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A NOTE ON EXACTNESS AND STABILITY IN HOMOTOPICAL ALGEBRA

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Abstract.

Exact sequences are a well known notion in homological algebra. We investigate here the more vague properties of "homotopical exactness", appearing for instance in the fibre or cofibre sequence of a map. Such notions of exactness can be given for very general "categories with homotopies" having *homotopy* kernels and cokernels, but become more interesting under suitable "stability" hypotheses, satisfied - in particular - by chain complexes. It is then possible to measure the default of homotopical exactness of a sequence by the homotopy type of a certain object, a sort of "homotopical homology".

Introduction

The purpose of this work is to investigate the notion of "homotopically exact" sequence in categories equipped with homotopies, pursuing a project of developing homotopical algebra as an *enriched version* of homological algebra [6, 7]. Well known instances of such sequences are:

(a) the *cofibre sequence* of a map $f\colon A \to B$, or Puppe sequence [18], for topological spaces or pointed spaces

$$A \to B \to \mathbf{C}f \to \Sigma A \to \Sigma B \to \Sigma \mathbf{C}f \to \ \dots \qquad (1)$$

($\mathbf{C}f$ is the *h-cokernel* of f, or standard homotopy cokernel, or mapping cone; Σ denotes the suspension) where every map is, up to homotopy equivalence, an h-cokernel of the preceding one;

(b) the *fibre sequence* of a map of pointed spaces, which has a dual construction and properties;

(c) the *fibre-cofibre sequence* of a map $f\colon A \to B$ of chain complexes ($\mathbf{K}f$ is the *h-kernel*)

$$\dots \ \to \Omega A \to \Omega B \to \mathbf{K}f \to A \to B \to \mathbf{C}f \to \Sigma A \to \Sigma B \to \ \dots \qquad (2)$$

where both the aforementioned exactness conditions are satisfied, and each three-term part is homotopy equivalent to a *componentwise-split short exact sequence of complexes.* The drastic simplification of exactness properties in the last example is a product of the

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homotopical *stability* of chain complexes: the suspension and loop endofunctor (which in general just form an adjunction $\Sigma \dashv \Omega$), are *inverse* and take the sequence (2) to itself, by a three-step shift forward or backward, so that the properties of its left part reflect on the right part and vice versa. Triangulated categories abstract these facts in the notion of "exact triangle" [19, 20, 21, 11].

For the sake of simplicity, let us go on considering this simple but relevant situation: the category of chain complexes $\mathrm{Ch}_*\mathbf{D}$ over an *additive* category, even though the following notions are studied below in a much more general frame. An *h-differential sequence* $(f, g; \alpha)$ consists of two consecutive maps of chain complexes f, g together with a nullhomotopy α of their composite (represented by a dotted arc)

$$B \xrightarrow{f} A \xrightarrow{g} C \qquad \alpha\colon 0 \simeq gf\colon B \to C. \tag{3}$$

This sequence $(f, g; \alpha)$ will be said to be *h-exact* if the h-kernel of g is homotopically equivalent to the h-kernel of the h-cokernel of f, or equivalently if the dual condition is satisfied; other conditions, of *left* and *right* h-exactness, are equivalent to the previous one in the stable case (thm. 2.3). One can measure the default of exactness by the homotopy type of a chain complex $\mathbf{H}(f, g; \alpha)$, called the *homotopical homology* of the sequence (2.5, 3.4).

The construction is a *homotopical version* of an obvious construction of ordinary homology (in an abelian category), as presented in the left diagram below for a differential sequence (f, g); the construction only uses kernels and cokernels, and the sequence is exact if and only if $H(f, g) = 0$

$$\begin{array}{ccccc} B & = & B & \to & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathrm{Ker}g & \to & A & \xrightarrow{g} & C \\ \downarrow & & \downarrow & & \| \\ H(f,g) & \to & \mathrm{Cok}f & \to & C \end{array} \qquad \begin{array}{ccccc} B & = & B & \to & 0 \\ {\scriptstyle u_\alpha}\downarrow & & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \alpha} & \downarrow \\ \mathbf{K}g & \xrightarrow{kg} & A & \xrightarrow{g} & C \\ {\scriptstyle cu_\alpha}\downarrow & & {\scriptstyle cf}\downarrow & & \| \\ \mathbf{H}(f,g;\alpha) & \xrightarrow{kv_\alpha} & \mathbf{C}f & \xrightarrow{v_\alpha} & C \end{array} \tag{4}$$

In a similar way, we can construct in $\mathrm{Ch}_*\mathbf{D}$ the right diagram above, replacing (co)kernels with standard *homotopy* (co)kernels: we start now from the h-differential sequence $(f, g; \alpha)$, construct the h-kernel $\mathbf{K}g$ (with a structural nullhomotopy $0 \simeq g{\cdot}kg$), the h-cokernel $\mathbf{C}f$, two canonical maps $u_\alpha\colon B \to \mathbf{K}g$ and $v_\alpha\colon \mathbf{C}f \to C$ (coherently with all previous maps *and* homotopies), and, rather surprisingly, find that there is *one* chain complex which is at the same time the h-kernel of $\mathbf{C}f \to C$ and the h-cokernel of $B \to \mathbf{K}g$, namely:

$$\begin{gathered} (\mathbf{H}(f, g; \alpha))_n = B_{n-1} \oplus A_n \oplus C_{n+1}, \\ \partial(b, a, c) = (-\partial b, -fb + \partial a, -\alpha b + ga - \partial c). \end{gathered} \tag{5}$$

Now, the sequence $(f, g; \alpha)$ is h-exact if and only if $\mathbf{H}(f, g; \alpha)$ is contractible. Note that, *in contrast with ordinary homology*, all this requires the *additive* structure of $\mathbf{D}$, instead of its kernels and cokernels; moreover, monics are replaced with *fibrations* and epis with *cofibrations*.

The abstract frame we will use for "categories with homotopies" is a notion of *homotopical category*, developed in previous papers and recalled in Subsections 1.1, 1.2: it is a sort of lax 2-category with suitable comma and cocomma squares. The main reference, [7], is cited as Part I (and I.7 means Section 7 therein).

The links of this setting with triangulated categories or the Freyd embedding are dealt with in [8, 10], respectively; its *parallelism* with homological algebra in [6]. Finally, it can be noted that the Freyd embedding of a stable homotopy category into an abelian category [3, 4, 5] provides a notion of exactness for a sequence (f, g) of maps with $gf \simeq 0$, which is too weak for the present purposes (1.5, 2.1).

Outline. Section 1 introduces our notions of homotopical exactness of an h-differential sequence. Then, in Section 2, various properties of stability are considered (2.1, 2.2), with their links with h-exactness of sequences (2.3) and the introduction of the "homotopical homology" (2.5). Section 3 deals with the categories of chain complexes (a stable case) and positive complexes (left h-stable); these homotopical categories have a homotopical homology, which characterises h-exactness in both cases (3.4-3.6). The last two sections are devoted to bounded complexes between fixed degrees: $\mathrm{Ch}_0^p\mathbf{D}$ is an h-semistable homotopical category. In Section 4 we consider the very particular case $\mathrm{Ch}_0^1\mathbf{D} = \mathbf{D^2}$ of maps of $\mathbf{D}$, viewed as chain complexes and *equipped with chain homotopies*; again, homotopical homology measures the default of h-exactness. Finally, Section 5 studies the general bounded case $\mathrm{Ch}_0^p\mathbf{D}$ on an abelian basis $\mathbf{D}$; here, similar results hold up to *weak equivalences* (the chain maps which induce isomorphism in homology): the weak homotopy type of $\mathbf{H}(f, g; \alpha)$ measures the weak exactness of the sequence.

1. Exactness in homotopical categories

After a study of homotopy (co)kernels and of homotopical exactness of sequences (1.5), we prove that the fibre and cofibre sequences of a map are, respectively, left and right h-exact (1.7).

1.1. Homotopical categories. In the whole paper, $\mathbf{A}$ is a *pointed homotopical category*, as defined in I.7 (i.e., Section 7 of [7]); we just review its main aspects, also to fix the (slightly different) notation used here.

To begin with, $\mathbf{A}$ is a sort of lax 2-category, quite different from bicategories. It has objects, maps, and *homotopies* (2-cells) $\alpha: f \to g: A \to B$, with part of the usual structure of 2-dimensional categories, plus an equivalence relation $\alpha \simeq_2 \alpha'$ between parallel homotopies $\alpha, \alpha': f \to g$, called 2-homotopy (and corresponding to the relative homotopy of paths, with fixed endpoints). The vertical composition of homotopies (or *concatenation*) will be written additively, $\alpha+\beta: f \to h$ (for $\beta: g \to h$); thus, the vertical identity (or *trivial homotopy*) of a map is written as $0: f \to f$ or 0_f, and a *reverse* homotopy as $-\alpha: g \to f$. The vertical structure behaves categorically ("groupoidally") *up to 2-homotopy.* There is no assigned horizontal composition of cells, but only a *whisker composition* $v\cdot\alpha\cdot u$ for homotopies and maps, with reduced interchange property *up to* $\simeq_2$. (A *strict* interchange would allow us to derive *one* horizontal composition from the previous operations; but, even for chain complexes, this is not the case.)

The homotopy relation $f \simeq g$ (meaning that there is some homotopy $\alpha: f \to g$) is a congruence; the quotient $\mathrm{Ho}\mathbf{A} = \mathbf{A}/\!\simeq$ is the *homotopy category*, while $\mathrm{Ho}_2\mathbf{A} = \mathbf{A}/\!\simeq_2$ (same objects, same maps, and classes of homotopies up to 2-homotopy) is the *track 2-category* of $\mathbf{A}$, with invertible cells. Generally, one cannot reduce the study of homotopy to $\mathrm{Ho}_2\mathbf{A}$, somehow in the same way as higher dimensional category theory cannot be reduced to 2-categories.

Homotopies can usually be represented by a cylinder functor, or dually by a path functor; often by both via their adjunction $I \dashv P$; then, all *higher homotopies* are automatically produced by their powers. This gives a more powerful abstract frame, studied in [9] in a form which will be marginally used here.

1.2. Homotopy kernels. As a second main aspect, the pointed homotopical category $\mathbf{A}$ is assumed to have *regular* homotopy kernels and homotopy cokernels, with respect to a zero object 0. The latter is defined by a 2-dimensional universal property: every object X has precisely one map $t: X \to 0$ and one homotopy $t \to t$ (necessarily 0_t); and dually.

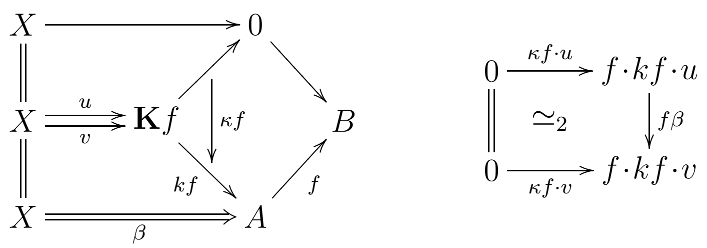
Also to fix the present notation, the *standard homotopy kernel*, or *h-kernel*, of the map $f: A \to B$ is a triple $\mathrm{hker} f = (\mathbf{K}f, kf, \kappa f)$, as in the left diagram below, determined up to isomorphism by the following universal property (of comma squares)

(6)

for every similar triple (X, x, ξ), where $x: X \to A$ and $\xi: 0 \to fx: X \to B$, there is a unique $u: X \to \mathbf{K}f$ such that $kf\cdot u = x$ and $\kappa f\cdot u = \xi$.

In particular, for $A = 0$ (as in the right diagram above), we obtain the *loop-object* $\Omega B = \mathbf{K}(0 \to B)$, with a structural homotopy $\omega B: 0 \to 0: \Omega B \to B$.

The h-kernel is assumed to be *regular*, i.e. to satisfy also the following *2-dimensional property* (called h4-regularity in I.2.5): given two maps $u, v: X \to \mathbf{K}f$ and a *coherent* homotopy $\beta: kf\cdot u \to kf\cdot v$ ($\kappa f\cdot u + f\beta \simeq_2 \kappa f\cdot v$), there is *some* homotopy $\alpha: u \to v$ which lifts β (i.e., $kf\cdot\alpha = \beta$)



(7)

The dual universal properties define the *h-cokernel* $\mathrm{hcok}f = (\mathbf{C}f, cf, \gamma f)$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \overset{\gamma f}{\nearrow} & \downarrow{\scriptstyle cf} \\ 0 & \longrightarrow & \mathbf{C}f \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & \overset{\sigma A}{\nearrow} & \downarrow \\ 0 & \longrightarrow & \Sigma A \end{array} \tag{8}$$

which reduces to the suspension ΣA when $B = 0$.

The h-kernel of $A \to 0$ is $1\colon A \to A$. But note that $f \simeq 0$ does not imply that $kf\colon \mathbf{K}f \to A$ be a homotopy equivalence, as shown by $\Omega B = \mathbf{K}(0 \to B)$. If $f\colon A \to B$ is a homotopy equivalence, then $\mathbf{K}f$ and $\mathbf{C}f$ are contractible, i.e. homotopy equivalent to 0 (I.3.7; the converse holds under stability hypotheses, 2.3). Thus, for every object A, the cocone $\mathbf{K}A = \mathbf{K}(\mathrm{id}A)$ and the cone $\mathbf{C}A = \mathbf{C}(\mathrm{id}A)$ are contractible

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle kA}\uparrow & \overset{\kappa A}{\nwarrow} & \uparrow \\ \mathbf{K}A & \longrightarrow & 0 \end{array} \qquad\qquad \begin{array}{ccc} A & = & A \\ \downarrow & \overset{\gamma A}{\nearrow} & \downarrow{\scriptstyle cA} \\ 0 & \longrightarrow & \mathbf{C}A \end{array} \tag{9}$$

1.3. Elements and coelements. The kernel-cokernel adjunction of abelian categories corresponds here to an adjunction given by h-kernels and h-cokernels. We consider first its strict version, then - in the next subsection - its coherent version.

Let us work in the category $\mathbf{A^2}$ of morphisms of $\mathbf{A}$: an object is an $\mathbf{A}$-map $x\colon X' \to X''$, a morphism $f = (f', f'')\colon x \to y$ is a commutative square of $\mathbf{A}$ (as in the diagram below); the composition is obvious: a pasting of commutative squares.

By h-kernels and h-cokernels, the morphism $f\colon x \to y$ yields a commutative diagram in $\mathbf{A}$

$$\begin{array}{ccccccc} \mathbf{K}x & \xrightarrow{kx} & X' & \xrightarrow{x} & X'' & \xrightarrow{cx} & \mathbf{C}x \\ {\scriptstyle \mathbf{K}(f)}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle \mathbf{C}(f)} \\ \mathbf{K}y & \xrightarrow[ky]{} & Y' & \xrightarrow[y]{} & Y'' & \xrightarrow[cy]{} & \mathbf{C}y \end{array} \tag{10}$$

where $\mathbf{K}(f)$ and $\mathbf{C}(f)$ are defined by

$$\begin{array}{ll} ky\cdot\mathbf{K}(f) = f'\cdot kx, & \kappa y\cdot\mathbf{K}(f) = f''\cdot\kappa x, \\ \mathbf{C}(f)\cdot cx = cy\cdot f'', & \mathbf{C}(f)\cdot\gamma x = \gamma y\cdot f'. \end{array} \tag{11}$$

This gives two adjoint endofunctors $c \dashv k\colon \mathbf{A^2} \to \mathbf{A^2}$, whose action on maps is obvious (and displayed in the diagram above): $k(f) = (\mathbf{K}(f), f')\colon kx \to ky$ and $c(f) = (f'', \mathbf{C}(f))\colon cx \to cy$. The unit and counit are determined as follows

$$(u_x, 1)\colon x \to kcx, \qquad\qquad (1, v_y)\colon cky \to y, \tag{12}$$

$$\begin{array}{ccccc} \bullet & \xrightarrow{x} & A & \xrightarrow{cx} & \mathbf{C}x \\ {\scriptstyle u_x}\downarrow & \nearrow{\scriptstyle kcx} & & & \\ \mathbf{K}cx & & & & \end{array} \qquad\qquad \begin{array}{ccccc} \mathbf{K}y & \xrightarrow{ky} & A & \xrightarrow{y} & \bullet \\ & & & {\scriptstyle cky}\searrow & \uparrow{\scriptstyle v_y} \\ & & & & \mathbf{C}ky \end{array}$$

$$kcx.u_x = x, \qquad \kappa cx.u_x = \gamma x, \qquad v_y.cky = y, \qquad v_y.\gamma ky = \kappa y. \tag{13}$$

(The action of the functors c, k on arrows - $c(f), k(f)$ - is written with parentheses, to avoid ambiguity with the structural morphisms cx, ky.) Keeping fixed the object A, all this restricts to an adjunction between the slice categories of *elements* and *coelements* of A, denoted as

$$\mathrm{El}A = \mathbf{A}/A = (\mathbf{A} \downarrow A), \qquad\qquad \mathrm{Cl}A = \mathbf{A}\backslash A = (A \downarrow \mathbf{A}), \tag{14}$$

$$c : \mathrm{El}A \rightleftarrows \mathrm{Cl}A : k, \qquad u_x\colon x \to kcx, \qquad v_y\colon cky \to y, \tag{15}$$

where (with the usual abuse of notation) c takes the morphism $f = (f, 1_A)\colon x \to x'$ to $c(f) = \mathbf{C}(f, 1_A)\colon cx \to cx'$, and k takes $g = (1_A, g)\colon y \to y'$ to $k(g) = \mathbf{K}(1_A, g)\colon ky \to ky'$

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \xrightarrow{cx} & \mathbf{C}x \\ {\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle \mathbf{C}(f,1)} \\ X' & \xrightarrow[x']{} & A & \xrightarrow[cx']{} & \mathbf{C}x' \end{array} \qquad \begin{array}{ccccc} \mathbf{K}Y & \xrightarrow{ky} & A & \xrightarrow{y} & Y \\ {\scriptstyle \mathbf{K}(1,g)}\downarrow & & \| & & \downarrow{\scriptstyle g} \\ \mathbf{K}Y' & \xrightarrow[ky']{} & A & \xrightarrow[y']{} & Y' \end{array} \tag{16}$$

For $A = 0$, we get $\mathrm{El}(0) = \mathbf{A} = \mathrm{Cl}(0)$ and (15) becomes the classical suspension-loop adjunction $\Sigma \dashv \Omega$

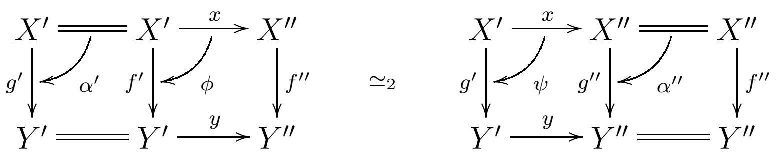
$$\Sigma : \mathbf{A} \rightleftarrows \mathbf{A} : \Omega, \qquad u_X\colon X \to \Omega\Sigma X, \qquad v_Y\colon \Sigma\Omega Y \to Y. \tag{17}$$

1.4. Homotopy elements. The previous point just requires "1-dimensional homotopical properties" of $\mathbf{A}$ (I.7.1). But the fact that $\mathbf{A}$ is homotopical allows for a more interesting version of elements, "up to homotopy" (cf. [12, 13], for coherent categories of objects over a space).

Let us replace $\mathbf{A^2} = (\mathbf{2}, \mathbf{A})$ with the *coherent homotopy category of morphisms* $[\mathbf{2}, \mathbf{A}]$: an object is still an $\mathbf{A}$-map $x\colon X' \to X''$, but a morphism $[f] = [f', f''; \phi] : x \to y$ derives from a triple $f = (f', f''; \phi)$ forming a homotopy-commutative square $\phi\colon f''x \simeq yf'$

$$\begin{array}{ccc} X' & \xrightarrow{x} & X'' \\ {\scriptstyle f'}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle f''} \\ Y' & \xrightarrow{y} & Y'' \end{array} \tag{18}$$

up to identifying $[f', f''; \phi] = [g', g''; \psi]$ when there exists a *coherent* pair of homotopies $\alpha'\colon f' \to g', \alpha''\colon f'' \to g''$, as in the diagram below

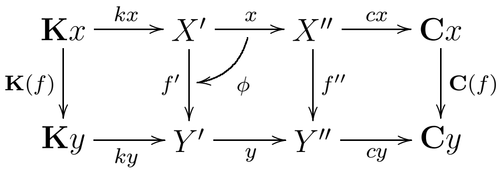

(19)

As a relevant fact, well known for the coherent category of objects over a fixed space ([12], 1.3), and related to a classical theorem of Dold ([2], 6.1), the map $[f', f''; \phi]: x \to y$ is an isomorphism if and only if f' and f'' are homotopy equivalences. In fact, in this case, one can choose an *h-adjoint equivalence* for f'

$$\alpha': 1 \to g'f',\ \beta': f'g' \to 1 \qquad (f'\alpha' + \beta'f' \simeq_2 0_{f'},\ \alpha'g' + g'\beta' \simeq_2 0_{g'}),$$

(by Vogt's Lemma [22]: given an arbitrary equivalence, replace the homotopy β' with $(-\beta'f'g' - f'\alpha g') + \beta'$ and verify the triangle identities); and similarly for f''. Finally, one constructs an inverse $[g', g'', \psi]$ with a suitable homotopy $\psi: g''y \to xg'$; namely, $\psi = (-g''y\beta' - g''\phi g') - \alpha''xg'$.

Replacing the diagram (10) with the following one, where $f = (f', f''; \phi)$, $\mathbf{K}(f)$ is defined below, and $\mathbf{C}(f)$ is similarly defined


(20)

$$ky{\cdot}\mathbf{K}(f) = f'{\cdot}kx, \qquad \kappa y{\cdot}\mathbf{K}(f) = f''{\cdot}\kappa x + \phi{\cdot}kx, \tag{21}$$

we have two adjoint endofunctors $c, k: [\mathbf{2}, \mathbf{A}] \to [\mathbf{2}, \mathbf{A}]$, with (strict) unit and counit defined essentially as above, $[u_x, 1; 0]: x \to kcx$ and $[1, v_y; 0]: cky \to y$.

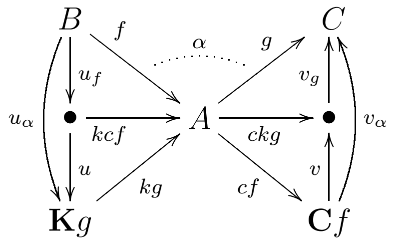
For a fixed A, we have a restricted adjunction between h-elements $x: \bullet \to A$ and h-coelements $y: A \to \bullet$

$$c : \mathrm{hEl}A \rightleftarrows \mathrm{hCl}A : k, \qquad [u_x]: x \to kcx, \qquad [v_y]: cky \to y; \tag{22}$$

and again, for $A = 0$, this reduces to a suspension-loop adjunction (induced by (17)), for the homotopy category $\mathrm{Ho}\mathbf{A} = \mathrm{hEl}(0) = \mathrm{hCl}(0)$

$$\Sigma : \mathrm{Ho}\mathbf{A} \rightleftarrows \mathrm{Ho}\mathbf{A} : \Omega, \qquad [u_X]: X \to \Omega\Sigma X, \qquad [v_Y]: \Sigma\Omega Y \to Y. \tag{23}$$

1.5. Homotopical exactness. An *h-differential sequence* $(f, g; \alpha)$ consists of a nullhomotopy $\alpha: 0 \simeq gf$. The properties we want to investigate concern a diagram constructed via h-kernels and h-cokernels; the vertical maps are defined below



$$\alpha \colon 0 \simeq gf \colon B \to C \tag{24}$$

$$u = \mathbf{K}(v_\alpha), \quad v = \mathbf{C}(u_\alpha),$$

$$\begin{array}{llll} kg \cdot u_\alpha = f, & \kappa g \cdot u_\alpha = \alpha, & v_\alpha \cdot cf = g, & v_\alpha \cdot \gamma f = \alpha, \\ kcf \cdot u_f = f, & \kappa cf \cdot u_f = \gamma f, & v_g \cdot ckg = g, & v_g \cdot \gamma kg = \kappa g, \\ kg \cdot u = kcf, & \kappa g \cdot u = v_\alpha \cdot \kappa cf, & v \cdot cf = ckg, & v \cdot \gamma f = \gamma kg \cdot u_\alpha, \end{array} \tag{25}$$

and the diagram commutes: $u \cdot u_f = u_\alpha$, $v_g \cdot v = v_\alpha$. (Note that $u_f = u_{\gamma f}$, $v_f = v_{\kappa f}$).

We say that the h-differential sequence $(f, g; \alpha)$ is:

(a) *left h-exact* if the map $[u_\alpha] \colon f \to kg$ (in hEl$\mathbf{A}$) determined by the universal property of the h-kernel is an isomorphism; or, equivalently, if $u_\alpha \colon B \to \mathbf{K}g$ is a homotopy equivalence in $\mathbf{A}$;

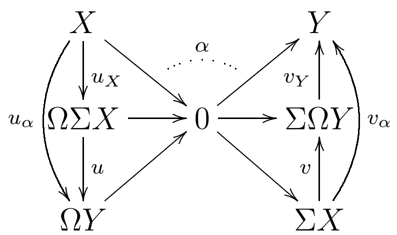
(b) *right h-exact* if, dually, $v_\alpha \colon \mathbf{C}f \to C$ is a homotopy equivalence;

(c) *strongly h-exact* if it is both left and right h-exact;

(d) *h-exact* if $k[v_\alpha] \colon kcf \to kg$ and $c[u_\alpha] \colon cf \to ckg$ are isomorphisms; or, equivalently, if $\mathbf{K}(v_\alpha) \colon \mathbf{K}cf \to \mathbf{K}g$ and $\mathbf{C}(u_\alpha) \colon \mathbf{C}f \to \mathbf{C}kg$ are homotopy equivalences of $\mathbf{A}$.

Strong h-exactness implies h-exactness. The condition (c) is often too strong to be of interest, but is of use in the stable case (2.3); we shall see that the last condition becomes simpler, and is implied by left (or right) h-exactness, as soon as $\mathbf{A}$ is homotopically "semistable" (2.3a).

It is interesting to note what happens when the middle object is zero: in this case (which would be trivial in homological algebra), our data reduce to a triple $(X, Y; \alpha)$, formed of two objects and a nullhomotopy α



$$\alpha \colon 0 \simeq 0 \colon X \to Y \tag{26}$$

$$u = \Omega(v_\alpha), \quad v = \Sigma(u_\alpha).$$

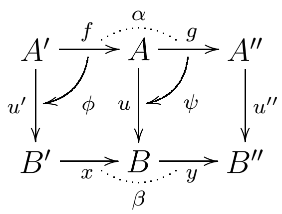
The typical left h-exact sequence of this type is $(\Omega Y, Y; \omega Y)$ and the typical right h-exact one is $(X, \Sigma X; \sigma X)$. This case also shows that a right h-exact sequence $(X, \Sigma X; \sigma X)$

need not be h-exact: in the homotopical category $\mathbf{Top}_*$ of pointed spaces, take $X = \mathbf{S}^0$; then $\Sigma X = \mathbf{S}^1$ is not homotopy equivalent to $\Sigma\Omega\mathbf{S}^1$ (apply H_1).

Marginally, we shall also consider a condition of *pseudo exactness*, meaning that $gf \simeq 0$ and $cf{\cdot}kg \simeq 0$, which just concerns a pair (f, g) of consecutive arrows; any h-exact sequence is also pseudo exact, but the converse is far from being true: a sequence $X \to 0 \to Y$ is trivially pseudo exact, *but* (for any choice of α) *cannot be left h-exact unless* X *is homotopy equivalent to* ΩY. In the stable case, this notion will be shown to be equivalent to ordinary exactness within the associated Freyd's abelian category (2.1).

1.6. Definition and Theorem. (Coherence)

Two h-differential sequences $(f, g; \alpha)$ *and* $(x, y; \beta)$ *will be said to be* coherently equivalent *if there is a coherent diagram linking them*

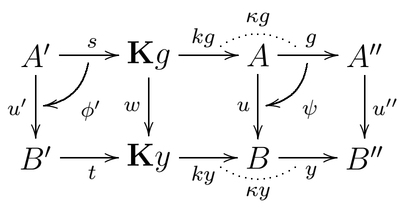


$$\alpha\colon 0 \simeq gf, \qquad \beta\colon 0 \simeq yx,$$
$$\phi\colon uf \simeq xu', \qquad \psi\colon u''g \simeq yu, \qquad (27)$$
$$\beta u' - y\phi \ \simeq_2\ u''\alpha + \psi f,$$

where the vertical arrows u, u', u'' *are homotopy equivalences.*

In these hypotheses, the upper row is left h-exact if and only if the lower row is so (and similarly for all the h-exactness conditions considered in 1.5).

Proof. Let us assume, for instance, that the upper row is left h-exact and prove that also the lower one is so. The map $[u, u''; \psi]\colon g \to y$ is iso in $[\mathbf{2}, \mathbf{A}]$, whence $k[u, u''; \psi]$ is iso, and $w = \mathbf{K}(u, u''; \psi)$ is a homotopy equivalence. We form the diagram



(28)

$$\begin{aligned} &ky{\cdot}w = u{\cdot}kg, && && \kappa y{\cdot}w = u''{\cdot}\kappa g + \psi{\cdot}kg, \\ &kg{\cdot}s = f, && \kappa g{\cdot}s = \alpha, && ky{\cdot}t = x, \qquad \kappa y{\cdot}t = \beta, \end{aligned} \qquad (29)$$

where $s = u_\alpha$ is a homotopy equivalence, by hypothesis, and we want to prove that also $t = u_\beta$ is so; plainly, it suffices to prove the existence of a homotopy $\phi'\colon ws \to tu'$, which can be derived from the 2-dimensional property of the h-pullback $\mathrm{hker}(y) = (\mathbf{K}y, ky, \kappa y)$. In fact, the maps $ws, tu'\colon A' \to \mathbf{K}y$ have a homotopic projection on B, coherently with $\kappa y\colon 0 \simeq y{\cdot}ky$ (as a consequence of the coherence hypothesis, in (27))

$$\begin{aligned} &ky{\cdot}ws = u{\cdot}kg{\cdot}s = uf, \qquad ky{\cdot}tu' = xu', \qquad \phi\colon ky{\cdot}ws \to ky{\cdot}tu', \\ &\kappa y{\cdot}ws + y\phi = (u''{\cdot}\kappa g + \psi{\cdot}kg){\cdot}s + y\phi = (u''\alpha + \psi f) + y\phi \\ &\qquad\qquad \simeq_2\ \beta u' = \kappa y{\cdot}tu'. \end{aligned} \qquad (30)$$

∎

1.7. Theorem. (The fibre and cofibre sequences of a map)
If **A** *is pointed homotopical, the* cofibre sequence *of a map* $f: A \to B$

$$A \xrightarrow{f} B \xrightarrow{cf} \mathbf{C}f \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma(cf)} \Sigma\mathbf{C}f \longrightarrow \cdots \qquad (\gamma f,\ 0,\ \sigma f,\ \gamma' f) \tag{31}$$

is right h-exact at any point, with non-canonical *nullhomotopies*

$$\sigma f: 0 \simeq \Sigma f \cdot \delta, \qquad \gamma' f: 0 \simeq \Sigma(cf) \cdot \Sigma f$$

and so on. In the stronger assumption of a pointed IP4-homotopical *structure for* **A** *[9], there are canonical nullhomotopies* $\sigma f, \Sigma(\gamma f)$*,... having that effect.*

Dually, the fibre sequence of a map *is left h-exact*

$$\cdots \longrightarrow \Omega\mathbf{K}f \xrightarrow{\Omega(kf)} \Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\partial} \mathbf{K}f \xrightarrow{kf} A \xrightarrow{f} B \qquad (\kappa' f,\ \omega f,\ 0,\ \kappa f) \tag{32}$$

Proof. The first part of the statement is essentially proved in I.5 (and only needs a *right* homotopical category). To begin with, it is proved (I.5.4.11) that there exists *some* nullhomotopy σf satisfying

$$\sigma f: 0 \simeq \Sigma f \cdot \delta: \mathbf{C}f \to \Sigma B, \qquad \sigma f \cdot cf = \sigma B: 0 \simeq 0: B \to \Sigma B. \tag{33}$$

Then, applying the homotopy invariant functor Σ (I.4.5) we obtain, again in a non-canonical way, the subsequent nullhomotopies $\gamma' f$,... We have now the *contracted cofibre diagram* (I.5.6.3)

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{cf} & \mathbf{C}f & \xrightarrow{\delta} & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B & \xrightarrow{\Sigma(cf)} & \Sigma\mathbf{C}f & \longrightarrow & \cdots \\
\| & & \| & & \| & & \uparrow u_0 & \simeq & \uparrow u_1 & \simeq & \uparrow u_2 & & \\
A & \xrightarrow[f]{} & B & \xrightarrow[x_1]{} & B_2 & \xrightarrow[x_2]{} & B_3 & \xrightarrow[x_3]{} & B_4 & \xrightarrow[x_4]{} & B_5 & \longrightarrow & \cdots
\end{array} \tag{34}$$

linking the cofibre sequence of f to the sequence of its iterated h-cokernels

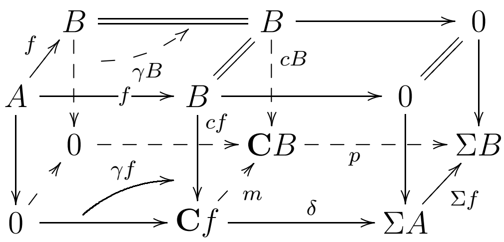
$$\begin{array}{ll}
x_1 = cf, & x_{i+1} = cx_i, \\
\mathrm{hcok}(x_i) = (B_{i+1}, x_{i+1}, \gamma x_i: 0 \simeq x_{i+1} \cdot x_i) & (i \geq 1).
\end{array} \tag{35}$$

All squares are h-commutative, all vertical arrows are homotopy equivalences. Moreover, the diagram is coherent (as in (27)), with respect to the nullhomotopies of the upper row (as in (31) and the structural nullhomotopies γx_i of the lower one: this follows from the construction of u_i's in I.5.4-6.

Finally, if **A** is pointed IP4-homotopical, in the sense of [9], the cone functor $\mathbf{C}: \mathbf{A} \to \mathbf{A}$ inherits an induced monad (as proved more in detail in [8], 3.7)

$$c\colon 1 \to \mathbf{C}, \qquad \mathbf{g}\colon \mathbf{C}^2 \to \mathbf{C} \qquad\qquad (\mathbf{g}\cdot c\mathbf{C} = id = \mathbf{g}\cdot\mathbf{C}c), \tag{36}$$

whose operation $\mathbf{g}$ (induced by a "connection" of the cylinder functor) represents a natural nullhomotopy of the cone $\mathbf{C}X$, via $\mathbf{g}X\cdot\gamma\mathbf{C}X\colon 0 \simeq 1\colon \mathbf{C}X \to \mathbf{C}X$. We can now deduce a structural nullhomotopy $\sigma f\colon 0 \simeq \Sigma f\cdot\delta$; in fact, $\Sigma f\cdot\delta$ factors through $\mathbf{C}B$, as proved by the following (canonical) diagram



(37)

In the left cube, the front and back face are h-pushouts (the h-cokernels of f and 1_B); in the right cube, the front and back face are ordinary pushouts, so that the front and back rectangle are h-pushouts, by the pasting property I.2.2: $\sigma A = \delta\cdot\gamma f$ and $\sigma B = p\cdot\gamma B$; thus $\Sigma f\cdot\delta = (\mathbf{C}f \to \mathbf{C}B \to \Sigma B) = pm$, and we can define

$$\begin{aligned} &\sigma f = p\cdot\mathbf{g}B\cdot\gamma\mathbf{C}B\cdot m\colon\ 0 \simeq pm,\\ &\sigma f\cdot cf = p\cdot\mathbf{g}B\cdot\gamma\mathbf{C}B\cdot m\cdot cf = p\cdot\mathbf{g}B\cdot\gamma\mathbf{C}B\cdot cB = p\cdot\mathbf{g}B\cdot\mathbf{C}cB\cdot\gamma B\\ &\qquad = p\cdot\gamma B = \sigma B. \end{aligned} \tag{38}$$

Finally, $\Sigma\colon \mathbf{A} \to \mathbf{A}$ has a canonical extension to homotopies (as proved for the cylinder functor, in [10], 2.9). ∎

2. The links between stability and exactness

Under suitable stability hypotheses (2.1, 2.2), the exactness properties behave in a simple way (2.3) and the default of h-exactness can be measured by the homotopy type of suitable objects (2.5).

2.1. Stability. Homotopical stability essentially requires that the suspension-loop adjunction be an equivalence. As in Part I, we will use a stronger definition, better related with (co)fibre sequences. Let $\mathbf{A}$ be pointed homotopical and $f\colon A \to B$ a map. Its fibre-cofibre sequence can be inserted as the central row of a homotopy-commutative *adjunction fibre-cofibre diagram* (I.7.5)

$$\begin{array}{ccccccccccccccc}
\cdots \to & \Sigma\Omega^2 B & \to & \Sigma\Omega\mathbf{K}f & \to & \Sigma\Omega A & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega B & \xrightarrow{\Sigma\partial} & \Sigma\mathbf{K}f & \xrightarrow{\Sigma kf} & \Sigma A & \xrightarrow{\Sigma f} & \cdots \\
& \downarrow{\scriptstyle v_{\Omega B}} & & \downarrow{\scriptstyle v_{\mathbf{K}f}} & & \downarrow{\scriptstyle v_A} & & \downarrow{\scriptstyle v_B}\ \simeq & & \downarrow{\scriptstyle V_f}\ \simeq & & \| & & \\
\cdots \xrightarrow{\Omega f} & \Omega B & \xrightarrow{\partial} & \mathbf{K}f & \xrightarrow{kf} & A & \xrightarrow{f} & B & \xrightarrow{cf} & \mathbf{C}f & \xrightarrow{\delta} & \Sigma A & \xrightarrow{\Sigma f} & \cdots \\
& \| & & \simeq\ \downarrow{\scriptstyle U_f} & \simeq & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle u_B} & & \downarrow{\scriptstyle u_{\mathbf{C}f}} & & \downarrow{\scriptstyle u_{\Sigma A}} & & \\
\cdots \xrightarrow{\Omega f} & \Omega B & \xrightarrow{\Omega\mathbf{C}f} & \Omega\mathbf{C}f & \xrightarrow{\Omega\delta} & \Omega\Sigma A & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma B & \to & \Omega\Sigma\mathbf{C}f & \to & \Omega\Sigma^2 A & \to & \cdots
\end{array} \tag{39}$$

whose upper and lower rows are obtained by letting the functors Σ and Ω operate on the original sequence, and shifting the result of three steps. The vertical arrows consist of the (co)units u, v of the adjunction $\Sigma \dashv \Omega$, together with the adjoint maps U_f, V_f provided by the following structural nullhomotopy ρ_f

$$\begin{aligned} &U_f\colon \mathbf{K}f \to \Omega\mathbf{C}f, \qquad\qquad V_f\colon \Sigma\mathbf{K}f \to \mathbf{C}f, \\ &\rho_f = cf\cdot\kappa f - \gamma f\cdot kf\colon 0 \to 0\colon \mathbf{K}f \to \mathbf{C}f. \end{aligned} \tag{40}$$

A is defined to be *h-stable* (I.7.8) if all these maps U_f, V_f are homotopy equivalences. Then also the (co)unit-maps u_A and v_A are so: in fact (I.7.6)

$$u_A \simeq U_f \quad (f\colon A \to 0), \qquad v_A \simeq V_g \quad (g\colon 0 \to A), \tag{41}$$

with coincidence whenever the trivial homotopies are strict identities for concatenation, as it happens within chain complexes. ($\mathrm{Ch}_*\mathbf{D}$ is even *strictly stable*: all the vertical arrows of (39) are isos and all the squares commute.)

In the h-stable case, the homotopy category Ho**A** has a canonical embedding in an *abelian* category $\mathbf{F} = Fr(\mathrm{Ho}\mathbf{A})$, introduced by Freyd [3, 4, 5]; this provides a notion of exactness which coincides with *pseudo exactness*, as defined at the end of 1.5: the sequence (f, g) of **A** is pseudo exact if and only if $([f], [g])$ is exact in **F**.

(We only sketch the proof of this fact, which will not be used here. The last property means that $\mathrm{Im}_\mathbf{F}[f] = \mathrm{Ker}_\mathbf{F}[g]$; but kernels in **F** derive from weak kernels in Ho**A**, and the latter from h-kernels of **A**, so that $\mathrm{Ker}_\mathbf{F}[g] = \mathrm{Im}_\mathbf{F}[kg]$. Finally, subobjects in **F** amount to *weak subobjects* in Ho**A** [10]; thus, $\mathrm{Im}_\mathbf{F}[f] = \mathrm{Im}_\mathbf{F}[kg]$ means that $[f]$ factors through $[kg]$ in Ho**A**, i.e. $gf \simeq 0$ in **A**, and conversely, i.e. $cf\cdot kg \simeq 0$.)

2.2. Other stability properties. The adjunction $c \dashv k$ between h-elements $f\colon \bullet \to A$ and $g\ A \to \bullet$ (1.4) yields other stability conditions, which we prove below to be weaker than h-stability (2.4). The pointed homotopical category **A** will be said to be:

(a) *left h-stable* if, for every object A, the adjunction $c \dashv h$ ($c\colon \mathrm{hEl}A \to \mathrm{hCl}A$) is a coreflection, i.e. all components $u_f\colon \mathrm{Dom}(f) \to \mathbf{K}cf$ are homotopy equivalences;

(b) *right h-stable* if the dual condition holds, i.e. all components $v_g\colon \mathbf{C}kg \to \mathrm{Cod}(g)$ are homotopy equivalences (if both conditions hold, the adjunction $c \dashv h$ is an equivalence);

(c) *h-semistable* if, for every object A, the adjunction $c \dashv h$ is *idempotent*, i.e. the following equivalent conditions hold:

- all components $\mathbf{C}(u_f)\colon \mathbf{C}f \to \mathbf{C}kcf$ are homotopy equivalences,
- all components $v_{cf}\colon \mathbf{C}kcf \to \mathbf{C}f$ are homotopy equivalences,
- all components $u_{kg}\colon \mathbf{K}g \to \mathbf{K}ckg$ are homotopy equivalences,

- all components $\mathbf{K}(v_g)\colon \mathbf{K}ckg \to \mathbf{K}g$ are homotopy equivalences.

The equivalence comes from a general result (of 2-category theory): in an adjunction, the four natural transformations which appear in the triangle identities are invertible whenever any of them is so ([16], Part 0, Lemma 4.3); such adjunctions are called "idempotent" (or "exact") because they are also characterised by the idempotence of the associated monad, cf. [1], Section 6. (Any adjunction between ordered sets is so, as well as any reflection or coreflection between categories.)

Plainly, left (or right) h-stable implies h-semistable. There are parallel strict notions (*left stable*, etc.), for strict elements and coelements, where these components are required to be isos.

Recall that the adjunction $\Sigma \dashv \Omega$ is a particular instance of the adjunction $c \dashv k$ (22); therefore, if $\mathbf{A}$ is left h-stable (resp. left and right h-stable, h-semistable), then, *at the level of the homotopy category* Ho$\mathbf{A}$, the $\Sigma\Omega$-adjunction is a coreflection (resp. an endoequivalence, an idempotent adjunction).

(We shall see that $\mathrm{Ch}_*\mathbf{D}$, being stable, is also left and right h-stable. Its subcategory of *positive* complexes is just left h-stable (3.6); *negative* complexes yield a right h-stable case and the *bounded* ones an h-semistable category.)

2.3. Theorem. (Stability and exactness)
(a) If $\mathbf{A}$ is h-semistable, the two conditions defining h-exactness (1.5d) are equivalent: $\mathbf{K}(v_\alpha)$ is a homotopy equivalence iff $\mathbf{C}(u_\alpha)$ is so; therefore, left or *right h-exact implies h-exact.*
(b) If $\mathbf{A}$ is left h-stable a sequence is h-exact iff it is left h-exact. $\mathbf{A}$ is left h-stable iff all sequences $(f, cf; \gamma f)$ are strongly h-exact, iff every right h-exact sequence is strongly h-exact. If $\mathbf{A}$ is left h-stable, a map f is a homotopy equivalence iff the object $\mathbf{C}f$ is contractible, and this implies that $\mathbf{K}f$ is also so.
(c) $\mathbf{A}$ is left and right h-stable iff all the conditions of h-exactness of 1.5 are equivalent. In this case, any fibre-cofibre sequence is strongly h-exact; a map f is a homotopy equivalence iff $\mathbf{K}f$ is contractible, iff $\mathbf{C}f$ is so.

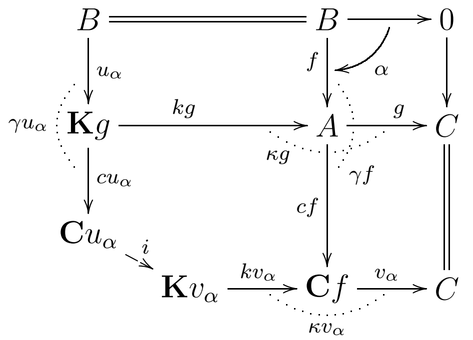
Proof. It suffices to prove (a) and (b). We use the notation of diagram (24); recall the relations $u{\cdot}u_f = u_\alpha$, $v_g{\cdot}v = v_\alpha$.
(a) By hypothesis, $\mathbf{C}(u_f)$ is a homotopy equivalence; if also u is so, the same is true of $v = \mathbf{C}(u_\alpha) = \mathbf{C}(u){\cdot}\mathbf{C}(u_f)$. And dually.
(b) If $\mathbf{A}$ is left h-stable, $u_f\colon B \to \mathbf{K}cf$ is a homotopy equivalence; then, $u_\alpha = u{\cdot}u_f$ shows that the sequence $(f, g; \alpha)$ is h-exact if and only if it is left h-exact. Thus, also by (a), if $\mathbf{A}$ is left h-stable, right h-exact implies strongly h-exact. In this case, any sequence $(f, cf; \gamma f)$ is strongly h-exact. But, if this holds, the canonical map $u_f\colon B \to \mathbf{K}cf$ is a homotopy equivalence and $\mathbf{A}$ is left h-stable.

Finally, let $\mathbf{A}$ be h-stable; if f is a homotopy equivalence, we know that $\mathbf{C}f$ is contractible (1.2); conversely, in this case, the invariance of h-exactness (1.6) shows that kcf is a homotopy equivalence, whence also $f = kcf.u_f$ is so. ■

2.4. Corollary. *If the pointed homotopical category* $\mathbf{A}$ *is h-stable, then it is also left and right h-stable.*

Proof. Let $\mathbf{A}$ be h-stable. Because of this, the fibre-cofibre sequence of f is strongly h-exact, as follows from the adjunction fibre-cofibre diagram (39), the invariance of h-exactness (1.6), and the fact that Σ and Ω, being quasi-reciprocal, preserve left and right h-exactness. Thus, the sequences $(f, cf; \gamma f)$ and $(kf, f; \kappa f)$ are strongly h-exact, which implies the thesis, by 2.3c. ∎

2.5. Homotopical homology. The default of h-exactness of h-differential sequences can be measured by the homotopy type of suitable objects. Starting from an h-differential sequence $(f, g; \alpha)$, there is a commutative, coherent diagram (for i, see below)



$$\alpha\colon 0 \simeq gf$$
$$\kappa g\colon 0 \simeq g\cdot kg$$
$$\gamma f\colon 0 \simeq cf\cdot f \qquad (42)$$
$$\kappa v_\alpha\colon 0 \simeq v_\alpha\cdot kv_\alpha$$
$$\gamma u_\alpha\colon 0 \simeq cu_\alpha\cdot u_\alpha.$$

It provides the *left homotopical homology* and the *right homotopical homology* of the sequence

$$\mathbf{H}^-(f, g; \alpha) = \mathbf{C}u_\alpha, \qquad \mathbf{H}^+(f, g; \alpha) = \mathbf{K}v_\alpha, \qquad (43)$$

linked by a non-canonical *comparison* i satisfying the following relations

$$kv_\alpha\cdot i\cdot cu_\alpha = cf\cdot kg, \qquad kv_\alpha\cdot i\cdot \gamma u_\alpha = \gamma f, \qquad \kappa v_\alpha\cdot i\cdot cu_\alpha = \kappa g; \qquad (44)$$

(its existence is proved below; a precise determination is possible in the stronger setting of IP4-homotopical categories [10]).

In the *left h-stable case*, the homotopy type of $\mathbf{H}^-(f, g; \alpha)$ measures the default of h-exactness of $(f, g; \alpha)$: the sequence is h-exact iff it is left h-exact (2.3c), iff u_α is a homotopy equivalence, iff $\mathbf{C}u_\alpha$ is *contractible* (2.3c, again). Dually, in the right h-stable case, the object $\mathbf{H}^+(f, g; \alpha)$ measures the same default of h-exactness (which coincides now with right h-exactness). Finally, if $\mathbf{A}$ is left and right h-stable, both objects $\mathbf{H}^-(f, g; \alpha)$ and $\mathbf{H}^+(f, g; \alpha)$ measure the default of h-exactness.

It is therefore of interest to study the property that *the canonical map* i *be a homotopy equivalence.* When this is the case, $\mathbf{H}(f, g; \alpha)$ will denote the homotopy type of $\mathbf{H}^-(f, g; \alpha)$ and $\mathbf{H}^+(f, g; \alpha)$; or their isomorphism type, if i is an iso, as happens for chain complexes (3.4, 3.5).

Now, to prove the existence of i, consider first that there is one map $x\colon \mathbf{K}g \to \mathbf{K}v_\alpha$ such that $kv_\alpha\cdot x = cf\cdot kg$, $\kappa v_\alpha\cdot x = \kappa g$. Second, $xu_\alpha \simeq 0$, by the 2-dimensional property of the h-kernel $(\mathbf{K}v_\alpha, kv_\alpha, \kappa v_\alpha)$; in fact, $kv_\alpha \cdot xu_\alpha = cf \cdot kg \cdot u_\alpha = cf \cdot f$, and the nullhomotopy $\gamma f\colon 0 \simeq cf \cdot f$ is (strictly) coherent with $\kappa v_\alpha\colon 0 \simeq v_\alpha \cdot kv_\alpha$, in the sense that $\kappa v_\alpha \cdot xu_\alpha = \kappa g \cdot u_\alpha = \alpha = v_\alpha \cdot \gamma f$ (and $\kappa v_\alpha \cdot 0$ is the trivial endohomotopy).

There is thus *some* homotopy $\rho\colon 0 \simeq xu_\alpha$ such that $kv_\alpha \rho = \gamma f$. (In the IP4-homotopical case, we may determine ρ via a *double homotopy* produced by the homotopy $\kappa v_\alpha \cdot xu_\alpha = v_\alpha \cdot \gamma f$ together with a connection of the cylinder or path functor.) Finally, this homotopy ρ produces a map $i\colon \mathbf{C}u_\alpha \to \mathbf{K}v_\alpha$ such that $i \cdot cu_\alpha = x$, $i \cdot \gamma u_\alpha = \rho$, and (44) holds.

3. Stability and homotopical exactness for chain complexes

This section deals with the usual homotopical structure of the category $\mathrm{Ch}_*\mathbf{D}$ of unbounded chain complexes over an additive category $\mathbf{D}$; positive chain complexes are briefly considered (3.5).

3.1. Notation. A morphism $f\colon \bigoplus A_i \to \bigoplus B_j$ between two finite biproducts in the additive category $\mathbf{D}$ is determined by its components $f_{ji}\colon A_i \to B_j$ and can be written as a matrix $f = (f_{ji})$. But we prefer to write it as an "expression in m variables", as one would do in a category of modules

$$f(a_1, a_2, \ldots,\ a_m) = (\textstyle\sum f_{1i}a_i, \sum f_{2i}a_i, \ldots,\ \sum f_{ni}a_i). \tag{45}$$

(Viewing a_i as the i-th projection of $A = \bigoplus A_i$, the expression is correct: $(a_1, a_2, \ldots,\ a_m)$ is the identity of A, with specified names for projections.)

3.2. The homotopical structure. The category $\mathrm{Ch}_*\mathbf{D}$ of (unbounded) chain complexes over the additive category $\mathbf{D}$ is equipped with the usual homotopies of chain maps

$$\begin{array}{ll} \alpha = (f, g, (\alpha_n))\colon f \to g\colon A \to B, & \\ \alpha_n\colon A_n \to B_{n+1}, & -f_n + g_n = \alpha_{n-1}\partial_n + \partial_{n+1}\alpha_n. \end{array} \tag{46}$$

These come from an IP4-homotopical structure, based on well known cylinder and path functors (cf. [9], 6.5-6.8). The opposite structure is isomorphic to $\mathrm{Ch}_*\mathbf{D}^{op}$; this duality provides a choice of h-kernels and h-cokernels which reduces the structural isomorphisms to identities.

Let $f\colon A \to B$ be a map of chain complexes. The left homotopical structure of $\mathrm{Ch}_*\mathbf{D}$ is computed as follows (note that any "shift of degree" leads to a change of sign)

$$\begin{array}{ll} (\mathbf{K}f)_n = A_n \oplus B_{n+1}, & \partial(a, b) = (\partial a, fa - \partial b), \\ k\colon \mathbf{K}f \to A, & k(a, b) = a, \\ \kappa\colon 0 \simeq fk\colon \mathbf{K}f \to B, & \kappa(a, b) = b, \end{array} \tag{47}$$

$$\Omega A = \mathbf{K}(0 \to A), \qquad (\Omega A)_n = A_{n+1}, \qquad \partial^\Omega_n = -\partial_{n+1}. \tag{48}$$

Analogously, the right homotopical structure is described by:

$$\begin{array}{ll} (\mathbf{C}f)_n = A_{n-1} \oplus B_n, & \partial(a,b) = (-\partial a, -fa + \partial b), \\ c\colon B \to \mathbf{C}f, & c(b) = (0,b), \\ \gamma\colon 0 \simeq cf\colon A \to \mathbf{C}f, & \gamma(a) = (-a, 0), \end{array} \tag{49}$$

$$\Sigma A = \mathbf{C}(A \to 0), \qquad (\Sigma A)_n = A_{n-1}, \qquad \partial_n^{\Sigma} = -\partial_{n-1}. \tag{50}$$

3.3. Stability. This homotopical structure is strictly stable, as is well known (for the fact that Σ and Ω are inverse) or follows easily from the previous expressions (for the additional condition in 2.1). Thus, $\mathrm{Ch}_*\mathbf{D}$ is also left and right h-stable (2.4); note, however, that it is *not* left stable in the strict sense: the previous computations show that $u_f\colon A \to \mathbf{K}cf$ is not an isomorphism, generally.

By 2.3, the fibre-cofibre sequence of the map f is strongly h-exact and sent to itself by Σ and Ω. Each h-exact sequence is thus coherently equivalent to a sequence $B \to \mathbf{C}f \to \Sigma A$, which means a componentwise-split short exact sequence of complexes (as in the triangulated structure).

3.4. Theorem. (Homotopical homology of chain complexes)

In the category $\mathrm{Ch}_*\mathbf{D}$ of chain complexes over an additive category, the left and the right homotopical homology of an h-differential sequence (2.5) coincide, yielding a chain complex whose homotopy type vanishes if and only if the sequence is h-exact (or, equivalently, strongly h-exact)

$$\begin{array}{c} (\mathbf{H}(f,g;\alpha))_n = B_{n-1} \oplus A_n \oplus C_{n+1}, \\ \partial(b,a,c) = (-\partial b, -fb + \partial a, -\alpha b + ga - \partial c). \end{array} \tag{51}$$

Proof. It is an easy computation, based on 3.2. ■

3.5. Positive chain complexes. Let the additive category $\mathbf{D}$ have *finite limits.* The category $\mathrm{Ch}_p\mathbf{D}$ of positive chain complexes (null in negative degree) is also homotopical (actually IP4-homotopical).

The h-cokernel of the chain map $f\colon A \to B$ in $\mathrm{Ch}_p\mathbf{D}$ is computed as in $\mathrm{Ch}_*\mathbf{D}$, since $(\mathbf{C}f)_n = A_{n-1} \oplus B_n$ is null in negative degree; the h-kernel is computed by coreflection and differs in degree 0, where we get a pullback

$$(\mathbf{K}f)_n = A_n \oplus B_{n+1}\ (n > 0), \qquad (\mathbf{K}f)_0 = \mathrm{pb}(f, \partial). \tag{52}$$

$\mathrm{Ch}_p\mathbf{D}$ is *left* h-stable but *not* right h-stable, as we prove below (3.6). The relation $\mathbf{C}u_\alpha = \mathbf{K}v_\alpha$ providing the homotopical homology $\mathbf{H}(f,g;\alpha)$ of an h-differential sequence (2.5) holds also here, with

$$\begin{array}{l} \mathbf{H}_0(f,g;\alpha) = (\mathbf{K}g)_0 = (\mathbf{C}u_\alpha)_0 = (\mathbf{K}v_\alpha)_0 \\ = \mathrm{pb}(g_0 : A_0 \to C_0 \leftarrow C_1 : \partial_1). \end{array} \tag{53}$$

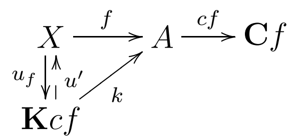
From left h-stability, we deduce that *the h-differential sequence $(f,g;\alpha)$ is h-exact iff it is left h-exact, iff $\mathbf{H}(f,g;\alpha)$ is contractible* (2.3b).

But being *right* h-exact is a stronger condition (even up to weak equivalence). For instance, the typical left h-exact sequence $\Omega A \to 0 \to A$ (with nullhomotopy ωA) is right h-exact iff the counit $v_A: \Sigma\Omega A \to A$ is a homotopy equivalence, which requires that $H_0(A) = H_0(\Sigma\Omega A) = 0$. On the other hand, the homotopical homology $\mathbf{H}(\Omega A, A; \omega A)$ of that sequence consists of the h-cokernel of $\mathrm{id}: \Omega A \to \Omega A$, i.e. the cone $\mathbf{C}(\Omega A)$, which is always contractible, independently of right h-exactness.

3.6. Proposition. $\mathrm{Ch}_p\mathbf{D}$ *is left h-stable. It is not right h-stable, unless* $\mathbf{D}$ *is trivial.*

Proof. Given $f: X \to A$, we show that $u_f: X \to \mathbf{K}cf$ is a deformation retract, where $K = \mathbf{K}cf$ is computed below, as a pullback in degree 0 (and $\mathbf{C}f$ in 3.2.4)

$$\begin{array}{l} K_n = A_n \oplus X_n \oplus A_{n+1} \quad (n > 0), \\ K_0 = \mathrm{pb}((cf)_0, \partial_1) \subset A_0 \oplus X_0 \oplus A_1, \\ \partial(a, x, a') = (\partial a, \partial x, a - \partial a' + fx), \\ k(a, x, a') = a, \quad \kappa(a, x, a') = (x, a'), \qquad u_f(x) = (fx, -x, 0), \end{array} \tag{54}$$



$$\gamma = \gamma f: 0 \simeq cf \cdot f: X \to \mathbf{C}f,$$

$$\kappa = \kappa cf: 0 \simeq cf \cdot k: K \to \mathbf{C}f.$$

Now, take $u': K \to X$, $u'(a, x, a') = -x$. It is a chain map, with $u'u_f = 1$; and there is a homotopy $\sigma: u_f u' \simeq 1_K$, namely $\sigma(a, x, a') = (a', 0, 0)$.

On the other hand, $\mathrm{Ch}_p\mathbf{D}$ is not right h-stable: the coreflexive adjunction $\Sigma \dashv \Omega$ ($\Omega\Sigma = 1$) is not an equivalence, even up to homotopy, since $(\Sigma\Omega A)_0 = 0$ and $H_0(\Sigma\Omega A) = 0$. ■

3.7. Negative and bounded chain complexes. Dually, the category of negative chain complexes, on an additive category $\mathbf{D}$ with finite colimits, is pointed homotopical and right h-stable.

A sequence $(f, g; \alpha)$ (as in 2.5) is h-exact iff it is right h-exact, iff $\mathbf{H}(f, g; \alpha)$ is contractible; left h-exactness is a stronger condition. The homotopical homology is computed as a pushout in degree 0

$$\mathbf{H}_0(f, g; \alpha) = (\mathbf{C}f)_0 = (\mathbf{C}u_\alpha)_0 = (\mathbf{K}v_\alpha)_0 = \mathrm{po}(B_{-1} \leftarrow B_0 \to A_0). \tag{55}$$

If $\mathbf{D}$ is finitely complete and cocomplete, the category $\mathrm{Ch}_0^p\mathbf{D}$ of chain complexes concentrated in degrees $n \in [0, p]$ $(p \geq 1)$ is homotopical and h-semistable, as it follows from computations similar to the previous ones. The object $\mathbf{H}(f, g; \alpha) = \mathbf{C}u_\alpha = \mathbf{K}v_\alpha$ is still well-defined, and computed by a pullback in degree 0 and a pushout in degree p

$$\begin{array}{c} \mathbf{H}_0(f, g; \alpha) = \mathrm{pb}(A_0 \to C_0 \leftarrow C_1), \\ \mathbf{H}_p(f, g; \alpha) = \mathrm{po}(B_{p-1} \leftarrow B_p \to A_p). \end{array} \tag{56}$$

The particular case $p = 1$ (where a complex is reduced to a map) will be treated in detail in the next section, where we show that the homotopy type of $\mathbf{H}(f, g; \alpha)$ still measures the default of h-exactness of the sequence. For $p > 1$, this holds in a weaker sense, as we show in Section 5.

Finally, let us also consider the "degenerate" case $\mathbf{A} = \mathrm{Ch}_0^0\mathbf{D} = \mathbf{D}$, where homotopies are trivial (i.e., identities), and h-(co)kernels are ordinary (co)kernels. If $\mathbf{D}$ is pointed, with kernels and cokernels, the adjunction $c \dashv k$ between elements and coelements is always idempotent (and induces the usual duality between normal subobjects and normal quotients): in other words, $\mathbf{A}$ is h-semistable. If $\mathbf{D}$ is abelian (but, actually, much more generally), left and right homology coincide and determine exact sequences; the present analysis of homotopical exactness reduces to pure homological algebra.

4. Categories of morphisms

The case $\mathrm{Ch}_0^1\mathbf{D} = \mathbf{D^2}$ gives a simple h-semistable category having homotopical homology.

4.1. The homotopy structure. In this section, $\mathbf{D}$ is always an additive category with finite limits and colimits. $\mathbf{A} = \mathrm{Ch}_0^1\mathbf{D} = \mathbf{D^2}$ is the category of morphisms of $\mathbf{D}$, *equipped with chain homotopies*, as already considered above (3.7) and examined below; this makes it a pointed homotopical category.

($\mathbf{D}$ can be an abelian category, or the category of topological abelian groups, or Banach spaces, etc. Note also that $\mathbf{Ab^2}$ is equivalent to the category of internal categories in abelian groups, by the Dold-Kan theorem [14]; one can show that,in this case, the present analysis of h-exactness agrees with notions recently introduced for *symmetric cat-groups*, in [15].)

An object will be written as $A = \partial_A\colon A' \to A''$ and a morphism as $f = (f', f'')\colon A \to B$. A nullhomotopy $a\colon 0 \simeq f$ is determined by a *diagonal* map $\alpha\colon A'' \to B'$

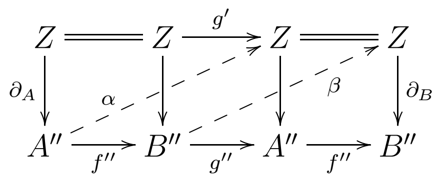
$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle\partial}\downarrow & \nearrow_{\alpha} & \downarrow{\scriptstyle\partial} \\ A'' & \xrightarrow[f'']{} & B'' \end{array} \qquad\qquad \begin{array}{l} f' = \alpha\partial, \\ \\ f'' = \partial\alpha, \end{array} \tag{57}$$

while a homotopy $\alpha\colon f \simeq g\colon A \to B$ is given by a nullhomotopy $\alpha\colon 0 \simeq -f + g$ (i.e. $-f' + g' = \alpha\partial,\ -f'' + g'' = \partial\alpha$). As an exception in the family $\mathrm{Ch}_0^p\mathbf{D}$, we have here a 2-category (essentially because 2-dimensional homotopies vanish, being produced by morphisms of degree 2): given two consecutive nullhomotopies $\alpha\colon 0 \simeq f$ and $\beta\colon 0 \simeq g$, the reduced interchange holds strictly: $g\alpha = \beta f$ (both homotopies are determined by the same diagonal $g'\alpha = \beta\partial\alpha = \beta f''\colon A'' \to C''$).

Plainly, an object $\partial_A\colon A' \to A''$ is contractible if and only if ∂_A is an isomorphism, if and only if $H_*(A) = 0$ (where H_* is the graded homology group, with two components). Thus, a homotopy equivalence need not be an iso; however, there is a relevant case where the two notions coincide.

4.2. Lemma. *If, in the morphism $f\colon A \to B$, either f' or f'' is an iso, then f is a homotopy equivalence if and only if it is an isomorphism.*

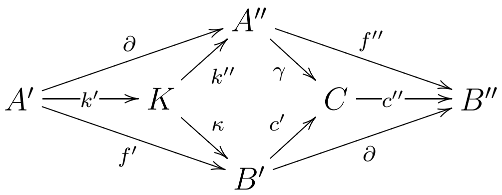
PROOF. Let $f' = 1_Z$ $(Z = A' = B')$ and choose an adjoint equivalence between A and B



$$\alpha\colon 1_A \simeq gf, \qquad \beta\colon fg \simeq 1_B,$$
$$f\alpha + \beta f = 0, \qquad \alpha g + g\beta = 0. \tag{58}$$

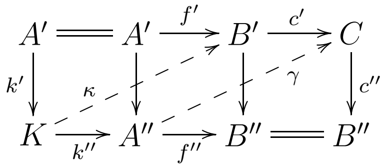
Then f'' admits an inverse mapping: $h = g'' + \partial_A\beta$. In fact, $f''h = f''g'' + f''\partial_A\beta = 1 - \partial_B\beta + \partial_B\beta = 1$ and $hf'' = g''f'' + \partial_A\beta f'' = 1 + \partial_A a - \partial_A a = 1$ (use the triangle identity $f\alpha + \beta f = 0$). Similarly, if $f'' = 1_Z$ $(Z = A'' = B'')$, then f' has an inverse mapping, namely $g' - a\partial_B\colon B' \to A'$. ■

4.3. HOMOTOPY KERNELS AND COKERNELS. If $f\colon A \to B$ is the commutative square (57), form the pullback K of (f'', ∂_B) and the pushout C of (∂_A, f'); this yields an inner bicartesian square



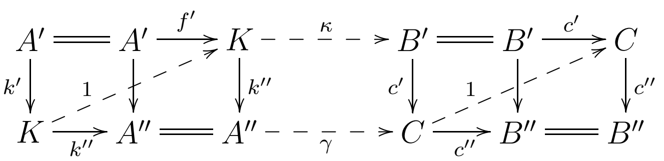
(59)

from which the h-kernel and the h-cokernel of f are constructed



$$\mathbf{K}f = k',\ kf = (1, k''),\ \kappa f = \kappa,$$
$$\mathbf{C}f = c'',\ cf = (c', 1),\ \gamma f = \gamma. \tag{60}$$

Now, the h-cokernel of kf (the *h-normal coimage* of f) is $ckf = (k', 1)\colon A \to k''$, and its h-kernel is again $kf\colon k' \to A$ (left solid diagram below). This shows that the adjunction between h-elements and h-coelements of $\mathbf{A}$ is exact (2.2c), and $\mathbf{A}$ is h-semistable



(61)

Symmetrically, the right solid diagram shows how the h-kernel of cf (the *h-normal image* of f) is $kcf = (1, c'')\colon c' \to B$, whose h-cokernel is again $cf\colon B \to c''$. Inserting the dotted morphism in the middle, the three central squares are a factorisation of $f\colon A \to B$, the *h-normal factorisation* (much in the same way as for the ordinary normal factorisation, in a pointed category with kernels and cokernels).

4.4. Left and right exactness. Given an h-differential sequence $(f, g; \alpha)$

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & A' & \xrightarrow{g'} & Y' \\ {\scriptstyle\partial}\downarrow & {\scriptstyle\alpha} & \downarrow & & \downarrow{\scriptstyle\partial} \\ X'' & \xrightarrow[f'']{} & A'' & \xrightarrow[g'']{} & Y'' \end{array} \qquad \alpha\colon 0 \simeq gf\colon X \to Y, \tag{62}$$

its properties of h-exactness are characterised as follows, up to coherent equivalence (1.6)

(a) in the typical left h-exact sequence, (X'', α, f'') is the pullback of (g'', ∂_Y) and $f' = 1_{A'}$,

(b) in the typical right h-exact sequence, (Y', α, g') is the pushout of (f', ∂_X) and $g'' = 1''_A$,

(c) the typical strongly h-exact sequence is of the following type, determined by an arbitrary factorisation $\partial_A = h''h'$

$$\begin{array}{ccccc} A' & = & A' & \xrightarrow{h'} & H \\ {\scriptstyle h'}\downarrow & {\scriptstyle 1} & \downarrow & & \downarrow{\scriptstyle h''} \\ H & \xrightarrow[h'']{} & A'' & = & A'' \end{array} \tag{63}$$

4.5. Theorem. (Homotopical homology of morphisms)

Given an h-differential sequence $a\colon 0 \simeq gf$, its homotopical homology exists and is computed as

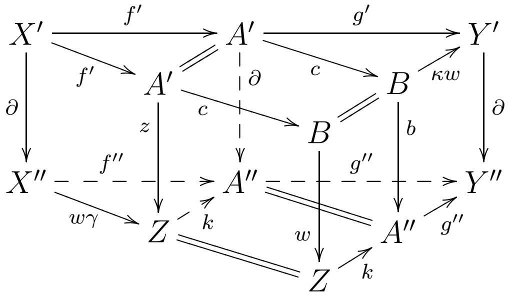
$$\mathbf{H}(f, g; \alpha) = \mathbf{C}u_\alpha = \mathbf{K}v\alpha = (w\colon B \to Z), \tag{64}$$

from the diagram below, where (B, c, γ) is the pushout of (∂_X, f'), while (Z, k, κ) is the pullback of (∂_Y, g'') and $w\colon B \to Z$ is the induced morphism (think of B as h-boundaries, *of Z as* h-cycles*)*

$$\begin{array}{ccccccc} X' & & \xrightarrow{f'} & A' & \xrightarrow{g'} & & Y' \\ & & {\scriptstyle c}\swarrow & & \searrow{\scriptstyle z} & {\scriptstyle\kappa}\nearrow & \\ {\scriptstyle\partial}\downarrow & & B & \overset{w}{\dashrightarrow} {\scriptstyle\partial}\downarrow \dashrightarrow & Z & & \downarrow{\scriptstyle\partial} \\ & {\scriptstyle\gamma}\nearrow & {\scriptstyle b}\searrow & & \swarrow{\scriptstyle k} & & \\ X'' & & \xrightarrow[f'']{} & A'' & \xrightarrow[g'']{} & & Y'' \end{array} \tag{65}$$

Moreover, our sequence is h-exact iff $\mathbf{K}(v_\alpha) = (1_{A'}, w)$ is a homotopy equivalence, iff w is an isomorphism of $\mathbf{D}$, iff $\mathbf{H}(f, g; \alpha)$ is contractible, iff its homology is null: $H_*(\mathbf{H}(f, g; \alpha)) = 0$.

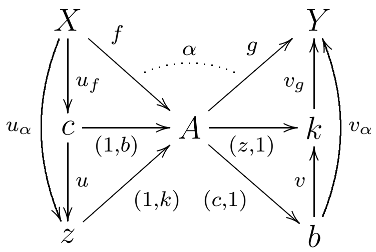
Proof. We already know that $\mathbf{H}(f, g; \alpha)$ exists and how to compute it (3.7), but let us write down everything, in this simple case. Computing h-kernels and h-cokernels, as in 4.3, the following diagram shows that $\mathbf{C}u_\alpha$ and $\mathbf{K}v_\alpha$ are indeed the object $w\colon B \to Z$



(66)

$$
\begin{array}{ll}
\mathrm{hker}(g) = (z, (1,k)\colon z \to \partial_A, \kappa), & \mathrm{hcok}(f) = (b, (c,1)\colon \partial_A \to b, \gamma), \\
u_\alpha = (f', w\gamma)\colon X \to z, & v_\alpha = (\kappa w, g'')\colon b \to Y, \\
\mathrm{hcok}(u_\alpha) = (w, (c,1)\colon z \to w, \gamma), & \mathrm{hker}(v_\alpha) = (w, (1,k)\colon w \to b, \kappa).
\end{array} \tag{67}
$$

Finally, the diagram (24) becomes



$$
u_f = (f', \gamma), \quad u = (1, w), \qquad v_g = (\kappa, g''), \quad v = (w, 1). \tag{68}
$$

This proves that our sequence is h-exact iff $\mathbf{K}(v_\alpha) = (1_{A'}, w)$ is a homotopy equivalence. The other equivalent conditions follow immediately from 4.2 and 4.1. ∎

5. Bounded chain complexes and weak exactness

For the category $\mathbf{A} = \mathrm{Ch}_0^p\mathbf{D}$ bounded chain complexes on an abelian category $\mathbf{D}$, a notion of *weak* exactness, controlled by homology, is more effective.

5.1. Homology. Let $\mathbf{D}$ be abelian. Recall that the category $\mathbf{A} = \mathrm{Ch}_0^p\mathbf{D}$ of chain complexes concentrated in degrees $n \in [0, p]$ $(p \geq 1)$ is homotopical and h-semistable (3.7).

The ordinary homology functors $H_n\colon \mathbf{A} \to \mathbf{D}$ $(0 \leq n \leq p)$ satisfy the following self-dual axioms, combining properties of homology and homotopy theories

(i) *homotopy invariance*: if $f \simeq g$ in $\mathbf{A}$, then $H_n(f) = H_n(g)$;

(ii) *exactness*: for every map $f\colon A \to B$ in $\mathbf{A}$, the sequences

$$
H_0(\mathbf{K}f) \to H_0(A) \to H_0(B), \quad H_p(A) \to H_p(B) \to H_p(\mathbf{C}f), \tag{69}
$$

are exact in $\mathbf{D}$;

(iii) *stability*: there are two natural *iso*morphisms

$$\omega\colon H_n(\Omega A) \to H_{n+1}(A), \quad \sigma\colon H_n(A) \to H_{n+1}(\Sigma A) \quad (0 \le n < p), \tag{70}$$

which are coherent with the morphisms U_f, V_f (for every map $f\colon A \to B$), i.e. form a commutative diagram (in **Ab**)

$$\begin{array}{ccc} H_n(\mathbf{K}f) & \xrightarrow{H_n(U_f)} & H_n(\Omega\mathbf{C}f) \\ {\scriptstyle \sigma\mathbf{K}}\downarrow & \searrow^{i_f} & \downarrow{\scriptstyle \omega\mathbf{C}} \\ H_{n+1}(\Sigma\mathbf{K}f) & \xrightarrow[H_{n+1}(V_f)]{} & H_{n+1}(\mathbf{C}f) \end{array} \tag{71}$$

yielding a natural transformation $i_f\colon H_n(\mathbf{K}f) \to H_{n+1}(\mathbf{C}f)$ $(0 \le n < p)$.

Abstracting from this situation, we shall consider the following setting: a pointed homotopical category $\mathbf{A}$ with $\Omega^{p+1} = 0 = \Sigma^{p+1}$, equipped with a theory $H_n\colon \mathbf{A} \to \mathbf{E}$ with values in an abelian category and satisfying the axioms above for $0 \le n \le p$. (This can be easily adapted to the positive or unbounded cases.)

5.2. Theorem. *In this setting:*
(a) ω and σ are also coherent with the morphisms induced in homology by the unit u_A and counit v_A of the adjunction $\Sigma \dashv \Omega$: they form commutative diagrams in $\mathbf{E}$

$$\begin{array}{ccc} H_n(A) & \xrightarrow{H_n(u_A)} & H_n(\Omega\Sigma A) \\ & {\scriptstyle \sigma A}\searrow \quad \swarrow{\scriptstyle \omega\Sigma A} & \\ & H_{n+1}(\Sigma A) & \end{array} \qquad \begin{array}{ccc} H_{n+1}(\Sigma\Omega A) & \xrightarrow{H_{n+1}(v_A)} & H_{n+1}(A) \\ & {\scriptstyle \omega\Sigma A}\nwarrow \quad \nearrow{\scriptstyle \omega A} & \\ & H_n(\Omega A) & \end{array} \tag{72}$$

(b) The morphisms $i_f, H_n(U_f), H_{n+1}(V_f), H_n(u_A), H_{n+1}(v_A)$ *are iso* $(0 \le n < p)$.
(c) every map $f\colon A \to B$ produces two exact sequences in $\mathbf{E}$

$$\begin{array}{ccccccccccc} H_p(\mathbf{K}f) \rightarrowtail & H_p(A) & \cdots & H_1(B) & \to & H_0(\mathbf{K}f) & \to & H_0(A) & \to & H_0(B) & \\ & \| & & \| & & \downarrow{\scriptstyle i_f} & & \| & & \| & \\ & H_p(A) & \cdots & H_1(B) & \to & H_1(\mathbf{C}f) & \to & H_0(A) & \to & H_0(B) & \twoheadrightarrow H_0(\mathbf{C}f) \end{array} \tag{73}$$

which are actually one, modulo the isomorphisms i_f and the commutative squares above: the fibre-cofibre homology sequence *of f.*

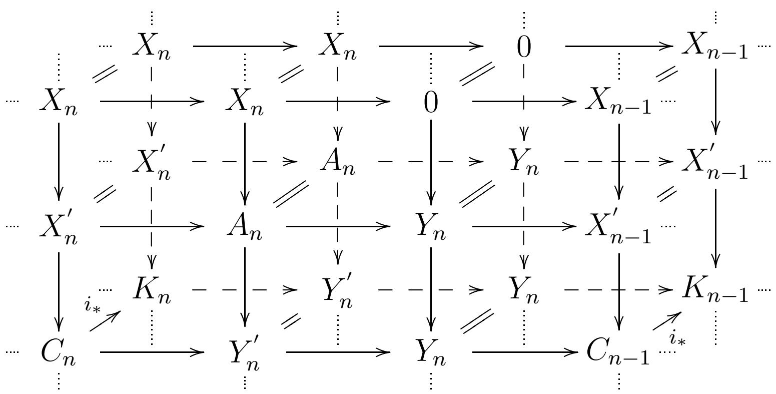
Proof. (a) The diagrams (72) derive from (71), writing u_A and v_A as in (41).
(c) For the upper row, apply H_0 to the fibre sequence of f, taking into account that $H_n(A) \cong H_0(\Omega^n A)$ for $0 \le n \le p$ and that $\Omega^{p+1} = 0$; exactness follows from the properties of the fibre sequence (1.7), together with the axioms of exactness and homotopy invariance for H_0. Similarly for the lower row. The commutativity of (73) follows from the commutativity of the adjunction fibre-cofibre diagram (39).
(b) The Five Lemma in $\mathbf{E}$ proves now that all i_f are iso; the rest follows from the commutative diagrams (71), (72). ∎

5.3. Weak equivalences. Now, say that a map $f\colon A \to B$ is a *weak equivalence* (with respect to the theory H_*) if $H_n(f)$ is iso for all n; similarly, the object A is *weakly null* (or *weakly contractible*) if $H_n(A) = 0$ for all n. Plainly, by the usual properties of exact sequences in abelian categories:
(a) $\mathbf{K}f$ is weakly null if and only if $H_n(f)$ is iso for $n > 0$ and mono for $n = 0$,
(b) $\mathbf{C}f$ is weakly null if and only if $H_n(f)$ is iso for $n < p$ and epi for $n = p$.

Thus, for $p = 2$, the complex $A = (A_2 \to A_1 \to A_0)$ is a weakly null object iff it forms a short exact sequence; but it is easy to see that it is nullhomotopic iff this sequence *splits*.

5.4. Lemma. *In the general setting mentioned at the end of (5.1), let an h-differential sequence* $(f, g; \alpha)$ *be given. Then the map* $i\colon \mathbf{C}u_\alpha = \mathbf{K}v_\alpha$ *(2.5) is a weak equivalence.*

Proof. The diagram (42) gives a commutative diagram in the abelian category $\mathbf{E}$



(74)

with $C_n = H_n(\mathbf{C}u_\alpha)$ and $K_n = H_n(\mathbf{K}v_\alpha)$.

Its rows and columns are exact, except - possibly - the lowest front row and the left back column, which are just known to be of order two. Moreover, in each row or column, each term C_n or K_n has at least two terms on each side (possibly null). By a sort of variation of the 3×3-Lemma, it follows that all $H_n(i)$ are iso. (Of course, one only needs to verify this for modules, by diagram chasing.) ■

5.5. Definition and Theorem. (Weak exactness)

In the general setting mentioned at the end of 5.1, let an h-differential sequence $(f, g; \alpha)$ *be given, and consider its diagram (24). The following conditions are equivalent:*

(a) the map $u\colon \mathbf{K}cf \to \mathbf{K}g$ *is a weak equivalence (with respect to* H_**),*

(b) $H_n(v_\alpha)\colon H_n(\mathbf{C}f) \to Hn(C)$ *is iso for* $n > 0$*, and mono for* $n = 0$*,*

(c) the object $H^+(f, g; \alpha) = \mathbf{K}v_\alpha$ *is weakly null.*

(d) the map $v\colon \mathbf{C}f \to \mathbf{C}kg$ *is a weak equivalence (with respect to* H_**),*

(e) $H_n(u_\alpha)\colon H_n(B) \to H_n(\mathbf{K}g)$ *is iso for* $n < p$*, and epi for* $n = p$*,*

(f) the object $H^-(f,g;\alpha) = \mathbf{C}u_\alpha$ is weakly null,

If they are satisfied, the sequence $(f,g;\alpha)$ will be said to be w-exact. This notion is strictly weaker than h-exactness, provided that $p \geq 2$ and $\mathbf{E}$ has some non-split short exact sequence.

PROOF. The preceding lemma shows that (c) and (f) are equivalent, while the equivalence of (b) and (c) has been considered in 5.3. Therefore, we only need to prove that (a) and (b) are equivalent. This follows from the Five Lemma in $\mathbf{E}$: take the diagram

$$\begin{array}{ccccc} \mathbf{K}cf & \xrightarrow{kcf} & A & \xrightarrow{cf} & \mathbf{C}f \\ {\scriptstyle u}\downarrow & & \| & & \downarrow{\scriptstyle v_\alpha} \\ \mathbf{K}g & \xrightarrow[kg]{} & A & \xrightarrow[g]{} & C \end{array} \tag{75}$$

and apply the (natural) fibre-cofibre homology sequence (73), to cf and g

$$\begin{array}{cccccccccccccc} 0 \to & H_p(\mathbf{K}cf) & \to & H_p(A) & \cdots & H_0(\mathbf{K}cf) & \to & H_0(A) & \to & H_0(\mathbf{C}f) & \longrightarrow & 0 & & \\ & \downarrow{\scriptstyle u_*} & & \| & & \downarrow{\scriptstyle u_*} & & \| & & \downarrow{\scriptstyle (v_\alpha)_*} & & \downarrow & & \\ 0 \to & H_p(\mathbf{K}g) & \to & H_p(A) & \cdots & H_0(\mathbf{K}g) & \to & H_0(A) & \to & H_0(C) & \to & H_0(\mathbf{C}g) & \to & 0 \end{array} \tag{76}$$

Finally, it suffices to consider a sequence $0 \to A \to 0$, which is h-exact (resp. w-exact) if and only if $\mathbf{A}$ is nullhomotopic (resp. w-null), which - under our assumptions - are indeed non-equivalent conditions by a remark at the end of 5.3. ■

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