Theory and Applications of Categories, Vol. 45, No. 41, 2026, pp. 1732–1747.

A NOTE ON THE HUQ-COMMUTATIVITY OF NORMAL MONOMORPHISMS

JAMES RICHARD ANDREW GRAY AND TAMAR JANELIDZE-GRAY

Abstract. We give a new criterion for when two Bourn-normal monomorphisms with the same codomain in a unital category Huq-commute. We use this to prove that in a unital category, satisfying the (additional) condition that a morphism is a monomorphism if and only if its kernel is a zero morphism, two Bourn-normal monomorphisms with the same codomain Huq-commute as soon as they have trivial pullback. As corollaries, we show that several facts known only in the protomodular context are in fact true in more general contexts.

1. Introduction

It is well known and easy to prove that if K and L are normal subgroups of a group G and $K \cap L = 0$, then each element in K commutes with each element of L. This fact has several known generalizations to categories.

An immediate generalization is obtained in the context where there is a suitable notion of a commutator $[-,-]$ defined for normal subobjects (which is commutative and) satisfies the property that if K, L are normal subobjects of an object X, then $[K, L] \leq K$. In this context, if K and L are normal subobjects of X, then $[K, L] \leq K \wedge L$. Therefore if $K \wedge L$ is trivial it immediately follows that $[K, L]$ is trivial, which implies that K and L commute. This holds for the Huq commutator in a normal unital finitely cocomplete category (see [2] for unital categories, [16] for normal categories, and e.g. [17] for the existence of Huq commutators in this context).

A different generalization was obtained by D. Bourn (Theorem 11 in [4]) in the context of pointed protomodular categories [3] (also introduced by D. Bourn): he proved that if k and l are Bourn-normal monomorphisms with the same codomain and the meet of k and l is 0, then k and l Huq-commute [12]. Recall that in a pointed finitely complete category, a Bourn-normal monomorphism is essentially the zero class of an internal equivalence relation, that is, $n : X \to Y$ is Bourn-normal if there exists an equivalence relation $r_1, r_2 : R \to Y$ such that $n = r_2 \mathrm{ker}(r_1)$—where $\mathrm{ker}(r_1) : \mathrm{Ker}(r_1) \to R$ is the kernel of r_1 (see Definition 6 of [4]).

One of our main aims in this paper is to show that there is a single categorical context, and single proof that implies these two different known facts. In fact we show (Theorem

Received by the editors 2022-06-27 and, in final form, 2026-06-28.
Transmitted by Tim van der Linden. Published on 2026-07-02.
2020 Mathematics Subject Classification: 18E13, 18E08, 18E99.
Key words and phrases: unital category, Huq-commutes, subtractive category, normal category.

4.9) that Bourn's result is true in the wider context of a unital category satisfying Condition 4.1, which simply requires a morphism to be a monomorphism as soon as its kernel is zero. This context is sufficiently wide so that it also includes every normal unital category, which implies that the former result also becomes a special case. In doing so, we produce a new criterion (Theorem 3.2) for when a pair of Bourn-normal monomorphisms commute in a unital category, which closely resembles Proposition 2.6.13 of [2]. Note that (i) in the normal unital (non-exact) context, Theorem 4.9 is stronger than the categorical generalization mentioned in the second paragraph, since not every Bourn-normal monomorphism is normal; (ii) there are examples of categories where our results are applicable which are not regular nor protomodular, and hence neither of the known results apply (see Example 4.8 below).

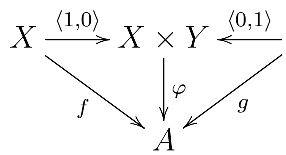
We briefly study Condition 4.1, and in particular: (i) we explain that it is a special case of a known condition (see Remark 4.2) and that it together with regularity is easily equivalent to normality (Proposition 4.4); (ii) we give examples of strongly unital [3] categories which satisfy Condition 4.1 (see Examples 4.7 and 4.8 below), some of which are not normal categories; (iii) we characterize Condition 4.1 in terms of the fibration of points (Proposition 4.5). Using in part this characterization, we show that in a pointed Mal'tsev category [7] satisfying Condition 4.1, the join of Bourn-normal monomorphisms, with the same codomain and trivial meet, exists and is Bourn-normal (Theorem 4.12). In addition, we show that the characterization of abelian objects, via the normality of their diagonal in the product (Proposition 3.2.14 of [2]), lifts from pointed protomodular categories to strongly unital categories [2] satisfying Condition 4.1 (see Proposition 4.13). In particular this means that the above mentioned fact holds in the quasi-variety consisting of those algebras which have a unique constant 0 and two binary operations s and p satisfying the following (quasi) identities: $s(x,0)=x$, $s(x,x)=0$, $p(x,0)=x=p(0,x)$ and $s(x,y)=0 \Rightarrow x=y$.

2. Preliminaries

In this section we recall the necessary definitions and preliminary facts, and introduce the notation we will use.

For a pointed category $\mathbb{C}$ we write 0 for the zero object as well as for each zero morphism between each pair of objects. For objects X and Y, we will often write $\pi_1 : X\times Y \to X$ and $\pi_2 : X\times Y \to Y$ for the first and second product projections (when they exist), and for morphisms $f : W \to X$ and $g : W \to Y$, we will write $\langle f,g\rangle : W \to X\times Y$ for the unique morphism with $\pi_1\langle f,g\rangle = f$ and $\pi_2\langle f,g\rangle = g$. Recall that a category $\mathbb{C}$ is unital if it is pointed, finitely complete, and for each pair of objects X and Y the unique morphisms $\langle 1,0\rangle : X \to X\times Y$ and $\langle 0,1\rangle : Y \to X\times Y$ are jointly strongly epimorphic.

We say that morphisms $f : X \to A$ and $g : Y \to A$ in a unital category $\mathbb{C}$ Huq-commute if there exists a morphism $\varphi : X \times Y \to A$ making the diagram



(1)

commute. The morphism φ is called the cooperator of f and g, which is unique since $\mathbb{C}$ is unital. A morphism $f : X \to A$ is called a central monomorphism if it is a monomorphism and it Huq-commutes with 1_A.

We will also need the following lemmas (see e.g., [2] and the references therein):

2.1. Lemma. *For $u : X' \to X$, $v : Y' \to Y$, $f : X \to A$ and $g : Y \to A$ morphisms in a unital category $\mathbb{C}$ and $m : A \to B$ a monomorphism.*

(i) The morphisms f and g Huq-commute if and only if the morphisms g and f Huq-commute;

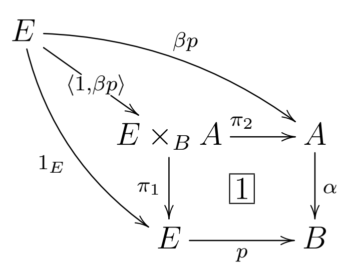
(ii) The morphisms mf and mg Huq-commute if and only if the morphisms f and g Huq-commute;

(iii) If the morphisms f and g Huq-commute, then so do the morphisms fu and gv.

2.2. Lemma. *For $f : X \to A$, $g : Y \to A$, $f' : X' \to A'$ and $g' : Y' \to A'$ in a unital category $\mathbb{C}$, the morphisms $f \times f'$ and $g \times g'$ Huq-commute if and only if both the morphisms f and g, and the morphisms f' and g' Huq-commute.*

2.3. Lemma. *For $f : X \to A$, $g : Y \to A$, $f' : X \to A'$ and $g' : Y \to A'$ in a unital category $\mathbb{C}$, the morphisms $\langle f, f' \rangle$ and $\langle g, g' \rangle$ Huq-commute if and only if f and g Huq-commute, and f' and g' Huq-commute.*

Recall that for a category $\mathbb{X}$ and an object B in $\mathbb{X}$, the category $\mathbf{Pt}_{\mathbb{X}}(B)$ of points, in the sense of D. Bourn, has as objects triples (A, α, β), where A is an object in $\mathbb{X}$, and $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{X}$ such that $\alpha\beta = 1_B$. A morphism f from (A, α, β) to (A', α', β') in $\mathbf{Pt}_{\mathbb{X}}(B)$ is a morphism $f : A \to A'$, such that $\alpha' f = \alpha$ and $f\beta = \beta'$. Furthermore, a morphism $p : E \to B$ in $\mathbb{X}$ determines a pullback functor $p^* : \mathbf{Pt}_{\mathbb{X}}(B) \to \mathbf{Pt}_{\mathbb{X}}(E)$ which sends (A, α, β) in $\mathbf{Pt}_{\mathbb{X}}(B)$ to $(E \times_B A, \pi_1, \langle 1, \beta p \rangle)$ in $\mathbf{Pt}_{\mathbb{X}}(E)$, with objects and morphisms defined as in the following commutative diagram



in which $\boxed{1}$ is a pullback. When $\mathbb{X}$ is a pointed category, pullback functors along morphisms of the form $0 \to B$ are essentially the same as kernel functors $\mathrm{Ker}_B : \mathbf{Pt}_{\mathbb{X}}(B) \to \mathbb{X}$. Recall that a finitely complete category $\mathbb{C}$ is protomodular [3] if for each morphism p the pullback functor p^* reflects isomorphisms.

3. Huq-commutativity of normal monomorphisms in unital categories

Throughout this section we assume that $\mathbb{C}$ is a unital category, $k : X \to A$ and $l : Y \to A$ are monomorphisms, $r_1, r_2 : R \to A$ and $s_1, s_2 : S \to A$ are equivalence relations, and $\kappa : X \to R$ and $\lambda : Y \to S$ are morphisms in $\mathbb{C}$ such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\kappa} & R \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \langle r_1, r_2 \rangle} \\ A & \xrightarrow[\langle 1,0 \rangle]{} & A \times A \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\lambda} & S \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \langle s_1, s_2 \rangle} \\ A & \xrightarrow[\langle 0,1 \rangle]{} & A \times A \end{array} \tag{2}$$

are pullbacks. Note that this amounts to saying k and l are Bourn-normal (see e.g. the introduction of [10] for an explanation of why this is the case). This includes, in particular, the case when k and l are the kernels of some morphisms f and g: in this case, r_1, r_2 and s_1, s_2 can be constructed as the kernel pairs of f and g respectively, and κ and λ are the unique morphisms with $r_1\kappa = k$, $r_2\kappa = 0$, $s_1\lambda = 0$, and $s_2\lambda = l$.

We will need the relation $R\square^0 S$, which can be seen as a pointed version of $R\square S$ introduced by A. Carboni, M.C. Pedicchio and N. Pirovano in [8]. If $\mathbb{C}$ were a variety of universal algebras, then

$$R\square S = \{(a,b,c,d) \in A^4 | (a,b),(c,d) \in R \text{ and } (a,c),(b,d) \in S\}$$

and an element (a,b,c,d) in $R\square S$ can be displayed as

$$\begin{array}{ccc} a & \xrightarrow{R} & b \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ c & \xrightarrow[R]{} & d. \end{array}$$

In the same context

$$R\square^0 S = \{(x,a,y) \in X \times A \times Y \,|\, (k(x),a) \in S \text{ and } (a,l(y)) \in R\}.$$

Note that an element (x,a,y) in $R\square^0 S$ can, after identifying $k(x)$ and x, and $l(y)$ and y, be displayed as follows

$$\begin{array}{ccc} x & \xrightarrow{R} & 0 \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ a & \xrightarrow[R]{} & y. \end{array}$$

Categorically the relation $R\square^0 S$ can be built via the pullbacks

$$\begin{array}{ccccccccc} & & & & P & & & & \\ & & & \swarrow^{p_1} & & \searrow^{p_2} & & & \\ & & S & & & & R & & \\ & \swarrow^{s_1} & & \searrow^{s_2} & & \swarrow^{r_1} & & \searrow^{r_2} & \\ A & & & & A & & & & A \end{array} \tag{3}$$

$$\begin{array}{ccc} R\square^0 S & \xrightarrow{\theta} & X \times Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle k\times l} \\ P & \xrightarrow[\langle s_1p_1, r_2p_2\rangle]{} & A \times A \end{array} \tag{4}$$

or directly as the limit of the outer arrows of what is easily seen to be a limiting cone

$$\begin{array}{ccccc} A & \xleftarrow{k} & X & & \\ {\scriptstyle s_1}\uparrow & & \uparrow{\scriptstyle \pi_1\theta} & & \\ S & \xleftarrow{p_1\psi} & R\square^0 S & \xrightarrow{\pi_2\theta} & Y \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle p_2\psi} & & \downarrow{\scriptstyle l} \\ A & \xleftarrow{r_1} & R & \xrightarrow{r_2} & A. \end{array} \tag{5}$$

Let $\alpha : X \to R\square^0 S$ and $\beta : Y \to R\square^0 S$ be the unique cone morphisms induced by the cones

$$\begin{array}{ccccc} A & \xleftarrow{k} & X & & \\ {\scriptstyle s_1}\uparrow & & \uparrow{\scriptstyle 1_X} & & \\ S & \xleftarrow{e_S k} & X & \xrightarrow{0} & Y \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle l} \\ A & \xleftarrow{r_1} & R & \xrightarrow{r_2} & A \\ A & \xleftarrow{k} & X & & \\ {\scriptstyle s_1}\uparrow & & \uparrow{\scriptstyle 0} & & \\ S & \xleftarrow{\kappa} & Y & \xrightarrow{1_Y} & Y \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle e_R l} & & \downarrow{\scriptstyle l} \\ A & \xleftarrow{r_1} & R & \xrightarrow{r_2} & A. \end{array} \tag{6}$$

Note that, in particular, it follows that α and β are morphisms making the two triangles in the diagram

$$\begin{array}{ccccc} & & R\square^0 S & & \\ & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle\theta} & \overset{\beta}{\nwarrow} & \\ X & \xrightarrow[\langle 1,0\rangle]{} & X\times Y & \xleftarrow[\langle 0,1\rangle]{} & Y, \end{array}$$

where θ is defined as in Diagram 4, commute. Since $\mathbb{C}$ is a unital category, this implies (see e.g. Theorem 1.2.12 of [2]):

3.1. Proposition. *The morphism θ in (3) is a strong epimorphism.* ■

It seems worth mentioning that if $\mathbb{C}$ is a variety, α and β are the maps defined by

$$x \overset{\alpha}{\longmapsto} \begin{array}{ccc} x & \xrightarrow{R} & 0 \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ x & \xrightarrow[R]{} & 0 \end{array} \qquad\qquad y \overset{\beta}{\longmapsto} \begin{array}{ccc} 0 & \xrightarrow{R} & 0 \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ y & \xrightarrow[R]{} & y. \end{array}$$

Using in part the previous fact, we are now ready to state and prove our criterion for the Huq-commutativity of Bourn-normal monomorphisms.

3.2. Theorem. *In a unital category $\mathbb{C}$ the following conditions are equivalent:*

(a) $k : X \to A$ and $l : Y \to A$, as defined in (2), Huq-commute;

(b) $\alpha : X \to R\square^0 S$ and $\beta : Y \to R\square^0 S$, as defined in (6), Huq-commute;

(c) $\theta : R\square^0 S \to X \times Y$ is a split epimorphism of cospans with domain $(R\square^0 S, \alpha, \beta)$ and codomain $(X \times Y, \langle 1, 0\rangle, \langle 0, 1\rangle)$.

Proof. Let $m : R\square^0 S \to X \times A \times Y$ be the morphism defined by $m = \langle \pi_1\theta, s_2p_1\psi, \pi_2\theta\rangle$. An easy calculation shows that m is a monomorphism. Noting that $m\alpha = \langle 1, k, 0\rangle$ and $m\beta = \langle 0, l, 1\rangle$, it follows from Lemma 2.1 that α and β Huq-commute if and only if $\langle 1, k, 0\rangle$ and $\langle 0, l, 1\rangle$ Huq-commute. However, by Lemma 2.3 this latter condition is equivalent to requiring k and l to Huq-commute. This proves $(a) \Leftrightarrow (b)$. To prove that $(b) \Rightarrow (c)$ we note that (b) is equivalent to requiring that there is a morphism $\sigma : X \times Y \to R\square^0 S$ making the upper part of the diagram

$$\begin{array}{ccccc} & & X\times Y & & \\ & \overset{\langle 1,0\rangle}{\nearrow} & \downarrow{\scriptstyle\sigma} & \overset{\langle 0,1\rangle}{\nwarrow} & \\ X & \xrightarrow{\alpha} & R\square^0 S & \xleftarrow{\beta} & Y \\ & \underset{\langle 1,0\rangle}{\searrow} & \downarrow{\scriptstyle\theta} & \underset{\langle 0,1\rangle}{\swarrow} & \\ & & X\times Y & & \end{array} \tag{7}$$

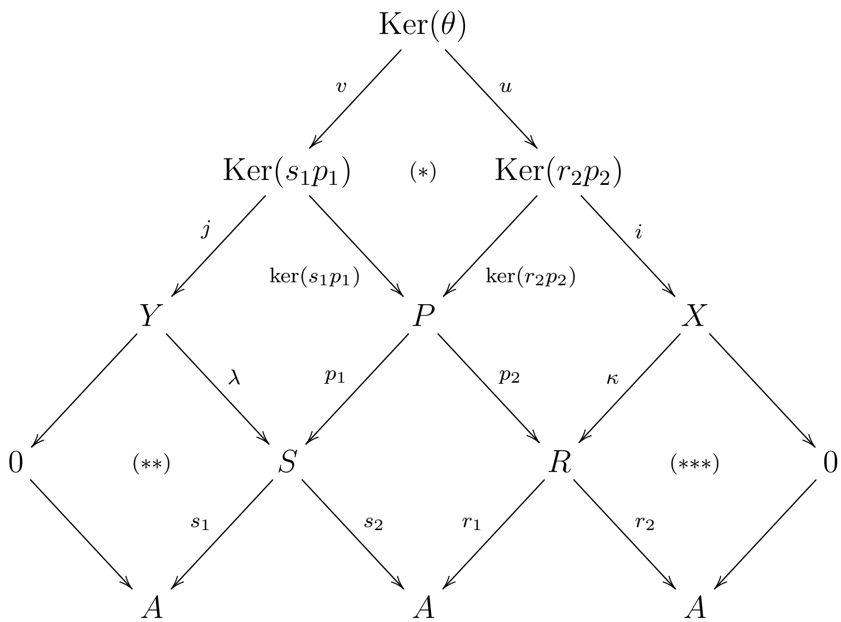
commute. However, since $\langle 1,0\rangle$ and $\langle 0,1\rangle$ are jointly epimorphic any such morphism must satisfy $\theta\sigma = 1_{X\times Y}$ and so (c) holds. The converse is immediate, since (c) implies that there is a morphism σ making the upper part of (7) commute, and as mentioned (b) is equivalent to the existence of such a morphism. ■

3.3. Lemma. *The objects* $\mathrm{Ker}(\theta)$ *and* $X\times_A Y$*, where* $X\times_A Y$ *is the pullback of* $k : X \to A$ *and* $l : Y \to A$*, are isomorphic.*

Proof. Note that since (4) is a pullback, it follows that

$$\mathrm{Ker}(\theta) \cong \mathrm{Ker}(\langle s_1p_1, r_2p_2\rangle) \cong \mathrm{Ker}(s_1p_1) \wedge \mathrm{Ker}(r_2p_2). \tag{8}$$

Now consider the diagram



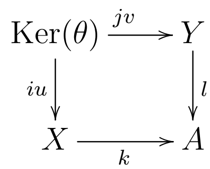
(9)

consisting of the diagram (3) and in which:

- i and j are the unique morphisms such that $\lambda j = p_1\mathrm{ker}(s_1p_1)$ and $\kappa i = p_2\mathrm{ker}(r_2p_2)$ (which exists because the squares $(**)$ and $(***)$ are pullbacks – see the diagrams in (2) above);

- u and v are the unique morphisms making $(*)$ in the diagram above, commute (which exists due to the isomorphism (8)).

Since each diamond in (9) is a pullback and $r_1\kappa = k$ and $s_2\lambda = l$, it follows that the diagram, which is essentially $X \wedge Y$ in A,



is also a pullback, and therefore, $\mathrm{Ker}(\theta) \cong X \times_A Y$ as desired. ■

4. A weakening of normality

Recall that a regular category [1] is normal [16] if and only if every regular epimorphism is a normal epimorphism. The aim of this section is to study a weakening of normality defined for pointed categories.

Let $\mathbb{X}$ be a pointed category. Consider the condition:

4.1. CONDITION. *A morphism $f : A \to B$ in $\mathbb{X}$ is a monomorphism if and only if* 0 *is the kernel of f.*

4.2. REMARK. *Condition 4.1 is not new: a pointed category $\mathbb{X}$ satisfies Condition 4.1 if and only if each reflexive relation in $\mathbb{X}$ satisfies what was called Condition $(*\pi_0)$ in [9], with respect to the ideal of zero morphisms.*

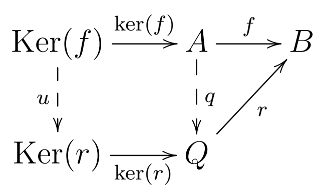
Rephrasing Proposition 3.1.21 of [2] we obtain:

4.3. PROPOSITION. *Every pointed protomodular category satisfies Condition 4.1.*

The following proposition follows from Corollary 2.3 of [9], however we give a direct proof in order to avoid introducing notation and terminology that would not otherwise be needed in this paper.

4.4. PROPOSITION. *A pointed regular category $\mathbb{X}$ with cokernels is normal if and only if it satisfies Condition 4.1.*

PROOF. It is immediate that a normal category satisfies Condition 4.1. To see why suppose $f = me$ where m is a monomorphism, e is a normal epimorphism, and f has trivial kernel. Then e has trivial kernel and hence must be an isomorphism. It remains to prove the converse. Suppose $f : A \to B$ is a regular epimorphism and consider the diagram



in which q is the cokernel of $\ker(f)$, r is the unique morphism with $rq = f$, and u is the unique morphism with $\ker(r)u = q\ker(f)$. It easily follows that the left hand square is a pullback, therefore u is a regular epimorphism. Since $\ker(r)u = q\ker(f) = 0$, it follows that $\ker(r) = 0$, and therefore r is a monomorphism. Since r is also a regular epimorphism, the latter implies that r is an isomorphism. ∎

4.5. PROPOSITION. *For a pointed finitely complete category $\mathbb{X}$ the following conditions are equivalent:*

(a) The category $\mathbb{X}$ satisfies Condition 4.1;

(b) For each object B in $\mathbb{X}$ the functor Ker_B reflects terminal objects;

(c) For each object B in $\mathbb{X}$ the functor Ker_B *reflects monomorphisms;*

(d) For each object B in $\mathbb{X}$ the category $\mathbf{Pt}_{\mathbb{X}}(B)$ satisfies Condition 4.1;

(e) For each morphism $p : E \to B$ in $\mathbb{X}$ the functor $p^* : \mathbf{Pt}_{\mathbb{X}}(B) \to \mathbf{Pt}_{\mathbb{X}}(E)$ reflects terminal objects;

(f) For each morphism $p : E \to B$ in $\mathbb{X}$ the functor $p^* : \mathbf{Pt}_{\mathbb{X}}(B) \to \mathbf{Pt}_{\mathbb{X}}(E)$ reflects monomorphisms.

Proof. For a morphism $f : A \to B$ in $\mathbb{X}$, note that:

(i) $f : A \to B$ is a monomorphism if and only if in the pullback diagram

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

π_1 is an isomorphism.

(ii) The morphism π_1 is an isomorphism whenever $(A \times_B A, \pi_1, \langle 1, 1\rangle)$ is a terminal object in $\mathbf{Pt}_{\mathbb{X}}(A)$;

(iii) The kernel of f is isomorphic to the kernel of π_1.

Combining these observations we see that (a)⇔(b). For any functor F between pointed categories which preserves terminal objects, since morphisms into the terminal object are necessarily split epimorphisms, one easily shows that if F reflects monomorphisms, then it reflects terminal objects. Therefore (f)⇒(e) and (c)⇒(b). Recalling that if a composite of functors FG reflects some property and F preserves it, then G reflects it, and noting that kernel functors certainly preserve terminal objects, one easily sees that (b)⇒(e) (just note that for each morphism $p : E \to B$ the functor $\mathrm{Ker}_E \circ p^*$ is isomorphic to Ker_B). Since each pullback functor between points along a morphism in a category of points of $\mathbb{X}$ is up to isomorphism a pullback functor between points for $\mathbb{X}$ it follows that (e)⇒(d). For a functor F between pointed finitely complete categories satisfying Condition 3.4, preserving limits and reflecting terminal objects, if $F(f)$ is a monomorphism then $F(\mathrm{Ker}(f)) \cong \mathrm{Ker}(F(f)) \cong 0$ and hence $\mathrm{Ker}(f) \cong 0$ which forces f to be a monomorphism. This proves (e)⇒(f) since we already know that (e)⇒(d). The proof is completed by noting that trivially (f)⇒(c) and (d)⇒(a). ∎

Recall that a pointed finitely complete category $\mathbb{X}$ is subtractive [15] if every relation (R, r_1, r_2) on an object X in $\mathbb{X}$ has the property that if $\langle 1, 0\rangle : X \to X \times X$ and $\langle 1, 1\rangle : X \to X \times X$ factor through $\langle r_1, r_2\rangle : R \to X \times X$, then so does $\langle 0, 1\rangle : X \to X \times X$. A pointed finitely complete category $\mathbb{X}$ is strongly unital [2] if for each object X in $\mathbb{X}$, the morphisms $\langle 1, 0\rangle : X \to X \times X$ and $\langle 1, 1\rangle : X \to X \times X$ are jointly strongly epimorphic. If $\mathcal{V}$ is a (quasi)-variety of universal algebras and $\mathbb{X}$ is a category, then we write $\mathcal{V}(\mathbb{X})$ for the category of internal $\mathcal{V}$-algebras in $\mathbb{X}$.

4.6. Proposition. *Let $\mathcal{V}$ be a (quasi)-variety of universal algebras considered as a category, and let $\mathbb{X}$ be a category with finite limits. If $\mathcal{V}$ satisfies Condition 4.1, is unital, or is subtractive, then $\mathcal{V}(\mathbb{X})$ satisfies Condition 4.1, is unital, or is subtractive, respectively.*

Proof. Since the Yoneda embedding $Y : \mathbb{X} \to \mathbf{Set}^{\mathbb{X}^{\mathrm{op}}}$ is full and faithful, and preserves and reflects limits and $\mathcal{V}(\mathbf{Set}^{\mathbb{X}^{\mathrm{op}}}) = \mathcal{V}^{\mathbb{X}^{\mathrm{op}}}$, there is an induced functor $\tilde{Y} : \mathcal{V}(\mathbb{X}) \to \mathcal{V}^{\mathbb{X}^{\mathrm{op}}}$ which is full and faithful, and preserves and reflects limits. The claim now follows by noting that (i) limits in functor categories are componentwise (provided the limits of components exist); (ii) a morphism in a functor category is an isomorphism whenever each one of its components is an isomorphism; and (iii) the above mentioned conditions can be phrased in the form, for each diagram where certain parts are limits, a certain morphism is an isomorphism.

■

Recall that a category is strongly unital if and only if it is unital and subtractive (Proposition 3 [15]). Recall also that a category is ideal determined [13] if it is (what is now know as) normal and the regular image of normal monomorphisms along regular epimorphisms are normal monomorphisms in it. Recall also that an ideal determined variety is necessarily a subtractive variety [11] and hence a subtractive category.

4.7. Example. *Recall that an implication algebra is a triple $(X, \to, 1)$ where X is a set, $\to$ is a binary operation and 1 is constant satisfying the axioms: $(x \to y) \to x = x$, $(x \to y) \to y = (y \to x) \to x$, $x \to (y \to z) = y \to (x \to z)$, $1 \to 1 = 1$. H. P. Gumm and A. Ursini showed in [11] that the variety of implication algebras form an ideal determined variety of universal algebras which is not congruence permutable. This means that the category of implication algebras is ideal determined but not Mal'tsev [13]. Since the two element Boolean algebra $2 = (\{0, 1\}, \to, 1)$ forms an implication algebra and $\{(0, 1), (1, 0), (1, 1)\}$ is a sub-algebra of 2×2 containing $\langle 0, 1\rangle(X) = \{(1, 0), (1, 1)\}$ and $\langle 1, 0\rangle(X) = \{(0, 1), (1, 1)\}$, we see that it is not a unital category. However, adding an independent binary operation $*$ satisfying $x * 1 = 1 * x = x$ will produce a unital (via Theorem 1.2.15 of [2]) ideal determined category, and hence a strongly unital normal category. We leave as open problems whether this latter variety is Mal'tsev or not and if there exists a normal strongly unital variety which is not Mal'tsev. On the other hand the previous proposition tells us that such internal algebras in a category with finite limits always produce a category which is strongly unital and satisfies Condition 4.1.*

4.8. Example. *Let $\mathcal{V}$ be the quasi-variety of universal algebras, with terms $p(x, y)$ and $s(x, y)$ satisfying $p(x, 0) = p(0, x) = x$, $s(x, 0) = x$, $s(x, x) = 0$, and $s(x, y) = 0 \Rightarrow x = y$. According to Proposition 3 of [15] $\mathcal{V}$ is a strongly unital category. Since for a morphism f in $\mathcal{V}$ with trivial kernel, $f(x) = f(y) \Rightarrow f(s(x, y)) = 0 \Rightarrow s(x, y) \in \mathrm{Ker}(f) \Rightarrow s(x, y) = 0 \Rightarrow x = y$ it follows that $\mathcal{V}$ is normal (this is closely related to Corollary 1.7 of [11]). In fact, it turns out that every regular epimorphism in $\mathcal{V}$ is an effective descent morphism, and $\mathcal{V}$ is not Mal'tsev. For the first fact, by Corollary 2.7.2 of [14] it is sufficient to* *consider a pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

in the variety generated by $\mathcal{V}$, where $E \times_B A$, E and B are in $\mathcal{V}$, and show that A is in $\mathcal{V}$. Suppose that a_1, a_2 are in A and $s(a_1, a_2) = 0$. It follows that $s(f(a_1), f(a_2)) = 0$ and hence $f(a_1) = f(a_2)$. Since p is surjective it follows that there exists e in E such that $p(e) = f(a_1) = f(a_2)$. Therefore (e, a_1) and (e, a_2) are in $E \times_B A$. Since $s((e, a_1), (e, a_2)) = (s(e, e), s(a_1, a_2)) = (0, 0)$ it follows that $(e, a_1) = (e, a_2)$ and hence $a_1 = a_2$. To show that it is not Mal'tsev, let A be the algebra with underlying set $\mathbb{N} \times \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) and with

$$p(x, y) = s(x, y) = \begin{cases} x + y & x \neq y \\ 0 & x = y \end{cases}$$

where $+$ is defined componentwise, and let

$$R = \{((a + k + c + 2w, b + c + 2x), (a + d + 2y, b + k + d + 2z)) \\ \mid a, b, c, d, k, w, x, y, z \in \mathbb{N}\}.$$

Now note that R is reflexive relation (just set $k = c = d = w = x = y = z = 0$) and closed under componentwise addition. It is not symmetric since $((1, 0), (0, 1))$ is in R (just set $a = b = c = d = w = x = y = z = 0$ and $k = 1$), but $((0, 1), (1, 0))$ is not it R. To see why note that if $a + k + c + 2w = 0$ and $b + k + d + 2z = 0$, then in particular $b = c = 0$. This means that $b + c + 2x = 2x \neq 1$. Now suppose that for some $a, b, c, d, k, w, y, z, a', b', c', d', k', w', x', y', z' \in \mathbb{N}$

$$(a + k + c + 2w, b + c + 2x) = (a' + k' + c' + 2w', b' + c' + 2x'), \tag{10}$$

and let

$$\begin{aligned} r &= (a + k + c + 2w, b + c + 2x) \\ r' &= (a' + k' + c' + 2w', b' + c' + 2x') \\ s &= (a + d + 2y, b + k + d + 2z) \\ s' &= (a' + d' + 2y', b' + k' + d' + 2z') \\ j &= ((0, 0), s + s'). \end{aligned}$$

Since $p((r, s), (r', s'))$ is in $\{((0, 0), (0, 0)), j\}$, to show that R is a subalgebra of the product, it is sufficient (up to symmetry) to show that j is in R. From (10) it follows that

$$a' = a + k - k' + c - c' + 2(w - w') \text{ and } b' = b + c - c' + 2(x - x'),$$

and hence

$$\begin{aligned} s+s' &= (a+a'+d+d'+2(y+y'), b+b'+k+k'+d+d'+2(z+z')) \\ &= (k-k'+c-c'+d+d'+2(a+y+y'+w-w'), \\ &\quad\; k-k'+c-c'+d+d'+2(b+k'+z+z'+x-x')) \\ &= (\bar{d}+2\bar{y}, \bar{d}+2\bar{z}) \end{aligned}$$

where $\bar{d} = k-k'+c-c'+d+d'$, $\bar{y} = a+y+y'+w-w'$ *and* $\bar{z} = b+k'+z+z'+x-x'$. *Although we only know that* $\bar{d}$, $\bar{y}$ *and* $\bar{z}$ *are in* $\mathbb{Z}$, *since* $\bar{d}+2\bar{y} \geq 0$ *and* $\bar{d}+2\bar{z} \geq 0$ *one can trivially choose* $\tilde{d}$, $\tilde{y}$ *and* $\tilde{z}$ *in* $\mathbb{N}$ *such that* $\bar{d}+2\bar{y} = \tilde{d}+2\tilde{y}$ *and* $\bar{d}+2\bar{z} = \tilde{d}+2\tilde{z}$, *(simply set* $\bar{d} = 2p+\tilde{d}$ *where* $p \in \mathbb{Z}$ *and* $\tilde{d} \in \{0,1\}$, *and then set* $\tilde{y} = \bar{y}+p$ *and* $\tilde{z} = \bar{z}+p$*), proving that* j *is in* R.

As before, by Proposition 4.6, we obtain that such internal algebras in a finitely complete category will produce strongly unital categories satisfying Condition 4.1. In particular, if the base category is the product of the category of sets with the opposite category of the quasi-variety $\mathcal{W}$ *of abelian groups satisfying* $4x = 0 \Rightarrow 2x = 0$, *then the resulting category will, on the one hand, not be Mal'tsev since* $\mathcal{V}$ *is not, and on the other hand not be regular (and hence not normal) since* $\mathcal{V}(\mathcal{W}^{\mathrm{op}}) = \mathcal{W}^{\mathrm{op}}$ *which is not regular.*

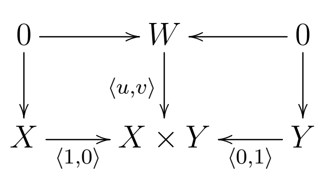
4.9. Theorem. *Let* k *and* l *be Bourn-normal monomorphisms in a unital category* $\mathbb{C}$ *satisfying Condition 4.1. If* k *and* l *have trivial pullback, then* k *and* l *commute.*

Proof. If k and l have trivial pullback, then by Lemma 3.3 the morphism θ has trivial kernel and hence it is a monomorphism, because of Condition 4.1. Moreover, since by Proposition 3.1 the morphism θ is a strong epimorphism it follows that it is an isomorphism. The claim now follows from Theorem 3.2. ∎

For $f, g : X \to A$ morphisms in a unital category $\mathbb{C}$ such that f and g Huq-commute, we defined $f+g : X \to A$ to be the composite $\varphi\langle 1,1\rangle$ where φ is the cooperator of f and g. It is easily verified that for a morphism $h : A \to B$, the morphisms hf and hg Huq-commute and $h(f+g) = hf+hg$. If $\mathbb{C}$ is strongly unital and $f : X \to A$ is central, then there exists a unique morphism $g : X \to A$ such that $f+g = 0$ (see Theorem 1.8.19 of [2]). We write $-f$ for this unique morphism.

4.10. Lemma. *In a strongly unital category* $\mathbb{C}$ *satisfying Condition 4.1, let* $k : X \to A$ *and* $l : Y \to A$ *be monomorphisms which commute, and let* $\varphi : X \times Y \to A$ *be their cooperator. If* $\langle u, v\rangle : W \to X \times Y$ *is the kernel of* φ, *then* u *and* v *are central monomorphisms and* $ku = l(-v)$.

Proof. Since each of the squares in the following diagram



are pullbacks, it follows by Condition 4.1 that $\langle 1, 0\rangle$ and $\langle u, v\rangle$, as well as $\langle 0, 1\rangle$ and $\langle u, v\rangle$, Huq-commute. It follows from Lemma 2.2, that u and v are central. To complete the proof just note that $\langle u, 0\rangle = \langle u, v\rangle + \langle 0, -v\rangle$ and therefore

$$\begin{aligned} ku &= \varphi\langle u, 0\rangle \\ &= \varphi(\langle u, v\rangle + \langle 0, -v\rangle) \\ &= \varphi\langle u, v\rangle + \varphi\langle 0, -v\rangle \\ &= \varphi\langle 0, -v\rangle \\ &= l(-v). \end{aligned}$$

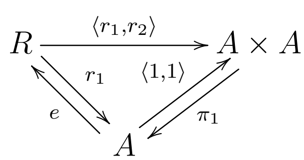
■

Recall that in a category $\mathbb{C}$ the join of monomorphisms $u : S \to A$ and $v : T \to A$ is simply the join in the preorder of monomorphisms into A (where $(S, u) \leq (T, v)$ if u factors through v). When $\mathbb{C}$ has pullbacks (of monomorphisms) this preorder admits meets and one easily checks that (I, j) is the join of (S, u) and (T, v) if it is minimal amongst those (I', j') which are larger than both (S, u) and (T, v).

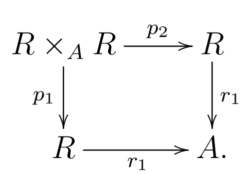
4.11. Theorem. *Let k and l be Bourn-normal monomorphisms in a strongly unital category $\mathbb{C}$ satisfying Condition 4.1. If k and l have trivial pullback, then k and l commute and their cooperator $\varphi : X \times Y \to A$ is a monomorphism, which is also their join.*

Proof. By Theorem 4.9 we know that k and l commute. It remains to show that their cooperator φ is a monomorphism and is their join. Let $\langle u, v\rangle : W \to X \times Y$ be the kernel of φ. Lemma 4.10 implies that $ku = l(-v)$ and hence $u = 0$ and $v = 0$ (since the pullback of k and l is trivial). Therefore $\langle u, v\rangle = 0$ and hence by Condition 4.1 φ is a monomorphism. The final point follows immediately from the fact that $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are jointly strongly epimorphic. ■

Recall that in the Mal'tsev context an equivalence relation $r_1, r_2 : R \to A$ is essentially the same thing as a monomorphism



in the category $\mathbf{Pt}(A)$ (see e.g. the proof of Theorem 1.2.12 of [2]). Moreover such a monomorphism $\langle r_1, r_2\rangle$ is necessarily Bourn-normal (see e.g. the introduction of [6]). To recall why, consider the pullback diagram



It follows that $\langle r_1p_1, r_2p_1\rangle, \langle r_1p_1, r_2p_2\rangle : (R\times_A R, r_1p_1, \langle e, e\rangle) \to (A\times A, \pi_1, \langle 1, 1\rangle)$ (where e is the splitting of r_1 and r_2) is an equivalence relation and the diagrams

$$\begin{array}{ccc} A\times(A\times A) & \xrightarrow{1\times\pi_2} & A\times A \\ {\scriptstyle 1\times\pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ A\times A & \xrightarrow[\pi_1]{} & A \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\langle 1, r_1e\rangle} & R\times_A R \\ {\scriptstyle \langle r_1, r_2\rangle}\downarrow & & \downarrow{\scriptstyle \langle r_1\pi_1, \langle r_2p_1, r_2p_2\rangle\rangle} \\ A\times A & \xrightarrow[1\times\langle p_2, p_1\rangle]{} & A\times(A\times A) \end{array}$$

are pullbacks.

4.12. THEOREM. *Let k and l be Bourn-normal monomorphisms in a Mal'tsev category $\mathbb{C}$ satisfying Condition 4.1. If k and l have trivial pullback, then k and l commute and their cooperator $\varphi : X\times Y \to A$ is a Bourn-normal monomorphism which is also their join.*

PROOF. By Proposition 4.5 we see that $X\wedge Y = 0$ implies $R\wedge S = 0$ when considered as subobjects of $(A\times A, \pi_1, \langle 1, 1\rangle)$ in the category of points over A. It now follows from Theorem 4.11 that $R\times S$ (in $\mathbf{Pt}(A)$) is a subobject of $(A\times A, \pi_1, \langle 1, 1\rangle)$, and hence is an equivalence relation with zero class the cooperator of k and l. ∎

Recall that an object X in a strongly unital category is called abelian if it admits the structure of an internal abelian group. Recall that this structure is unique when it exists and that it exists as soon as the morphism 1_X Huq-commutes with itself (see e.g. Corollary 1.8.20 of [2]).

4.13. PROPOSITION. *Let $\mathbb{C}$ be a strongly unital category satisfying Condition 4.1. For an object X in $\mathbb{C}$ the following conditions are equivalent:*

(a) X is abelian;

(b) $\langle 1, 1\rangle : X \to X\times X$ is a normal monomorphism;

(c) $\langle 1, 1\rangle : X \to X\times X$ is a Bourn-normal monomorphism.

PROOF. The implications (a) ⇒ (b)⇒ (c) are immediate. It is therefore sufficient to show that (a) follows from (c). Suppose X is an object in $\mathbb{C}$ and $\langle 1, 1\rangle$ is Bourn-normal. Since $\langle 1, 0\rangle$ and $\langle 1, 1\rangle$ have trivial pullback, it follows that they commute and hence we obtain a morphism ψ making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1,0\rangle} & X\times X & \xleftarrow{\langle 0,1\rangle} & X \\ \| & & \downarrow{\scriptstyle \psi} & & \| \\ X & \xrightarrow{\langle 1,0\rangle} & X\times X & \xleftarrow{\langle 1,1\rangle} & X \end{array}$$

commute. The claim now follows from Corollary 1.8.20 of [2], since $\pi_1\psi$ is a cooperator for 1_X and 1_X. ∎

4.14. Remark. *Given that abelianness is a property in a subtractive category, and abelianization is obtained by forming the cokernel of the diagonal in a regular subtractive category (provided the cokernel exists) [5], one expects that the above proposition is true in a wider context.*

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James Richard Andrew Gray
Department of Mathematical Sciences
University of Stellenbosch
Stellenbosch
South Africa.
email: jamesgray@sun.ac.za

Tamar Janelidze-Gray
Department of Mathematics and Applied Mathematics
University of Cape Town
Cape Town
South Africa.
email: tamar.janelidze-gray@uct.ac.za

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