for arbitrary functors. There are, however, cases in which indexing by infinite posets needs to be considered. For example, finite posets, relevant for a given situation, can share certain common features that can be expressed by these posets being subposets of an infinite or continuous poset J. By taking the left Kan extensions along these subposet inclusions,

A. Jin and F. Tombari were supported by the Wallenberg AI, Autonomous System and Software Program (WASP) funded by Knut and Alice Wallenberg Foundation. W. Chachólski was partially supported by VR, the Wallenberg AI, Autonomous System and Software Program (WASP) funded by Knut and Alice Wallenberg Foundation, and MultipleMS funded by the European Union under the Horizon 2020 program, grant agreement 733161, and dBRAIN collaborative project at digital futures at KTH.

Received by the editors 2026-04-17 and, in final form, 2026-08-10.

Transmitted by Julie Bergner. Published on 2026-08-18.

2020 Mathematics Subject Classification: 06A12, 18G10, 18A25. Secondary: 55N31, 55P99.

Key words and phrases: upper semilattice, dimension, functor category, Kan extension, free resolution, Koszul complex.

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we obtain functors indexed by J called tame. In such a situation, the full subcategory, $\mathrm{Tame}(J,\mathcal{C}) \subset \mathrm{Fun}(J,\mathcal{C})$, of tame functors, provides a convenient place in which functors indexed by different finite posets can be compared and studied together. Our aim is to introduce a construction of continuous posets, called realisation, for which tame functors have controllable homological properties. The poset $[0,\infty)^k$ with the product order is the prototype for realisations. Encoding information by tame functors indexed by $[0,\infty)^k$ is attractive for three reasons:

- metric properties of $[0,\infty)^k$ can be used to define and study distances on tame functors indexed by $[0,\infty)^k$ (see for example [CCGGO09, L15]), which are essential for addressing stability and can be used for hierarchical stabilisations (see [CR20, GC17, SCLRÖ17]);
- the poset $[0,\infty)^k$ has well behaved discrete approximations given by sublattices of the form $\mathbb{N}^k \hookrightarrow [0,\infty)^k$, which can be used to provide finite approximations of $[0,\infty)^k$;
- the mentioned discretisations and approximations have well studied homological properties, as the path algebra of $\mathbb{N}^k$ is isomorphic to the multigraded polynomial ring in k variables.

The key message of this paper is: *realisations form a rich family of* ***posets*** *satisfying analogues to the above properties of the poset* $[0,\infty)^k$.

The realisation construction transforms a poset I into a poset $\mathcal{R}(I)$. As a set, the realisation is given by the disjoint union of cubes $\mathcal{R}(I) = \coprod_{a,S}(-1,0)^{|S|}$ where a ranges over all the elements of I and S ranges over all finite subsets of parents of a that have ancestors. For example, if S is an empty subset of parents of a, then $(-1,0)^0$ is of size 1 and we identify its element with a. In this way we obtain an inclusion $I \subset \mathcal{R}(I)$. If $S = \{p\}$ is of size 1, then we think about the associated subposet $(-1,0) \subset \mathcal{R}(I)$ as the time to go back from a to its parent p. There exist similar constructions in the literature involving linear posets and quivers (see, for example, [PRY24, S16]). Figure 1 illustrates three simple examples of realisations of finite posets.

It turns out that the realisation of $\mathbb{N}^k$ with the product order is $[0,\infty)^k$, also equipped with the product order. If I is a finite-type distributive upper semilattice (or more generally, it is consistent, see 4.2), an analogous description of a partial order on its realisation $\mathcal{R}(I)$ can be obtained, so that the inclusions $I \subset \mathcal{R}(I) \supset (-1,0)^{|S|}$ preserve the poset relations. However, defining such a partial order on the realisation of an arbitrary poset is a nontrivial challenge. In fact, the discovery that this can be achieved not only for upper semilattices but for any poset has been one of the main inspirations for this article. Our strategy in this general case is to identify the realisation as a subposet of the Grothendieck construction of a certain lax functor, which may fail to be a functor in the case I is not consistent. The fact that for an arbitrary poset this functor is essentially lax is the main difficulty in expressing the poset relation of the realisation in simple terms. In particular, the realisation construction is not functorial with respect to I.

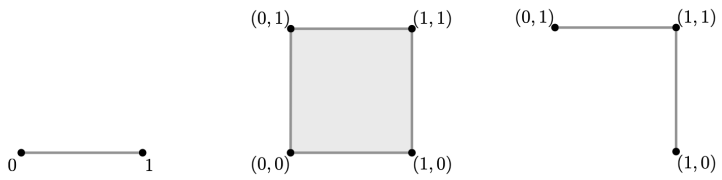


Figure 1: On the left, the realisation of the finite poset $\{0 < 1\}$, in the center, the one of $\{0 < 1\}^2$ and, on the right, the one of $\{(1, 0), (0, 1) < (1, 1)\}$. The different tones of gray represent the different dimensions of cube assembled together: 0-dimensional in black, 1-dimensional in gray, and 2-dimensional in lightgray.

Main results: Our story about realisations is divided into two parts. The first part is about combinatorial properties of realisations (Section 4). Our key results here are devoted to show that, despite the lack of functoriality of the realisation construction, the realisation $\mathcal{R}(I)$ inherits a poset structure from I, extending the poset relation of I itself. Furthermore, we show that realisations share certain regularity conditions with upper semilattices, such as that *dimension* and *parental dimension* of elements coincide, under finiteness assumptions (Corollary 4.18), and that the property of I being an upper semilattice is preserved by taking its realisation $\mathcal{R}(I)$, if I is distributive (Theorem 6.1), otherwise we show an example of this theorem failing (Example 6.9).

The second part is centered on tame functors. We show in Theorem 9.3 that a model structure on the category $\mathcal{C}$ naturally extends to a model structure on $\mathrm{Tame}(J, \mathcal{C})$ if J is an upper semilattice. The main difficulty here is to show that $\mathrm{Tame}(J, \mathcal{C})$ is closed under finite limits (Proposition 8.4). There are situations, however, when we would like to understand homological properties of tame functors indexed by realisations which are not upper semilattices. We show how to use Koszul complexes to compute Betti diagrams in the minimal free resolution of a vector space valued tame functor indexed by upper semilattices and realisations. Our main result in this direction is Theorem 10.19. It is noticeable there, not only that we can use Koszul complexes to compute Betti diagrams for tame functors indexed by realisations, as we can do for upper semilattices, but we can also bound their homological dimensions by the parental dimension of the elements of the indexing realisation posets. This bridges together the combinatorial study of realisations done in the first part of the article and their use as indexing posets of tame functors and their homological algebra.

2. Basic notions

In this section, we recall and introduce some basic concepts that might already be familiar. The reader may choose to skip this section and come back to it when a specific reference is made.

2.1. The standard poset of real numbers is denoted by $\mathbb{R}$. Its subposet of non-negative real numbers is denoted by $[0,\infty)$, of natural numbers $\{0,1,\ldots\}$ by $\mathbb{N}$, the first $n+1$ natural numbers by $[n]$, $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$, and $(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$.

The product of posets $(I,\leq)$ and $(J,\leq)$ is the poset $(I \times J, \leq)$, where $(x,y) \leq (x_1,y_1)$ if $x \leq x_1$ and $y \leq y_1$. For a set S and a poset I, the symbol I^S denotes the poset of all functions $f\colon S \longrightarrow I$ with $f \leq g$ if $f(x) \leq g(x)$ for every x in S. The poset I^S is isomorphic to the $|S|$-fold product $I^{|S|}$.

The poset $[1]^S$ is called the **discrete cube** of dimension $|S|$ and the poset $[-1,0]^S$ is called the **geometric cube** of dimension $|S|$.

The inclusion poset of all subsets of S is denoted by 2^S. The function mapping $f\colon S \longrightarrow [1]$ in $[1]^S$ to the subset $f^{-1}(1) \subset S$ is an isomorphism between the posets $[1]^S$ and 2^S. This function is used to identify these two posets, and 2^S is also referred to as the discrete cube of dimension $|S|$.

For a set S, consider a disjoint union $S \coprod \{-\infty,\infty\}$ and the following relation on this enlarged set: elements in S are incomparable and $-\infty \leq s \leq \infty$ for all s in S. This poset is denoted by ΣS and called the **suspension** of S.

2.2. Let I be a poset and a be its element. For a subset $S \subset I$, define $S \leq a := \{s \in S \mid s \leq a\}$, $S < a := \{s \in S \mid s < a\}$, and $a \leq S := \{s \in S \mid a \leq s\}$.

A poset I is called of **finite-type** if $I \leq a$ is finite for every a in I. A finite poset is of finite-type. The poset $\mathbb{N}$ is infinite and of finite-type. The poset $\mathbb{R}$ is not of finite-type. An element x in I is called a **parent** of a if $x < a$ and there is no element y in I such that $x < y < a$. The term a **covers** x is also commonly used to say that x is a parent of a. The symbol $\mathcal{P}_I(a)$ or $\mathcal{P}(a)$, if I is understood, denotes the set of all parents of a. For example, every element in $\mathbb{R}$ has an empty set of parents. In the poset $[1]$, $\mathcal{P}_{[1]}(0) = \emptyset$ and $\mathcal{P}_{[1]}(1) = \{0\}$. A poset I is of finite-type if and only if, for every a in I, the sets $\mathcal{P}_I(a)$ and $\{n \mid \text{there is a sequence } x_0 < \cdots < x_n = a \text{ in } I\}$ are finite.

Let I and J be posets and $f\colon I \longrightarrow J$ be a function (not necessarily preserving the poset relations). For an element a in J, define

$$f \leq a := \{x \in I \mid f(x) \leq a\} \qquad a \leq f := \{x \in I \mid a \leq f(x)\}.$$

For example, $(\mathrm{id}_I \leq a) \subset I$ coincides with $(I \leq a) \subset I$.

If $f \leq a$ is finite for every a in J, then $f\colon I \longrightarrow J$ is called of **finite-type**. For example, if I is finite, then every function $f\colon I \longrightarrow J$ is of finite-type.

2.3. Let $S \subset I$ be a subset of a poset I. If $(a \leq S) = S$ (i.e. $a \leq s$ for all s in S), then a is called an **ancestor**, or lower bound, of S. If $(a < S) = S$ (i.e. $a < s$ for all s in S), then a is called a **proper ancestor** of S. Every element is a proper ancestor of the empty subset.

If a is an ancestor of S for which there is no other ancestor b of S such that $a < b$, then a is called an **inf**, or greatest lower bound, of S. For example, an element a in I is an inf of the empty subset if and only if there is no b in I for which $a < b$. Such an element is also called **maximal** in I. For instance, an element is an inf of S if and only

if it is maximal in the subposet $\{a \in I \mid (a \leq S) = S\} \subset I$. An element a in I is an inf of the entire I if and only if $a \leq x$ for every x in I. If such an element exists, then it is unique and is called the **global minimum** of I. Two different inf elements of a subset are not comparable.

In general, S can have many inf elements. In the case S has only one inf element, then this element is called the **product**, or meet, of S and is denoted either by $\bigwedge_I S$ or $\bigwedge S$, if I is clear form the context. Explicitly, the product of S is an element ν in I such that $(\nu \leq S) = S$ and, for every a in I for which $(a \leq S) = S$, the relation $a \leq \nu$ holds. The product $\bigwedge_I \{x, y\}$ is also denoted as $x \wedge_I y$ or $x \wedge y$. For example, if the product of the empty subset of I exists, then I has a unique maximal element given by $\bigwedge_I \emptyset$. This element may fail however to be the global maximum of I. The element $\bigwedge_I I$, if it exists, is the global minimum of I.

2.4. Let $S \subset I$ be a subset of a poset I. If $(S \leq a) = S$ (i.e. $s \leq a$ for all s in S), then a is called a **descendent**, or upper bound, of S. Every element is a descendent of the empty subset.

If a is a descendent of S for which there is no other descendent b of S such that $b < a$, then a is called a **sup**, or lowest upper bound, of S. For example, an element a in I is a sup of the empty subset if and only if there is no b in I for which $b < a$. Such an element is also called **minimal** in I. For instance, an element is a sup of S if and only if it is minimal in the subposet $\{a \in I \mid (S \leq a) = S\} \subset I$. An element a in I is a sup of the entire I if and only if $x \leq a$ for every x in I. If such an element exists, then it is unique and is called the **global maximum** of I. Two different sup elements of a subset are not comparable.

In general, S can have many sup elements. In the case S has only one sup element, then this element is called the **coproduct**, or join, of S and is denoted either by $\bigvee_I S$ or $\bigvee S$, if I is clear form the context. Explicitly, the coproduct of S is an element ν in I such that $(S \leq \nu) = S$, and for every a in I for which $(S \leq a) = S$, the relation $\nu \leq a$ holds. The coproduct $\bigvee_I \{x, y\}$ is also denoted as $x \vee_I y$ or $x \vee y$. For example, if the coproduct of the empty subset of I exists, then I has a unique minimal element given by $\bigvee_I \emptyset$. This element may fail however to be the global minimum of I. The element $\bigvee_I I$, if it exists, is the global maximum of I.

A product can be expressed as a coproduct: $\bigwedge_I S = \bigvee_I \left(\bigcap_{x \in S} (I \leq x)\right)$, where the equality should be read as follows: the coproduct on the right exists if and only if the product on the left exists, in which case they are equal. Thus, if every subset of I has the coproduct, then every subset has also the product.

2.5. All subsets of $[0, \infty)$ have products. All subsets of $[-1, 0]$ have products and coproducts.

Let S be a set. If all subsets of a poset I have products, then the same is true for all subsets of I^S, where the product of $T \subset I^S$ is given by the function mapping x in S to the product $\bigwedge_I \{f(x) \mid f \in T\}$ in I. If all subsets of I have coproducts, then the same is true for all subsets of I^S, where the coproduct of $T \subset I^S$ is given by the function

$x \mapsto \bigvee_I \{f(x) \mid f \in T\}$.

In the suspension ΣS (see 2.1), the element $-\infty$ is its global minimum, and the element ∞ is its global maximum. The elements $-\infty$ and ∞ are respectively the product and coproduct of any subset $U \subset S$ of size at least 2. The suspension ΣS is of finite-type if and only if S is finite.

Every subset Z of the discrete cube 2^S (see 2.1) has coproduct given by the union $\bigvee Z = \bigcup_{\sigma \in Z} \sigma$, and product given by the intersection $\bigwedge Z = \bigcap_{\sigma \in Z} \sigma$. The subset $\emptyset \subset S$ is the global minimum of 2^S and $S \subset S$ is the global maximum. Note also that every parent of $U \subset S$ in 2^S is of the form $U \setminus \{x\}$ for x in U. Thus, the function $x \mapsto U \setminus \{x\}$ is a bijection between U and $\mathcal{P}_{2^S}(U)$ (see 2.2).

2.6. A poset that has a global minimum is called **unital**. For example $[-1, 0]$ is unital and $\mathbb{R}$ is not. If I is unital, then so is I^S, for every set S, with its global minimum given by the constant function mapping every s in S to the global minimum of I. A function $f\colon I \longrightarrow J$ between unital posets is called **unital** if it maps the global minimum in I to the global minimum in J.

For a poset I, the symbol I_* denotes the poset formed by adding an additional element $-\infty$ to I and setting $-\infty < x$ for all x in I. The element $-\infty$ in I_* is its global minimum.

A function $f\colon I \longrightarrow J$ extends uniquely to a unital function $f_*\colon I_* \longrightarrow J_*$ for which the diagram

$$\begin{array}{ccc} I & \xrightarrow{f} & J \\ \downarrow & & \downarrow \\ I_* & \xrightarrow{f_*} & J_* \end{array}$$

commutes. Note that $(f_* \leq -\infty) = \{-\infty\}$, and $(f_* \leq a) = (f \leq a) \cup \{-\infty\}$ if a is in J. Thus, a function $f\colon I \longrightarrow J$ is of finite-type if and only if $f_*\colon I_* \longrightarrow J_*$ is of finite-type. Furthermore, $f_* \leq a$ is non-empty for every a in J_*. Moreover, the function $J^I \longrightarrow J_*^{I_*}$, $f \mapsto f_*$, is injective and preserves the poset relations: if $f \leq g$, then $f_* \leq g_*$.

2.7. For a poset $(I, \leq)$, the same symbol I denotes also the category whose set of objects is I and where $\mathrm{mor}_I(x, y)$ is either empty, if $x \not\leq y$, or has cardinality 1, if $x \leq y$.

A **functor** between poset categories I and J is a function $f\colon I \longrightarrow J$ preserving the poset relations: if $x \leq y$ in I, then $f(x) \leq f(y)$ in J. The category whose objects are posets and morphisms are functors is denoted by Posets.

A function $f\colon I \longrightarrow J$ is called a **subposet inclusion** if, $x \leq y$ happens in I if and only if $f(x) \leq f(y)$ in J. A subposet inclusion is always an injective functor. Not all injective functors however are subposet inclusions. A subposet inclusion $f\colon I \longrightarrow J$ induces an isomorphism between I and the subposet $f(I) \subset J$.

The symbol $\mathrm{Fun}(I, J)$ denotes the subposet of J^I (see 2.1) whose elements are functors. Since in this poset $f \leq g$ if $f(x) \leq g(x)$ for all x in I, the category $\mathrm{Fun}(I, J)$ coincides with the category whose morphisms are natural transformations. More generally, for a poset I and a category $\mathcal{C}$, the symbol $\mathrm{Fun}(I, \mathcal{C})$ denotes the category of functors $I \longrightarrow \mathcal{C}$ with natural transformations as morphisms. If $f\colon I \longrightarrow J$ is a functor between posets, then the precomposition with f functor is denoted by $(-)^f\colon \mathrm{Fun}(J, \mathcal{C}) \longrightarrow \mathrm{Fun}(I, \mathcal{C})$.

The global minimum (see 2.3) of a poset coincides with the initial object in the associated category. If I is a unital poset (it has the global minimum) and $\mathcal{C}$ is a category with an initial object, then a functor $f\colon I \longrightarrow \mathcal{C}$ is called **unital** if it maps the global minimum in I to an initial object in $\mathcal{C}$. The symbol $\mathrm{Fun}_*(I,\mathcal{C})$ denotes the category whose objects are unital functors from I to $\mathcal{C}$ and morphisms are natural transformations.

Let I be a poset and $\mathcal{C}$ be a category with a unique initial object. Any functor $F\colon I \longrightarrow \mathcal{C}$ can be extended uniquely to a unital functor $F_*\colon I_* \longrightarrow \mathcal{C}$ whose composition with $I \subset I_*$ is F. The association $F \mapsto F_*$ is an isomorphism of categories between $\mathrm{Fun}(I,\mathcal{C})$ and $\mathrm{Fun}_*(I_*,\mathcal{C})$. We use this isomorphism to identify these categories.

2.8. Let I be a poset. A **lax functor** $T\colon I \rightsquigarrow$ Posets ([S72, Definition 1]) assigns to every element a in I a poset T_a and to every relation $a \leq b$ in I a functor $T_{a\leq b}\colon T_a \longrightarrow T_b$. These functors are required to satisfy two conditions. First, $T_{a\leq a}$ is the identity for all a in I. Second, $T_{a\leq c} \leq T_{b\leq c}T_{a\leq b}$ for all $a \leq b \leq c$ in I. For example, every functor is a lax functor, as in this case the equality $T_{a\leq c} = T_{b\leq c}T_{a\leq b}$ holds.

Let $T\colon I \rightsquigarrow$ Posets be a lax functor. Define $\mathrm{Gr}_I T := \{(a,f) \mid a \in I, f \in T_a\}$. For (a,f) and (b,g) in $\mathrm{Gr}_I T$, set $(a,f) \leq (b,g)$ if $a \leq b$ in I and $T_{a\leq b}(f) \leq g$ in T_b. For example, $(a,f) \leq (a,g)$ if and only if $f \leq g$ in T_a. The conditions required to be satisfied by lax functors guarantee the transitivity of the relation $\leq$ on $\mathrm{Gr}_I T$. The poset $(\mathrm{Gr}_I T, \leq)$ is called **Grothendieck construction**. In the case T is the constant functor with value J, then $\mathrm{Gr}_I T$ is isomorphic to the product $I \times J$.

The function $\pi\colon \mathrm{Gr}_I T \longrightarrow I$, mapping (a,f) to a, is a functor called the **standard projection**. For a in I, the function $\mathrm{in}_a\colon T_a \longrightarrow \mathrm{Gr}_I T$, mapping f to (a,f), is also a functor called the **standard inclusion**. The functor in_a is a subposet inclusion (see 2.7).

2.9. Proposition. *Let I be a poset, $T\colon I \rightsquigarrow$ Posets a lax functor, and $\emptyset \neq S \subset \mathrm{Gr}_I T$.*

1. *Assume $\pi(S) = \{a\}$. Then an element (b,g) is a sup of S in $\mathrm{Gr}_I T$ if and only if $b = a$ and g is a sup of $\{f \mid (a,f) \in S\}$ in T_a.*
2. *Let b be a sup of $\pi(S)$ in I. Then an element (b,m) is a sup of S in $\mathrm{Gr}_I T$ if and only if m is a sup of $\{T_{a\leq b}f \mid (a,f)$ in $S\}$ in T_b.*

Proof. Statement (1) is a particular case of (2). Statement (2) is a consequence of the equivalence: $T_{a\leq b}f \leq g \leq m$ in T_b, for every (a,f) in S, if and only if $(a,f) \leq (b,g) \leq (b,m)$ in $\mathrm{Gr}_I T$, for every (a,f) in S.

2.10. Let $f\colon I \longrightarrow J$ be a function (not necessarily a functor). Define $J[f] \subset 2^I$ to be the subposet whose elements are subsets of the form $(f \leq a) \subset I$, for a in J. Let $f\leq\colon J \longrightarrow J[f]$ be the function mapping a to $f \leq a$. Note that if $a \leq b$ in J, then $(f \leq a) \subset (f \leq b)$, which means $f\leq$ is a functor. An important property of being a functor is that its fibers $(f\leq)^{-1}(f \leq a) = \{b \in J \mid (f \leq b) = (f \leq a)\} \subset J$, for a in J, satisfy the following property: if $x \leq y$ in J belong to $(f\leq)^{-1}(f \leq a)$, then so does any z in J such that $x \leq z \leq y$. Recall that such subposets of J are called **intervals** or **convex**.

2.11. Let $f\colon I \longrightarrow J$ be a functor of posets and $\mathcal{C}$ a category. Recall that a functor $F\colon J \longrightarrow \mathcal{C}$ is called a **left Kan extension** of a functor $G\colon I \longrightarrow \mathcal{C}$ along f (see [M98]), if there is a natural transformation $\alpha\colon G \longrightarrow Ff$ satisfying the following universal property: for every functor $H\colon J \longrightarrow \mathcal{C}$, the function $\mathrm{Nat}_J(F,H) \longrightarrow \mathrm{Nat}_I(G,Hf)$, mapping $\phi\colon F \longrightarrow H$ to the following composition, called the **adjoint** to ϕ, is a bijection:

$$G \xrightarrow{\alpha} Ff \xrightarrow{\phi^f} Hf$$

This universal property has two consequences. The first is the uniqueness of the left Kan extension: if F and F' are two left Kan extensions of G along $f\colon I \longrightarrow J$, and $\alpha\colon G \longrightarrow Ff$ and $\alpha'\colon G \longrightarrow F'f$ are natural transformations satisfying the above universal property, then there is a unique isomorphism $\psi\colon F \longrightarrow F'$ for which $\alpha' = \psi^f\alpha$. Because of this uniqueness, if it exists, a left Kan extension of $G\colon I \longrightarrow \mathcal{C}$ along $f\colon I \longrightarrow J$ is denoted by $f^kG\colon J \longrightarrow \mathcal{C}$. Functoriality of left Kan extensions is another consequence of the universal property. Let $\phi\colon G \longrightarrow G'$ be a natural transformation between two functors $G, G'\colon I \longrightarrow \mathcal{C}$. If these functors admit left Kan extensions along f, then there is a unique natural transformation $f^k\phi\colon f^kG \longrightarrow f^kG'$ for which the following square commutes:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ (f^kG)f & \xrightarrow{(f^k\phi)^f} & (f^kG')f \end{array}$$

If all functors in $\mathrm{Fun}(I,\mathcal{C})$ admit left Kan extensions along f, then the assignment $f^k\colon \mathrm{Fun}(I,\mathcal{C}) \longrightarrow \mathrm{Fun}(J,\mathcal{C})$ is a functor which is left adjoint to the precomposition with f functor $(-)^f\colon \mathrm{Fun}(J,\mathcal{C}) \longrightarrow \mathrm{Fun}(I,\mathcal{C})$. For example, this happens if $\mathcal{C}$ is closed under finite colimits and $f\colon I \longrightarrow J$ is of finite-type. In this case, the left Kan extension of G along f is given by $a \mapsto \mathrm{colim}_{f\leq a}G$ (see [M98]). This description of the left Kan extension, in the case f is of finite-type and $\mathcal{C}$ is closed under finite colimits, has several consequences. For example, f^kG is isomorphic to a functor given by a composition (see 2.10):

$$J \xrightarrow{f\leq} J[f] \longrightarrow \mathcal{C}$$

The restrictions of f^kG to the fibres $(f\leq)^{-1}(f \leq a) = \{b \in J \mid (f \leq b) = (f \leq a)\} \subset J$, for a in J, are therefore isomorphic to constant functors. When I is finite, then, since $J[f]$ is also finite, the left Kan extension f^kG factors through a finite poset $J[f]$. Such functors have been considered for example in [M20, Definition 2.11].

2.12. Let $f\colon I \longrightarrow J$ be a functor between posets. A functor $G\colon I \longrightarrow \mathcal{C}$ is called f-**right invertible** (or right invertible with respect to f) if its left Kan extension $f^kG\colon J \longrightarrow \mathcal{C}$ exists and the natural transformation $G \longrightarrow (f^kG)f$, adjoint to the identity $f^kG \longrightarrow f^kG$, is an isomorphism. A functor $F\colon J \longrightarrow \mathcal{C}$ is called f-**left invertible** (or left invertible with respect to f) if it is the left Kan extension of its restriction $Ff\colon I \longrightarrow \mathcal{C}$, i.e. if the natural transformation $f^k(Ff) \longrightarrow F$, adjoint to the identity $Ff \longrightarrow Ff$, is an isomorphism. Note

that if $G\colon I \longrightarrow \mathcal{C}$ is f-right invertible, then its left Kan extension $f^kG\colon J \longrightarrow \mathcal{C}$ is f-left invertible. Similarly, if $F\colon J \longrightarrow \mathcal{C}$ is f-left invertible, then its restriction $Ff\colon I \longrightarrow \mathcal{C}$ is f-right invertible.

If $G\colon I \longrightarrow \mathcal{C}$ is f-right invertible, then, for every $H\colon I \longrightarrow \mathcal{C}$ for which $f^kH\colon J \longrightarrow \mathcal{C}$ exits, the function $f^k\colon \mathrm{Nat}_I(H,G) \longrightarrow \mathrm{Nat}_J(f^kH, f^kG)$, mapping ϕ to $f^k\phi$, is a bijection. Similarly, if $F\colon J \longrightarrow \mathcal{C}$ is f-left invertible, then, for every $H\colon J \longrightarrow \mathcal{C}$, the function $(-)^f\colon \mathrm{Nat}_J(F,H) \longrightarrow \mathrm{Nat}_I(Ff, Hf)$, mapping ϕ to ϕ^f, is a bijection. It follows that a natural transformation $\phi\colon H \longrightarrow G$, between f-right invertible functors $H, G\colon I \longrightarrow \mathcal{C}$, is an isomorphism if and only if its left Kan extension $f^k\phi\colon f^kH \longrightarrow f^kG$ is an isomorphism. Similarly, a natural transformation $\phi\colon F \longrightarrow H$, between f-left invertible functors $F, H\colon J \longrightarrow \mathcal{C}$, is an isomorphism if and only if its restriction $\phi^f\colon Ff \longrightarrow Hf$ is an isomorphism.

A category $\mathcal{C}$ is called f-**right-invertible** if every functor $G\colon I \longrightarrow \mathcal{C}$ is f-right invertible. If every functor $F\colon J \longrightarrow \mathcal{C}$ is f-left invertible, then $\mathcal{C}$ is called f-**left invertible**. If a category $\mathcal{C}$ is f-left invertible, then it is also f-right invertible. For example, if $\mathcal{C}$ is a category closed under finite colimits, then it is right invertible with respect to every subposet inclusion $f\colon I \subset J$, with I poset of finite-type (see 2.7). This is because every a in I is the terminal object in $f \leq f(a)$, and consequently, for every functor $G\colon I \longrightarrow \mathcal{C}$, the morphism $G(a) \longrightarrow \mathrm{colim}_{f\leq f(a)}G = (f^kG)(f(a))$ is an isomorphism.

2.13. Proposition. *Let $g\colon I \longrightarrow L$ and $h\colon L \longrightarrow J$ be poset functors, and let $F\colon J \longrightarrow \mathcal{C}$ be the left Kan extension of $G\colon I \longrightarrow \mathcal{C}$ along the composition $f = hg$. Then F is h-left invertible if and only if $Fh\colon L \longrightarrow \mathcal{C}$ is h-right invertible.*

Proof. If F is h-left invertible, then, directly from the definition, Fh is h-right invertible. Assume Fh is h-right invertible. This means that the left Kan extension $h^k(Fh)$ exists and the natural transformation $\varphi\colon Fh \longrightarrow (h^k(Fh))h$, adjoint to the identity $\mathrm{id}\colon h^k(Fh) \longrightarrow h^k(Fh)$, is an isomorphism. To show that F is h-left invertible, we need to prove that the natural transformation $\mu\colon h^k(Fh) \longrightarrow F$, adjoint to the identity $\mathrm{id}\colon Fh \longrightarrow Fh$, is also an isomorphim. Since the composition of φ with the restriction $\mu^h\colon (h^k(Fh))h \longrightarrow Fh$ is the identity, this restriction μ^h is an isomorphism. Consider next the following commutative diagram of sets where the horizontal arrows are the bijections expressing the fact that F is the left Kan extension of G:

$$\begin{array}{ccc} \mathrm{Nat}_I(G,\ (h^k(Fh))hg) & \cong & \mathrm{Nat}_J(F,\ h^k(Fh)) \\ {\scriptstyle \mathrm{Nat}_I(G,\ \mu^{hg})}\downarrow & & \downarrow{\scriptstyle \mathrm{Nat}_I(F,\ \mu)} \\ \mathrm{Nat}_I(G, Fhg) & \cong & \mathrm{Nat}_J(F,F) \end{array}$$

Since μ^h is an isomorphism, the left vertical arrow in the above diagram is a bijection. It follows that the right vertical arrow is also a bijection, and consequently there exists $\psi\colon F \longrightarrow h^k(Fh)$ for which the composition $\mu\psi\colon F \longrightarrow F$ is the identity. Consider the composition $\psi\mu\colon h^k(Fh) \longrightarrow h^k(Fh)$. It is a natural transformation between h-left invertible

functors, since Fh is assumed to be h-right invertible. Moreover its restriction $(\psi\mu)^h$ is an isomorphism since μ^h is an isomorphism. This implies that $\psi\mu$ is also an isomorphism.

3. Dimension and parental dimension

Let $(I, \leq)$ be a poset and x be an element in I. In this section we propose two ways of measuring complexity of expressing x as a sup (see 2.4).

3.1. Let

$$\Phi_I(x) := \left\{ U \mid \begin{array}{c} U \subset I \text{ is finite, has a proper ancestor, } x \text{ is a sup of it, and} \\ x \text{ is not a sup of any } S \text{ such that } \emptyset \neq S \subsetneq U \end{array} \right\}.$$

Since $\emptyset$ belongs to $\Phi_I(x)$, this collection is non-empty. The following extended (containing ∞) number is called the **dimension** of x:

$$\dim_I(x) := \sup\{|U| \mid U \in \Phi_I(x)\}.$$

An element U in $\Phi_I(x)$, for which $\dim_I(x) = |U|$, is said to **realise** $\dim_I(x)$.

Since any subset $U \subset I$, for which x is a sup, is contained in $I \leq x$, there is an inequality $\dim_I(x) \leq |I \leq x|$. In particular, if I is of finite-type (see 2.2), then $\dim_I(x)$ is finite for every x.

The empty set realises $\dim_I(x)$ if and only if $\dim_I(x) = 0$, which happens if and only if x is minimal in I (see 2.4). The set $\{x\}$ is the only set realising $\dim_I(x)$ if and only if $\dim_I(x) = 1$. If $\dim_I(x) > 1$, then every set realising $\dim_I(x)$ cannot contain x.

Here are some examples:

- In the poset $[n]$, $\dim_{[n]}(k) = \begin{cases} 0 & \text{if } k = 0, \\ 1 & \text{if } k > 0. \end{cases}$
- In the poset $[-1, 0]$, $\dim_{[-1,0]}(x) = \begin{cases} 0 & \text{if } x = -1, \\ 1 & \text{if } x > -1. \end{cases}$
- In the poset $\mathbb{R}$, $\dim_{\mathbb{R}}(x) = 1$ for every x in $\mathbb{R}$.
- In the poset 2^S (see 2.1), $\dim_{2^S}(\sigma) = |\sigma|$ and this dimension is realised by, for example, $\{\{x\} \mid x \in \sigma\}$.
- In the suspension poset ΣS (see 2.1),

$$\dim_{\Sigma S}(x) = \begin{cases} 2 & \text{if } x = \infty \text{ and } |S| \geq 2, \\ 1 & \text{if either } x \in S, \text{ or } x = \infty \text{ and } |S| < 2, \\ 0 & \text{if } x = -\infty. \end{cases}$$

  If $|S| \geq 2$, then every subset of S of size 2 realises $\dim_{\Sigma S}(\bigvee S) = 2$.

The dimension of an element x depends on the global properties of the subposet $I \leq x$. This dimension can be approximated by a more tractable notion of dimension, called the parental dimension, which, for elements in a finite-type poset, depends only on their parents (see Proposition 3.5).

3.2. Let

$$\Psi_I(x) := \left\{ U \mid \begin{smallmatrix} U \subset (I<x) \text{ is finite, has an ancestor, and} \\ x \text{ is a sup of every two element subset of } U \end{smallmatrix} \right\}.$$

Since $\emptyset$ belongs to $\Psi_I(x)$, this collection is non-empty. The following extended number is called the **parental dimension** of x:

$$\text{par-dim}_I(x) := \sup\{|U| \mid U \in \Psi_I(x)\}.$$

An element U in $\Psi_I(x)$, for which $\text{par-dim}_I(x) = |U|$, is said to **realise** $\text{par-dim}_I(x)$. Here are some examples:

- If I is $[n]$, or $[-1, 0]$, or $\mathbb{R}$, then $\text{par-dim}_I(x) = \dim_I(x)$ for all x in I.
- In the poset 2^S, for every $\sigma \subset S$, $\text{par-dim}_I(x) = |\sigma| = \dim_I(x)$ and the parental dimension is realised, for example, by $\{\sigma \setminus \{x\} \mid x \in \sigma\}$.
- In the suspension poset ΣS, $\text{par-dim}_{\Sigma S}(\infty) = \max\{|S|, 1\}$, and, if S is non-empty, then S realises $\text{par-dim}_{\Sigma S}(\infty)$. If S has at least three elements, then there is a strict inequality $\dim_{\Sigma S}(\infty) = 2 < |S| = \text{par-dim}_{\Sigma S}(\infty)$.

3.3. Proposition. *Let I be a poset.*

1. $\dim_I(x) = 0$ *if and only if* $\text{par-dim}_I(x) = 0$.
2. $\dim_I(x) = 1$ *if and only if* $\text{par-dim}_I(x) = 1$.
3. $\dim_I(x) \geq 2$ *if and only if* $\text{par-dim}_I(x) \geq 2$.

Proof. The empty set realises $\text{par-dim}_I(x)$ if and only if $\text{par-dim}_I(x) = 0$, which happens if and only if x is minimal in I. This proves (1). To show (3), assume $\dim_I(x) \geq 2$, and let S in $\Phi_I(x)$ (see 3.1) be such that $|S| \geq 2$. Choose s in S. Since x is not a sup of $S \setminus \{s\}$, there is $z < x$ for which $(S \setminus \{s\}) \subset (I \leq z)$. Moreover, $s < x$ and x is a sup of $\{s, z\}$, showing that $\{s, z\}$ belongs to $\Psi_I(x)$. Thus, $\text{par-dim}_I(x) \geq 2$. Since (1) and (3) hold, (2) must also hold.

One reason why parental dimension is easier to calculate is the following:

3.4. Lemma. *Let I be a poset, x its element, and U be in $\Psi_I(x)$. Assume $\alpha\colon U \longrightarrow I$ is a function such that $u \leq \alpha(u) < x$, for every u in U. Then α is injective, in particular $|\alpha(U)| = |U|$, and its image $\alpha(U)$ belongs to $\Psi_I(x)$.*

Proof. Let s and u be elements in U. If $\alpha(s) = \alpha(u)$, then the relations $s \leq \alpha(s) = \alpha(u) \geq u$ and $\alpha(s) < x$ imply that x is not a sup of $\{s, u\}$. This can happen only if $s = u$, as U belongs to $\Psi_I(x)$. Thus α is injective and $|U| = |\alpha(U)|$. If $\alpha(s) \neq \alpha(u)$, then a consequence of the relations $s \leq \alpha(s) < x > \alpha(u) \geq u$ and the fact that x is a sup of $\{s, u\}$ is that x is also a sup of $\{\alpha(s), \alpha(u)\}$. Furthermore, any ancestor of U is also an ancestor of $\alpha(U)$. Consequently $\alpha(U)$ is in $\Psi_I(x)$.

Lemma 3.4 can be used to prove the following proposition which is the reason behind choosing the name parental dimension.

3.5. Proposition. *If I is a poset of finite-type, then, for every x in I,*

$$\text{par-dim}_I(x) = \max\{|S| \mid S \subset \mathcal{P}(x) \text{ and } S \text{ has an ancestor}\}.$$

Proof. Since x is a sup of every two elements subset of $\mathcal{P}(x)$, the right side of the claimed equality is smaller or equal than the left side. To show the opposite inequality, consider U in $\Psi_I(x)$. Since I is of finite-type, for all u in U, there is $\alpha(u)$ in $\mathcal{P}(x)$ for which $u \leq \alpha(u)$. According to Lemma 3.4, $|\alpha(U)| = |U|$, which concludes the proof.

There are two numbers assigned to x in I, its dimension $\dim_I(x)$ and its parental dimension $\text{par-dim}_I(x)$. According to Proposition 3.3, if one of these dimensions is 0 or 1 then so is the other. In general, the parental dimension always bounds the dimension, as the following result shows.

3.6. Proposition. *For every x in a poset I, $\dim_I(x) \leq \text{par-dim}_I(x)$.*

Proof. The cases $\dim_I(x)$ is 0 and 1 follow from Proposition 3.3. Let U be in $\Phi_I(x)$, with $|U| \geq 2$. Since, for every u in U, the element x is not a sup of $U \setminus \{u\}$, there is s_u in I such that $u' \leq s_u < x$ for every u' in $U \setminus \{u\}$. If u_0 and u_1 are different elements in U, then x being a sup of U implies that s_{u_0} and s_{u_1} are also different and x is a sup of $\{s_{u_0}, s_{u_1}\}$. Moreover, any proper ancestor of U is also an ancestor of $S := \{s_u \mid u \in U\}$. Thus, S has the same size as U and belongs to $\Psi_I(x)$. For every element U in $\Phi_I(x)$, we have constructed an element S in $\Psi_I(x)$ of the same size as U.

The suspension example in 3.2 shows that the inequality in Proposition 3.6 can be strict.

3.7. Proposition. *Let I and J be posets. For every x in I and y in J:*

1. $\dim_{I\times J}(x,y) = \dim_I(x) + \dim_J(y)$,
2. $\text{par-dim}_{I\times J}(x,y) = \text{par-dim}_I(x) + \text{par-dim}_J(y)$.

Proof. (1) First, we show $\dim_{I\times J}(x,y) \geq \dim_I(x) + \dim_J(y)$. If $\dim_I(x) = 0$, then x is minimal in I. In this case, every element of $\Phi_{I\times J}(x,y)$ is of the form $\{x\} \times U$, where U belongs to $\Phi_J(y)$ and hence the inequality is clear. Assume $\dim_I(x) \geq 1$ and $\dim_J(y) \geq 1$. Let S and U belong respectively to $\Phi_I(x)$ and $\Phi_J(y)$. Let x' be a proper ancestor of S and y' be a proper ancestor of U. Then $|S| + |U| = |(S \times \{y'\}) \cup (\{x'\} \times U)|$. Since $(S \times \{y'\}) \cup (\{x'\} \times U) \subset I \times J$ belongs to $\Phi_{I\times J}(x,y)$, we get $\dim_{I\times J}(x,y) \geq |S| + |U|$, which gives the desired inequality.

To show $\dim_{I\times J}(x,y) \leq \dim_I(x) + \dim_J(y)$, consider W in $\Phi_{I\times J}(x,y)$. For every element (a,b) in W, the set W cannot contain a subset of the form $\{(a',b),(a,b')\}$, where $a' \neq a$ and $b' \neq b$, otherwise (x,y) would be a sup of $W \setminus \{(a,b)\}$. Consequently, $|W| \leq |\text{pr}_I(W)| + |\text{pr}_J(W)|$, where pr_I and pr_J denote the projections. Moreover, $\text{pr}_I(W)$ and $\text{pr}_J(W)$ have, respectively, x and y as sup, so $|\text{pr}_I(W)| + |\text{pr}_J(W)| \leq \dim_I(x) + \dim_J(y)$.

(2) As before, we start by showing $\text{par-dim}_{I\times J}(x,y) \geq \text{par-dim}_I(x) + \text{par-dim}_J(y)$. We argue in the same way. Let S and U belong respectively to $\Psi_I(x)$ and $\Psi_J(y)$. Then

$(S \times \{y\}) \cup (\{x\} \times U) \subset I \times J$ belongs to $\Psi_{I\times J}(x, y)$ and is of size $|S| + |U|$, which gives the desired inequality.

To show the inequality $\text{par-dim}_{I\times J}(x, y) \leq \text{par-dim}_I(x) + \text{par-dim}_J(y)$, consider W in $\Psi_{I\times J}(x, y)$. Then every (a, b) in W is such that $(a, b) < (x, y)$ and we define $\alpha(a, b) := (a, y)$ if $a < x$, and $\alpha(a, b) := (a, b)$ if $a = x$. These elements are chosen so that $(a, b) \leq \alpha(a, b) < (x, y)$ for every (a, b) in W. By Lemma 3.4, $\alpha(W)$ also belongs to $\Psi_{I\times J}(x, y)$. Since $|\alpha(W)| \leq |\text{pr}_I \alpha(W)| + |\text{pr}_J \alpha(W)|$, we get the desired inequality.

Let S be a finite set. Since $[-1, 0]^S$ and $[-1, 0]^{|S|}$ are isomorphic (see 2.1), Proposition 3.7 gives $\dim_{[-1,0]^S} f = \text{par-dim}_{[-1,0]^S} f = |\{s \in S \mid f(s) > -1\}|$.

3.8. Neither the dimension nor the parental dimension are monotonic in the following sense. If $I \subset J$ is a subposet inclusion (see 2.7) and $x \in I$, then in general the following inequalities may fail: $\dim_I(x) \leq \dim_J(x)$ and $\text{par-dim}_I(x) \leq \text{par-dim}_J(x)$. For example, consider:

$$I = \{(0, 0), (1, 0), (0, 1), (2, 2)\} \subset \mathbb{N}^2$$

$$J = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 2)\} \subset \mathbb{N}^2$$

Then $\dim_I(2, 2) = \text{par-dim}_I(2, 2) = 2$, $\dim_J(2, 2) = \text{par-dim}_J(2, 2) = 1$. For this monotonicity to hold, additional assumptions need to be made, for example:

3.9. Proposition. *Let $I \subset J$ be a subposet inclusion (see 2.7). Assume an element x in I has the following property: for every finite subset $S \subset I$, if x is a sup of S in I, then x is a sup of S in J. Under this assumption* $\dim_I(x) \leq \dim_J(x)$ *and* $\text{par-dim}_I(x) \leq \text{par-dim}_J(x)$.

Proof. The assumption on x implies that $\Phi_I(x) \subset \Phi_J(x)$ (see 3.1) and $\Psi_I(x) \subset \Psi_J(x)$ (see 3.2), which gives the claimed inequalities.

4. Realising posets

In this section we introduce a construction that transforms posets into posets, mimicking the relation between $\mathbb{N}^r$ and $[0, \infty)^r$. Let $(I, \leq)$ be a poset.

4.1. Consider a in I and its set of parents $\mathcal{P}(a)$. Table 1 describes how a partitions I into three disjoint subsets of a-**inconsistent**, a-**dependent**, and a-**independent** elements. We say that x is a-independent if it is not a-dependent ($a \not\leq x$), but it is p-dependent ($p \leq x$) for some p in $\mathcal{P}(a)$. Observe that every parent p of a is a-independent. An element which is either a-dependent or a-independent is called a-**consistent**. This terminology is inspired by linear algebra. We think of this partition as solutions of certain poset conditions, which in the case of an upper semilattice (see 6.2) behave as solutions of a linear system: an inconsistent system has no solutions, an independent system has a unique solution, and a dependent system has many solutions.

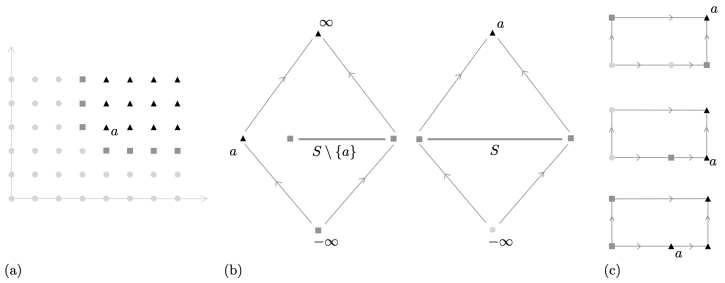


Figure 2: Black triangles are a-dependent, dark grey squares are a-independent, and light grey dots are a-inconsistent

x is a-consistent	x is a-dependent	$a \leq x$
	x is a-independent	$a \not\leq x$ and $(\mathcal{P}(a) \leq x) \neq \emptyset$
x is a-inconsistent		$a \not\leq x$ and $(\mathcal{P}(a) \leq x) = \emptyset$

Table 1:

Figure 2 illustrates the partitions of $\mathbb{N}^2$, ΣS, and a poset of size 5, into a-inconsistent, a-dependent, and a-independent blocks for some choices of a.

A consequence of the transitivity of a partial order is that being dependent is preserved by the poset relation: if $a \leq b$, then a b-dependent element is also a-dependent. Being independent does not have this property: a b-independent element may fail to be a-independent for some $a \leq b$. For example, consider the subposet $I = \{(0,0), (2,0), (3,0), (0,2), (3,2)\} \subset [0,\infty)^2$. Then, in I, the element $(0,2)$ is $(3,2)$-independent, however it is not $(3,0)$-independent. The element $(0,2)$ is neither $(3,0)$-dependent and hence neither consistency is in general preserved by the poset relation.

4.2. Define I to be **consistent** if, for every $a \leq b$ in I, every b-independent element, having a common ancestor with a, is also a-consistent. Thus, I being consistent is equivalent to the following condition for every $a \leq b$ in I: a b-consistent element x is a-consistent if and only if there is an element y in I such that $x \geq y \leq a$.

Distributive upper semilattices of finite-type are key examples of consistent posets (see 6.4). For example, the poset $\mathbb{N}^r$ is consistent. If $|S| \geq 3$, then the suspension ΣS (see 2.1) is an example of a consistent upper semilattice which is not distributive (see 5.1).

4.3. Define $\mathcal{G}(I) := \coprod_{a \in I} [-1,0]^{\mathcal{P}(a)}$. We identify elements of $\mathcal{G}(I)$ with pairs (a, f) consisting of an element a in I and a function $f\colon \mathcal{P}(a) \longrightarrow [-1,0]$. The set $\operatorname{supp}(f) := \{p \in \mathcal{P}(a) \mid f(p) < 0\}$ is called the **support** of f. If $\mathcal{P}(a)$ is empty, then the set $[-1,0]^{\mathcal{P}(a)}$ contains only one element, denoted by 0, in which case $(a,0)$ is the only element in $\mathcal{G}(I)$ with the first coordinate equal to a.

For (a, f) and (b, g) in $\mathcal{G}(I)$, set $(a, f) \leq (b, g)$ if

(a) $a \leq b$ in I;

(b) all elements in $\operatorname{supp}(g)$ are a-consistent: if $g(x) < 0$, then either x is a-dependent ($a \le x$) or x is a-independent ($a \not\le x$ and $(\mathcal{P}(a) \le x) \neq \emptyset$);

(c) $\bigwedge_{[-1,0]}\{f(y) \mid y \in (\mathcal{P}(a) \le x)\} \le g(x)$ for all a-independent x in $\mathcal{P}(b)$.

For example, $(a, f) \le (a, g)$ in $\mathcal{G}(I)$ if and only if $f \le g$ in $[-1, 0]^{\mathcal{P}(a)}$. Thus the inclusion $[-1, 0]^{\mathcal{P}(a)} \subset \mathcal{G}(I)$, assigning to f the pair (a, f), is a subposet inclusion (2.7). If 0 is the constant function with value 0, then $(a, 0) \le (b, 0)$ in $\mathcal{G}(I)$ if and only if $a \le b$ in I. Thus the inclusion $I \subset \mathcal{G}(I)$, assigning to a the pair $(a, 0)$, is also a subposet inclusion. The projection $\pi\colon \mathcal{G}(I) \longrightarrow I$, assigning to a pair (a, f) the element a, is a functor of posets.

We think about an element (a, f) in $\mathcal{G}(I)$ as an intermediate point between a and its parents, where the value of $f(p)$ describes the time needed to go back in the direction of the parent p.

4.4. Proposition. *If $(a, f) \le (b, g)$ in $\mathcal{G}(I)$, then every $w \le a$ which is an ancestor of* $\operatorname{supp}(f)$ *is also an ancestor of* $\operatorname{supp}(g)$.

Proof. Let $w \le a$ be an ancestor of $\operatorname{supp}(f)$, and x be in $\operatorname{supp}(g)$. Then x is a-consistent, and, thus, either $a \le x$, implying $w \le x$, or there exists y in $\mathcal{P}(a) \le x$. Since $\bigwedge\{f(y) \mid y \in (\mathcal{P}(a) \le x)\} \le g(x) < 0$, we have $f(y) < 0$. For such a y, $w \le y \le x$.

To understand the relation $\le$ on $\mathcal{G}(I)$ it is convenient to describe it in an alternative way using translations.

4.5. Definition. For a in I, define $T_a := [-1, 0]^{\mathcal{P}(a)}$ (see 2.1). For $a \le b$ in I, define $T_{a\le b}\colon T_a \longrightarrow T_b$ to map $f\colon \mathcal{P}(a) \longrightarrow [-1, 0]$ to $T_{a\le b}f\colon \mathcal{P}(b) \longrightarrow [-1, 0]$ where

$$(T_{a\le b}f)(x) := \begin{cases} -1 & \text{if } x \text{ is } a\text{-dependent,} \\ \bigwedge_{[-1,0]}\{f(y) \mid y \in (\mathcal{P}(a) \le x)\} & \text{if } x \text{ is } a\text{-independent,} \\ 0 & \text{if } x \text{ is } a\text{-inconsistent.} \end{cases}$$

The function $T_{a\le b}$ is called **translation** along $a \le b$.

4.6. Lemma. *Let $a \le b \le c$ be in I.*

1. *$T_{a\le a}\colon T_a \longrightarrow T_a$ is the identity,*
2. *$T_{a\le b}\colon T_a \longrightarrow T_b$ is a functor of posets (see 2.7),*
3. *$T_{a\le c} \le T_{b\le c}T_{a\le b}$.*

Proof. Statements (1) and (2) are direct consequences of the definition.

To prove statement (3) choose a function $f\colon \mathcal{P}(a) \longrightarrow [-1, 0]$ and an element x in $\mathcal{P}(c)$. If x is a-dependent, then $(T_{a\le c}f)(x) = -1 \le (T_{b\le c}T_{a\le b}f)(x)$. Assume x is not a-dependent. Then x is not b-dependent. If x is b-inconsistent, then $(T_{a\le c}f)(x) \le 0 = (T_{b\le c}T_{a\le b}f)(x)$. Let x be b-independent. Then $(T_{b\le c}T_{a\le b}f)(x) = \bigwedge\{(T_{a\le b}f)(y) \mid y \in$

$(\mathcal{P}(b) \leq x)\}$, and the desired inequality $(T_{a\leq c}f)(x) \leq (T_{b\leq c}T_{a\leq b}f)(x)$ would then follow if $(T_{a\leq c}f)(x) \leq (T_{a\leq b}f)(y)$ for all y in $\mathcal{P}(b) \leq x$. Let y be in $\mathcal{P}(b) \leq x$. Then y cannot be a-dependent, since x is not a-dependent. If y is a-independent, then x is also a-independent, and since $(\mathcal{P}(a) \leq y) \subset (\mathcal{P}(a) \leq x)$, then $(T_{a\leq c}f)(x) = \bigwedge\{f(z) \mid z \in (\mathcal{P}(a) \leq x)\} \leq \bigwedge\{f(z) \mid z \in (\mathcal{P}(a) \leq y)\} = (T_{a\leq b}f)(y)$. Finally, if y is a-inconsistent, then $(T_{a\leq c}f)(x) \leq 0 = (T_{a\leq b}f)(y)$.

According to Lemma 4.6, translations form a lax functor (see 2.8), denoted by $T\colon I \rightsquigarrow$ Posets and called ***I*-translation**. One should be aware that in general T may fail to be a functor:

4.7. Example. Consider the subposet $I = \{x = (0,0), a = (2,0), b = (3,0), y = (0,2), c = (3,2)\} \subset [0,\infty)^2$. Then $\mathcal{P}(a) = \{x\}$, $\mathcal{P}(b) = \{a\}$, and $\mathcal{P}(c) = \{y, b\}$. Note that y is b-inconsistent. If we denote by -0.5 the function $\mathcal{P}(a) \longrightarrow [-1,0]$ mapping x to -0.5, then

- $T_{a\leq b}(-0.5)\colon \mathcal{P}(b) \longrightarrow [-1,0]$ is the constant function -1,
- $T_{b\leq c}(T_{a\leq b}(-0.5))\colon \mathcal{P}(c) \longrightarrow [-1,0]$ maps y to 0 and b to -1,
- $T_{a\leq c}(-0.5)\colon \mathcal{P}(c) \longrightarrow [-1,0]$ maps y to -0.5 and b to -1.

In this case, $T_{a\leq c}(-0.5) < T_{b\leq c}T_{a\leq b}(-0.5)$.

Since $T\colon I \rightsquigarrow$ Posets is a lax functor, we can form its Grothendieck construction $\mathrm{Gr}_I T$. As sets, $\mathcal{G}(I)$ and $\mathrm{Gr}_I T$ are identical. The next lemma states that $\mathcal{G}(I)$ and $\mathrm{Gr}_I T$ also coincide as posets (see 2.8, for the poset relation on the latter).

4.8. Lemma. *Let (a,f) and (b,g) be elements in $\mathcal{G}(I)$.*

1. *$(a,f) \leq (b,g)$ if and only if $a \leq b$ in I and $T_{a\leq b}f \leq g$ in $[-1,0]^{\mathcal{P}(b)}$.*
2. *If $a \leq b$, then $(a,f) \leq (b, T_{a\leq b}f)$.*
3. *Elements in $\mathrm{supp}(T_{a\leq b}f)$ are a-consistent.*

Proof. Statement (3) is a direct consequence of (2) and (2) is a direct consequence of (1). To show (1), choose x in $\mathcal{P}(b)$. If x is a-dependent, then $(T_{a\leq b}f)(x) = -1 \leq g(x)$. If x is a-independent, then $(T_{a\leq b}f)(x) = \bigwedge\{f(y) \mid y \in (\mathcal{P}(a) \leq x)\}$ and hence $(T_{a\leq b}f)(x) \leq g(x)$ is equivalent to condition (c) in 4.3. If x is a-inconsistent, then $T_{a\leq b}f(x) = 0$ and hence $T_{a\leq b}f(x) \leq g(x)$ if and only if x is not in the support of g. These equivalences are what is needed to show (1).

Since the posets $\mathcal{G}(I)$ and $\mathrm{Gr}_I T$ coincide, Proposition 2.9 gives the following explicit description of some sup elements in $\mathcal{G}(I)$.

4.9. Corollary. *Let $S \subset \mathcal{G}(I)$ be a non-empty subset.*

1. *If $\pi(S)$ is equal to $\{a\}$, then S admits a coproduct: $\bigvee_{\mathcal{G}(I)} S = (a, g)$, where $g = \bigvee_{[-1,0]^{\mathcal{P}(a)}} \{f \mid (a, f) \in S\}$.*
2. *If b is a sup of $\pi(S)$ in I, then there is a unique sup of S in $\mathcal{G}(I)$ of the form (b, m), where $m = \bigvee_{[-1,0]^{\mathcal{P}(b)}} \{T_{a\leq b} f \mid (a, f) \text{ in } S\}$.*

In this article, we are primarily interested in a subposet of $\mathcal{G}(I)$, called the realisation of I.

4.10. Definition. For an element a in I, define $\mathcal{R}_a(I)$ to be the subposet of $\mathcal{G}(I)$ consisting of all the pairs (a, f) satisfying the following conditions:

- $f(x) > -1$, for every x in $\mathcal{P}(a)$;
- $\operatorname{supp}(f)$ is finite;
- $\operatorname{supp}(f)$ has an ancestor in I.

The subposet $\mathcal{R}(I) := \bigcup_{a\in I} \mathcal{R}_a(I) \subset \mathcal{G}(I)$ is called the **realisation** of I.

For example, for every a in I, the element $(a, 0)$ belongs to $\mathcal{R}_a(I) \subset \mathcal{R}(I)$. The function $I \longrightarrow \mathcal{R}(I)$, mapping a to $(a, 0)$, is a subposet inclusion, and we identify I with its image in $\mathcal{R}(I)$.

Let (a, f) be in $\mathcal{R}(I)$ and $(a, f) \leq (b, g)$ in $\mathcal{G}(I)$. Then, according to Proposition 4.4, $\operatorname{supp}(g)$ has an ancestor. Thus, to show that (b, g) is in $\mathcal{R}(I)$ only the first two conditions in Definition 4.10 need to be verified: the values of g are strictly bigger than -1 and the support of g is finite.

Let a be in I. For a finite subset $S \subset \mathcal{P}(a)$ having an ancestor and a function $f\colon S \longrightarrow (-1, 0)$, denote by $\overline{f}\colon \mathcal{P}(a) \longrightarrow (-1, 0]$ the extension:

$$\overline{f}(x) := \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

The element $(a, \overline{f})$ belongs to $\mathcal{R}_a(I) \subset \mathcal{R}(I)$. The function $(-1, 0)^S \longrightarrow \mathcal{R}_a(I)$, mapping f to $(a, \overline{f})$, is a subposet inclusion. Since these subposets are disjoint, as a set, $\mathcal{R}_a(I)$ can be identified with the disjoint union $\coprod_S (-1, 0)^S$ where S ranges over finite subsets of $\mathcal{P}(a)$ that have ancestors. Thus $\mathcal{R}(I)$ can be identified with the disjoint union $\coprod_a \coprod_S (-1, 0)^S$ where a ranges over all elements of I and, for each such a, S ranges over all finite subsets of $\mathcal{P}(a)$ that have ancestors.

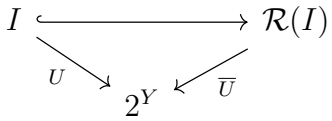
4.11. The reason we are interested in the realisation is that it generalises the relation $\mathbb{N}^r \subset [0, \infty)^r$. Consider $\alpha\colon \mathcal{R}(\mathbb{N}^r) \longrightarrow [0, \infty)^r$ mapping (a, f) to $a + \sum_{(a-e_i)\in\mathcal{P}(a)} f(a - e_i)e_i$ where e_i is the i-th vector in the standard basis of $\mathbb{R}^r$. This function is an isomorphism of posets. We use this to identify the realisation $\mathcal{R}(\mathbb{N}^r)$ with $[0, \infty)^r$. The composition of the function $\mathbb{N}^r \longrightarrow \mathcal{R}(\mathbb{N}^r)$ mapping a to $(a, 0)$ with α is the inclusion $\mathbb{N}^r \subset [0, \infty)^r$. The same formula $(a, f) \mapsto a + \sum_{(a-e_i)\in\mathcal{P}(a)} f(a - e_i)e_i$ gives a poset isomorphism between $\mathcal{R}([1]^r)$ and $[0, 1]^r$.

4.12. Let P be a set. Consider the inclusion poset 2^P of subsets of P (see 2.1). Since 2^P has a global minimum (see 2.3), given by the empty subset, all subsets of 2^P have an ancestor. Every parent of S in 2^P is of the form $S \setminus \{x\}$ for x in S, and hence the function mapping x in S to $S \setminus \{x\}$ is a bijection between S and the set of its parents $\mathcal{P}_{2^P}(S)$ in 2^P. We use this bijection to identify $\mathcal{P}_{2^P}(S)$ with S (see 2.5). The realisation $\mathcal{R}(2^P)$ can be identified with a subposet of $[-1,0]^P$. By definition, $\mathcal{R}(2^P)$ consists of pairs $(S, f\colon S \longrightarrow (-1,0])$ (here we use the identification between S and $\mathcal{P}_{2^P}(S)$) where the support of f is finite. For such an element in $\mathcal{R}(2^P)$, define $\underline{f}\colon P \longrightarrow [-1,0]$:

$$\underline{f}(x) := \begin{cases} f(x) & \text{if } x \in S \\ -1 & \text{if } x \notin S \end{cases}$$

The function $\mathcal{R}(2^P) \longrightarrow [-1,0]^P$, mapping $(S, f\colon S \longrightarrow (-1,0])$ to $\underline{f}$, is a subposet inclusion (see 2.7). Its image consists of functions $g\colon P \longrightarrow [-1,0]$ for which $g^{-1}((-1,0)) = \{x \in P \mid -1 < g(x) < 0\}$ is finite. The subposet in $[-1,0]^P$ of such functions is therefore isomorphic to $\mathcal{R}(2^P)$. For example, if P is finite, then $\mathcal{R}(2^P)$ is isomorphic to $[-1,0]^P$: the realisation of the discrete cube of finite dimension is isomorphic to the geometric cube of the same dimension (see 2.1).

4.13. The realisation $\mathcal{R}(I)$ has the following extension property, which is convenient for constructing functors indexed by it. Let $U\colon I \longrightarrow 2^Y$ be a functor of posets (see 2.7) where Y is a set and 2^Y is the inclusion poset of all subsets of Y (see 2.1). Choose a distance d on Y whose values are bounded by a real number m. When all the values of U are non-empty, we are going to use this distance to extend U along $I \subset \mathcal{R}(I)$ to form a commutative diagram of poset functors:



For (a, f) in $\mathcal{R}(I)$, define:

$$\overline{U}(a,f) := U(a) \cap \bigcap_{p \in \mathcal{P}(a)} B(U(p), (1 + f(p))m)$$

where $B(V,r) = \{y \in Y \mid d(x,y) < r \text{ for some } x \text{ in } V\}$. Note that $\overline{U}(a,0) = U(a)$. To prove the functoriality of $\overline{U}$ all the conditions describing the poset relation on $\mathcal{R}(I)$ (see 4.3) are needed.

4.14. Proposition. *Let Y be a set and $U\colon I \longrightarrow 2^Y$ be a functor of posets whose values are non-empty subsets of Y. If $(a,f) \subset (b,g)$ in $\mathcal{R}(I)$, then $\overline{U}(a,f) \subset \overline{U}(b,g)$ in 2^Y. Moreover all values of $\overline{U}$ are non-empty.*

PROOF. Let u be in $\overline{U}(a,f)$. By definition, this means: u is in $U(a)$ and, for all p in $\mathcal{P}(a)$, there is u_p in $U(p)$ for which $d(u,u_p) < (1+f(p))m$. We need to show that u is in $\overline{U}(b,g)$, which is equivalent to: u being in $U(b)$ and, for all x in $\mathcal{P}(b)$, there is v_x in $U(x)$ for which $d(u,v_x) < (1+g(x))m$.

The relation $(a,f) \leq (b,g)$ yields $a \leq b$ (condition (a) in 4.3), and hence $U(a) \subset U(b)$, implying u is in $U(b)$. Choose x in $\mathcal{P}(b)$. If $g(x) = 0$, then since all distances in Y are bounded by m, we can choose v_x to be any element in $U(x)$ (which exists since $U(x)$ is non-empty by assumption). Assume $g(x) < 0$. By condition (b) in 4.3, either x is a-dependent ($a \leq x$) or x is a-independent ($a \not\leq x$ and $(\mathcal{P}(a) \leq x) \neq \emptyset$). If $a \leq x$, then we can take $v_x = u$, since $U(a) \subset U(x)$. If x is a-independent, then the relation $(a,f) \leq (b,g)$ implies the existence of p in $\mathcal{P}(a) \leq x$ for which $f(p) \leq g(x)$ (here we use the fact that $\operatorname{supp}(f)$ is finite so that $\bigwedge_{[-1,0]}\{f(y) \mid y \in (\mathcal{P}(a) \leq x)\}$ is realised by p, see condition (c) in 4.3). In this case we can take $v_x = u_p$, as $d(u_p,x) \leq (1+f(p))m \leq (1+g(x))m$.

To show non-emptiness of $\overline{U}(a,f)$, note that $\operatorname{supp}(f)$ has an ancestor w. Thus $U(w) \subset U(a)$ and $U(w) \subset U(p)$ for every p in $\operatorname{supp}(f)$ and consequently $U(w) \subset \overline{U}(a,f)$.

Since $\mathcal{R}(I)$ is a subposet of $\mathcal{G}(I)$, one may also construct some sup elements in $\mathcal{R}(I)$ analogously to Corollary 4.9.

4.15. COROLLARY. *Let $S \subset \mathcal{R}(I)$ be a non-empty subset.*

1. *If $\pi(S) = \{a\}$, then S has a coproduct, $\bigvee_{\mathcal{R}(I)} S$, in $\mathcal{R}(I)$ and it is equal to (a,g), where $g = \bigvee_{[-1,0]^{\mathcal{P}(a)}}\{f \mid (a,f) \in S\}$.*

2. *Assume I is a poset whose every element has a finite set of parents. If b is a sup of $\pi(S)$ in I, then there is a unique sup of S in $\mathcal{R}(I)$ of the form (b,m), where $m = \bigvee_{[-1,0]^{\mathcal{P}(b)}}\{T_{a\leq b}f \mid (a,f) \text{ in } S\}$.*

PROOF. Since $\mathcal{R}(I)$ is a subposet of $\mathcal{G}(I)$, to prove the result it is enough to show that (a,g) and (b,m) belong to $\mathcal{R}(I)$.

(1): For every (a,f) in S, since $f \leq g$, then $g(x) > -1$ for all x in $\mathcal{P}(a)$, and $\operatorname{supp}(g) \subset \operatorname{supp}(f)$. Thus, $\operatorname{supp}(g)$ is finite and every ancestor of $\operatorname{supp}(f)$ is also an ancestor of $\operatorname{supp}(g)$.

(2): The equality $m(x) = -1$ holds if and only if $(T_{a\leq b}f)(x) = -1$ for all (a,f) in S. Thus, $m(x) = -1$ if and only if $a \leq x$ (x is a-dependent), for every a in $\pi(S)$. The equality $m(x) = -1$ would then contradict the assumption that b is a sup of $\pi(S)$, since x is in $\mathcal{P}(b)$. The values of the function m belong, therefore, to $(-1,0]$. The finiteness of $\operatorname{supp}(m)$ is guaranteed by the finiteness assumption on the sets of parents in I, and the existence of its ancestor is guaranteed by Lemma 4.4.

According to Corollary 4.15, if the set of parents of every element in I is finite, then elements in $\mathcal{R}(I)$ of the form $(b, \bigvee\{T_{a\leq b}f \mid (a,f) \text{ in } S\})$, where b is a sup of $\pi(S)$ in I, are sups of $S \subset \mathcal{R}(I)$. One should be aware, however, that an element (c,h) in $\mathcal{R}(I)$ may be a sup of S even though c is not a sup of $\pi(S)$ in I as Example 6.9 illustrates.

4.16. An important aspect of realisations is that they admit explicit grid-like discretisations. These discretisations are particularly useful to describe properties of tame functors, such as their homological dimension (see 10.19).

For subposets $D \subset I$ and $V \subset (-1,0)$, denote by $\mathcal{R}_D(I,V) \subset \mathcal{R}(I)$ the following subposet:

$$\mathcal{R}_D(I,V) := \{(a,f) \in \mathcal{R}(I) \mid a \in D \text{ and } f(\operatorname{supp}(f)) \subset V\}.$$

An element (a,f) in $\mathcal{R}(I)$ belongs to $\mathcal{R}_D(I,V)$ if and only if a is in D and all non-zero values of f belong to V. In particular, $(a,0)$ belongs to $\mathcal{R}_D(I,V)$ if and only if a is in D, which means that the following inclusions hold:

$$\begin{array}{ccc} D & \hookrightarrow & \mathcal{R}_D(I,V) \\ \downarrow & & \downarrow \\ I & \hookrightarrow & \mathcal{R}(I) \end{array}$$

Recall that $\mathcal{R}(I)$ can be identified with the disjoint union $\coprod_a \coprod_S (-1,0)^S$, where for every a in I, $S \subset \mathcal{P}(a)$ is finite and have an ancestor (see the paragraph after 4.10). Via this identification, $\mathcal{R}_D(I,V)$ corresponds to $\coprod_a \coprod_S V^S$, where a ranges over all elements in D and, for each such a, S ranges over all finite subsets of $\mathcal{P}(a)$ that have ancestors.

If $D = \{a\}$, then $\mathcal{R}_D(I,V)$ is also denoted as $\mathcal{R}_a(I,V)$, if $D = I$, then $\mathcal{R}_D(I,V)$ is also denoted as $\mathcal{R}(I,V)$, and if $V = (-1,0)$, then $\mathcal{R}_D(I,V)$ is also denoted as $\mathcal{R}_D(I)$. For example, choose $V = \{-0.5\} \subset (-1,0)$. Recall that the realisation $\mathcal{R}(\mathbb{N}^r)$ can be identified with the poset $[0,\infty)^r$ (see 4.11). Via this identification, $\mathcal{R}(\mathbb{N}^r, V) \subset \mathcal{R}(\mathbb{N}^r)$ corresponds to $0.5\mathbb{N}^r \subset [0,\infty)^r$.

If D and V are finite, then so is $\mathcal{R}_D(I,V)$. Furthermore, for any finite subposet $S \subset \mathcal{R}(I)$, there is a finite $D \subset I$ and a finite $V \subset (-1,0)$ for which $S \subset \mathcal{R}_D(I,V)$. In case I is of finite-type, we could choose D to be of the form $I \leq a$, for some a in I.

If I is of finite-type, then the dimension and the parental dimension of every element in the realisation $\mathcal{R}(I)$ coincide and can be calculated in an analogous way as in Proposition 3.5.

4.17. Theorem. *Let $V \subset (-1,0)$ be a subset, I be a poset of finite-type, and $D \subset I$ a subposet such that $(I \leq d) \subset D$ for every d in D. Assume (a,f) is in $\mathcal{R}_D(I,V)$ and there is ε in V such that $\varepsilon < f(x)$, for all x in $\mathcal{P}(a)$. Then the numbers* $\text{par-dim}_{\mathcal{R}_a(I,V)}(a,f)$, $\text{par-dim}_{\mathcal{R}_D(I,V)}(a,f)$, $\dim_{\mathcal{R}_a(I,V)}(a,f)$ *and* $\dim_{\mathcal{R}_D(I,V)}(a,f)$ *coincide and are equal to*

$$\max\{|S| \mid \operatorname{supp}(f) \subset S \subset \mathcal{P}(a) \text{ and } S \text{ has an ancestor}\}.$$

Proof. The vertical inequalities in the following diagram follow from Proposition 3.6. The horizontal inequalities are a consequence of Proposition 3.9, where the required assumption is given by Corollary 4.15.(1):

$$\begin{array}{ccc} \text{par-dim}_{\mathcal{R}_D(I,V)}(a,f) & \geq & \text{par-dim}_{\mathcal{R}_a(I,V)}(a,f) \\ \vee & & \vee \\ \dim_{\mathcal{R}_D(I,V)}(a,f) & \geq & \dim_{\mathcal{R}_a(I,V)}(a,f) \end{array}$$

Let $\mathcal{A} := \{S \mid \operatorname{supp}(f) \subset S \subset \mathcal{P}(a) \text{ and } S \text{ has an ancestor}\}$. Next we show that $\dim_{\mathcal{R}_a(I,V)}(a,f) \geq \max\{|S| \mid S \in \mathcal{A}\}$. Since I is of finite-type, the collection $\mathcal{A}$ is finite and all its elements are finite. Let S be in $\mathcal{A}$. Define $\varepsilon_S \colon \mathcal{P}(a) \longrightarrow (-1,0]$ and $f_p \colon \mathcal{P}(a) \longrightarrow (-1,0]$, for every p in S, by the formulas:

$$\varepsilon_S(x) = \begin{cases} \varepsilon & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \qquad f_p(x) = \begin{cases} f(x) & \text{if } x = p \\ \varepsilon & \text{if } x \in S \setminus \{p\} \\ 0 & \text{if } x \notin S \end{cases}$$

These functions are chosen so that, for every p in S, $-1 < \varepsilon_S < f_p < f$. Since $\operatorname{supp}(f_p) \subset S = \operatorname{supp}(\varepsilon_S)$, both sets $\operatorname{supp}(f_p)$ and $\operatorname{supp}(\varepsilon_S)$ have ancestors in I and thus (a, f_p) and (a, ε_S) belong to $\mathcal{R}_a(I,V)$. The relation $\varepsilon_S < f_p$, for every p, implies that (a, ε_S) is a proper ancestor of $U := \{(a, f_p) \mid p \in S\}$ in $\mathcal{R}_a(I,V)$. If $p \neq q$ in S, then $f_p \neq f_q$, and hence $|U| = |S|$. Furthermore, $\bigvee_{\mathcal{R}_a(I,V)} U = (a,f)$ and $\bigvee_{\mathcal{R}_a(I,V)}(U \setminus \{(a,f_p)\}) < (a,f)$ for every p in S. The set U belongs, therefore, to the collection $\Phi_{\mathcal{R}_a(I,V)}(a,f)$ used to define the dimension (see 3.1), and, consequently, $\dim_{\mathcal{R}_a(I,V)}(a,f) \geq |U| = |S|$. As this happens for every S in $\mathcal{A}$, we get the desired inequality.

To finish the proof we show $\max\{|S| \mid S \in \mathcal{A}\} \geq \text{par-dim}_{\mathcal{R}_D(I,V)}(a,f)$. Let U be an element in $\Psi_{\mathcal{R}_D(I,V)}(a,f)$ (see 3.2). In particular, U has an ancestor (c,h) in $\mathcal{R}_D(I,V)$. For (b,g) in U, consider $\alpha(b,g) := (T_{b \leq a} g) \vee \varepsilon$. We claim $(a, \alpha(b,g))$ belongs to $\mathcal{R}_D(I,V)$ and $\alpha(b,g) < f$. This is clear if $b = a$. If $b < a$, since I is of finite-type, there is a parent p of a such that $b \leq p < a$. In this case $T_{b \leq a} g(p) = -1$, and, consequently, $\alpha(b,g)(p) = \varepsilon < f(p)$. Furthermore, any ancestor of $\operatorname{supp}(h)$ is also an ancestor of $\operatorname{supp}(\alpha(b,g))$. In particular, $S := \bigcup_{(b,g) \in U} \operatorname{supp}(\alpha(b,g))$ has an ancestor, and it contains $\operatorname{supp}(f)$, as $(T_{b \leq a} g \vee \varepsilon) \leq f$. Thus, S is in $\mathcal{A}$.

For every (b,g) in U, let $p_{(b,g)}$ be a parent of a in I for which $\alpha(b,g)(p_{(b,g)}) < f(p_{(b,g)})$. If $\alpha(b,g) \neq \alpha(b',g')$, then, since $\alpha(b,g) \vee \alpha(b',g') = f$, the elements $p_{(b,g)}$ and $p_{(b',g')}$ have to be different. Thus $|\{p_{(b,g)} \mid (b,g) \in U\}| = |U|$, and consequently $|S| \geq |U|$ as $\{p_{(b,g)} \mid (b,g) \in U\} \subset S$. As this happens for every U in $\Psi_{\mathcal{R}_D(I,V)}(a,f)$, we get $\max\{|S| \mid S \in \mathcal{A}\} \geq \text{par-dim}_{\mathcal{R}_D(I,V)}(a,f)$.

Choosing $V = (-1,0)$ and $D = I$ in Theorem 4.17 (see also the text before the statement), we obtain the following corollary, where the assumption on the existence of ε is guaranteed by I being of finite-type.

4.18. Corollary. *Let I be a poset of finite-type. Then, for every element (a,f) in $\mathcal{R}(I)$, the numbers* $\text{par-dim}_{\mathcal{R}_a(I)}(a,f)$, $\text{par-dim}_{\mathcal{R}(I)}(a,f)$, $\dim_{\mathcal{R}_a(I)}(a,f)$ *and* $\dim_{\mathcal{R}(I)}(a,f)$ *coincide and are equal to*

$$\max\{|S| \mid \operatorname{supp}(f) \subset S \subset \mathcal{P}(a) \text{ and } S \text{ has an ancestor}\}.$$

4.19. Let I be a poset of finite-type and (a,f) be in $\mathcal{R}(I)$. By Theorems 3.5 and 4.17, $\text{par-dim}_I(a) = \text{par-dim}_{\mathcal{R}(I)}(a,0)$ and $\text{par-dim}_{\mathcal{R}(I)}(a,f) \leq \text{par-dim}_I(a)$. This last inequality can be sharp. For example, consider the subposet $I = \{(1,0,0),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\} \subset \mathbb{R}^3$. Let $a = (1,1,1)$ and $f \colon \mathcal{P}(a) \longrightarrow (-1,0]$ be the function which is 0 except for $f(0,1,1) \neq 0$. In this case, $\text{par-dim}_{\mathcal{R}(I)}(a,f) = 1$ and $\text{par-dim}_I(a) = 2$.

4.20. We now aim to give a visual representation of realisations of some finite posets. Consider the following elements in $\mathbb{R}^2$: $a = (0,0)$, $b = (3,0)$, $c = (0,2)$, $d = (3,2)$, $h = (2,0)$, and $k = (1,2)$, and the following subposets of $\mathbb{R}^2$: $I_1 = \{a,b,c,d\}$, $I_2 = \{a,b,c,d,h\}$, and $I_3 = \{a,b,c,d,h,k\}$. We demonstrate how $\mathcal{R}(I_2)$ is obtained here, starting by $\mathcal{G}(I_2)$. As a set $\mathcal{G}(I_2)$ is given by the disjoint union of $[-1,0]^{\emptyset}$, $[-1,0]^a$, where a is the unique parent of h, $[-1,0]^a$, where a is the unique parent of c, $[-1,0]^h$, $[-1,0]^{\{b,c\}}$. To exemplify how the poset relation works here, consider the pairs $(b, f\colon h \mapsto -0.2)$, $(d, g\colon (b \mapsto -0.5, c \mapsto 0))$ and $(d, k\colon (b \mapsto -0.5, c \mapsto -0.5))$. These points are, respectively, on the line segment between h and b, on the line segment between d and b, and in the grey square. In particular, the second and third points lie on the same horizontal line. Note that $b \leq d$ in I_2. Moreover, $\operatorname{supp}(g) = \{b\}$ which is b-consistent, and $\operatorname{supp}(k)$ contains c which is not b-consistent. Thus, $(b,f) \leq (d,g)$, and $(b,f) \not\leq (d,k)$. Instead, $(d,k) \leq (d,g)$ because $k(b) \leq g(b)$ and $k(c) \leq g(c)$. In this case, in which for every pair (x,f) in $\mathcal{G}(I_2)$, $\operatorname{supp}(f)$ is finite and has an ancestor whenever it is not empty, the realisation of I_2 is obtained by $\mathcal{G}(I_2)$ by removing the pairs (x,f) with $f(y) = -1$ for any $y \in \mathcal{P}(x)$.

The realisations of I_1, I_2 and I_3 illustrated in Figure 3 are represented as subposets of $\mathbb{R}^2$. We have already seen that in the second subfigure, the points in the grey square are not comparable to the points in the line segment from h to b. Analogously, in the third subfigure, the points in the grey square are not comparable to the points in the line segments from c to k and from h to b. This is the reason why missing areas appear in the representations.

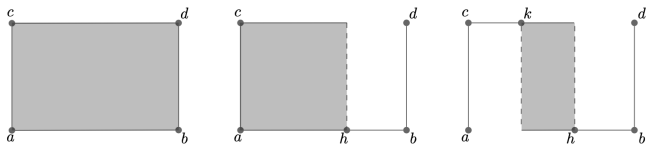


Figure 3: The dashed lines are not part of the realisation poset.

5. Upper semilattices

5.1. By definition an **upper semilattice** is a poset whose every non-empty finite subset has a coproduct (see 2.4). To verify if a poset is an upper semilattice it is enough to prove the existence of the coproduct of every two non-comparable elements.

An upper semilattice I is called **distributive** if, for every a, b, x in I for which the products $a \wedge x$ and $b \wedge x$ exist, the product $(a \vee b) \wedge x$ also exists and $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x)$.

A poset I is an upper semilattice if and only if I_* (see 2.6) is an upper semilattice. Moreover, an upper semilattice I is distributive if and only if I_* is distributive.

The product $I \times J$ of posets (see 2.1) is an upper semilattice if and only if I and J are upper semilattices. If I and J are upper semilattices, then $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2,\ y_1 \vee y_2)$ for (x_1, y_1), (x_2, y_2) in $I \times J$. The product $I \times J$ is a distributive upper semilattice if and only if I and J are distributive upper semilattices (compare with 5.4). If I is a (distributive) upper semilattice, then so is I^S for every set S.

The poset $\mathbb{R}$ is a distributive upper semilattice which is not unital. The posets $[n]$, $\mathbb{N}^r$, $[-1, 0]^S$, and $[0, \infty)^r$ are unital distributive upper semilattices.

Let S be a set. The suspension ΣS (see 2.1) is a consistent and unital upper semilattice. It is distributive if and only if $|S| \leq 2$. For example, if $S = \{s_1, s_2, s_3\}$, then $(s_1 \vee s_2) \wedge s_3 = s_3$, while $(s_1 \wedge s_3) \vee (s_2 \wedge s_3) = -\infty$. Thus, if $|S| > 2$, then ΣS is a consistent unital upper semilattice which is not distributive.

For every element a in I, the poset $\mathcal{R}_a(I)$ (see 4.10) is a distributive upper semilattice. Thus, the realisation $\mathcal{R}(I) = \bigcup_{a \in I} \mathcal{R}_a(I)$ is a union of distributive upper semilattices. The realisation $\mathcal{R}(I)$, however, may fail to be an upper semilattice, even if I is an upper semilattice (see Example 6.9).

5.2. Let I be an upper semilattice and J a poset. A function $f\colon I \longrightarrow J$ is called a **homomorphism** if, for every non-empty finite subset $S \subset I$, the element $f(\bigvee_I S)$ is a sup of $f(S)$ in J. If $f\colon I \longrightarrow J$ is a homomorphism, then it is a functor since if $a \leq b$ in I, then $f(b) = f(a \vee b)$ is a sup of $\{f(a), f(b)\}$, which implies $f(a) \leq f(b)$ in J. If J is an upper semilattice, then a function $f\colon I \longrightarrow J$ is a homomorphism if and only if $f(x \vee y) = f(x) \vee f(y)$ for all elements x and y in I. If every two elements in I are comparable (if I is a totally ordered set), then a function $f\colon I \longrightarrow J$ is a homomorphism if and only if it is a functor. For example, $f\colon \mathbb{N} \longrightarrow [0, \infty)^r$ is a homomorphism if and only if it is a functor.

The inclusion $I \subset I_*$ is a homomorphism (see 2.6). Furthermore, a function $f\colon I \longrightarrow J$ is a homomorphism if and only if $f_*\colon I_* \longrightarrow J_*$ is a homomorphism. The standard inclusion $\mathrm{in}\colon I \longrightarrow \mathcal{R}(I)$ (see 4.10) is also a homomorphism. For every a in I, the poset $\mathcal{R}_a(I)$ is an upper semilattice (see 5.1) and according to Corollary 4.9 the inclusion $\mathcal{R}_a(I) \subset \mathcal{R}(I)$ is a homomorphism. For every a in I, $T_a = [-1, 0]^{\mathcal{P}(a)}$ is an upper semilattice and, for every $a \leq b$ in I, the translation $T_{a \leq b}$ (see 4.5) is a homomorphism.

5.3. Example. Let $f\colon [1]^2 \longrightarrow [0, \infty)^2$ be a function defined as follows:

$$f(0,0) = (0,0), \quad f(1,0) = (1,0), \quad f(0,1) = (0,1), \quad f(1,1) = (2,2).$$

This function is a unital functor, but it is not a homomorphism since:

$$\bigvee_{[1]^2}\{(1,0),(0,1)\} = (1,1) \qquad \bigvee_{[0,\infty)^2}\{(1,0),(0,1)\} = (1,1)$$

$$f\Big(\bigvee_{[1]^2}\{(1,0),(0,1)\}\Big) = f(1,1) = (2,2) \neq (1,1) = \bigvee_{[0,\infty)^2}\{(1,0),(0,1)\}$$

5.4. To construct upper semilattices, the Grothendieck construction (see 2.8) can be used. Let I be an upper semilattice and $T\colon I \longrightarrow$ Posets be a functor (and not just a lax functor) such that T_a is an upper semilattice and $T_{a\leq b}$ is a homomorphism, for every $a \leq b$ in I. Then $\mathrm{Gr}_I T$ is also an upper semilattice. To see this, consider (a, x) and (b, y) in $\mathrm{Gr}_I T$. We claim $(a \vee b, T_{a\leq a\vee b}x \vee T_{b\leq a\vee b}y)$ is their coproduct in $\mathrm{Gr}_I T$. Let $(a, x) \leq (c, z) \geq (b, y)$, which means $a \leq c \geq b$ in I and $T_{a\leq c}x \leq z \geq T_{b\leq c}y$ in T_c. Since I is an upper semilattice, $a \vee b \leq c$, and, since T is a functor, $T_{a\vee b\leq c}T_{a\leq a\vee b}x = T_{a\leq c}x \leq z \geq T_{b\leq c}y = T_{a\vee b\leq c}T_{b\leq a\vee b}y$. Consequently, $(T_{a\vee b\leq c}T_{a\leq a\vee b}x) \vee (T_{a\vee b\leq c}T_{b\leq a\vee b}y) \leq z$. As $T_{a\vee b\leq c}$ is a homomorphism, $T_{a\vee b\leq c}(T_{a\leq a\vee b}x \vee T_{b\leq a\vee b}y) \leq z$, which implies $(a \vee b, T_{a\leq a\vee b}x \vee T_{b\leq a\vee b}y) \leq (c, z)$.

5.5. The dimension (see 3.1) and the parental dimension (see 3.2) of an element x in an upper semilattice I can be described using coproducts:

$$\dim_I(x) = \sup\left\{|U| \mid \begin{smallmatrix} U\subset I \text{ is finite, has a proper ancestor, } \bigvee U = x, \\ \text{and } \bigvee S < x \text{ for every set } S \text{ such that } \emptyset \neq S \subsetneq U \end{smallmatrix}\right\},$$

$$\text{par-}\dim_I(x) = \sup\left\{|U| \mid \begin{smallmatrix} U\subset (I<x) \text{ is finite, has an ancestor,} \\ \text{and } a \vee b = x, \text{ for every } a \neq b \text{ in } U \end{smallmatrix}\right\}.$$

5.6. Let I be an upper semilattice of finite-type (see 2.2) and $S \subset I$ be non-empty. If the product of S exists, then it is an ancestor of S. On the other hand, if S has an ancestor, then this ancestor belongs to $\bigcap_{x\in S}(I \leq x)$, which is finite by the finite-type assumption. Thus, if S has an ancestor, then the coproduct $\bigvee\left(\bigcap_{x\in S}(I \leq x)\right)$ exists (I is an upper semilattice), which implies the existence of the product $\bigwedge S$ and the equality $\bigwedge S = \bigvee\left(\bigcap_{x\in S}(I \leq x)\right)$ (see the end of 2.2). In conclusion, the product of a subset of a finite-type upper semilattice exists if and only if this subset has an ancestor. In particular, if $\bigwedge S$ exists in I, then so does $\bigwedge U$ for every non-empty subset $U \subset S$.

From this discussion and Proposition 3.5 it follows that the parental dimension of an element x in I can be described using products:

$$\text{par-}\dim_I(x) = \max\{|S| \mid S \subset \mathcal{P}(x) \text{ for which } \textstyle\bigwedge S \text{ exists}\}.$$

5.7. Let J be a poset. A subposet inclusion $f\colon I \subset J$ is called a **sublattice** if I is an upper semilattice and f is a homomorphism (see 5.2). For example $[0, \infty) \subset \mathbb{R} \supset \mathbb{N}$ are sublattices. If J is an upper semilattice, then for every a in J, the inclusion $(J \leq a) \subset J$ is a sublattice. A sublattice $I \subset J$ satisfies the assumption of Proposition 3.9 and hence $\dim_I(x) \leq \dim_J(x)$ and $\text{par-}\dim_I(x) \leq \text{par-}\dim_J(x)$, for all x in I.

If J is an upper semilattice, then the intersection of sublattices in J is also a sublattice (note that this may fail if J is not an upper semilattice). Thus, if J is an upper semilattice, then the intersection of all the sublattices of J containing a subset $U \subset J$ is the smallest sublattice of J containing U. This intersection is denoted by $\langle U\rangle$ and called the sublattice **generated** by U. A sublattice generated by a subset $U \subset J$ is only defined when J is an upper semilattice and this is automatically assumed whenever $\langle U\rangle$ is discussed. The sublattice $\langle U\rangle$ can be described explicitly $\langle U\rangle = U \cup \{\bigvee_J S \mid S \subset U \text{ is finite and non-empty}\}$. Thus if U is finite, then so is $\langle U\rangle$.

In Proposition 3.6 it was shown that the parental dimension of an element bounds its dimension. This bound can be strict as the suspension example (see 3.2) illustrates. However, for distributive upper semilattices of finite-type dimension and parental dimension are equal.

5.8. Proposition. *Let I be a finite-type distributive upper semilattice. Then* $\dim_I(x) = \text{par-dim}_I(x)$*, for every x in I.*

Proof. The cases when $\dim_I(x)$ equals 0 and 1 follow from Proposition 3.3. Assume $\text{par-dim}_I(x) \geq 2$ and $S \subset \mathcal{P}(x)$ realises $\text{par-dim}_I(x)$. This means S is a maximal subset of $\mathcal{P}(x)$ whose product $\bigwedge S$ exists. The assumption $\text{par-dim}_I(x) \geq 2$ guarantees $|S| \geq 2$. For s in S, set $u_s := \bigwedge(S \setminus \{s\})$ and $U := \{u_s \mid s \in S\}$. We claim: (a) $\bigwedge U = \bigwedge S$, (b) $(\bigvee U) = x$, (c) $|U| = |S|$, and (d) $(\bigvee U') < x$ if $\emptyset \neq U' \subsetneq U$. These properties imply that U belongs to $\Phi_I(x)$ (see 2.2) and hence $\dim_I(x) \geq |U| = |S| = \text{par-dim}_I(x)$, which together with Proposition 3.6 gives the desired equality.

Property (a) is clear. Assume (b) does not hold and there is p in $\mathcal{P}(x)$ such that $\bigvee U \leq p < x$, which means $u_s \leq p$ for every s in S. If p is not in S, then $\bigwedge S \leq \bigwedge(S \setminus \{s\}) \leq p$ would imply existence of the product $\bigwedge(S \cup \{p\})$ contradicting the maximality of S. The element p is therefore in S, and $\bigwedge(S \setminus \{p\}) = u_p \leq p$, leading to $u_p \vee p = p$. Since $s \vee p = x$ for every s in $S \setminus \{p\}$, distributivity gives a contradiction $x = \bigwedge_{s \in S \setminus \{p\}}(s \vee p) \leq u_p \vee p = p$.

To prove property (c) we need to show that if $s \neq s'$, then $u_s \neq u_{s'}$. Assume this is not the case and $u_s =_{s'}$. Distributivity and the already proven property (b) leads to a contradiction:

$$s' = x \wedge s' = (\bigvee U) \wedge s' = \bigvee_s (u_s \wedge s') = (u_{s'} \wedge s') \vee (\bigvee_{s \neq s'} (u_s \wedge s')) = \\ (\bigwedge S) \vee (\bigvee_{s \neq s'} u_s) = \bigvee_{s \neq s'} u_s = \bigvee_s u_s = \bigvee U = x$$

Thus, U and S have the same size, which coincide with property (c).

It remains to show (d): $\bigvee U' < x$ if $\emptyset \neq U' \subsetneq U$. Assume by contradiction that there is s' in S such that $\bigvee_{s \neq s'} u_s = x$. Then, again, distributivity and property (b) leads to a contradiction:

$$s' = x \wedge s' = (\bigvee_{s \neq s'} u_s) \wedge s' = \bigvee_{s \neq s'} (u_s \wedge s') = \bigvee_{s \neq s'} u_s = x.$$

5.9. We finish this section with a characterisation of finite-type posets whose elements have dimension at most one. A poset I, that is connected and such that every pair of non-comparable elements x and y in I has no common ancestor, is called a **tree**.

5.10. Proposition.

1. *An upper semilattice whose every element has dimension (see 3.1) at most* 1 *is a tree.*

2. *A tree of finite-type is an upper semilattice whose every element has dimension at most* 1.

PROOF. (1): First, observe that upper semilattices are connected. Let x and y be non-comparable elements in an upper semilattice I whose every element has dimension at most 1. If x and y had a common ancestor, then $\{x, y\}$ would belong to $\Phi_I(x \vee y)$ (see 3.1) leading to a contradiction $\dim_I(x \vee y) \geq 2$.

(2): Let I be a tree of finite-type and x, y be its elements. We need to show the existence of $x \vee y$. First, assume that x and y have a common descendent. Then, since I is of finite-type, $\{x, y\}$ has a sup. We claim that this sup is unique, otherwise two such sup elements would be non-comparable and have x as a common ancestor contradicting the assumption I is a tree.

Assume x and y do not have a common descendent. Since I is connected, there is a sequence $C = \{c_0 = x, c_1, \ldots, c_n = y\}$ in I where c_i is comparable to c_{i+1} for every $0 \leq i < n$. This set C does not have a descendent, otherwise $\{x, y\}$ would also have it. Thus C must have at least two maximal elements. Among these, there are c_i and c_j such that $(C \leq c_i) \cap (C \leq c_j) \neq \emptyset$, otherwise C would not be connected. The elements c_i and c_j are not comparable and have no common ancestor. This contradicts again the assumption I is a tree.

Finally, for every x in I, $\dim_I(x) \leq 1$, since any subset of size at least 2 with x as sup cannot have a common ancestor by hypothesis.

6. Realisations of upper semilattices

The realisation of an upper semilattice may fail to be an upper semilattice (see Example 6.9). This section is devoted to discussing some of the reasons for this and what assumptions can eliminate them. Our aim is to prove:

6.1. THEOREM. *If I is a distributive upper semilattice of finite-type, then its realisation $\mathcal{R}(I)$ (see 4.3) is also a distributive upper semilattice.*

An upper semilattice whose elements have dimension at most 1 is distributive. In particular, finite-type trees are distributive upper semilattices (see 5.10.(2)). Theorem 6.1 implies therefore that the realisation of a finite-type tree is a distributive upper semilattice, and since its elements have dimensions at most 1 it is a tree (see 5.10.(1)).

6.2. Let I be an upper semilattice and a, x be in I. If $\mathcal{P}(a) \leq x$ contains two different elements y_1 and y_2, then $y_1 \vee y_2 = a$, which implies $a \leq x$, and hence $(\mathcal{P}(a) \leq x) = \mathcal{P}(a)$. Thus, there are three possibilities: the set $\mathcal{P}(a) \leq x$ is empty, consist of only one element, or is the entire set $\mathcal{P}(a)$. Furthermore, if $\mathcal{P}(a) \leq x$ contains only one element and $a \not\leq x$, then this element has to be the product $a \wedge x$. These facts are used to give in Table 2 a characterisation of the blocks in the partition of I described in 4.1.

This table can be used to describe the translation operation (see 4.5) more explicitly.

x is a-consistent	x is a-dependent	$a \leq x$. In this case $(\mathcal{P}(a) \leq x) = \mathcal{P}(a)$
	x is a-independent	$a \not\leq x$ and the product $a \wedge x$ exists and belongs to $\mathcal{P}(a)$. In this case $\lvert\mathcal{P}(a) \leq x\rvert = 1$.
x is a-inconsistent		$a \not\leq x$ and either the product $a \wedge x$ does not exist or it exists but does not belong to $\mathcal{P}(a)$. In this case $(\mathcal{P}(a) \leq x) = \emptyset$.

Table 2:

Let $a \leq b$ in I and $f\colon \mathcal{P}(a) \longrightarrow [-1, 0]$ be a function. Then:

$$(T_{a\leq b} f)(x) = \begin{cases} -1 & \text{if } x \text{ is } a\text{-dependent,} \\ f(a \wedge x) & \text{if } x \text{ is } a\text{-independent,} \\ 0 & \text{if } x \text{ is } a\text{-inconsistent.} \end{cases}$$

Similarly, a simplified characterisation of an upper semilattice of finite-type to be consistent (see 4.1) can be given.

6.3. Lemma. *Let I be an upper semilattice of finite-type. Then I is consistent if and only if the following condition is satisfied for every $a \leq b$ in I: if x is b-independent and the product $a \wedge x$ exists, then x is a-consistent, in which case either $a \wedge x = a$ (x is a-dependent) or $a \wedge x$ is a parent of a (x is a-independent).*

Proof. Consider elements $a \leq b$ in I. Assume I is a consistent upper semilattice and x is b-independent. If the product $a \wedge x$ exists, then $a \wedge x$ is a common ancestor of a and x, and consistency of I implies that x is a-consistent.

Assume the condition in the lemma holds. Let x be b-independent and let it have a common ancestor with a. Since I is of finite-type, the product $a \wedge x$ exists. The a-consistency of x is then given by the assumed condition.

6.4. Proposition. *A distributive upper semilattice of finite-type is consistent.*

Proof. To prove the proposition we use Lemma 6.3. Let $a \leq b$ and x be an element which is b-independent for which the product $a \wedge x$ exists. Let p be a parent of b such that $p \leq x$. Then $p \leq p \vee (a \wedge x) \leq b$, and since $b \not\leq x$, the element b cannot be equal to $p \vee (a \wedge x)$. Thus, $a \wedge x \leq p \vee (a \wedge x) = p$. There are three options: either $a \wedge x = p$, or $a \wedge x$ is a parent of a, or there is y such that $a \wedge x < y < a$. The first two options give a-consistency of x proving the proposition. We claim that the third option is not possible. For y such that $a \wedge x < y < a$, we have $y \not\leq p$, otherwise $y \leq x$ implying $y \leq a \wedge x$. Since $y \not\leq p$, then $p \vee y = b$. We can then use distributivity to obtain a contradiction $a = b \wedge a = (p \vee y) \wedge a = (p \wedge a) \vee (y \wedge a) \leq (a \wedge x) \vee y \leq y$.

The key reason why we are interested in consistent upper semilattices is the following:

6.5. Lemma. *If I is a consistent upper-semilattice, then the I-translation is a functor: $T_{a\leq c} = T_{b\leq c}T_{a\leq b}$ for every $a \leq b \leq c$ in I.*

Proof. Consider a function $f\colon \mathcal{P}(a) \longrightarrow [-1, 0]$. According to Lemma 4.6 we just need to show $(T_{a\leq c}f)(x) \geq (T_{b\leq c}T_{a\leq b}f)(x)$ for all x in $\mathcal{P}(c)$. Let x be in $\mathcal{P}(c)$. If x is a-inconsistent, then $(T_{a\leq c}f)(x) = 0$ and the desired inequality is clear.

Assume x is a-consistent. Then $a \wedge x$ exists and either $a = a \wedge x$ or $a \wedge x$ is a parent of a. Since x is a parent of c, it is c-consistent. Consistency of I applied to $b \leq c$ implies the b-consistency of x. Thus, either x is b-dependent, or b-independent. If x is b-dependent, then $(T_{b\leq c}T_{a\leq b}f)(x) = -1$ and the desired inequality is clear. Assume x is b-independent. Then $(T_{b\leq c}T_{a\leq b}f)(x) = (T_{a\leq b}f)(b \wedge x)$. The product $b \wedge x$ is a-dependent if and only if x is a-dependent. If this happens, then $(T_{a\leq c}f) = -1 = (T_{a\leq b}f)(b \wedge x)$ and the desired equality holds in this case as well. The product $b \wedge x$ is a-independent if and only if x is a-independent. If this happens, then $(T_{a\leq c}f) = f(a \wedge x) = f(a \wedge b \wedge x) = (T_{a\leq b}f)(b \wedge x)$ and the desired equality also holds.

For every $a \leq b$ in a poset I, the translation $T_{a\leq b}$ is a homomorphism of upper semilattices. If, in addition, I is a consistent upper semilattice, then by Lemma 6.5, the I-translation is a functor, and consequently (see 5.4):

6.6. Proposition. *If I is a consistent upper semilattice, then the Grothendieck construction $\mathrm{Gr}_I T = \mathcal{G}(I)$ is an upper semilattice.*

For the realisation $\mathcal{R}(I) \subset \mathcal{G}(I)$ (see 4.3) to be an upper semilattice we need a further finiteness restriction:

6.7. Proposition. *Let I be a consistent upper semilattice whose every element has a finite set of parents. Then $\mathcal{R}(I)$ is an upper semilattice.*

Proof. Let (a, f), (b, g) be in $\mathcal{R}(I)$ and $m := \bigvee\{T_{a\leq a\vee b}f, T_{b\leq a\vee b}g\}$. The poset $\mathcal{G}(I)$ is an upper semilattice (see 6.6). Thus $(a \vee b, m)$ is the coproduct of (a, f) and (b, g) in $\mathcal{G}(I)$ (see 4.9). Since $(a \vee b, m)$ belongs to $\mathcal{R}(I)$ (see 4.15), it is the coproduct of (a, f) and (b, g) in $\mathcal{R}(I)$.

6.8. Corollary. *Let I be a consistent upper semilattice whose every element has a finite set of parents. Choose an element d in I and a subposet $V \subset (-1, 0)$. Then the inclusion $\alpha\colon \mathcal{R}_{I\leq d}(I, V) \subset \mathcal{R}(I)$ is a sublattice (see 5.7), i.e., $\mathcal{R}_{I\leq d}(I, V)$ is an upper semilattice, and α is a homomorphism.*

Proof. Just note that if (a, f) and (b, g) belong to $\mathcal{R}_{I\leq d}(I, V)$, then so does their coproduct $(a \vee b, m)$ described in the proof of 6.7.

We are now ready to prove Theorem 6.1.

Proof. [Proof of Theorem 6.1] Let I be a distributive upper semilattice of finite-type. Its realisation $\mathcal{R}(I)$ is an upper semilattice because of Propositions 6.4 and 6.7. It remains to show that $\mathcal{R}(I)$ is distributive. This requires understanding of products in $\mathcal{R}(I)$. Consider the elements $(a, f) \geq (c, m) \leq (x, h)$ in $\mathcal{R}(I)$. Then $a \geq c \leq x$ and hence the product $a \wedge x$ exists (see discussion in 5.6). Let y be a parent of $a \wedge x$. If p_1 and p_2 are different parents

of a for which $p_1 \wedge x = y = p_2 \wedge x$, then, by the distributivity, we would get a contradiction $a \wedge x = (p_1 \vee p_2) \wedge x = (p_1 \wedge x) \vee (p_2 \wedge x) = y$. Thus, the set $\{p \in \mathcal{P}(a) \mid p \wedge x = y\}$ contains at most one element. If $\{p \in \mathcal{P}(a) \mid p \wedge x = y\}$ is non-empty, then its unique element is denoted by yx^{-1}. Define $R(f)\colon \mathcal{P}(a \wedge x) \longrightarrow (-1, 0]$

$$R(f)(y) := \begin{cases} f(yx^{-1}) & \text{if } \{p \in \mathcal{P}(a) \mid p \wedge x = y\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Let w be an ancestor of $\operatorname{supp}(f)$. For every y in $\operatorname{supp}(R(f))$, the set $\{p \in \mathcal{P}(a) \mid p \wedge x = y\}$ is non-empty and $R(f)(y) = f(yx^{-1}) < 0$. Thus, $w \leq yx^{-1}$ which implies $w \wedge x \leq y$. The element $w \wedge x$ is therefore an ancestor of $\operatorname{supp}(R(f))$ and $(a \wedge x, R(f))$ is in $\mathcal{R}(I)$.

We claim that the following relations hold $(c, m) \leq (a \wedge x, R(f)) \leq (a, f)$. Let y be in $\operatorname{supp}(R(f))$. In particular, $f(yx^{-1}) < 0$. The relation $(c, m) \leq (a, f)$ has two consequences. First, the product $yx^{-1} \wedge c = y \wedge c$ exists and y is c-consistent (see Table 2). All the elements of $\operatorname{supp}(R(f))$ are therefore c-consistent. Second, if y in addition is c-independent, then yx^{-1} is also c-independent and $m(y \wedge c) = m(yx^{-1} \wedge c) \leq f(yx^{-1}) = R(f)(y)$. These two consequences give the relation $(c, m) \leq (a \wedge x, R(f))$. By direct calculation:

$$T_{a\wedge x \leq a} R(f)(p) = \begin{cases} -1 & \text{if } p \text{ is } a \wedge x\text{-dependent} \\ f(p) & \text{if } p \text{ is } a \wedge x\text{-independent} \\ 0 & \text{if } p \text{ is } a \wedge x\text{-inconsistent} \end{cases}$$

If p in $\mathcal{P}(a)$ is $a \wedge x$-inconsistent, then it is also c-inconsistent (see Table 2). Thus, $T_{a\wedge x \leq a} R(f) \leq f$, as all the elements in $\operatorname{supp}(f)$ are c-consistent. This shows $(a \wedge x, R(f)) \leq (a, f)$ (see 4.8).

In an analogous way $(c, m) \leq (a \wedge x, R(h)) \leq (x, h)$ and the obtained relations imply:

$$\begin{array}{ccccc} (a, f) & \geq & (a \wedge x, \bigwedge\{R(f), R(h)\}) & \leq & (x, h) \\ & & \mid\vee & & \\ & & (c, m) & & \end{array}$$

This proves $(a \wedge x, \bigwedge\{R(f), R(h)\})$ is the product of (a, f) and (x, h). In particular, the product of two elements in $\mathcal{R}(I)$ exists if and only if these elements have a common ancestor.

We are now ready to prove distributivity of $\mathcal{R}(I)$. Let (a, f), (b, g), (x, h) be in $\mathcal{R}(I)$ for which the products $(a, f) \wedge (x, h)$ and $(b, g) \wedge (x, h)$ exist. Since $(a, f) \wedge (x, h)$ is an ancestor of both $(a, f) \vee (b, g)$ and (x, h), the product $((a, f) \vee (b, g)) \wedge (x, h)$ exists. Moreover, this product is of the form $((a \wedge x) \vee (b \wedge x), m)$, where:

$$m = \bigwedge\{R(\bigvee\{T_{a\leq a\vee b} f, T_{b\leq a\vee b} g\}), R(h)\}.$$

By direct verification $m = \bigvee\{\bigwedge\{R(T_{a\leq a\vee b} f), R(h)\}, \bigwedge\{R(T_{b\leq a\vee b} g),\ R(h)\}\}$, which gives the desired equality $((a, f) \vee (b, g)) \wedge (x, h) = ((a, f) \wedge (x, h)) \vee ((b, g) \wedge (x, h))$.

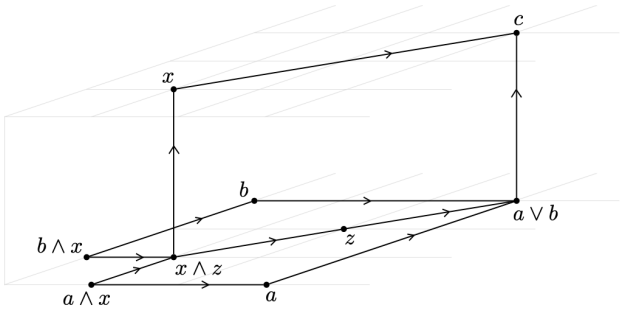


Figure 4: The arrows go from smaller to bigger elements of the upper semilattice.

6.9. Examples. The subposet $I = \{a = (0,0), b = (3,0), c = (0,2), h = (2,0), d = (3,2)\} \subset [0,\infty)^2$ is an example of a non-consistent upper semilattice, which according to Proposition 6.4, cannot be distributive. Indeed, b is d-independent, but it is not c-consistent, even though $c \leq d$ and $b \wedge c = a$ exists. However, its realisation is still an upper semilattice (see 4.20).

Consider the subposet $I \subset [0,\infty)^3$ given by the points $\{a \wedge x = (1,0,0), b \wedge x = (0,1,0), a = (3,0,0), b = (0,3,0), x \wedge z = (1,1,0), z = (2,2,0), a \vee b = (3,3,0), x = (1,1,2), c = (3,3,2)\}$, where the names of the points are chosen to help understanding the relations between them (see Figure 4). Analogously, the poset I is a non-consistent upper semilattice. Furthermore, its realisation $\mathcal{R}(I)$ is not an upper semilattice. This can be seen by taking the points $(a, f\colon \{a\wedge x\} \longrightarrow (-1,0])$ and $(b, g\colon \{b\wedge x\} \longrightarrow (-1,0])$ of $\mathcal{R}(I)$, where $f(a\wedge x) = -0.25$ and $g(b\wedge x) = -0.5$. We can define $(a\vee b, m\colon \{a,b,z\} \longrightarrow (-1,0])$, where $m(a) = 0$, $m(b) = 0$ and $m(z) = f(a \wedge x) \vee g(b \wedge x) = -0.25$. Consider also $(c, h\colon \{a \vee b, x\} \longrightarrow (-1,0])$, with $h(a \vee b) = 0$ and $h(x) = -0.25$. The two elements $(a \vee b, m)$ and (c, h) are both sups of $\{(a, f), (b, g)\}$. They are indeed not comparable, since x in $\operatorname{supp}(h)$ is $(a \vee b)$-inconsistent.

In this part we introduce tame functors indexed by posets. An important property of a tame functor is that to describe it only a finite amount of information is needed. For example, every functor indexed by a finite poset is tame. Tameness is therefore an interesting property for functors indexed by infinite posets, for example by realisations.

7. Transfers and left Kan extensions

7.1. Let I be an upper semilattice (see 5.1) and $f\colon I \longrightarrow J$ a function of finite-type (see 2.2). Consider the unital function $f_*\colon I_* \longrightarrow J_*$ (see 2.6). Since I_* is also an upper semilattice and $f_* \leq a$ is non-empty and finite, we can form the coproduct $\bigvee_{I_*}(f_* \leq a)$ for all a in J_*. Note that

$$\bigvee_{I_*}(f_* \leq a) = \begin{cases} -\infty & \text{if } a = -\infty, \text{ or } a \text{ is in } J \text{ and } (f \leq a) = \emptyset, \\ \bigvee_I(f \leq a) & \text{if } a \text{ is in } J \text{ and } (f \leq a) \neq \emptyset. \end{cases}$$

If $a \leq b$ in J_*, then $\bigvee_{I_*}(f_* \leq a) \leq \bigvee_{I_*}(f_* \leq b)$ since $(f_* \leq a) \subset (f_* \leq b)$. The function mapping a in J_* to $\bigvee_{I_*}(f_* \leq a)$ in I_* is therefore a unital functor, which we denote by $f^!\colon J_* \longrightarrow I_*$ and call the **transfer** of f. The transfer $f^!\colon J_* \longrightarrow I_*$ is only defined when $f\colon I \longrightarrow J$ is a function of finite-type between an upper semilattice I and a poset J. Whenever the transfer $f^!$ is considered, we automatically assume that these conditions are satisfied.

Since $f_* \leq f_*(a)$ contains a, the relation $a \leq \bigvee_{I_*}(f_* \leq f_*(a)) = f^!f_*(a)$ holds for every a in I_*. Consequently, $\mathrm{id}_{I_*} \leq f^!f_*$ holds in the poset ${I_*}^{I_*}$.

In general, $f^!(a) = \bigvee_{I_*}(f_* \leq a)$ may fail to belong to $f_* \leq a$, i.e., the relation $f_*f^!(a) = f_*(\bigvee_{I_*}(f_* \leq a)) \leq a$ may fail to hold. For example, when $f\colon [1]^2 \longrightarrow [0,\infty)^2$ is given as in Example 5.3 and $a = (1,1)$ in $[0,\infty)^2$. However, if f is a homomorphism of finite-type, then the following results hold. These are immediate consequences of the fact that, in this case, the transfer $f^!$ is the right adjoint of f_*, however, we report the proofs for self-containment.

7.2. Proposition. *Let $f\colon I \longrightarrow J$ be a homomorphism (see 5.2) of finite-type between an upper semilattice I and a poset J (see 5.2).*

1. *$f_*f^! \leq \mathrm{id}_{J_*}$ in ${J_*}^{J_*}$.*
2. *For every a in J_*, the element $f^!(a)$ belongs to $f_* \leq a$ and it is its global maximum.*
3. *$f^!f_*f^! = f^!$.*
4. *$f_*f^!f_* = f_*$.*
5. *For every a in J_*, the element $f_*f^!(a)$ belongs to $\{b \in J_* \mid (f_* \leq a) = (f_* \leq b)\}$ and it is its global minimum.*
6. *For a and b in J, $(f \leq a) = (f \leq b)$ if and only if $f^!(a) = f^!(b)$.*
7. *If f is injective, then $f^!f_* = \mathrm{id}_{I_*}$.*
8. *If f is surjective, then $f_*f^! = \mathrm{id}_{J_*}$.*

Proof. (1): Let a be in J_*. Since f is a homomorphism, then so is f_*, and therefore $f_*f^!(a) = f_*(\bigvee_{I_*}(f_* \leq a))$ is a sup of $\{f_*(x) \mid x \in (f_* \leq a)\}$ in J_*. The desired relation, $f_*f^!(a) \leq a$, is then a consequence of the inclusion $\{f_*(x) \mid x \in (f_* \leq a)\} \subset (J_* \leq a)$.
(2): By (1), $f_*f^!(a) \leq a$, and hence $f^!(a)$ is in $f_* \leq a$. The element $f^!(a)$ is the global maximum of $f_* \leq a$ since it belongs to it and it is its coproduct.
(3): The relation $f^! \leq f^!f_*f^!$ holds in general and does not require f to be a homomorphism. The reverse relation $f^!f_*f^! \leq f^!$ follows from (1) and the fact that $f^!$ is a functor (see 5.2).
(4): The relation $\mathrm{id}_{I_*} \leq f^!f_*$ and f_* being a functor imply $f_* \leq f_*f^!f_*$. The reverse relation $f_*f^!f_* \leq f_*$ is a particular case of (1).

(5): The inclusion $(f_* \leq f_*f^!(a)) \subset (f_* \leq a)$ follows from (1). If x in I_* is such that $f_*(x) \leq a$, then, since $f^!$ and f_* are homomorphisms, $f_*f^!f_*(x) \leq f_*f^!(a)$. Statement (4) implies therefore $f_*(x) \leq f_*f^!(a)$. This shows $(f_* \leq a) \subset (f_* \leq f_*f^!(a))$ and consequently $(f_* \leq a) = (f_* \leq f_*f^!(a))$, proving the first part of the statement. According to (2), if $(f \leq a) = (f \leq b)$, then $f^!(a)$ belongs to $f \leq b$, and consequently $f_*f^!(a) \leq b$, showing the minimality of $f_*f^!(a)$.
(6): If $(f \leq a) = (f \leq b)$, then $(f_* \leq a) = (f_* \leq b)$ and hence the coproducts $f^!(a) = \bigvee_I(f_* \leq a)$ and $f^!(b) = \bigvee_I(f_* \leq b)$ are equal. If $f^!(a) = f^!(b)$, then $f_*f^!(a) = f_*f^!(b)$ and consequently $(f_* \leq a) = (f_* \leq f_*f^!(a))$ and $(f_* \leq b) = (f_* \leq f_*f^!(b))$ are also equal, which implies $(f \leq a) = (f \leq b)$.
(7, 8): These two statements are direct consequences of (4).

We can use Proposition 7.2 to interpret the transfer of a homomorphism of finite-type as a localization which is a process involving inverting morphisms.

7.3. Corollary. *Let $f\colon I \longrightarrow J$ be a homomorphism of finite-type between an upper semilattice I and a poset J, and $\mathcal{C}$ a category with an initial object. Then the following statements about a functor $F\colon J_* \longrightarrow \mathcal{C}$ are equivalent:*

- *for all a in J, the morphism $F(f_*f^!(a) \leq a)$ is an isomorphism (F inverts morphisms of the form $f_*f^!(a) \leq a$);*
- *there is a functor $G\colon I_* \longrightarrow \mathcal{C}$ for which F is isomorphic to $Gf^!$.*

Proof. Assume $F(f_*f^!(a) \leq a)$ is an isomorphism for all a in J. Thus, the natural transformation $(Ff_*)f^! \longrightarrow F$, given by $\{F(f_*f^!(a) \leq a)\}_{a\in J}$, is an isomorphism, and we could take Ff_* for G.

Assume F is isomorphic to $Gf^!$. Since $f^!f_*f^! = f^!$ (see 7.2.(3)), for all a in J, the morphism $Gf^!(f_*f^!(a) \leq a)$ is an isomorphism. The same is therefore true for $F(f_*f^!(a) \leq a)$.

The transfer is convenient for constructing certain left adjoints even in cases when colimits cannot be performed.

7.4. Proposition. *Let $\mathcal{C}$ be a category with an initial object and $f\colon I \longrightarrow J$ a homomorphism of finite-type between an upper semilattice I and a poset J. Then the functor $(-)^{f^!}\colon \mathrm{Fun}_*(I_*, \mathcal{C}) \longrightarrow \mathrm{Fun}_*(J_*, \mathcal{C})$ is left adjoint to $(-)^{f_*}\colon \mathrm{Fun}_*(J_*, \mathcal{C}) \longrightarrow \mathrm{Fun}_*(I_*, \mathcal{C})$.*

Proof. Let $G\colon I_* \longrightarrow \mathcal{C}$ and $F\colon J_* \longrightarrow \mathcal{C}$ be unital functors. We need to show there is a bijection, functorial in G and F, between the sets of natural transformations $\mathrm{Nat}_{J_*}(Gf^!, F)$ and $\mathrm{Nat}_{I_*}(G, Ff_*)$. For a natural transformation $\phi = \{\phi_a\colon Gf^!(a) \longrightarrow F(a)\}_{a\in J_*}$ in $\mathrm{Fun}_*(J_*, \mathcal{C})$, define:

$$\overline{\phi} := \left\{ G(x) \xrightarrow{G(x\leq f^!f_*(x))} Gf^!f_*(x) \xrightarrow{\phi_{f_*(x)}} Ff_*(x) \right\}_{x\in I_*}$$

Then $\overline{\phi}$ is a natural transformation between G and Ff_* in $\mathrm{Fun}_*(I_*, \mathcal{C})$.

For a natural transformation $\psi = \{\psi_x \colon G(x) \longrightarrow Ff_*(x)\}_{x\in I_*}$ in $\mathrm{Fun}_*(I_*,\mathcal{C})$, define:

$$\widehat{\psi} := \left\{ Gf^!(a) \xrightarrow{\psi_{f^!(a)}} Ff_*f^!(a) \xrightarrow{F(f_*f^!(a)\leq a)} F(a) \right\}_{a\in J_*}$$

Then $\widehat{\psi}$ is a natural transformation between $Gf^!$ and F in $\mathrm{Fun}_*(J_*,\mathcal{C})$.

Statements (2) and (3) of Proposition 7.2 imply $\widehat{\overline{\phi}} = \phi$ and $\overline{\widehat{\psi}} = \psi$. The assignments $\phi \mapsto \overline{\phi}$ and $\psi \mapsto \widehat{\psi}$ are therefore inverse bijections.

Recall that there is a commutative square of functors where the vertical functors are isomorphism of categories (see 2.7):

$$\begin{array}{ccc} \mathrm{Fun}(J,\mathcal{C}) & \xrightarrow{(-)^f} & \mathrm{Fun}(I,\mathcal{C}) \\ {\scriptstyle -_*}\downarrow & & \downarrow{\scriptstyle -_*} \\ \mathrm{Fun}_*(J_*,\mathcal{C}) & \xrightarrow{(-)^{f_*}} & \mathrm{Fun}_*(I_*,\mathcal{C}) \end{array}$$

Using these vertical isomorphisms, Proposition 7.4 can be rephrased as:

7.5. Corollary. *Let $\mathcal{C}$ be a category with an initial object and let $f\colon I \longrightarrow J$ be a homomorphism of finite-type between an upper semilattice I and a poset J. Then the precomposition with f, $(-)^f\colon \mathrm{Fun}(J,\mathcal{C}) \longrightarrow \mathrm{Fun}(I,\mathcal{C})$, has a left adjoint given by $(-)^{f^!}$.*

7.6. The left adjoint to $(-)^f\colon \mathrm{Fun}(J,\mathcal{C}) \longrightarrow \mathrm{Fun}(I,\mathcal{C})$ is also called the **left Kan extension** along f (see 2.11). Typically such left adjoints are constructed via colimits in $\mathcal{C}$. However, according to Corollary 7.5, when $\mathcal{C}$ has an initial object e and $f\colon I \longrightarrow J$ is a homomorphism of finite-type between an upper semilattice I and a poset J, the left Kan extension $f^kG\colon J \longrightarrow \mathcal{C}$ of $G\colon I \longrightarrow C$ along f exists and it is isomorphic to $(-)^{f^!}$, as a consequence of the universal property of left Kan extensions (see 2.11). In fact, it can be constructed explicitly (see also [BCM20, Proposition 5.6]):

$$f^kG(a) \text{ is isomorphic to } \begin{cases} e & \text{if } (f\leq a) = \emptyset, \\ Gf^!(a) = G(\bigvee_I (f\leq a)) & \text{if } (f\leq a) \neq \emptyset. \end{cases}$$

The above explicit description of the left Kan extension has consequences that are generically not true for arbitrary left Kan extensions. For example, next result states that, under the hypothesis of this section on f, every category $\mathcal{C}$ with an initial object is f-right and left invertible even though $\mathcal{C}$ is not closed under finite colimits (confront with the discussion in 2.12).

7.7. Proposition. *Let $f\colon I \longrightarrow J$ be a homomorphism of finite-type between an upper semilattice I and a poset J, and $\mathcal{C}$ a category with an initial object.*

1. *If f is injective, then $\mathcal{C}$ is f-right invertible.*
2. *If f is surjective, then $\mathcal{C}$ is f-left invertible.*
3. *If $\mathcal{C}$ is closed under taking (finite) limits, then the functor $f^k\colon \mathrm{Fun}(I,\mathcal{C}) \longrightarrow \mathrm{Fun}(J,\mathcal{C})$ commutes with (finite) limits.*

PROOF. Statement (1) follows from Proposition 7.2.(7), and (2) from Proposition 7.2.(8).

To prove (3), consider a functor $\phi\colon D \longrightarrow \mathrm{Fun}(I,\mathcal{C})$ indexed by a (finite) small category D. Since $\mathcal{C}$ is closed under taking (finite) limits, then so is $\mathrm{Fun}(I,\mathcal{C})$. According to the above description of the left Kan extension and the fact that the limits in $\mathrm{Fun}(I,\mathcal{C})$ are constructed objectwise:

$$f^k(\lim_D\phi)(a) \text{ is isomorphic to } \begin{cases} e & \text{if } (f\leq a)=\emptyset \\ (\lim_D\phi)(\bigvee_I(f\leq a)) & \text{if } (f\leq a)\neq\emptyset \end{cases}$$

$$\lim_D(f^k\phi)(a) \text{ is isomorphic to } \begin{cases} e & \text{if } (f\leq a)=\emptyset \\ \lim_D(\phi(\bigvee_I(f\leq a))) & \text{if } (f\leq a)\neq\emptyset \end{cases}$$

from which it follows that $\lim_D(f^k\phi)$ and $f^k(\lim_D\phi)$ are isomorphic.

Here is a characterisation of functors isomorphic to left Kan extensions along a homomorphism of finite-type based on Corollary 7.3 and the discussion in 7.6.

7.8. COROLLARY. *Let $f\colon I \longrightarrow J$ be a homomorphism of finite-type between an upper semilattice I and a poset J, and $\mathcal{C}$ a category with an initial object e. Then the following statements about a functor $F\colon J \longrightarrow \mathcal{C}$ are equivalent:*

- *for all a in J, if $(f\leq a)\neq\emptyset$, then $F(f(\bigvee_I(f\leq a))\leq a)$ is an isomorphism, and if $(f\leq a)=\emptyset$, then $F(a)$ is isomorphic to e;*
- *there is a functor $G\colon I \longrightarrow \mathcal{C}$ for which F is isomorphic to f^kG;*
- *F is isomorphic to $f^k(Ff)$;*
- *there is a unital functor $G\colon I_* \longrightarrow \mathcal{C}$ for which F is isomorphic to the composition $J \hookrightarrow J_* \xrightarrow{f^!} I_* \xrightarrow{G} \mathcal{C}$.*

7.9. Recall that every function $f\colon I \longrightarrow J$ leads to a functor $f{\leq}\colon J \longrightarrow J[f]$, where $J[f]\subset 2^I$ is the subposet of subsets of the form $(f\leq a)\subset I$ for a in J, and $f{\leq}$ maps a to $f\leq a$ (see 2.10). Since $f{\leq}$ is a functor, its fibers $(f{\leq})^{-1}(f\leq a)\subset J$, for a in J, are convex subposets of J (see 2.10).

As noted in 2.11, if f is a functor of finite-type and $\mathcal{C}$ is a category closed under finite colimits, then, for every $G\colon I \longrightarrow \mathcal{C}$, the left Kan extension $f^kG\colon J \longrightarrow \mathcal{C}$ is isomorphic to a functor that factors through $f{\leq}\colon J \longrightarrow J[f]$. This implies that the restrictions of f^kG to the fibers $(f{\leq})^{-1}(f\leq a)\subset J$ of $f{\leq}$, for a in J, are isomorphic to constant functors. In general these fibers, although being convex, can still be rather complex subposets of J.

Assume $f\colon I \longrightarrow J$ is a homomorphism of finite-type between an upper semilattice I and a poset J. In this case, the only assumption we need to make about $\mathcal{C}$ is that it has an initial object e. Under these assumptions the fibers of $f{\leq}$ can be described using the transfer. Both functors, the transfer $f^!\colon J_* \longrightarrow I_*$ and $f{\leq}\colon J \longrightarrow J[f]$, fit into the

commutative diagram:

$$\begin{array}{ccccc} I & \xrightarrow{f} & J & \xrightarrow{f\leq} & J[f] \\ & & \downarrow & & \downarrow V \\ & & J_* & \xrightarrow{f^!} & I_* \end{array}$$

where $V(f \leq a) := f^!(a) = \bigvee_{I_*}(f_* \leq a)$. According to Proposition 7.2.(6), $V\colon J[f] \longrightarrow I_*$ is a monomorphism. Since $f\leq$ is a surjection, V is a poset isomorphism between $J[f]$ and the image $f^!(J) \subset I_*$. Moreover, the fibers of $f\leq$ can be expressed using the fibers of the transfer:

$$(f\leq)^{-1}(f \leq a) = \begin{cases} (f^!)^{-1}(-\infty) \cap J & \text{if } (f \leq a) = \emptyset, \\ (f^!)^{-1}(f^!(a)) & \text{if } (f \leq a) \neq \emptyset. \end{cases}$$

Thus, if $(f \leq a) \neq \emptyset$, then, according to Proposition 7.2.(5), the fiber $(f\leq)^{-1}(f \leq a) = (f^!)^{-1}(f^!(a))$ is a poset with a global minimum given by $f(\bigvee_I(f \leq a))$. In this case, for every $G\colon I \longrightarrow \mathcal{C}$, the left Kan extension $f^kG\colon J \longrightarrow \mathcal{C}$ is constant on subposets of J that are not only convex but also have global minima.

7.10. Assume I is a consistent upper semilattice of finite-type. According to Proposition 6.7, its realisation $\mathcal{R}(I)$ is also an upper semilattice. Choose an element d in I and a finite subset $V \subset (-1, 0)$. According to Corollary 6.8, the inclusion $\alpha\colon \mathcal{R}_{I\leq d}(I, V) \subset \mathcal{R}(I)$ is a sublattice (see 5.7). It consists of these pairs (a, f) for which $a \leq d$, and the non-zero values of f are in V.

We are going to describe the transfer $\alpha^!\colon \mathcal{R}(I)_* \longrightarrow \mathcal{R}_{I\leq d}(I, V)_*$. Let (a, f) be in $\mathcal{R}(I)$. By definition (see 7.1):

$$\alpha^!(a, f) = \begin{cases} -\infty & \text{if } \mathcal{R}_{I\leq d}(I, V) \leq (a, f) \text{ is empty,} \\ \bigvee(\mathcal{R}_{I\leq d}(I, V)) \leq (a, f)) & \text{otherwise.} \end{cases}$$

Thus, to describe $\alpha^!(a, f)$, the set $\mathcal{R}_{I\leq d}(I, V) \leq (a, f)$ needs to be discussed. We claim that it is non-empty if and only if d and $\mathrm{supp}(f)$ have a common ancestor. Here is a proof. If there is (b, g) in $\mathcal{R}_{I\leq d}(I, V)$ for which $(b, g) \leq (a, f)$, then, according to Proposition 4.4, any ancestor of $\mathrm{supp}(g)$ is also an ancestor of $\mathrm{supp}(f)$. The relation $b \leq d$ implies therefore that any common ancestor of b and $\mathrm{supp}(g)$ is also a common ancestor of d and $\mathrm{supp}(f)$.

Assume d and $\mathrm{supp}(f)$ have a common ancestor. If V is empty, set $v_0 := 0$, if V is non-empty, set v_0 to be the minimal element in V. Define $S := \{x \in \mathcal{P}(a) \mid f(x) < v_0\}$. Since S is a subset of $\mathrm{supp}(f)$, it has a common ancestor with d, and consequently the following definition of an element c in I makes sense as the necessary products exist due to I being an upper semilattice of finite-type:

$$c := \begin{cases} (\bigwedge S) \wedge d & \text{if } S \neq \emptyset, \\ a \wedge d & \text{if } S = \emptyset. \end{cases}$$

Any ancestor of $\{d, \mathrm{supp}(f), a\}$ is also an ancestor of c. Let y be a parent of c. Consider the subset $(y \leq \mathcal{P}(a)) \subset \mathcal{P}(a)$. There are two possibilities, either c is an ancestor of

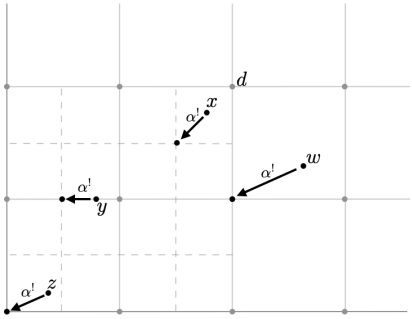


Figure 5: The intersections of the solid lines correspond to points in $\mathbb{N}^2$. The intersections of all lines correspond to points in $\mathcal{R}_{\mathbb{N}^2\leq(2,2)}(\mathbb{N}^2,\{-0.5\})$.

the set $y \leq \mathcal{P}(a)$ or not. In the second case, for every x in $(y \leq \mathcal{P}(a)) \setminus (c \leq I)$, two things happen: first, $f(x) \geq v_0$ (x does not belong to S) and, second, there is an equality $y = c \wedge x$. We use the first inequality to define a function $h\colon \mathcal{P}(c) \longrightarrow (-1, 0]$ by the following formula:

$$h(y) := \begin{cases} 0 & \text{if } c \text{ is an ancestor of } y \leq \mathcal{P}(a), \\ \max\{v \in V \mid v \leq f\left((y \leq \mathcal{P}(a)) \setminus (c \leq I)\right)\} & \text{otherwise.} \end{cases}$$

According to the above formula, if $h(y) < 0$, then there is x in $\mathcal{P}(a)$ for which $y = c \wedge x$ and $f(x) < 0$. Any common ancestor of c and $\mathrm{supp}(f)$ is therefore also an ancestor of y. As this happens for every y for which $h(y) < 0$, the support $\mathrm{supp}(h)$ has an ancestor, and hence (c, h) belongs to the realisation $\mathcal{R}(I)$. Since h has values in V, the element (c, h) belongs to $\mathcal{R}_{I\leq d}(I, V)$.

Consider the translation $T_{c\leq a}h$. Let x be a parent of a for which $f(x) < 0$. Consistency of I and the fact c and $\mathrm{supp}(f)$ have a common ancestor, imply that the product $c \wedge x$ exists and is either c or is a parent of c. If $c \leq x$, then $T_{c\leq a}h(x) = -1 \leq f(x)$, and, if $c \wedge x$ is a parent of c, then $T_{c\leq c}h(x) = h(c \wedge x) \leq f(x)$ by the formula defining h. These relations imply $(c, h) \leq (a, f)$, and hence the set $\mathcal{R}_{I\leq d}(I, V) \leq (a, f)$ is non-empty.

We are going to prove $(c, h) = \bigvee(\mathcal{R}_{I\leq d}(I, V)) \leq (a, f))$, which gives: $\alpha^!(a, f) = (c, h)$. Let (b, g) be in $\mathcal{R}_{I\leq d}(I, V) \leq (a, f)$. We need to show $(b, g) \leq (c, h)$. The relation $b \leq c$ holds since b is an ancestor of $\{d, \mathrm{supp}(f), a\}$. Consider $T_{b\leq c}g$. Let y be a parent of c for which $h(y) < 0$. Then there is x in $\mathcal{P}(a)$ such that $y = c \wedge x$ and $f(x) < 0$. By the assumption $T_{b\leq c}g(y) \leq f(x)$. Since the values of $T_{b\leq c}g$ are in V, then $T_{b\leq c}g(y) \leq h(y)$, which gives $(b, g) \leq (c, h)$.

Consider $V = \{-0.5\}$ and $d = (2, 2)$ in $\mathbb{N}^2$. Recall that the subposet inclusion $\alpha\colon \mathcal{R}_{\mathbb{N}^2\leq d}(\mathbb{N}^2, V) \hookrightarrow \mathcal{R}(\mathbb{N}^2)$ can be identified with $0.5[2]^2 \subset [0, \infty)^2$ (see 4.16). Figure 5 illustrates the effect of the transfer of α via this identification.

8. Tame functors

8.1. Definition. Let $\mathcal{C}$ be a category and J a poset. A functor $F\colon J \longrightarrow \mathcal{C}$ is called **tame** or **discretisable** if there is a finite poset I and functors $f\colon I \longrightarrow J$ and $G\colon I \longrightarrow \mathcal{C}$ for which F is isomorphic to the left Kan extension f^kG of G along f (see 2.11), in which case F is also said to be **discretised** by f, and G is called a **discretisation** of F.

The symbol $\mathrm{Tame}(J,\mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(J,\mathcal{C})$ with tame functors as objects.

If J is a finite poset, then any functor $F\colon J \longrightarrow \mathcal{C}$ is tame. The notion of tameness is therefore restrictive and meaningful only when J is infinite. To determine if $F\colon J \longrightarrow \mathcal{C}$ is tame we need to look for two things: a functor $f\colon I \longrightarrow J$ that discretises F and a discretisation $G\colon I \longrightarrow \mathcal{C}$ of F.

How can we search for a discretising f? In many situations it is enough to consider only finite subposet inclusions $I \subset J$. This is the case if for example $\mathcal{C}$ is closed under finite colimits (for instance the category of vector spaces over a given field). The closure of $\mathcal{C}$ under finite colimits guarantees the existence of the left Kan extension $f^k\colon \mathrm{Fun}(I,\mathcal{C}) \longrightarrow \mathrm{Fun}(J,\mathcal{C})$ for every functor $f\colon I \longrightarrow J$ with finite I. This has the following consequences:

8.2. Proposition. *Let $\mathcal{C}$ be a category closed under finite colimits and J a poset.*

1. *Consider the following commutative diagram of poset functors with I_0 and I_1 finite:*

$$I_0 \xrightarrow{g} I_1 \xrightarrow{f_1} J, \qquad I_0 \xrightarrow{f_0} J$$

   If $F\colon J \longrightarrow \mathcal{C}$ is discretised by f_0, then it is also discretised by f_1.

2. *If $F\colon J \longrightarrow \mathcal{C}$ is discretised by $f\colon I \longrightarrow J$, then it is also discretised by every finite subposet $I' \subset J$ for which $f(I) \subset I'$.*

3. *Let $F_1, \ldots, F_n\colon J \longrightarrow \mathcal{C}$ be a finite sequence of tame functors. Then there is a finite subposet $I \subset J$ that discretises F_i, for all i.*

4. *If $F\colon J \longrightarrow \mathcal{C}$ is discretised by a subposet inclusion $f\colon I \subset J$ with I finite, then F is f-left invertible (the morphism $f^k(Ff) \longrightarrow F$, adjoint to id_{Ff}, is an isomorphism, see 2.12).*

5. *Let L be a finite category and $\phi\colon L \longrightarrow \mathrm{Fun}(J,\mathcal{C})$ a functor. Assume $\phi(l)$ is tame for every object l in L. Then the functor $\mathrm{colim}_L\phi$ is also tame.*

Proof. (1) As long as they exist, Kan extensions commute with compositions: f_0^k and $f_1^k g^k$ are naturally isomorphic. Thus, if F is isomorphic to f_0^kG (is discretised by f_0), then it is also isomorphic to $f_1^k(g^kG)$ (is discretised by f_1).

(2) It is a direct consequence of (1).

(3) Let $I_i \subset J$ be a finite subposet discretising F_i. Then, according to (2), $\cup_{i=1}^{n} I_i \subset I$ discretises F_i, for every i.

(4) Consider $G\colon I \longrightarrow \mathcal{C}$ such that there is an isomorphism $\alpha\colon f^k G \longrightarrow F$. Since the category C is closed under finite colimits, it is f-right invertible, and therefore so is the functor G. Consequently, $f^k G$ is f-left invertible (see 2.12).

(5) Use (3) to choose a finite subposet inclusion $f\colon I \subset J$ that discretises $\phi(l)$ for every l in L. By (4), the natural transformation $f^k(\phi f) \longrightarrow \phi$, adjoint to the identity $\mathrm{id}\colon \phi f \longrightarrow \phi f$, is an isomorphism. Kan extensions commute with colimits and hence $\mathrm{colim}_L \phi$ is isomorphic to $f^k \mathrm{colim}_L(\phi f)$, proving its tameness.

According to Proposition 8.2, if $\mathcal{C}$ is closed under finite colimits, the search for discretising functors can be restricted to finite subposet inclusions $I \subset J$, in which case a discretisation is given by restricting to I (see 2.11). That is a considerable simplification. For example, a tame functor indexed by a realisation $\mathcal{R}(I)$ with values in a category closed under finite colimits can always be discretised by a subposet of the form $\mathcal{R}_D(I, V)$, for some finite $D \subset I$ and a finite $V \subset (-1, 0)$ (see 4.16). However not all the categories $\mathcal{C}$ for which we would like to have a simpler way of verifying tameness are closed under finite colimits. For example, the category of simplicial complexes fails to have this property. In such a case, Corollary 7.8 can be used where the only assumption on $\mathcal{C}$ is that it has an initial object e. It gives a characterisation of functors which are discretised by a homomorphism $f\colon I \longrightarrow J$ out of a finite upper semilattice I. According to this corollary, a functor $F\colon J \longrightarrow \mathcal{C}$ is discretised by f if and only if, for every a in J, if $(f \leq a) \neq \emptyset$, then $F(f(\bigvee_I (f \leq a)) \leq a)$ is an isomorphism, and if $(f \leq a) = \emptyset$, then $F(a)$ is isomorphic to e. For an arbitrary J, not all tame functors are discretised by homomorphisms. To make sure all tame functors are discretised by a homomorphism, J itself needs to be an upper semilattice. The rest of this section is devoted to discussing tameness under this assumption on J and when $\mathcal{C}$ is a category with an initial object e.

8.3. Proposition. *Let J be an upper semilattice and $\mathcal{C}$ a category with an initial object.*

1. *A functor $F\colon J \longrightarrow \mathcal{C}$ is tame if and only if it is discretised by a finite sublattice $I \subset J$ (see 5.7).*
2. *Let $I_0 \subset I_1 \subset J$ be finite sublattices of J. If $F\colon J \longrightarrow \mathcal{C}$ is discretised by $I_0 \subset J$, then it is also discretised by $I_1 \subset J$.*
3. *Let $F_1, \ldots, F_n\colon J \longrightarrow \mathcal{C}$ be a finite sequence of tame functors. Then there is a finite sublattice $I \subset J$ that discretises F_i, for all i.*

Proof. (1): Let $F\colon J \longrightarrow \mathcal{C}$ be tame. Choose a finite poset I' and functors $f\colon I' \longrightarrow J$ and $G\colon I' \longrightarrow \mathcal{C}$ for which there is an isomorphism $f^k G \longrightarrow F$. Let $\phi\colon G \longrightarrow Ff$ be the natural transformation adjoint to this isomorphism. Since $f(I') \subset J$ is finite, the sublattice generated by this image $I := \langle f(I') \rangle \subset J$ is also finite (see 5.7). Let us denote the inclusion $I \subset J$ by g and by $h\colon I' \longrightarrow I$ the functor that maps x to $f(x)$. Since g is a homomorphism of finite-type between upper semilattices, the left Kan extension

$g^k\colon \mathrm{Fun}(I,\mathcal{C}) \longrightarrow \mathrm{Fun}(J,\mathcal{C})$ exists (see Section 7). In particular, $g^k(Fg)$ exists and, hence, by Proposition 7.2.(4), the natural transformation $Fg \longrightarrow (g^k(Fg))g$ is an isomorphism, meaning that Fg is g-right invertible. By Propostion 2.13, F is g-left invertible, implying that F is discretised by the finite sublattice inclusion $g\colon I \subset J$.

(2): Exactly the same argument as in Proposition 8.2.(2) can be used to prove this statement since the left Kan extensions along $I_0 \subset I_1$, $I_0 \subset J$, and $I_1 \subset J$ exist by Corollary 7.5.

(3): For every i choose a finite sublattice $I_i \subset J$ discretising F_i. Define $I := \langle I_1 \cup \cdots \cup I_n \rangle \subset J$ (see 5.7). Since I is finite, according to (2), F_i is discretised by the inclusion $I \subset J$ for every i.

For example, let J be an upper semilattice of finite-type. Then for every finite subset $I \subset J$, there is b (for example $\bigvee_J I$) for which $I \subset (J \leq b) \subset J$. Consequently, according to Proposition 8.3, a functor $F\colon J \longrightarrow \mathcal{C}$ is tame if and only if it is discretised by the sublattice $(J \leq b) \subset J$, for some b in J. Thus, according to Corollary 7.8, F is tame if and only if, there is b in J such that, for every a in J, if $(J \leq b) \cap (J \leq a) \neq \emptyset$, then $F\left((\bigvee(J \leq b) \cap (J \leq a)) \leq a\right)$ is an isomorphism, and if $(J \leq b) \cap (J \leq a) = \emptyset$, then $F(a)$ is isomorphic to e.

If $\mathcal{C}$ is closed under taking finite colimits and limits, then so is the category of all functors $\mathrm{Fun}(J,\mathcal{C})$ for every poset J. In this case, the category $\mathrm{Tame}(J,\mathcal{C})$ is also closed under finite colimits (see Proposition 8.2.(4)). In general, we do not know if $\mathrm{Tame}(J,\mathcal{C})$ is closed under finite limits. A sufficient assumption to guarantee this is, again, that J is an upper semilattice.

8.4. Proposition. *Let J be an upper semilattice, $\mathcal{C}$ a category closed under finite limits, L a finite category, and $\phi\colon L \longrightarrow \mathrm{Fun}(J,\mathcal{C})$ a functor. Assume $\phi(l)$ is tame for every object l in L. Then the functor $\lim_L \phi$ is also tame.*

Proof. Let $f\colon I \subset J$ be a finite sublattice that discretises $\phi(l)$ for every object l in L (see Proposition 8.3.(3)). Then the natural transformation $f^k(\phi f) \longrightarrow \phi$, adjoint to the identity $\mathrm{id}\colon \phi f \longrightarrow \phi f$, is an isomorphism (see Corollary 7.8). Consequently, $\lim_L \phi$ is isomorphic to $\lim_L f^k(\phi f)$ and hence, by Proposition 7.7, it is also isomorphic to $f^k(\lim_L \phi f)$.

According to Propositions 8.2.(4) and 8.4, if J is an upper semilattice and $\mathcal{C}$ is closed under finite limits and colimits, then so is the category of tame functors $\mathrm{Tame}(J,\mathcal{C})$. Furthermore, the inclusion $\mathrm{Tame}(J,\mathcal{C}) \subset \mathrm{Fun}(J,\mathcal{C})$ preserves finite limits and colimits.

9. Homotopy theory of tame functors

It is well known that if I is a finite poset, $\mathrm{Fun}(I,\mathcal{M})$ inherits a model category structure from the model category $\mathcal{M}$ (see [DS95]). It is, however, unlikely that $\mathrm{Fun}(J,\mathcal{M})$ has a model category structure for an infinite poset J, for example, if $\mathcal{M}$ is the category of non-negative chain complexes over finite dimensional vector spaces. In this section, we show how a model category structure on $\mathrm{Fun}(I,\mathcal{M})$, with I finite, can be extended to a

model category structure on $\mathrm{Tame}(J, \mathcal{M})$, for any upper semilattice J. This extends the result of [CGL20], where J is the poset of non-negative real numbers. The assumption on J being an upper semilattice is crucial, otherwise $\mathrm{Tame}(J, \mathcal{M})$ may not even be closed under finite limits (see Proposition 8.4).

Let $\mathcal{M}$ be a model category as defined in [DS95]. This means three classes of morphisms in $\mathcal{M}$ are chosen: **weak equivalences** ($\xrightarrow{\sim}$), **fibrations** ($\twoheadrightarrow$), and **cofibrations** ($\hookrightarrow$), which are required to satisfy the following axioms:

MC1 Finite limits and colimits exist in $\mathcal{M}$.

MC2 If f and g are morphisms in $\mathcal{M}$ for which gf is defined and if two of the three morphisms f, g, gf are weak equivalences, then so is the third.

MC3 The three classes of morphisms are preserved by retracts.

MC4 Consider a commutative square in $\mathcal{M}$ consisting of the solid morphisms:

$$\begin{array}{ccc} X & \longrightarrow & E \\ {\scriptstyle\alpha}\downarrow & \nearrow & \downarrow{\scriptstyle\beta} \\ Y & \longrightarrow & B \end{array}$$

Then a morphism, depicted by the dotted arrow, making this diagram commutative, exists under either of the following two assumptions: (i) α is a cofibration and a weak equivalence and β is a fibration, or (ii) α is a cofibration and β is a fibration and a weak equivalence.

MC5 Every morphism in $\mathcal{M}$ can be factored in two ways: (i) $\beta\alpha$, where α is a cofibration and β is a fibration and a weak equivalence, and (ii) $\beta\alpha$, where α is a cofibration and a weak equivalence and β is a fibration.

A guiding example for a model category in this section is the one of non-negative chain complexes over finite dimensional vector space, $\mathrm{ch}(\mathrm{vect}_K)$, where weak equivalences are homology isomorphisms.

Axiom MC1 guarantees the existence of the initial object $\mathrm{colim}_\emptyset F$, denoted by e, and of the terminal object $\lim_\emptyset F$, denoted by $*$. An object X in $\mathcal{M}$ is called **cofibrant** if the morphism $e \rightarrowtail X$ is a cofibration. If the morphism $X \twoheadrightarrow *$ is a fibration, then X is called **fibrant**.

Assume I is a finite poset. Then the following choices of weak equivalences, fibrations, and cofibrations in $\mathrm{Fun}(I, \mathcal{M})$ gives a model structure (see for example [DS95, R74]). A natural transformation $\varphi : F \Longrightarrow G$ in $\mathrm{Fun}(I, \mathcal{M})$ is

- a weak equivalence (resp. a fibration) if, for all a in I, the morphism $\varphi_a : F(a) \Longrightarrow G(a)$ is a weak equivalence (resp. a fibration) in $\mathcal{M}$;

- a cofibration if, for all a in I, the morphism

$$\operatorname{colim}\left(\operatorname{colim}_{I<a} G \leftarrow \operatorname{colim}_{I<a} F \to F(a)\right) \longrightarrow G(a),$$

induced by the following commutative diagram, is a cofibration in $\mathcal{M}$:

$$\begin{array}{ccc} \operatorname{colim}_{I<a} F & \longrightarrow & F(a) \\ {\scriptstyle \operatorname{colim}_{I<a}\varphi}\downarrow & & \downarrow{\scriptstyle \varphi_a} \\ \operatorname{colim}_{I<a} G & \longrightarrow & G(a) \end{array}$$

The described model structures on functors indexed by finite posets are compatible in the following sense. Let $f\colon I_0 \longrightarrow I_1$ be a functor between finite posets. If $\phi\colon F \longrightarrow G$ is a cofibration (resp. a cofibration and a weak equivalence) in $\operatorname{Fun}(I_0, \mathcal{M})$, then so is $f^k\phi\colon f^k F \longrightarrow f^k G$ in $\operatorname{Fun}(I_1, \mathcal{M})$. This is a consequence of the universal property of left Kan extensions and the axiom (MC4). However, left Kan extensions in general fail to preserve weak equivalences and fibrations (compare with Proposition 9.4).

To verify if a natural transformation in $\operatorname{Fun}(I, \mathcal{M})$ is a cofibration, we need to perform colimits over subposets $(I < a) \subset I$. In general, these subposets can be large and constructing colimits over them may require performing a lot of identifications. When I is an upper semilattice however, we can be more efficient. For a in a finite upper semilattice I, define a sublattice:

$$I_a := \{\textstyle\bigwedge S \mid S \subset \mathcal{P}(a) \text{ has an ancestor}\} \subset (I < a).$$

For all b in $I < a$, the set $b \leq \mathcal{P}(a)$ is non-empty and $b \leq \bigwedge(b \leq \mathcal{P}(a))$. The element $\bigwedge(b \leq \mathcal{P}(a))$ is therefore the initial object in $b \leq I_a$ and consequently $b \leq I_a$ is contractible (see [BK72]). As this happens for all b in $I < a$, the inclusion $I_a \subset (I < a)$ is cofinal (see [M98, BK72]), which proves:

9.1. Proposition. *Let I be a finite upper semilattice and a its element. For every functor $F\colon (I < a) \longrightarrow \mathcal{M}$, the morphism $\operatorname{colim}_{I_a} F \longrightarrow \operatorname{colim}_{I<a} F$ is an isomorphism and $\operatorname{hocolim}_{I_a} F \longrightarrow \operatorname{hocolim}_{I<a} F$ is a weak equivalence.*

According to Proposition 9.1, when I is a finite upper semilattice, to verify if a natural transformation in $\operatorname{Fun}(I, \mathcal{M})$ is a cofibration, we need only to perform colimits over the subposets $I_a \subset I$ for a in I. How can such colimits be calculated? One way is to consider, for a in I, the subposet of the the discrete cube (see 2.1):

$$C_a := \{S \subset \mathcal{P}(a) \mid \mathcal{P}(a) \setminus S \text{ has an ancestor}\} \subset 2^{\mathcal{P}(a)}$$

and the functor $\bigwedge^c\colon C_a \longrightarrow I_a$ mapping S to $\bigwedge(\mathcal{P}(a) \setminus S)$. Since, for every b in I_a, the subset $\mathcal{P}(a) \setminus (b \leq \mathcal{P}(a)) \subset \mathcal{P}(a)$ is the initial object in the poset $b \leq \bigwedge^c$ (see 2.7), this poset is contractible. The functor $\bigwedge^c$ is therefore cofinal. Thus, for every $F\colon I_a \longrightarrow \mathcal{M}$, the morphism $\operatorname{colim}_{C_a}(F \bigwedge^c) \longrightarrow \operatorname{colim}_{I_a} F$ is an isomorphism and $\operatorname{hocolim}_{C_a}(F \bigwedge^c) \longrightarrow \operatorname{hocolim}_{I_a} F$ is a weak equivalence.

The assumption on the indexing poset being an upper semilattice is not only helpful in verifying if a natural tranformation is a cofibration. It is also crucial in proving that the following choices of weak equivalences, fibrations, and cofibrations in $\mathrm{Tame}(J,\mathcal{M})$ satisfy the axioms of a model structure.

9.2. Definition. Let J be an upper semilattice and $\mathcal{M}$ a model category. A natural transformation $\phi\colon F \longrightarrow G$ in $\mathrm{Tame}(J,\mathcal{M})$ is called:

- a weak equivalence (resp. fibration) if, for all a in J, the morphism $\phi_a\colon F(a)\longrightarrow G(a)$ is a weak equivalence (resp. fibration) in $\mathcal{M}$;
- a cofibration if there is a finite subposet inclusion $f\colon I \subset J$ discretising F and G, and for which $\phi^f\colon Ff \longrightarrow Gf$ is a cofibration in $\mathrm{Fun}(I,\mathcal{M})$.

9.3. Theorem. *Let J be an upper semilattice and $\mathcal{M}$ a model category. The choices described in Definition 9.2 satisfy the axioms of a model structure on* $\mathrm{Tame}(J,\mathcal{M})$.

Before we prove Theorem 9.3, we first show:

9.4. Proposition. *Let $\mathcal{M}$ be a model category and $f\colon I\longrightarrow J$ a homomorphism of finite-type between upper semilattices. If a natural transformation $\phi\colon F\longrightarrow G$ in* $\mathrm{Fun}(I,\mathcal{M})$ *is a weak equivalence (resp. fibration), then so is its left Kan extension $f^k\phi\colon f^kF\longrightarrow f^kG$ in* $\mathrm{Fun}(J,\mathcal{M})$.

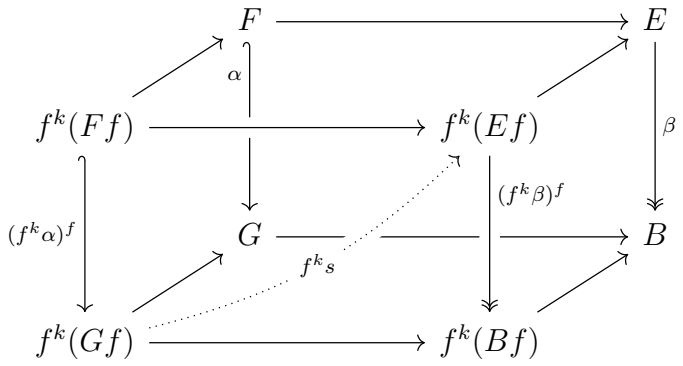
Proof. Let $\phi\colon F\longrightarrow G$ be a weak equivalence (resp. fibration). Since f is a finite-type homomorphism between upper semilattices, the morphism $(f^k\phi)_a\colon f^kF(a)\longrightarrow f^kG(a)$ is isomorphic to either $\mathrm{id}\colon e\longrightarrow e$, if $(f\leq a)=\emptyset$, or to $\phi_{\bigvee_I(f\leq a)}\colon F(\bigvee_I(f\leq a))\longrightarrow G(\bigvee_I(f\leq a))$, if $(f\leq a)\neq\emptyset$ (see 7.6). As these morphisms are weak equivalences (resp. fibrations), for all a, then so is the left Kan extension $f^k\phi$.

Proof. [Proof of Theorem 9.3] Requirement MC1 follows from Proposition 8.4. MC2 and MC3 are clear since $\mathcal{M}$ satisfies them.

MC4: Let $\alpha\colon F\hookrightarrow G$ be a cofibration in $\mathrm{Tame}(J,\mathcal{M})$. Choose a finite sublattice $I\subset J$ and a functor $f: I\longrightarrow J$ that discretises both F and G and for which $\alpha^f\colon Ff\longrightarrow Gf$ is a cofibration in $\mathrm{Fun}(I,\mathcal{M})$. Assume α is part of a commutative square in $\mathrm{Tame}(J,\mathcal{M})$ depicted in (1) on the left, where β is a fibration. By applying $(-)^f$ to this square we get a commutative square in $\mathrm{Fun}(I,\mathcal{M})$, depicted by the solid arrows square on the right in (1), where the natural transformations are a cofibration and a fibration in $\mathrm{Fun}(I,\mathcal{M})$. If, in addition, α or β is a weak equivalence, then the lift, depicted by the dotted arrow on the right of (1), exists since $\mathrm{Fun}(I,\mathcal{M})$ is a model category.

$$\begin{array}{ccc} F & \longrightarrow & E \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ G & \longrightarrow & B \end{array} \qquad\qquad \begin{array}{ccc} Ff & \longrightarrow & Ef \\ {\scriptstyle\alpha^f}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle\beta^f} \\ Gf & \longrightarrow & Bf \end{array} \tag{1}$$

By applying the left Kan extension f^k to the square on the right of (1) and comparing the result to the original square on the left, we can form the following commutative diagram in $\mathrm{Tame}(J, \mathcal{M})$:



Since the natural transformation $f^k(Gf) \longrightarrow G$ is an isomorphism, a desired lift exists in the right square of (1) under the additional assumption that either α or β is a weak equivalence.

MC5: Let $\phi\colon F \longrightarrow G$ be a natural transformation in $\mathrm{Tame}(J, \mathcal{M})$. Choose a finite sublattice $f\colon I \subset J$ that discretises both F and G. Factor the natural transformation $\phi^f\colon Ff \longrightarrow Gf$ in $\mathrm{Fun}(I, \mathcal{M})$ as $\phi^f = \beta\alpha$ where $\alpha\colon Ff \longrightarrow H$ is a cofibration, $\beta\colon H \longrightarrow Gf$ is a fibration, and either α or β is a weak equivalence. By applying the left Kan extension f^k to these factorisations we obtain a commutative diagram in $\mathrm{Tame}(J, \mathcal{M})$ where the vertical natural transformations are isomorphisms:

$$\begin{array}{ccccc} f^k(Ff) & \xrightarrow{f^k\alpha} & f^kH & \xrightarrow{f^k\beta} & f^k(Gf) \\ \downarrow & & & & \downarrow \\ F & & \xrightarrow{\quad\phi\quad} & & G \end{array}$$

Since $(f^k\alpha)^f$ is isomorphic to α, it is a cofibration in $\mathrm{Fun}(I, \mathcal{M})$ and consequently $f^k\alpha$ is a cofibration in $\mathrm{Tame}(J, \mathcal{M})$. According to Proposition 9.4, $f^k\beta$ is a fibration in $\mathrm{Tame}(J, \mathcal{M})$. The same proposition assures also that if α (resp. β) is a weak equivalence, then so is $f^k\alpha$ (resp. $f^k\beta$). This gives the desired factorisation of ϕ.

10. Betti diagrams of vector spaces valued functors.

In this section, we show that upper semilattices and realisations share similar homological algebra properties. For example, they both allow to reduce the computation of Betti diagrams in the minimal resolution of a tame functor to convenient finite subsets, and this can be done using Koszul complexes. As a consequence, we show that, in both cases, the length of a minimal resolution of a functor is bounded above by the parental-dimension of elements of a subposet of the indexing poset, providing a Hilbert syzygies-type theorem.

10.1. *[Freeness] Let $(J,\leq)$ be a poset and $vect_K$ the category of finite dimensional K-vector spaces. Consider the poset $(J,=)$ with the trivial poset relation on the set J, where two elements are related if and only if they are equal. The identity function $\iota\colon (J,=)\longrightarrow J$, mapping a to $\iota(a)=a$, is a functor. A functor $V\colon (J,=)\longrightarrow vect_K$ is just a sequence $\{V_a\}_{a\in J}$ of finite dimensional K-vector spaces. The set $\{a\in J \mid V_a\neq 0\}$ is called the support of V and is denoted by $supp(V)$.*

A functor $F\colon J\longrightarrow vect_K$ is called ***free finitely generated*** *if it is isomorphic to the left Kan extension along $\iota\colon (J,=)\longrightarrow J$ of some $V=\{V_a\}_{a\in J}$ whose support is finite. Since all free functors considered in this article are finitely generated, we use the term* ***free****, without mentioning finite generation, to describe such functors. The name free is justified by the universal property of the left Kan extension, which gives a linear isomorphism between $Nat_J(\iota^k V,H)$ and $\prod_{a\in J} Hom(V_a,H(a))$, for every functor $H\colon J\longrightarrow vect_K$. For example, for b in J and a finite dimensional vector space U, consider the simple functor $U[b]\colon J\longrightarrow vect_K$ defined as*

$$U[b](a):=\begin{cases}0 & \text{if } a\neq b,\\ U & \text{if } a=b.\end{cases}$$

Then the vector spaces $Nat_J(\iota^k V,U[b])$ and $Hom(V_b,U)$ are isomorphic. By varing b in J and taking U to be K, we can conclude that free functors $\iota^k V$ and $\iota^k W$ are isomorphic if and only if V_a and W_a are isomorphic for all a in J. Thus, a free functor F determines a unique sequence $\beta F=\{(\beta F)_a\}_{a\in J}$ of vector spaces whose support is finite, called the ***Betti diagram*** *of F, for which F is isomorphic to the left Kan extension $\iota^k(\beta F)$. If $supp(\beta F)=\{a\}$, then F is called* ***homogeneous*** *and is also denoted by the symbol $F(a)[a,-)$. The restriction of $F(a)[a,-)$ to the subposet $(a\leq J)\subset J$ is isomorphic to the constant functor with value $F(a)$. Its restriction to $\{x\in J \mid a\not\leq x\}$ is isomorphic to the constant functor with value 0. If F is free, then it is isomorphic to the direct sum $\bigoplus_{a\in J}(\beta F)_a[a,-)$ and, consequently, F is isomorphic to the left Kan extension of $\{(\beta F)_a\}_{a\in supp(\beta F)}$ along $(supp(\beta F),=)\hookrightarrow J$. Thus, every free functor F is tame and is discretised by the subposet inclusion $supp(\beta F)\subset J$. Moreover, every collection $V=\{V_a\}_{a\in J}$ with finite support is the Betti diagram of a free functor.*

If $G\colon I\longrightarrow vect_K$ is free, then, for every functor $f\colon I\longrightarrow J$, the left Kan extension $f^kG\colon J\longrightarrow vect_K$ is also free and $(\beta f^kG)_a=0$ if $f^{-1}(a)$ is empty, and $(\beta f^kG)_a$ is isomorphic to $\bigoplus_{b\in f^{-1}(a)}(\beta G)_b$, if $f^{-1}(a)$ is non-empty.

10.2. *[*Resolutions] Let J be a poset and $F\colon J\longrightarrow \mathrm{vect}_K$ a functor. An exact sequence $P_{n-1}\longrightarrow\cdots\longrightarrow P_0\longrightarrow F\longrightarrow 0$ in $\mathrm{Fun}(J,\mathrm{vect}_K)$, with P_i free for every i, is called an **n-resolution** of F. A 1-resolution $P_0\longrightarrow F\longrightarrow 0$ is also called a **cover** of F. A 2-resolution $P_1\longrightarrow P_0\longrightarrow F\longrightarrow 0$ is also called a **presentation** of F. An infinite exact sequence $\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow F\longrightarrow 0$, with P_i free for every i, is called an **∞-resolution** of F. An n-resolution of F is also denoted, as a map of chain complexes, by $P\longrightarrow F$, where F is a chain complex concentrated in degree 0, and either $P=(P_{n-1}\longrightarrow\cdots\longrightarrow P_0)$ or $P=(\cdots\longrightarrow P_1\longrightarrow P_0)$, depending if n is finite.

A functor $F\colon J \longrightarrow \mathrm{vect}_K$ is called **n-resolvable** if it has an n-resolution. Functors that are 1-resolvable are also called **finitely generated**. Functors that are 2-resolvable are also called **finitely presented**. If the indexing poset J is finite, then all functors are ∞-resolvable. This is because all such functors are finitely generated, and hence by taking covers of successive kernels, an ∞-resolution can be constructed.

If J is infinite, then not all n-resolvable functors have to be $(n+1)$-resolvable. For example, let $J = [0,2) \coprod \{a,b\} \coprod (2,3]$, with a and b incomparable and $x < a < y$ and $x < b < y$, for x in $[0,2)$ and y in $(2,3]$. Consider the subposet $I = \{1,a,b\} \subset J$ and a functor $G\colon I \longrightarrow \mathrm{vect}_K$ where both $G(1<a)$ and $G(1<b)$ are given by $K \longrightarrow 0$. Let $F\colon J \longrightarrow \mathrm{vect}_K$ be the left Kan extension of G along $I \subset J$. Then F is finitely presented (2-resolvable), however it is not 3-resolvable.

Here is a characterisation of finitely generated and finitely presented functors.

10.3. Proposition. *Let J be a poset and $F\colon J \longrightarrow \mathrm{vect}_K$ a functor.*

1. *F is finitely generated if and only if there is a finite poset I and a functor $f\colon I \longrightarrow J$ for which the natural transformation $\mu\colon f^k(Ff) \longrightarrow F$, adjoint to the identity $\mathrm{id}\colon Ff \longrightarrow Ff$, is surjective.*

2. *F is finitely presented if and only if it is tame, i.e., if and only if there is a finite poset I and a functor $f\colon I \longrightarrow J$ such that F is f-left invertible (for which the natural transformation $\mu\colon f^k(Ff) \longrightarrow F$, adjoint to the identity $\mathrm{id}\colon Ff \longrightarrow Ff$, is an isomorphism).*

Proof. (1): Let $\pi\colon P_0 \longrightarrow F$ be a cover. Since P_0 is tame, there is a functor $f\colon I \longrightarrow J$, from a finite I, discretising P_0 (for example $I = \mathrm{supp}(\beta P_0) \subset J$). By Proposition 8.2.(4), the morphism $f^k(P_0 f) \longrightarrow P_0$, adjoint to the identity, is an isomorphism. Thus, the commutativity of the following square and the fact that left Kan extensions preserve surjections imply the surjectivity of μ:

$$\begin{array}{ccc} f^k(P_0 f) & \longrightarrow & P_0 \\ {\scriptstyle f^k(\pi f)}\downarrow & & \downarrow{\scriptstyle \pi} \\ f^k(Ff) & \xrightarrow{\mu} & F \end{array}$$

Let I be a finite poset, $f\colon I \longrightarrow J$ a functor for which $\mu\colon f^k(Ff) \longrightarrow F$ is surjective, and $\pi\colon P_0 \longrightarrow Ff$ a cover. The surjectivity of μ implies the surjectivity of the following composition, which is then a cover of F:

$$f^k(P_0) \xrightarrow{f^k\pi} f^k(Ff) \xrightarrow{\mu} F$$

(2): If $P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$ is a 2-resolution, then $\mathrm{colim}(0 \leftarrow P_1 \longrightarrow P_0)$ is isomorphic to F. Since tameness is preserved by finite colimits (see Proposition 8.2.(4)), F is tame.

Assume F is discretised by a subposet inclusion $f\colon I \subset J$ with finite I. Consider a 2-resolution $P_1 \longrightarrow P_0 \longrightarrow Ff \longrightarrow 0$, which exists since I is finite. As before, $f^k(Ff)$, and hence F, is isomorphic to $\operatorname{colim}(0 \leftarrow f^kP_1 \longrightarrow f^kP_0)$. Since f^kP_1 and f^kP_0 are free, F is finitely presented.

Proposition 10.3 characterises n-resolvable functors for $n \leq 2$. We do not have a similar characterisation for $n > 2$, only a partial result:

10.4. Proposition. *Let J be a poset. Assume $F\colon J \longrightarrow \mathrm{vect}_K$ is discretised by a functor $f\colon I \longrightarrow J$ from a finite poset I and for which $f^k\colon \mathrm{Fun}(I, \mathrm{vect}_K) \longrightarrow \mathrm{Fun}(J, \mathrm{vect}_K)$ is exact. Then F is ∞-resolvable.*

Proof. Let $G\colon I \longrightarrow \mathrm{vect}_K$ be such that f^kG is isomorphic to F. Choose an ∞-resolution $\pi\colon P \longrightarrow G$. Exactness of f^k means that the left Kan extension $f^k\pi\colon f^kP \longrightarrow f^kG$ is an ∞-resolution f^kG.

Here is a condition guaranteeing exactness of the left Kan extension:

10.5. Lemma. *Assume $f\colon I \longrightarrow J$ is a functor of posets from a finite poset I satisfying the following property: for every a in J, every pair of elements in $f \leq a$ that have an ancestor also has a descendent in $f \leq a$ (such posets are called weakly directed in [BFVZ21]). Then $f^k\colon \mathrm{Fun}(I, \mathrm{vect}_K) \longrightarrow \mathrm{Fun}(J, \mathrm{vect}_K)$ is exact.*

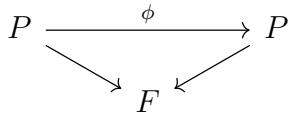
Proof. We show $\operatorname{colim}_{f\leq a}\colon \mathrm{Fun}(f \leq a, \mathrm{vect}_K) \longrightarrow \mathrm{vect}_K$ is exact for all a in J. Let $M \subset (f \leq a)$ consists of all the maximal elements. The assumption on f implies $M \subset (f \leq a)$ is cofinal and hence $\operatorname{colim}_{f\leq a}F$ is isomorphic to $\bigoplus_{x\in M} F(x)$. The lemma follows from the exactness of direct sums.

10.6. Corollary. *Let J be a poset.*

1. *Every functor $F\colon J \longrightarrow \mathrm{vect}_K$ discretised by a homomorphism $f\colon I \longrightarrow J$ from a finite upper semilattice I is ∞-resolvable.*
2. *If J is an upper semilattice, then a functor $F\colon J \longrightarrow \mathrm{vect}_K$ is tame if and only if it is ∞-resolvable.*

Proof. In both (1) and (2) the poset $f \leq a$, for all a in J, has a terminal object given by its coproduct in I. Thus, by Lemma 10.5, the conclusion of Proposition 10.4 holds.

10.7. *[Minimality and Betti diagrams] An n-resolution $\pi\colon P \longrightarrow F$ is called* ***minimal*** *if every chain map $\phi\colon P \longrightarrow P$, for which the following triangle commutes, is an isomorphism:*



If $P \longrightarrow F$ and $Q \longrightarrow F$ are minimal n-resolutions of F, then P and Q are isomorphic. Thus, if $P \longrightarrow F$ is a minimal n-resolution, then the isomorphism type of P_i, for $i < n$, is uniquely determined by the isomorphism type of F, and its Betti diagram βP_i is called

the i-th ***Betti diagram*** *of F, and is denoted by $\beta^i F$. For example if F is free, then $id\colon F \longrightarrow F$ is a minimal ∞-resolution of F and, hence, for every a in J, the vector space $(\beta^0 F)_a$ is isomorphic to $(\beta F)_a$, and $(\beta^i F)_a = 0$ for $i > 0$.*

A minimal 1-resolution $P_0 \longrightarrow F$ is also called a ***minimal cover*** *of F. An n-resolution $P \longrightarrow F$ is minimal if and only if, for every $i < n$, the following natural transformations are minimal covers: $P_0 \longrightarrow F$, $P_1 \longrightarrow ker(P_0 \longrightarrow F)$, $\ldots$, $P_i \longrightarrow ker(P_{i-1} \longrightarrow P_{i-2})$. Thus, if $P_0 \longrightarrow F$ is a minimal cover of an n-resolvable functor, then, for $1 \leq i < n$, $\beta^i F$ is isomorphic to $\beta^{i-1} ker(P_0 \longrightarrow F)$. Also, if $P \longrightarrow F$ is a minimal n-resolution, then $\beta^0 F$ is isomorphic to βP_0, $\beta^1 F$ is isomorphic to $\beta^0 ker(P_0 \longrightarrow F)$, and for $i > 1$, $\beta^i F$ is isomorphic to $\beta^0 ker(P_{i-1} \longrightarrow P_{i-2})$.*

Our strategy for constructing minimal resolutions of n-resolvable functors indexed by an arbitrary poset is to reduce this problem to the case when the indexing poset is finite.

10.8. Proposition. Let J be a poset and $F\colon J \longrightarrow \mathrm{vect}_K$ a functor. Assume $P \longrightarrow F$ is a n-resolution which is discretised by a subposet inclusion $f\colon I \subset J$, with I finite. If $Q \longrightarrow Ff$ is a minimal n-resolution of Ff, then its adjoint $f^k Q \longrightarrow F$ is a minimal resolution of F and, for $i < n$ and a in J,

$$(\beta^i F)_a \text{ is isomorphic to } \begin{cases} (\beta^i(Ff))_a = (\beta Q_i)_a & \text{if } a \in I, \\ 0 & \text{if } a \notin I. \end{cases}$$

Proof. *By restricting the n-resolution $P \longrightarrow F$ of F along f, we obtain a resolution $Pf \longrightarrow Ff$ of Ff. Let $Q \longrightarrow Ff$ be a minimal n-resolution. Then there are chain maps $\phi\colon Q \longrightarrow Pf$ and $\psi\colon Pf \longrightarrow Q$ for which the composition $\psi\phi\colon Q \longrightarrow Q$ is an isomorphism and the diagram on the left commutes:*

$$\begin{array}{ccccc} Q & \xrightarrow{\phi} & Pf & \xrightarrow{\psi} & Q \\ & \searrow & \downarrow & \swarrow & \\ & & Ff & & \end{array} \qquad \begin{array}{ccccc} f^k Q & \xrightarrow{f^k\phi} & f^k(Pf) & \xrightarrow{f^k\psi} & f^k Q \\ & \searrow & \downarrow & \swarrow & \\ & & F & & \end{array}$$

By taking the adjoints of the vertical maps in the left diagram, we obtain a commutative diagram on the right. Since $P = f^k(Pf) \longrightarrow F$ is an n-resolution, then so is its retract $f^k Q \longrightarrow F$. Its minimality follows from the minimality of $Q \longrightarrow Ff$ and the fact that $f^k\colon Nat_I(Q_i, Q_i) \longrightarrow Nat_J(f^k Q_i, f^k Q_i)$ is a bijection for every i (see 2.12).

Based on Proposition 10.8, the first step to construct a minimal n-resolution of a functor $F\colon J \longrightarrow \mathrm{vect}_K$ is to find a finite subposet inclusion $f\colon I \subset J$ for which there is an n-resolution $P \longrightarrow F$ with P being discretised by f (the natural transformation $f^k(Pf) \longrightarrow P$ is an isomorphism). For $n = 1$, according to the proof of Proposition 10.3.(1), such a subposet inclusion is given by any $f\colon I \subset J$ for which $f^k(Ff) \longrightarrow F$ is a surjection. For $n = 2$, according to the proof of Proposition 10.3.(2), such a subposet inclusion is given by any $f\colon I \subset J$ that discretises F. We do not have a similar statement for $n > 2$.

The second step is to construct a minimal n-resolution $Q \longrightarrow Ff$ of the restriction Ff. The adjoint of this minimal resolution $f^k Q \longrightarrow F$ is then the desired minimal n-resolution

of F. This process reduces finding a minimal n-resolution of F to finding a minimal n-resolution of Ff which is a functor indexed by a finite poset. Constructing minimal resolutions of functors indexed by finite posets is standard and involves radicals (see for example [ARS97, CE99]).

10.9. [Radical] Let I be a finite poset. The **radical** of $G\colon I \longrightarrow \mathrm{vect}_K$ is a subfunctor $\mathrm{rad}(G) \subset G$ given by $\mathrm{rad}(G)(a) = \mathrm{im}\left(\bigoplus_{s<a} G(s) \longrightarrow G(a)\right)$ for a in I. The quotient functor $G/\mathrm{rad}(G)$ is semisimple as it is isomorphic to a direct sum $\bigoplus_{a\in I} U_a[a]$ of simple functors (see 10.1), where $U_a := (G/\mathrm{rad}(G))(a)$. For example, for a free functor $G = \bigoplus_{a\in I}(\beta G)_a[a,-)$, the quotient $G/\mathrm{rad}(G)$ is isomorphic to $\bigoplus_{a\in I}(\beta G)_a[a]$.

A key property of quotienting by the radical, when the indexing poset is finite, is the surjectivity detection: a natural transformation $H \longrightarrow G$ is surjective if and only if its composition with the quotient $G \longrightarrow G/\mathrm{rad}(G)$ is surjective. The surjectivity detection can be used to construct minimal covers. Consider the quotient $G/\mathrm{rad}(G)$ of $G\colon I \longrightarrow \mathrm{vect}_K$. Set $P_0 := \bigoplus_{a\in I} U_a[a,-)$, where $U_a = (G/\mathrm{rad}(G))(a)$. Note that there is an isomorphism $P_0/\mathrm{rad}(P_0) \longrightarrow G/\mathrm{rad}(G)$. Let $\pi\colon P_0 \longrightarrow G$ be any natural transformation fitting into the following commutative square, where the bottom horizontal arrow represents the chosen isomorphism:

$$\begin{array}{ccc} P_0 & \xrightarrow{\quad\pi\quad} & G \\ \downarrow & & \downarrow \\ P_0/\mathrm{rad}(P_0) & \longrightarrow & G/\mathrm{rad}(G) \end{array}$$

Such π exists since P_0 is free. The composition $P_0 \longrightarrow G/\mathrm{rad}(G)$ is surjective, and hence so is π. The same argument gives the surjectivity of every $\phi\colon P_0 \longrightarrow P_0$ for which $\pi\phi = \pi$. Since the values of P_0 are finite dimensional, every such ϕ is therefore an isomorphism, and hence $\pi\colon P_0 \longrightarrow G$ is a minimal cover. This shows:

10.10. Proposition. *Let I be a finite poset and $G\colon I \longrightarrow \mathrm{vect}_K$ a functor. A natural transformation $P_0 \longrightarrow G$ is a minimal cover if and only if P_0 is free and the induced natural transformation $P_0/\mathrm{rad}(P_0) \longrightarrow G/\mathrm{rad}(G)$ is an isomorphism. Furthermore, $(\beta^0 G)_a$ is isomorphic to $(G/\mathrm{rad}(G))(a)$ for all a.*

Since all the functors indexed by a finite poset I are finitely generated, by taking minimal covers of successive kernels, every functor indexed by I admits a minimal ∞-resolution. This inductive step-wise construction of a minimal ∞-resolution can be used for a step-wise inductive procedure of calculating the Betti diagrams. However, can the Betti diagrams be retrieved in one step without the need of an inductive procedure? The answer to this question is: yes, it can standardly be done by using Koszul complexes.

10.11. Let I be a finite poset and a its element. Choose a linear ordering $\prec$ on the set of parents $\mathcal{P}(a)$ of a. For every functor $G\colon I \longrightarrow \mathrm{vect}_K$, we define a non-negative chain complex denoted by $\mathcal{K}_a G$ and called the **Koszul complex** of G at a. Let k be a natural

number. Define:

$$(\mathcal{K}_a G)_k := \begin{cases} G(a) & \text{if } k = 0, \\ \bigoplus\limits_{\substack{S\subset\mathcal{P}(a),\ |S|=k \\ S \text{ has an ancestor}}} \operatorname{colim}_{\cap_{s\in S}(I\leq s)} G & \text{if } k > 0. \end{cases}$$

For example, $(\mathcal{K}_a G)_1 = \bigoplus_{s\in\mathcal{P}(a)} G(s)$ and $(\mathcal{K}_a G)_k = 0$ if $k > \text{par-dim}_I(a)$ (see Proposition 3.5).

Define $\partial\colon (\mathcal{K}_a G)_{k+1} \longrightarrow (\mathcal{K}_a G)_k$ as follows:

- If $k = 0$, then $\partial\colon (\mathcal{K}_a G)_1 \longrightarrow (\mathcal{K}_a G)_0 = G(a)$ is the linear function which on the summand $G(s)$ in $(\mathcal{K}_a G)_1$, indexed by s in $\mathcal{P}(a)$, is given by $G(s < a)$.
- Let $k > 0$. For $k \geq j \geq 0$, let $\partial_j\colon (\mathcal{K}_a G)_{k+1} \longrightarrow (\mathcal{K}_a G)_k$ be the linear function mapping the summand $\operatorname{colim}_{\cap_{s\in S}(I\leq s)} G$ in $(\mathcal{K}_a G)_{k+1}$, indexed by $S = \{s_0 \prec \cdots \prec s_k\} \subset \mathcal{P}(a)$, to the summand $\operatorname{colim}_{\cap_{s\in S\setminus\{s_j\}}(I\leq s)} G$ in $(\mathcal{K}_a G)_k$, indexed by $S \setminus \{s_j\} \subset \mathcal{P}(a)$, via the function of the colimits $\operatorname{colim}_{\cap_{s\in S}(I\leq s)} G \longrightarrow \operatorname{colim}_{\cap_{s\in S\setminus\{s_j\}}(I\leq s)} G$ induced by the poset inclusion $\cap_{s\in S}(I \leq s) \subset \cap_{s\in S\setminus\{s_j\}}(I \leq s)$. Define $\partial\colon (\mathcal{K}_a G)_{k+1} \longrightarrow (\mathcal{K}_a G)_k$ to be the alternating sum $\partial = \sum_{j=0}^{k} (-1)^j \partial_j$.

The linear functions ∂ form a chain complex as it is standard to verify that composition of two consecutive such functions is the 0 function.

For a natural transformation $\phi\colon F \longrightarrow G$, define:

$$(\mathcal{K}_a F)_k \xrightarrow{(\mathcal{K}_a\phi)_k} (\mathcal{K}_a G)_k := \begin{cases} \phi_a & \text{if } k = 0, \\ \bigoplus\limits_{\substack{S\subset\mathcal{P}(a),\ |S|=k \\ S \text{ has an ancestor}}} \operatorname{colim}_{\cap_{s\in S}(I\leq s)} \phi & \text{if } k > 0. \end{cases}$$

These linear functions, for all k, form the chain map $\mathcal{K}_a\phi\colon \mathcal{K}_a F \longrightarrow \mathcal{K}_a G$. The association $\phi \mapsto \mathcal{K}_a\phi$ is a functor.

We observe that the image of the differential $\partial\colon (\mathcal{K}_a G)_1 \longrightarrow (\mathcal{K}_a G)_1 = G(a)$ coincides with $\operatorname{rad}(G)(a)$. Consequently, the vector spaces $H_0(\mathcal{K}_a G)$ and $(G/\operatorname{rad}(G))(a)$ are isomorphic, and, according to Proposition 10.10, they are also isomorphic to $(\beta^0 G)_a$.

10.12. Consider an element a in a finite poset I satifying: every $S \subset \mathcal{P}(a)$ having an ancestor also has the product $\bigwedge S$ in I. For example, if I is an upper semilattice, then all its elements satisfy this property. Under this assumption, for every subset $S \subset \mathcal{P}(a)$ that has an ancestor, the product $\bigwedge S$ is the terminal object in the category $\cap_{s\in S}(I \leq s)$ and consequently, for $k > 0$,

$$(\mathcal{K}_a G)_k = \bigoplus_{\substack{S\subset\mathcal{P}(a),\ |S|=k \\ S \text{ has an ancestor}}} G(\bigwedge S).$$

10.13. Let I be a finite poset and a its element. Since colimits commute with direct sums, so does the functor $\mathcal{K}_a$, i.e., the natural transformation $\mathcal{K}_aF \oplus \mathcal{K}_aG \longrightarrow \mathcal{K}_a(F \oplus G)$ is an isomorphism.

As the colimit operation is right exact, so is our Koszul complex construction: if $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ is an exact sequence in $\mathrm{Fun}(I, \mathrm{vect}_K)$, then $\mathcal{K}_aF \longrightarrow \mathcal{K}_aG \longrightarrow \mathcal{K}_aH \longrightarrow 0$ is an exact sequence of chain complexes. In general colimits do not preserve monomorphisms, and hence one does not expect the Koszul complex construction to preserve monomorphisms in general either. Let n be an extended positive natural number (containing ∞). An element a in I is called **Koszul n-exact** if, for every exact sequence of functors $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ and $k < n$, the following sequence of vector spaces is exact:

$$0 \longrightarrow (\mathcal{K}_aF)_k \longrightarrow (\mathcal{K}_aG)_k \longrightarrow (\mathcal{K}_aH)_k \longrightarrow 0$$

For example all elements in I turn out to be Koszul 2-exact.

10.14. Proposition. *Let I be a finite poset.*

1. *Then every element a in I is Koszul 2-exact.*
2. *Assume an element a in I has the following property: every subset $S \subset \mathcal{P}(a)$ which has an ancestor has the product $\bigwedge S$ in I. Then a is Koszul ∞-exact.*
3. *If I is an upper semilattice, then all its elements are Koszul ∞-exact.*

Proof. Statement (1) is a consequence of the fact that direct sums preserves exactness. Under the assumption of statement (2), for a functor F and $k > 0$, the vector space $(\mathcal{K}_aF)_k$ can also be described as (see 10.12):

$$\bigoplus_{\substack{S \subset \mathcal{P}(a),\ |S|=k \\ S \text{ has an ancestor}}} F(\bigwedge S).$$

In this case, the statement also follows from the exactness of direct sums. Finally statement (3) is a particular case of (2).

Proposition 10.14 translates into the homology exact sequence:

10.15. Corollary. *Let I be a finite poset and $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ be an exact sequence in $\mathrm{Fun}(I, \mathrm{vect}_K)$.*

1. *For every element a in I, there is an exact sequence of vector spaces:*

$$\begin{array}{ccccccc} & & & & H_2(\mathcal{K}_aG) & \longrightarrow & H_2(\mathcal{K}_aH) \\ \hookrightarrow & H_1(\mathcal{K}_aF) & \longrightarrow & H_1(\mathcal{K}_aG) & \longrightarrow & H_1(\mathcal{K}_aH) & \\ \hookrightarrow & H_0(\mathcal{K}_aF) & \longrightarrow & H_0(\mathcal{K}_aG) & \longrightarrow & H_0(\mathcal{K}_aH) & \longrightarrow 0 \end{array}$$

2. *Assume an element a in I has the following property: every subset $S \subset \mathcal{P}(a)$ which has an ancestor has the product $\bigwedge S$ in I. Then there is an exact sequence of vector spaces:*

$$\begin{array}{ccccccc} & & & & & & \cdots \\ \hookrightarrow & H_2(\mathcal{K}_a F) & \longrightarrow & H_2(\mathcal{K}_a G) & \longrightarrow & H_2(\mathcal{K}_a H) & \\ \hookrightarrow & H_1(\mathcal{K}_a F) & \longrightarrow & H_1(\mathcal{K}_a G) & \longrightarrow & H_1(\mathcal{K}_a H) & \\ \hookrightarrow & H_0(\mathcal{K}_a F) & \longrightarrow & H_0(\mathcal{K}_a G) & \longrightarrow & H_0(\mathcal{K}_a H) & \longrightarrow 0 \end{array}$$

10.16. Consider a homogeneous free functor $F = V[b, -)\colon I \longrightarrow \mathrm{vect}_K$, where I is a finite poset. We claim that $\mathcal{K}_a F$ has the following homology:

$$H_i(\mathcal{K}_a F) \text{ is isomorphic to } \begin{cases} V & \text{if } i = 0 \text{ and } b = a, \\ 0 & \text{otherwise.} \end{cases}$$

To prove the claim, recall that $F(x) = 0$ if $b \not\leq x$, and F restricted to $(b \leq I) \subset I$ is isomorphic to the constant functor with value V. Thus, if $b \not\leq a$, then $\mathcal{K}_a F = 0$, and the claim holds. If $b = a$, then $(\mathcal{K}_a F)_0 = F(a) = V$ and $(\mathcal{K}_a F)_i = 0$ for $i > 0$, and again the claim holds. Assume $b < a$. Then $(\mathcal{K}_a F)_0 = F(a)$, which is isomorphic to V. Moreover, for a subset that has an ancestor $S \subset \mathcal{P}(a)$ with $|S| > 0$, the colimit $\mathrm{colim}_{\cap_{s\in S}(I\leq s)} F$ is isomorphic to $\mathrm{colim}_{\cap_{s\in S}(b\leq I\leq s)} F$, which is either isomorphic to V, in the case $S \subset (b \leq \mathcal{P}(a))$, or is 0 otherwise. Consequently, the complex $\mathcal{K}_a F$ is isomorphic to $L \otimes V$, where L is the augmented chain complex of the standard $|b \leq \mathcal{P}(a)|$-dimensional simplex whose homology is trivial in all degrees:

$$L := \left(\cdots \longrightarrow \bigoplus_{\substack{S\subset(b\leq\mathcal{P}(a)) \\ |S|=2}} K \longrightarrow \bigoplus_{\substack{S\subset(b\leq\mathcal{P}(a)) \\ |S|=1}} K \longrightarrow K \right).$$

Since Koszul complexes commute with direct sums, if F is free and isomorphic to $\bigoplus_{b\in I} (\beta F)_b [b, -)$, then:

$$H_i(\mathcal{K}_a F) \text{ is isomorphic to } \begin{cases} (\beta F)_a & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

We are now ready to state the key fact connecting the homology of the Koszul complexes of a functor with its Betti diagrams.

10.17. THEOREM. *Let I be a finite poset and $G\colon I \longrightarrow \mathrm{vect}_K$ a functor.*

1. *For every a in I and $i = 0, 1, 2$, the vector spaces $(\beta^i G)_a$ and $H_i(\mathcal{K}_a G)$ are isomorphic.*

2. *Assume an element a in I has the following property: every subset $S \subset \mathcal{P}(a)$ which has an ancestor has the product $\bigwedge S$ in I. Then, for every i, the vector spaces $(\beta^i G)_a$ and $H_i(\mathcal{K}_a G)$ are isomorphic.*

PROOF. The proof relies on Corollary 10.15. Since the arguments for statements (1) and (2) are analogous, we show only (1). The case $i = 0$ follows from Proposition 10.10 and the fact that $H_0(\mathcal{K}_a G)$ is isomorphic to $(G/\mathrm{rad}(G))(a)$.

Consider an exact sequence $0 \longrightarrow S_1 \longrightarrow P_0 \xrightarrow{\pi} G \longrightarrow 0$ where π is a minimal cover. It leads to an exact sequence of homologies (see Corollary 10.15):

$$
\begin{array}{ccccccccc}
 & & & & H_2(\mathcal{K}_a P_0) & \longrightarrow & H_2(\mathcal{K}_a G) & \rightarrowtail & \\
 & & & \overset{\alpha_2}{\nearrow} & & & & \searrow & \\
 & \rightarrowtail & H_1(\mathcal{K}_a S_1) & \longrightarrow & H_1(\mathcal{K}_a P_0) & \longrightarrow & H_1(\mathcal{K}_a G) & \rightarrowtail & \\
 & & & \overset{\alpha_1}{\nearrow} & & & & \searrow & \\
 & \rightarrowtail & H_0(\mathcal{K}_a S_1) & \longrightarrow & H_0(\mathcal{K}_a P_0) & \xrightarrow{H_0(\mathcal{K}_a \pi)} & H_0(\mathcal{K}_a G) & \longrightarrow & 0
\end{array}
$$

Minimality of π is equivalent to $H_0(\mathcal{K}_a \pi)$ being an isomorphism. Since P_0 is free, $H_1(\mathcal{K}_a P_0) = H_2(\mathcal{K}_a P_0) = 0$ (see 10.16). These two observations imply $H_1(\mathcal{K}_a G)$ is isomorphic to $H_0(\mathcal{K}_a S_1)$, and $H_2(\mathcal{K}_a G)$ is isomorphic to $H_1(\mathcal{K}_a S_1)$. By the already proven case $i = 0$, $H_0(\mathcal{K}_a S_1)$ is isomorphic to $(\beta^0 S_1)_a$ which is isomorphic to $(\beta^1 G)_a$. This gives the case $i = 1$. Applying this case to S_1, we get that $H_1(\mathcal{K}_a S_1)$ is isomorphic to $(\beta^1 S_1)_a$, which is isomorphic to $(\beta^2 G)_a$, and the case $i = 2$ also holds.

Here are some consequences of the presented statements, which are proven by the same strategy: first dicretise and then use the Koszul complex construction.

10.18. COROLLARY. *Let J be a poset and $F\colon J \longrightarrow \mathrm{vect}_K$ a functor.*

1. *Assume F is n-resolvable and $f\colon I \subset J$ is a subposet inclusion with finite I discretising an n-resolution of F. Then $\mathrm{supp}(\beta^i F) \subset I$ for all $0 \leq i < n$. Moreover, for $0 \leq i < \min(3, n)$ and a in I, $(\beta^i F)_a$ is isomorphic to $H_i(\mathcal{K}_a(Ff))$.*

2. *Assume F is discretised by a subposet inclusion $f\colon I \subset J$ which is a homomorphism out of a finite upper semilattice I. Then F is ∞-resolvable and $\mathrm{supp}(\beta^i F) \subset I$ for all $i \geq 0$. Moreover, for $i \geq 0$ and a in I, $(\beta^i F)_a$ is isomorphic to $H_i(\mathcal{K}_a(Ff))$.*

3. *Assume J is an upper semilattice and F a tame functor. Then F is ∞-resolvable and, for a in J with $\mathrm{par\text{-}dim}_J(a) < i$, $(\beta^i F)_a = 0$.*

PROOF. Statement (1) is a consequence of Proposition 10.8 and Theorem 10.17. Statement (2) is a consequence of Corollary 10.6.(1), and again Proposition 10.8 and Theorem 10.17. Statement (3) follows from statement (2) and Proposition 3.9 since tame functors indexed by an upper semilattice can be discretised by a finite sublattice (see Proposition 8.3.(1)).

As a consequence, if we consider the upper semilattice $J = \mathbb{R}^k$, then we get the well-known result stating that the minimal resolution of a tame (i.e., finitely presented, by Theorem 10.3) functor indexed by J has at most length k, which is the maximal parental dimension of its elements (see the third example in 3.1, the first example in 3.2 and Proposition 3.7).

We finish this article with our key theorem describing how to determine Betti diagrams of functors indexed by realisations of finite-type posets that admit discretisable resolutions. For such functors, Koszul complexes can be used, similarly to the case of functors indexed by upper semilattices (see Corollary 10.18). The main reason for this is the following fact: functors indexed by realisations have natural grid-like discretisations that can be refined in a way that every set of parents of an element having an ancestor also has a product in the refinement.

10.19. THEOREM. *Let I be a finite-type poset and $F\colon \mathcal{R}(I) \longrightarrow \mathrm{vect}_K$ an n-resolvable functor. Assume d is an element in I and V is a finite subset of $(-1,0)$ for which $\mathrm{supp}(\beta^j F) \subset \mathcal{R}_{I\leq d}(I,V) \overset{\alpha}{\hookrightarrow} \mathcal{R}(I)$ for all $j \leq n$.*

1. *Let $0 \leq i < \min(3,n)$ and (a,f) be in $\mathrm{supp}(\beta^i F)$. Then $(\beta^i F)_{(a,f)}$ is isomorphic to $H_i(\mathcal{K}_{(a,f)}(F\alpha))$.*
2. *Let $i < n$ and (a,f) be in $\mathrm{supp}(\beta^i F)$ for which there is ε in V such that $f(x) > \varepsilon$ for all x in $\mathcal{P}(a)$. Then $(\beta^i F)_{(a,f)}$ is isomorphic to $H_i(\mathcal{K}_{(a,f)}(F\alpha))$.*
3. *If $\text{par-dim}_{\mathcal{R}(I)}(a,f) < i < n$, then $(\beta^i F)_{(a,f)} = 0$.*

PROOF. (1) The assumption implies that a minimal $\min(3,n)$-resolution of F is discretised by $\mathcal{R}_{I\leq d}(I,V) \subset \mathcal{R}(I)$. This statement is then a particular case of Corollary 10.18.(1).
(2) The assumption implies that a minimal $i+1$ resolution of F is discretised by $\mathcal{R}_{I\leq d}(I,V) \subset \mathcal{R}(I)$. Moreover the product of every set of parents of (a,f) in $\mathcal{R}_{I\leq d}(I,V)$ exists. This statement is then a particular case of Theorem 10.17.(2).
(3) Since $\text{par-dim}_{\mathcal{R}_{I\leq d}(I,V)}(a,f) \leq \text{par-dim}_{\mathcal{R}(I)}(a,f)$ for every (a,f) in $\mathcal{R}_{I\leq d}(I,V)$, this statement follows from (2).

References

[ARS97] M. Auslander, I. Reiten and S.O. Smalø, *Representation theory of Artin algebras*. Cambridge University Press, Cambridge in Cambridge Studies in Advanced Mathematics, 36, xiv–425 (1997). ISBN: 0-521-41134-3; 0-521-59923-7.

[BFVZ21] M. Bergomi, M. Ferri, P. Vertechi and L. Zuffi, *Beyond topological persistence: Starting from networks*. Mathematics, 9,23 (2021). DOI: 10.3390/math9233079.

[BCM20] M.B. Botnan, J. Curry and E. Munch, *A Relative Theory of Interleavings*. ArXiv:2004.14286 (2020).

[BK72] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*. Springer-Verlag, Berlin-New York in Lecture Notes in Mathematics, 304, v–348 (1972).

[CE99] H. Cartan and S. Eilenberg, *Homological algebra*. Princeton University Press, Princeton, NJ in Princeton Landmarks in Mathematics, xvi–390. ISBN: 0-691-04991-2.

[CGL20] W. Chachólski, B. Giunti and C. Landi, *Invariants for tame parametrised chain complexes*. Homology, Homotopy and Applications, 23, 2, 183–213 (2020).

[CR20] W. Chachólski and H. Riihimäki, *Metrics and stabilization in one parameter persistence*. SIAM Journal on Applied Algebra and Geometry, 4, 1, 69–98. DOI:10.1137/19M1243932.

[CCGGO09] F. Chazal, D. Cohen-Steiner, M. Glisse, L.J. Guibas and S.Y. Oudot, *Proximity of persistence modules and their diagrams*. Proceedings of the 25th annual symposium on Computational geometry, 237–246 (2009).

[DS95] W.G. Dwyer and J. Spaliński, *Homotopy theories and model categories*. Handbook of algebraic topology, 73–126 (1995). DOI:10.1016/B978-044481779-2/50003-1.

[L15] M. Lesnick, *The Theory of the Interleaving Distance on Multidimensional Persistence Modules*. Foundations of Computational Mathematics, 15, 613-650 (2015).

[M98] S. Mac Lane, *Categories for the working mathematician*. Springer-Verlag, New York in Graduate Texts in Mathematics, 5, xii–314 (1998). ISBN:0-387-98403-8.

[M20] E. Miller, *Essential graded algebra over polynomial rings with real exponents*. ArXiv:2008.03819 (2020).

[GC17] O. Gäfvert and W. Chachólski, *Stable Invariants for Multidimensional Persistence*. arXiv:1703.03632 (2017).

[PRY24] C. Paquette, D.J. Rock and E. Yıldırım, *Categories of generalized thread quivers*. ArXiv:2410.14656 (2024).

[R74] C. Reedy, *Homotopy theory of model categories*. Unpublished manuscript (1974).

[S16] B. Schröder, *Ordered sets*, Springer, 328 (2016).

[SCLRÖ17] M. Scolamiero, W. Chachólski, A. Lundman, R. Ramanujam and S. Öberg, *Multidimensional persistence and noise*. Foundations of Computational Mathematics, 17, 6, 1367–1406 (2017). DOI:10.1007/s10208-016-9323-y.

[S72] R. Street, *Two constructions on lax functors*. Cahiers de Topologie et Géométrie Différentielle, 13, 217–264 (1972). ISSN: 0008-0004.

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