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HOMOLOGICAL ALGEBRA OF PRO-LIE POLISH ABELIAN GROUPS

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ABSTRACT. In this paper, we initiate the study of pro-Lie Polish abelian groups from the perspective of homological algebra. We extend to this context the type-decomposition of locally compact Polish abelian groups of Hoffmann and Spitzweck, and prove that the category **proLiePAb** of pro-Lie Polish abelian groups is a thick subcategory of the category of Polish abelian groups. We completely characterize injective and projective objects in **proLiePAb**. We conclude that **proLiePAb** has enough projectives but not enough injectives and homological dimension 1. We also completely characterize injective and projective objects in the category of non-Archimedean Polish abelian groups, concluding that it has enough injectives and projectives and homological dimension 1. Injective objects are also characterized for the categories of topological torsion Polish abelian groups and for Polish abelian topological p-groups, showing that these categories have enough injectives and homological dimension 1.

1. Introduction

In this paper, we initiate the study of pro-Lie Polish abelian groups from the viewpoint of homological algebra. *Pro-Lie groups* [HM07] are the topological groups that can be written as limits of an inverse system of Lie groups. We will focus on pro-Lie groups that are furthermore *Polish*, in which case they can written as limits of an inverse sequence of Lie groups. The class of pro-Lie Polish groups is a natural extension of the class of Lie groups, and it satisfies desirable closure properties, as it is closed under countable products, closed subgroups, and quotients by closed subgroups within the class of Polish groups.

We will focus on the case of *abelian* pro-Lie Polish groups, as they form a category enriched over abelian groups, where the machinery from homological algebra can be applied. The category of abelian Polish pro-Lie groups contains all the *locally compact* Polish

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abelian groups, as well as the *non-Archimedean* Polish abelian groups (which are inverse limits of countable discrete groups). In fact, the class of pro-Lie Polish abelian groups is the smallest class of Polish abelian groups that contains all locally compact Polish abelian groups and non-Archimedean Polish abelian groups and it is closed under extensions.

The study of homological invariants in the context of topological abelian groups followed rapidly their introduction in the purely algebraic setting, going as far back as the 1940s [CE48, Nag49]; see also [Cal51, Mac57a, Bro71, FG71, Ful72, Ful70, Mos67]. This study has been most commonly undertaken for *locally compact* topological Hausdorff (Polish) abelian groups. Injective and projective objects in this category were characterized by Moskowitz in [Mos67], showing in particular that this category does *not* have enough injective and projective objects. The Yoneda Ext groups in terms of extensions were introduced and studied by Fulp and Griffith in [FG71], where it is proved that Ext^2 vanishes. The functor Ext for locally compact Hausdorff topological abelian groups was later studied from the viewpoint of derived categories by Hoffmann and Spitzweck [HS07], who recognized it as the cohomological right derived functor of Hom in the sense of Verdier [Ver77], despite the category of locally compact Hausdorff topological groups not having enough injectives or projectives.

Derived categories and functors had been traditionally studied in the context of *abelian categories*. However, this framework does not include the category of locally compact topological Hausdorff (Polish) abelian groups, as the inclusion map $\mathbb{Q} \to \mathbb{R}$ of the rationals with the discrete topology into the reals with the Euclidean topology is an arrow that is both monic and epic but not an isomorphism. This and other categories of algebraic structures endowed with a topology can be seen as *quasi-abelian categories*, where the axioms of abelian categories (such as the requirement that every epimorphism be the cokernel of its kernel) are replaced with suitable relaxations. Most of the notions and constructions usually considered in the context of abelian categories can be generalized to the quasi-abelian case. Furthermore, Schneiders describes in [Sch99], building on work of Beilinson–Bernstein–Deligne [BBD82], a canonical way to enlarge a given quasi-abelian category $\mathcal{A}$ to an abelian category $\mathrm{LH}(\mathcal{A})$ called its *left heart*.

The quasi-abelian viewpoint affords Hoffmann and Spitzwek to *refine* the functor Ext for locally compact topological (Polish) abelian groups, by seeing it as a functor to the *left heart* of the category of topological (Polish) abelian groups. The category **PAb** of Polish abelian groups, or any of its full subcategories closed by taking closed subgroups, quotients by closed subgroups, and extensions, is easily seen to be quasi-abelian. Using tools from descriptive set theory, an explicit description of its left heart as concrete category was recently provided in [Lup24] in terms of *groups with a Polish cover* and *Borel-definable group homomorphisms*; see also [BLP24].

Building on the work of Hoffmann and Spitzweck, [Lup25] gives a short purely homological proof of the fact that the functor Hom on the category **LCPAb** of locally compact Polish abelian groups has a *total* right derived functor, from which Ext^n for $n \geq 0$ is recovered by taking cohomology, and the homological dimension of **LCPAb** is 1. The latter result recovers the theorem that $\mathrm{Ext}^n = 0$ for $n \geq 2$ that had previously been obtained by

Fulp and Griffith via a much more laborious argument. The enrichment of Ext as group with a Polish cover is used in [Lup25] to completely characterize injective and projective objects in the left heart of **LCPAb** and many of its salient thick subcategories, showing in particular that LH (**LCPAb**) has no nonzero injectives.

In this paper, we extend the applications of Borel-definable methods to the category of pro-Lie Polish abelian groups. We begin with showing that the category **proLiePAb** of pro-Lie Polish abelian groups is quasi-abelian, and in fact a thick subcategory of **PAb**. In order to prove that an extension of pro-Lie Polish abelian groups is pro-Lie, we use recently developed tools from the theory of groups with a Polish cover and Borel and continuous cocycles, including a continuous selection theorem from [BLP24]. We then completely characterize projective and injective objects in **proLiePAb**, showing in particular that **proLiePAb** has enough projectives (but not enough injectives) and homological dimension 1. Thus **proLiePAb** is in some ways better-behaved than **LCPAb** from the viewpoint of homological algebra, and the total derived functor of Hom on **proLiePAb** can be defined in terms of projective resolutions, as in the case of discrete groups. In the case when the first argument is locally compact, and the second one merely pro-Lie, we prove that Ext can be regarded as a functor to the left heart of **PAb**, and hence enriched with the structure of group with a Polish cover, extending from the locally compact case the analysis in [Lup25].

We perform a similar analysis on the category $\mathbf{PAb}_{\mathrm{nA}}$ of *non-Archimedean* Polish abelian groups, which is also a thick subcategory of **PAb**. In this case, we completely characterize injective and projective objects, and show that $\mathbf{PAb}_{\mathrm{nA}}$ has enough injectives and projectives and homological dimension 1. We also completely characterize injective and projective objects in the categories of pro-p pro-Lie Polish abelian groups and topological torsion pro-Lie Polish abelian groups, and show that these categories have enough injectives and homological dimension 1.

We also extend from the locally compact case—considered by Hoffmann and Spitzweck in [HS07]—to pro-Lie Polish abelian groups the notions of type $\mathbb{Z}$, type $\mathbb{A}$, and type $\mathbb{S}^1$ groups, as well as the decomposition of an arbitrary pro-Lie Polish abelian groups in terms of groups of these types.

Finally, we show that the functor Hom : $\mathbf{LCPAb} \times \mathbf{proLiePAb} \to \mathbf{proLiePAb}$ admits a total right derived functor, extending a result from [Lup25] in the locally compact case.

The rest of this paper is divided into three sections. In Section 2 we recall some preliminary notions from category theory, including the notions of exact and quasi-abelian category and the corresponding notions of derived category and functor. In Section 3 we obtain our main results outlined above concerning the category of pro-Lie Polish abelian groups. Finally, in Section 4 we consider several natural thick (or just fully exact) subcategories of **proLiePAb**, characterizing injective and projective objects in each of them, and determining whether they have enough injectives or projectives.

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2. Category theory background

2.1. Exact categories. An *additive category* [ML98, Section VIII.2] is a category enriched over the category **Ab** of abelian groups that also has a terminal object (which is necessarily also initial, and called the *zero object*) and binary products (which are necessarily also coproducts, and called *biproducts*). A kernel-cokernel pair in an additive category $\mathcal{A}$ is a pair (f, g) of arrows in $\mathcal{A}$ such that f is the kernel of g and g is the cokernel of f. This can be seen as an object in the double arrow category $\mathcal{A}^{\cdot\to\cdot\to\cdot}$ of $\mathcal{A}$.

An *exact structure* $\mathcal{E}$ on $\mathcal{A}$ [Bü10, Definition 2.1] is a collection of short exact sequences closed under isomorphism in $\mathcal{A}^{\cdot\to\cdot\to\cdot}$ that satisfies the following axioms as well as their duals obtained by reversing all the arrows. In order to formulate the axioms, we say that a morphism f is an admissible monic (or inflation) or an admissible epic (or deflation) if it appears in an exact sequence in $\mathcal{E}$:

1. for every object A of $\mathcal{A}$, the corresponding identity arrow 1_A is an admissible monic;
2. the collection of admissible monics is closed under composition;
3. the push-out of an admissible monic along an arbitrary morphism exists and it is an admissible monic.

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is an exact structure on $\mathcal{A}$. A short exact sequence in $(\mathcal{A}, \mathcal{E})$ is by definition an element of $\mathcal{E}$. In an exact category, an isomorphism is an admissible monic and an admissible epic; see [Bü10, Remark 2.3]. Furthermore, by [Bü10, Remark 2.3, Proposition 2.15] we have the following—see [Awo06, Sections 5.2, 5.3] for the notion of pull-back along a morphism:

2.2. Lemma. *In an exact category, the pull-back of an admissible monic along an admissible epic is an admissible monic.*

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are exact categories, and $F : \mathcal{A} \to \mathcal{B}$ is a functor. One says that F is *exact* if it maps short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathcal{B}$ [Bü10, Remark 2.3, Proposition 2.15]. By [Bü10, Proposition 5.2], this implies that F preserves push-outs along admissible monics, and pull-backs along admissible epics.

2.3. Remark. In the context of finitely complete and finitely cocomplete categories, a stronger notion of exact functor is considered. If $\mathcal{C}$ and $\mathcal{D}$ are categories that are finitely complete and finitely cocomplete (i.e., have finite limits and finite colimits), then a functor $F : \mathcal{C} \to \mathcal{D}$ is called *exact* if it is finitely continuous and finitely cocontinuous (i.e., commutes with finite limits and finite colimits); see [KS06, Section 3.3]. An exact functor in the sense of [Bü10, Proposition 5.2] between finitely complete and finitely cocomplete exact categories is not necessarily exact in the sense of [KS06, Section 3.3].

Suppose that $\mathcal{A}$ is an exact category. A *fully exact* subcategory [Bü10, Definition 10.21] of $\mathcal{A}$ is a full subcategory $\mathcal{B}$ of $\mathcal{A}$ such that, whenever $0 \to B' \to A \to B'' \to 0$ is a short exact sequence in $\mathcal{A}$ with B' and B'' in $\mathcal{B}$, one also has that A is isomorphic to an object of $\mathcal{B}$. In this case, we have that $\mathcal{B}$ is an exact category, where a short exact sequence in $\mathcal{B}$ is a short exact sequence in $\mathcal{A}$ whose morphisms are in $\mathcal{B}$ [Bü10, Definition 10.20].

Let $\mathcal{A}$ be an exact category and f be a morphism in $\mathcal{A}$. Then we say that f is *admissible* [Bü10, Definition 8.1] if $f = m \circ e$ for some admissible epic e and admissible monic m. The pair (e, m) is called an admissible factorization of f and its essentially unique [Bü10, Lemma 8.4]. The notion of admissible morphism recovers the notion of admissible monic (respectively, epic) in the particular case when the given morphism is monic (respectively, epic) [Bü10, Remark 8.3]. The class of admissible morphisms is closed under push-out along admissible monics and pull-back along admissible epics [Bü10, Lemma 8.7].

2.4. Definition. *A sequence of admissible morphisms*

$$A \xrightarrow{f} B \xrightarrow{g} A'$$

in an exact category $\mathcal{A}$ is exact if

$$\bullet \xrightarrow{m} B \xrightarrow{e'} \bullet$$

is short exact, where $f = m \circ e$ and $g = m' \circ e'$ are admissible factorizations.

2.5. Derived categories. Let $\mathcal{A}$ be an additive category. One can then define the category $\mathrm{Ch}(\mathcal{A})$ of complexes, and the homotopy category $\mathrm{K}(\mathcal{A})$; see [Bü10, Section 9]. We have that $\mathrm{K}(\mathcal{A})$ has a canonical structure of *triangulated category* [Bü10, Remark 9.8], graded by the translation functor T defined by $TA = A[1]$ where $A[k]^n = A^{n+k}$ for $n, k \in \mathbb{Z}$. The distinguished triangles in $\mathrm{K}(A)$ are those isomorphic to a strict triangle, which is one of the form

$$(A, B, \mathrm{cone}(f), f, i, j)$$

for some morphism of complexes $f : A \to B$, where $i : B \to \mathrm{cone}(f)$ and $j : \mathrm{cone}(f) \to TA$ are the canonical morphisms of complexes.

A morphism $f : A \to A$ in $\mathcal{A}$ is idempotent if $f = f \circ f$. The category $\mathcal{A}$ is idempotent-complete if every idempotent morphism has a kernel. Suppose now that $\mathcal{A}$ is an *idempotent-complete exact category*. A complex A over $\mathcal{A}$ is *acyclic* [Bü10, Section 10] if it has admissible differentials and it is exact at each degree as in Definition 2.4. We let $\mathrm{N}(\mathcal{A})$ be the full subcategory of $\mathrm{K}(\mathcal{A})$ spanned by acyclic complexes. Then we have that $\mathrm{N}(\mathcal{A})$ is a *thick subcategory* [Nee01, Definition 2.1.6] of the triangulated category $\mathrm{K}(\mathcal{A})$ [Bü10, Corollary 10.11]. This means that $\mathrm{N}(\mathcal{A})$ is a *triangulated subcategory* of $\mathrm{K}(\mathcal{A})$ [Nee01, Definition 1.5.1] and contains all the direct summands of its objects.

One can thus define the corresponding Verdier quotient $\mathrm{D}(\mathcal{A}) := \mathrm{K}(\mathcal{A}) / \mathrm{N}(\mathcal{A})$ [Nee01, Theorem 2.1.8], which is called the *derived category* of $\mathcal{A}$ [Bü10, Section 10]. By definition, this is the localization $\mathrm{K}(\mathcal{A})[\Sigma^{-1}]$ where Σ is the multiplicative system in $\mathrm{K}(\mathcal{A})$

consisting of homotopy classes of chain maps whose mapping cone is acyclic, called *quasi-isomorphisms* [Bü10, Definition 10.16]. If $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is the quotient functor, then a morphism in $\mathrm{K}(\mathcal{A})$ is a quasi-isomorphism if and only if its image in $\mathrm{D}(\mathcal{A})$ is invertible, i.e. the multiplicative system Σ is *saturated* [Bü10, Remark 10.19].

The full subcategories $\mathrm{K}^{+}(\mathcal{A})$, $\mathrm{K}^{-}(\mathcal{A})$, and $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ of $\mathrm{K}(\mathcal{A})$ and the corresponding full subcategories $\mathrm{D}^{+}(\mathcal{A})$, $\mathrm{D}^{-}(\mathcal{A})$, and $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ of $\mathrm{D}(\mathcal{A})$ are defined in a similar fashion, by considering only left-bounded, right-bounded, and bounded complexes respectively.

2.6. Abelian and quasi-abelian categories. A *quasi-abelian category* [Bü10, Definition 4.1] is a finitely complete and finitely cocomplete additive category such that the push-out of a kernel along an arbitrary morphism is a kernel, and dually the pull-back of a cokernel along an arbitrary morphism is a cokernel. If $\mathcal{A}$ is a quasi-abelian category, then $(\mathcal{A}, \mathcal{E})$ is an exact category, where $\mathcal{E}$ is the collection of all the kernel-cokernel pairs in $\mathcal{A}$ [Bü10, Proposition 4.4]. In what follows, we will regard every quasi-abelian category as an exact category with respect to such a canonical (maximal) exact structure. An *abelian category* is a quasi-abelian category where every morphism is admissible or, equivalently, every monomorphism is a kernel and every epimorphism is a cokernel [Bü10, Remark 4.7].

Recall that a subcategory $\mathcal{B}$ of $\mathcal{A}$ is strictly full if it is full and whenever an object of $\mathcal{A}$ is isomorphic to an object of $\mathcal{B}$ then it belongs to $\mathcal{B}$. The following definitions generalize [KS06, Definition 8.3.21], where they are stated in the context of abelian categories.

2.7. Definition. *A strictly full subcategory $\mathcal{B}$ of a quasi-abelian category $\mathcal{A}$ is:*

- *a* quasi-abelian subcategory *of $\mathcal{A}$ if $\mathcal{B}$ is a quasi-abelian category and the inclusion functor $\mathcal{B} \to \mathcal{A}$ is finitely continuous and finitely cocontinuous;*
- thick *if it is a strictly full and fully exact quasi-abelian subcategory (Notice that this is different from the notion of thick subcategory of a triangulated category from [Nee01, Definition 2.1.6].);*
- closed by extensions *in $\mathcal{A}$ if whenever*

$$0 \to B_0 \to A_1 \to B_2 \to 0$$

  is a short exact sequence in $\mathcal{A}$, if B_0, B_2 are in $\mathcal{B}$ then also A_1 is in $\mathcal{B}$;
- closed by kernels *if for any morphism f in $\mathcal{B}$,* $\ker_{\mathcal{A}}(f)$ *is in $\mathcal{B}$;*
- closed by cokernels *if for any morphism f in $\mathcal{B}$,* $\operatorname{coker}_{\mathcal{A}}(f)$ *is in $\mathcal{B}$.*

The next lemma provides a characterization of thick subcategories; see also [KS06, Remark 8.3.22(iii)].

2.8. Lemma. *Let $\mathcal{A}$ be a quasi-abelian category, and let $\mathcal{B}$ be a strictly full subcategory of $\mathcal{A}$ containing the zero object of $\mathcal{A}$. The following assertions are equivalent:*

1. *$\mathcal{B}$ is a thick subcategory of $\mathcal{A}$;*
2. *$\mathcal{B}$ is closed by extensions, kernels, and cokernels.*

PROOF. (1)⇒(2) Since $\mathcal{B}$ is a fully exact subcategory of $\mathcal{A}$, it is closed by extensions. Suppose that f is an arrow in $\mathcal{B}$. Since $\mathcal{B}$ is a quasi-abelian category, f has a kernel $\ker_{\mathcal{B}}(f)$ in $\mathcal{B}$. Since the inclusion $\mathcal{B} \to \mathcal{A}$ is finitely continuous, we have that $\ker_{\mathcal{B}}(f)$ is also the kernel of f in $\mathcal{A}$. This shows that $\ker_{\mathcal{A}}(f)$ is in $\mathcal{B}$, and $\mathcal{B}$ is closed under kernels in $\mathcal{A}$. By duality, it follows that $\mathcal{B}$ is also closed under cokernels.

(2)⇒(1) By assumption, $\mathcal{B}$ is closed under extensions and, in particular, under biproducts. This shows that $\mathcal{B}$ has biproducts, and the inclusion $\mathcal{B} \to \mathcal{A}$ preserves biproducts. If f is an arrow in $\mathcal{B}$, then by hypothesis $\ker_{\mathcal{A}}(f)$ is in $\mathcal{B}$, and is easily seen to be the kernel of f in $\mathcal{B}$. Considering the expression of finite limits in terms of products and equalizers, it follows that $\mathcal{B}$ is a finitely complete category, and the inclusion $\mathcal{B} \to \mathcal{A}$ is finitely continuous. By duality, we also have that $\mathcal{B}$ is finitely cocomplete, and the inclusion $\mathcal{B} \to \mathcal{A}$ is finitely cocontinuous.

Consider a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{k_0} & B_0 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow[k_1]{} & C \end{array}$$

in $\mathcal{B}$ (and, hence, in $\mathcal{A}$), where k_0 is a kernel in $\mathcal{B}$. Then k_0 is also a kernel in $\mathcal{A}$. Since $\mathcal{A}$ is quasi-abelian, it follows that k_1 is a kernel in $\mathcal{A}$, and

$$k_1 = \ker_{\mathcal{A}}(\operatorname{coker}_{\mathcal{A}}(k_1))$$

by [Sch99, Proposition 1.1.4]. Since $\mathcal{B}$ is closed under cokernels, it follows that k_1 is a kernel in $\mathcal{B}$. This and the dual result show that $\mathcal{B}$ is a quasi-abelian category. Finally, $\mathcal{B}$ is fully exact in $\mathcal{A}$ by hypothesis. ∎

Let $\mathcal{A}$ be a quasi-abelian category and $\mathrm{D}(\mathcal{A})$ be its derived category. The *left heart* of $\mathcal{A}$ is by definition the heart of $\mathrm{D}(\mathcal{A})$ with respect to its canonical left truncation structure [Sch99, Definition 1.2.18]. Concretely, $\mathrm{LH}(\mathcal{A})$ is the full subcategory spanned by complexes A with $A^n = 0$ for $n \in \mathbb{Z} \setminus \{-1, 0\}$ and such that the differential $A^{-1} \to A^0$ is monic. Then $\mathrm{LH}(\mathcal{A})$ is an abelian category that contains $\mathcal{A}$ as a subcategory [Sch99, Proposition 1.2.29], and the inclusion functor $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ is fully faithful and finitely continuous. Furthermore, the inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ satisfies the following universal property: for every abelian category $\mathcal{M}$ and finitely continuous and exact functor $F : \mathcal{A} \to \mathcal{M}$ there exists an essentially unique functor $\mathrm{LH}(\mathcal{A}) \to \mathcal{M}$ whose restriction to $\mathcal{A}$ is isomorphic to F [Sch99, Proposition 1.2.34]. The inclusion $\mathcal{A} \to \mathrm{LH}(\mathcal{A})$ extends to an equivalence of categories $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathrm{LH}(\mathcal{A}))$ [Sch99, Proposition 1.2.32], and it has a left adjoint $\kappa : \mathrm{LH}(\mathcal{A}) \to \mathcal{A}$ [Sch99, Definition 1.2.26 and Proposition 1.2.27].

The canonical left truncation structure on $\mathrm{D}(\mathcal{A})$ yields the *cohomology functors* $\mathrm{H}^n : \mathrm{D}(\mathcal{A}) \to \mathrm{LH}(\mathcal{A})$ for $n \in \mathbb{Z}$ [Sch99, Definition 1.2.18]. We have that a morphism f in $\mathrm{D}(\mathcal{A})$ is an isomorphism if and only if $\mathrm{H}^n(f)$ is an isomorphism for every $n \in \mathbb{Z}$.

Recall that a *torsion pair* [Tat21, Definition 2.4] in a quasi-abelian category $\mathcal{M}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories, where $\mathcal{T}$ is called a torsion class and $\mathcal{F}$ called a torsion-free class, such that:

1. for all objects T of $\mathcal{T}$ and F of $\mathcal{F}$, $\mathrm{Hom}\,(T, F) = 0$;
2. for all objects M of $\mathcal{M}$ there exists a short exact sequence

$$0 \to {}_{\mathcal{T}}M \to M \to M_{\mathcal{F}} \to 0$$

where ${}_{\mathcal{T}}M$ is in $\mathcal{T}$ and $M_{\mathcal{F}}$ is in $\mathcal{F}$.

In this case, we have that an object X of $\mathcal{M}$ is in $\mathcal{T}$ if and only if $\mathrm{Hom}\,(X, C) = 0$ for all C in $\mathcal{F}$, and it is in $\mathcal{F}$ if and only if $\mathrm{Hom}\,(T, X) = 0$ for all T in $\mathcal{T}$. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{M}$, then $\mathcal{T}$ and $\mathcal{F}$ are full subcategories of $\mathcal{M}$ (essentially) closed under extensions [Rum01, Theorem 2]. Conversely, if $\mathcal{A}$ is a quasi-abelian category, let $\kappa : \mathrm{LH}\,(\mathcal{A}) \to \mathcal{A}$ be the left adjoint of the inclusion $\mathcal{A} \to \mathrm{LH}\,(\mathcal{A})$. Thus, we have that

$$\mathrm{Hom}_{\mathcal{A}}\,(\kappa\,(X)\,, B) \cong \mathrm{Hom}_{\mathrm{LH}(\mathcal{A})}\,(X, B)$$

for all objects B of $\mathcal{A}$. Define $\mathcal{T}$ to be the full subcategory of $\mathrm{LH}\,(\mathcal{A})$ spanned by the objects X such that $\kappa\,(X) = 0$, and define $\mathcal{F}$ to be equal to $\mathcal{A}$. Then we have that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathrm{LH}\,(\mathcal{A})$ [Rum01, Theorem 2].

We want to consider functors with target (the left heart of) categories of topological modules, as in [CL25]. In order to extend the theory of (quasi-)abelian categories to include categories enriched over categories of topological modules, the abstract notion of "category of modules" has been introduced in [CL25, Definition 2.19]; see also [CL25, Section 2.11 and Section 2.12].

2.9. Definition. *A* category of modules *is a locally small tensor category $\mathcal{M}$ such that:*

1. *$\mathcal{M}$ has finite products, and every object has a unique group object structure;*
2. *with respect to the induced* **Ab***-category structure and the given tensor structure, $\mathcal{M}$ is a quasi-abelian tensor category with enough projectives;*
3. *the tensor identity R is projective and a generator;*
4. *for projective objects P and Q, $P \otimes Q$ is projective;*
5. *every projective object P is flat, i.e., the functor $P \otimes -$ is exact.*

As remarked in [CL25, Section 2.15], if $\mathcal{M}$ is a category of modules, then its *category of epimorphic towers* $\mathbf{\Pi}\,(\mathcal{M})$ is also a category of modules; see [CL25, Section 2.14].

Every category of modules is, in particular, an **Ab**-category. The **Ab**-enrichment is induced by the unique group object structure on the objects, and makes all morphisms group homomorphisms. The corresponding notion of (quasi-)abelian category of modules is defined in the obvious way; see [CL25, Section 2.14].

If $\mathcal{M}$ is a category of modules, we can define the notion of $\mathcal{M}$-category, as a particular instance of the usual notion of enriched category. In this context, we can also define the notion of (quasi-)abelian $\mathcal{M}$-category, and triangulated $\mathcal{M}$-category; see [CL25, Section 2.14]. It is also observed in [CL25, Section 2.15] that the left heart of a category of modules is also a category of modules.

For example, notice that the category **LieAb** of abelian Lie groups and continuous group homomorphisms is a category of modules. Considering that every abelian Lie group is of the form

$$T \otimes F \otimes V$$

where T is a torus group, F is a finite-rank free abelian group, and V is a finite-dimensional vector group, the *tensor product* bifunctor is defined in the obvious way. Such a tensor product, even for more general locally compact Polish groups, was already considered in [Mos67, Section IV]; see also [Gar66]. Indeed, if G, H are Lie abelian Polish groups, then $\mathrm{Hom}(G, H)$ endowed with the compact-open topology is also a Lie abelian Polish group. In particular, the Pontryagin dual $G^{\vee} := \mathrm{Hom}(G, \mathbb{T})$ is also a Lie abelian Polish group. One can use this duality and self-enrichment to define the tensor product $G \otimes H$ to be the Pontryagin dual of $\mathrm{Hom}(G, H^{\vee})$, where $H^{\vee}$ is the Pontryagin dual of H. In this way, we have

$$\mathrm{Hom}(G \otimes H, \mathbb{T}) \cong \mathrm{Hom}(G, \mathrm{Hom}(H, \mathbb{T}))$$

and more generally for any other Lie abelian Polish group L,

$$\mathrm{Hom}(G \otimes H, L) \cong \mathrm{Hom}(G, \mathrm{Hom}(H, L)).$$

In turn, this renders the category **proLiePAb** of pro-Lie Polish abelian groups a category of modules, being (equivalent to) the category of epimorphic towers in **LieAb**.

This is one of the advantages of **proLiePAb** over its full subcategory **LCPAb** of locally compact Polish abelian groups, which does not have a canonical monoidal structure. Other ways in which **proLiePAb** is superior to **LCPAb** consists in having enough projectives, as it will be shown in this paper, which is not true for **LCPAb**. Since furthermore subobjects of projectives are projectives, it follows that the category LH (**proLiePAb**) is an *abelian* category with enough projectives of homological dimension 1.

2.10. Derived functors. Suppose that $\mathcal{A}$ and $\mathcal{R}$ are exact categories, and $F : \mathcal{A} \to \mathcal{R}$ is a functor. Then F induces a *triangulated* functor $\mathrm{K}^{+}(\mathcal{A}) \to \mathrm{K}^{+}(\mathcal{R})$ [Nee01, Definition 2.1.1], which we still denote by F. A *total right derived functor* for $F : \mathrm{K}^{+}(\mathcal{A}) \to \mathrm{K}^{+}(\mathcal{R})$ [Bü10, Section 10.6]—see also [Sch99, Definition 1.3.1]—is a triangulated functor $\mathrm{R}F : \mathrm{D}^{+}(\mathcal{A}) \to \mathrm{D}^{+}(\mathcal{R})$ together with a morphism $\mu : Q_{\mathcal{R}} \circ F \Rightarrow \mathrm{R}F \circ Q_{\mathcal{A}}$ of triangulated functors that satisfies the following universal property: for every triangulated functor $G : \mathrm{D}^{+}(\mathcal{A}) \to \mathrm{D}^{+}(\mathcal{R})$ and morphism $\nu : Q_{\mathcal{R}} \circ F \to G \circ Q_{\mathcal{A}}$ of triangulated functors, there exists a unique morphism of triangulated functors $\sigma : \mathrm{R}F \Rightarrow G$ such that $\nu = (\sigma Q_{\mathcal{A}}) \circ \mu$.

If $\mathcal{R}$ is quasi-abelian, then $\mathrm{H}^0 \circ F$ is a *cohomological functor* [Ver77, Section 1.1, Definition 3.1]; see also [Nee01, Definition 1.1.7]. A *right cohomological derived functor* for $\mathrm{H}^0 \circ F$ is defined in a similar fashion, via a suitable universal property [Ver77, Section 2.2, Definition 1.4].

2.11. Definition. *Suppose that $\mathcal{A}$ is an exact category, and $\mathcal{C}$ is a full subcategory of $\mathcal{A}$. We say that $\mathcal{C}$ is:*

1. generating *[KS06, Definition 8.3.21(v)] if for each object A of $\mathcal{A}$ there exists an admissible epimorphism $C \to A$ with C in $\mathcal{C}$;*
2. cogenerating *if $\mathcal{C}^{\mathrm{op}}$ is generating in $\mathcal{A}^{\mathrm{op}}$;*
3. closed under quotients *if for all short exact sequences*

$$0 \to A' \to C \to A'' \to 0$$

in $\mathcal{A}$ with C in $\mathcal{C}$, A'' is isomorphic to an object of $\mathcal{C}$;

4. closed under subobjects *if $\mathcal{C}^{\mathrm{op}}$ is closed under quotients in $\mathcal{A}^{\mathrm{op}}$.*

We have the following result [KS06, Proposition 13.2.2(b)]; see also [Bü10, Theorem 10.22 and Remark 10.23], [KS06, Proposition 13.2.2], [Sch99, Lemma 1.3.3 and Lemma 1.3.4], and [HS07, Corollary 3.10].

2.12. Lemma. *Suppose that $\mathcal{A}$ is an exact category and $\mathcal{C}$ is a cogenerating fully exact subcategory of $\mathcal{A}$ closed under quotients. Then for any bounded complex A in $\mathcal{A}$ there exists a bounded complex C in $\mathcal{C}$ and a quasi-isomorphism $\eta : A \to C$ with $\eta^k : A^k \to C^k$ an admissible monic for every $k \in \mathbb{Z}$.*

Furthermore, the inclusion $\mathcal{C} \to \mathcal{A}$ induces an equivalence of categories

$$\mathrm{D}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{A}).$$

As a consequence of Lemma 2.12 we have the following; see [Bü10, Section 10.6], [KS06, Definition 10.3.2, Proposition 10.3.3, Corollary 13.3.8], [Sch99, Proposition 1.3.5], and [HS07, Remark 4.10].

2.13. Proposition. *Suppose that $F : \mathcal{A} \to \mathcal{R}$ is a functor between exact categories, and $\mathcal{C}$ is an cogenerating full subcategory of $\mathcal{A}$ closed under quotients such that $F|_{\mathcal{C}}$ is exact. For a bounded complex A over $\mathcal{A}$, pick a bounded complex C_A over $\mathcal{C}$ together with a quasi-isomorphism $\eta_A : A \to C_A$. For bounded complexes A, B over $\mathcal{A}$ and morphisms $f : A \to B$ in $\mathrm{D}^b(\mathcal{A})$, define*

$$(\mathrm{R}F)(A) := F(C_A)$$

and

$$(\mathrm{R}F)(f) := Q_{\mathcal{R}}(F(g)) \circ Q_{\mathcal{R}}(F(\sigma))^{-1},$$

where $\sigma : C \to C_A$ and $g : C \to C_B$ are morphisms in $\mathrm{K}(\mathcal{C})$ such that

$$Q_{\mathcal{A}}(g)\, Q_{\mathcal{A}}(\sigma)^{-1} = Q_{\mathcal{A}}(\eta_B) \circ f \circ Q_{\mathcal{A}}(\eta_A)^{-1}.$$

This yields a triangulated functor $\mathrm{R}F : \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{R})$. Defining $\mu_A := F(\eta_A) : F(A) \to (\mathrm{R}F)(A)$ for each bounded complex A over $\mathcal{A}$ yields a morphism $\mu : Q_{\mathcal{R}}F \Rightarrow$

$(\mathrm{R}F)\,Q_{\mathcal{A}}$ *of triangulated functors. We have that* $(\mathrm{R}F,\mu)$ *is the total right derived functor of* F*. Furthermore, if* $\mathcal{R}$ *is a quasi-abelian category, then* $\mathrm{H}^0\circ\mathrm{R}F$ *is a cohomological right derived functor of* $\mathrm{H}^0\circ F$*.*

Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ are exact categories, and $F:\mathcal{A}^{\mathrm{op}}\times\mathcal{B}\to\mathcal{R}$ is a functor. Given bounded complexes A over $\mathcal{A}$ and B over $\mathcal{B}$, we have a corresponding double complex $F(A,B)$ over $\mathcal{R}$. We let $F^\bullet(A,B)$ be the total complex associated with $F(A,B)$, which is well-defined and bounded since A and B are bounded and hence $F(A,B)$ has only finitely many nonzero entries in each diagonal. This defines a triangulated functor $F^\bullet:\mathrm{K}^b(\mathcal{A})^{\mathrm{op}}\times\mathrm{K}^b(\mathcal{B})\to\mathrm{K}^b(\mathcal{R})$; see [KS06, Proposition 11.6.4]. The same proof as Proposition 2.13 gives the following.

2.14. Proposition. *Suppose that* $F:\mathcal{A}^{\mathrm{op}}\times\mathcal{B}\to\mathcal{R}$ *is a functor between exact categories,* $\mathcal{C}$ *is an generating full subcategory of* $\mathcal{A}$ *closed under subobjects, and* $\mathcal{D}$ *is a cogenerating full subcategory of* $\mathcal{B}$ *closed under quotients. Assume that for every object* C *of* $\mathcal{C}$ *and* D *of* $\mathcal{D}$*,* $F(C,-)$ *is exact on* $\mathcal{D}$ *and* $F(-,D)$ *is exact on* $\mathcal{C}$*. For bounded complexes* A *and* B *over* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively, pick bounded complexes* C_A *and* D_B *over* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively, together with quasi-isomorphisms* $\eta_A:C_A\to A$ *and* $\eta_B:B\to D_B$*, and define*

$$(\mathrm{R}F)(A,B):=F(C_A,D_B)\,.$$

This yields a triangulated functor $\mathrm{R}F^\bullet:\mathrm{D}^b(\mathcal{A})^{\mathrm{op}}\times\mathrm{D}^b(\mathcal{B})\to\mathrm{D}^b(\mathcal{R})$*. Defining* $\mu_A:=F(\eta_A,\eta_B):F^\bullet(A,B)\to(\mathrm{R}F)(A)$ *for each bounded complex* A *over* $\mathcal{A}$ *yields a morphism* $\mu:Q_{\mathcal{R}}F^\bullet\Rightarrow(\mathrm{R}F)(Q_{\mathcal{A}}\times Q_{\mathcal{B}})$ *of triangulated functors. We have that* $(\mathrm{R}F,\mu)$ *is the total right derived functor of* $F^\bullet$*. Furthermore, if* $\mathcal{R}$ *is a quasi-abelian category, then* $\mathrm{H}^0\circ\mathrm{R}F^\bullet$ *is a cohomological right derived functor of* $\mathrm{H}^0\circ F^\bullet$*.*

2.15. Injective and projective objects. Suppose that $\mathcal{A}$ is an exact category. An object I of $\mathcal{A}$ is called *injective* [Bü10, Definition 11.1] if the functor $\mathrm{Ext}(-,I):\mathcal{A}^{\mathrm{op}}\to\mathbf{Ab}$ is exact; see also [Bü10, Proposition 11.3]. The category $\mathcal{A}$ *has enough injectives* [Bü10, Definition 11.9] if for every object A of $\mathcal{A}$ there exists an admissible monic $m:A\to I$ where I is injective. Let $\mathcal{A}$ be an exact category with enough injectives, and let $\mathcal{I}$ be the class of injective objects in $\mathcal{A}$. Then for every left-bounded complex A in $\mathcal{A}$ there exists a left-bounded complex I_A over $\mathcal{I}$ and a quasi-isomorphism $\mu_A:A\to I_A$ [Bü10, Theorem 12.7]. In particular, when A is an object of $\mathcal{A}$, regarded as a complex concentrated in degree zero, then I_A is called an *injective resolution* of A [Bü10, Definition 12.1].

Furthermore, the inclusion $\mathcal{I}\to A$ induces an equivalence of triangulated categories $\mathrm{K}^+(\mathcal{I})\to\mathrm{D}^+(\mathcal{A})$. If $\mathcal{B}$ is an exact category, and $F:\mathcal{A}\to\mathcal{B}$ is a functor, then F has a total right derived functor $\mathrm{R}F:\mathrm{D}^+(\mathcal{A})\to\mathrm{D}^+(\mathcal{B})$. This is defined by setting $(\mathrm{R}F)(A):=F(I_A)$ and $(\mathrm{R}F)(f)=Q_{\mathcal{B}}(F(g))$ for left-bounded complexes A and B over $\mathcal{A}$ and morphism $f:A\to B$ in $\mathrm{D}^+(\mathcal{A})$, where $g:I_{\mathcal{A}}\to I_{\mathcal{B}}$ is a morphism in $\mathrm{K}(\mathcal{A})$ satisfying $Q_{\mathcal{A}}(g)=Q_{\mathcal{A}}(\mu_B)\circ f\circ Q_{\mathcal{A}}(\mu_A)^{-1}$.

For objects A and B of $\mathcal{A}$ and $n\in\mathbb{Z}$, one defines $\mathrm{Ext}^n(A,B):=\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A,B[n])$. One says that $\mathcal{A}$ has *finite homological dimension* if there exists $d\in\mathbb{Z}$ such that $\mathrm{Ext}^n(A,$

$B) = 0$ for all $n > d$ and objects A and B of $\mathcal{A}$. In this case, the least such d is called the *homological dimension* $\mathrm{hd}(\mathcal{A})$ of $\mathcal{A}$ [KS06, Exercise 13.8]. If $\mathcal{A}$ has enough injectives and $d \geq 0$, then $\mathrm{hd}(\mathcal{A}) \leq d$ if and only if for every exact sequence $X^0 \to X^1 \to \cdots \to X^d \to 0$ in $\mathcal{A}$ with $X^0, X^1, \dots, X^{d-1}$ injective, we have that X^d is also injective [KS06, Exercise 13.8]. In this case, every object of $\mathcal{A}$ has an injective resolution I_A with $I_A^k = 0$ for $k \in \mathbb{Z} \setminus \{0, 1, \dots, d\}$.

If $\mathcal{A}$ has enough injectives and finite homological dimension, then for every (bounded) complex A over $\mathcal{A}$ there exists a quasi-isomorphism $\mu_A : A \to I_A$ where I_A is a (bounded) complex over $\mathcal{I}$ [Ive86, Corollary I.7.7]. Furthermore, the inclusion $\mathcal{I} \to \mathcal{A}$ induces an equivalence of categories $\mathrm{K}(\mathcal{I}) \to \mathrm{D}(\mathcal{A})$, which restricts to an equivalence of categories $\mathrm{K}^b(\mathcal{I}) \to \mathrm{D}^b(\mathcal{A})$ [Ive86, Proposition IX.2.12]; see also [KS06, Proposition 13.2.2]. As a consequence of Proposition 2.14 one obtains the following:

2.16. Proposition. *Let $\mathcal{A}$ and $\mathcal{R}$ be exact categories. Let $F : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{R}$ be a functor. Suppose that either:*

- *$\mathcal{A}$ has enough injectives, and $F(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathcal{R}$ is exact for every injective object I of $\mathcal{A}$, or*
- *$\mathcal{A}$ has enough projectives, and $F(P, -) : \mathcal{A}^{\mathrm{op}} \to \mathcal{R}$ is exact for every projective object P of $\mathcal{A}$.*

Suppose furthermore that $\mathcal{A}$ has homological dimension at most 1. Then $F^\bullet : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{R})$ has a total right derived functor $\mathrm{R}F : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{R})$. If $\mathcal{R}$ is quasi-abelian, then $\mathrm{H}^0 \circ \mathrm{R}F$ is a cohomological derived functor of $\mathrm{H}^0 \circ F^\bullet$.

As a particular instance of Proposition 2.14, one has the following.

2.17. Corollary. *Suppose that $\mathcal{M}$ is a quasi-abelian category of modules, and $\mathcal{A}$ is a quasi-abelian $\mathcal{M}$-category. Assume that $\mathcal{A}$ has enough injectives or enough projectives, and homological dimension at most 1. Then $\mathrm{Hom}^\bullet : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{K}^b(\mathcal{M})$ has a total right derived functor $\mathrm{RHom} : \mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{M})$, and $\mathrm{Ext}^0 := \mathrm{H}^0 \circ \mathrm{RHom}$ is a cohomological derived functor of $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})} : \mathrm{K}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M})$. Furthermore, for bounded complexes A and B over $\mathcal{A}$, $\mathrm{Ext}^0(A, B)$ is naturally isomorphic as an abelian group to $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B)$. This isomorphism turns $\mathrm{D}^b(\mathcal{A})$ into a triangulated category enriched over $\mathrm{LH}(\mathcal{M})$.*

Proof. The first assertion is a particular instance of Proposition 2.14.

For the second assertion, observe that by [KS06, Proposition 11.7.3], we have that $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}$ is naturally isomorphic to $\mathrm{H}^0 \circ \mathrm{Hom}^\bullet$. Therefore, by the first assertion we have that RHom is a total right derived functor of $\mathrm{Hom}^\bullet$, and $\mathrm{Ext}^0 := \mathrm{H}^0 \circ \mathrm{RHom}$ is a cohomological right derived functor of $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})} \cong \mathrm{H}^0 \circ \mathrm{Hom}^\bullet$. Furthermore, we have that Ext^0 is naturally isomorphic to $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}$ by [KS06, Proposition 11.7.3]. This allows one to regard $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{A})}$ as a functor to $\mathrm{D}^b(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}^b(\mathcal{A}) \to \mathrm{LH}(\mathcal{M})$, which turns $\mathrm{D}^b(\mathcal{A})$ into a triangulated category enriched over $\mathrm{LH}(\mathcal{M})$. ∎

Under the hypotheses of Corollary 2.17, one sets $\mathrm{Ext}^n(A,B) := \mathrm{Ext}^0(A,B[n])$ for complexes A and B over $\mathcal{A}$ and $n \in \mathbb{Z}$. Then we have that $\mathrm{Ext}^n = 0$ for $n \geq 2$. Furthermore, for objects A,B of $\mathcal{A}$, $\mathrm{Ext}(A,B) := \mathrm{Ext}^1(A,B)$ is isomorphic to the group $\mathrm{Ext}_{\mathrm{Yon}}(A,B)$ of *isomorphism classes of extensions* of A by B; see [Ive86, Section XI.4]. Notice that if $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$, then for objects A and B of $\mathcal{B}$, the group $\mathrm{Ext}_{\mathrm{Yon}}(A,B)$ is unchanged whether it is computed in $\mathcal{B}$ or in $\mathcal{A}$; see [KS06, Exercise 13.17].

2.18. The left heart of thick categories of Polish abelian groups. Let **PAb** be the category of Polish abelian groups and continuous group homomorphisms. This is a quasi-abelian category [Lup24, Lemma 6.1]. An explicit description of LH(**PAb**) was given in [Lup24] in terms of *groups with a Polish cover*. An abelian group with a Polish cover is a group G explicitly presented as a quotient $\hat{G}/N$, where $\hat{G}$ is a Polish abelian group and $N \subseteq \hat{G}$ is a Polishable subgroup of $\hat{G}$. This means that N is a Polish group with respect to some Polish topology such that the inclusion $N \to \hat{G}$ is continuous. The Polish topology on N is in general not the subspace topology inherited from $\hat{G}$ (unless N is closed in $\hat{G}$). Nonetheless, the topology on N is in some sense *induced* by the topology on $\hat{G}$, as it is the unique Polish topology on N whose open sets are Borel in $\hat{G}$. Every Polish abelian group G can be identified with the group with a Polish cover $\hat{G}/N$ where $G = \hat{G}$ and N is the trivial subgroup of G. A group homomorphism $\varphi : G \to H$ between groups with a Polish cover $G = \hat{G}/N$ and $H = \hat{H}/M$ is *Borel-definable* if has a *lift* to a Borel function $f : \hat{G} \to \hat{H}$, such that $\varphi(x+N) = f(x) + M$ for every $x \in \hat{G}$.

The category LH(**PAb**) is (equivalent to) the category of groups with a Polish cover and Borel-definable group homomorphisms [Lup24, Theorem 6.2]. More generally, if $\mathcal{B}$ is a thick subcategory of **PAb**, then LH($\mathcal{B}$) is (equivalent to) the full subcategory of the category of abelian groups with a Polish cover spanned by *groups with a* $\mathcal{B}$-cover, which are the groups with a Polish cover of the form $\hat{G}/N$ where both $\hat{G}$ and N belong to $\mathcal{B}$ [Lup24, Theorem 6.13 and Proposition 6.15]. As it is remarked therein, a Borel-definable homomorphism between groups with a $\mathcal{B}$-cover is an isomorphism in LH($\mathcal{B}$) if and only if it is a bijection.

2.19. Lemma. *Let $\mathcal{B}$ be a strictly full subcategory of* **PAb**. *If $\mathcal{B}$ is closed under taking closed subgroups, quotients of closed subgroups, and extensions, then $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$.*

Proof. This is an immediate consequence of the characterization of thick subcategories from Lemma 2.8. ∎

An abelian Polish group is *non-Archimedean* if it has a basis of zero neighborhoods consisting of subgroups. This is equivalent to the assertion that A is isomorphic to an inverse limit of countable groups. Non-Archimedean Polish abelian groups form a thick subcategory of the category of Polish abelian groups; see [Lup24, Theorem 6.17].

3. Homological algebra for pro-Lie Polish abelian groups

3.1. Borel cocycles for Polish groups. Let G, H be abelian Polish groups. A (symmetric) 2-*cocycle* on G with values in H is a function $c : G \times G \to H$ satisfying the following identities for all $x, y, z \in G$:

- $c(x+y, z) + c(x, y) = c(x, y+z) + c(y, z)$;
- $c(x, y) = c(y, x)$;
- $c(0, x) = 0$.

We say that a 2-cocycle is Borel (respectively, continuous) if it is Borel (respectively, continuous) as a function $G \times G \to H$. Given a function $t : G \to H$, we define $\delta t : G \times G \to H$ to be the function $(x, y) \mapsto t(x) + t(y) - t(x+y)$. A Borel 2-cocycle c on G with values in H is a coboundary if there exists a Borel function $t : G \to H$ such that $c = \delta t$. Borel 2-cocycles on G with values in H form a group $\mathrm{Z}^1(G, H)$ with respect to pointwise addition. Coboundaries form a subgroup $\mathrm{B}^1(G, H)$ of $\mathrm{Z}^1(G, H)$. We define $\mathrm{Ext}_{\mathrm{c}}(G, H)$ to be the quotient group $\mathrm{Z}^1(G, H)/\mathrm{B}^1(G, H)$. This defines a functor $\mathrm{Ext}_{\mathrm{c}} : \mathbf{PAb}^{\mathrm{op}} \times \mathbf{PAb} \to \mathbf{Ab}$.

We let $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ be the group whose elements are the isomorphism classes of abelian Polish group extensions $A \to X \to C$, where the group operation is induced by Baer sum of extensions and the trivial element is the class of split extensions. A short exact sequence $A \to X \to C$ in $\mathbf{PAb}$ yields an element of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ as follows. Pick a Borel right inverse $t : C \to X$ for the map $X \to C$, which exists by [Kec95, Theorem 12.17]. Identifying A with a closed subgroup of X, we have that for every $x, y \in C$, $\kappa(x, y) := t(x+y) - t(x) - t(y)$ belongs to A. This defines a Borel 2-cocycle on C with values in A, whose corresponding element of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ is independent on the choice of t. This assignment defines an injective group homomorphism $\mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{c}}(C, A)$.

3.2. Lemma. *Suppose that C and A are locally compact Polish abelian groups. Then the group homomorphism*

$$\mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{c}}(C, A)$$

is an isomorphism.

Proof. It suffices to show that it is surjective. Given a Borel cocycle $\kappa : C \times C \to A$ one can define a corresponding extension $A \to X \to C$ as follows. One let X be the group that has $A \times C$ as set of objects, and group operation defined by

$$(a, c) + (a', c') = (a + a' + \kappa(c, c'), c + c').$$

We also endow $A \times C$ with the product Borel structure and the Borel measure defined as the product of Haar measures on A and G. Then we have that this is a σ-finite invariant Borel measure, and so by [Mac57b, Theorem 1] there exists a unique locally compact Polish group topology on X that is compatible with its Borel structure and has the given measure as a Haar measure. When endowed with this topology, the canonical maps $A \to X \to C$ give a short exact sequence in **PAb**. The image of the corresponding element in $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ under the homomorphism $\mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{c}}(C, A)$ is equal to the element of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ represented by κ. ∎

3.3. LEMMA. *Suppose that C and A are Polish abelian groups, where C is non-Archimedean. Then the image of the group homomorphism $\mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{c}}(C, A)$ is equal to the subgroup of elements represented by* continuous *cocycles.*

PROOF. Suppose that $A \to X \to C$ is a short exact sequence in **PAb**, where C is non-Archimedean. Then by [BLP24, Proposition 4.6] we have that the map $X \to C$ has a *continuous* right inverse $t : C \to X$. This yields a cocycle δt that represents an element of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ that is in the image of the element of $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ represented by the given extension.

Conversely if $\kappa : C \times C \to A$ is a continuous cocycle, then one can define the extension $A \to X \to C$ letting X be the group that has $A \times C$ as set of objects, endowed with the product topology, and group operation defined by

$$(a, c) + (a', c') = (a + a' + \kappa(c, c'), c + c').$$

∎

The following lemma is [Lup25, Lemma 4.7].

3.4. LEMMA. *Suppose that C is an abelian Polish group and A is an abelian Polish group. Suppose that $c : C \times C \to A$ is a continuous cocycle, and $t : C \to A$ is a function such that $c = \delta t$. If t is Borel, then t is continuous, and c is a coboundary.*

PROOF. Define the Polish group $X := A \rtimes_c C$ obtained by endowing the product $A \times C$ with the product topology and the group operation defined by $(a, x) + (a', x') := (a + a' + c(x, x'), x + x')$ for $a, a' \in A$ and $x, x' \in C$. Observe that the map $a \mapsto (a, 0)$ is an inclusion of A into X as a closed subgroup, giving an extension $A \to X \to C$.

Let also $A \times C$ be the product Polish group. We can define a Borel group homomorphism $\varphi : X \to A \times C$ by setting

$$\varphi(a, x) := (a + t(x), x).$$

Being a Borel group homomorphism, we must have that φ is continuous. Hence, $t : C \to A$ is continuous. ∎

Given a short exact sequence $A \to B \to C$ with the corresponding cocycle κ as above, and abelian Polish groups X and Y, we define the corresponding *boundary homomorphisms* $\mathrm{Hom}(X, C) \to \mathrm{Ext}_{\mathrm{c}}(X, A)$, $\varphi \mapsto \kappa \circ (\varphi \times \varphi) + \mathrm{B}^1(X, A)$ and $\mathrm{Hom}(A, Y) \to \mathrm{Ext}_{\mathrm{c}}(C, Y)$, $\psi \mapsto \psi \circ \kappa + \mathrm{B}^1(C, Y)$. The same proof as in the case of discrete groups gives the following lemma; see [Fuc70, Chapter IX]. When X, C are non-Archimedean, one can choose κ to be continuous, whence the boundary homomorphism takes values in $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$.

3.5. Lemma. *Let X, Y be abelian Polish groups, and let $A \to B \overset{\pi}{\to} C$ be a short exact sequence of abelian Polish groups.*

1. *We have exact sequences of abelian groups:*

$$\begin{aligned} 0 &\to \operatorname{Hom}(X,A) \to \operatorname{Hom}(X,B) \to \operatorname{Hom}(X,C) \\ &\to \operatorname{Ext}_{\mathrm{c}}(X,A) \to \operatorname{Ext}_{\mathrm{c}}(X,B) \to \operatorname{Ext}_{\mathrm{c}}(X,C) \end{aligned}$$

2. *If X, C are non-Archimedean, then we have an exact sequence of abelian groups:*

$$\begin{aligned} 0 &\to \operatorname{Hom}(X,A) \to \operatorname{Hom}(X,B) \to \operatorname{Hom}(X,C) \\ &\to \operatorname{Ext}_{\mathrm{Yon}}(X,A) \to \operatorname{Ext}_{\mathrm{Yon}}(X,B) \to \operatorname{Ext}_{\mathrm{Yon}}(X,C) \end{aligned}$$

3. *If A, B, C are non-Archimedean, then we have an exact sequence of abelian groups:*

$$\begin{aligned} 0 &\to \operatorname{Hom}(C,Y) \to \operatorname{Hom}(B,Y) \to \operatorname{Hom}(A,Y) \\ &\to \operatorname{Ext}_{\mathrm{Yon}}(C,Y) \to \operatorname{Ext}_{\mathrm{Yon}}(B,Y) \to \operatorname{Ext}_{\mathrm{Yon}}(A,Y) \end{aligned}$$

Proof. Let us fix a Borel right inverse $\sigma : C \to B$ for π, and let $\kappa = \delta\sigma : C \times C \to A$. We identify A with a closed subgroup of B. By [BLP24, Proposition 4.6], if C is non-Archimedean, then one can choose σ to be continuous.

(1) and (2): It is clear that the image of $\operatorname{Hom}(X,C) \to \operatorname{Ext}_{\mathrm{c}}(X,A)$ is contained in the kernel of $\operatorname{Ext}_{\mathrm{c}}(X,A) \to \operatorname{Ext}_{\mathrm{c}}(X,B)$. We prove the converse implication. Suppose that $c : X \times X \to A$ is a Borel cocycle such that there exists a function $t : X \to B$ such that $c = \delta t$. Then we have that $\pi t : X \to C$ is a group homomorphism. Then $\kappa \circ (\pi t \times \pi t)$ is a cocycle cohomologous to c, as we are about to show. Define $g : X \to A$, $x \mapsto \sigma\pi t(x) - t(x)$. For $x, y \in X$,

$$\begin{aligned} (\kappa \circ (\pi t \times \pi t))(x,y) &= \kappa(\pi t(x), \pi t(y)) \\ &= \sigma(\pi t(x)) + \sigma(\pi t(y)) - \sigma(\pi t(x+y)) \\ &= t(x) + t(y) - t(x+y) + \delta g(x,y) \\ &= c(x,y) + \delta g(x,y). \end{aligned}$$

This concludes the proof that the image of $\operatorname{Hom}(X,C) \to \operatorname{Ext}_{\mathrm{c}}(X,A)$ is equal to the kernel of $\operatorname{Ext}_{\mathrm{c}}(X,A) \to \operatorname{Ext}_{\mathrm{c}}(X,B)$.

We now prove that the kernel of $\operatorname{Ext}_{\mathrm{c}}(X,B) \to \operatorname{Ext}_{\mathrm{c}}(X,C)$ is contained in the image of $\operatorname{Ext}_{\mathrm{c}}(X,A) \to \operatorname{Ext}_{\mathrm{c}}(X,B)$. Suppose that $c : X \times X \to B$ is a Borel cocycle such that there is a Borel function $f : X \to C$ such that $\pi c = \delta f$. Then we have that $c_0 := c - \delta(\sigma f)$ is a Borel cocycle $X \times X \to A$ such that $c = c_0 + \delta(\sigma f)$ and hence c, c_0 are cohomologous as Borel cocycles $X \times X \to B$. This shows that the kernel of $\operatorname{Ext}_{\mathrm{c}}(X,B) \to \operatorname{Ext}_{\mathrm{c}}(X,C)$ is contained in the image of $\operatorname{Ext}_{\mathrm{c}}(X,A) \to \operatorname{Ext}_{\mathrm{c}}(X,B)$.

When X, C are non-Archimedean, σ is continuous. If c is continuous, then f is continuous, and hence c_0 is continuous as well. This shows that the kernel of $\operatorname{Ext}_{\mathrm{Yon}}(X,B) \to$

$\mathrm{Ext}_{\mathrm{Yon}}(X,C)$ is contained in the image of $\mathrm{Ext}_{\mathrm{Yon}}(X,A)\to\mathrm{Ext}_{\mathrm{Yon}}(X,B)$ when X,C are non-Archimedean.

(3) Suppose now that $c:C\times C\to Y$ is a Borel cocycle that represents an element of the kernel of $\mathrm{Ext}_{\mathrm{c}}(C,Y)\to\mathrm{Ext}_{\mathrm{c}}(B,Y)$. Thus, we have that $c\circ(\pi\times\pi)$ is a coboundary, and there exists a Borel function $f:B\to Y$ such that $\delta f=c\circ(\pi\times\pi)$. Hence,

$$f(b+b')=f(b)+f(b')+c(\pi(b),\pi(b'))$$

for $b,b'\in B$. This implies that

$$f(0)=0$$

and

$$f(b)+f(-b)=-c(\pi(b),-\pi(b)).$$

Then we have that $\varphi:=f|_A:A\to Y$ is a continuous group homomorphism. We have that $\varphi\circ\kappa$ is a Borel cocycle cohomologous to c. Indeed, define $g:=f\sigma:C\to Y$. For $x,y\in C$ we have that

$$\begin{aligned}
&(\varphi\circ\kappa)(x,y)\\
=\;&\varphi(\sigma(x)+\sigma(y)-\sigma(x+y))\\
=\;&f(\sigma(x)+\sigma(y)-\sigma(x+y))\\
=\;&f(\sigma(x)+\sigma(y))+f(-\sigma(x+y))+c(x+y,-(x+y))\\
=\;&f(\sigma(x))+f(\sigma(y))+c(x,y)-f(\sigma(x+y))\\
&-c(x+y,-(x+y))+c(x+y,-(x+y))\\
=\;&(c+\delta g)(x,y).
\end{aligned}$$

This shows that the kernel of $\mathrm{Ext}_{\mathrm{c}}(C,Y)\to\mathrm{Ext}_{\mathrm{c}}(B,Y)$ is contained in the image of $\mathrm{Hom}(A,Y)\to\mathrm{Ext}_{\mathrm{c}}(C,Y)$. In particular, we have that the kernel of $\mathrm{Ext}_{\mathrm{Yon}}(C,Y)\to\mathrm{Ext}_{\mathrm{Yon}}(B,Y)$ is contained in the image of $\mathrm{Hom}(A,Y)\to\mathrm{Ext}_{\mathrm{Yon}}(C,Y)$.

We now prove that the kernel of $\mathrm{Ext}_{\mathrm{Yon}}(B,Y)\to\mathrm{Ext}_{\mathrm{Yon}}(A,Y)$ is contained in the image of $\mathrm{Ext}_{\mathrm{Yon}}(C,Y)\to\mathrm{Ext}_{\mathrm{Yon}}(B,Y)$. Suppose that $Y\to H\overset{p}{\to}B$ is an extension that represents an element of the kernel of $\mathrm{Ext}_{\mathrm{Yon}}(B,Y)\to\mathrm{Ext}_{\mathrm{Yon}}(A,Y)$. This means that the induced extension $Y\to p^{-1}(A)\to A$ splits. Thus, there exists a continuous group homomorphism $\xi:A\to H$ with closed image such that $p\xi$ is the inclusion of A in B. In particular, we have that $\pi p\xi=0$. Thus, πp induces a continuous group homomorphism $\lambda:H/\xi(A)\to C$ such that $\lambda(h+\xi(A))=\pi(p(h))$ for $h\in H$. This yields a commuting diagram

$$\begin{array}{ccccccccc}
 & & & & A & = & A & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & H & \to & B & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & H/\xi(A) & \to & C & \to & 0
\end{array}$$

where the diagram

$$\begin{array}{ccc} H & \to & B \\ \downarrow & & \downarrow \\ H/\xi(A) & \to & C \end{array}$$

is a push-out. This yields a short exact sequence $Y \to H/\xi(A) \to C$ that represents an element of $\mathrm{Ext}_{\mathrm{Yon}}(C,Y)$ whose image in $\mathrm{Ext}_{\mathrm{Yon}}(B,Y)$ is the element represented by $Y \to H \to B$. ■

Suppose that G, H are abelian Polish groups, where $H = \prod_{n\in\alpha} H_n$ for some $\alpha \leq \omega$. It is clear from the definition that

$$\mathrm{Ext_c}(G,H) \cong \prod_{n\in\alpha} \mathrm{Ext_c}(G,H_n).$$

Furthermore, if $\alpha < \omega$ then, since $\mathrm{Ext_c}(-,G)$ is an additive functor, we also have

$$\mathrm{Ext_c}(H,G) \cong \prod_{n\in\alpha} \mathrm{Ext_c}(H_n,G)$$

3.6. Preliminaries on pro-Lie groups. In this section, we recall fundamental facts about the theory of pro-Lie groups as can be found in [HM07]. We will only consider abelian Polish groups. The category **PAb** of abelian Polish groups and continuous group homomorphisms is a quasi-abelian category [Lup24, Theorem 6.2]. Locally compact Polish abelian groups form a thick subcategory **LCPAb** of **PAb** [Lup24, Theorem 6.17].

An abelian Polish group is a *Lie group* if and only if it is of the form $V \oplus T \oplus D$ where D is countable discrete, V is a finite-dimensional vector group (isomorphic to $\mathbb{R}^n$ for some $n \in \omega$), and T is a finite-dimensional torus (isomorphic to $\mathbb{T}^d$ for some $d \in \omega$); see [HM13, Exercise E5.18]. Lie groups form a thick subcategory **LiePAb** of the category of locally compact abelian Polish groups; see [Lup24, Theorem 6.17].

A Polish abelian group G has *no small subgroups* if and only if it has a zero neighborhood U such that for every subgroup N of G contained in U, one has that $N = \{0\}$ [Mos67]. When G is locally compact, this is equivalent to the assertion that the Pontryagin dual $G^{\vee}$ is compactly generated, as well as to the assertion that G is a Lie group [Mos67, Theorem 2.4 and Corollary 1].

Let G be an abelian Polish group. A closed subgroup N of G is co-Lie if G/N is a Lie group. Following [HM07], we let $\mathcal{N}(G)$ be the collection of co-Lie closed subgroups of G.

3.7. Definition. *An abelian Polish group is* pro-Lie *if every zero neighborhood in G contains an element of* $\mathcal{N}(G)$.

An abelian Polish group G is a *pro-Lie group* if every zero neighborhood of G contains a closed subgroup N such that G/N is a Lie group. This implies that $\mathcal{N}(G)$ is closed under intersections, and hence a filter basis [HM07, page 148]. It also implies that $G \cong \lim_{N\in\mathcal{N}(G)} G/N$. For a decreasing chain $(N_k)_{k\in\omega}$ in $\mathcal{N}(G)$, we say that $N_k \to 0$ if for every zero neighborhood U in G, N_k is contained in U eventually. Notice that such a sequence exists for a pro-Lie Polish abelian group (as a Polish abelian group has a countable basis of zero neighborhoods).

3.8. Lemma. *Suppose that G is a pro-Lie abelian Polish group. Let $(N_k)_{k\in\omega}$ be a decreasing chain in $\mathcal{N}(G)$. Then we have that $N_k \to 0$ if and only if $(N_k)_{k\in\omega}$ is cofinal in $\mathcal{N}(G)$. If this holds, then $G \cong \lim_k G/N_k$.*

Proof. Suppose that $N_k \to 0$. For $M \in \mathcal{N}(G)$, we have that G/M is a Lie group. Thus, it has a zero neighborhood U that does not contain any subgroup. Since $N_k \to 0$, we have that there exists $k \in \omega$ such that $\pi(N_k) \subseteq U$, where $\pi : G \to G/M$ is the quotient map. This implies that $\pi(N_k) = \{0\}$ and $N_k \subseteq M$. The converse implication is obvious. The second assertion follows from the fact that $G \cong \lim_{N\in\mathcal{N}(G)} G/N$. ∎

It is proved in [HM07, Theorem 3.39] that a Polish abelian group is pro-Lie if and only if it is the inverse limit of an inverse system of Polish abelian Lie groups.

A Polish group G is *almost connected* if $G/c(G)$ is compact, where $c(G)$ is the connected component of the trivial element of G (which is a closed subgroup of G) [HM07, Definition 5.6]. If G is an abelian pro-Lie group, then $G/c(G)$ is non-Archimedean [HM07, Theorem 5.20(iii)].

The class of abelian Polish pro-Lie groups contains all locally compact abelian Polish groups [HM07, Example 5.1], all non-Archimedean abelian Polish groups, and all almost connected abelian Polish groups [Yam53a, Yam53b], and is closed within the category of Polish groups under the following operations [HM07, Chapter 3]:

- taking closed subgroups;
- taking quotients by closed subgroups;
- taking countable limits.

An abelian pro-Lie group is called:

- a *vector group* if it is isomorphic to $\mathbb{R}^\alpha$ for some $\alpha \leq \omega$;
- a *torus group* if it is isomorphic to $\mathbb{T}^\beta$ for some $\beta \leq \omega$.

We also say that an abelian Polish pro-Lie group is *vector-free* if it has no nonzero closed subgroups that are vector groups. By [HM07, Theorem 5.19] we have the following:

3.9. Lemma. *Let G be an abelian Polish pro-Lie group and H a closed subgroup of G. If H is isomorphic to an abelian Polish pro-Lie group of the form $T \oplus V$ where T is a torus group and V is a vector group, then the short exact sequence $H \to G \to G/H$ splits.*

An abelian Polish pro-Lie group G admits a closed subgroup V, called *maximal vector subgroup* (or vector group complement), that is a vector group, and such that $H := G/V$ is vector-free; see [HM07, Theorem 5.20]. In this case, by Lemma 3.9 we have that $G \cong V \oplus H$. Weil's Lemma for pro-Lie groups asserts the following; see [HM07, Theorem 5.3].

3.10. Lemma. *Let G be an abelian Polish pro-Lie group and let E be either $\mathbb{Z}$ or $\mathbb{R}$. If $f : E \to G$ is a continuous homomorphism, then exactly one of the following alternatives holds: either the image of f has compact closure, or f is injective with closed image.*

Let G be an abelian Polish group. Define $\mathrm{comp}\,(G)$ to be the subgroup of $g \in G$ such that the subgroup generated by g has compact closure. If G is an abelian Polish pro-Lie group, then $\mathrm{comp}\,(G)$ is a closed subgroup of G [HM07, Theorem 5.5]. One says that G is *elementwise compact* if $G = \mathrm{comp}\,(G)$ and *compact-free* if $\mathrm{comp}\,(G) = \{0\}$; see [HM07, Definition 5.4]. We have that $G/\mathrm{comp}\,(G) \cong V \oplus S$ where V is a maximal vector subgroup and S is non-Archimedean and compact-free [HM07, Theorem 5.20(iv)]. By [HM07, Proposition 5.43], we have the following:

3.11. Lemma. *If G is an abelian Polish group, then there exists a non-Archimedean closed subgroup D of G such that G/D is a torus group.*

The following result is established in [HM07, Theorem 4.1].

3.12. Lemma. *Suppose that G is a pro-Lie group, and H is a closed subgroup of G. Suppose that (N_k) is a cofinal sequence in $\mathcal{N}\,(G)$. Then G/H is pro-Lie, and $((\overline{N_k + H})/H)_{k\in\omega}$ is a cofinal sequence in $\mathcal{N}\,(G/H)$.*

We can describe the groups of morphisms between abelian pro-Lie groups as follows.

3.13. Lemma. *Suppose that G is a pro-Lie Polish abelian group and H is a Lie abelian group. Then* $\mathrm{Hom}\,(G, H) \cong \mathrm{colim}_{N\in\mathcal{N}(G)}\mathrm{Hom}\,(G/N, H)$.

Proof. Since H has no small subgroups, for every continuous group homomorphism $\varphi : G \to H$ there exists $N \in \mathcal{N}\,(G)$ such that $N \subseteq \mathrm{Ker}\,(\varphi)$ and hence φ factors through a homomorphism $G/N \to H$. ■

3.14. Proposition. *Suppose that G and H are pro-Lie Polish abelian groups. Then*

$$\mathrm{Hom}\,(G, H) \cong \lim_{M\in\mathcal{N}(H)}\mathrm{colim}_{N\in\mathcal{N}(G)}\mathrm{Hom}\,(G/N, H/M)\,.$$

Proof. Since $H \cong \lim_{M\in\mathcal{N}(H)} H/M$, by the universal property of the limit we have

$$\mathrm{Hom}\,(G, H) \cong \lim_{M\in\mathcal{N}(H)}\mathrm{Hom}\,(G, H/M)\,.$$

By definition, for $M \in \mathcal{N}\,(H)$, H/M is a Polish Lie abelian group. Thus, the conclusion follows from Lemma 3.13. ■

We regard pro-Lie Polish abelian groups as a full subcategory **proLiePAb** of the quasi-abelian category **PAb** of Polish abelian groups. We will soon prove that **proLiePAb** is in fact a thick subcategory of **PAb**.

When G is a locally compact Polish abelian group and H is a pro-Lie Polish abelian group, we have that $\mathrm{Hom}\,(G, H)$ is a *Polish* abelian group endowed with the compact-open topology [Gao09, Exercise 1.1.6], and the isomorphism

$$\mathrm{Hom}\,(G, H) \cong \lim_{M\in\mathcal{N}(H)}\mathrm{Hom}\,(G, H/M)$$

is as topological groups.

3.15. LEMMA. *If V is a Polish $\mathbb{R}$-vector space that is a pro-Lie abelian Polish group, then $V \cong \mathbb{R}^{\alpha}$ for some $\alpha \leq \omega$.*

PROOF. Let (N_k) be a cofinal sequence in $\mathcal{N}(V)$. Since V is a Polish $\mathbb{R}$-vector space, the function $\mathrm{Hom}(\mathbb{R}, V) \to V$, $\varphi \mapsto \varphi(1)$ is a topological isomorphism. We have that

$$\mathrm{Hom}(\mathbb{R}, V) \cong \lim_k \mathrm{Hom}(\mathbb{R}, V/N_k)$$

Since V/N_k is a connected Polish Lie abelian group, we have that $V/N_k \cong \mathbb{R}^{d_k} \oplus \mathbb{T}^{m_k}$, and $W_k := \mathrm{Hom}(\mathbb{R}, V/N_k)$ is a finite-dimensional $\mathbb{R}$-vector space. Since $\mathrm{Ker}(W_{k+1} \to W_k)$ is a closed $\mathbb{R}$-subspace of W_{k+1}, it is a finite-dimensional $\mathbb{R}$-vector space and a direct summand of W_{k+1}. Thus, we have that

$$V \cong \mathrm{Hom}(\mathbb{R}, V) \cong \lim_k W_k$$

is isomorphic to a product of finite-dimensional $\mathbb{R}$-vector spaces, and hence to $\mathbb{R}^{\alpha}$ for some $\alpha \leq \omega$. ∎

By Lemma 3.15, the vector groups are precisely the pro-Lie Polish abelian groups that are Polish $\mathbb{R}$-vector spaces. These are also called *weakly complete topological Polish* $\mathbb{R}$-*vector spaces in* [HM07, Proposition 5.43]. We have that if V, W are vector groups, and $\varphi : V \to W$ is a continuous homomorphism, then it is $\mathbb{R}$-linear and its image is a closed subspace of W which is a direct summand [HM07, Theorem A2.12].

3.16. EXTENSIONS OF ABELIAN POLISH PRO-LIE GROUPS. The goal of this section is to prove the following result.

3.17. THEOREM. *Pro-Lie Polish abelian groups form a thick subcategory of the category of Polish abelian groups.*

In view of the results of [HM07, Chapter 3], in order to establish Theorem 3.17 it suffices to prove that the class of pro-Lie Polish abelian groups is closed under extensions.

3.18. LEMMA. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups, where $A = \prod_{n\in\omega} A_n$ for Polish groups A_n for $n \in \omega$. Then we have an injective continuous homomorphism with closed image $\eta : B \to \prod_{n\in\omega} B_n$ where, for every $n \in \omega$, there is a short exact sequence of abelian Polish groups $A_n \to B_n \to C$. Furthermore, we have a commuting diagram*

$$\begin{array}{ccc} A & = & \prod_n A_n \\ \downarrow & & \downarrow \\ B & \overset{\eta}{\to} & \prod_n B_n \end{array}$$

where the map

$$\prod_{n\in\omega} A_n \to \prod_{n\in\omega} B_n$$

is induced by the maps $A_n \to B_n$ for $n \in \omega$.

Proof. We identify A with a closed subgroup of B. Let $\pi : B \to C$ be the quotient map. For $n \in \omega$, consider the pushout

$$\begin{array}{ccc} A & \to & B \\ p_n \downarrow & & \downarrow \eta_n \\ A_n & \overset{\sigma_n}{\to} & B_n \end{array}$$

where $p_n : A \to A_n$ is the canonical projection. Explicitly, we have that

$$B_n := A_n \oplus_A B$$

is the quotient of $A_n \oplus B$ by the closed subgroup

$$\Xi_n := \{(-p_n(a), a) : a \in A\}.$$

The map $\eta_n : B \to B_n$ is given by $b \mapsto (0, b) + \Xi_n$. Then we have a short exact sequence

$$A_n \to B_n \overset{\pi_n}{\to} C$$

where

$$\pi_n((t, b) + \Xi_n) = \pi(b).$$

Define the continuous group homomorphism $\eta : B \to \prod_n B_n$, $b \mapsto (\eta_n(b))_{n\in\omega}$. We claim that η is injective and has closed image. Indeed, suppose that $b \in B$ is such that $\eta(b) = 0$. This gives that $b \in A$ and $p_n(b) = 0$ for every $n \in \omega$, and hence $b = 0$. This shows that η is injective.

Consider the continuous homomorphism $\tau : \prod_{n\in\omega} B_n \to C^\omega$, $(x_n) \mapsto (\pi_n(x_n))$, and let

$$\Delta_C = \left\{(c_n)_{n\in\omega} \in C^\omega : \forall n \in \omega, c_n = c_0\right\} \subseteq C^\omega.$$

Then we have that the image of η is equal to the preimage of Δ_C under τ. Indeed, it is clear that $\tau \circ \eta$ has image contained in Δ_C. Conversely, suppose that

$$((t_n, b_n) + \Xi_n)_{n\in\omega} \in \prod_{n\in\omega} B_n$$

is mapped to Δ_C by τ. This implies that $\pi(b_n) = \pi(b_0)$ for every $n \in \omega$. Thus, for every $n \in \omega$ there exists $a_n \in A$ such that

$$b_n = b_0 + a_n.$$

Define now $s_n = p_n(a_n)$ for $n \in \omega$, $s := (s_n) \in A$, and $t = (t_n) \in A$. We have

$$\begin{aligned} ((t_n, b_n) + \Xi_n)_{n\in\omega} &= ((t_n, b_0 + a_n) + \Xi_n)_{n\in\omega} \\ &= ((t_n + s_n, b_0) + \Xi_n)_{n\in\omega} \\ &= ((0, b_0 + t + s) + \Xi_n)_{n\in\omega} = \eta(b_0 + t + s) \end{aligned}$$

This concludes the proof. ∎

3.19. LEMMA. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is isomorphic to a closed subgroup of an abelian Polish group A', then there exists a short exact sequence of abelian Polish groups $A' \to B' \to C$ such that B is isomorphic to a closed subgroup of B'.*

PROOF. It suffices to consider the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & \to & B' \end{array}$$

where the vertical map $A \to A'$ is the inclusion of A into A' as a closed subgroup. ∎

3.20. LEMMA. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is non-Archimedean and C is locally compact, then B is pro-Lie.*

PROOF. If A is countable, then B is locally compact, and hence pro-Lie. If $A \cong \prod_{n\in\omega} A_n$ where, for every $n \in \omega$, A_n is countable, then the conclusion follows from Lemma 3.18. Finally, if A is isomorphic to a closed subgroup of $\prod_{n\in\omega} A_n$ where, for every $n \in \omega$, A_n is countable, then the conclusion follows from Lemma 3.19. ∎

3.21. LEMMA. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is pro-Lie and C is locally compact, then B is pro-Lie.*

PROOF. By Lemma 3.11, we have a short exact sequence $D \to A \to T$ where D is non-Archimedean and T is a torus group. Considering the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ T & \to & B_T \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} D & \to & D & \to & 0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & B_T & \to & C \end{array}$$

whose rows and columns are short exact sequences. Since C and T are locally compact, we have that B_T is locally compact. Since D is non-Archimedean, we have that B is pro-Lie by Lemma 3.20. ∎

If X is a countable set, then for a bounded function $f : X \to \mathbb{R}$ we set

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

3.22. LEMMA. *For every $\varepsilon > 0$, countable discrete group A, and bounded 2-cocycle $c : A \times A \to \mathbb{R}$ with $\|c\|_\infty \le \varepsilon$, there exists a bounded $t : A \to \mathbb{R}$ with $\|t\|_\infty \le \varepsilon$ and $\delta t = c$.*

PROOF. Since A is abelian and, in particular, amenable, we have that the bounded cohomology group $\mathrm{H}^2_{\mathrm{b}}(A, \mathbb{R})$ is trivial by [Fri21, Theorem 3.6]. Furthermore, A has n-th vanishing modulus of 1; see [FFLM10, Definition 4.10 and Example 4.11]. The statement of the lemma is just a reformulation of this fact. ∎

3.23. LEMMA. *Suppose that $V \to X \xrightarrow{\pi} C$ is a short exact sequence of abelian Polish groups. If $V \cong \mathbb{R}$ and $C \cong \prod_{n\in\omega} A_n$ where A_n is a countable abelian group for every $n \in \omega$, then the sequence $V \to X \to C$ splits.*

PROOF. We identify C with $\prod_{n\in\omega} A_n$, and we identify V with $\mathbb{R}$ and with a closed subgroup of X. For $n \in \omega$, we write

$$C_{>k} = \{x \in C : \forall i \le k, x_i = 0\}.$$

Fix a compatible complete invariant metric d on X [Gao09, Exercise 2.2.9]. For $x \in V$ we let $|x|$ be its absolute value in $\mathbb{R}$.

By [BLP24, Proposition 4.6], there exists a continuous right inverse $\varphi : C \to X$ for π. Define by recursion a strictly increasing sequence $(k_n)_{n\in\omega}$ in ω such that, for all $n \in \omega$ and $x, y \in C_{>k_n}$, $d(\varphi(x), 0) \le 2^{-n}$ and $|\delta\varphi(x, y)| \le 2^{-n}$.

After replacing A_n with $\prod_{i=k_n+1}^{k_{n+1}} A_i$ for $n \in \omega$, we can assume without loss of generality that $k_n = n$. We have that $\|\delta\varphi|_{A_n\times A_n}\|_\infty \le 2^{-n}$. By Lemma 3.22, we have that exists a bounded function $t_n : A_n \to V \cong \mathbb{R}$ such that $\delta t_n = \delta\varphi|_{A_n\times A_n}$ and $\|t_n\|_\infty \le 2^{-n}$.

We define now $t : C \to V$ by setting

$$t(x) = \sum_{n\in\omega} t_n(x_n),$$

where we identify x_n as an element of A_n. Define also the continuous function $\psi : C \to X$ by setting

$$\psi(x) = \sum_{n\in\omega} \varphi(x_n)$$

We notice that ψ is a right inverse for π. Furthermore, we have that, for $x, y \in C$,

$$\delta\psi(x, y) = \sum_{n\in\omega} \delta\varphi(x_n, y_n) = \sum_{n\in\omega} \delta t_n(x_n, y_n) = \delta t(x, y).$$

Therefore, we have that $\eta := \psi - t : C \to X$ is a continuous homomorphism that is a right inverse for π. This shows that the short exact sequence $V \to X \to C$ splits. ∎

3.24. Lemma. *Suppose that $\mathbb{T} \to X \overset{\pi}{\to} C$ is a short exact sequence of abelian Polish groups. If $C \cong \prod_{n\in\omega} A_n$ where A_n is a countable abelian group for every $n \in \omega$, then the sequence $\mathbb{T} \to X \to C$ splits.*

Proof. We identify $\mathbb{T}$ with a closed subgroup of X and with $\mathbb{R}/\mathbb{Z}$. We let $d_{\mathbb{T}}$ be a compatible complete invariant metric on $\mathbb{T}$. For $\varepsilon > 0$ define $\mathbb{T}_\varepsilon = \{x \in \mathbb{T} : d_{\mathbb{T}}(x, 0) < \varepsilon\}$. Fix $\varepsilon > 0$ such that there exists a continuous function $\rho : \mathbb{T}_{3\varepsilon} \to \mathbb{R}$ such that $\rho(z)+\mathbb{Z} = z$ for every $z \in \mathbb{T}_{3\varepsilon}$, and $\rho(x + y) = \rho(x) + \rho(y)$ for all $x, y \in \mathbb{T}_\varepsilon$.

By [BLP24, Proposition 4.6], we have a continuous right inverse $\varphi : C \to X$ for π. Define the continuous $\kappa : C \times C \to \mathbb{T}$ by $\kappa(x, y) = \varphi(x + y) - \varphi(x) - \varphi(y)$. Since κ is continuous, there exists $n_0 \in \omega$ such that $d_{\mathbb{T}}(\kappa(x, y), 0) < \varepsilon$ for every $x \in C_{\geq n_0} := \{x \in C : \forall i < n_0, x_i = 0\}$.

Since $\mathrm{Ext}_{\mathrm{Yon}}(C_{<n_0}, \mathbb{T}) = 0$ by injectivity of $\mathbb{T}$ in the category of locally compact abelian Polish groups, where

$$C_{<n_0} = \{x \in C : \forall i \geq n_0, x_i = 0\},$$

it suffices to show that the function $\pi|_{\pi^{-1}(C_{\geq n})} : \pi^{-1}(C_{\geq n}) \to C_{\geq n}$ has a right inverse that is a continuous group homomorphism. Thus, we can assume without loss of generality that $n = 0$ and $d_{\mathbb{T}}(\kappa(x, y), 0) < \varepsilon$ for every $x \in C$.

This implies that, setting $c := \rho\circ\kappa$, one obtains a continuous 2-cocycle $c : C \times C \to \mathbb{R}$ such that $c(x, y) + \mathbb{Z} = \kappa(x, y)$ for every $x, y \in C$. By Lemma 3.23 and Lemma 3.3, we have that $\mathrm{Ext}_{\mathrm{Yon}}(C, \mathbb{R}) = 0$. Therefore, c is a coboundary, and hence κ is a coboundary as well. This implies that φ has a right inverse that is a continuous homomorphism. ∎

3.25. Lemma. *Suppose that C is a non-Archimedean abelian Polish group. Then there exists a continuous surjective homomorphism $\left(\mathbb{Z}^{(\omega)}\right)^\omega \to C$.*

Proof. We have that C is isomorphic to a closed subgroup of $\prod_{n\in\omega} A_n$ where, for every $n \in \omega$, A_n is countable. For every $n \in \omega$, there exists a surjective homomorphism $\mathbb{Z}^{(\omega)} \to A_n$. Hence, C is isomorphic to a quotient of a closed subgroup $\hat{C}$ of $\left(\mathbb{Z}^{(\omega)}\right)^\omega$. Let S be a pruned tree on $\mathbb{Z}^{(\omega)}$ such that $\hat{C}$ is equal to closed subgroup $[S] \subseteq \left(\mathbb{Z}^{(\omega)}\right)^\omega$ consisting of the branches of S. For every $n \in \omega$, let $B_n := S \cap \left(\mathbb{Z}^{(\omega)}\right)^n$, which is a subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^n$. Since S is pruned, we have that the projection map $B_{n+1} \to B_n$ is onto. Furthermore, we have that $\hat{C} \cong [S] \cong \lim_n B_n$. Since $B_n \cong \mathbb{Z}^{(\omega)}$ and $\mathbb{Z}^{(\omega)}$ is projective for countable abelian groups, we have that $B_{n+1} \cong B_n \oplus \mathrm{Ker}(\pi_{n+1})$. Thus, we have that $\hat{C} \cong B_0 \oplus \prod_{n\geq 1} \mathrm{Ker}(\pi_n)$. Since for every countable group B there exists a surjective homomorphism $\mathbb{Z}^{(\omega)} \to B$, the conclusion follows. ∎

3.26. Lemma. *Suppose that $L \to X \overset{\pi}{\to} C$ is a short exact sequence of abelian Polish groups. If $L \cong \mathbb{R}^\alpha \oplus \mathbb{T}^\beta$ for $\alpha, \beta \leq \omega$ and C is non-Archimedean, then the sequence $L \to X \to C$ splits.*

PROOF. By Lemma 3.18 it suffices to consider the case when $L = \mathbb{R}$ or $L = \mathbb{T}$. In this case, Lemma 3.23 and Lemma 3.24 prove the result when C is product of countable groups. If C is an arbitrary non-Archimedean Polish abelian group, then by Lemma 3.25 there exists a surjective continuous homomorphism $g : \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to C$. Considering the pullback diagram

$$\begin{array}{ccc} Y & \to & \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \\ \downarrow & & \downarrow g \\ X & \to & C \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} 0 & \to & \ker(g) & \to & \ker(g) \\ \downarrow & & \downarrow & & \downarrow \\ L & \to & Y & \to & \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \\ \downarrow & & \downarrow & & \downarrow \\ L & \to & X & \to & C \end{array}$$

whose rows and columns are short exact sequences. Then the exact sequence $L \to Y \to \left(\mathbb{Z}^{(\omega)}\right)^{\omega}$ splits as explained at the beginning of the proof of this lemma, hence $Y \cong L \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\omega}$ is pro-Lie. Therefore, X is pro-Lie, being quotient of a pro-Lie group by a closed subgroup. (Notice that a surjective continuous homomorphism between Polish groups is a quotient mapping by the Open Mapping Theorem for Polish groups; see [Gao09, Corollary 2.3.4].) By Lemma 3.9, we conclude that the short exact sequence $L \to X \to C$ splits. ■

3.27. LEMMA. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. If A is pro-Lie and C is non-Archimedean, then B is pro-Lie.*

PROOF. As in the proof of Lemma 3.21, by Lemma 3.11 we have a short exact sequence $D \to A \to T$ where D is non-Archimedean and T is a torus group. Considering the pushout diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ T & \to & B_T \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} D & \to & D & \to & 0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & B_T & \to & C \end{array}$$

whose rows and columns are short exact sequences. Since C is non-Archimedean and T is a torus group, we have that the short exact sequence $T \to B_T \to C$ splits by Lemma

3.26. Consider the commuting diagram

$$\begin{array}{ccccc} D & \to & B_0 & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \to & B & \to & B_T \cong T \oplus C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & T & \to & T \end{array}$$

with short exact rows and columns. We have that B_0 is non-Archimedean since D and C are non-Archimedean. Hence, B is pro-Lie by Lemma 3.20. ∎

3.28. Theorem. *Suppose that $A \to B \to C$ is a short exact sequence of abelian Polish groups. Then we have that B is pro-Lie if and only if A and C are pro-Lie.*

Proof. We just need to prove that if A and C are pro-Lie, then B is pro-Lie, as the other implication is established in [HM07, Chapter 3]. By Lemma 3.11, we have a short exact sequence $D \to C \to T$ where D is non-Archimedean and T is a torus group. Considering the pullback diagram

$$\begin{array}{ccc} B_D & \to & D \\ \downarrow & & \downarrow \\ B & \to & C \end{array}$$

we obtain a commuting diagram

$$\begin{array}{ccccc} A & \to & B_D & \to & D \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & T & \to & T \end{array}$$

with exact rows and columns. Since D is non-Archimedean, we have that B_D is pro-Lie by Lemma 3.27. Since T is locally compact, we have that B is pro-Lie by Lemma 3.21. ∎

Theorem 3.17 is an immediate consequence of Theorem 3.28.

3.29. Type decomposition for pro-Lie Polish abelian groups. We recall the type decomposition for locally compact Polish abelian groups as described by Hoffmann and Spitzweck in [HS07, Section 2].

3.30. Definition. *Let A be a locally compact Polish abelian group. Then we say that:*

- *A is a topological p-group if, for every $x \in A$, the sequence $(p^n x)_{n\in\omega}$ is vanishing or, equivalently, A has a basis of zero neighborhoods consisting of open subgroups U such that A/U is a p-group [Arm81, Chapter 2];*
- *A is a topological torsion group if, for every $x \in A$, the sequence $(n!x)_{n\in\omega}$ is vanishing or, equivalently, A has a basis of zero neighborhoods consisting of open subgroups U such that A/U is a torsion group [Arm81, Chapter 3];*

- *A is type $\mathbb{Z}$ if it is discrete and torsion-free;*
- *A is type $\mathbb{S}^1$ if it is compact and connected or, equivalently, its Pontryagin dual $A^\vee$ is type $\mathbb{Z}$;*
- *A is type $\mathbb{R}$ if it is a vector group;*
- *A is type $\mathbb{A}$ if $A \cong V \oplus B$ where V is a vector group and B is a topological torsion group.*

For a locally compact Polish abelian group A, it is proved in [HS07, Proposition 2.2] that there exist canonical short exact sequences $F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$ and $A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$ where $A_{\mathbb{Z}}$, $A_{\mathbb{S}^1}$, and $A_{\mathbb{A}}$ have type $\mathbb{Z}$, $\mathbb{S}^1$, and $\mathbb{A}$, respectively. Furthermore, we have $A_{\mathbb{A}} = A_{\mathbb{R}} \oplus A_{\mathrm{t}}$ where $A_{\mathbb{R}}$ is a finite-dimensional vector group and A_{t} is a topological torsion group.

If P is any of the properties from Definition 3.30, we say a pro-Lie Polish abelian group A satisfies P if and only if

$$\{N \in \mathcal{N}(A) : A/N \text{ satisfies } P\}$$

is cofinal in $\mathcal{N}(A)$ (ordered by reverse inclusion). We say that A has sub-type $\mathbb{Z}$ if it is a Polishable subgroup of a type $\mathbb{Z}$ pro-Lie Polish abelian group.

3.31. Lemma. *Suppose that A is an abelian Polish pro-Lie group.*

1. *A is a topological p-group if and only if A is isomorphic to a closed subgroup of $\mathbb{Z}(p^\infty)^\omega$ if and only if the sequence $(p^n x)_{n\in\omega}$ is vanishing for every $x \in A$;*
2. *A is a topological torsion group if and only if A is isomorphic to a closed subgroup of $\prod_p \mathbb{Z}(p^\infty)^\omega$ if and only if the sequence $(n!x)_{n\in\omega}$ is vanishing for every $x \in A$;*
3. *A is type $\mathbb{Z}$ if and only if A is isomorphic to a closed subgroup of $(\mathbb{Q}^{(\omega)})^\omega$;*
4. *A is type $\mathbb{S}^1$ if and only if A is compact connected;*
5. *A is type $\mathbb{R}$ if and only if it is a vector group;*
6. *A is type $\mathbb{A}$ if and only if $A \cong V \oplus B$ where B is a vector group and B is a topological torsion group.*

Proof. We give details for (6), the other assertions being easy to see.

Suppose that A is type $\mathbb{A}$. Then we have that $A \cong V \oplus B$ where V is a vector group and B is vector-free. We claim that B is a topological torsion group. If $N \in \mathcal{N}(B)$ then there exist $L \in \mathcal{N}(V)$ and $M \in \mathcal{N}(B)$ contained in N such that

$$\frac{V \oplus B}{L \oplus M} \cong (V/L) \oplus (B/M)$$

is type $\mathbb{A}$, and hence B/M is type $\mathbb{A}$. We can write $B/M = W \oplus S$ where W is a vector group and S is a topological torsion group. Since B is vector-free, we must have that the composition $B \to B/M \to W$ where $B \to B/M$ is the quotient map and $B/M \to W$ is the coordinate projection, is the zero homomorphism. This shows that $W = 0$ and B/M is a topological torsion group.

The converse implication follows from (1) and (5). ■

For locally compact abelian Polish groups A and B, we have that $\mathrm{Hom}(A, B) = 0$ whenever:

- A has type $\mathbb{S}^1$ and B has type $\mathbb{A}$ or $\mathbb{Z}$;
- A has type $\mathbb{A}$ and B has type $\mathbb{Z}$.

It follows from this and Proposition 3.14 that the same conclusions hold when A and B are pro-Lie Polish abelian groups.

If A is a pro-Lie Polish abelian group, then by [HM07, Theorem 5.20] we have that A has a largest closed vector subgroup, which we denote by $A_{\mathbb{R}}$. Then we have that $A = A_{\mathbb{R}} \oplus B$ where B is vector-free. We also let $A_{\mathbb{S}^1}$ be the connected component of the trivial element in B, which is *compact* [HM07, Theorem 24(ii)]. We let $c(A)$ be the connected component of the trivial element in A.

3.32. Lemma. *Suppose that* $\left(A^{(k)}\right)$ *is a tower of compact connected Lie groups. Then* $\lim_k A^{(k)}$ *is connected.*

Proof. Define $B^{(k)} = \bigcap_{n>k} \mathrm{Ran}\left(A^{(n)} \to A^{(k)}\right) \subseteq A^{(k)}$ for $k \in \omega$. Then, by compactness, $\left(B^{(k)}\right)$ is an epimorphic tower of compact connected Lie groups with $\lim_k B^{(k)} = \lim_k A^{(k)}$. Thus, without loss of generality we can assume that $\left(A^{(k)}\right)$ is an epimorphic tower. If $U \subseteq \lim_k A^{(k)}$ is an open subgroup, then for every $k \in \omega$, $\pi^{(k)}(U) \subseteq A^{(k)}$ is an open subgroup. Thus, $\pi^{(k)}(U) = A^{(k)}$ and hence U is dense in $\lim_k A^{(k)}$. This implies that $U = \lim_k A^{(k)}$, showing that $\lim_k A^{(k)}$, being a compact group, is connected. ■

3.33. Lemma. *If V is a* locally compact *vector group, then V is projective in the category of pro-Lie Polish abelian groups.*

Proof. Suppose that Y is a pro-Lie Polish abelian group and $\varphi : Y \to V$ is a surjective continuous group homomorphism. Since V is a Lie group, we can find a tower $(Y_n, \eta_n : Y_{n+1} \to Y_n)$ of abelian Lie groups and a continuous group homomorphism $\varphi_0 : Y_0 \to V$ such that $Y = \lim_n Y_n$ and $\varphi = \varphi_0 \circ \pi_0$ where $\pi_0 : Y \to Y_0$ is the canonical map. By projectivity of V in the category of locally compact Polish abelian groups, we can find a continuous homomorphism $\psi_0 : V \to Y_0$ that is a right inverse for φ_0. Since the image of ψ_0 is a closed subgroup of Y_0 isomorphic to V, we can find a continuous homomorphism $\psi_1 : \psi_0(V) \to Y_1$ that is a right inverse for $\eta_0|_{\eta_0^{-1}(\psi_0(V))}$, whence $\psi_1 \circ \psi_0$ is a continuous homomorphism that is a right inverse for $\varphi_0 \circ \eta_0$. Proceeding in this fashion, we can define recursively a continuous homomorphisms $\psi_n : (\psi_{n-1} \circ \cdots \circ \psi_0)(V) \to Y_n$ that is a right inverse for $\eta_n|_{\eta_n^{-1}((\psi_n\circ\cdots\circ\psi_0)(V))}$, whence $\psi_n \circ \cdots \circ \psi_0$ is a continuous homomorphism that is a right inverse for $\varphi_0 \circ \eta_0 \circ \cdots \circ \eta_{n-1}$.

One can then define the continuous group homomorphism

$$\psi : V \to Y,\, x \mapsto ((\psi_n \circ \cdots \circ \psi_0)(x))_{n\in\omega},$$

which is a right inverse for φ. ∎

3.34. Corollary. *If Y is a vector-free pro-Lie Polish abelian group, then for every $N \in \mathcal{N}(Y)$, Y/N is a vector-free abelian Lie group.*

3.35. Lemma. *Suppose that $\pi : B \to A$ is a continuous surjective homomorphism between Lie Polish abelian groups. Then its restriction $B_0 \to A_0$ is surjective.*

Proof. We can write $B = B_0 \oplus C$ where C is countable. Thus, we have that $\pi(B_0)$ is an analytic subgroup of A_0 of countable index, whence it is open. Since A_0 is connected, this implies that $\pi(B_0) = A_0$. ∎

3.36. Lemma. *Suppose that A is a pro-Lie Polish abelian group. Let $\left(A^{(k)}\right)$ be an epimorphic tower with $A = \lim_k A^{(k)}$. Then $(c(A^{(k)}))$ is an epimorphic tower.*

Proof. This follows from Lemma 3.35. ∎

3.37. Lemma. *Let A be a vector-free pro-Lie Polish abelian group. Suppose that $\left(A^{(k)}\right)$ is an epimorphic tower of Lie groups with $A = \lim_k A^{(k)}$. Then $c(A) = \lim_k c\left(A^{(k)}\right)$ is a compact connected subgroups of A.*

Proof. Notice that, for every $k \in \omega$, $A^{(k)}$ is vector-free. We have that $\lim_k c(A^{(k)})$ is compact, and connected by Lemma 3.32. This shows that $\lim_k c(A^{(k)}) \subseteq c(A)$. The converse inclusion follow from functoriality of $c(A)$. ∎

3.38. Lemma. *Suppose that A is a vector-free pro-Lie Polish abelian group. Then $A/c(A)$ is a non-Archimedean Polish abelian group.*

Proof. Let $\left(A^{(k)}\right)$ be an epimorphic tower of (necessarily vector-free) Lie Polish abelian groups with $A = \lim_k A^{(k)}$. For every $k \in \omega$, we have a short exact sequence

$$c(A^{(k)}) \to A^{(k)} \to C^{(k)}$$

where $C^{(k)}$ is countable. Since $(c(A)^{(k)})$ is an epimorphic tower by Lemma 3.36, this implies that we have a short exact sequence

$$\lim_k c(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}$$

Since $c(A) = \lim_k c(A^{(k)})$ by Lemma 3.36, $A = \lim_k A^{(k)}$, and $\lim_k C^{(k)}$ is non-Archimedean, the conclusion follows. ∎

3.39. Lemma. *Suppose that A is a non-Archimedean Polish group. Then A has a largest topological torsion closed subgroup A_{t}. Furthermore, A/A_{t} has sub-type $\mathbb{Z}$.*

Proof. Let $\left(A^{(k)}\right)$ be an epimorphic tower of countable groups with $A = \mathrm{lim}_k A^{(k)}$. For $k \in \omega$, let $T\left(A^{(k)}\right)$ be the torsion subgroup of $A^{(k)}$, and set $C^{(k)} := A^{(k)}/T\left(A^{(k)}\right)$. Then we have an exact sequence

$$0 \to \mathrm{lim}_k T(A^{(k)}) \to \mathrm{lim}_k A^{(k)} \to \mathrm{lim}_k C^{(k)}.$$

Setting $A_{\mathrm{t}} := \mathrm{lim}_k T\left(A^{(k)}\right)$ and $C := \mathrm{lim}_k C^{(k)}$, we have that A_{t} is a topological torsion group and C has type $\mathbb{Z}$. The above argument shows that A/A_{t} is a Polishable subgroup of C, and hence of sub-type $\mathbb{Z}$.

If B is a topological torsion closed subgroup of A, then $\mathrm{Hom}(B, C) = 0$ and hence $B \subseteq A_{\mathrm{t}}$. ∎

3.40. Lemma. *Suppose that $\left(A^{(k)}\right)$ is a tower of connected Lie groups. Then $\mathrm{lim}_k A^{(k)}$ is connected.*

Proof. As in the proof of Lemma 3.32, we can assume without loss of generality that $\left(A^{(k)}\right)$ is an epimorphic tower. We can write $A^{(k)} = V^{(k)} \oplus T^{(k)}$ where $V^{(k)}$ is a finite-dimensional vector group and $T^{(k)}$ is a finite-dimensional torus group. Furthermore, since $\mathrm{Hom}\left(T^{(k+1)}, V^{(k)}\right) = 0$ for every $k \in \omega$, $\left(V^{(k)}\right)$ is an epimorphic tower. Therefore, we have an exact sequence

$$0 \to \mathrm{lim}_k T^{(k)} \to \mathrm{lim}_k A^{(k)} \to \mathrm{lim}_k V^{(k)} \to \mathrm{lim}^1_k T^{(k)}.$$

Notice that, for every $k \in \omega$, if d_k is the dimension of $T^{(k)}$, then $d_k + 1$ is the maximum length of a chain of compact connected subgroups of $T^{(k)}$. It follows that the chain of subgroups $\left(\mathrm{Ran}\left(T^{(n)} \to T^{(k)}\right)\right)_{n>k}$ stabilizes. As this holds for every $k \in \omega$, $\mathrm{lim}^1_k T^{(k)} = 0$. By projectivity of vector groups, and since $\mathrm{Ran}\left(V^{(k+1)} \to V^{(k)}\right)$ is a closed vector subgroup of $V^{(k)}$ for every $k \in \omega$ by [HM07, Theorem A2.12], we have that $V := \mathrm{lim}_k V^{(k)}$ is a vector group. Hence, by projectivity of vector groups, we have that $\mathrm{lim}_k A^{(k)} \cong \mathrm{lim}_k T^{(k)} \oplus V$, where $\mathrm{lim}_k T^{(k)}$ is connected by the previous lemma. ∎

3.41. Corollary. *Let A be a pro-Lie Polish abelian group.*

1. *Suppose that $\left(A^{(k)}\right)$ is an epimorphic tower of Lie groups with $A = \mathrm{lim}_k A^{(k)}$. Then $c(A) = \mathrm{lim}_k c\left(A^{(k)}\right)$.*
2. *$A/c(A)$ is a non-Archimedean Polish abelian group.*

Proof. The proofs of Lemma 3.37 and Lemma 3.38 apply verbatim with Lemma 3.32 replaced with Lemma 3.40. ∎

3.42. LEMMA. *Suppose that A is a pro-Lie Polish abelian group. Then $A/c(A)$ is a non-Archimedean Polish abelian group.*

PROOF. Let $\left(A^{(k)}\right)$ be an epimorphic tower of Lie Polish abelian groups with $A = \lim_k A^{(k)}$. For every $k \in \omega$, we have a short exact sequence

$$c\left(A^{(k)}\right) \to A^{(k)} \to C^{(k)}$$

where $C^{(k)}$ is countable. Since $(c(A)^{(k)})$ is an epimorphic tower by Lemma 3.36, we have a short exact sequence

$$\lim_k c(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}$$

By Corollary 3.41, $c(A) = \lim_k c(A^{(k)})$. Hence, $A/c(A) \cong \lim_k C^{(k)}$ is non-Archimedean. ∎

3.43. LEMMA. *Suppose that A is a non-Archimedean Polish group. Then A has a largest topological torsion closed subgroup A_{t}. Furthermore, A/A_{t} has sub-type $\mathbb{Z}$.*

PROOF. Let $\left(A^{(k)}\right)$ be an epimorphic tower of countable groups with $A = \lim_k A^{(k)}$. For $k \in \omega$, let $T\left(A^{(k)}\right)$ be the torsion subgroup of $A^{(k)}$, and set $C^{(k)} := A^{(k)}/T\left(A^{(k)}\right)$. Then we have an exact sequence

$$0 \to \lim_k T(A^{(k)}) \to \lim_k A^{(k)} \to \lim_k C^{(k)}.$$

Setting $A_{\mathrm{t}} := \lim_k T\left(A^{(k)}\right)$ and $C := \lim_k C^{(k)}$, we have that A_{t} is a topological torsion group and C has type $\mathbb{Z}$. The above argument shows that A/A_{t} is a Polishable subgroup of C, and hence of sub-type $\mathbb{Z}$.

If B is a topological torsion closed subgroup of A, then $\mathrm{Hom}(B, C) = 0$ and hence $B \subseteq A_{\mathrm{t}}$. ∎

3.44. LEMMA. *We have that $\mathbb{R}^\omega$ is projective in* **proLiePAb**.

PROOF. Suppose that X is a pro-Lie Polish abelian group and $\varphi : X \to \mathbb{R}^\omega$ is a continuous surjective homomorphism. We prove that it has a right inverse that is a continuous group homomorphism. We can write $X = V \oplus Y$ where V is a vector group and Y has no nonzero closed vector subgroups. We have short exact sequences $Y_{\mathbb{S}^1} \to F_{\mathbb{Z}}Y \to Y_{\mathrm{t}}$ and $F_{\mathbb{Z}}Y \to Y \to Y_{\mathbb{Z}}$, where $\varphi|_{Y_{\mathbb{S}^1}} = 0$, and the induced function $Y_{\mathrm{t}} \to \mathbb{R}^\omega$ is also trivial. Thus, if $\psi : Y_{\mathbb{Z}} \to \mathbb{R}^\omega$ is the continuous homomorphism induced by $\varphi|_Y$, then $\mathrm{Ran}(\psi) = \mathrm{Ran}(\varphi|_Y)$.

By [HM07, Theorem A2.12], $\mathrm{Ran}(\varphi|_V)$ is a closed subgroup and a topological direct summand of $\mathbb{R}^\omega$.

Let $\left(N_k^V\right)$ be a cofinal sequence in $\mathcal{N}(V)$, $\left(N_k^Y\right)$ be a cofinal sequence in $\mathcal{N}(Y)$, and set $N_k := N_k^V \oplus N_k^Y$ for $k \in \omega$. Thus, (N_k) is a cofinal sequence in $\mathcal{N}(X)$. For $n \in \omega$, set

$$M_n = \{x \in \mathbb{R}^\omega : \forall i \leq n,\, x_i = 0\} \subseteq \mathbb{R}^\omega.$$

We claim that $\mathrm{Ran}(\varphi|_V) = \mathbb{R}^\omega$. It suffices to prove that, for every $n \in \omega$, $\pi_{M_n} \circ \varphi|_V : V \to \mathbb{R}^n$ is surjective. We have that $\pi_{M_n} \circ \varphi : X \to \mathbb{R}^n$ factors through X/N_k for some $k \in \omega$. Set $Z_k := X/N_k$ and let $\pi_{N_k} : X \to Z_k$ be the quotient map. Then we have that there exists a continuous homomorphism $\psi : Z_k \to \mathbb{R}^n$ such that $\psi\pi_{N_k} = \pi_{M_n}\varphi$. We have $Z_k \cong V/N_k^V \oplus Y/N_k^Y$. Since Y has no nonzero vector subgroups, it is easily seen considering the type decompositions of Y and Y/N_k that the image of $\psi|_{Y/N_k^Y} : Y/N_k^Y \to \mathbb{R}^n$ is countable. Since $\mathbb{R}^n$ has no nontrivial analytic subgroups of countable index (as any such a subgroup must be open), it follows that $\psi|_{V/N_k^V}$ is surjective, and hence $\pi_{M_n}\varphi|_V$ is surjective.

We have therefore shown that $\varphi|_V : V \to \mathbb{R}^\omega$ is surjective. Since $\mathrm{Ker}(\varphi|_V)$ is a closed $\mathbb{R}$-subspace of V, it is a vector group. Therefore, the short exact sequence $\mathrm{Ker}(\varphi|_V) \to V \to \mathbb{R}^\omega$ splits, and there exists a continuous homomorphism $\psi : \mathbb{R}^\omega \to V \subseteq X$ that is a right inverse for $\varphi|_V$. Thus, ψ is also a right inverse for φ, if regarded as a continuous group homomorphism with codomain X. ■

If A is a pro-Lie Polish abelian group, we define A_{t} to be the largest topological torsion closed subgroup of $A/c(A)$, and $F_{\mathbb{Z}}A$ to be the preimage of A_{t} under the quotient map $A \to A/c(A)$. Define also $A_{\mathbb{Z}} := A/F_{\mathbb{Z}}A$ and $A_{\mathbb{S}^1} = \mathrm{comp}(c(A))$, which is the set of elements of $c(A)$ that generate a subgroup with compact closure.

By Lemma 3.42 and Lemma 3.44, we have that $c(A) = A_{\mathbb{R}} \oplus A_{\mathbb{S}^1}$ where $A_{\mathbb{R}}$ is the largest closed vector subgroup of A. We also set $A_{\mathbb{A}} = A_{\mathbb{R}} \oplus A_{\mathrm{t}}$. Then we have canonical exact sequences

$$A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$$

and

$$F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$$

where $A_{\mathbb{S}^1}$, $A_{\mathbb{A}}$ have type $\mathbb{S}^1$ and $\mathbb{A}$, respectively, and $A_{\mathbb{Z}}$ has sub-type $\mathbb{Z}$.

3.45. Remark. In general it is not the case that $A_{\mathbb{Z}}$ has type $\mathbb{Z}$. For example, consider the reduced product

$$G := \prod_n (\mathbb{Z} : 2\mathbb{Z})$$

consisting of sequences of integers that are eventually even. Then G is a Polishable subgroup of $\mathbb{Z}^\omega$ such that every open subgroup U of G is such that G/U has nontrivial torsion subgroup. Indeed, there exists $n \in \omega$ such that the set

$$U_n := \{x \in G : \forall i < \omega,\, x_i \in 2\mathbb{Z} \text{ and } \forall i < n,\, x_i = 0\}$$

is contained U, and hence G/U is a quotient of $G/U_n \cong \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{(\omega)}$.

It easily follows that, if A is a pro-Lie Polish abelian groups, then setting $A_{\mathbb{S}^1} := \lim_{N \in \mathcal{N}(G)} (A/N)_{\mathbb{S}^1}$ etcetera, we obtain short exact sequences $A_{\mathbb{S}^1} \to F_{\mathbb{Z}}A \to A_{\mathbb{A}}$ and $F_{\mathbb{Z}}A \to A \to A_{\mathbb{Z}}$. Furthermore, we have that $A_{\mathbb{S}^1}$ is the largest closed subgroup of A of type $\mathbb{S}^1$ and the smallest closed subgroup of $F_{\mathbb{Z}}A$ such that $F_{\mathbb{Z}}A/A_{\mathbb{S}^1}$ is of type $\mathbb{A}$, while $F_{\mathbb{Z}}A$ is the smallest closed subgroup of A such that $A/F_{\mathbb{Z}}A$ is of sub-type $\mathbb{Z}$.

This decomposition for pro-Lie Polish abelian groups can be reformulated in terms of *torsion pars*, showing that pro-Lie Polish abelian groups of type $\mathbb{A}$, type $\mathbb{S}^1$, and subtype $\mathbb{Z}$ form fully exact subcategories of the category **proLiePAb**.

3.46. Injective and projective pro-Lie Polish abelian groups. In this section, we completely characterize the projective objects in **proLiePAb**, and obtain the following.

3.47. Theorem. *The quasi-abelian category* **proLiePAb** *has enough projectives but not enough injectives, and homological dimension* 1*.*

Recall that an abelian Lie group A is of the form $V \oplus T \oplus D$ where V is a finite-dimensional vector group, T is a finite-dimensional torus, and D is discrete. (This is equivalent to the assertion that A is a locally compact Polish abelian group *with no small subgroups*; see [Mos67, Theorem 2.4].) It easily follows that there exists a surjective homomorphism $\mathbb{R}^n \oplus \mathbb{Z}^{(\omega)} \to A$ for some $n \in \omega$. Recall that the projective objects in **LiePAb** are precisely those of the form $V \oplus F$ where V is a finite-dimensional vector group and F is a countable free abelian group [Mos67, Theorem 3.3]. The injective objects in **LiePAb** are precisely those of the form $V \oplus T$ where V is a finite-dimensional vector group and T is a finite-dimensional torus [Mos67, Theorem 3.2].

3.48. Lemma. *Suppose that $\mathcal{A}$ be a quasi-abelian subcategory of* **PAb** *closed under countable products, and let $\mathcal{D}$ be a class of projective objects in $\mathcal{A}$ closed under direct sums. Suppose that (G_n) is an inverse sequence of objects of $\mathcal{A}$ with surjective continuous homomorphisms $\pi_{n+1} : G_{n+1} \to G_n$ as bonding maps. We also set $G_{-1} = 0$ and $\pi_0 = 0$. Suppose that for every $n \in \omega$ there exists a surjective continuous homomorphism $P \to \mathrm{Ker}\,(\pi_n : G_n \to G_{n-1})$ for some $P \in \mathcal{D}$. Then there exists a surjective continuous homomorphism $\mathrm{lim}_n D_n \to \mathrm{lim}_n G_n$ where $D = (D_n)$ is an inverse sequence with $D_n \in \mathcal{D}$ and surjective continuous homomorphisms $p_{n+1} : D_{n+1} \to D_n$ as bonding maps.*

Proof. We define by recursion objects D_n of $\mathcal{D}$, continuous homomorphisms $p_{n+1} : D_{n+1} \to D_n$, and $\varphi_n : D_n \to G_n$ such that $\pi_{n+1}\varphi_{n+1} = \varphi_n p_{n+1}$. We have that φ_0 exists by hypothesis. Suppose that φ_i and p_i have been defined for $i \leq n$. Then we consider a pushout diagram

$$\begin{array}{ccc} Y_{n+1} & \to & G_{n+1} \\ \downarrow & & \downarrow \\ D_n & \to & G_n \end{array}$$

Since $D_n \to G_n$ and $G_{n+1} \to G_n$ are continuous surjective homomorphisms, the same holds for $Y_{n+1} \to D_n$ and $Y_{n+1} \to G_n$. Since D_n is projective, we have that $Y_{n+1} \cong D_n \oplus \mathrm{Ker}\,(\pi_{n+1})$. By the inductive hypothesis, and since $\mathcal{D}$ is closed under direct sums, there exists a surjective continuous homomorphism $D_{n+1} \to Y_{n+1}$. This concludes the recursive construction.

One then has that $\mathrm{lim}_n \varphi_n : \mathrm{lim}_n D_n \to \mathrm{lim}_n G_n$ is a continuous surjective homomorphism. ∎

3.49. Lemma. *Let G be a pro-Lie Polish abelian group. Then there exists a surjective continuous homomorphism $\mathbb{R}^\omega \oplus \left(\mathbb{Z}^{(\omega)}\right)^\omega \to G$.*

Proof. By Lemma 3.48 it suffices to prove that the conclusion holds when G is an abelian Lie group, in which case the conclusion follows from the above remarks. ■

3.50. Lemma. *If G is a closed subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^\omega$, then $G \cong \mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta$ for some $\alpha, \beta \in \omega + 1$.*

Proof. Define $V_n = \left\{x \in \left(\mathbb{Z}^{(\omega)}\right)^\omega : \forall i < n, x_i = 0\right\}$. Then (V_n) is a basis of zero neighborhoods for $\left(\mathbb{Z}^{(\omega)}\right)^\omega$, and $(G \cap V_n)$ is a basis of zero neighborhoods of G. For $n \in \omega$, $G/(G \cap V_n)$ is a subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^n$, and hence free abelian. Since countable free abelian groups are projective, we have that $G \cong \lim_n G/(G \cap V_n)$ is isomorphic to a product of free abelian groups. The conclusion easily follows. ■

3.51. Lemma. *If G is a projective object in* **proLiePAb**, *then $G \cong \mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for some $\alpha, \beta, \gamma \in \omega + 1$.*

Proof. By Lemma 3.49, we have a surjective epimorphism $\left(\mathbb{Z}^{(\omega)}\right)^\omega \oplus \mathbb{R}^\omega \to G$. If G is projective, then we have that G is a topological direct summand of $\left(\mathbb{Z}^{(\omega)}\right)^\omega \oplus \mathbb{R}^\omega$. This implies that $G_{\mathbb{S}^1} = 0$, $G_{\mathrm{t}} = 0$, and hence $G \cong V \oplus G_0$ where V is a vector group and G_0 is subtype $\mathbb{Z}$. Since V is injective in **proLiePAb** and $\mathrm{Hom}\left(V, \left(\mathbb{Z}^{(\omega)}\right)^\omega\right) = 0$, we can assume without loss of generality that $V = 0$ and $G = G_0$ is subtype $\mathbb{Z}$. In this case, we have that $\mathrm{Hom}\left(\mathbb{R}^\omega, G\right) = 0$ and $G \subseteq \left(\mathbb{Z}^{(\omega)}\right)^\omega$. The conclusion thus follows from Lemma 3.50. ■

3.52. Lemma. *Suppose that $(C_i)_{i\in\omega}$ is a sequence of countable abelian groups, $C = \prod_{i\in\omega} C_i$, and A is a countable abelian group. If*

$$\mathrm{Ext}_{\mathrm{Yon}}\left(C_i, A\right) = 0$$

for every $i \in \omega$, then

$$\mathrm{Ext}_{\mathrm{Yon}}\left(C, A\right) = 0.$$

Proof. By Lemma 3.3 we can identify $\mathrm{Ext}_{\mathrm{Yon}}\left(C, A\right)$ with the subgroup of $\mathrm{Ext}_{\mathrm{c}}\left(C, A\right)$ corresponding to continuous cocycles. For $n \in \omega$, define

$$C^{>n} = \{x \in C : \forall i \leq n, x_i = 0\}$$

and

$$C^{\leq n} = \{x \in C : \forall i > n, x_i = 0\}.$$

Suppose that $c : C \times C \to A$ is a continuous cocycle. Since c is continuous, there exists $n_0 \in \omega$ such that $c\left(x, y\right) = 0$ whenever $x, y \in C^{>n}$. Since

$$\mathrm{Ext}_{\mathrm{Yon}}\left(C^{\leq n}, A\right) \cong \mathrm{Ext}\left(C_0, A\right) \oplus \cdots \oplus \mathrm{Ext}\left(C_n, A\right) \cong 0,$$

it follows that the inclusion $C^{>n} \to C$ induces an isomorphism

$$\mathrm{Ext}_{\mathrm{Yon}}\left(C, A\right) \to \mathrm{Ext}_{\mathrm{Yon}}\left(C^{>n}, A\right).$$

Thus, we have that c is a coboundary. ■

3.53. LEMMA. *Suppose that C and A are non-Archimedean Polish abelian groups. Suppose that (V_k) is a basis of zero neighborhoods of A with $V_0 = A$. If* $\mathrm{Ext}_{\mathrm{Yon}}(C, V_k/V_{k+1}) = 0$ *for every $k \in \omega$, then* $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$.

PROOF. Suppose that $c : C \times C \to A$ is a continuous cocycle. By hypothesis, there exists a continuous function $\varphi_0 : C \to A$ such that $c_0 := \delta\varphi_0 + c$ defines an element of $\mathrm{Ext}_{\mathrm{c}}(C, V_1)$. Proceeding in this fashion, we can define a sequence of continuous 2-cocycles $c_n : C \times C \to V_n$ and continuous functions $\varphi_n : C^\omega \to V_n$ with $c_0 = c$ such that $c_{n+1} = \delta\varphi_n + c_n$ for every $n \in \omega$.

Setting $\varphi := \sum_{n\in\omega} \varphi_n$ we obtain a continuous function $C \to A$ such that $\delta\varphi + c = 0$, concluding the proof. ∎

3.54. LEMMA. *Suppose that $(C_i)_{i\in\omega}$ is a sequence of countable abelian groups, and A is a non-Archimedean abelian Polish group. Suppose that (V_k) is a basis of zero neighborhoods of A with $V_0 = A$. Set also $C := \prod_{i\in\omega} C_i$. If* $\mathrm{Ext}_{\mathrm{Yon}}(C_i, V_k/V_{k+1}) = 0$ *for every $k, i \in \omega$, then* $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$.

PROOF. This is obtained combining Lemma 3.52 and Lemma 3.53. ∎

3.55. LEMMA. *Suppose that $(C_i)_{i\in\omega}$ is a sequence of countable free abelian groups. Then, setting $C := \prod_{k\in\omega} C_k$, we have that C is projective for pro-Lie Polish abelian groups.*

PROOF. By Lemma 3.3, if A is a pro-Lie Polish abelian group, then we can identify $\mathrm{Ext}_{\mathrm{Yon}}(C, A)$ with the subgroup of $\mathrm{Ext}_{\mathrm{c}}(C, A)$ corresponding to continuous cocycles. By Lemma 3.9, since pro-Lie Polish abelian groups form a thick subcategory of **PAb**, we have that $\mathrm{Ext}_{\mathrm{Yon}}(C, T) = 0$ for every torus T. Furthermore, by Lemma 3.54 we have that $\mathrm{Ext}_{\mathrm{Yon}}(C, A) = 0$ for every non-Archimedean Polish abelian group A. If B is an arbitrary pro-Lie abelian Polish group, then by Lemma 3.11 we have a short exact sequence $A \to B \to T$ where A is non-Archimedean and T is a torus. By Lemma 3.5(2), this induces an exact sequence

$$0 = \mathrm{Ext}_{\mathrm{Yon}}(C, A) \to \mathrm{Ext}_{\mathrm{Yon}}(C, B) \to \mathrm{Ext}_{\mathrm{Yon}}(C, T) = 0,$$

showing that $\mathrm{Ext}_{\mathrm{Yon}}(C, B) = 0$. ∎

3.56. THEOREM. *Let G be an abelian pro-Lie Polish group. The following assertions are equivalent:*

1. *G is projective in* **proLiePAb***;*
2. *G is isomorphic to $\mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for some $\alpha, \beta, \gamma \in \omega + 1$;*
3. *$\{N \in \mathcal{N}(G) : G/N$ is projective in* **LieAb**$\}$ *is cofinal in $\mathcal{N}(G)$.*

Proof. (2)⇒(1) We have that $\mathbb{R}^\omega$ is projective by Lemma 3.44. We have that $\left(\mathbb{Z}^{(\omega)}\right)^\omega$ is projective by Lemma 3.55. Any group of the form $\mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for $\alpha, \beta, \gamma \in \omega + 1$ is a direct summand of $\mathbb{R}^\omega \oplus \left(\mathbb{Z}^{(\omega)}\right)^\omega$, and hence projective.

(1)⇒(2) Any projective object is isomorphic to one of this form by Lemma 3.51.

(3)⇒(1) Let (N_k) be a cofinal sequence in $\mathcal{N}(G)$ such that $G^{(k)} := G/N_k$ is projective in **LieAb** for every $k \in \omega$. Then by Lemma 3.41 we have that $c(G) = G_\mathbb{R}$ and $G/c(G) \cong \lim_k G_\mathbb{Z}^{(k)}$ where $(G_\mathbb{Z}^{(k)})_{k\in\omega}$ is an epimorphic tower of free abelian groups. Whence, $G_\mathbb{Z}$ is isomorphic to a product of free abelian groups and hence projective in **proLiePAb**. Hence, $G \cong G_\mathbb{R} \oplus G_\mathbb{Z}$ is projective in **proLiePAb**.

(2)⇒(3) This follows from the fact that the projective objects in **LieAb** are of the form $\mathbb{R}^n \oplus \mathbb{Z}^{(\alpha)}$ for $n < \omega$ and $\alpha \leq \omega$. ■

3.57. Theorem. *The category* **proLiePAb** *has enough projectives and homological dimension* 1.

Proof. We have that **proLiePAb** has enough projectives by Theorem 3.56 and Lemma 3.49. We now prove that closed subgroups of projectives are projective. Suppose that G is a closed subgroup of $(F \oplus \mathbb{R})^\omega$ where F is a countable free abelian group of infinite rank. Then we have that $G_{\mathbb{S}^1} = 0$ and $G_\mathrm{t} = 0$. Hence, we have that $G \cong G_\mathbb{R} \oplus G_\mathbb{Z}$. Since $G_\mathbb{R}$ is injective, we can assume without loss of generality that $G = G_\mathbb{Z}$.

Set
$$N_k := \{x \in (F \oplus \mathbb{R})^\omega : \forall i < k,\, x_i = 0\}.$$
Then (N_k) is a vanishing sequence in $\mathcal{N}((F \oplus \mathbb{R})^\omega)$, and $(M_k)_{k\in\omega}$ is a vanishing sequence in $\mathcal{N}(G)$, where $M_k := N_k \cap G$. We have that G/M_k is isomorphic to a closed subgroup L of
$$(F \oplus \mathbb{R})^\omega / N_k \cong (F \oplus \mathbb{R})^k \cong F^k \oplus \mathbb{R}^k.$$
We consider the cohomological derived functor of $\mathrm{H}^0 \circ \mathrm{Hom}^\bullet$ for locally compact Polish abelian groups introduced in [Lup25]; see also [HS07, Section 4]. For every locally compact Polish abelian group A, we have an exact sequence
$$0 = \mathrm{Ext}\left(F^k \oplus \mathbb{R}^k, A\right) \to \mathrm{Ext}(L, A) \to \mathrm{Ext}^2\left(\left(F^k \oplus \mathbb{R}^k\right)/L, A\right) = 0.$$
Therefore, we have that L is a projective locally compact Polish abelian group. Since L is also countable, we have that is a free abelian group.

Thus, we have that $G \cong \lim_k G/M_k$ where, for every $k \in \omega$, G/M_k is a countable free abelian group. This shows that G is isomorphic to a product of countable free abelian groups, whence projective in **proLiePAb**. ■

3.58. Corollary. *The functor*
$$\mathrm{Hom}^\bullet : \mathrm{K}^b(\mathbf{proLiePAb})^{\mathrm{op}} \times \mathrm{K}^b(\mathbf{proLiePAb}) \to \mathrm{K}^b(\mathbf{Ab})$$
admits a total right derived functor
$$\mathrm{RHom} : \mathrm{D}^b(\mathbf{proLiePAb})^{\mathrm{op}} \times \mathrm{D}^b(\mathbf{proLiePAb}) \to \mathrm{D}^b(\mathbf{Ab}).$$

3.59. COROLLARY. *A pro-Lie Polish abelian group G is projective if and only if* Ext(G, $\mathbb{Z} = 0$ *and* Ext $\left(G, \mathbb{Z}^{(\omega)}\right) = 0$.

PROOF. We prove sufficiency, as necessity is obvious. By Theorem 3.57, we have a short exact sequence $P^0 \to P^1 \to G$ where P^0 and P^1 are projectives. By Theorem 3.56, we have that $P^0 \cong \mathbb{Z}^\alpha \oplus \left(\mathbb{Z}^{(\omega)}\right)^\beta \oplus \mathbb{R}^\gamma$ for some $\alpha, \beta, \gamma \leq \omega$. We have that $\mathrm{Ext}\,(G, \mathbb{R}^\gamma) = \mathrm{Ext}\,(G, \mathbb{R})^\gamma = 0$ since $\mathbb{R}$ is injective. By assumption, we have $\mathrm{Ext}\,(G, \mathbb{Z}^\alpha) \cong \mathrm{Ext}\,(G, \mathbb{Z})^\alpha = 0$ and $\mathrm{Ext}(G, \left(\mathbb{Z}^{(\omega)}\right)^\beta) \cong \mathrm{Ext}\left(G, \mathbb{Z}^{(\omega)}\right)^\beta = 0$ by hypothesis. This implies that $\mathrm{Ext}\,(G, P^0) = 0$. Thus, G is a topological direct summand of P^1, and hence projective. ■

3.60. LEMMA. *Every countable divisible group is injective in the category of non-Archimedean Polish abelian groups.*

PROOF. Let D be a countable divisible group, and A be a non-Archimedean Polish abelian group. Then there exists a sequence (C_i) of countable abelian groups such that, setting

$$C := \prod_n C_n,$$

there exists an injective continuous homomorphism $A \to C$. This induces a surjective homomorphism

$$\mathrm{Ext}\,(C, D) \to \mathrm{Ext}\,(A, D)\,.$$

As D is injective in the category of countable abelian groups, for every $i \in \omega$ we have that $\mathrm{Ext}\,(C_i, D) = 0$. By Lemma 3.52 this implies that $\mathrm{Ext}\,(C, D) = 0$ and hence $\mathrm{Ext}\,(A, D) = 0$. ■

For pro-Lie Polish abelian groups A and B, we set $\mathrm{Ext}^n\,(A, B) = \mathrm{H}^n\,(\mathrm{RHom}\,(A, B))$. We thus have that $\mathrm{Ext}^n\,(A, B) = 0$ for $n \geq 2$, while $\mathrm{Ext}^1\,(A, B)$ (which we simply denote by $\mathrm{Ext}\,(A, B)$) is isomorphic to the group $\mathrm{Ext}^1_{\mathrm{Yon}}\,(A, B)$ (which we have been denoting by $\mathrm{Ext}_{\mathrm{Yon}}\,(A, B)$) of isomorphism classes of (1-fold) extensions of A by B.

Notice that, if C is a countable discrete group and A is a pro-Lie Polish abelian group, then any extension $A \to X \to C$ in **Ab** can be turned uniquely into an extension in **proLiePAb**. Whence, we have that $\mathrm{Ext}\,(C, A) \cong \mathrm{Ext}\,(C, A_{\mathrm{disc}})$, where A_{disc} is the group A endowed with the discrete topology.

We now characterize the injective objects in the category of pro-Lie Polish abelian groups.

3.61. LEMMA. *Every abelian Polish pro-Lie group is isomorphic to a closed subgroup of $\mathbb{R}^\omega \oplus \mathbb{T}^\omega \oplus U^\omega$ where $U = \mathbb{Q}^{(\omega)} \oplus \bigoplus_p \mathbb{Z}\,(p^\infty)^{(\omega)}$.*

PROOF. Let A be a pro-Lie abelian Polish group. Then A is isomorphic to a closed subgroup of $\prod_{k \in \omega} A_k$ where, for every $k \in \omega$, A_k is an abelian Polish Lie group. Thus, A_k is isomorphic to $\mathbb{R}^{n_k} \oplus \mathbb{T}^{m_k} \oplus D_k$ for some $n_k, m_k \in \omega$ and countable discrete group D_k. Thus, A_k is isomorphic to a closed subgroup of $\mathbb{R}^\omega \oplus \mathbb{T}^\omega \oplus U$. The conclusion about A follows. ■

3.62. Theorem. *Let G be an abelian Polish pro-Lie group. The following assertions are equivalent:*

1. *G is an* injective object *in* **proLiePAb***;*
2. *G is isomorphic to $\mathbb{R}^{\alpha} \oplus \mathbb{T}^{\beta}$ for some $\alpha, \beta \in \omega + 1$;*
3. *G is path-connected;*
4. $\mathrm{Ext}\,(\mathbb{T}, G) = 0$.

Proof. (2)⇒(1) This follows from Lemma 3.9 considering that

proLiePAb

is a thick subcategory of **PAb** by Theorem 3.17.

(4)⇒(3) Suppose that $\mathrm{Ext}\,(\mathbb{T}, G) = 0$. Then we have an exact sequence

$$\mathrm{Hom}\,(\mathbb{R}, G) \to \mathrm{Hom}\,(\mathbb{Z}, G) \to \mathrm{Ext}\,(\mathbb{T}, G) = 0$$

This shows that G is path-connected.

(3)⇒(2) Since G is connected, we have that $G \cong G_{\mathbb{R}} \oplus G_{\mathbb{S}^1}$. Furthermore, $G_{\mathbb{S}^1}$ is path-connected, and hence a torus by [Arm81, Theorem 8.27]. ∎

3.63. Corollary. *The class of injective objects in* **proLiePAb** *is closed under quotients.*

3.64. Corollary. *If D is a nonzero countable discrete divisible group, then there exists no injective object I in* **proLiePAb** *such that D is isomorphic to a closed subgroup of I. Hence,* **proLiePAb** *does not have enough injectives.*

Proof. Suppose that D is a countable discrete divisible group that is a closed subgroup of $\mathbb{R}^{\omega} \oplus \mathbb{T}^{\omega}$. Then we have that $\mathbb{T}^{\omega} \cap D$ is a divisible subgroup of D. Hence, we have that $D \cong H \oplus (\mathbb{T}^{\omega} \cap D)$ for some (necessarily divisible) subgroup H of D. The composition $H \to \mathbb{R}^{\omega} \oplus \mathbb{T}^{\omega} \to \mathbb{R}^{\omega}$ is injective. Since $\mathbb{T}^{\omega}$ is compact and H is closed in $\mathbb{T}^{\omega} \oplus \mathbb{R}^{\omega}$, we have that $\mathbb{T}^{\omega} + H$ is also closed in $\mathbb{T}^{\omega} \oplus \mathbb{R}^{\omega}$. This implies that the image of the function $H \to \mathbb{R}^{\omega} \oplus \mathbb{T}^{\omega} \to \mathbb{R}^{\omega}$ is a closed subgroup of $\mathbb{R}^{\omega}$. Every closed divisible subgroup of $\mathbb{R}^{\omega}$ is a closed $\mathbb{R}$-subspace. Since H is countable, this implies that $H = 0$.

Thus, we have that $D = D \cap \mathbb{T}^{\omega} \subseteq \mathbb{T}^{\omega}$. Thus, we have that the inclusion $D \to \mathbb{T}^{\omega}$ induces a surjective homomorphism $\mathbb{Z}^{(\omega)} \to D^{\vee}$. Since $D^{\vee}$ is compact and connected, it is uncountable whenever it is nonzero. Hence, $D^{\vee} = 0$ and $D = 0$. ∎

4. Thick categories of pro-Lie Polish abelian groups

4.1. Pro-p Polish abelian groups. Let us say that a *pro-p* Polish group is a pro-Lie Polish abelian topological p-group. We let $\mathbf{PAb}(p)$ be the category of pro-p Polish abelian groups, which is easily seen to be a thick subcategory of **proLiePAb**. A locally compact Polish abelian group G is pro-p if and only if it has a basis of zero neighborhoods consisting of subgroups U such that G/U is a p-group. A pro-Lie Polish abelian group is a pro-p group if and only if G/N is pro-p for every $N \in \mathcal{N}(G)$.

For an abelian group G, we define $pG = \{px : x \in G\}$. An abelian group G is p-divisible if $pG = G$. We also define recursively for every ordinal α:

$$p^0 G = G$$

and, for $\alpha > 0$,

$$p^\alpha G = \bigcap_{\beta<\alpha} p\left(p^\beta G\right).$$

If σ is the least ordinal such that $p^\sigma G = p^{\sigma+1} G$, then $p^\sigma G$ is the largest p-divisible subgroup of G. Recall that a group G is p-local if it is q-divisible for every prime number q other than p. When G is p-local, σ is the Ulm rank of G and $p^\sigma G$ is the largest divisible subgroup $d(G)$ of G. When G is a Polish group, the least ordinal σ such that $p^\sigma G = p^{\sigma+1} G$ is countable. We define, for an ordinal α,

$$L^p_\alpha(G) = \lim_{\beta<\alpha} G/p^\beta G$$

and $E^p_\alpha(G)$ to be the quotient of $L^p_\alpha(G)$ by the image of G under the canonical homomorphism $G \to L^p_\alpha(G)$.

For an ordinal α, as a particular instance of [Nun06, Theorem 1.5] in the case of the cotorsion functor with enough projectives $G \mapsto p^\alpha G$, we have the following:

4.2. Lemma. *Let A, G be groups. The short exact sequences*

$$p^\alpha A \to A \to A/p^\alpha A$$

and

$$p^\alpha G \to G \to G/p^\alpha G$$

induce homomorphisms

$$\mathrm{Ext}(A, p^\alpha G) \to \mathrm{Ext}(p^\alpha A, p^\alpha G)$$

and

$$\mathrm{Hom}(A, G/p^\alpha G) \to \mathrm{Ext}(A, p^\alpha G).$$

These induce an isomorphism

$$\eta : \frac{\mathrm{Ext}(A, p^\alpha G)}{\mathrm{Ran}(\mathrm{Hom}(A, G/p^\alpha G) \to \mathrm{Ext}(A, p^\alpha G))} \to \mathrm{Ext}(p^\alpha A, p^\alpha G)$$

Furthermore, the exact sequence $p^{\alpha}G \to G \to G/p^{\alpha}G$ *induces an exact sequence*

$$\begin{aligned}\mathrm{Hom}\left(A, G/p^{\alpha}G\right) &\to \mathrm{Ext}\left(A, p^{\alpha}G\right) \\ &\to p^{\alpha}\mathrm{Ext}\left(A, G\right) \to p^{\alpha}\mathrm{Ext}\left(A, G/p^{\alpha}G\right) \to 0\end{aligned}$$

which induces an exact sequence

$$\begin{aligned}0 &\to \frac{\mathrm{Ext}\left(A, p^{\alpha}G\right)}{\mathrm{Ran}\left(\mathrm{Hom}\left(A, G/p^{\alpha}G\right) \to \mathrm{Ext}\left(A, p^{\alpha}G\right)\right)} \overset{\rho}{\to} p^{\alpha}\mathrm{Ext}\left(A, G\right) \\ &\to p^{\alpha}\mathrm{Ext}\left(A, G/p^{\alpha}G\right) \to 0.\end{aligned}$$

Consider the homomorphism

$$r := \rho \circ \eta^{-1} : \mathrm{Ext}\left(p^{\alpha}A, p^{\alpha}G\right) \to p^{\alpha}\mathrm{Ext}\left(A, G\right).$$

We have an exact sequence

$$0 \to \mathrm{Ext}\left(p^{\alpha}A, p^{\alpha}G\right) \overset{r}{\to} p^{\alpha}\mathrm{Ext}\left(A, G\right) \to p^{\alpha}\mathrm{Ext}\left(A, G/p^{\alpha}G\right) \to 0$$

and r *restricts to an isomorphism*

$$\gamma : \mathrm{PExt}\left(p^{\alpha}A, p^{\alpha}G\right) \to u_1\left(p^{\alpha}\mathrm{Ext}\left(A, G\right)\right)$$

where u_1 *is the first Ulm subgroup and* $\mathrm{PExt}\left(p^{\alpha}A, p^{\alpha}G\right) = u_1\left(\mathrm{Ext}\left(p^{\alpha}A, p^{\alpha}G\right)\right)$.

4.3. Theorem. *Suppose that* G *is a pro-p Polish abelian group. The following assertions are equivalent:*

1. G *is injective in* $\mathbf{PAb}\left(p\right)$*;*
2. G *is divisible and it has a basis of zero neighborhoods consisting of divisible subgroups;*
3. $G \cong \mathbb{Z}\left(p^{\infty}\right)^{\alpha} \oplus (\mathbb{Z}\left(p^{\infty}\right)^{(\omega)})^{\beta}$ *for* $\alpha, \beta \leq \omega$*;*
4. $\mathrm{Ext}(\mathbb{Z}\left(p^{\infty}\right)^{(\omega)}, G) = 0$.

Proof. (2)⇒(3) Let $(V_k)_{k\in\omega}$ be a decreasing basis of zero neighborhoods of G consisting of divisible subgroups such that $V_0 = G$. For $n \in \omega$ define $H_n := V_n/V_{n+1}$. Since V_{n+1} is divisible, the short exact sequence $V_{n+1} \to V_n \to V_n/V_{n+1}$ splits. Let $\pi_n : V_n \to V_n/V_{n+1}$ be the quotient map and $\rho_n : V_n/V_{n+1} \to V_n$ be a group homomorphism that is a right inverse for π_n. Given $g \in G$ one defines $g_n \in V_n$ for $n \in \omega$ recursively by setting $g_0 = g$ and $g_{n+1} = g_n - \rho_n\pi_n\left(g_n\right)$. The continuous group homomorphism $G \to \prod_{n\in\omega} H_n$, $g \mapsto (g_n + V_{n+1})_{n\in\omega}$ is a group isomorphism. The conclusion now follows from the structure theorem for countable divisible p-groups.

(3)⇒(1) Suppose that, $G = \prod_{n\in\omega} D_n$ where, for every $n \in \omega$, D_n is a countable divisible group. If H is a pro-p Polish abelian group, we have that

$$\mathrm{Ext}\,(H, G) \cong \prod_{n\in\omega} \mathrm{Ext}\,(H, D_n)\,.$$

For $n \in \omega$, we have that $\mathrm{Ext}\,(H, D_n) = 0$ by Lemma 3.60.

(1)⇒(2) If G is not divisible, then $\mathrm{Ext}\,(\mathbb{Z}/p\mathbb{Z}, G) \neq 0$. Since $\mathbb{Z}/p\mathbb{Z}$ is a closed subgroup of $\mathbb{Z}\,(p^\infty)^{(\omega)}$, it follows that $\mathrm{Ext}(\mathbb{Z}\,(p^\infty)^{(\omega)}, G) \neq 0$.

Suppose that G does not have a basis of zero neighborhoods consisting of divisible subgroups. Then it has a basis $(V_k)_{k\in\omega}$ of open subgroups that are not divisible and such that $d\,(V_k)$ is not open. For $k \in \omega$, let $\rho_k \geq 1$ be the least countable ordinal such that $p^{\rho_k} V_k$ is *not* open. Set $H := V_0$ and $\alpha := \rho_0$.

Suppose initially that $\alpha = \beta + 1$ is a successor ordinal. Then after replacing H with $p^\beta H$ we can assume that $\alpha = 1$. Thus, we have that H is an open subgroup such that pH is not open. Then we have that, for every $k \geq 1$, V_k is not contained in pH. Thus, for every $k \geq 1$,

$$\mathrm{Ran}\,(V_k/pV_k \to H/pH) \cong \mathrm{Ran}\,(\mathrm{Ext}\,(\mathbb{Z}/p\mathbb{Z}, V_k) \to \mathrm{Ext}\,(\mathbb{Z}/p\mathbb{Z}, H)) \neq 0.$$

Thus, for every $k \geq 1$ there exists a cocycle $c_k : \mathbb{Z}/p\mathbb{Z} \to V_k$ that is not a coboundary even when regarded as a cocycle with values in H. Define the continuous cocycle $c : (\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \to H$ by setting

$$c\,((x_i)\,, (y_i)) = \sum_i c_i\,(x_i, y_i)\,.$$

We claim that c is not a coboundary even when regarded as a cocycle with values in G. We have an exact sequence

$$\mathrm{Hom}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, G/H) \to \mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, H) \to \mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, G)$$

Thus, it suffices to prove that c defines an element $[c]$ of $\mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, H)$ that does not belong to

$$\mathrm{Ran}\left(\mathrm{Hom}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, G/H) \to \mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, H)\right).$$

Considering the isomorphism

$$\mathrm{Hom}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, G/H) \cong \mathrm{Hom}\,(\mathbb{Z}/p\mathbb{Z}, G/H)^{(\mathbb{N})}$$

due to the fact that G/H is countable, it suffices to prove that, for every $k \in \omega$, the image of $[c]$ in $\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, H)$ induced by the inclusion $\mathbb{Z}/p\mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}$ in the k-th coordinate, is nonzero. As this image is the element $[c_k]$ represented by the cocycle c_k, we have that it is nontrivial by the choice of c_k. This shows that $\mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}, G) \neq 0$.

Suppose now that α is a limit ordinal. For $k \geq 1$ let α_k be the least countable ordinal such that V_k is *not* contained in $p^{\alpha_k} H$. Notice that, since α is a limit ordinal, $\alpha_k < \alpha$.

After passing to a subsequence of $(V_k)_{k\geq 1}$, we can assume that $(\alpha_k)_{k\geq 1}$ is nondecreasing. We also have that $\alpha = \sup_k \alpha_k$. We notice that, for $k \geq 1$, α_k must be a successor ordinal, and we let β_k be its predecessor.

Fix $k \geq 1$ and let T_k be a countable p-group with $p^{\beta_k} T_k = \mathbb{Z}/p\mathbb{Z}$. (Notice that this exists by Ulm's classification of countable p-groups; see [Fuc73, Chapter XII].) Consider the exact sequence

$$0 \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H\right) \xrightarrow{r} p^{\beta_k}\mathrm{Ext}\left(T_k, H\right) \to p^{\beta_k}\mathrm{Ext}\left(T_k, H/p^{\beta_k} H\right) \to 0.$$

The inclusion $V_k \subseteq p^{\beta_k} H$ induces an exact sequence

$$\mathrm{Ext}\left(p^{\beta_k} T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H/V_k\right) \to 0$$

that is isomorphic to

$$V_k/pV_k \to p^{\beta_k} H/p^{\alpha_k} H \to p^{\beta_k} H/(p^{\alpha_k} H + V_k) \to 0$$

Since V_k is not contained in $p^{\alpha_k} H$, we have that the homomorphism

$$p^{\beta_k} H/p^{\alpha_k} H \to p^{\beta_k} H/(p^{\alpha_k} H + V_k)$$

is not injective, and hence the homomorphism

$$V_k/pV_k \to p^{\beta_k} H/p^{\alpha_k} H$$

is nonzero. Thus, the homomorphism

$$\mathrm{Ext}\left(p^{\beta_k} T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H\right)$$

is nonzero. The inclusion $p^{\beta_k} T_k \to T_k$ induces a surjective homomorphism

$$\mathrm{Ext}\left(T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, V_k\right).$$

In view of the above remarks, there exists a cocycle $c_k : T_k \times T_k \to V_k$ that represents an element of $\mathrm{Ext}\left(T_k, V_k\right)$ whose image under the composition

$$\mathrm{Ext}\left(T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, V_k\right) \to \mathrm{Ext}\left(p^{\beta_k} T_k, p^{\beta_k} H\right) \xrightarrow{r} p^{\beta_k}\mathrm{Ext}\left(T_k, H\right) \subseteq \mathrm{Ext}\left(T_k, H\right)$$

is nonzero. Thus, c_k is not a coboundary when regarded as a cocycle with values in H.

Proceeding as above, we can conclude that, setting $T := \prod_{k\geq 1} T_k$, $\mathrm{Ext}\left(T, H\right)$ is nonzero.

(4)⇔(1) This follows from the fact that every object of $\mathbf{PAb}\left(p\right)$ is isomorphic to a closed subgroup of $(\mathbb{Z}\left(p^\infty\right)^{(\omega)})^\omega$. ∎

4.4. Remark. The proof of Theorem 4.3 shows that $\mathrm{Ext}((\mathbb{Z}/p\mathbb{Z})^{\omega}, \mathbb{Q}_p)$ is nonzero.

Let us denote by $\mathbf{LCPAb}(p)$ the category of *locally compact* pro-p Polish abelian groups.

4.5. Theorem. *Suppose that G is a pro-p locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{LCPAb}(p)$;*
2. *G is injective in $\mathbf{PAb}(p)$;*
3. *G is a countable divisible p-group.*

Proof. (1)⇒(2) By Theorem 4.3, it suffices to prove that G is divisible and has a basis of zero neighborhoods consisting of divisible subgroups. As in the proof of (1)⇒(2) of Theorem 4.3, we have that G is divisible. Suppose that G does not have a basis of zero neighborhoods consisting of divisible subgroups. Then it has a basis $(V_k)_{k\in\omega}$ of *compact* open subgroups that are not divisible and such that $d(V_k)$ is not open. For $k \in \omega$, let $\rho_k \geq 1$ be the least countable ordinal such that $p^{\rho_k} V_k$ is *not* open. By compactness of V_k, we have that $\rho_k \leq \omega$. Indeed, if $\rho_k > \omega$, then we have that $\{p^n V \setminus p^{n+1} V : n \in \omega\}$ is an infinite clopen partition of the compact set $V \setminus \rho^{\omega} V$, which is impossible. Thus, proceeding as in the proof of (1)⇒(2) of Theorem 4.3, one obtains that there exists a sequence (T_k) of *finite* p-groups such that, setting $T := \prod_{k\in\omega} T_k$, one has that $\mathrm{Ext}(T, G) \neq 0$, which is a contradiction.

(2)⇒(3) This follows from (1)⇒(3) in Theorem 4.3, since G is by hypothesis locally compact. ∎

4.6. Corollary. *Suppose that G is a pro-p locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is projective in $\mathbf{LCPAb}(p)$;*
2. *G is a compact torsion-free p-group.*

Proof. We have that Pontryagin duality establishes an equivalence between $\mathbf{LCPAb}(p)$ and its opposite. Thus, it maps injective objects to projective objects and vice versa. Furthermore, by [Arm81, Theorem 4.15], a locally compact Polish abelian group G is compact and torsion-free if and only if its dual is countable and divisible. ∎

4.7. Corollary. *The quasi-abelian category* $\mathbf{PAb}(p)$ *has enough injectives and homological dimension* 1*.*

Proof. Let G be a pro-p Polish abelian group. Then G is isomorphic to a closed subgroup of $\prod_{n\in\omega} G_n$ for some sequence (G_n) of countable p-groups. In turn, for every $n \in \omega$, G_n is a subgroup of the countable divisible p-group $D := \mathbb{Z}(p^{\infty})^{(\omega)}$. Thus, G is isomorphic to a closed subgroup of D^{ω}, which is injective by Theorem 4.3. This shows that $\mathbf{PAb}(p)$ has enough injectives.

Suppose that A is an injective pro-p Polish abelian group, B is a Polish abelian group, and $\pi : A \to B$ is a surjective continuous homomorphism. Since A is divisible, B is also divisible. If (V_k) is a basis of zero neighborhoods of A consisting of divisible subgroups, then $(\pi(V_k))$ is a basis of zero neighborhoods of B consisting of divisible subgroups by the Open Mapping Theorem for Polish groups. Thus, B is an injective pro-p Polish abelian group. This shows that quotients of injectives are injective in $\mathbf{PAb}(p)$, and hence $\mathbf{PAb}(p)$ has homological dimension at most 1. ∎

4.8. Problem. *Characterize the projective objects in* $\mathbf{PAb}(p)$*. Does* $\mathbf{PAb}(p)$ *have enough projectives?*

4.9. Topological torsion groups. Recall that a pro-Lie Polish abelian group G is topological torsion if and only if it is an inverse limit of countable torsion groups. We thus say that a Polish abelian group is topological torsion if it is an inverse limit of countable torsion groups. Topological torsion Polish abelian groups form a thick subcategory **proTorPAb** of **proLiePAb**. We denote by **proTorLCPAb** the category of locally compact topological torsion Polish abelian groups. These are precisely the locally compact Polish abelian groups that are totally disconnected and whose Pontryagin dual is also totally disconnected [Arm81, Theorem 3.5].

Given a topological torsion Polish abelian group, we let G_p be the subgroup of G consisting of elements x such that $\lim_{n\to\infty} p^n x = 0$ [Arm81, Definition 2.1], called the *p-component* of G. Then we have that G_p is a closed subgroup of G [Arm81, Lemma 3.8]. We can write $G \cong G_p \oplus G_p^{\#}$ where $G_p^{\#}$ has trivial p-component, and

$$\mathrm{Hom}\left(G_p^{\#}, T\right) = \mathrm{Hom}\left(T, G_p^{\#}\right) = 0$$

for any topological p-group T. It follows that, when G and H are topological torsion groups, that

$$\mathrm{Hom}(G, H) \cong \mathrm{Hom}(G_p, H_p) \oplus \mathrm{Hom}\left(G_p^{\#}, H_p^{\#}\right)$$

and

$$\mathrm{Ext}(G, H) \cong \mathrm{Ext}(G_p, H_p) \oplus \mathrm{Ext}\left(G_p^{\#}, H_p^{\#}\right).$$

The same proofs as the ones in Section 4.1 give the following results.

4.10. Theorem. *Suppose that G is a topological torsion Polish abelian group. The following assertions are equivalent:*

1. *G is injective in* **proTorPAb***;*
2. *G is divisible and it has a basis of zero neighborhoods consisting of divisible subgroups;*
3. *G is a product of countable divisible abelian groups;*
4. $\mathrm{Ext}((\mathbb{Z}(p^{\infty})^{(\omega)})^{\omega}, G) = 0$ *for every prime p;*
5. *G_p is injective in* $\mathbf{PAb}(p)$ *for every prime p.*

4.11. Theorem. *Suppose that G is a topological torsion locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is injective in* **proTorLCPAb***;*
2. *G is injective in* **proTorPAb***;*
3. *G is a countable and divisible.*

4.12. Corollary. *Suppose that G is a topological torsion locally compact Polish abelian group. The following assertions are equivalent:*

1. *G is projective in* **proTorLCPAb***;*
2. *G is compact and torsion-free.*

4.13. Corollary. *The category* **proTorPAb** *has enough injectives and homological dimension* 1.

4.14. Problem. *Characterize the projective objects in* **proTorPAb**. *Does* **proTorPAb** *have enough projectives?*

4.15. Injective non-Archimedean Polish abelian groups. In this section, we characterize injective and projective objects in the category of non-Archimedean Polish abelian groups. Recall that a Polish abelian group G is non-Archimedean if it has a basis of zero neighborhoods consisting of subgroups. This is equivalent to the assertion that G is isomorphic to a closed subgroup of U^{ω}, where U is the universal countable discrete group $\mathbb{Q}^{(\omega)} \oplus \bigoplus_p \mathbb{Z}(p^{\infty})^{(\omega)}$. Non-Archimedean Polish abelian groups form a thick subcategory $\mathbf{PAb}_{\mathrm{nA}}$ of the quasi-abelian category of Polish abelian groups.

Recall that a locally compact Polish abelian group has type $\mathbb{Z}$ if and only if it is discrete and torsion-free. A pro-Lie Polish abelian group G has type $\mathbb{Z}$ if

$$\{N \in \mathcal{N}(G) : G/N \text{ has type } \mathbb{Z}\}$$

is cofinal in $\mathcal{N}(G)$. Notice that every type $\mathbb{Z}$ Polish abelian group is isomorphic to a subgroup of $(\mathbb{Q}^{(\omega)})^{\omega}$.We let $\mathbf{proLiePAb}_{\mathbb{Z}}$ be the fully exact subcategory of **proLiePAb** consisting of Polish abelian groups of type $\mathbb{Z}$.

4.16. THEOREM. *Let A be a non-Archimedean Polish abelian group. The following assertions are equivalent:*

1. *A is of type $\mathbb{Z}$, and projective in* **proLiePAb***;*
2. *A is projective in* $\mathbf{proLiePAb}_{\mathbb{Z}}$*;*
3. *A is isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$;*
4. $\mathrm{Ext}\,(A, \mathbb{Z}) = 0$ *and* $\mathrm{Ext}\left(A, \mathbb{Z}^{(\omega)}\right) = 0$.

PROOF. The equivalence (3)⇔(1) follows from Theorem 3.56, while the implications (1)⇒(4) and (1)⇒(2) is obvious.

(2)⇒(1) Suppose that A is projective in $\mathbf{proLiePAb}_{\mathbb{Z}}$. Then by Theorem 3.57 there exists a surjective homomorphism $P \oplus V \to A$ where P is isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$ and V is a vector group. Since A is type $\mathbb{Z}$, $\mathrm{Hom}\,(V, A) = 0$. Thus, there exists a surjective homomorphism $P \to A$. Since A is projective in $\mathbf{proLiePAb}_{\mathbb{Z}}$, we have that A is a direct summand of P. Since P is projective in **proLiePAb** by Theorem 3.56, also A is projective in **proLiePAb**, concluding the proof.

(4)⇒(1) For $\alpha \leq \omega$ we have that $\mathrm{Ext}\,(A, \mathbb{Z}^{\alpha}) \cong \mathrm{Ext}\,(A, \mathbb{Z})^{\alpha} = 0$ and $\mathrm{Ext}\left(A, \left(\mathbb{Z}^{(\omega)}\right)^{\alpha}\right) \cong \mathrm{Ext}\left(A, \left(\mathbb{Z}^{(\omega)}\right)\right)^{\alpha} = 0$. By Lemma 3.49 there exists a surjective epimorphism $\mathbb{R}^{\omega} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to A$. Since $\mathrm{Hom}\,(\mathbb{R}^{\omega}, A) = 0$, there exists a surjective epimorphism $\pi : \left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to A$. Let $C := \mathrm{Ker}\,(\pi)$ and observe that C is a closed subgroup of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$, and hence non-Archimedean. Furthermore, C is projective in **proLiePAb**, and hence isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$. Thus, by the above remarks we have that $\mathrm{Ext}\,(A, C) = 0$ and hence A is isomorphic to a topological direct summand of $\left(\mathbb{Z}^{(\omega)}\right)^{\omega}$. ∎

4.17. THEOREM. *Let A be a non-Archimedean pro-Lie Polish abelian group. The following assertions are equivalent:*

1. *A is projective in* $\mathbf{PAb}_{\mathrm{nA}}$*;*
2. *A is isomorphic to $\mathbb{Z}^{\alpha} \oplus \left(\mathbb{Z}^{(\omega)}\right)^{\beta}$ for some $\alpha, \beta \leq \omega$;*
3. $\mathrm{Ext}\,(A, \mathbb{Z}) = 0$ *and* $\mathrm{Ext}\left(A, \mathbb{Z}^{(\omega)}\right) = 0$.

PROOF. This is an immediate consequence of Theorem 4.16. ∎

The same proof as Theorem 4.3 gives the following.

4.18. THEOREM. *Suppose that G is a non-Archimedean Polish abelian group. The following assertions are equivalent:*

1. *G is injective in $\mathbf{PAb}_{\mathrm{nA}}$;*
2. *G is divisible and it has a basis of zero neighborhoods consisting of divisible subgroups;*
3. *G is a product of countable divisible abelian groups;*
4. $\mathrm{Ext}((\mathbb{Z}(p^{\infty})^{(\omega)})^{\omega}, G) = 0$ *for every prime p.*

4.19. THEOREM. *Let A be a type $\mathbb{Z}$ pro-Lie Polish abelian group. The following assertions are equivalent:*

1. *A is injective in $\mathbf{proLiePAb}_{\mathbb{Z}}$;*
2. *A is divisible;*
3. *$A \cong \mathbb{Q}^{\alpha} \oplus (\mathbb{Q}^{(\omega)})^{\beta}$ for some $\alpha, \beta \leq \omega$.*

PROOF. (1)⇒(2) Suppose that A is an injective type $\mathbb{Z}$ pro-Lie Polish abelian group. Then A is isomorphic to a closed subgroup of $(\mathbb{Q}^{(\omega)})^{\omega}$. Since A is injective, it is a direct summand of $(\mathbb{Q}^{(\omega)})^{\omega}$, and hence it is divisible.

(2)⇒(3) If A is divisible, then $A \cong \lim_k A_k$ where, for every $k \in \omega$, A_k is countable, divisible, and torsion-free and $A_{k+1} \to A_k$ is surjective with divisible kernel (notice that the kernel of $A_{k+1} \to A_k$ is a pure subgroup of A_{k+1} since A_k is torsion-free, and since A_{k+1} is torsion-free divisible, the same holds for the kernel of $A_{k+1} \to A_k$). Since countable divisible groups are injective for countable abelian groups, it follows that A is isomorphic to the product of countable divisible groups.

(3)⇒(1) We show that if D is a countable divisible group and $\alpha \leq \omega$, then D^{α} is injective in $\mathbf{proLiePAb}_{\mathbb{Z}}$. For a type $\mathbb{Z}$ pro-Lie Polish abelian group A, we have that

$$\mathrm{Ext}\left(A, D^{\alpha}\right) \cong \mathrm{Ext}\left(A, D\right)^{\alpha}.$$

Thus, it suffices to consider the case when $\alpha = 1$. In this case, we have a short exact sequence $A \to B \to C$ where $B = \prod_k B_k$ is a product of countable abelian groups. This induces a surjective homomorphism

$$\mathrm{Ext}\left(B, D\right) \to \mathrm{Ext}\left(A, D\right).$$

Since D is injective for countable abelian groups, we have that $\mathrm{Ext}\left(B, D\right) = 0$ by Lemma 3.54. ■

4.20. Corollary. *We have that:*

1. *The category* $\mathbf{PAb}_{\mathrm{nA}}$ *has enough injective objects and enough projective objects, and homological dimension* 1*;*
2. *The category* $\mathbf{proLiePAb}_{\mathbb{Z}}$ *has enough projective objects but not enough injective objects, and homological dimension* 1*. An object in* $\mathbf{proLiePAb}_{\mathbb{Z}}$ *has an injective resolution if and only if it is injective.*

Proof. (1) The same proof as Lemma 3.49 shows that, for every non-Archimedean Polish abelian group A there exists a surjective continuous homomorphism $\left(\mathbb{Z}^{(\omega)}\right)^{\omega} \to A$. By Theorem 3.57 this implies that $\mathbf{PAb}_{\mathrm{nA}}$ has enough projective objects and homological dimension 1. The same proof as Corollary 4.7 shows that $\mathbf{PAb}_{\mathrm{nA}}$ has enough injective objects.

(2) We have that $\mathbf{proLiePAb}_{\mathbb{Z}}$ has enough projectives by Lemma 3.48 and Theorem 4.16. Furthermore, it has homological dimension 1 by Lemma 3.50. Suppose that A is a type $\mathbb{Z}$ Polish abelian group. Suppose that $A \to D \to B$ is a short exact sequence where D and B are type $\mathbb{Z}$ Polish abelian groups, and D is injective in $\mathbf{proLiePAb}_{\mathbb{Z}}$. By Theorem 4.19, we have that D is divisible. Thus, for $p \in \mathbb{Z}$ we have that $\mathrm{Hom}\left(\mathbb{Z}/p\mathbb{Z}, B\right) = 0$ and $\mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, D\right) = 0$. This implies that $\mathrm{Ext}\left(\mathbb{Z}/p\mathbb{Z}, A\right) = 0$ and A is divisible. ∎

4.21. Ext as a group with a Polish cover. Suppose that A, B are countable abelian groups. Then we have that $\mathrm{Hom}\left(A, B\right)$ is a Polish abelian group when endowed with the compact-open topology. We let $\mathbf{PAb}_{\aleph_0}$ be the thick subcategory of $\mathbf{proLiePAb}$ consisting of countable abelian groups. Since $\mathbf{PAb}_{\aleph_0}$ has enough injective and projective objects and homological dimension 1, we have that the functor

$$\mathrm{Hom} : \mathbf{PAb}_{\aleph_0}^{\mathrm{op}} \times \mathbf{PAb}_{\aleph_0} \to \mathbf{PAb}$$

admits a total right derived functor.

Suppose now more generally that A is a *locally compact* Polish abelian group, and B is a pro-Lie Polish abelian group. Then we still have that $\mathrm{Hom}\left(A, B\right)$ is a Polish abelian group with respect to the compact-open topology. In fact, we have the following:

4.22. Proposition. *Let* A, B *be pro-Lie Polish abelian groups. Suppose that* A *is locally compact. Then* $\mathrm{Hom}\left(A, B\right)$ *is a pro-Lie Polish abelian groups when endowed with the compact-open topology.*

Proof. Since B is Polish, $\mathrm{Hom}\left(A, B\right)$ is Polish. We need to show that it is pro-Lie. Since B is pro-Lie, we can write $B = \mathrm{lim}_n B_n$ where B_n is a Lie abelian group for every $n \in \omega$. Whence

$$\mathrm{Hom}\left(A, B\right) \cong \mathrm{lim}_n \mathrm{Hom}\left(A, B_n\right).$$

Thus, we can assume that B is either $\mathbb{R}$ or $\mathbb{T}$ or a countable discrete abelian groups. We have that

$$\mathrm{Hom}\left(A, \mathbb{R}\right) = \mathrm{Hom}\left(A_{\mathbb{R}}, \mathbb{R}\right) \cong A_{\mathbb{R}}$$

$$\mathrm{Hom}\,(A,\mathbb{T})\cong A^{\vee}$$

is the Pontryagin dual of A (which is locally compact). Thus, we can assume that B is countable discrete. Considering the type decomposition, it suffices to prove the statement for locally compact Polish groups $A_{\mathbb{Z}}$, $A_{\mathbb{S}^1}$, and $A_{\mathbb{A}}=A_{\mathbb{R}}\oplus A_{\mathrm{t}}$ of type $\mathbb{Z}$, $\mathbb{S}^1$, and $\mathbb{A}$, respectively. When B is countable discrete, we have that

$$\mathrm{Hom}\,(A_{\mathbb{S}^1},B)\cong\mathrm{Hom}\,(A_{\mathbb{R}},B)\cong 0$$

and

$$\mathrm{Hom}\,(A_{\mathbb{Z}},B)$$

is isomorphic to a closed subgroup of B^{ω}. We have that $A_{\mathrm{t}}=\mathrm{colim}_n C_n$ where C_n is pro-finite. Thus,

$$\mathrm{Hom}\,(A_{\mathrm{t}},B)\cong\lim_n\mathrm{Hom}\,(C_n,B)$$

For $n\in\omega$, as C_n is pro-finite and B is countable discrete,

$$\mathrm{Hom}\,(C_n,B)$$

is countable and locally finite. Thus, $\mathrm{Hom}\,(A_{\mathrm{t}},B)$ is pro-countable. This concludes the proof. ■

We let **LCPAb** be the thick subcategory of **proLiePAb** consisting of locally compact Polish abelian groups. By the previous proposition, **LCPAb** is a quasi-abelian **proLiePAb**-category, where **proLiePAb** is regarded as a category of modules; see Definition 2.9. More generally, we can consider the functor

$$\mathrm{Hom}:\mathbf{LCPAb}^{\mathrm{op}}\times\mathbf{proLiePAb}\to\mathbf{proLiePAb}.$$

We will show that this functor admits a total right derived functor.

4.23. Definition. *A pro-Lie Polish abelian group G is* essentially injective *if it has a closed subgroup U injective in* **PAb** *such that G/U is non-Archimedean and injective in* $\mathbf{PAb}_{\mathrm{nA}}$.

By Theorem 3.62, a pro-Lie Polish abelian group is essentially injective if and only if G is isomorphic to $T\oplus V\oplus A$ where T is a torus, V is a vector group, and A is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. We let $\mathcal{D}$ be the collection of essentially injective pro-Lie Polish abelian groups. By Lemma 3.61 and Theorem 4.18, every pro-Lie Polish abelian group is isomorphic to a closed subgroup of an element of $\mathcal{D}$. We will now show that the class $\mathcal{D}$ is closed under taking quotients by closed subgroups. Towards this goal, we isolate a few lemmas concerning Ext of countable abelian groups, regarded as a group with a Polish cover.

4.24. Lemma. *Suppose that T is a countable torsion group and K is a countable torsion-free group. Then* $\mathrm{Ext}\,(T,K)$ *is a Polish group isomorphic to* $\mathrm{Hom}\,(T,D/K)$ *where D is the divisible hull of K.*

Proof. The short exact sequence $K \to D \to D/K$ induces an exact sequence

$$\mathrm{Hom}\,(T, D) \to \mathrm{Hom}\,(T, D/K) \to \mathrm{Ext}\,(T, K) \to \mathrm{Ext}\,(T, D)\,.$$

Since T is torsion and D is torsion-free, $\mathrm{Hom}\,(T, D) = 0$. Since D is divisible, $\mathrm{Ext}\,(T, D) = 0$. The conclusion follows. ∎

4.25. Lemma. *Suppose that A is a countable abelian group. If $\mathrm{Ext}\,(A, \mathbb{Z})$ is a Polish group, then $A_{\mathbb{Z}}$ is free abelian.*

Proof. Since A is countable, $F_{\mathbb{Z}}A = A_{\mathrm{t}}$ is the torsion subgroup of A. Consider the exact sequence

$$0 = \mathrm{Hom}\,(A_{\mathrm{t}}, \mathbb{Z}) \to \mathrm{Ext}\,(A_{\mathbb{Z}}, \mathbb{Z}) \to \mathrm{Ext}\,(A, \mathbb{Z}) \to \mathrm{Ext}\,(A_{\mathrm{t}}, \mathbb{Z}) \to 0.$$

Since A_{t} is torsion, we have that $\mathrm{Ext}\,(A_{\mathrm{t}}, \mathbb{Z})$ is Polish by Lemma 4.24. It follows that $\mathrm{Ext}\,(A_{\mathbb{Z}}, \mathbb{Z})$ is Polish as well.

Since $A_{\mathbb{Z}}$ is torsion-free, it has no finite subgroups. Thus, by [EM42, Corollary 11.6], $\{0\}$ is dense in $\mathrm{Ext}\,(A_{\mathbb{Z}}, \mathbb{Z})$. This implies that $\mathrm{Ext}\,(A_{\mathbb{Z}}, \mathbb{Z}) = 0$. Hence, $A_{\mathbb{Z}}$ is a free abelian group by [FS13, Theorem 3.2]. ∎

4.26. Lemma. *The class $\mathcal{D}$ of essentially injective pro-Lie Polish abelian groups is closed under taking quotients by closed subgroups.*

Proof. Let H be a pro-Lie Polish group. Then we have that the connected component $c\,(H)$ of zero in H is $H_{\mathbb{S}^1} \oplus H_{\mathbb{R}}$, and $H/c\,(H)$ is non-Archimedean. Thus, we have that $H = V \oplus A$ for some vector group A and non-Archimedean abelian Polish group A if and only if $H_{\mathbb{S}^1} = 0$.

Suppose that G is an essentially injective pro-Lie Polish abelian group and N is a closed subgroup of G. Set $H := G/N$. Since G is essentially injective, we have that $G = T \oplus V \oplus A$ where V is a vector group, T is a torus, and A is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. We need to prove that G/N is essentially injective.

We have that $T/\,(T \cap N)$ is injective, being the quotient of a torus group by a closed subgroup. Furthermore, letting $\pi : G \to V \oplus A$ be the canonical quotient map, we have that $\pi^{-1}\,(\pi\,(N)) = N + T$ is closed in G, and hence $\pi\,(N)$ is closed is $V \oplus A$. Thus, we have a short exact sequence

$$\frac{T}{T \cap N} \to G/N \to \frac{V \oplus A}{\pi\,(N)}$$

that splits by injectivity of $T/\,(T \cap N)$. It follows that, after replacing G with G/T, we can assume without loss of generality that $T = 0$.

Since N is a closed subgroup of $G = V \oplus A$, we have that $c\,(N) \subseteq V = c(G)$ and hence $c\,(N) = N_{\mathbb{R}}$ is a vector group and $N_{\mathbb{S}^1} = 0$. Thus, we can write $N = N_{\mathbb{R}} \oplus \Xi$ where Ξ is non-Archimedean. Since $N_{\mathbb{R}} \subseteq V$ is a vector group, we can assume without loss of generality, possibly after replacing G with $G/N_{\mathbb{R}}$, that $N_{\mathbb{R}} = 0$ and $N = \Xi$ is non-Archimedean.

Then we have that $N \cap V$ is a closed subgroup of V and $V/(N \cap V)$ is injective in **PAb**. Thus, $V/(N \cap V) = W \oplus T$ where T is a torus group and W is a vector group. Since W is projective in **PAb**, we can assume without loss of generality that $W = 0$ and $V/(N \cap V)$ is a torus group.

Thus, we have that $N/(N \cap V)$ is a closed subgroup of G/N. Being also injective in **PAb**, it is a direct summand. Thus, after replacing G with G/V and N with $N/(N \cap V)$, we can assume that $G = A$ is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. In this case, we have that also G/N is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$, concluding the proof. ∎

Recall that a locally compact Polish group G is *codivisible* if its Pontryagin dual $G^{\vee}$ is divisible. We let $\mathcal{C}$ be the collection of locally compact groups G that are codivisible and satisfy $G_{\mathbb{S}^1} = 0$. Such a group G can be written as $V \oplus A$ where V is a vector group and A is totally disconnected. Clearly, $\mathcal{C}$ is closed under taking closed subgroups.

4.27. Lemma. *If C is a locally compact Polish group with $C_{\mathbb{Z}} = 0$, then there exists a short exact sequence $C \to D \to D/C$ where D is divisible with $D_{\mathbb{Z}} = 0$ and D/C is a countable torsion group.*

Proof. Without loss of generality, we can assume that C has no nonzero closed vector subgroups. By [HS07, Proposition 3.8], there exists a locally compact Polish group D containing C as an open subgroup such that D/C is a countable torsion group. Consider the pushout diagram

$$\begin{array}{ccc} C & \rightarrow & C_{\mathrm{t}} \\ \downarrow & & \downarrow \\ D & \rightarrow & H \end{array}$$

This gives rise to a commutative diagram

$$\begin{array}{ccccc} C_{\mathbb{S}^1} & \rightarrow & C & \rightarrow & C_{\mathrm{t}} \\ \downarrow & & \downarrow & & \downarrow \\ C_{\mathbb{S}^1} & \rightarrow & D & \rightarrow & H \\ & & \downarrow & & \downarrow \\ & & D/C & \rightarrow & H/C_{\mathrm{t}} \end{array}$$

where $D/C \to H/C_{\mathrm{t}}$ is an isomorphism. Since C_{t} is a topological torsion group and $H/C_{\mathrm{t}} \cong D/C$ is a countable torsion group (and in particular a topological torsion group), we conclude that H is a topological torsion group, whence $D_{\mathbb{Z}} = 0$. ∎

4.28. Corollary. *If C is a locally compact Polish group with $C_{\mathbb{S}^1} = 0$, then there exists a short exact sequence $A \to D \to C$ where D is a codivisible locally compact Polish abelian group with $D_{\mathbb{S}^1} = 0$ and A is a profinite Polish group.*

4.29. Lemma. *For every locally compact Polish group C there exists a continuous surjective homomorphism $G \to C$ for some element G of $\mathcal{C}$.*

Proof. Without loss of generality, we can assume that C has no nonzero closed vector groups as a direct summand, whence $C/C_{\mathbb{S}^1}$ is totally disconnected. We have that $C_{\mathbb{S}^1}^{\vee}$ is a countable torsion-free group. Thus there exists a short exact sequence $E \to C_{\mathbb{S}^1}^{\vee} \to S$ where E is a countable free abelian group and S is a countable torsion group. By duality, this gives a short exact sequence $A \to C_{\mathbb{S}^1} \to T$, where $A = S^{\vee}$ is a profinite abelian Polish group and $T = E^{\vee}$ is a torus group. Considering the inclusion $C_{\mathbb{S}^1} \to C$ one obtains by pushout a diagram

$$\begin{array}{ccc} C_{\mathbb{S}^1} & \to & T \\ \downarrow & & \downarrow \\ C & \to & H \end{array}$$

which gives a commutative diagram

$$\begin{array}{ccccc} A & \to & C_{\mathbb{S}^1} & \to & T \\ \downarrow & & \downarrow & & \downarrow \\ A & \to & C & \to & H \\ & & \downarrow & & \downarrow \\ & & C/C_{\mathbb{S}^1} & \to & T/H \end{array}$$

where $A \to C \to H$ is a short exact sequence and the continuous group homomorphism $T \to H$ is injective with closed image, and the continuous group homomorphism $C/C_{\mathbb{S}^1} \to T/H$ is an isomorphism. By injectivity of T, we have that $H \cong T \oplus C/C_{\mathbb{S}^1}$. Since T is a locally compact torus group, there exists a continuous surjective homomorphism $V \to T$ for some locally compact vector group V, which induces a continuous surjective homomorphism $V \oplus C/C_{\mathbb{S}^1} \to H$. We consider the pullback diagram

$$\begin{array}{ccc} C & \to & H \\ \uparrow & & \uparrow \\ C' & \to & V \oplus C/C_{\mathbb{S}^1} \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccc} A & \to & C & \to & H \\ \uparrow & & \uparrow & & \uparrow \\ A & \to & C' & \to & V \oplus C/C_{\mathbb{S}^1} \end{array}$$

By projectivity of V, we can write $C' = V \oplus C''$, where we have an extension $A \to C'' \to C/C_{\mathbb{S}^1}$. Hence, C'' is totally disconnected. The conclusion thus follows from Corollary 4.28. ■

4.30. Proposition. *We have that, for a locally compact Polish abelian group $C \in \mathcal{C}$ and a pro-Lie Polish abelian group $D \in \mathcal{D}$,* $\mathrm{Ext}(C, D) = 0$.

Proof. We can write $D = T \oplus V \oplus A$ where T is a torus, V is a vector group, and A is non-Archimedean and injective in $\mathbf{PAb}_{\mathrm{nA}}$. We can also write $C = W \oplus B$ where W is a vector group and B is totally disconnected. By injectivity of T and V, and projectivity of W, in the category of pro-Lie Polish abelian groups, we have that $\mathrm{Ext}(C,D) \cong \mathrm{Ext}(B,A)$. By Theorem 4.18, we have that $A \cong \prod_{n\in\omega} A_n$ where A_n is a countable divisible group. Thus,

$$\mathrm{Ext}(B,A) \cong \prod_{n\in\omega} \mathrm{Ext}(B,A_n) = 0.$$

This concludes the proof. ∎

It follows from Proposition 2.14 in the case when $\mathcal{A} = \mathbf{LCPAb}$, $\mathcal{B} = \mathbf{proLiePAb}$, and $\mathcal{C}$ and $\mathcal{D}$ are the classes defined above, Proposition 4.30, Lemma 4.29, and Lemma 4.26, that

$$\mathrm{Hom}^{\bullet} : \mathrm{K}^b(\mathbf{LCPAb}) \times \mathrm{K}^b(\mathbf{proLiePAb}) \to \mathrm{K}^b(\mathbf{proLiePAb})$$

has a total right derived functor

$$\mathrm{RHom} : \mathrm{D}^b(\mathbf{LCPAb}) \times \mathrm{D}^b(\mathbf{proLiePAb}) \to \mathrm{D}^b(\mathbf{proLiePAb}),$$

and

$$\mathrm{H}^0 \circ \mathrm{RHom} : \mathrm{D}^b(\mathbf{LCPAb}) \times \mathrm{D}^b(\mathbf{proLiePAb}) \to \mathrm{LH}(\mathbf{proLiePAb})$$

is a cohomological derived functor of

$$\mathrm{H}^0 \circ \mathrm{Hom}^{\bullet} : \mathrm{K}^b(\mathbf{LCPAb}) \times \mathrm{K}^b(\mathbf{proLiePAb}) \to \mathrm{LH}(\mathbf{proLiePAb}).$$

The same argument as in the proof of [Lup25, Proposition 4.13] yields the following description of $\mathrm{Ext}^1(G,A)$ for locally compact Polish abelian group G and pro-Lie Polish abelian group A. Recall that if (X,μ) is a standard probability space and A is a Polish space, then we let $L^0(X,A)$ be the Polish space of μ-a.e. classes of functions $X \to A$ endowed with the topology of convergence in measure.

4.31. Proposition. *Suppose that G is a locally compact abelian Polish group and A is a pro-Lie abelian Polish group. Define $\mathrm{Z}(\mathrm{Ext}_{\mathrm{Yon}}(G,A))$ to be the group of Borel 2-cocycles on G with coefficients in A.*

Endow $\mathrm{Z}(\mathrm{Ext}_{\mathrm{Yon}}(G,A))$ with the topology given by letting a net (c_i) converge to c if and only if $(c_i(x,y,z))$ converges to $c(x,y,z)$ in A for every $x,y,z \in G$. Define $\mathrm{B}(\mathrm{Ext}_{\mathrm{Yon}}(G,A))$ to be the subgroup of $\mathrm{Z}(\mathrm{Ext}_{\mathrm{Yon}}(G,A))$ consisting of Borel cocycles of the form $\delta f(x,y) = f(y) - f(x+y) + f(x)$ for some Borel function $f : G \to A$. The function

$$\Psi : \mathrm{Z}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G,A)\right) \to L^0\left(G^2,A\right)$$

defined by mapping c to its a.e.-class establishes a continuous isomorphism with a closed subgroup of $L^0(G^2,A)$. Furthermore,

$$\mathrm{B}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G,A)\right)$$

is a Polishable subgroup of

$$\mathrm{Z}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G,A)\right).$$

Consider the group with a Polish cover

$$\mathrm{Ext}^1_{\mathrm{Yon}}(G,A) := \mathrm{Z}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G,A)\right)/\mathrm{B}\left(\mathrm{Ext}^1_{\mathrm{Yon}}(G,A)\right).$$

Then $\mathrm{Ext}^1_{\mathrm{Yon}}(G,A)$ *is naturally Borel-definably isomorphic to the group with a Polish cover* $\mathrm{Ext}^1(G,A)$.

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