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PRODUCTS IN DOUBLE CATEGORIES, REVISITED

EVAN PATTERSON

ABSTRACT. Products in double categories, as found in cartesian double categories, are an elegant concept with numerous applications, yet also have a few puzzling aspects. In this paper, we revisit double-categorical products from an unbiased perspective, following up an original idea by Paré to employ a double-categorical analogue of the family construction, or free product completion. Defined in this way, double categories with finite products are strictly more expressive than cartesian double categories, while being governed by a single universal property that is no more difficult to work with. We develop the basic theory and examples of such products and, by duality, of coproducts in double categories. As an application, we introduce finite-product double theories, a categorification of finite-product theories that extends recent work by Lambert and the author on cartesian double theories, and we construct the virtual double category of models of a finite-product double theory.

Contents

1. Introduction

A cartesian double category is a double category with binary and nullary products [Aleiferi, 2018], as defined by the general theory of limits in double categories [Grandis and Paré, 1999; Grandis, 2019]. This means that a double category $\mathbb{D}$ is **cartesian** when there exist right adjoints to the diagonal double functor $\Delta : \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and the unique double

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functor $! : \mathbb{D} \to \mathbb{1}$. Equivalently, a cartesian double category is a cartesian object in the 2-category of double categories.

Cartesian double categories aim to capture through a simple universal property what in a bicategory can only be described through far more elaborate structures, such as cartesian bicategories [Carboni and Walters, 1987; Carboni et al., 2008]. Recent works in pure and applied category theory have sought to exploit this feature of double categories. For example, Lambert has given axioms for "a double category of relations" [Lambert, 2022] simplifying those of "a bicategory of relations" [Carboni and Walters, 1987]. The author has used the dual notion of a cocartesian double category to streamline the theory of structured cospans [Patterson, 2023], a framework for modeling open systems [Fiadeiro and Schmitt, 2007; Baez and Courser, 2020]. Finally, cartesian double categories are the point of departure for "cartesian double theories," a categorification of finite-product theories whose models are categorical doctrines [Lambert and Patterson, 2024].

Yet, despite their elegance and success in applications, there are good reasons to doubt that cartesian double categories are the right notion of a "double category with finite products." In ordinary category theory, binary and nullary products famously generate all finite products, hence it is standard to identify a "category with binary and nullary products" with a "category with finite products." Surprisingly, this identification, rarely given a second thought, breaks down in certain ways for double categories.

We begin with a striking disanalogy between cocartesian categories and cocartesian double categories. The category of sets and functions plays a distinguished role in category theory through the Yoneda lemma and the representability arguments that the lemma enables. In developing the Yoneda theory of double categories [Paré, 2011], Paré has convincingly argued that the role played in category theory by Set is played in double category theory by $\mathbb{S}\mathsf{pan}$, the double category of sets, functions, and spans. For this reason, Paré and others even call this double category $\mathbb{S}\mathsf{et}$. The category Set is also characterized as the free cocompletion of the terminal category; similarly, FinSet is the free cocartesian category (category with finite coproducts) on the terminal category. Yet the free cocartesian double category on the terminal double category is not $\mathbb{F}\mathsf{inSpan}$, as one would expect under this analogy, but the degenerate fragment of it whose proarrows are identity spans.

There is also disharmony between cartesian double categories and cartesian bicategories. Although Aleiferi motivates her introduction of cartesian double categories through cartesian bicategories [Aleiferi, 2018], hence the parallel terminology, no formal comparison between the two concepts has been made. One might hope that a cartesian double category has an underlying cartesian bicategory, but that seems unlikely since a generic cartesian double category has no means of transferring the cartesian structure from arrows to proarrows. A cartesian *equipment* does have such a means. So, a cartesian equipment is expected to have an underlying cartesian bicategory, and, at least in the locally posetal case, Lambert has proved this [Lambert, 2022, Proposition 3.1]. But, even supposing that this is true in general, a puzzle remains: a cartesian equipment is defined by two independent universal properties, and the universal property of being an equipment has, on the face of it, nothing to do with products, except insofar as it involves a "mapping into" universal property.[1] Is this really necessary to obtain a cartesian bicategory?

So it appears that a cartesian double category possess too little structure to fulfill the role of a "double category with finite products," whereas a cartesian equipment possesses too much. In this paper, we explore an intermediate conception of products in a double category, based on a double-categorical version of the family construction. This idea was first proposed by Robert Paré in a talk at the Category Theory conference [Paré, 2009] but has never been systematically developed in print. It is the aim of this paper to do so, while exhibiting its virtues and drawing connections with the related ideas surveyed above.

The essence of the idea is easily explained. As reviewed in Section 2, the free coproduct completion of a category C is a category, often called $\mathsf{Fam}(\mathsf{C})$, whose objects are indexed families of objects in C. This logic can be turned around. Forgetting the definition of coproducts and starting from the family construction, one can define a category C to have coproducts just when the embedding $\Delta : \mathsf{C} \to \mathsf{Fam}(\mathsf{C})$ that sends objects in C to singleton families has a left adjoint, $\Sigma : \mathsf{Fam}(\mathsf{C}) \to \mathsf{C}$. Products can then be defined by dualization. We carry out an analogous procedure for double categories, constructing a double category of families, $\mathbb{F}\mathsf{am}(\mathbb{D})$, on a double category $\mathbb{D}$ and then defining $\mathbb{D}$ to have (colax) coproducts just when the embedding $\Delta : \mathbb{D} \to \mathbb{F}\mathsf{am}(\mathbb{D})$ has a (colax) left adjoint $\Sigma : \mathbb{F}\mathsf{am}(\mathbb{D}) \to \mathbb{D}$, in the sense of Grandis and Paré's theory of adjunctions between double categories [Grandis and Paré, 2004; Grandis, 2019].

In a hypothetical double category of families, objects should clearly be indexed by sets, as in the one-dimensional case, but proarrows admit at least two possible indexing schemes. One could insist that an indexed family of proarrows go between indexed families of objects with the same indexing set. That choice leads to cocartesian double categories and their infinitary analogues. However, having adopted the unbiased perspective, enforcing equality of indexing sets seems artificial, even uncategorical insofar as it asks for equality between objects. A more flexible approach takes a family of proarrows to be indexed by an arbitrary *span* between the object indexing sets. Thus, given a double category $\mathbb{D}$, a proarrow in $\mathbb{F}\mathsf{am}(\mathbb{D})$ from an indexed family of objects $(x_i)_{i\in I}$ to another $(y_j)_{j\in J}$ consists of a span $I \xleftarrow{\ell} A \xrightarrow{r} J$ of sets together with a family of proarrows in $\mathbb{D}$ of the form

$$m_a : x_{\ell(a)} \nrightarrow y_{r(a)}, \qquad a \in A.$$

The double family construction is developed along these lines in Section 3. In Section 4, we define **colax** or **strong coproducts** in a double category $\mathbb{D}$ to be a colax or pseudo left adjoint to the embedding $\Delta : \mathbb{D} \to \mathbb{F}\mathsf{am}(\mathbb{D})$, and we verify that two fundamental classes of double categories, those of spans and matrices, have strong coproducts under appropriate assumptions.

[1] Even this comparison is arguably misleading since equipments have a dual characterization in terms of a "mapping out" universal property [Shulman, 2008, Theorem 4.1], so are, in any analogy with limits or colimits, equally close to either.

The new definition's most immediate advantage is in giving a more flexible notion of coproduct. Coproducts of proarrows indexed by an identity span $I \xleftarrow{1} I \xrightarrow{1} I$ are the **parallel coproducts** familiar from cocartesian double categories. Coproducts of proarrows with common source and target, indexed by a span $1 \xleftarrow{!} I \xrightarrow{!} 1$, are **local coproducts**. These are, in particular, local coproducts in the underlying bicategory but have a stronger universal property.

Once the notion of coproduct-preserving double functor has been established in Section 5, we formulate and prove the expected result that $\mathbb{F}\mathsf{am}(\mathbb{D})$ is the free coproduct completion of the double category $\mathbb{D}$. In particular, $\mathbb{S}\mathsf{pan}$ is the free coproduct completion of the terminal double category, since we have an isomorphism of double categories $\mathbb{F}\mathsf{am}(\mathbb{1}) \cong \mathbb{S}\mathsf{pan}$ essentially by construction. Similarly, $\mathbb{F}\mathsf{in}\mathsf{Fam}(\mathbb{1}) \cong \mathbb{F}\mathsf{inSpan}$ is the free completion of the terminal double category with finite coproducts. Thus the first conceptual problem with cocartesian double categories is resolved.

In Section 6, we turn to products in double categories. Of course, products in a double category $\mathbb{D}$ can be defined as coproducts in the opposite double category $\mathbb{D}^{\mathrm{op}}$, where $(\mathbb{D}^{\mathrm{op}})_i = (\mathbb{D}_i)^{\mathrm{op}}$ for each $i = 0, 1$. Yet there is an interesting asymmetry in the main examples that has no counterpart for one-dimensional categories. Key examples of double categories, such as those of spans and of matrices, have *strong* coproducts but *lax* products, a situation first noticed by Paré in the double category of relations [Paré, 2009]. Products in these double categories are not entirely lax: products commute with external composition when the adjacent legs of the indexing spans are bijections. We isolate this condition as *iso-strong products* in Section 7 and check that double categories of spans and of matrices have iso-strong products under the expected hypotheses.

A less immediate, but very important, consequence of the defining universal property is that double categories with products automatically possess certain *restriction* cells (also known as *cartesian* cells). Namely, they have restrictions along any structure arrow between products, such as projections and diagonals. Restrictions, companions, and conjoints in double categories with products are investigated in Section 8, culminating in the characterization of double categories with iso-strong finite products as cartesian double categories that have restrictions along structure arrows between products. We thus obtain inclusions

$$\begin{aligned}\text{cartesian equipments} &\subset \text{double categories with iso-strong finite products}\\ &\subset \text{cartesian double categories}\end{aligned}$$

and the inclusions are strict. The same sequence of inclusions holds for precartesian equipments, double categories with lax finite products, and precartesian double categories, where a *precartesian* double category is like a cartesian double category except that the pseudo double functors comprising the right adjoints are replaced by lax double functors [Aleiferi, 2018, §4.1].

Double categories with products possess *just enough* restrictions to have underlying cartesian bicategories, allowing the structure arrows between products to be transferred to proarrows but in general nothing more. To be more precise, the underlying bicategory of a double category with iso-strong finite products is cartesian. We defer the proof of this theorem to another work [Patterson, 2024, Theorem 5.7], since it involves techniques for transposing structure in double categories of a rather different spirit than the present paper. Nevertheless, this result resolves the second conceptual problem raised above. As a corollary, it follows that cartesian equipments have underlying cartesian bicategories.

The last two sections of the paper concern a different application, double-categorical logic, which was actually the author's original motivation to revisit products in double categories.[2] In recent work [Lambert and Patterson, 2024], Michael Lambert and the author proposed cartesian double categories as a double-categorical approach to doctrines, defining a **cartesian double theory** to be a small, strict cartesian double category and a **model** of a cartesian double theory to be a cartesian lax functor out of it. Cartesian equipments were not considered viable as double theories since lax functors seem not to interact well with all of the abundant structure present in an equipment. Double categories with finite products, in the sense developed here, are a useful compromise, more expressive than a cartesian double category yet fully controlled by a single universal property that lax functors respect.

In Section 10, we define a **finite-product double theory** to be a small, strict double category with strong finite products and a **model** of a finite-product double theory to be a product-preserving lax functor out of it. We present several examples of finite-product double theories illustrating their extra flexibility compared to cartesian double theories. Finally, in Section 11, we construct a unital virtual double category of models of a finite-product double theory and prove a series of simple but necessary lemmas showing that morphisms and higher morphisms of models behave properly.

Our thesis is, in short, that the canonical notion of products in double categories is the one investigated here, at least for double categories with "object-like" proarrows, such as those of spans, matrices, relations, and profunctors. A natural question for the future is whether the same unbiased perspective can be usefully applied to general limits and colimits in double categories.

Organization This paper naturally divides into two parts, concerning coproducts and products. The two parts are largely, though not completely, independent of each other. In Sections 2–5, we review the family construction for ordinary categories, extend it to a double-categorical family construction, formulate coproducts in a double category $\mathbb{D}$ as a left adjoint to the inclusion double functor $\Delta : \mathbb{D} \to \mathbb{F}\mathsf{am}(\mathbb{D})$, and then explicitly construct coproducts in several important classes of double categories. In Sections 6–11, we dualize to study products in double categories; distinguish between lax, iso-strong, and strong products in key examples; and finally apply these notions to formulate finite-product double theories and their virtual double categories of models. The reader primarily interested in a direct description of products, or in their applications to formal category theory, can begin at Section 6, referring occasionally to earlier sections for technical results.

[2] In fact, double-categorical logic was also Paré's original motivation [Paré, 2009], although he considers both a different type of a double theory and a different type of model than we do.

Conventions Double categories and double functors are assumed to be pseudo unless otherwise stated. Every double category $\mathbb{D}$ has two **underlying categories**, the category $\mathbb{D}_0$ of objects and arrows and the category $\mathbb{D}_1$ of proarrows and cells. A double category also has an **underlying 2-category** of objects, arrows, and cells bounded by identity proarrows, and an **underlying bicategory** of objects, proarrows, and **globular** cells, which are bounded by identity arrows. We always write applications of the external composition $\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ in a double category in diagrammatic order, and we denote the external identity by $\mathrm{id} : \mathbb{D}_0 \to \mathbb{D}_1$.

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2. Review of the family construction

In this expository section, we review the family construction, or free coproduct completion, of a category, as we will develop the family construction for double categories in analogy with it. Much is known about the category of families in a given category, which enjoys surprisingly many properties besides simply being the free coproduct completion. Notable references for these properties include the papers [Carboni and Johnstone, 1995; Hu and Tholen, 1995; Adámek and Rosickỳ, 2020]. The category of families apparently originates with Bénabou's study of the foundations of naive category theory [Bénabou, 1985, §3.3].

The family construction can be defined as the Grothendieck construction of a restricted representable functor. Given the simplicity of the category of families, this may seem a circuitous route but it has the advantage of immediately exhibiting the fibered structure of families. More to the point, it will motivate an analogous approach to families in double categories.

2.1. Construction. [Families] The **category of families** in a category C, denoted $\mathsf{Fam}(\mathsf{C})$, is the Grothendieck construction of the functor $\mathbf{Cat}(-, \mathsf{C})|_{\mathsf{Set}} : \mathsf{Set}^{\mathrm{op}} \to \mathbf{Cat}$:

$$\mathsf{Fam}(\mathsf{C}) \coloneqq \int^{I \in \mathsf{Set}} \mathbf{Cat}(I, \mathsf{C}).$$

In the definition, the sets I are regarded as discrete categories by a common abuse of notation. Thus, the category $\mathsf{Fam}(\mathsf{C})$ has

- as objects $(I, \underline{x})$, an **indexing set** I together with an I-indexed family $\underline{x} : I \to \mathsf{C}$ of objects in C, whose elements are denoted $x_i \coloneqq \underline{x}(i)$;
- as morphisms $(f_0, f) : (I, \underline{x}) \to (J, \underline{y})$, a function $f_0 : I \to J$ between indexing sets together with an I-indexed family $f : \underline{x} \Rightarrow \underline{y} \circ f_0$ of morphisms in C, having components of the form $f_i : x_i \to y_{f_0(i)}$.

By the fundamental relationship between indexed categories and fibrations, the canonical projection $\mathsf{Fam}(\mathsf{C}) \to \mathsf{Set}$ sending a family of objects to its indexing set is a fibration.

There is also a **category of finite families** in C, denoted $\mathsf{FinFam}(\mathsf{C})$. It is the full subcategory of $\mathsf{Fam}(\mathsf{C})$ spanned by families of objects indexed by *finite* sets; alternatively, it is the Grothendieck construction of the functor $\mathbf{Cat}(-,\mathsf{C})|_{\mathsf{FinSet}} : \mathsf{FinSet}^{\mathrm{op}} \to \mathbf{Cat}$. Everything that we will say about the family construction and coproducts remains true when "family" is replaced by "finite family" and "coproduct" by "finite coproduct." Often we will not bother to say so explicitly.

It would be conventional at this point to observe that $\mathsf{Fam}(\mathsf{C})$ is the free completion of C to a category with coproducts, but this presupposes the definition of coproducts, whereas we intend to use a double family construction to *define* coproducts in a double category. Mirroring this logic, we recall how to recover the usual definition of a coproduct in a category from the family construction. Denote by $\Delta \coloneqq \Delta_{\mathsf{C}} : \mathsf{C} \to \mathsf{Fam}(\mathsf{C})$ the functor that sends each object x in C to the singleton family $(1, x)$, indexed by the terminal set.

2.2. Proposition. [Coproducts as left adjoints] *A category* C *has all (small) coproducts if and only if the functor* $\Delta : \mathsf{C} \to \mathsf{Fam}(\mathsf{C})$ *has a left adjoint:*

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathsf{Fam}(\mathsf{C}) \arrow[l, bend right, "\Sigma"'] \end{tikzcd} .$$

Proof. The most expeditious proof uses the equivalence between adjunctions and universal arrows; see, for example, [Mac Lane, 1998, §IV.1] or [Grandis, 2019, §1.5.1]. Thus, the existence of a functor $\Sigma : \mathsf{Fam}(\mathsf{C}) \to \mathsf{C}$ along with an adjunction $\Sigma \dashv \Delta$ is equivalent to the existence merely of, for every family of objects $(I, \underline{x})$ in C, a universal arrow from $(I, \underline{x})$ to the functor Δ. Such a universal arrow consists of an object $\Sigma\underline{x} \coloneqq \Sigma(I, \underline{x})$ in C along with a morphism $(\iota_0, \iota) : (I, \underline{x}) \to \Delta\Sigma\underline{x}$ in $\mathsf{Fam}(\mathsf{C})$. It satisfies the universal property that, for any object $y \in \mathsf{C}$ and morphism $(f_0, f) : (I, \underline{x}) \to \Delta y$ in $\mathsf{Fam}(\mathsf{C})$, there exists a unique morphism $h : \Sigma\underline{x} \to y$ in C making the triangle commute:

$$\begin{tikzcd} (I, \underline{x}) \arrow[r, "{(\iota_0, \iota)}"] \arrow[dr, "{(f_0, f)}"'] & \Delta\Sigma\underline{x} \arrow[d, "\Delta h"] \\ & \Delta y \end{tikzcd} .$$

Now, since f_0 and ι_0 must both be the unique map $! : I \to 1$, we recover the usual definition of a coproduct: there is an object $\Sigma\underline{x}$ in C, conventionally denoted $\sum_{i \in I} x_i$, along with **coprojection** morphisms $\iota_i : x_i \to \Sigma\underline{x}$ for each $i \in I$. These satisfy the universal property that for any object $y \in \mathsf{C}$ and family of morphisms $(f_i : x_i \to y)_{i \in I}$, there exists a unique morphism $h : \Sigma\underline{x} \to y$ making all the triangles commute:

$$\begin{tikzcd} x_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & \Sigma\underline{x} \arrow[d, "h"] \\ & y \end{tikzcd}, \qquad i \in I.$$

■

Functors that preserve coproducts can also be characterized using the family construction. Notice that the assignment $\mathsf{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ is a functor (even a 2-functor, although we won't need that). Specifically, given a functor $F : \mathsf{C} \to \mathsf{D}$, the functor $\mathsf{Fam}(F) : \mathsf{Fam}(\mathsf{C}) \to \mathsf{Fam}(\mathsf{D})$ results by applying the Grothendieck construction, which is functorial [Peschke and Tholen, 2020, §6.2], to the natural transformation

$$\mathbf{Cat}(-, F)|_{\mathsf{Set}} : \mathbf{Cat}(-, \mathsf{C})|_{\mathsf{Set}} \Rightarrow \mathbf{Cat}(-, \mathsf{D})|_{\mathsf{Set}} : \mathsf{Set}^{\mathrm{op}} \to \mathbf{Cat}.$$

In concrete terms, $\mathsf{Fam}(F)$ acts on indexed families by post-composition: for a family of objects $(I, \underline{x})$ in C, we have $\mathsf{Fam}(F)(I, \underline{x}) = (I, F\underline{x})$, so that $(F\underline{x})_i = F(x_i)$ for each $i \in I$, and similarly for families of morphisms. To avoid notational clutter, we will often simply write $F\underline{x}$ for this action.

2.3. Proposition. [Coproduct-preserving functors] *A functor $F : \mathsf{C} \to \mathsf{D}$ between categories with coproducts preserves all coproducts if and only if the morphism of adjunctions*

$$\begin{tikzcd} \mathsf{Fam}(\mathsf{C}) \arrow[r, "\Sigma \dashv \Delta", "\shortmid" marking] \arrow[d, "\mathsf{Fam}(F)"'] \arrow[dr, phantom, "(\Phi{,}1)"] & \mathsf{C} \arrow[d, "F"] \\ \mathsf{Fam}(\mathsf{D}) \arrow[r, "\Sigma \dashv \Delta"', "\shortmid" marking] & \mathsf{D} \end{tikzcd}$$

defined by the pair of functors $(\mathsf{Fam}(F), F)$ *is strong.*

Proof. A *morphism of adjunctions* is a cell in the double category of adjunctions [Grandis, 2019, §3.1.6]. Less standardly, the cell being *strong* means that the identity transformation on the left

$$\begin{tikzcd} \mathsf{Fam}(\mathsf{C}) \arrow[d, "\mathsf{Fam}(F)"'] & \mathsf{C} \arrow[l, "\Delta_{\mathsf{C}}"'] \arrow[d, "F"] \arrow[dl, phantom, "\Downarrow 1"] \\ \mathsf{Fam}(\mathsf{D}) & \mathsf{D} \arrow[l, "\Delta_{\mathsf{D}}"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{Fam}(\mathsf{C}) \arrow[r, "\Sigma_{\mathsf{C}}"] \arrow[d, "\mathsf{Fam}(F)"'] \arrow[dr, phantom, "\Phi"] & \mathsf{C} \arrow[d, "F"] \\ \mathsf{Fam}(\mathsf{D}) \arrow[r, "\Sigma_{\mathsf{D}}"'] & \mathsf{D} \end{tikzcd}$$

has a mate cell Φ as shown on the right that is a natural isomorphism. Unwinding the definitions, the component of Φ at a family $(I, \underline{x})$ is the canonical comparison

$$\textstyle\sum_{i \in I} Fx_i = \Sigma F\underline{x} \xrightarrow{\Phi_{\underline{x}}} F\Sigma\underline{x} = F(\sum_{i \in I} x_i)$$

provided by the universal property of the coproduct in D. By definition, the functor F preserves the coproduct of a family $\underline{x}$ if and only if the component $\Phi_{\underline{x}}$ is an isomorphism. ■

We now recall the central property of the family construction.

2.4. Theorem. [Free coproduct completion] *For any category* C*, the category* $\mathsf{Fam}(\mathsf{C})$ *is the free coproduct completion of* C.

Proof. Given a family of objects $(I, (\underline{U}, \underline{\underline{x}}))$ in $\mathsf{Fam}(\mathsf{C})$, i.e., an object of $\mathsf{Fam}(\mathsf{Fam}(\mathsf{C}))$, a coproduct of it in $\mathsf{Fam}(\mathsf{C})$ is

$$\Sigma(\underline{U}, \underline{\underline{x}}) := (\sqcup\underline{U}, \underline{x}), \qquad \text{where} \qquad \sqcup\,\underline{U} := \bigsqcup_{i\in I} U_i, \qquad \underline{x} := [\underline{x}_i]_{i\in I} : \sqcup\underline{U} \to \mathsf{C}.$$

In other words, $\sqcup\underline{U}$ is the disjoint union or coproduct of the indexing sets and $\underline{x}$ is the copairing of the indexed families given by the universal property of the coproduct in Set. The coprojections in $\mathsf{Fam}(\mathsf{C})$ are the morphisms

$$(\iota_i, 1_{\underline{x}_i}) : (U_i, \underline{x}_i) \to (\sqcup U, \underline{x}), \qquad i \in I,$$

defined by the inclusions $\iota_i : U_i \to \sqcup\underline{U}$ in Set and the family of identity morphisms $(1_{\underline{x}_i})_u : x_{i,u} \to x_{\iota_i(u)}$ in C, for each $u \in U_i$. The universal property of the coproduct in $\mathsf{Fam}(\mathsf{C})$ follows directly from that of the coproduct in Set.

So the functor $\Delta : \mathsf{C} \to \mathsf{Fam}(\mathsf{C})$ embeds C into a category with coproducts. That $\mathsf{Fam}(\mathsf{C})$ is the *free* coproduct completion of C is the universal property that for any other functor $F : \mathsf{C} \to \mathsf{D}$ into a category D with coproducts, there exists a coproduct-preserving functor $\hat{F} : \mathsf{Fam}(\mathsf{C}) \to \mathsf{D}$ making the triangle

$$\begin{tikzcd} & \mathsf{Fam}(\mathsf{C}) \arrow[d, dashed, "\hat{F}"] \\ \mathsf{C} \arrow[ur, "\Delta"] \arrow[r, "F"'] & \mathsf{D} \end{tikzcd}$$

commute, and $\hat{F}$ is unique up to natural isomorphism. Indeed, a functor $\hat{F} : \mathsf{Fam}(\mathsf{C}) \to \mathsf{D}$ making the triangle commute is uniquely determined on singleton families and maps between them by F. But an arbitrary family of objects in C is a coproduct of singleton families, so if $\hat{F}$ is to preserve coproducts, it must send the family to a coproduct of fixed objects in D. Since D is assumed to have coproducts, we can make such a choice of $\hat{F}$, and since coproducts are unique up to unique isomorphism commuting with coprojections, any such choice of $\hat{F}$ is unique up to unique natural isomorphism preserving the triangle formed with Δ and F. ∎

3. The family construction for double categories

As reviewed in the previous section, families in a category can be defined as the Grothendieck construction of the restriction of a representable functor. We will now define families in a double category analogously using the double Grothendieck construction, a tool recently introduced by Cruttwell, Lambert, Pronk, and Szyld in their study of double fibrations [Cruttwell et al., 2022]. The impatient reader may wish to skip to the concrete description of the double family construction following Construction 3.3.

Besides the double Grothendieck construction, we will need a bit of the technology of *double 2-categories* developed by Cruttwell et al. A **double 2-category** is a pseudocategory in **2Cat**, the 2-category of 2-categories, 2-functors, and 2-natural transformations [Cruttwell et al., 2022, Definition 3.2]. So, a double 2-category $\mathbb{E}$ is underlied by a 2-category $\mathbb{E}_0$ of objects, arrows, and 2-cells between arrows as well as a 2-category $\mathbb{E}_1$ of proarrows, cells, and 2-cells between cells. The fundamental example of a double 2-category is nothing other than the receiving object for the lax double pseudofunctors that are the input to the double Grothendieck construction.

3.1. Example. [Spans of categories] As a particular case of [Cruttwell et al., 2022, Construction 3.6], the double category of spans of categories upgrades to a double 2-category $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$. Its 2-category of objects is simply $\mathbb{S}\mathsf{pan}(\mathbf{Cat})_0 = \mathbf{Cat}$, the 2-category of categories. More interestingly, its 2-category of morphisms $\mathbb{S}\mathsf{pan}(\mathbf{Cat})_1$ has spans of categories as objects; has maps of spans as morphisms; and has as 2-cells

$$\begin{tikzcd} \mathsf{X} \arrow[d, "F"'] & \mathsf{S} \arrow[l, "L"'] \arrow[r, "R"] \arrow[d, "H"'] & \mathsf{Y} \arrow[d, "G"] \\ \mathsf{W} & \mathsf{T} \arrow[l, "L"] \arrow[r, "R"'] & \mathsf{Z} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathsf{X} \arrow[d, "F'"'] & \mathsf{S} \arrow[l, "L"'] \arrow[r, "R"] \arrow[d, "H'"'] & \mathsf{Y} \arrow[d, "G'"] \\ \mathsf{W} & \mathsf{T} \arrow[l, "L"] \arrow[r, "R"'] & \mathsf{Z} \end{tikzcd},$$

triples of natural transformations $\alpha : F \Rightarrow F'$, $\beta : G \Rightarrow G'$, and $\sigma : H \Rightarrow H'$ satisfying the two "cylinder conditions"

$$\begin{tikzcd} \mathsf{X} \arrow[d, bend right, "F"'] \arrow[d, bend left, "F'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathsf{S} \arrow[l, "L"'] \arrow[d, bend left, "H'"] \\ \mathsf{W} & \mathsf{T} \arrow[l, "L"] \end{tikzcd} = \begin{tikzcd} \mathsf{X} \arrow[d, bend right, "F"'] & \mathsf{S} \arrow[l, "L"'] \arrow[d, bend right, "H"'] \arrow[d, bend left, "H'"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] \\ \mathsf{W} & \mathsf{T} \arrow[l, "L"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{S} \arrow[r, "R"] \arrow[d, bend right, "H"'] \arrow[d, bend left, "H'"] \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & \mathsf{Y} \arrow[d, bend left, "G'"] \\ \mathsf{T} \arrow[r, "R"'] & \mathsf{Z} \end{tikzcd} = \begin{tikzcd} \mathsf{S} \arrow[r, "R"] \arrow[d, bend right, "H"'] & \mathsf{Y} \arrow[d, bend right, "G"'] \arrow[d, bend left, "G'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\ \mathsf{T} \arrow[r, "R"'] & \mathsf{Z} \end{tikzcd}.$$

The source and target 2-functors $\mathbb{S}\mathsf{pan}(\mathbf{Cat})_1 \rightrightarrows \mathbf{Cat}$ are the obvious ones that extract the left and right feet, legs, and cells between legs. External composition is by 2-pullback in **Cat**.

Any double category $\mathbb{D}$ has an underlying span of categories $\mathbb{D}_0 \xleftarrow{s} \mathbb{D}_1 \xrightarrow{t} \mathbb{D}_0$, an object of $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$. Any span of sets can be regarded as a span of discrete categories, hence also as an object of $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$. These observations motivate the following construction of a doubly indexed category. We will also need some special notation for elements of spans of sets, inspired by [Paré, 2011, §3.7]. For any span $I \xleftarrow{\ell} A \xrightarrow{r} J$ in Set, we will generally label the left and right legs as ℓ and r and then conflate the apex of the span with the span itself, writing $A : I \nrightarrow J$. We then write $a : i \nrightarrow j$ for an element $a \in A$ such that $\ell(a) = i$ and $r(a) = j$.

3.2. CONSTRUCTION. [Double indexing of a double category] Given a double category $\mathbb{D}$, we construct a lax double functor suggestively denoted

$$F = \mathbb{S}\mathsf{pan}(\mathbf{Cat})(-, \mathbb{D})|_{\mathbb{S}\mathsf{pan}} : \mathbb{S}\mathsf{pan}^{\mathrm{op}} \to \mathbb{S}\mathsf{pan}(\mathbf{Cat}).$$

The underlying functor $F_0 \coloneqq \mathbf{Cat}(-, \mathbb{D}_0)|_{\mathsf{Set}} : \mathsf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ is that used previously in Construction 2.1. It sends a set I to the functor category $\mathbf{Cat}(I, \mathbb{D}_0)$ having as objects, I-indexed families $\underline{x} : I \to \mathbb{D}_0$ of objects in $\mathbb{D}$ and as morphisms $\underline{x} \to \underline{x}'$, transformations $f : \underline{x} \Rightarrow \underline{x}'$. The second underlying functor is

$$F_1 \coloneqq \mathbb{S}\mathsf{pan}(\mathbf{Cat})_1(-, \mathbb{D})|_{\mathbb{S}\mathsf{pan}_1} : \mathbb{S}\mathsf{pan}_1^{\mathrm{op}} \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})_1.$$

It sends a span of sets $I \xleftarrow{\ell} A \xrightarrow{r} J$ to the span of categories whose apex is the category

$$\mathbb{S}\mathsf{pan}(\mathbf{Cat})_1(I \xleftarrow{\ell} A \xrightarrow{r} J,\ \mathbb{D}_0 \xleftarrow{s} \mathbb{D}_1 \xrightarrow{t} \mathbb{D}_0)$$

from Example 3.1. Thus, the objects of this category are indexed families of objects and proarrows in $\mathbb{D}$ of the form

$$\begin{tikzcd} I \arrow[d, "\underline{x}"'] & A \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "\underline{m}"'] & J \arrow[d, "\underline{y}"] \\ \mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathbb{D}_0 \end{tikzcd},$$

which we abbreviate as $\underline{m} : \underline{x} \nrightarrow \underline{y}$; and its morphisms from $\underline{m} : \underline{x} \nrightarrow \underline{y}$ to $\underline{m}' : \underline{x}' \nrightarrow \underline{y}'$ are families of arrows $f : \underline{x} \Rightarrow \underline{x}'$ and $g : \underline{y} \Rightarrow \underline{y}'$ together with a family of cells $\alpha : \underline{m} \Rightarrow \underline{m}'$ of form

$$\begin{tikzcd} x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "f_i"'] \arrow[dr, phantom, "\alpha_a"] & y_j \arrow[d, "g_j"] \\ x'_{f_0(i)} \arrow[r, "m'_{\alpha_0(a)}"', "\shortmid"{marking}] & y'_{g_0(j)} \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

The left and right legs of this span of categories are the obvious ones.

To complete the construction, the laxators and unitors of the lax double functor F are furnished by the external composition and identities in $\mathbb{D}$. Given spans of sets $A : I \nrightarrow J$ and $B : J \nrightarrow K$, the laxator $F_{A,B}$ is defined by the functor $FA \times_{FJ} FB \to F(A \times_J B)$ that sends a pair of proarrow families $\underline{m} : \underline{x} \nrightarrow \underline{y}$ and $\underline{n} : \underline{y} \nrightarrow \underline{z}$ to the family $\underline{m} \odot \underline{n} : \underline{x} \nrightarrow \underline{z}$ with components

$$(\underline{m} \odot \underline{n})(a, b) \coloneqq m_a \odot n_b : x_i \nrightarrow z_k, \qquad (i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k) : (I \overset{A}{\nrightarrow} J \overset{B}{\nrightarrow} K). \tag{1}$$

and similarly for families of cells. Given a set I, the unitor $F_I : \mathrm{id}_{FI} \to F(\mathrm{id}_I)$ is defined by the functor sending each I-indexed object family $\underline{x}$ to the proarrow family $\mathrm{id}_{\underline{x}} : \underline{x} \to \underline{x}$ with components $\mathrm{id}_{x_i} : x_i \nrightarrow x_i$ for $i \in I$, and similarly for families of arrows.

When $\mathbb{D}$ is a *strict* double category, we obtain a genuine lax double functor, where the associativity and unitality axioms for the laxators and unitors are immediate from the corresponding axioms for the external composition and identity in $\mathbb{D}$. In general, though, what we have is a lax double *pseudo*functor [Cruttwell et al., 2022, Definition 3.12], whose associativity and unitality modifications are given by the associators and unitors of $\mathbb{D}$. The coherence axioms for these modifications correspond to the coherence axioms for the pseudo double category. In either case, the naturality axioms for the laxators and unitors amount to the statements that reindexing commutes with composing indexed families of proarrows.

3.3. Construction. [Double families] The **double category of families** in a double category $\mathbb{D}$, denoted $\mathbb{F}\mathsf{am}(\mathbb{D})$, is the double Grothendieck construction of the lax double pseudofunctor from Construction 3.2:

$$\mathbb{F}\mathsf{am}(\mathbb{D}) := \int^{S \in \mathbb{S}\mathsf{pan}} \mathbb{S}\mathsf{pan}(\mathbf{Cat})(S, \mathbb{D}).$$

The **double category of finite families** in $\mathbb{D}$, denoted $\mathbb{F}\mathsf{inFam}(\mathbb{D})$, is defined similarly by replacing $\mathbb{S}\mathsf{pan}$ with $\mathbb{F}\mathsf{inSpan}$, the double category of spans of finite sets.

By construction, the canonical projection $\mathbb{F}\mathsf{am}(\mathbb{D}) \to \mathbb{S}\mathsf{pan}$, a strict double functor, is a double fibration. In fact, if $\mathbb{S}\mathsf{q}(\mathsf{C})$ denotes the double category of commutative squares in a category C, then $\mathbb{F}\mathsf{am}(\mathbb{S}\mathsf{q}(\mathsf{C}))$ is the double category denoted $\mathbb{F}\mathsf{am}(\mathsf{C})$ and called the "double family fibration" by Cruttwell et al [Cruttwell et al., 2022, Examples 2.29 and 3.19]. Any double category of families is also an example of a "double category of decorated spans" in the generalized sense of [Patterson, 2023, §3].

It is indispensable to have a fully concrete description of the double category of families. Given a double category $\mathbb{D}$, its double category of families, $\mathbb{F}\mathsf{am}(\mathbb{D})$, has

- as objects, an **indexing set** I together with an I-indexed family $\underline{x} : I \to \mathbb{D}_0$, consisting of objects x_i in $\mathbb{D}$ for $i \in I$;
- as arrows $(I, \underline{x}) \to (J, \underline{y})$, a function $f_0 : I \to J$ together with an I-indexed family f of arrows in $\mathbb{D}$ of the form $f_i : x_i \to y_{f_0(i)}$ for $i \in I$;
- as proarrows $(I, \underline{x}) \nrightarrow (J, \underline{y})$, an **indexing span** $I \xleftarrow{\ell} A \xrightarrow{r} J$ together with an A-indexed family $\underline{m} : A \to \mathbb{D}_1$, consisting of proarrows in $\mathbb{D}$ of the form $m_a : x_i \nrightarrow y_j$ for each $a : i \nrightarrow j$;
- as cells $\begin{array}{ccc} (I,\underline{x}) & \xrightarrow{(A,\underline{m})} & (J,\underline{y}) \\ {\scriptstyle (f_0,f)}\downarrow & & \downarrow{\scriptstyle (g_0,g)} \\ (K,\underline{w}) & \xrightarrow[(B,\underline{n})]{} & (L,\underline{z}) \end{array}$, a map of spans (f_0, α_0, g_0) as shown on the left below together with an A-indexed family α of cells in $\mathbb{D}$ of the form on the right:

$$\begin{tikzcd} I \arrow[d, "f_0"'] & A \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "\alpha_0"] & J \arrow[d, "g_0"] \\ K & B \arrow[l, "\ell"] \arrow[r, "r"'] & L \end{tikzcd}, \qquad \begin{tikzcd} x_i \arrow[r, "m_a", "|" marking] \arrow[d, "f_i"'] \arrow[dr, phantom, "\alpha_a"] & y_j \arrow[d, "g_j"] \\ w_{f_0(i)} \arrow[r, "n_{\alpha_0(a)}"', "|" marking] & z_{g_0(j)} \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

To compose proarrows in $\mathbb{F}\mathsf{am}(\mathbb{D})$, first compose the indexing spans by pullback as usual, then compose the families of proarrows componentwise in $\mathbb{D}$ as defined in Equation (1).

The definition of the double category of families is lent further support by the several senses in which the 1-categorical and double-categorical family constructions commute. First of all, we have the expected compatibility $\mathbb{F}\mathsf{am}(\mathbb{D})_0 = \mathsf{Fam}(\mathbb{D}_0)$ for any double category $\mathbb{D}$. Less immediately, we also have:

3.4. Proposition. [Families of spans] *Let* S *be a category with pullbacks. Then there is an isomorphism of double categories*

$$\mathbb{F}\mathsf{am}(\mathbb{S}\mathsf{pan}(\mathsf{S})) \cong \mathbb{S}\mathsf{pan}(\mathsf{Fam}(\mathsf{S})).$$

Proof. As observed in [Adámek and Rosickỳ, 2020, Remark 2.3], when the category S has pullbacks, so does $\mathsf{Fam}(\mathsf{S})$. The pullback of a cospan $(I,\underline{x}) \xrightarrow{(f_0,f)} (K,\underline{z}) \xleftarrow{(g_0,g)} (J,\underline{y})$ in $\mathsf{Fam}(\mathsf{S})$ is computed by, first, taking the pullback of the indexing sets

$$\begin{tikzcd}
& I \times_K J \arrow[dl, "p_0"'] \arrow[dr, "q_0"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
I \arrow[dr, "f_0"'] & & J \arrow[dl, "g_0"] \\
& K &
\end{tikzcd}$$

and then, for each pair $(i,j) \in I \times_K J$, taking the pullback in S

$$\begin{tikzcd}
& w_{(i,j)} \arrow[dl, "p_{(i,j)}"'] \arrow[dr, "q_{(i,j)}"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
x_i \arrow[dr, "f_i"'] & & y_j \\
& z_{f_0(i)} = z_{g_0(j)} & \arrow[from=2-3, to=3-2, "g_j"]
\end{tikzcd} .$$

The span $(I,\underline{x}) \xleftarrow{(p_0,p)} (I \times_K J, \underline{w}) \xrightarrow{(q_0,q)} (J,\underline{y})$ is then a pullback in $\mathsf{Fam}(\mathsf{S})$. This proves that the double category $\mathbb{S}\mathsf{pan}(\mathsf{Fam}(\mathsf{S}))$ is well defined. Comparing with the external composition in Construction 3.3, it also shows that not only are the proarrows in $\mathbb{F}\mathsf{am}(\mathbb{S}\mathsf{pan}(\mathsf{S}))$ and $\mathbb{S}\mathsf{pan}(\mathsf{Fam}(\mathsf{S}))$ the same, they have the same composition law. The remaining identifications are straightforward. ∎

As a corollary, we obtain an isomorphism that is also easy to see directly.

3.5. Corollary. *Denoting the terminal double category by* $\mathbb{1}$*, there is an isomorphism*

$$\mathbb{F}\mathsf{am}(\mathbb{1}) \cong \mathbb{S}\mathsf{pan}.$$

Proof. Taking $\mathsf{S} = 1$ to be the terminal category, we have $\mathbb{S}\mathsf{pan}(1) \cong \mathbb{1}$ and $\mathsf{Fam}(1) \cong \mathsf{Set}$. ∎

It will be important later to know that $\mathbb{F}\mathsf{am}(\mathbb{D})$ inherits extension cells, including companions and conjoints, whenever they exist in $\mathbb{D}$. The general theory of companions, conjoints, and equipments is established in [Grandis and Paré, 2004; Shulman, 2008, 2010]. In this paper, we use the terms **restriction** and **extension cell** as more evocative synonyms for cartesian and opcartesian cells in a double category, respectively.

3.6. Proposition. [Extensions of families] *Let $\mathbb{D}$ be a double category. Suppose given a co-niche in $\mathbb{F}\mathsf{am}(\mathbb{D})$ of the form*

$$\begin{tikzcd} (I,\underline{x}) \arrow[r, "(A,\underline{m})", "\shortmid"{marking}] \arrow[d, "(f_0,f)"'] & (J,\underline{y}) \arrow[d, "(g_0,g)"] \\ (K,\underline{w}) & (L,\underline{z}) \end{tikzcd}$$

such that each constituent co-niche in $\mathbb{D}$

$$\begin{tikzcd} x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "f_i"'] & y_j \arrow[d, "g_j"] \\ w_{f_0(i)} & z_{g_0(j)} \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

can be filled by an extension cell. Then the co-niche in $\mathbb{F}\mathsf{am}(\mathbb{D})$ is also fillable by an extension cell.

Proof. From the extension in $\mathbb{S}\mathsf{pan}$

$$\begin{tikzcd} I \arrow[d, "f_0"'] & A \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, equal] & J \arrow[d, "g_0"] \\ K & A \arrow[l, "f_0 \circ \ell"] \arrow[r, "g_0 \circ r"'] & L \end{tikzcd}$$

together with the family of extensions in $\mathbb{D}$

$$\begin{tikzcd} x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "f_i"'] \arrow[dr, phantom, "\mathrm{ext}_a"] & y_j \arrow[d, "g_j"] \\ w_{f_0(i)} \arrow[r, "n_a"', "\shortmid"{marking}] & z_{g_0(j)} \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

we obtain a family of cells in $\mathbb{D}$:

$$\begin{tikzcd} (I,\underline{x}) \arrow[r, "(A,\underline{m})", "\shortmid"{marking}] \arrow[d, "(f_0,f)"'] \arrow[dr, phantom, "(1_A,\mathrm{ext})"] & (J,\underline{y}) \arrow[d, "(g_0,g)"] \\ (K,\underline{w}) \arrow[r, "(A,\underline{n})"', "\shortmid"{marking}] & (L,\underline{z}) \end{tikzcd}.$$

We verify that this family satisfies the universal property of an extension cell in $\mathbb{F}\mathsf{am}(\mathbb{D})$. Given a cell in $\mathbb{F}\mathsf{am}(\mathbb{D})$ as on the left

$$\begin{tikzcd}
(I,\underline{x}) \arrow[r, "{(A,\underline{m})}", "\shortmid"{marking}] \arrow[d, "{(f_0,f)}"'] & (J,\underline{y}) \arrow[d, "{(g_0,g)}"] \\
(K,\underline{w}) \arrow[r, phantom, "{(\alpha_0,\alpha)}"] \arrow[d, "{(h_0,h)}"'] & (L,\underline{z}) \arrow[d, "{(k_0,k)}"] \\
(K',\underline{w}') \arrow[r, "{(B,\underline{p})}"', "\shortmid"{marking}] & (L',\underline{z}')
\end{tikzcd}
=
\begin{tikzcd}
(I,\underline{x}) \arrow[r, "{(A,\underline{m})}", "\shortmid"{marking}] \arrow[d, "{(f_0,f)}"'] \arrow[dr, phantom, "{(1_A,\mathrm{ext})}"] & (J,\underline{y}) \arrow[d, "{(g_0,g)}"] \\
(K,\underline{w}) \arrow[r, "{(A,\underline{n})}"', "\shortmid"{marking}] \arrow[d, "{(h_0,h)}"'] \arrow[dr, phantom, "\exists!"] & (L,\underline{z}) \arrow[d, "{(k_0,k)}"] \\
(K',\underline{w}') \arrow[r, "{(B,\underline{p})}"', "\shortmid"{marking}] & (L',\underline{z}')
\end{tikzcd},$$

we must show that there is a unique factorization as on the right. But this follow directly by applying the universal properties of each of the constituent extension cells:

$$\begin{tikzcd}
x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "f_i"'] & y_j \arrow[d, "g_j"] \\
w_{f_0(i)} \arrow[r, phantom, "\alpha_a"] \arrow[d, "h_{f_0(i)}"'] & z_{g_0(j)} \arrow[d, "k_{g_0(j)}"] \\
w'_{h_0(f_0(i))} \arrow[r, "p_{\alpha_0(a)}"', "\shortmid"{marking}] & z'_{k_0(g_0(j))}
\end{tikzcd}
=
\begin{tikzcd}
x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "f_i"'] \arrow[dr, phantom, "\mathrm{ext}_a"] & y_j \arrow[d, "g_j"] \\
w_{f_0(i)} \arrow[r, "n_a"', "\shortmid"{marking}] \arrow[d, "h_{f_0(i)}"'] \arrow[dr, phantom, "\exists!"] & z_{g_0(j)} \arrow[d, "k_{g_0(j)}"] \\
w'_{h_0(f_0(i))} \arrow[r, "p_{\alpha_0(a)}"', "\shortmid"{marking}] & z'_{k_0(g_0(j))}
\end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

∎

3.7. Corollary. *If a double category $\mathbb{D}$ is an equipment, then $\mathbb{F}\mathsf{am}(\mathbb{D})$ is also an equipment.*

Proof. This follows immediately from Proposition 3.6 using the characterization of an equipment as a double category whose source-target pairing is an opfibration [Shulman, 2008, Theorem 4.1]. ∎

3.8. Proposition. [Companions and conjoints of families] *Suppose $(f_0, f) : (I, \underline{x}) \to (J, \underline{y})$ is a family of arrows in a double category $\mathbb{D}$ such that each component $f_i : x_i \to y_{f_0(i)}$ has a companion $(f_i)_! : x_i \nrightarrow y_{f_0(i)}$ in $\mathbb{D}$. Then the family of arrows (f_0, f) has a companion in $\mathbb{F}\mathsf{am}(\mathbb{D})$, namely the family of proarrows $((f_0)_!, f_!) : (I, \underline{x}) \nrightarrow (J, \underline{y})$, where $(f_0)_! = (I = I \xrightarrow{f_0} J)$ is the companion of f_0 in $\mathbb{S}\mathsf{pan}$ and $(f_!)_i \coloneqq (f_i)_!$ for each $i \in I$.*

Dually, if each component f_i has a conjoint $f_i^* : y_{f_0(i)} \nrightarrow x_i$ in $\mathbb{D}$, then the family of arrows (f_0, f) has a conjoint in $\mathbb{F}\mathsf{am}(\mathbb{D})$, namely the family of proarrows (f_0^*, f^*) indexed by the conjoint span $f_0^* = (J \xleftarrow{f_0} I = I)$.

Proof. Since companions and conjoints are special kinds of extension cells, this result is a direct consequence of Proposition 3.6. We sketch a direct proof anyway since it is useful to have formulas for the binding cells of companions and conjoints in $\mathbb{F}\mathsf{am}(\mathbb{D})$. The unit and counit cells for a companion pair in $\mathbb{F}\mathsf{am}(\mathbb{D})$,

$$
\begin{tikzcd}
(I,\underline{x}) \arrow[r, "\mathrm{id}_{(I,\underline{x})}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "{(1_I,\eta)}"] & (I,\underline{x}) \arrow[d, "{(f_0,f)}"] \\
(I,\underline{x}) \arrow[r, "{((f_0)_!,f_!)}"', "\shortmid"{marking}] & (J,\underline{y})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(I,\underline{x}) \arrow[r, "{((f_0)_!,f_!)}", "\shortmid"{marking}] \arrow[d, "{(f_0,f)}"'] \arrow[dr, phantom, "{(f_0,\varepsilon)}"] & (J,\underline{y}) \arrow[d, equal] \\
(J,\underline{y}) \arrow[r, "\mathrm{id}_{(J,\underline{y})}"', "\shortmid"{marking}] & (J,\underline{y})
\end{tikzcd}
,
$$

are the families of units and counits in $\mathbb{D}$

$$
\begin{tikzcd}
x_i \arrow[r, "\mathrm{id}_{x_i}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\eta_i"] & x_i \arrow[d, "f_i"] \\
x_i \arrow[r, "(f_i)_!"', "\shortmid"{marking}] & y_{f_0(i)}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
x_i \arrow[r, "(f_i)_!", "\shortmid"{marking}] \arrow[d, "f_i"'] \arrow[dr, phantom, "\varepsilon_i"] & y_{f_0(i)} \arrow[d, equal] \\
y_{f_0(i)} \arrow[r, "\mathrm{id}_{y_{f_0(i)}}"', "\shortmid"{marking}] & y_{f_0(i)}
\end{tikzcd}
, \qquad i \in I.
$$

The equations for the companion pair in $\mathbb{F}\mathsf{am}(\mathbb{D})$ follow immediately by applying the companion equations componentwise in $\mathbb{D}$. Dually, the unit and counit for a conjoint pair in $\mathbb{F}\mathsf{am}(\mathbb{D})$,

$$
\begin{tikzcd}
(I,\underline{x}) \arrow[r, "\mathrm{id}_{(I,\underline{x})}", "\shortmid"{marking}] \arrow[d, "{(f_0,f)}"'] \arrow[dr, phantom, "{(1_I,\eta)}"] & (I,\underline{x}) \arrow[d, equal] \\
(J,\underline{y}) \arrow[r, "{(f_0^*,f^*)}"', "\shortmid"{marking}] & (I,\underline{x})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(J,\underline{y}) \arrow[r, "{(f_0^*,f^*)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "{(f_0,\varepsilon)}"] & (I,\underline{x}) \arrow[d, "{(f_0,f)}"] \\
(J,\underline{y}) \arrow[r, "\mathrm{id}_{(J,\underline{y})}"', "\shortmid"{marking}] & (J,\underline{y})
\end{tikzcd}
,
$$

are the families of units and counits in $\mathbb{D}$

$$
\begin{tikzcd}
x_i \arrow[r, "\mathrm{id}_{x_i}", "\shortmid"{marking}] \arrow[d, "f_i"'] \arrow[dr, phantom, "\eta_i"] & x_i \arrow[d, equal] \\
y_{f_0(i)} \arrow[r, "f_i^*"', "\shortmid"{marking}] & x_i
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
y_{f_0(i)} \arrow[r, "f_i^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon_i"] & x_i \arrow[d, "f_i"] \\
y_{f_0(i)} \arrow[r, "\mathrm{id}_{y_{f_0(i)}}"', "\shortmid"{marking}] & y_{f_0(i)}
\end{tikzcd}
, \qquad i \in I.
$$

∎

4. Coproducts in double categories

Following the analogy now established between families in categories and in double categories, we define coproducts in a double category. The characterization of coproducts in a category C as left adjoint to the embedding $\Delta : \mathsf{C} \to \mathsf{Fam}(\mathsf{C})$ (Proposition 2.2) becomes the *definition* of coproducts in a double category. Given a double category $\mathbb{D}$, denote by $\Delta := \Delta_{\mathbb{D}} : \mathbb{D} \to \mathbb{F}\mathsf{am}(\mathbb{D})$ the double functor sending each object $x \in \mathbb{D}$ to the singleton family $(1, x)$ and each proarrow $m : x \nrightarrow y$ in $\mathbb{D}$ to the singleton family $(1 \xleftarrow{!} 1 \xrightarrow{!} 1, m) : (1, x) \nrightarrow (1, y)$.

4.1. Definition. [Double coproducts] *A double category* $\mathbb{D}$ ***has colax coproducts*** *if the double functor* $\Delta : \mathbb{D} \to \mathbb{F}\mathsf{am}(\mathbb{D})$ *has a colax left adjoint:*

$$\begin{tikzcd} \mathbb{D} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathbb{F}\mathsf{am}(\mathbb{D}) \arrow[l, bend right, "\Sigma"'] \end{tikzcd} .$$

The double category $\mathbb{D}$ ***has strong coproducts*** *when the left adjoint* $\Sigma : \mathbb{F}\mathsf{am}(\mathbb{D}) \to \mathbb{D}$ *is a* pseudo *double functor. Colax and strong* ***finite coproducts*** *in* $\mathbb{D}$ *are defined analogously, replacing the double category* $\mathbb{F}\mathsf{am}(\mathbb{D})$ *with* $\mathbb{F}\mathsf{inFam}(\mathbb{D})$*.*

Recall that a double adjunction is in general between a colax double functor on the left and a lax double functor on the right [Grandis and Paré, 2004], [Grandis, 2019, §4.3]. In this case, the right adjoint is pseudo, so we can see the adjunction as being in the 2-category $\mathbf{Dbl}_{\mathrm{clx}}$ of double categories, colax double functors, and natural transformations. On the other hand, nothing requires the left adjoint to be pseudo and we will see examples where it is not.

Another description of double coproducts involves less data than the definition above and hence is more convenient to check in examples.

4.2. Proposition. [Double coproducts via universal arrows] *A double category* $\mathbb{D}$ *has colax coproducts if and only if*

(i) for every object family $(I, \underline{x})$ *in* $\mathbb{D}$*, there is a choice of object* $\Sigma\underline{x} \coloneqq \Sigma(I, \underline{x})$ *in* $\mathbb{D}$ *and a universal arrow*

$$(!, \iota) : (I, \underline{x}) \to \Delta\Sigma\underline{x}$$

from $(I, \underline{x})$ *to the functor* $\Delta_0 : \mathbb{D}_0 \to \mathsf{Fam}(\mathbb{D}_0)$*;*

(ii) for every proarrow family $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ *in* $\mathbb{D}$*, there is a choice of proarrow* $\Sigma\underline{m} \coloneqq \Sigma(A, \underline{m}) : \Sigma\underline{x} \nrightarrow \Sigma\underline{y}$ *in* $\mathbb{D}$*, compatible with the choices on objects, and a universal arrow*

$$\begin{tikzcd} (I, \underline{x}) \arrow[r, "(A,\underline{m})", "\shortmid"{marking}] \arrow[d, "(!,\iota)"'] \arrow[dr, phantom, "(!,\iota)"] & (J, \underline{y}) \arrow[d, "(!,\iota)"] \\ \Delta\Sigma\underline{x} \arrow[r, "\Delta\Sigma\underline{m}"', "\shortmid"{marking}] & \Delta\Sigma\underline{y} \end{tikzcd}$$

from $(A, \underline{m})$ *to the functor* $\Delta_1 : \mathbb{D}_1 \to \mathbb{F}\mathsf{am}(\mathbb{D})_1$*.*

Moreover, in this case, the coproducts in $\mathbb{D}$ *are strong if and only if, for every pair of composable families* $(I, \underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J, \underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K, \underline{z})$ *and for every family* $(I, \underline{x})$*, the canonical comparison cells*

$$\begin{tikzcd} \Sigma\underline{x} \arrow[rr, "\Sigma(\underline{m}\odot\underline{n})", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Sigma_{\underline{m},\underline{n}}"] & & \Sigma\underline{z} \arrow[d, equal] \\ \Sigma\underline{x} \arrow[r, "\Sigma\underline{m}"', "\shortmid"{marking}] & \Sigma\underline{y} \arrow[r, "\Sigma\underline{n}"', "\shortmid"{marking}] & \Sigma\underline{z} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Sigma\underline{x} \arrow[r, "\Sigma(\mathrm{id}_{\underline{x}})", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Sigma_{\underline{x}}"] & \Sigma\underline{x} \arrow[d, equal] \\ \Sigma\underline{x} \arrow[r, "\mathrm{id}_{\Sigma\underline{x}}"', "\shortmid"{marking}] & \Sigma\underline{x} \end{tikzcd}$$

are isomorphisms in $\mathbb{D}_1$*, where the composite family* $(A \times_J B, \underline{m} \odot \underline{n})$ *has been defined in Equation* (1)*.*

PROOF. This is a direct application of the equivalence between adjunctions and universal arrows for double categories, stated for right adjoints in [Grandis and Paré, 2004, Theorem 3.6] and [Grandis, 2019, Theorem 4.3.6]. ∎

Note that condition (i) in Proposition 4.2 says nothing more than that the underlying category $\mathbb{D}_0$ has coproducts (cf. Proposition 2.2). Condition (ii) is more elaborate. It implies, but is stronger than, the statement that $\mathbb{D}_1$ has coproducts. First, it must be possible to choose coproducts in $\mathbb{D}_1$ compatibly with coproducts in $\mathbb{D}_0$. Moreover, depending on the indexing span, condition (ii) encompasses notions of coproduct going beyond coproducts in $\mathbb{D}_1$, such as local coproducts. Stated explicitly, the universal property in (ii) says that for any family of cells indexed by the span $A : I \nrightarrow J$ and having the form on the left,

$$\begin{tikzcd} x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "f_i"'] \arrow[dr, phantom, "\alpha_a"] & y_j \arrow[d, "g_j"] \\ w \arrow[r, "n"', "\shortmid"{marking}] & z \end{tikzcd} \quad = \quad \begin{tikzcd} x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "\iota_i"'] \arrow[dr, phantom, "\iota_a"] & y_j \arrow[d, "\iota_j"] \\ \Sigma\underline{x} \arrow[r, "\Sigma\underline{m}", "\shortmid"{marking}] \arrow[d, dashed] \arrow[dr, phantom, "\Downarrow"] & \Sigma\underline{y} \arrow[d, dashed] \\ w \arrow[r, "n"', "\shortmid"{marking}] & z \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

there exists a unique cell factoring each cell α_a through the corresponding coprojection ι_a, as shown on the right.

The canonical comparison cells used in the final statement of Proposition 4.2 are the unique solutions, given by the universal properties of universal arrows, to the equations

$$\begin{tikzcd} x_i \arrow[rr, "m_a \odot m_b", "\shortmid"{marking}] \arrow[d, "\iota_i"'] \arrow[drr, phantom, "\iota_{(a,b)}"] & & z_k \arrow[d, "\iota_k"] \\ \Sigma\underline{x} \arrow[rr, "\Sigma(\underline{m} \odot \underline{n})", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Sigma_{\underline{m},\underline{n}}"] & & \Sigma\underline{z} \arrow[d, equal] \\ \Sigma\underline{x} \arrow[r, "\Sigma\underline{m}"', "\shortmid"{marking}] & \Sigma\underline{y} \arrow[r, "\Sigma\underline{n}"', "\shortmid"{marking}] & \Sigma\underline{z} \end{tikzcd} = \begin{tikzcd} x_i \arrow[r, "m_a", "\shortmid"{marking}] \arrow[d, "\iota_i"'] \arrow[dr, phantom, "\iota_a"] & y_j \arrow[r, "m_b", "\shortmid"{marking}] \arrow[d, "\iota_j"] \arrow[dr, phantom, "\iota_b"] & z_k \arrow[d, "\iota_k"] \\ \Sigma\underline{x} \arrow[r, "\Sigma\underline{m}"', "\shortmid"{marking}] & \Sigma\underline{y} \arrow[r, "\Sigma\underline{n}"', "\shortmid"{marking}] & \Sigma\underline{z} \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k) : (I \overset{A}{\nrightarrow} J \overset{B}{\nrightarrow} K),$$

where the cells $\iota_{(a,b)}$ are the hypothesized coprojections into $\Sigma(A \times_J B, \underline{m} \odot \underline{n})$, along with the equations

$$\begin{tikzcd} x_i \arrow[r, "\mathrm{id}_{x_i}", "\shortmid"{marking}] \arrow[d, "\iota_i"'] \arrow[dr, phantom, "\iota_i"] & x_i \arrow[d, "\iota_i"] \\ \Sigma\underline{x} \arrow[r, "\Sigma(\mathrm{id}_{\underline{x}})", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Sigma_{\underline{x}}"] & \Sigma\underline{x} \arrow[d, equal] \\ \Sigma\underline{x} \arrow[r, "\mathrm{id}_{\Sigma\underline{x}}"', "\shortmid"{marking}] & \Sigma\underline{x} \end{tikzcd} = \begin{tikzcd} x_i \arrow[r, "\mathrm{id}_{x_i}", "\shortmid"{marking}] \arrow[d, "\iota_i"'] \arrow[dr, phantom, "\mathrm{id}_{\iota_i}"] & x_i \arrow[d, "\iota_i"] \\ \Sigma\underline{x} \arrow[r, "\mathrm{id}_{\Sigma\underline{x}}"', "\shortmid"{marking}] & \Sigma\underline{x} \end{tikzcd}, \qquad i \in I.$$

When the assignments in Proposition 4.2 are extended to form a colax double functor $\Sigma : \mathbb{F}\mathsf{am}(\mathbb{D}) \to \mathbb{D}$, the cells above become the composition and identity comparison cells for the colax functor.

Under mild conditions, double categories of spans have coproducts.

4.3. Theorem. [Coproducts of spans] *Let S be a category with pullbacks and (finite) coproducts. Then the double category $\mathbb{S}\mathsf{pan}(\mathsf{S})$ has colax (finite) coproducts.*

Moreover, if S is an (finitary or infinitary) extensive category, then the (finite or arbitrary) coproducts in $\mathbb{S}\mathsf{pan}(\mathsf{S})$ are strong.

Proof. We first show that $\mathbb{S}\mathsf{pan}(\mathsf{S})$ has colax coproducts. By assumption, the category $\mathbb{S}\mathsf{pan}(\mathsf{S})_0 = \mathsf{S}$ has coproducts. Suppose that $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ is a family of spans in S, say of the form $m_a = (x_i \xleftarrow{\ell_a} s_a \xrightarrow{r_a} y_j)$ for each $a : i \nrightarrow j$. We take the coproduct $\Sigma\underline{s} = \sum_{a \in A} s_a$ in S of the apexes and then form the span $\Sigma\underline{x} \leftarrow \Sigma\underline{s} \to \Sigma\underline{y}$ by taking the copairing in S of the morphisms

$$\Sigma\underline{x} \xleftarrow{\iota_i} x_i \xleftarrow{\ell_a} s_a \xrightarrow{r_a} y_j \xrightarrow{\iota_j} \Sigma\underline{y}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

The accompanying family of coprojections is

$$\begin{tikzcd}
(I,\underline{x}) \arrow[r, "{(A,\underline{m})}", "|"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\iota"] & (J,\underline{y}) \arrow[d, "\iota"] \\
\Delta\Sigma\underline{x} \arrow[r, "\Delta\Sigma\underline{m}"', "|"] & \Delta\Sigma\underline{y}
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
x_i \arrow[d, "\iota_i"'] & s_a \arrow[l, "\ell_a"'] \arrow[r, "r_a"] \arrow[d, "\iota_a"'] & y_j \arrow[d, "\iota_j"] \\
\Sigma\underline{x} & \Sigma\underline{s} \arrow[l] \arrow[r] & \Sigma\underline{y}
\end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

As for the universal property, suppose that $n = (w \xleftarrow{\ell} t \xrightarrow{r} z)$ is another span in S along with a family of span maps

$$\begin{tikzcd}
(I,\underline{x}) \arrow[r, "{(A,\underline{m})}", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "h"] & (J,\underline{y}) \arrow[d, "g"] \\
\Delta w \arrow[r, "\Delta n"', "|"] & \Delta z
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
x_i \arrow[d, "f_i"'] & s_a \arrow[l, "\ell_a"'] \arrow[r, "r_a"] \arrow[d, "h_a"'] & y_j \arrow[d, "g_i"] \\
w & t \arrow[l, "\ell"] \arrow[r, "r"'] & z
\end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

Taking the copairings $f \coloneqq [f_i]_{i \in I}$, $g \coloneqq [g_j]_{j \in J}$, and $h \coloneqq [h_a]_{a \in A}$, we get a unique map of spans

$$\begin{tikzcd}
\Sigma\underline{x} \arrow[d, "f"'] & \Sigma\underline{s} \arrow[l] \arrow[r] \arrow[d, "h"'] & \Sigma\underline{y} \arrow[d, "g"] \\
w & t \arrow[l, "\ell"] \arrow[r, "r"'] & z
\end{tikzcd}$$

that factors through the coprojections. Therefore, by Proposition 4.2, the double category $\mathbb{S}\mathsf{pan}(\mathsf{S})$ has colax coproducts.

Now suppose that S is an extensive category. Under the assumption that S has coproducts and pullbacks, extensivity amounts to the properties that coproducts in S are disjoint and stable under pullback [Lack and Sobociński, 2006, Proposition 17], cf. [Carboni et al., 1993, Proposition 2.14]. We need to see that $\mathbb{S}\mathsf{pan}(\mathsf{S})$ has *strong* coproducts. Given composable families of spans $(I,\underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J,\underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K,\underline{z})$, with families of apexes $\underline{s}$ and $\underline{t}$, we must show that the canonical comparison map in S

$$
\begin{tikzcd}
\Sigma(\underline{s} \times_{\underline{y}} \underline{t}) \arrow[d] \\
\Sigma\underline{s} \times_{\Sigma\underline{y}} \Sigma\underline{t}
\end{tikzcd}
\quad\leftrightsquigarrow\quad
\begin{tikzcd}
 & & s_a \times_{y_j} t_b \arrow[dll, "\pi_1"'] \arrow[d, dashed, "\sigma_{a,b}"'] \arrow[drr, "\pi_2"] & & \\
s_a \arrow[d, "\iota_a"'] \arrow[drr, "r_a"'] & & \Sigma\underline{s} \times_{\Sigma\underline{y}} \Sigma\underline{t} \arrow[dll] \arrow[drr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & t_b \arrow[d, "\iota_b"] \arrow[dll, "\ell_b"] \\
\Sigma\underline{s} \arrow[drr] & & y_j \arrow[d, "\iota_j"'] & & \Sigma\underline{t} \arrow[dll] \\
 & & \Sigma\underline{y} & &
\end{tikzcd}
, \quad (i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k) : (I \overset{A}{\nrightarrow} J \overset{B}{\nrightarrow} K).
$$

on the left, which is determined by the family of comparisons $\sigma_{a,b}$ on the right, is an isomorphism. This we do by decomposing it into several parts

$$
\begin{tikzcd}
\sum\limits_{i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k} s_a \times_{y_j} t_b \arrow[r, "\cong"] \arrow[d, dashed] & \sum\limits_{i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k} s_a \times_{\Sigma\underline{y}} t_b \arrow[d, "\cong"] \\
\Sigma\underline{s} \times_{\Sigma\underline{y}} \Sigma\underline{t} & \sum\limits_{\substack{a:i \nrightarrow j \\ b:j' \nrightarrow k}} s_a \times_{\Sigma\underline{y}} t_b \arrow[l, "\cong"]
\end{tikzcd}
\quad\leftrightsquigarrow\quad
\begin{tikzcd}
s_a \times_{y_j} t_b \arrow[r, "\phi_{a,b}"] \arrow[d, dashed, "\sigma_{a,b}"'] & s_a \times_{\Sigma\underline{y}} t_b \arrow[d, equal] \\
\Sigma\underline{s} \times_{\Sigma\underline{y}} \Sigma\underline{t} & s_a \times_{\Sigma\underline{y}} t_b \arrow[l, "\psi_{a,b}"]
\end{tikzcd}
$$

that we individually verify to be isomorphisms. Starting from the end, the final comparison morphism in this sequence

$$
\begin{tikzcd}
\sum\limits_{\substack{a:i \nrightarrow j \\ b:j' \nrightarrow k}} s_a \times_{\Sigma\underline{y}} t_b \arrow[d, "\cong"'] \\
\Sigma\underline{s} \times_{\Sigma\underline{y}} \Sigma\underline{t}
\end{tikzcd}
\quad\leftrightsquigarrow\quad
\begin{tikzcd}
 & s_a \arrow[dl, "r_a"'] \arrow[d, "\iota_a"'] & s_a \times_{\Sigma\underline{y}} t_b \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\psi_{a,b}"'] & t_b \arrow[d, "\iota_b"] \arrow[dr, "\ell_b"] & \\
y_j \arrow[dr, "\iota_j"'] & \Sigma\underline{s} \arrow[d] & \Sigma\underline{s} \times_{\Sigma\underline{y}} \Sigma\underline{t} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, phantom, "\lrcorner" rotate=-45] & \Sigma\underline{t} \arrow[d] & y_{j'} \arrow[dl, "\iota_{j'}"] \\
 & \Sigma\underline{y} \arrow[rr, equal] & & \Sigma\underline{y} &
\end{tikzcd}
, \quad
\begin{array}{l} a : i \nrightarrow j \\ b : j' \nrightarrow k, \end{array}
$$

is an isomorphism because coproducts in S are stable under pullback. Next, we analyze the pullbacks $s_a \times_{\Sigma\underline{y}} t_b$ involved by cases. In the case that $j = j'$, i.e., $(a,b) \in A \times_J B$, we form the pasting of pullback squares

$$\begin{tikzcd}
s_a \times_{y_j} t_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & t_b \arrow[r, equal] \arrow[d, "\ell_b"] & t_b \arrow[d, "\ell_b"] \\
s_a \arrow[r, "r_a"'] \arrow[d, equal] & y_j \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & y_j \arrow[d, "\iota_j"] \\
s_a \arrow[r, "r_a"'] & y_j \arrow[r, "\iota_j"'] & \Sigma \underline{y}
\end{tikzcd},$$

where the bottom right square is a pullback, i.e., the coprojection $\iota_j : y_j \to \Sigma \underline{y}$ is monic, because coproducts in S are disjoint. Thus, the overall square is also a pullback and we obtain a canonical isomorphism $\phi_{a,b} : s_a \times_{y_j} t_b \xrightarrow{\cong} s_a \times_{\Sigma \underline{y}} t_b$. Otherwise, $j \neq j'$ and the pullback of $y_j \xrightarrow{\iota_j} \Sigma \underline{y} \xleftarrow{\iota_{j'}} y_{j'}$ is the initial object, again because coproducts in S are disjoint. This calculation verifies the first two isomorphisms in the decomposition and completes the proof. ■

4.4. Corollary. *The double category* $\mathbb{S}\mathsf{pan}$ *has strong coproducts, and* $\mathbb{F}\mathsf{inSpan}$ *has strong finite coproducts.*

Proof. The category $\mathsf{S} = \mathsf{Set}$ is infinitary extensive, and FinSet is finitary extensive. ■

Double categories of matrices also have strong coproducts. This is true without further assumptions as distributivity is needed to construct a double category in the first place. By a *distributive monoidal category*, we will mean a symmetric monoidal category with arbitrary coproducts over which the tensor distributes separately in each variable.

4.5. Proposition. [Coproducts of matrices] *For any distributive monoidal category* $\mathcal{V}$*, the double category of* $\mathcal{V}$*-matrices,* $\mathcal{V}$-$\mathbb{M}\mathsf{at}$*, has strong coproducts.*

Proof. Clearly, the underlying category $\mathcal{V}$-$\mathbb{M}\mathsf{at}_0 = \mathsf{Set}$ has coproducts. Given a family of $\mathcal{V}$-matrices $(A, \underline{M}) : (I, \underline{X}) \nrightarrow (J, \underline{Y})$, which have the form $M_a : X_i \times Y_j \to \mathcal{V}$ for each element $a : i \nrightarrow j$, the coproduct $\Sigma \underline{M} : \Sigma \underline{X} \nrightarrow \Sigma \underline{Y}$ is the $\mathcal{V}$-matrix $\Sigma \underline{M} : \Sigma \underline{X} \times \Sigma \underline{Y} \to \mathcal{V}$ defined pointwise by the coproducts in $\mathcal{V}$

$$\Sigma \underline{M}(x_i, y_j) \coloneqq \sum_{a : i \nrightarrow j} M_a(x_i, y_j), \qquad \begin{array}{l} i \in I,\ j \in J, \\ x_i \in X_i,\ y_j \in Y_j. \end{array}$$

Here, for *fixed* i and j, the sum ranges over all $a \in A$ such that $\ell(a) = i$ and $r(a) = j$. The accompanying coprojections

$$\begin{tikzcd}
X_i \arrow[r, "M_a", "\shortmid"{marking}] \arrow[d, "\iota_i"'] \arrow[dr, phantom, "\iota_a"] & Y_j \arrow[d, "\iota_j"] \\
\Sigma \underline{X} \arrow[r, "\Sigma \underline{M}"', "\shortmid"{marking}] & \Sigma \underline{Y}
\end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

are the maps of matrices with components given by coprojections in $\mathcal{V}$

$$M_a(x_i, y_j) \xrightarrow{\iota_a} \sum_{a':i \nrightarrow j} M_{a'}(x_i, y_j) = \Sigma \underline{M}(x_i, y_j), \qquad \begin{array}{l} a : i \nrightarrow j, \\ x_i \in X_i,\ y_j \in Y_j. \end{array}$$

The universal property of the coproduct in $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ is easily verified using the universal properties of coproducts in Set and in $\mathcal{V}$.

So $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ has colax coproducts. To see that it has strong coproducts, take composable families of $\mathcal{V}$-matrices $(I, \underline{X}) \overset{(A,\underline{M})}{\nrightarrow} (J, \underline{Y}) \overset{(B,\underline{N})}{\nrightarrow} (K, \underline{Z})$. The domain of the comparison cell $\Sigma(\underline{M} \odot \underline{N}) \to \Sigma\underline{M} \odot \Sigma\underline{N}$ has elements

$$\Sigma(\underline{M} \odot \underline{N})(x_i, z_k) = \sum_{i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k} (M_a \odot N_b)(x_i, z_k) = \sum_{i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k} \sum_{y_j \in Y_j} M_a(x_i, y_j) \otimes N_b(y_j, z_k)$$

for each choice of $i \in I$, $k \in K$, $x_i \in X_i$, and $z_k \in Z_k$, where the outer sum ranges over all $(a, b) \in A \times_J B$ such that $\ell(a) = i$ and $r(b) = k$. Rearranging the sums, the right-hand side is isomorphic in $\mathcal{V}$ to

$$\sum_{j \in J} \sum_{y_j \in Y_j} \sum_{a:i \nrightarrow j} \sum_{b:j \nrightarrow k} M_a(x_i, y_j) \otimes M_b(y_j, z_k).$$

Using the distributivity of tensors over coproducts in $\mathcal{V}$, this is in turn isomorphic to

$$\sum_{j \in J} \sum_{y_j \in Y_j} \Sigma\underline{M}(x_i, y_j) \otimes \Sigma\underline{N}(y_j, z_k) \cong (\Sigma\underline{M} \odot \Sigma\underline{N})(x_i, z_k).$$

The composite of these isomorphisms is one component of the comparison cell between matrices. Since all the components are isomorphisms, the comparison cell is itself an isomorphism. ■

From the equivalence $\mathbb{M}\mathsf{at} \simeq \mathbb{S}\mathsf{pan}$, we obtain an independent proof of the previous corollary.

4.6. Corollary. *The double category* $\mathbb{M}\mathsf{at}$ *has strong coproducts.*

Proof. The cartesian monoidal category $\mathcal{V} = \mathsf{Set}$ is distributive. ■

5. Functors between double categories with coproducts

We have yet to characterize $\mathbb{F}\mathsf{am}(\mathbb{D})$ as the free coproduct completion of the double category $\mathbb{D}$, as it must be. To even state such a result, we need a notion not just of coproducts but of double functors that preserve coproducts. Unlike in the one-dimensional case, choices must be made to define an ambient 2-category for the universal property of the free coproduct completion: between colax and strong coproducts, between colax and pseudo double functors. We begin by saying what it means for a colax functor to preserve coproducts, turning the characterization in Proposition 2.3 of preserving ordinary coproducts into the *definition* of preserving double coproducts.

5.1. Construction. [Functorality of double families] Given a colax double functor $F : \mathbb{D} \to \mathbb{E}$, there is another colax double functor

$$\mathsf{Fam}(F) : \mathsf{Fam}(\mathbb{D}) \to \mathsf{Fam}(\mathbb{E})$$

that acts on families of objects and proarrows in $\mathbb{D}$ by applying F elementwise to objects and proarrows, while preserving indexing sets and spans. Specifically, $\mathsf{Fam}(F)$ sends

- an object family $\underline{x} : I \to \mathbb{D}_0$ in $\mathbb{D}$ to the object family $F_0 \circ \underline{x} : I \to \mathbb{E}_0$ in $\mathbb{E}$;
- a proarrow family $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ in $\mathbb{D}$ to the proarrow family in $\mathbb{E}$:

$$\begin{tikzcd}
I & A \arrow[l, "\ell"'] \arrow[r, "r"] & J \\
\mathbb{D}_0 \arrow[from=u, "\underline{x}"'] & \mathbb{D}_1 \arrow[from=u, "\underline{m}"'] \arrow[l, "s"'] \arrow[r, "t"] & \mathbb{D}_0 \arrow[from=u, "\underline{y}"] \\
\mathbb{E}_0 \arrow[from=u, "F_0"'] & \mathbb{E}_1 \arrow[from=u, "F_1"'] \arrow[l, "s"] \arrow[r, "t"'] & \mathbb{E}_0 \arrow[from=u, "F_0"]
\end{tikzcd} .$$

Similarly, $\mathsf{Fam}(F)$ acts on families of arrows and cells by applying F elementwise to arrows and cells. The composition and identity comparisons for $\mathsf{Fam}(F)$ are families of composition and identity comparisons for F, to be described in greater detail in Section 9 when they are needed. In particular, $\mathsf{Fam}(F)$ is pseudo whenever F is.

Given that the double functor $\mathsf{Fam}(F)$ acts by post-composition with F, it is only a mild abuse of notation to write $F\underline{x} := \mathsf{Fam}(F)(I, \underline{x})$ and $F\underline{m} := \mathsf{Fam}(F)(A, \underline{m})$ for this "elementwise" application, and we will often do so.

5.2. Definition. [Preservation of double coproducts] *Let $\mathbb{D}$ and $\mathbb{E}$ be double categories with colax (possibly strong) coproducts. A colax functor $F : \mathbb{D} \to \mathbb{E}$* ***preserves coproducts*** *if the induced morphism of adjunctions*

$$\begin{tikzcd}
\mathsf{Fam}(\mathbb{D}) \arrow[r, "\Sigma \dashv \Delta", "\shortmid"{marking}] \arrow[d, "\mathsf{Fam}(F)"'] \arrow[dr, phantom, "(\Phi{,}1)"] & \mathbb{D} \arrow[d, "F"] \\
\mathsf{Fam}(\mathbb{E}) \arrow[r, "\Sigma \dashv \Delta"', "\shortmid"{marking}] & \mathbb{E}
\end{tikzcd}$$

is strong.

Here the morphism of adjunctions $(\Phi, 1)$ is a cell in the double category of adjunctions [Grandis, 2019, §3.1.5] in $\mathbf{Dbl}_{\text{clx}}$, the 2-category of double categories, colax double functors, and natural transformations. It consists of the mate pair

$$\begin{tikzcd}
\mathsf{Fam}(\mathbb{D}) \arrow[r, "\Sigma_{\mathbb{D}}"] \arrow[d, "\mathsf{Fam}(F)"'] & \mathbb{D} \arrow[d, "F"] \\
\mathsf{Fam}(\mathbb{E}) \arrow[r, "\Sigma_{\mathbb{E}}"'] \arrow[ur, phantom, "\Nearrow \Phi"] & \mathbb{E}
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\mathsf{Fam}(\mathbb{D}) \arrow[d, "\mathsf{Fam}(F)"'] \arrow[dr, phantom, "\Searrow 1"] & \mathbb{D} \arrow[l, "\Delta_{\mathbb{D}}"'] \arrow[d, "F"] \\
\mathsf{Fam}(\mathbb{E}) & \mathbb{E} \arrow[l, "\Delta_{\mathbb{E}}"]
\end{tikzcd} ,$$

where the transformation on the right is the identity. The components of Φ are, at an object family $(I, \underline{x})$ in $\mathbb{D}$, the usual coproduct comparison $\Phi_{\underline{x}} : \Sigma F\underline{x} \to F\Sigma\underline{x}$ and, at a proarrow family $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ in $\mathbb{D}$, the cell on the left

$$\begin{tikzcd} \Sigma F\underline{x} \arrow[r, "\Sigma F\underline{m}", "|"] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\Phi_{\underline{m}}"] & \Sigma F\underline{y} \arrow[d, "\Phi_{\underline{y}}"] \\ F\Sigma\underline{x} \arrow[r, "F\Sigma\underline{m}"', "|"] & F\Sigma\underline{y} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Fx_i \arrow[r, "Fm_a", "|"] \arrow[d, "F\iota_i"'] \arrow[dr, phantom, "F\iota_a"] & Fy_j \arrow[d, "F\iota_j"] \\ F\Sigma\underline{x} \arrow[r, "F\Sigma\underline{m}"', "|"] & F\Sigma\underline{y} \end{tikzcd}, \quad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

that is uniquely determined by the family of cells on the right, using the universal property of the coproduct of $F\underline{m}$ in $\mathbb{E}$. By definition, the morphism of adjunctions $(\Phi, 1)$ is **strong** just when Φ is a natural isomorphism. Thus, a colax functor F preserves coproducts just when the canonical comparisons $\Phi_{\underline{x}}$ and $\Phi_{\underline{m}}$ are all isomorphisms in $\mathbb{E}_0$ and $\mathbb{E}_1$, respectively.

5.3. Example. [Preserving coproducts of spans] As observed in [Grandis, 2019, §C3.11], for any functor $F : \mathsf{C} \to \mathsf{D}$ between categories with pullbacks, there is a unitary colax functor

$$\mathbb{S}\mathsf{pan}(F) : \mathbb{S}\mathsf{pan}(\mathsf{C}) \to \mathbb{S}\mathsf{pan}(\mathsf{D})$$

that applies F pointwise to spans and maps between them, and $\mathbb{S}\mathsf{pan}(F)$ is pseudo if and only if F preserves pullbacks.

Suppose that C and D have coproducts, so that $\mathbb{S}\mathsf{pan}(\mathsf{C})$ and $\mathbb{S}\mathsf{pan}(\mathsf{D})$ have colax coproducts (Theorem 4.3). Then the colax functor $\mathbb{S}\mathsf{pan}(F)$ preserves coproducts if and only if F preserves coproducts in the usual sense.

To express that $\mathbb{F}\mathsf{am}(\mathbb{D})$ is the free coproduct cocompletion of $\mathbb{D}$, we work in the 2-category of double categories with colax coproducts, colax functors that preserve coproducts, and natural transformations. However, $\mathbb{F}\mathsf{am}(\mathbb{D})$ itself has strong coproducts, which turns out to be important in the proof.

5.4. Theorem. [Free double coproduct completion] *For any double category $\mathbb{D}$, the double category $\mathbb{F}\mathsf{am}(\mathbb{D})$ has strong coproducts. Moreover, $\mathbb{F}\mathsf{am}(\mathbb{D})$ is the free coproduct completion of $\mathbb{D}$ in the sense that, for any colax functor $F : \mathbb{D} \to \mathbb{E}$ into a double category $\mathbb{E}$ with colax coproducts, there exists a coproduct-preserving colax functor $\hat{F} : \mathbb{F}\mathsf{am}(\mathbb{D}) \to \mathbb{E}$ making the triangle*

$$\begin{tikzcd} & \mathbb{F}\mathsf{am}(\mathbb{D}) \arrow[d, dashed, "\hat{F}"] \\ \mathbb{D} \arrow[ur, "\Delta"] \arrow[r, "F"'] & \mathbb{E} \end{tikzcd} \tag{2}$$

commute, and $\hat{F}$ is unique up to natural isomorphism.

PROOF. We first show that $\mathbb{F}\mathsf{am}(\mathbb{D})$ has strong coproducts, which are reminiscent of coproducts in $\mathbb{S}\mathsf{pan}(\mathsf{S})$ (Theorem 4.3). As reviewed in Theorem 2.4, the underlying category $\mathbb{F}\mathsf{am}(\mathbb{D})_0 = \mathsf{Fam}(\mathbb{D}_0)$ has coproducts and is in fact the free coproduct completion of $\mathbb{D}_0$. So suppose that

$$(A, (\underline{S}, \underline{\underline{m}})) : (I, (\underline{U}, \underline{\underline{x}})) \nrightarrow (J, (\underline{V}, \underline{\underline{y}}))$$

is a family of proarrows in $\mathbb{F}\mathsf{am}(\mathbb{D})$, i.e., a proarrow in $\mathbb{F}\mathsf{am}(\mathbb{F}\mathsf{am}(\mathbb{D}))$. This data comprises a span of sets $I \xleftarrow{\ell} A \xrightarrow{r} J$ together with a family of proarrows

$$(S_a, \underline{m}_a) : (U_i, \underline{x}_i) \nrightarrow (V_j, \underline{y}_j), \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

in $\mathbb{F}\mathsf{am}(\mathbb{D})$. Each of the latter consists, in turn, of a span of sets $U_i \xleftarrow{\ell_a} S_a \xrightarrow{r_a} V_j$ together with a family of proarrows

$$m_{a,s} : x_{i,u} \nrightarrow y_{j,v}, \qquad (u \overset{s}{\nrightarrow} v) : (U_i \overset{S_a}{\nrightarrow} V_j),$$

in $\mathbb{D}$. Now, a coproduct $\Sigma(\underline{S}, \underline{\underline{m}}) : \Sigma(\underline{U}, \underline{\underline{x}}) \nrightarrow \Sigma(\underline{V}, \underline{\underline{y}})$ of this family of families is constructed by taking the span $\sqcup\underline{U} \leftarrow \sqcup\underline{S} \rightarrow \sqcup\underline{V}$ defined by the copairings of the functions

$$\sqcup\underline{U} \xleftarrow{\iota_i} U_i \xleftarrow{\ell_a} S_a \xrightarrow{r_a} V_j \xrightarrow{\iota_j} \sqcup\underline{V}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

together with the family of proarrows in $\mathbb{D}$

$$\begin{tikzcd}
\sqcup\underline{U} \arrow[d, "\underline{x}"'] & \sqcup\underline{S} \arrow[l] \arrow[r] \arrow[d, "\underline{m}"'] & \sqcup\underline{V} \arrow[d, "\underline{y}"] \\
\mathbb{D}_0 & \mathbb{D}_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathbb{D}_0
\end{tikzcd},$$

where again the copairings $\underline{x} \coloneqq [\underline{x}_i]_{i\in I}$, $\underline{y} \coloneqq [\underline{y}_j]_{j\in J}$, and $\underline{m} \coloneqq [\underline{m}_a]_{a\in A}$ are defined using the universal property of coproducts in Set. For each element $a : i \nrightarrow j$, the coprojection

$$\begin{tikzcd}
(U_i, \underline{x}_i) \arrow[r, "(S_a, \underline{m}_a)", "\shortmid"{marking}] \arrow[d, "(\iota_i, 1_{\underline{x}_i})"'] \arrow[dr, phantom, "(\iota_a, 1_{\underline{m}_a})"] & (V_j, \underline{y}_j) \arrow[d, "(\iota_j, 1_{\underline{y}_j})"] \\
(\sqcup\underline{U}, \underline{x}) \arrow[r, "(\sqcup\underline{S}, \underline{m})"', "\shortmid"{marking}] & (\sqcup\underline{V}, \underline{y})
\end{tikzcd}$$

in $\mathbb{F}\mathsf{am}(\mathbb{D})$ consists of the map of spans

$$\begin{tikzcd}
U_i \arrow[d, "\iota_i"'] & S_a \arrow[l, "\ell_a"'] \arrow[r, "r_a"] \arrow[d, "\iota_a"'] & V_j \arrow[d, "\iota_j"] \\
\sqcup\underline{U} & \sqcup\underline{S} \arrow[l] \arrow[r] & \sqcup\underline{V}
\end{tikzcd}, \qquad
\begin{tikzcd}
x_{i,u} \arrow[r, "m_{a,s}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "1"] & y_{j,v} \arrow[d, equal] \\
x_{\iota_i(u)} \arrow[r, "m_{\iota_a(s)}"', "\shortmid"{marking}] & y_{\iota_j(v)}
\end{tikzcd}, \quad (u \overset{s}{\nrightarrow} v) : (U_i \overset{S_a}{\nrightarrow} V_j),$$

on the left and along with the family of identity cells in $\mathbb{D}$ on the right. The universal property of the coproduct in $\mathbb{F}\mathsf{am}(\mathbb{D})$ follows easily from that of coproducts in Set.

We only sketch the proof that coproducts in $\mathbb{F}\mathsf{am}(\mathbb{D})$ are strong, being similar to the proof that coproducts in $\mathbb{S}\mathsf{pan}(\mathsf{S})$ are strong (Theorem 4.3). For any composable pair of proarrows

$$(I, (\underline{U}, \underline{\underline{x}})) \overset{(A,(\underline{S},\underline{\underline{m}}))}{\longrightarrow\!\!\!\!\!\!\!+\;\;} (J, (\underline{V}, \underline{\underline{y}})) \overset{(B,(\underline{T},\underline{\underline{n}}))}{\longrightarrow\!\!\!\!\!\!\!+\;\;} (K, (\underline{W}, \underline{\underline{z}}))$$

in $\mathbb{F}\mathsf{am}(\mathbb{F}\mathsf{am}(\mathbb{D}))$, we have canonical bijections

$$\coprod_{(a,b)\in A\times_J B} S_a \times_{V_j} T_b \xrightarrow{\cong} \coprod_{(a,b)\in A\times B} S_a \times_{\sqcup \underline{V}} T_b \xrightarrow{\cong} \sqcup \underline{S} \times_{\sqcup \underline{V}} \sqcup \underline{T}$$

due to the extensivity of Set. The map between indexing spans in the canonical comparison cell is thus a bijection. Moreover, the components of that comparison cell are just the identities on the proarrow composites $m_{a,s} \odot n_{b,t}$.

It remains to prove the universal property of the free coproduct completion. Any colax functor $\hat{F} : \mathbb{F}\mathsf{am}(\mathbb{D}) \to \mathbb{E}$ making the triangle (2) commute is uniquely determined on singleton families of objects and proarrows, and on maps between them, by $F : \mathbb{D} \to \mathbb{E}$. But arbitrary families of objects and proarrows in $\mathbb{D}$ are coproducts in $\mathbb{F}\mathsf{am}(\mathbb{D})$ of singleton families, so if $\hat{F}$ is to preserve coproducts, it must send these to coproducts of fixed objects and proarrows in $\mathbb{E}$. By functorality and the universal property of coproducts, $\hat{F}$ is then also determined on arrows and cells in $\mathbb{F}\mathsf{am}(\mathbb{D})$. Finally, the composition and identity comparisons of $\hat{F}$ are uniquely determined by those of F, using the dual of Lemma 9.3, proved in detail later, along with the fact that coproducts in $\mathbb{F}\mathsf{am}(\mathbb{D})$ are strong. So, since $\mathbb{E}$ is assumed to have colax coproducts, we can choose coproducts in $\mathbb{E}$ to define $\hat{F}$, and such $\hat{F}$ is unique up to the choice of coproducts, hence up to natural isomorphism commuting with the triangle (2). ∎

5.5. Corollary. *The double category* $\mathbb{S}\mathsf{pan}$ *is the free coproduct completion of the terminal double category. Similarly,* $\mathbb{F}\mathsf{in}\mathsf{Span}$ *is the free finite coproduct completion of the terminal double category.*

Proof. Immediate from the preceding theorem and Corollary 3.5. ∎

6. Lax products in double categories

We turn now from coproducts to products in double categories. Of course, products are dual to coproducts, so for certain abstract purposes nothing further need be said. For other purposes that is not enough. Certainly, to understand any examples, a concrete description of products in double categories is needed, and we will give one. But there is a deeper divergence between double products and coproducts. Famously, the symmetry arising from duality is partly broken in category theory due to the special role played by the category Set, which is far from being self-dual. The same is true in double category theory due to the special role played by the double category $\mathbb{S}\mathsf{pan}$, but to an even greater degree. We will see that the prototypical double categories of spans and of matrices have strong coproducts, as shown in Section 4, but have *lax* products.

Let us recall the principle of duality in double category theory. The **oppositization** 2-functor $\mathrm{op} : \mathbf{Cat}^{\mathrm{co}} \to \mathbf{Cat}$ sends a category to its opposite category, preserving the orientation of functors and reversing that of natural transformations. It also preserves 2-pullbacks. Thus, the **opposite** $\mathbb{D}^{\mathrm{op}}$ of a double category $\mathbb{D}$ can be defined by applying the oppositization 2-functor to $\mathbb{D}$, viewed as a pseudocategory in **Cat**. So the categories underlying $\mathbb{D}^{\mathrm{op}}$ are $(\mathbb{D}^{\mathrm{op}})_0 = (\mathbb{D}_0)^{\mathrm{op}}$ and $(\mathbb{D}^{\mathrm{op}})_1 = (\mathbb{D}_1)^{\mathrm{op}}$. Similarly, the **opposite** $F^{\mathrm{op}} : \mathbb{D}^{\mathrm{op}} \to \mathbb{E}^{\mathrm{op}}$ of a lax or colax double functor $F : \mathbb{D} \to \mathbb{E}$ has underlying functors $(F^{\mathrm{op}})_0 = (F_0)^{\mathrm{op}}$ and $(F^{\mathrm{op}})_1 = (F_1)^{\mathrm{op}}$. But the orientations of the composition and identity comparisons are reversed, so that a lax functor has a colax opposite and vice versa. The orientations of natural transformations between double functors are also reserved. Altogether, then, there are **oppositization** 2-functors

$$\mathrm{op} : \mathbf{Dbl}_{\mathrm{lax}}^{\mathrm{co}} \to \mathbf{Dbl}_{\mathrm{clx}} \qquad \text{and} \qquad \mathrm{op} : \mathbf{Dbl}_{\mathrm{clx}}^{\mathrm{co}} \to \mathbf{Dbl}_{\mathrm{lax}}$$

that are inverse to each other. These both restrict on pseudo double functors to an involutive 2-functor $\mathrm{op} : \mathbf{Dbl}^{\mathrm{co}} \to \mathbf{Dbl}$.

6.1. Definition. [Contravariant double families] *The **contravariant double category of families** in a double category $\mathbb{D}$ is the double category $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ defined by the relation*

$$\mathbb{F}\mathsf{am}^*(\mathbb{D})^{\mathrm{op}} = \mathbb{F}\mathsf{am}(\mathbb{D}^{\mathrm{op}}).$$

*For contrast, $\mathbb{F}\mathsf{am}(\mathbb{D})$ can also be called the **covariant** double category of families.*

The covariant and contravariant double categories of families have the same objects and proarrows but different arrows and cells. Concretely, the double category $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ has

- as objects, an **indexing set** I together with an I-indexed family $\underline{x} : I \to \mathbb{D}_0$, consisting of objects x_i in $\mathbb{D}$ for $i \in I$;
- as arrows $(I, \underline{x}) \to (J, \underline{y})$, a function $f_0 : J \to I$ together with a J-indexed family f of morphisms in $\mathbb{D}$ of the form $f_j : x_{f_0(j)} \to y_j$ for $j \in J$;
- as proarrows $(I, \underline{x}) \nrightarrow (J, \underline{y})$, an **indexing span** $I \xleftarrow{\ell} A \xrightarrow{r} J$ together with an A-indexed family $\underline{m} : A \to \mathbb{D}_1$, consisting of proarrows in $\mathbb{D}$ of the form $m_a : x_i \nrightarrow y_j$ for each $a : i \nrightarrow j$;
- as cells $\begin{array}{ccc} (I,\underline{x}) & \xrightarrow{(A,\underline{m})} & (J,\underline{y}) \\ {\scriptstyle (f_0,f)}\downarrow & & \downarrow{\scriptstyle (g_0,g)} \\ (K,\underline{w}) & \xrightarrow[(B,\underline{n})]{} & (L,\underline{z}) \end{array}$, a map of spans (f_0, α_0, g_0) as shown on the left below

together with a B-indexed family α of cells in $\mathbb{D}$ of the form on the right:

$$\begin{tikzcd} I & A \arrow[l, "\ell"'] \arrow[r, "r"] & J \\ K \arrow[u, "f_0"] & B \arrow[l, "\ell"] \arrow[r, "r"'] \arrow[u, "\alpha_0"'] & L \arrow[u, "g_0"'] \end{tikzcd} \qquad \begin{tikzcd} x_{f_0(k)} \arrow[r, "m_{\alpha_0(b)}", "\shortmid"{marking}] \arrow[d, "f_k"'] \arrow[dr, phantom, "\alpha_b"] & y_{g_0(\ell)} \arrow[d, "g_\ell"] \\ w_k \arrow[r, "n_b"', "\shortmid"{marking}] & z_\ell \end{tikzcd}, \quad (k \overset{b}{\nrightarrow} \ell) : (K \overset{B}{\nrightarrow} L).$$

In this description, recall from Section 3 that given a span $I \xleftarrow{\ell} A \xrightarrow{r} J$, the notation $a : i \nrightarrow j$ or $(i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J)$ refers to any element $a \in A$ of the span's apex and asserts that $i = \ell(a)$ and $j = r(a)$.

The duality principle for double categories extends to double adjunctions. Like any 2-functor, the oppositization 2-functor $\mathrm{op} : \mathbf{Cat}^{\mathrm{co}} \to \mathbf{Cat}$ preserves adjunctions, but the "co" implies that left and right are exchanged. So an adjunction $F \dashv G$ becomes an adjunction $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$. Similarly, the oppositization 2-functor $\mathrm{op} : \mathbf{Dbl}^{\mathrm{co}}_{\mathrm{lax}} \to \mathbf{Dbl}_{\mathrm{clx}}$ turns a pseudo-lax adjunction $F \dashv G$ into a colax-pseudo adjunction $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$, whereas the 2-functor $\mathrm{op} : \mathbf{Dbl}^{\mathrm{co}}_{\mathrm{clx}} \to \mathbf{Dbl}_{\mathrm{lax}}$ turns a colax-pseudo adjunction into a pseudo-lax adjunction. The following definition is thus seen to be dual to Definition 4.1 in the sense that lax products in $\mathbb{D}$ are colax coproducts in $\mathbb{D}^{\mathrm{op}}$.

6.2. Definition. [Double products] *A double category $\mathbb{D}$ **has lax products** if the double functor $\Delta : \mathbb{D} \to \mathbb{F}\mathsf{am}^*(\mathbb{D})$ has a lax right adjoint:*

$$\begin{tikzcd} \mathbb{D} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbb{F}\mathsf{am}^*(\mathbb{D}) \arrow[l, bend left, "\Pi"] \end{tikzcd}.$$

*The double category $\mathbb{D}$ **has strong products** if the right adjoint $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{D}$ is pseudo.*

As noted in the Introduction, double products of this kind were first proposed in a talk by Robert Paré [Paré, 2009]. We proceed with some straightforward dualizations of Section 4.

6.3. Proposition. [Double products via universal arrows] *A double category $\mathbb{D}$ has lax products if and only if*

(i) for every object family $(I, \underline{x})$ in $\mathbb{D}$, there is a choice of object $\Pi\underline{x} \coloneqq \Pi(I, \underline{x})$ in $\mathbb{D}$ and a universal arrow

$$(!, \pi) : \Delta\Pi\underline{x} \to (I, \underline{x})$$

from the functor $\Delta_0 : \mathbb{D}_0 \to \mathsf{Fam}^*(\mathbb{D}_0)$ to $(I, \underline{x})$;

(ii) for every proarrow family $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ in $\mathbb{D}$, there is a choice of proarrow $\Pi\underline{m} \coloneqq \Pi(A, \underline{m}) : \Pi\underline{x} \nrightarrow \Pi\underline{y}$ in $\mathbb{D}$, compatible with the choices above, and a universal

arrow

$$\begin{tikzcd}
\Delta\Pi\underline{x} \arrow[r, "\Delta\Pi\underline{m}", "\shortmid" marking] \arrow[d, "{(!,\pi)}"'] \arrow[dr, phantom, "{(!,\pi)}"] & \Delta\Pi\underline{y} \arrow[d, "{(!,\pi)}"] \\
(I,\underline{x}) \arrow[r, "{(A,\underline{m})}"', "\shortmid" marking] & (J,\underline{y})
\end{tikzcd}$$

from the functor $\Delta_1 : \mathbb{D}_1 \to \mathbb{F}\mathsf{am}^*(\mathbb{D})_1$ *to* $(A, \underline{m})$.

Moreover, in this case, the products are strong if and only if, for every pair of composable families $(I,\underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J,\underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K,\underline{z})$ *and for every family* $(I,\underline{x})$*, the canonical comparison cells*

$$\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi\underline{m}", "\shortmid" marking] \arrow[d, equal] & \Pi\underline{y} \arrow[r, "\Pi\underline{n}", "\shortmid" marking] \arrow[d, phantom, "\Pi_{\underline{m},\underline{n}}"] & \Pi\underline{z} \arrow[d, equal] \\
\Pi\underline{x} \arrow[rr, "\Pi(\underline{m}\odot\underline{n})"', "\shortmid" marking] & & \Pi\underline{z}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\mathrm{id}_{\Pi\underline{x}}", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Pi_{\underline{x}}"] & \Pi\underline{x} \arrow[d, equal] \\
\Pi\underline{x} \arrow[r, "\Pi(\mathrm{id}_{\underline{x}})"', "\shortmid" marking] & \Pi\underline{x}
\end{tikzcd}$$

are isomorphisms in $\mathbb{D}_1$.

PROOF. This proposition is dual to Proposition 4.2; alternatively, it follows immediately from [Grandis and Paré, 2004, Theorem 3.6] or [Grandis, 2019, Theorem 4.3.6]. ∎

The canonical comparison cells in the final statement of the proposition are the unique solutions to the equations

$$\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi\underline{m}", "\shortmid" marking] \arrow[d, equal] & \Pi\underline{y} \arrow[r, "\Pi\underline{n}", "\shortmid" marking] \arrow[d, phantom, "\Pi_{\underline{m},\underline{n}}"] & \Sigma\underline{z} \arrow[d, equal] \\
\Pi\underline{x} \arrow[rr, "\Pi(\underline{m}\odot\underline{n})"', "\shortmid" marking] \arrow[d, "\pi_i"'] & \arrow[d, phantom, "\pi_{(a,b)}"] & \Pi\underline{z} \arrow[d, "\pi_k"] \\
x_i \arrow[rr, "m_a\odot m_b"', "\shortmid" marking] & & z_k
\end{tikzcd}
=
\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi\underline{m}", "\shortmid" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi\underline{y} \arrow[r, "\Pi\underline{n}", "\shortmid" marking] \arrow[d, "\pi_j"] \arrow[dr, phantom, "\pi_b"] & \Pi\underline{z} \arrow[d, "\pi_k"] \\
x_i \arrow[r, "m_a"', "\shortmid" marking] & y_j \arrow[r, "m_b"', "\shortmid" marking] & z_k
\end{tikzcd},
\qquad (i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k) : (I \overset{A}{\nrightarrow} J \overset{B}{\nrightarrow} K). \tag{3}$$

and

$$\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\mathrm{id}_{\Pi\underline{x}}", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Pi_{\underline{x}}"] & \Pi\underline{x} \arrow[d, equal] \\
\Pi\underline{x} \arrow[r, "\Pi(\mathrm{id}_{\underline{x}})"', "\shortmid" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & \Pi\underline{x} \arrow[d, "\pi_i"] \\
x_i \arrow[r, "\mathrm{id}_{x_i}"', "\shortmid" marking] & x_i
\end{tikzcd}
=
\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\mathrm{id}_{\Pi\underline{x}}", "\shortmid" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\mathrm{id}_{\pi_i}"] & \Pi\underline{x} \arrow[d, "\pi_i"] \\
x_i \arrow[r, "\mathrm{id}_{x_i}"', "\shortmid" marking] & x_i
\end{tikzcd},
\qquad i \in I, \tag{4}$$

afforded by the universal properties of the universal arrows. The comparison cells form the laxators and unitors of the lax double functor $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{D}$.

Double categories of spans are the prime example of double categories with lax products, and we will see that certain products of spans *are* genuinely lax. As preparation to describe products of spans, recall that the category of elements construction has a right adjoint [Paré, 1990, Proposition 8].

6.4. Construction. [Adjoint elements constructions] Fixing a category C, there is a functor $\int^{\mathsf{C}} : \mathsf{Set}^{\mathsf{C}} \to \mathsf{Cat}$ that sends each copresheaf X on C to its category of elements $\int X$. Conversely, for any category X, there is a copresheaf $\mathrm{El}_{/\mathsf{C}}(\mathsf{X})$ on C whose elements of type $c \in \mathsf{C}$ are the generalized elements of X of shape c/C. In other words,

$$\mathrm{El}_{/\mathsf{C}}(\mathsf{X}) := \mathsf{Cat}(-/\mathsf{C}, \mathsf{X}) : \mathsf{C} \to \mathsf{Set}.$$

This construction is functorial in X, and it is right adjoint to the category of elements:

$$\begin{tikzcd} \mathsf{Set}^{\mathsf{C}} \arrow[r, bend left, "\int^{\mathsf{C}}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "\mathrm{El}_{/\mathsf{C}}"] \end{tikzcd}.$$

6.5. Theorem. [Products of spans] *Let* S *be a (finitely) complete category. Then the double category* $\mathbb{S}\mathsf{pan}(\mathsf{S})$ *has lax (finite) products.*

Proof. Consider the infinitary case. By assumption, the underlying category $\mathbb{S}\mathsf{pan}(\mathsf{S})_0 = \mathsf{S}$ has products. Suppose, then, that $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ is a family of spans in S. Writing $\mathsf{span} := \{\bullet \leftarrow \bullet \rightarrow \bullet\}$ for the walking span, the span of indexing sets $I \leftarrow A \rightarrow J$ can be identified with a copresheaf $F : \mathsf{span} \to \mathsf{Set}$, and the family of spans in S on the left

$$\begin{tikzcd} I \arrow[d, "\underline{x}"'] & A \arrow[l] \arrow[r] \arrow[d, "\underline{m}"'] & J \arrow[d, "\underline{y}"] \\ \mathsf{S} & \mathbb{S}\mathsf{pan}(\mathsf{S})_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathsf{S} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathsf{span} \arrow[r, bend left, "F"] \arrow[r, bend right, "\mathrm{El}_{/\mathsf{span}}(\mathsf{S})"'] \arrow[r, phantom, "\underline{m}\Downarrow"] & \mathsf{Set} \end{tikzcd}$$

can be identified with a natural transformation as on the right. The latter is in turn equivalent, via the adjunction in Construction 6.4, to a diagram $M : \int F \to \mathsf{S}$ whose shape is the category of elements of $F : \mathsf{span} \to \mathsf{Set}$. We now construct the product of the spans $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ by taking limits of the diagrams in S shown on the left below:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[dr, "\underline{x}"'] & \int F \arrow[d, "M"'] & J \arrow[l, hook'] \arrow[dl, "\underline{y}"] \\ & \mathsf{S} & \end{tikzcd} \qquad \overset{\lim}{\longmapsto} \qquad \Pi\underline{x} \leftarrow \Pi\underline{m} \rightarrow \Pi\underline{y}.$$

Writing the original spans as $m_a = (x_i \xleftarrow{\ell_a} s_a \xrightarrow{r_a} y_j)$ for each $a : i \nrightarrow j$, the apex of the product span can be depicted schematically as

$$\Pi\underline{m} := \lim_{\int F} M = \lim_{\substack{i \text{ or } j \text{ or} \\ a:i' \nrightarrow j'}} \left(\begin{tikzcd} x_i & s_a \arrow[dl, "\ell_a"'] \arrow[dr, "r_a"] & y_j \\ \vdots & \vdots & \vdots \\ x_{i'} & & y_{j'} \end{tikzcd} \right).$$

The projection cells are the evident maps of spans

$$\begin{tikzcd} \Pi\underline{x} \arrow[d, "\pi_i"'] & \Pi\underline{m} \arrow[l] \arrow[r] \arrow[d, "\pi_a"'] & \Pi\underline{y} \arrow[d, "\pi_j"] \\ x_i & s_a \arrow[l, "\ell_a"] \arrow[r, "r_a"'] & y_k \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

where the squares commute because the apex $\Pi\underline{m}$ of the top span is, in particular, the apex of a cone over the diagram $M : \int F \to \mathsf{S}$.

To prove the universal property, suppose given families of maps $f : \Delta x \to \underline{x}$ and $g : \Delta y \to \underline{y}$ along with a family of maps of spans

$$\begin{tikzcd} x \arrow[d, "f_i"'] & s \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "h_a"'] & y \arrow[d, "g_j"] \\ x_i & s_a \arrow[l, "\ell_a"] \arrow[r, "r_a"'] & y_j \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

Taken together, the maps $f_i \circ l$, $g_j \circ r$, and h_a, constitute a cone over the diagram $M : \int F \to \mathsf{S}$ with apex s. By the universal property of the limit $\Pi\underline{m} = \lim M$ in S, there exists a unique map $h : s \to \Pi\underline{m}$ factoring each $f_i \circ \ell$ through π_i, each $g_j \circ r$ through π_j, and each h_a through π_a. Putting $f := \langle f_i \rangle_{i \in I}$ and $g := \langle g_j \rangle_{j \in J}$, this is equivalent to there being a unique map of spans

$$\begin{tikzcd} x \arrow[d, "f"'] & s \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, dashed, "h"'] & y \arrow[d, "g"] \\ \Pi\underline{x} & \Pi\underline{m} \arrow[l] \arrow[r] & \Pi\underline{y} \end{tikzcd}$$

factoring the span map (f_i, h_a, g_j) through the projection (π_i, π_a, π_j) for each $a : i \nrightarrow j$. ∎

6.6. Corollary. [Coproducts of cospans] *Let* S *be a (finitely) cocomplete category. Then the double category* $\mathbb{C}\mathsf{ospan}(\mathsf{S})$ *has colax (finite) coproducts.*

Proof. This follows by duality, since $\mathbb{C}\mathsf{ospan}(\mathsf{S})^{\mathrm{op}} = \mathbb{S}\mathsf{pan}(\mathsf{S}^{\mathrm{op}})$. ∎

6.7. Corollary. *The double category* $\mathbb{S}\mathsf{pan}$ *has lax products, and* $\mathbb{F}\mathsf{inSpan}$ *has lax finite products. Dually,* $\mathbb{C}\mathsf{ospan}$ *has colax coproducts and* $\mathbb{F}\mathsf{inCospan}$ *has colax finite coproducts.*

Proof. The category Set is complete, and FinSet is finitely complete. ∎

We now prove that double categories of matrices have lax products when the base category has products. We do not yet assume that the monoidal product in the base category is the cartesian product.

6.8. Proposition. [Products of matrices] *For any distributive monoidal category* $\mathcal{V}$ *with (finite) products, the double category* $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ *has lax (finite) products.*

PROOF. Clearly, the underlying category $\mathcal{V}\text{-}\mathbb{M}\mathsf{at}_0 = \mathsf{Set}$ has products. Given a family $(A, \underline{M}) : (I, \underline{X}) \nrightarrow (J, \underline{Y})$ of $\mathcal{V}$-matrices, the product $\Pi\underline{M} : \Pi\underline{X} \nrightarrow \Pi\underline{Y}$ is the $\mathcal{V}$-matrix defined pointwise by the products in $\mathcal{V}$

$$\Pi\underline{M}(\underline{x}, \underline{y}) := \prod_{a : i \nrightarrow j} M_a(x_i, y_j), \qquad \underline{x} \in \Pi\underline{X} = \prod_{i \in I} X_i,\ \underline{y} \in \Pi\underline{Y} = \prod_{j \in J} Y_j,$$

where each product in $\mathcal{V}$ ranges over all $(i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J)$. The associated projection cell shown on the left

$$\begin{tikzcd} \Pi\underline{X} \arrow[r, "\Pi\underline{M}", "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi\underline{Y} \arrow[d, "\pi_j"] \\ X_i \arrow[r, "M_a"', "\shortmid"{marking}] & Y_j \end{tikzcd} \quad \rightsquigarrow \quad \begin{array}{c} \Pi\underline{M}(\underline{x}, \underline{y}) \\ {\scriptstyle \pi_a}\downarrow \\ M_a(x_i, y_j) \\ \| \\ M_a(\pi_i(\underline{x}), \pi_j(\underline{y})) \end{array}, \quad \begin{array}{c} \underline{x} \in \Pi\underline{X} \\ \underline{y} \in \Pi\underline{Y} \end{array},$$

is given by the family of projection maps in $\mathcal{V}$ on the right. The universal property follows straightforwardly from that of products in $\mathcal{V}$. ■

Products in a double category specialize to reproduce a number of familiar concepts. Two especially important cases are parallel products and local products, both highlighted in Paré's original talk on double products [Paré, 2009]. We remark that parallel products have also been defined as limits of identity loose transformations [Grandis and Paré, 1999, §4.3], whereas local products have been defined using Kan extensions in double categories [Grandis and Paré, 2008, §7.5].

6.9. EXAMPLE. [Parallel products] A **parallel product** in a double category $\mathbb{D}$ is a product in $\mathbb{D}$ of a family of proarrows of form $(\mathrm{id}_I, \underline{m}) : (I, \underline{x}) \nrightarrow (I, \underline{y})$, where id_I is the identity span $I \xleftarrow{1} I \xrightarrow{1} I$. The product $\Pi\underline{m} : \Pi\underline{x} \to \Pi\underline{y}$ is traditionally denoted

$$\prod_{i \in I} m_i : \prod_{i \in I} x_i \nrightarrow \prod_{i \in I} y_i.$$

Let us isolate this special case by saying that a double category $\mathbb{D}$ **has lax parallel products** if, for every set I, the diagonal double functor $\Delta_I : \mathbb{D} \to \mathbb{D}^I$ has a lax right adjoint $\times_I : \mathbb{D}^I \to \mathbb{D}$, and **has strong parallel products** if the right adjoints are pseudo. With this terminology, a double category is **precartesian** in the sense of Aleiferi [Aleiferi, 2018, §4.1] just when it has lax finite parallel products, and it is **cartesian** just when it has strong finite parallel products.

In the double category $\mathbb{S}\mathsf{pan}(\mathsf{S})$, a parallel product of spans is computed pointwise as the products in S of the apexes and of the left and right feet. This can be seen directly, or as a special case of Theorem 6.5. In $\mathcal{V}\text{-}\mathbb{M}\mathsf{at}$, the product of matrices $M_i : X_i \nrightarrow Y_i$, $i \in I$, is

$$\Pi\underline{M}(\underline{x}, \underline{y}) = \prod_{i \in I} M_i(x_i, y_i), \qquad \underline{x} \in \prod_{i \in I} X_i,\ \underline{y} \in \prod_{i \in I} Y_i.$$

6.10. Example. [Local products] A **local product** in a double category $\mathbb{D}$ is a product in $\mathbb{D}$ of a proarrow family of form $(t_I, \underline{m}) : (1, x) \nrightarrow (1, y)$, where t_I is the span $1 \xleftarrow{!} I \xrightarrow{!} 1$. The local product $\Pi\underline{m} := \Pi(t_I, \underline{m})$ can be denoted $\prod_{i \in I} m_i : x \nrightarrow y$.

As the name suggests, a local product in a double category $\mathbb{D}$ is, in particular, a local product in the underlying bicategory of $\mathbb{D}$, i.e., a product in the hom-category $\mathbb{D}(x, y)$ of proarrows $x \nrightarrow y$ and globular cells between them. However, it is important to realize that the double-categorical universal property is stronger than the bicategorical one. The double-categorical universal property says that for any arrows $f : w \to x$ and $g : z \to y$ and any family of cells $\begin{smallmatrix} w & \overset{n}{\nrightarrow} & z \\ f\downarrow & \alpha_i & \downarrow g \\ x & \underset{m_i}{\nrightarrow} & y \end{smallmatrix}$ for $i \in I$, there exists a unique cell α factoring each cell α_i through the projection π_i:

$$\begin{tikzcd} w \arrow[r, "n", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha_i"] & z \arrow[d, "g"] \\ x \arrow[r, "m_i"', "\shortmid"{marking}] & y \end{tikzcd} = \begin{tikzcd} w \arrow[r, "n", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & z \arrow[d, "g"] \\ x \arrow[r, "\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & y \arrow[d, equal] \\ x \arrow[r, "m_i"', "\shortmid"{marking}] & y \end{tikzcd}, \qquad i \in I.$$

The bicategorical universal property restricts f and g to be identity arrows.

For example, in $\mathbb{S}\mathsf{pan}(\mathsf{S})$, the local product of spans $m_i = (x \xleftarrow{\ell_i} s_i \xrightarrow{r_i} y)$, $i \in I$, has as its apex the limit of the diagram in S

$$\begin{tikzcd} & s_i \arrow[dl, "\ell_i"'] \arrow[dr, "r_i"] & \\ x & \vdots & y \\ & s_{i'} \arrow[ul, "\ell_{i'}"] \arrow[ur, "r_{i'}"'] & \end{tikzcd}$$

(Theorem 6.5) or equivalently the product of the objects $(s_i, \langle \ell_i, r_i \rangle)_{i \in I}$ in the slice category $\mathsf{S}/(x \times y)$. In $\mathcal{V}$-$\mathbb{M}\mathsf{at}$, the local product of matrices $M_i : X \nrightarrow Y$, $i \in I$, is simply the pointwise product in $\mathcal{V}$:

$$\Pi\underline{M}(x, y) = \prod_{i \in I} M_i(x, y), \qquad x \in X,\ y \in Y.$$

Another special class of products in a double category, those built out of identity proarrows, will be studied Section 8. Before that, we need to determine when products can be expected to commute with external composition.

7. Iso-strong products in double categories

It is easy to find examples showing that products in a double category $\mathbb{S}$ of spans or matrices constitute a genuinely lax functor $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{S}) \to \mathbb{S}$. Indeed, a "generic" choice of proarrow families has this property. Here is a simple example involving local products. Let $\mathcal{V}$ be a distributive category and let

$$\begin{tikzcd} X \arrow[r, bend left, "M_1", "\shortmid"{marking}] \arrow[r, bend right, "M_2"', "\shortmid"{marking}] & Y \arrow[r, "N", "\shortmid"{marking}] & Z \end{tikzcd}$$

be $\mathcal{V}$-matrices. Then, on the one hand, we have

$$(\Pi \underline{M} \odot N)(x,z) = ((M_1 \times M_2) \odot N)(x,z) \cong \sum_{y \in Y} M_1(x,y) \times M_2(x,y) \times N(y,z) \tag{5}$$

for each $x \in X$ and $z \in Z$. On the other hand, by distributivity, we have

$$\begin{aligned} \Pi(\underline{M} \odot \Delta N)(x,z) &= ((M_1 \odot N) \times (M_2 \odot N))(x,z) \\ &\cong \sum_{(y,y') \in Y \times Y} M_1(x,y) \times N(y,z) \times M_2(x,y') \times N(y',z). \end{aligned} \tag{6}$$

In the canonical comparison, each constituent map in $\mathcal{V}$ sends the summand indexed by y in Equation (5) to the summand indexed by the diagonal pair (y,y) in (6). These maps are hardly ever isomorphisms.

The class of proarrow products expected to commute, up to isomorphism, with external composition and identities are singled out in the following definition.

7.1. Definition. [Iso-strong products] *A double category $\mathbb{D}$ has* ***iso-strong products*** *when*

(i) $\mathbb{D}$ has lax products;

(ii) for every pair of proarrow families $(I, \underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J, \underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K, \underline{z})$ in $\mathbb{D}$ such that the legs $A \to J$ and $J \leftarrow B$ are isomorphisms (bijections), the composition comparison cell

$$\Pi_{\underline{m},\underline{n}} : \Pi \underline{m} \odot \Pi \underline{n} \to \Pi(\underline{m} \odot \underline{n})$$

is an isomorphism; and

(iii) for every object family $(I, \underline{x})$ in $\mathbb{D}$, the identity comparison cell

$$\Pi_{\underline{x}} : \mathrm{id}_{\Pi \underline{x}} \to \Pi(\mathrm{id}_{\underline{x}})$$

is an isomorphism.

A double category with iso-strong finite products has, in particular, strong finite parallel products (Example 6.9) and hence is a cartesian double category in the sense of [Aleiferi, 2018].

7.2. Remark. [Normal and unitary products] The last condition in Definition 7.1 can be called having **normal lax products** since it is equivalent to the existence of a *normal* lax functor $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{D}$. It is reasonable to include in the definition of iso-strong products because it involves the identity family $\mathrm{id}_{(I,\underline{x})} = (\mathrm{id}_I, \mathrm{id}_{\underline{x}})$, whose indexing span has bijective legs.

As a practical matter, in double categories with normal lax products, it is straightforward to choose products so that $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{D}$ is a *unitary* lax functor [Grandis, 2019, §3.5.2], i.e., the equations $\Pi(\mathrm{id}_{\underline{x}}) = \mathrm{id}_{\Pi\underline{x}}$ hold strictly. A double category with such a choice of products is said to be equipped with **unitary lax products**. We will not assume that such a choice as been made, although it can be convenient to do so.

7.3. Proposition. [Iso-strong products of spans] *Let* S *be (finitely) complete category. Then the double category* $\mathbb{S}\mathsf{pan}(\mathsf{S})$ *has iso-strong (finite) products.*

Proof. That $\mathbb{S}\mathsf{pan}(\mathsf{S})$ has lax products has been shown in Theorem 6.5, and that they are normal is both obvious and known from the study of $\mathbb{S}\mathsf{pan}(\mathsf{S})$ as a cartesian double category. We explicitly construct the composition comparison $\Pi_{\underline{m},\underline{n}}$ for families of spans $(I,\underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J,\underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K,\underline{z})$.

By reducing the pullback of limits to a single limit, the composite of products $\Pi\underline{m} \odot \Pi\underline{n}$ can be described similarly to the proof of Theorem 6.5. Continuing to use that notation, let

$$\mathsf{span}_{\odot} \coloneqq \mathsf{span} +_1 \mathsf{span} = \{\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet\}$$

be the walking pair of composable spans. Then the given data on the left

$$\begin{tikzcd} I \arrow[d, "\underline{x}"'] & A \arrow[l] \arrow[r] \arrow[d, "\underline{m}"'] & J \arrow[d, "\underline{y}"'] & B \arrow[l] \arrow[r] \arrow[d, "\underline{n}"'] & K \arrow[d, "\underline{z}"] \\ \mathsf{S} & \mathbb{S}\mathsf{pan}(\mathsf{S})_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathsf{S} & \mathbb{S}\mathsf{pan}(\mathsf{S})_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathsf{S} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{span}_{\odot} \arrow[r, bend left, "F"] \arrow[r, bend right, "\mathrm{El}_{/\mathsf{span}_{\odot}}(\mathsf{S})"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Set} \end{tikzcd}$$

can be identified with a copresheaf $F : \mathsf{span}_{\odot} \to \mathsf{Set}$ along with a natural transformation as on the right, which is in turn equivalent to a diagram $D : \int F \to \mathsf{S}$ by Construction 6.4. The apex of the composite span $\Pi\underline{m} \odot \Pi\underline{n}$ is isomorphic to the limit of the diagam D in S.

The product of composites $\Pi(\underline{m} \odot \underline{n})$ can also be described in this way by reducing the limit of pullbacks to a single limit. As before, the data on the left

$$\begin{tikzcd} I \arrow[d, "\underline{x}"'] & A \times_J B \arrow[l] \arrow[r, equal] \arrow[d, "\underline{m}\circ\pi_A"'] & A \times_J B \arrow[r, equal] \arrow[d, "\underline{y}\circ\pi_J"'] & A \times_J B \arrow[r] \arrow[d, "\underline{n}\circ\pi_B"'] & K \arrow[d, "\underline{z}"] \\ \mathsf{S} & \mathbb{S}\mathsf{pan}(\mathsf{S})_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathsf{S} & \mathbb{S}\mathsf{pan}(\mathsf{S})_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathsf{S} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{span}_{\odot} \arrow[r, bend left, "F'"] \arrow[r, bend right, "\mathrm{El}_{/\mathsf{span}_{\odot}}(\mathsf{S})"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Set} \end{tikzcd}$$

is equivalent to a copresheaf F' and a natural transformation as on the right, which is equivalent to a diagram $D' : \int F' \to \mathsf{S}$. The apex of the span $\Pi(\underline{m} \odot \underline{n})$ is isomorphic to the limit of D' in S.

By the functorality of the elements construction, the morphism of copresheaves

$$\begin{tikzcd} I & A \arrow[l] \arrow[r] & J & B \arrow[l] \arrow[r] & K \\ I \arrow[u, equal] & A \times_J B \arrow[l] \arrow[u, "\pi_A"] \arrow[r, equal] & A \times_J B \arrow[u, "\pi_J"] \arrow[r, equal] & A \times_J B \arrow[u, "\pi_B"] \arrow[r] & K \arrow[u, equal] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{span}_\odot \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Uparrow"] & \mathsf{Set} \end{tikzcd}$$

induces a functor $\int F' \to \int F$ that is also a morphism $D' \to D$ in the slice category Cat/S of diagrams in S. The functorality of limits with respect to morphisms in $(\mathsf{Cat}/\mathsf{S})^{\mathrm{op}}$ yields the apex map in the comparison cell $\Pi_{\underline{m},\underline{n}} : \Pi\underline{m} \odot \Pi\underline{n} \to \Pi(\underline{m} \odot \underline{n})$. Now, if the legs $A \to J$ and $J \leftarrow B$ are isomorphisms, then so are all the projections π_A, π_B, and π_J out of the pullback $A \times_J B$. Thus, $F' \Rightarrow F$ is a natural isomorphism and, by the functorality of the subsequent constructions, the comparison cell $\Pi_{\underline{m},\underline{n}}$ is also an isomorphism. ∎

7.4. Corollary. [Iso-strong coproducts of cospans] *Let S be a (finitely) cocomplete category. Then the double category $\mathbb{C}\mathsf{ospan}(\mathsf{S})$ has iso-strong (finite) coproducts.*

The proof makes it clear that essentially the *only* condition under which products of spans, or dually coproducts of cospans, commute with external composition is the iso-strong condition in Definition 7.1.

Under reasonable hypotheses, double categories of matrices also have iso-strong products. By a *distributive category*, we will mean a category with finite products and arbitrary coproducts, over which the products distribute.

7.5. Proposition. [Iso-strong products of matrices] *For any distributive category $\mathcal{V}$, the double category $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ has iso-strong finite products.*

Proof. That $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ has lax finite products has been shown under weaker hypotheses in Proposition 6.8, and that they are normal under the present hypotheses follows from the isomorphisms $1 \times 1 \cong 1$ and $1 \times 0 \cong 0$ in $\mathcal{V}$. So suppose that $(I, \underline{X}) \xrightarrow{(A,\underline{M})} (J, \underline{Y}) \xrightarrow{(B,\underline{N})} (K, \underline{Z})$ are families of $\mathcal{V}$-matrices such that the legs $A \to J$ and $B \to J$ are isomorphisms. Denoting the inverse functions by $a : J \to A$ and $b : J \to B$, we have the isomorphisms

$$(\Pi\underline{M} \odot \Pi\underline{N})(\underline{x}, \underline{z}) \cong \sum_{\underline{y} \in \Pi\underline{Y}} \prod_{j \in J} M_{a(j)}(x_{\ell(a(j))}, y_j) \times N_{b(j)}(y_j, z_{r(b(j))})$$

and

$$\Pi(\underline{M} \odot \underline{N})(\underline{x}, \underline{z}) \cong \prod_{j \in J} \sum_{y_j \in Y_j} M_{a(j)}(x_{\ell(a(j))}, y_j) \times N_{b(j)}(y_j, z_{r(b(j))})$$

in $\mathcal{V}$ for each $\underline{x} \in \Pi\underline{X}$ and $\underline{y} \in \Pi\underline{Y}$. The canonical distributivity morphism

$$\sum_{\underline{y} \in \Pi\underline{Y}} \prod_{j \in J} \longrightarrow \prod_{j \in J} \sum_{y_j \in Y_j}$$

from the first object in $\mathcal{V}$ to the second, involving the dependent product $\Pi\underline{Y} = \prod_{j \in J} Y_j$ of sets, is an isomorphism since $\mathcal{V}$ is a distributive category. ∎

8. Restrictions in double categories with products

In any double category with lax products, one can ask about the proarrows that arise as products of identity proarrows for different choices of indexing span. Such proarrows and accompanying cells carry a surprisingly intricate structure, all following from the universal property of products. An example will illustrate the general situation.

8.1. Example. [Diagonal proarrows] Let $\mathbb{D}$ be a double category with lax finite products. The **diagonal proarrow** on an object $x \in \mathbb{D}$ is the product

$$\delta_x := \Pi\left(\begin{tikzcd}[row sep=small] & x \\ x \arrow[ur, "\mathrm{id}_x", "\shortmid"{marking}] \arrow[dr, "\mathrm{id}_x"', "\shortmid"{marking}] & \\ & x\end{tikzcd}\right) := \Pi(1 \xleftarrow{!} 2 = 2,\ (\mathrm{id}_x, \mathrm{id}_x)), \tag{7}$$

where the diagram of proarrows on the left is shorthand for the family of proarrows on the right. Besides the proarrow $\delta_x : x \nrightarrow x^2$, this product consists of two projection cells

$$\begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & x^2 \arrow[d, "\pi_i"] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd}, \qquad i = 1, 2.$$

From the diagonal δ_x and the parallel product id_x^2, we can form **unit** and **counit** cells

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & x \arrow[d, "\Delta_x"] \\ x \arrow[r, "\delta_x"', "\shortmid"{marking}] & x^2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, "\Delta_x"'] \arrow[dr, phantom, "\varepsilon"] & x^2 \arrow[d, equal] \\ x^2 \arrow[r, "\mathrm{id}_x^2"', "\shortmid"{marking}] & x^2 \end{tikzcd},$$

where $\Delta_x : x \to x^2$ is the usual diagonal map. The unit cell is the unique solution to the equations

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & x \arrow[d, "\Delta_x"] \\ x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & x^2 \arrow[d, "\pi_i"] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd} = 1_{\mathrm{id}_x} = \mathrm{id}_{1_x}, \qquad i = 1, 2,$$

given by the universal property of the product δ_x. Similarly, the counit cell is the unique solution to the equations

$$\begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, "\Delta_x"'] \arrow[dr, phantom, "\varepsilon"] & x^2 \arrow[d, equal] \\ x^2 \arrow[r, "\mathrm{id}_x^2", "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & x^2 \arrow[d, "\pi_i"] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd} = \begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & x^2 \arrow[d, "\pi_i"] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd}, \qquad i = 1, 2,$$

given by the universal property of the parallel product id_x^2.

Now suppose that the double category $\mathbb{D}$ has *normal* lax finite products (Remark 7.2). Then the codomain $\mathrm{id}_x^2 : x^2 \nrightarrow x^2$ of the counit cell can be interchanged with $\mathrm{id}_{x^2} : x^2 \nrightarrow x^2$. We obtain the data that would make the arrow $\Delta_x : x \to x^2$ and proarrow $\delta_x : x \nrightarrow x^2$ into a companion pair, provided the cells satisfy the two compatibility equations. Later, we will prove that they do by a more abstract approach.

Conversely, suppose that we start with a cartesian equipment $\mathbb{E}$. For simplicity, assume that we have made a unitary choice of finite parallel products, so that $\mathrm{id}_x^I = \mathrm{id}_{x^I}$ for each finite set I and object $x \in \mathbb{E}$. Define the **diagonal proarrow** $\delta_x : x \nrightarrow x^2$ to be a companion to the diagonal arrow $\Delta_x : x \to x^2$. Then there is a counit cell of the form

$$\begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, "\Delta_x"'] \arrow[dr, phantom, "\varepsilon"] & x^2 \arrow[d, equal] \\ x^2 \arrow[r, "\mathrm{id}_x^2"', "\shortmid"{marking}] & x^2 \end{tikzcd},$$

which is also the restriction of the parallel product $\mathrm{id}_x^2 = \mathrm{id}_{x^2}$ along the arrows Δ_x and 1_{x^2}. Using the projections for the product id_x^2, we can form a new pair of **projections**

$$\begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & x^2 \arrow[d, "\pi_i"] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd} \quad := \quad \begin{tikzcd} x \arrow[r, "\delta_x", "\shortmid"{marking}] \arrow[d, "\Delta_x"'] \arrow[dr, phantom, "\varepsilon"] & x^2 \arrow[d, equal] \\ x^2 \arrow[r, "\mathrm{id}_x^2", "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & x^2 \arrow[d, "\pi_i"] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd}, \qquad i = 1, 2.$$

We obtain the data that would make δ_x into a product in $\mathbb{E}$ as in Equation (7), provided the universal property is satisfied. That is indeed the case, as we will show later.

The rest of this section is dedicated to systematizing this example and explaining the connection between cartesian equipments and double categories with finite products. The key observations are that when a double category has lax products, restrictions are closed under products, and when the products are normal, companions and conjoints are also closed under products. The latter result is quicker to prove, so we begin with it.

8.2. Proposition. [Products of companions and conjoints] *Let $\mathbb{D}$ be a double category with normal lax products. Suppose that $(f_0, f) : (I, \underline{x}) \to (J, \underline{y})$ is a family of arrows $f_j : x_{f_0(j)} \to y_j$ in $\mathbb{D}$ each having a companion $(f_j)_! : x_{f_0(j)} \nrightarrow y_j$. Then the product arrow $\Pi(f_0, f) : \Pi\underline{x} \to \Pi\underline{y}$ also has a companion, namely the product $\Pi(f_0^*, f_!) : \Pi\underline{x} \nrightarrow \Pi\underline{y}$ of the family of companions $(f_!)_j := (f_j)_!$ indexed by the conjoint span $f_0^* = (I \xleftarrow{f_0} J = J)$.*

Dually, if each component f_j has a conjoint $f_j^* : y_j \nrightarrow x_{f_0(j)}$, then the product arrow $\Pi(f_0, f) : \Pi\underline{x} \to \Pi\underline{y}$ also has a conjoint, namely the product $\Pi((f_0)_!, f^*) : \Pi\underline{y} \nrightarrow \Pi\underline{x}$ of the family of conjoints $(f^*)_j := f_j^*$ indexed by the companion span $(f_0)_! = (J = J \xrightarrow{f_0} I)$.

PROOF. Since conjoints in a double category are companions in the opposite double category and vice versa, and since $\mathbb{F}\mathsf{am}^*(\mathbb{D})^{\mathrm{op}} = \mathbb{F}\mathsf{am}(\mathbb{D}^{\mathrm{op}})$, the dual of Proposition 3.8 states that the companion and conjoint of the arrow (f_0, f) in $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ are $(f_0^*, f_!)$ and $((f_0)_!, f^*)$, respectively. The result follows since the product operation $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{D}$, being a normal lax functor, preserves companions and conjoints [Dawson et al., 2010, Proposition 3.8]. ∎

8.3. COROLLARY. [Companions and conjoints of pairings] *Let $f_i : x \to y_i$ be a family of arrows with common domain in a double category with normal lax products. If each arrow f_i has a companion, then so does the pairing $\langle f_i \rangle_{i\in I} : x \to \prod_{i\in I} y_i$, and likewise for conjoints.*

PROOF. This is the special case of Proposition 8.2 with reindexing map of the form $f_0 : I \xrightarrow{!} 1$. ∎

8.4. COROLLARY. [Structure proarrows between products] *Let $\mathbb{D}$ be a double category with normal lax products. For any families of objects $(I, \underline{x})$ and $(J, \underline{y})$ in $\mathbb{D}$ and function between indexing sets $f_0 : J \to I$ such that $\underline{x} \circ f_0 = \underline{y}$, the product arrow $\Pi(f_0) \coloneqq \Pi(f_0, 1) : \Pi\underline{x} \to \Pi\underline{y}$ has both a companion $\Pi(f_0)_! = \Pi(f_0^*, \mathrm{id}) : \Pi\underline{x} \nrightarrow \Pi\underline{y}$ and a conjoint $\Pi(f_0)^* = \Pi((f_0)_!, \mathrm{id}) : \Pi\underline{y} \nrightarrow \Pi\underline{x}$.*

PROOF. This is another special case of Proposition 8.2 since, for any object x in a double category, the identity arrow $1_x : x \to x$ always has the identity proarrow $\mathrm{id}_x : x \nrightarrow x$ as both its companion and its conjoint [Shulman, 2010, Lemma 3.12]. ∎

Such results can be stated more succinctly using a special notation for reindexed families of objects and proarrows. Given a function $f_0 : J \to I$ and an I-indexed family of objects $\underline{x}$, denote by $f_0^*(\underline{x}) \coloneqq \underline{x} \circ f_0$ the J-indexed family obtained by precomposing $\underline{x}$ with f_0. Similarly, given an I-indexed parallel family of proarrows $\underline{m} : \underline{x} \nrightarrow \underline{y}$, denote by $f_0^*\underline{m} : f_0^*\underline{x} \nrightarrow f_0^*\underline{y}$ the J-indexed parallel family of proarrows obtained by precomposing all of $\underline{x}$, $\underline{y}$, and $\underline{m}$ with f_0.

With this notation, Corollary 8.4 can be restated as follows: for any function $f_0 : J \to I$, the **structure arrow** $\Pi(f_0) : \Pi\underline{x} \to \Pi f_0^*\underline{x}$ between products has both a companion $\Pi(f_0)_! : \Pi\underline{x} \nrightarrow \Pi f_0^*\underline{x}$ and a conjoint $\Pi(f_0)^* : \Pi f_0^*\underline{x} \nrightarrow \Pi\underline{x}$. This corollary encompasses familiar structure arrows like diagonals and projections. Specifically, in any double category with normal lax finite products,

- diagonals $\Delta_x : x \to x^2$ have companions $\delta_x : x \nrightarrow x^2$ and conjoints $\delta_x^* : x^2 \nrightarrow x$, via the unique function $f_0 : 2 \xrightarrow{!} 1$ as constructed directly in Example 8.1;
- deletion arrows $!_x : x \to 1$ have companions $\varepsilon_x : x \nrightarrow 1$ and conjoints $\varepsilon_x^* : 1 \nrightarrow x$, via the function $f_0 : 0 \xrightarrow{!} 1$;
- projections $\pi_x : x \times y \to x$ and $\pi_y : x \times y \to y$ have companions and conjoints, via the two inclusions $f_0 : 1 \hookrightarrow 2$; and

- braidings $\sigma_{x,y} : x \times y \to y \times x$ have companions and conjoints, via the swap function $f_0 : 2 \xrightarrow{\cong} 2$, which are mutually inverse up to globular isomorphism [Shulman, 2010, Lemma 3.20].

In fact, since companions and conjoints are functorial up to isomorphism, these special cases essentially exhaust the content of Corollary 8.4 for finite products.

As hinted by Example 8.1, the normality assumption in Proposition 8.2 and its corollaries can be discarded while retaining nearly the same universal property. We will see that structure proarrows such as the diagonal $\delta_x : x \nrightarrow x^2$ and codiagonal $\delta_x^* : x^2 \nrightarrow x$, while not necessarily companions or conjoints in a double category with merely lax finite products, are still restrictions of parallel products of identity proarrows.

8.5. Proposition. [Restricting along structure arrows between products] *Let $\mathbb{D}$ be a double category with lax products. Suppose given a niche in* $\mathbb{F}\mathsf{am}^*(\mathbb{D})$

$$\begin{tikzcd} (I,\underline{x}) \arrow[d, "{(f_0,f)}"'] & (J,\underline{y}) \arrow[d, "{(g_0,g)}"] \\ (K,\underline{w}) \arrow[r, "{(B,\underline{n})}"', "\shortmid" marking] & (L,\underline{z}) \end{tikzcd}$$

such that each constituent niche in $\mathbb{D}$

$$\begin{tikzcd} x_{f_0(k)} \arrow[d, "f_k"'] & y_{g_0(\ell)} \arrow[d, "g_\ell"] \\ w_k \arrow[r, "n_b"', "\shortmid" marking] & z_\ell \end{tikzcd}, \qquad (k \overset{b}{\nrightarrow} \ell) : (K \overset{B}{\nrightarrow} L),$$

has a restriction. Then $\mathbb{D}$ *has a restriction cell of the form*

$$\begin{tikzcd} \Pi\underline{x} \arrow[r, "{\Pi(\underline{n}(f,g))}", "\shortmid" marking] \arrow[d, "{\Pi(f_0,f)}"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{y} \arrow[d, "{\Pi(g_0,g)}"] \\ \Pi\underline{w} \arrow[r, "\Pi\underline{n}"', "\shortmid" marking] & \Pi\underline{z} \end{tikzcd}. \tag{8}$$

Here the domain of the restriction cell is the product of the proarrow family $(B, \underline{n}(f,g))$, indexed by the span $I \xleftarrow{f_0} K \leftarrow B \to L \xrightarrow{g_0} J$ and comprising the restricted proarrows $n_b(f_k, g_\ell) : x_{f_0(k)} \nrightarrow y_{g_0(\ell)}$ for each element $b : k \nrightarrow \ell$.

Proof. By the dual of Proposition 3.6, the family of restriction cells

$$\begin{tikzcd} (I,\underline{x}) \arrow[r, "{(B,\underline{n}(f,g))}", "\shortmid" marking] \arrow[d, "{(f_0,f)}"'] \arrow[dr, phantom, "{(1_B,\mathrm{res})}"] & (J,\underline{y}) \arrow[d, "{(g_0,g)}"] \\ (K,\underline{w}) \arrow[r, "{(B,\underline{n})}"', "\shortmid" marking] & (L,\underline{z}) \end{tikzcd}$$

is itself a restriction cell in $\mathbb{F}\mathsf{am}^*(\mathbb{D})$. We define the required cell (8) to be the product of this family of restrictions. We must show that the product is itself a restriction.

Now, *if* the double category $\mathbb{D}$ were an equipment, then $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ would also be an equipment by the dual of Corollary 3.7. The result would then follow immediately since the product operation $\Pi : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{D}$, like any lax functor between equipments, preserves restrictions [Shulman, 2008, Proposition 6.4]. However, we are *not* assuming that $\mathbb{D}$ is an equipment, and the cited proof does not go through when that assumption is dropped. We instead present a more specialized analysis that uses the universal properties of both products and restrictions.

We must show that any cell α of the form on the left

$$\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q"'] & y \arrow[d, "r"] \\
\Pi\underline{x} \arrow[r, phantom, "\alpha"] \arrow[d, "\Pi(f_0,f)"'] & \Pi\underline{y} \arrow[d, "\Pi(g_0,g)"] \\
\Pi\underline{w} \arrow[r, "\Pi\underline{n}"', "|" marking] & \Pi\underline{z}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q"'] \arrow[dr, phantom, "\exists!\ \bar{\alpha}"] & y \arrow[d, "r"] \\
\Pi\underline{x} \arrow[r, "\Pi\underline{n}(f,g)", "|" marking] \arrow[d, "\Pi(f_0,f)"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{y} \arrow[d, "\Pi(g_0,g)"] \\
\Pi\underline{w} \arrow[r, "\Pi\underline{n}"', "|" marking] & \Pi\underline{z}
\end{tikzcd}
\tag{9}$$

has a unique factorization through the claimed restriction cell as on the right. Decomposing the arrows $q = \langle q_i \rangle_{i \in I}$ and $r = \langle r_j \rangle_{j \in J}$ into pairings, the given cell α is the pairing of the cells

$$\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q_{f_0(k)}"'] & y \arrow[d, "r_{g_0(\ell)}"] \\
x_{f_0(k)} \arrow[r, phantom, "\alpha_b"] \arrow[d, "f_k"'] & y_{g_0(\ell)} \arrow[d, "g_\ell"] \\
w_k \arrow[r, "n_b"', "|" marking] & z_\ell
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & y \arrow[d] \\
\Pi\underline{w} \arrow[r, "\Pi\underline{n}", "|" marking] \arrow[d, "\pi_k"'] \arrow[dr, phantom, "\pi_b"] & \Pi\underline{z} \arrow[d, "\pi_\ell"] \\
w_k \arrow[r, "n_b"', "|" marking] & z_\ell
\end{tikzcd}
, \qquad (k \overset{b}{\nrightarrow} \ell) : (K \overset{B}{\nrightarrow} L),$$

with respect to the product $\Pi\underline{n} : \Pi\underline{w} \nrightarrow \Pi\underline{z}$. For each $b : k \nrightarrow \ell$, the universal property of the restriction $n_b(f_k, g_\ell)$ then gives a unique factorization

$$\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q_{f_0(k)}"'] & y \arrow[d, "r_{g_0(\ell)}"] \\
x_{f_0(k)} \arrow[r, phantom, "\alpha_b"] \arrow[d, "f_k"'] & y_{g_0(\ell)} \arrow[d, "g_\ell"] \\
w_k \arrow[r, "n_b"', "|" marking] & z_\ell
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q_{f_0(k)}"'] \arrow[dr, phantom, "\exists!\ \bar{\alpha}_b"] & y \arrow[d, "r_{g_0(\ell)}"] \\
x_{f_0(k)} \arrow[r, "n_b(f_k,g_\ell)", "|" marking] \arrow[d, "f_k"'] \arrow[dr, phantom, "\mathrm{res}_b"] & y_{g_0(\ell)} \arrow[d, "g_\ell"] \\
w_k \arrow[r, "n_b"', "|" marking] & z_\ell
\end{tikzcd}
.$$

Finally, we define the needed cell $\bar{\alpha}$ to be the pairing of the cells $\bar{\alpha}_b$ with respect to the product $\Pi(\underline{n}(f,g)) : \Pi\underline{x} \nrightarrow \Pi\underline{y}$, i.e., the unique solution to the equations

$$
\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q"'] \arrow[dr, phantom, "\bar{\alpha}"] & y \arrow[d, "r"] \\
\Pi\underline{x} \arrow[r, "\Pi\underline{n}(f{,}g)", "|" marking] \arrow[d, "\pi_{f_0(k)}"'] \arrow[dr, phantom, "\pi_b"] & \Pi\underline{y} \arrow[d, "\pi_{g_0(\ell)}"] \\
x_{f_0(k)} \arrow[r, "n_b(f_k{,}g_\ell)"', "|" marking] & y_{g_0(\ell)}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "m", "|" marking] \arrow[d, "q_{f_0(k)}"'] \arrow[dr, phantom, "\bar{\alpha}_b"] & y \arrow[d, "r_{g_0(\ell)}"] \\
x_{f_0(k)} \arrow[r, "n_b(f_k{,}g_\ell)"', "|" marking] & y_{g_0(\ell)}
\end{tikzcd}, \qquad (k \overset{b}{\nrightarrow} \ell) : (K \overset{B}{\nrightarrow} L).
$$

By the above relations, supplemented by the equation

$$
\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi\underline{n}(f{,}g)", "|" marking] \arrow[d, "\pi_{f_0(k)}"'] \arrow[dr, phantom, "\pi_b"] & \Pi\underline{y} \arrow[d, "\pi_{g_0(\ell)}"] \\
x_{f_0(k)} \arrow[r, "n_b(f_k{,}g_\ell)", "|" marking] \arrow[d, "f_k"'] \arrow[dr, phantom, "\mathrm{res}_b"] & y_{g_0(\ell)} \arrow[d, "g_\ell"] \\
w_k \arrow[r, "n_b"', "|" marking] & z_\ell
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi\underline{n}(f{,}g)", "|" marking] \arrow[d, "\Pi(f_0{,}f)"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{y} \arrow[d, "\Pi(g_0{,}g)"] \\
\Pi\underline{w} \arrow[r, "\Pi\underline{n}", "|" marking] \arrow[d, "\pi_k"'] \arrow[dr, phantom, "\pi_b"] & \Pi\underline{z} \arrow[d, "\pi_\ell"] \\
w_k \arrow[r, "n_b"', "|" marking] & z_\ell
\end{tikzcd}
$$

which is a consequence of the functorality of products, the post-composite of Equation (9) with the projection $\pi_b : \Pi\underline{n} \to n_b$ holds for every $b : k \nrightarrow \ell$, hence Equation (9) holds by the universal property of the product $\Pi\underline{n}$. This proves the existence part of the universal property of the restriction. Uniqueness follows from the uniqueness in each step of the above construction. ■

We can go further to characterize products in a double category in terms of restrictions and parallel products (Example 6.9). The preceding proposition gives one half of the proof.

8.6. Theorem. [Double products via restrictions] *A double category has lax (finite) products if and only if it has*

(i) lax (finite) parallel products, and

(ii) restrictions of parallel products along structure arrows between products.

Moreover, in this case, the products are iso-strong if and only if the parallel products are strong.

Proof. For the proof of the first characterization, let $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ be any family of proarrows in a double category $\mathbb{D}$, indexed by the span $I \xleftarrow{f_0} A \xrightarrow{g_0} J$.

Suppose $\mathbb{D}$ has lax products. Then the forward implication follows immediately from Proposition 8.5. However, it will be instructive to unpack this further. For every $a \in A$, the proarrow $m_a : x_{f_0(a)} \nrightarrow y_{g_0(a)}$ trivially has a restriction along identity arrows, namely itself. Thus, by Proposition 8.5, the product $\Pi\underline{m} : \Pi\underline{x} \nrightarrow \Pi\underline{y}$ is the restriction of the parallel product $\Pi\underline{m} : \Pi f_0^*\underline{x} \to \Pi g_0^*\underline{y}$ along the structure arrows $\Pi(f_0)$ and $\Pi(g_0)$:

$$\begin{tikzcd} \Pi\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, "\Pi(f_0)"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{y} \arrow[d, "\Pi(g_0)"] \\ \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}"', "|" marking] & \Pi g_0^*\underline{y} \end{tikzcd} . \tag{10}$$

We have shown that an *arbitrary* product in $\mathbb{D}$ admits a canonical decomposition as the restriction of a parallel product along structure arrows.

Now suppose $\mathbb{D}$ is a double category with lax parallel products and restrictions of these along structure arrows. We show that, conversely, the proarrow $\Pi\underline{m} : \Pi\underline{x} \nrightarrow \Pi\underline{y}$ now *defined* to be the restriction (10) is a product of the family $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (J, \underline{y})$ when it is equipped with the projection cells

$$\begin{tikzcd} \Pi\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi\underline{y} \arrow[d, "\pi_j"] \\ x_i \arrow[r, "m_a"', "|" marking] & y_j \end{tikzcd} \quad := \quad \begin{tikzcd} \Pi\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, "\Pi(f_0)"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{y} \arrow[d, "\Pi(g_0)"] \\ \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, "\pi_a"'] \arrow[dr, phantom, "\pi_a"] & \Pi g_0^*\underline{y} \arrow[d, "\pi_a"] \\ x_{f_0(a)} \arrow[r, "m_a"', "|" marking] & y_{g_0(a)} \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

To prove the universal property, suppose given a family of cells $\begin{tikzcd} w \arrow[r, "n", "|" marking] \arrow[d, "f_i"'] \arrow[dr, phantom, "\alpha_a"] & z \arrow[d, "g_j"] \\ x_i \arrow[r, "m_a"', "|" marking] & y_j \end{tikzcd}$ indexed by elements $a : i \nrightarrow j$. Abbreviating the families $h_a := f_{f_0(a)}$ and $k_a := g_{g_0(a)}$, we have equations

$$\Pi(f_0) \circ \langle f_i \rangle_{i \in I} = \langle h_a \rangle_{a \in A} \qquad \text{and} \qquad \Pi(g_0) \circ \langle g_j \rangle_{j \in J} = \langle k_a \rangle_{a \in A}.$$

We can therefore form the pairing with respect to the parallel product as on the left

$$\begin{tikzcd} w \arrow[r, "n", "|" marking] \arrow[d, "\langle h_a \rangle_{a \in A}"'] \arrow[dr, phantom, "\langle \alpha_a \rangle_{a \in A}"] & z \arrow[d, "\langle k_a \rangle_{a \in A}"] \\ \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}"', "|" marking] & \Pi g_0^*\underline{y} \end{tikzcd} \quad = \quad \begin{tikzcd} w \arrow[r, "n", "|" marking] \arrow[d, "\langle f_i \rangle_{i \in I}"'] \arrow[dr, phantom, "\exists!\ \alpha"] & z \arrow[d, "\langle g_j \rangle_{j \in J}"] \\ \Pi\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, "\Pi(f_0)"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{y} \arrow[d, "\Pi(g_0)"] \\ \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}"', "|" marking] & \Pi g_0^*\underline{y} \end{tikzcd} .$$

and then factorize it through the restriction cell to obtain a unique cell α as on the right.

By construction, the cell α is a solution to the equations

$$\begin{tikzcd} w \arrow[r, "n", "|" marking] \arrow[d, "\langle f_i\rangle_{i\in I}"'] \arrow[dr, phantom, "\alpha"] & z \arrow[d, "\langle g_j\rangle_{j\in J}"] \\ \Pi\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi\underline{y} \arrow[d, "\pi_j"] \\ x_i \arrow[r, "m_a"', "|" marking] & y_j \end{tikzcd} \quad = \quad \begin{tikzcd} w \arrow[r, "n", "|" marking] \arrow[d, "f_i"'] \arrow[dr, phantom, "\alpha_a"] & z \arrow[d, "g_j"] \\ x_i \arrow[r, "m_a"', "|" marking] & y_j \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J),$$

and α is the unique solution by the uniqueness property above. Using the description of lax products as universal arrows in Proposition 6.3, we conclude that $\mathbb{D}$ has lax products. This completes the proof characterizing lax products.

To characterize iso-strong products, we need only prove the reverse implication. Suppose that a double category $\mathbb{D}$ has strong parallel products and restrictions of these along structure arrows. Then, by the foregoing, $\mathbb{D}$ already has lax products, and the products are normal by assumption, so we just need to establish the iso-strong condition on composites (Definition 7.1).

First, notice that the general case of composing along bijective legs can be reduced to the special case of composing along legs that are identities. Given indexing spans $I \xleftarrow{\ell_A} A \xrightarrow{r_A} J$ and $J \xleftarrow{\ell_B} A \xrightarrow{r_B} K$, where r_A and ℓ_B are bijections as in Definition 7.1, the maps of spans

$$\begin{tikzcd} I \arrow[d, equal] & A \arrow[l, "\ell_A"'] \arrow[r, "r_A"] \arrow[d, "r_A"'] & J \arrow[d, equal] \\ I & J \arrow[l, "\ell_a \circ r_A^{-1}"] \arrow[r, equal] & J \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} J \arrow[d, equal] & B \arrow[l, "\ell_B"'] \arrow[r, "r_B"] \arrow[d, "\ell_B"] & K \arrow[d, equal] \\ J \arrow[r, equal] & J \arrow[r, "r_B \circ \ell_B^{-1}"'] & K \end{tikzcd}$$

are isomorphisms. Thus, they induce isomorphisms in $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ between families indexed by those spans and, by the functorality of products, between products of such families. By the naturality axiom for laxators, the composition comparisons for products indexed by the original spans are invertible precisely when the comparisons for products indexed by the specials spans $I \leftarrow J = J$ and $J = J \to K$ are invertible.

So assume that we have families of proarrows $(A, \underline{m}) : (I, \underline{x}) \nrightarrow (A, \underline{y})$ and $(A, \underline{n}) : (A, \underline{y}) \nrightarrow (J, \underline{z})$ indexed by spans $I \xleftarrow{f_0} A = A$ and $A = A \xrightarrow{g_0} J$, where f_0 and g_0 are arbitrary functions. We can express the restrictions giving the products of these families as

$$\begin{tikzcd} \Pi\underline{x} \arrow[r, "\Pi(f_0)_!", "|" marking] \arrow[d, "\Pi(f_0)"'] \arrow[dr, phantom, "\mathrm{res}"] & \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "1"] & \Pi\underline{y} \arrow[d, equal] \\ \Pi f_0^*\underline{x} \arrow[r, equal, "|" marking] & \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}"', "|" marking] & \Pi\underline{y} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \Pi\underline{y} \arrow[r, "\Pi\underline{n}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "1"] & \Pi g_0^*\underline{z} \arrow[r, "\Pi(g_0)^*", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{res}"] & \Pi\underline{z} \arrow[d, "\Pi(g_0)"] \\ \Pi\underline{y} \arrow[r, "\Pi\underline{n}"', "|" marking] & \Pi g_0^*\underline{z} \arrow[r, equal, "|" marking] & \Pi g_0^*\underline{z} \end{tikzcd},$$

using Corollary 8.4 and the formula for general restrictions in terms of companions and conjoints [Shulman, 2008, Equation 4.7]. But, by the same formula, the composite cell

$$\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi(f_0)_!", "\shortmid"{marking}] \arrow[d, "\Pi(f_0)"'] \arrow[dr, phantom, "\text{res}"] & \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, equal] & \Pi\underline{y} \arrow[r, "\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, phantom, "1"] & \Pi g_0^*\underline{z} \arrow[r, "\Pi(g_0)^*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\text{res}"] & \Pi\underline{z} \arrow[d, "\Pi(g_0)"] \\
\Pi f_0^*\underline{x} \arrow[r, equal, "\shortmid"{marking}] & \Pi f_0^*\underline{x} \arrow[r, "\Pi\underline{m}"', "\shortmid"{marking}] & \Pi\underline{y} \arrow[r, "\Pi\underline{n}"', "\shortmid"{marking}] & \Pi g_0^*\underline{z} \arrow[r, equal, "\shortmid"{marking}] & \Pi g_0^*\underline{z}
\end{tikzcd}$$

is a restriction of the composite of parallel products $\Pi\underline{m}\odot\Pi\underline{n} \cong \Pi(\underline{m}\odot\underline{n}) : \Pi f_0^*\underline{x} \nrightarrow \Pi g_0^*\underline{z}$. This completes the proof of the iso-strong condition and the theorem. ∎

8.7. Corollary. *Any precartesian equipment has lax finite products, and any cartesian equipment has iso-strong finite products.*

Since the double categories of spans in a finitely complete category, and of matrices valued in a distributive category, are known to be cartesian equipments, we obtain independent proofs of Propositions 7.3 and 7.5. Nevertheless, the proofs in Sections 6 and 7 are useful in giving unbiased formulas for products in these double categories. On the other hand, we have not directly shown that double categories of relations or of profunctors have iso-strong finite products. That follows from the corollary since these double categories are known to be cartesian equipments.

We emphasize the converse to the corollary is not true: a double category with products, even strong ones, need not be an equipment. Here is a simple counterexample. Let $\mathbb{D}$ be the strict double category freely generated by a nontrival arrow $f : x \to y$. Then f does not have a companion since $\mathbb{D}$ does not even have a proarrow $x \nrightarrow y$. Now the double category $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ has strong products, being the free product completion of $\mathbb{D}$ (Theorem 5.4), yet the image of f under the embedding $\Delta : \mathbb{D} \to \mathbb{F}\mathsf{am}^*(\mathbb{D})$ still does not have a companion. Again, $\mathbb{F}\mathsf{am}^*(\mathbb{D})$ does not even have a proarrow $\Delta x \nrightarrow \Delta y$. So the concepts of a cartesian equipment and a double category with finite products do not coincide.

9. Lax functors between double categories with products

Though related to cartesian equipments, as we have just seen, double categories with finite products turn out to be better behaved as domains of lax functors. A lax functor between equipments preserves restrictions, while dually a colax functor between equipments preserves extensions [Shulman, 2008, Proposition 6.4]. Only a pseudo double functor preserves the full range of operations available in an equipment. Specifically, the axioms of a lax functor between equipments seem to only weakly constrain the laxators for general composites of companions and conjoints. By contrast, although a generic double category with iso-strong products has certain companions and conjoints, the corresponding laxators are well controlled. The reason is that these special restrictions are actually products, as shown in Section 8, and lax functors interact well with products, as we will show in this section.

A lax functor F can be defined to preserve products when its opposite, the colax functor F^{op}, preserves coproducts (Definition 5.2). Or, more directly:

9.1. Definition. [Preservation of double products] *Let $\mathbb{D}$ and $\mathbb{E}$ be double categories with lax products. A lax functor $F : \mathbb{D} \to \mathbb{E}$* ***preserves products*** *if the morphism of adjunctions*

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\Delta \dashv \Pi", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "(1,\Phi)"] & \mathbb{F}\mathsf{am}^*(\mathbb{D}) \arrow[d, "\mathbb{F}\mathsf{am}^*(F)"] \\
\mathbb{E} \arrow[r, "\Delta \dashv \Pi"', "|" marking] & \mathbb{F}\mathsf{am}^*(\mathbb{E})
\end{tikzcd}$$

is strong.

To elaborate, the components of the natural transformation

$$\begin{tikzcd}
\mathbb{F}\mathsf{am}^*(\mathbb{D}) \arrow[r, "\Pi_{\mathbb{D}}"] \arrow[d, "\mathbb{F}\mathsf{am}^*(F)"'] \arrow[dr, phantom, "\swarrow \Phi"] & \mathbb{D} \arrow[d, "F"] \\
\mathbb{F}\mathsf{am}^*(\mathbb{E}) \arrow[r, "\Pi_{\mathbb{E}}"'] & \mathbb{E}
\end{tikzcd}$$

are, at the family of objects $(I, \underline{x})$ in $\mathbb{D}$, the usual canonical comparison $\Phi_{\underline{x}} : F\Pi\underline{x} \to \Pi F\underline{x}$ between products, and at the family of proarrows $(A, \underline{m}) : (I, \underline{x}) \to\!\!\!\!\!+ (J, \underline{y})$ in $\mathbb{D}$, the comparison cell $\Phi_{\underline{m}} : F\Pi\underline{m} \to \Pi F\underline{m}$ that is the unique solution to the equations

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "|" marking] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\Phi_{\underline{m}}"] & F\Pi\underline{y} \arrow[d, "\Phi_{\underline{y}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\underline{m}", "|" marking] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi F\underline{y} \arrow[d, "\pi_j"] \\
Fx_i \arrow[r, "Fm_a"', "|" marking] & Fy_j
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "|" marking] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "F\pi_a"] & F\Pi\underline{y} \arrow[d, "F\pi_j"] \\
Fx_i \arrow[r, "Fm_a"', "|" marking] & Fy_j
\end{tikzcd}, \qquad (i \overset{a}{\to\!\!\!\!\!+} j) : (I \overset{A}{\to\!\!\!\!\!+} J).$$

A lax functor $F : \mathbb{D} \to \mathbb{E}$ preserves products just when all components $\Phi_{\underline{x}}$ and $\Phi_{\underline{m}}$ are isomorphisms in $\mathbb{E}_0$ and $\mathbb{E}_1$, respectively.

9.2. Example. [Preserving products of spans] Recalling Example 5.3, any pullback-preserving functor $F : \mathsf{C} \to \mathsf{D}$ between categories with pullbacks induces a (pseudo) double functor

$$\mathbb{S}\mathsf{pan}(F) : \mathbb{S}\mathsf{pan}(\mathsf{C}) \to \mathbb{S}\mathsf{pan}(\mathsf{D})$$

between double categories of spans. Suppose that C and D also have products, i.e., are complete categories, so that $\mathbb{S}\mathsf{pan}(\mathsf{C})$ and $\mathbb{S}\mathsf{pan}(\mathsf{D})$ have iso-strong products. Then $\mathbb{S}\mathsf{pan}(F)$ preserves products if and only if F preserves products, i.e., is a continuous functor.

Let $F : \mathbb{D} \to \mathbb{E}$ be a lax functor between double categories. To state the next lemma, we need an explicit description of the comparisons for the lax functor

$$\mathbb{F}\mathsf{am}^*(F) : \mathbb{F}\mathsf{am}^*(\mathbb{D}) \to \mathbb{F}\mathsf{am}^*(\mathbb{E}),$$

given by dual of Construction 5.1, that applies F elementwise to families. The laxator of $\mathbb{F}\mathsf{am}^*(F)$ at a pair of proarrow families $(I, \underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J, \underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K, \underline{z})$ in $\mathbb{D}$, denoted $F_{\underline{m},\underline{n}}$ on the left

$$\begin{tikzcd}
F\underline{x} \arrow[r, "F\underline{m}", "\shortmid"{marking}] \arrow[d, equal] & F\underline{y} \arrow[r, "F\underline{n}", "\shortmid"{marking}] \arrow[d, phantom, "F_{\underline{m},\underline{n}}"] & F\underline{z} \arrow[d, equal] \\
F\underline{x} \arrow[rr, "F(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] & & F\underline{z}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
Fx_i \arrow[r, "Fm_a", "\shortmid"{marking}] \arrow[d, equal] & Fy_j \arrow[r, "Fn_b", "\shortmid"{marking}] \arrow[d, phantom, "F_{m_a,n_b}"] & Fz_k \arrow[d, equal] \\
Fx_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & Fz_k
\end{tikzcd}, \quad (i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k) : (I \overset{A}{\nrightarrow} J \overset{B}{\nrightarrow} K),$$

is the family of laxators of F on the right, indexed by pairs $(a, b) \in A \times_J B$. Similarly, the unitor of $\mathbb{F}\mathsf{am}^*(F)$ at an object family $(I, \underline{x})$ in $\mathbb{D}$, denoted $F_{\underline{x}}$ on the left

$$\begin{tikzcd}
F\underline{x} \arrow[r, "\mathrm{id}_{F\underline{x}}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{\underline{x}}"] & F\underline{x} \arrow[d, equal] \\
F\underline{x} \arrow[r, "F\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] & F\underline{x}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{x_i}"] & Fx_i \arrow[d, equal] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}, \quad i \in I,$$

is the family of unitors of F on the right. With this notation, we state an important technical lemma relating laxators and unitors of products to products of laxators and unitors.

9.3. Lemma. [Laxators and unitors for products] *Let $F : \mathbb{D} \to \mathbb{E}$ be a lax functor between double categories with lax products. For any composable families of proarrows $(I, \underline{x}) \overset{(A,\underline{m})}{\nrightarrow} (J, \underline{y}) \overset{(B,\underline{n})}{\nrightarrow} (K, \underline{z})$ in $\mathbb{D}$, we have*

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, equal] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, phantom, "F_{\Pi\underline{m},\Pi\underline{n}}"] & F\Pi\underline{z} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[rr, "F(\Pi\underline{m}\odot\Pi\underline{n})"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "F\Pi_{\underline{m},\underline{n}}"] & F\Pi\underline{z} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[rr, "F\Pi(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] & \arrow[d, phantom, "\Phi_{\underline{m}\odot\underline{n}}"] & F\Pi\underline{z} \arrow[d, "\Phi_{\underline{z}}"] \\
\Pi F\underline{x} \arrow[rr, "\Pi F(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] & & \Pi F\underline{z}
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\Phi_{\underline{m}}"] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, dashed, "\Phi_{\underline{y}}"] \arrow[dr, phantom, "\Phi_{\underline{n}}"] & F\Pi\underline{z} \arrow[d, "\Phi_{\underline{z}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\underline{m}"', "\shortmid"{marking}] \arrow[d, equal] & \Pi F\underline{y} \arrow[r, "\Pi F\underline{n}"', "\shortmid"{marking}] \arrow[d, phantom, "\Pi_{F\underline{m},F\underline{n}}"] & \Pi F\underline{z} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[rr, "\Pi(F\underline{m}\odot F\underline{n})"', "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Pi F_{\underline{m},\underline{n}}"] & \Pi F\underline{z} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[rr, "\Pi F(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] & & \Pi F\underline{z}
\end{tikzcd}. \tag{11}$$

Similarly, for any family of objects $(I, \underline{x})$ in $\mathbb{D}$, we have

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{\Pi\underline{x}}"] & F\Pi\underline{x} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[r, "F\,\mathrm{id}_{\Pi\underline{x}}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F\Pi_{\underline{x}}"] & F\Pi\underline{x} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[r, "F\Pi\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\Phi_{\mathrm{id}_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] & \Pi F\underline{x}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{id}_{\Phi_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\mathrm{id}_{\Pi F\underline{x}}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Pi_{F\underline{x}}"] & \Pi F\underline{x} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[r, "\Pi\,\mathrm{id}_{F\underline{x}}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Pi F_{\underline{x}}"] & \Pi F\underline{x} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[r, "\Pi F\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] & \Pi F\underline{x}
\end{tikzcd}
. \tag{12}$$

In particular, if F preserves products and the domain $\mathbb{D}$ has strong products, then the laxator $F_{\Pi\underline{m},\Pi\underline{n}}$ for a composite of products is uniquely determined by the product of laxators $\Pi F_{\underline{m},\underline{n}}$ and the unitor $F_{\Pi\underline{x}}$ for a product is uniquely determined by the product of unitors $\Pi F_{\underline{x}}$.

Proof. The proof is a direct calculation, even if the notation is heavy. Beginning with the laxators, choose elements $i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k$ of the indexing spans. Then post-composing the left-hand side of Equation (11) with the projection $\pi_{(a,b)}$ in $\mathbb{E}$ gives

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, phantom, "\mathrm{LHS}"] & F\Pi\underline{z} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[rr, "\Pi F(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[drr, phantom, "\pi_{(a,b)}"] & & \Pi F\underline{z} \arrow[d, "\pi_k"] \\
F x_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & F z_k
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, equal] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, phantom, "F_{\Pi\underline{m},\Pi\underline{n}}"] & F\Pi\underline{z} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[rr, "F(\Pi\underline{m}\odot\Pi\underline{n})"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "F\Pi_{\underline{m},\underline{n}}"] & & F\Pi\underline{z} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[rr, "F\Pi(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[drr, phantom, "F\pi_{(a,b)}"] & & F\Pi\underline{z} \arrow[d, "F\pi_k"] \\
F x_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & F z_k
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, equal] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, phantom, "F_{\Pi\underline{m},\Pi\underline{n}}"] & F\Pi\underline{z} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[rr, "F(\Pi\underline{m}\odot\Pi\underline{n})"', "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[drr, phantom, "F(\pi_a\odot\pi_b)"] & & F\Pi\underline{z} \arrow[d, "F\pi_k"] \\
F x_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & F z_k
\end{tikzcd}
,$$

where the last equality uses Equation (3). Meanwhile, post-composing the right-hand side of Equation (11) with this projection gives

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[drr, phantom, "\mathrm{RHS}"] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] & F\Pi\underline{z} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[rr, "\Pi F(\underline{m}\odot\underline{n})"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[drr, phantom, "\pi_{(a,b)}"] & & \Pi F\underline{z} \arrow[d, "\pi_k"] \\
Fx_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & Fz_k
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\Phi_{\underline{m}}"] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{y}}"] \arrow[dr, phantom, "\Phi_{\underline{n}}"] & F\Pi\underline{z} \arrow[d, "\Phi_{\underline{z}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\underline{m}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Pi_{F\underline{m},F\underline{n}}"] & \Pi F\underline{y} \arrow[r, "\Pi F\underline{n}"', "\shortmid"{marking}] & \Pi F\underline{z} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[rr, "\Pi(F\underline{m}\odot F\underline{n})"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[drr, phantom, "\pi_{(a,b)}"] & & \Pi F\underline{z} \arrow[d, "\pi_k"] \\
Fx_i \arrow[r, "Fm_a"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "F_{m_a,n_b}"] & Fy_j \arrow[r, "Fn_b"', "\shortmid"{marking}] & Fz_k \arrow[d, equal] \\
Fx_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & Fz_k
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\Phi_{\underline{m}}"] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{y}}"] \arrow[dr, phantom, "\Phi_{\underline{n}}"] & F\Pi\underline{z} \arrow[d, "\Phi_{\underline{z}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\underline{m}"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi F\underline{y} \arrow[r, "\Pi F\underline{n}"', "\shortmid"{marking}] \arrow[d, "\pi_j"] \arrow[dr, phantom, "\pi_b"] & \Pi F\underline{z} \arrow[d, "\pi_k"] \\
Fx_i \arrow[r, "Fm_a"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "F_{m_a,n_b}"] & Fy_j \arrow[r, "Fn_b"', "\shortmid"{marking}] & Fz_k \arrow[d, equal] \\
Fx_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & Fz_k
\end{tikzcd}$$

$$=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "F\pi_a"] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] \arrow[d, "F\pi_j"] \arrow[dr, phantom, "F\pi_b"] & F\Pi\underline{z} \arrow[d, "F\pi_k"] \\
Fx_i \arrow[r, "Fm_a"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "F_{m_a,n_b}"] & Fy_j \arrow[r, "Fn_b"', "\shortmid"{marking}] & Fz_k \arrow[d, equal] \\
Fx_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & Fz_k
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "F_{\Pi\underline{m},\Pi\underline{n}}"] & F\Pi\underline{y} \arrow[r, "F\Pi\underline{n}", "\shortmid"{marking}] & F\Pi\underline{z} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[rr, "F(\Pi\underline{m}\odot\Pi\underline{n})"', "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[drr, phantom, "F(\pi_a\odot\pi_b)"] & & F\Pi\underline{z} \arrow[d, "F\pi_k"] \\
Fx_i \arrow[rr, "F(m_a\odot n_b)"', "\shortmid"{marking}] & & Fz_k
\end{tikzcd},$$

where the last equation is the naturality of the laxators of F with respect to the projection cells π_a and π_b in $\mathbb{D}$. Thus, post-composite of Equation (11) with the projection $\pi_{(a,b)}$ in $\mathbb{E}$ is true for any choice of elements $i \overset{a}{\nrightarrow} j \overset{b}{\nrightarrow} k$. By the universal property of the product $\Pi F(\underline{m} \odot \underline{n})$, Equation (11) itself is true.

Turning to the unitors, fix an element $i \in I$. The post-composite of the left-hand side of Equation (12) with the projection π_i in $\mathbb{E}$ is

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{LHS}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & \Pi F\underline{x} \arrow[d, "\pi_i"] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{\Pi\underline{x}}"] & F\Pi\underline{x} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[r, "F\,\mathrm{id}_{\Pi\underline{x}}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F\Pi_{\underline{x}}"] & F\Pi\underline{x} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[r, "F\Pi\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "F\pi_i"] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{\Pi\underline{x}}"] & F\Pi\underline{x} \arrow[d, equal] \\
F\Pi\underline{x} \arrow[r, "F\,\mathrm{id}_{\Pi\underline{x}}"', "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "F\,\mathrm{id}_{\pi_i}"] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd},$$

where the last equality uses Equation (4). On the other hand, the post-composite with the right-hand side of Equation (12) is

$$
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{RHS}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi F\,\mathrm{id}_{\underline{x}}"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & \Pi F\underline{x} \arrow[d, "\pi_i"] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{id}_{\Phi_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\mathrm{id}_{\Pi F\underline{x}}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Pi_{F\underline{x}}"] & \Pi F\underline{x} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[r, "\Pi\,\mathrm{id}_{F\underline{x}}"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & \Pi F\underline{x} \arrow[d, "\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{x_i}"] & Fx_i \arrow[d, equal] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{id}_{\Phi_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\mathrm{id}_{\Pi F\underline{x}}"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\mathrm{id}_{\pi_i}"] & \Pi F\underline{x} \arrow[d, "\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{x_i}"] & Fx_i \arrow[d, equal] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "\mathrm{id}_{F\pi_i}"] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{x_i}"] & Fx_i \arrow[d, equal] \\
Fx_i \arrow[r, "F\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Fx_i
\end{tikzcd}
.
$$

But these are equal by the naturality of the unitors of F with respect to the projection arrow $\pi_i : \Pi\underline{x} \to x_i$ in $\mathbb{D}$. By the universal property of the product $\Pi F\,\mathrm{id}_{\underline{x}}$, Equation (12) holds. ■

An analogous lemma for lax functors between cartesian double categories has been used to study models of cartesian double theories [Lambert and Patterson, 2024, Lemma 5.2]. We will use Lemma 9.3 for the same purpose when we define finite-product double theories and their models in the next section.

10. Finite-product double theories

As an application of the tools developed so far, we extend the cartesian double theories recently proposed in [Lambert and Patterson, 2024] by allowing arbitrary finite products instead of only finite parallel products as in a cartesian double category. Double theories with finite products have greater expressivity, encompassing among other things an important class of enriched categories, yet require minimal modifications to the machinery producing a virtual double category of models. This development brings us full circle to Paré's early work on products in double categories [Paré, 2009], which was also motivated by double theories, albeit with different intended semantics and models.

10.1. Definition. [Finite-product double theories and models] *A **finite-product double theory** is a small, strict double category with strong finite products. A **morphism** of finite-product double theories is a strict double functor that preserves finite products.*

*A **model** of a finite-product double theory $\mathbb{T}$ in a double category $\mathbb{S}$ with lax finite products is a lax double functor $M : \mathbb{T} \to \mathbb{S}$ that preserves finite products. If left unstated, the semantics for a model is assumed to be $\mathbb{S} = \mathbb{S}\mathsf{pan}$.*

The name "finite-product double theory" is clearly intended to evoke the finite-product theories familiar from categorical logic. Indeed, they categorify that concept. The differing strengths of products in the syntax and the semantics is important. A typical semantics, such as spans or matrices, will have at most iso-strong products, whereas the theory must have strong products in order to fully control the comparison cells of its models, as shown by Lemma 9.3.

All of the cartesian double theories in [Lambert and Patterson, 2024] can be seen as finite-product double theories that happen to use only finite parallel products. The examples presented here take advantage of the extra flexibility afforded by arbitrary finite products. To avoid confusion with parallel products, we write local products using conjunctive notation, so that $m \wedge n : x \to y$ is the local product of two proarrows $m, n : x \to y$ with common source and target, and $\top \coloneqq \top_{x,y} : x \to y$ is the local terminal between two objects x, y. The same notation is used in the literature on cartesian bicategories [Carboni et al., 2008, §3.1].

10.2. Example. [Categories enriched in commutative monoids] The **theory of local commutative monoids** $\mathbb{T}_{\mathsf{lcMon}}$ is the finite-product double theory generated by a single object x and two globular cells, the **local multiplication** and **unit**

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x \wedge \mathrm{id}_x", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mu"] & x \arrow[d, equal] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} x \arrow[r, "\top", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & x \arrow[d, equal] \\ x \arrow[r, "\mathrm{id}_x"', "\shortmid"{marking}] & x \end{tikzcd},$$

subject to the usual equations of associativity, unitality, and commutativity:

$$\begin{tikzcd} \mathrm{id}_x \wedge \mathrm{id}_x \wedge \mathrm{id}_x \arrow[r, "\mu \wedge 1"] \arrow[d, "1 \wedge \mu"'] & \mathrm{id}_x \wedge \mathrm{id}_x \arrow[d, "\mu"] \\ \mathrm{id}_x \wedge \mathrm{id}_x \arrow[r, "\mu"'] & \mathrm{id}_x \end{tikzcd} \quad \begin{tikzcd} \top \wedge \mathrm{id}_x \arrow[r, "\eta \wedge 1"] \arrow[dr, "\cong"'] & \mathrm{id}_x \wedge \mathrm{id}_x \arrow[d, "\mu"'] & \mathrm{id}_x \wedge \top \arrow[l, "1 \wedge \eta"'] \arrow[dl, "\cong"] \\ & \mathrm{id}_x & \end{tikzcd} \quad \begin{tikzcd} \mathrm{id}_x \wedge \mathrm{id}_x \arrow[r, "\sigma_{\mathrm{id}_x, \mathrm{id}_x}"] \arrow[dr, "\mu"'] & \mathrm{id}_x \wedge \mathrm{id}_x \arrow[d, "\mu"] \\ & \mathrm{id}_x \end{tikzcd}.$$

We claim that a model of the theory of local commutative monoids, namely a finite-product-preserving lax functor $M : \mathbb{T}_{\mathsf{lcMon}} \to \mathbb{S}\mathsf{pan}$, is precisely a category enriched in commutative monoids. First, the data $(Mx, M\,\mathrm{id}_x, M_{x,x}, M_x)$, comprising the image of x and $\mathrm{id}_x : x \to x$ and corresponding laxator and unitor, is a category [Lambert and Patterson, 2024, §2], call it C. Moreover, Lemma 9.3 ensures that all of the other laxators and unitors of M are uniquely determined by products of $M_{x,x}$ and M_x. The images of μ and η under M are natural transformations [Lambert and Patterson, 2024, §2], conventionally written in additive notation as

$$+ : \mathsf{C}(a, b)^2 \to \mathsf{C}(a, b) \qquad \text{and} \qquad 0 : 1 \to \mathsf{C}(a, b)$$

for each pair of objects $a, b \in \mathsf{C}$. The three equational axioms on μ and η say that each hom-set of C is endowed with the structure of a commutative monoid. Finally, the naturality conditions say that for all morphisms $h : a' \to a$ and $k : b \to b'$ in C, the diagrams

$$\begin{tikzcd} \mathsf{C}(a,b)^2 \arrow[r, "+"] \arrow[d, "\mathsf{C}(h,k)^2"'] & \mathsf{C}(a,b) \arrow[d, "\mathsf{C}(h,k)"] \\ \mathsf{C}(a',b')^2 \arrow[r, "+"'] & \mathsf{C}(a',b') \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "0"'] & \mathsf{C}(a,b) \arrow[d, "\mathsf{C}(h,k)"] \\ & \mathsf{C}(a',b') \end{tikzcd}$$

commute, that is, for all morphisms $f, g : a \to b$ in C,

$$h \cdot (f+g) \cdot k = (h \cdot f \cdot k) + (h \cdot g \cdot k) \qquad \text{and} \qquad h \cdot 0_{a,b} \cdot k = 0_{a',b'}.$$

These equations are equivalent to the biadditivity that makes the category C be enriched in commutative monoids.

Enriched categories are usually defined for a base of enrichment that is a monoidal category or a multicategory. Perhaps surprisingly, the double theory in Example 10.2 makes no reference to the monoidal category of commutative monoids under its tensor product or to the multicategory of commutative monoids and multi-additive maps. Instead, the naturality requirements of a lax functor automatically ensure that composition is biadditive. That models of double theories are functorial and natural by construction is a key advantage compared with models of one-dimensional theories such as finite-limit theories.

The property of commutative monoids that makes Example 10.2 work is that the category of commutative monoids is algebraic. Recall that an **algebraic category** is a category concretely equivalent to the category of models of an algebraic theory, which we assume to be single-sorted; see [Adámek et al., 2010, Chapter 11] or [Borceux, 1994, Vol. 2, Chapter 3]. In general, given a commutative algebraic theory T, we can construct a finite-product double theory whose models are categories enriched in models of T. This includes the important example of categories enriched in abelian groups or, more generally, enriched in R-modules over a commutative ring R.

10.3. Construction. [Categories enriched in an algebraic category] Let T be an algebraic theory (not necessarily commutative) with generating object x. We define a finite-product double theory $\mathbb{T}$ generated by a single object, also denoted x, and by, for each morphism $f : x^m \to x^n$ in T, a globular cell $F(f) : \mathrm{id}_x^{\wedge m} \to \mathrm{id}_x^{\wedge n}$, where $\mathrm{id}_x^{\wedge n}$ denotes the n-fold local product $\mathrm{id}_x \wedge \cdots \wedge \mathrm{id}_x : x \nrightarrow x$ of the identity $\mathrm{id}_x : x \nrightarrow x$ with itself. We impose relations making the assignment $F : \mathsf{T} \to \mathbb{T}(x,x)$ into a finite-product-preserving functor from T into the hom-category $\mathbb{T}(x,x)$ of proarrows $x \nrightarrow x$ and globular cells. Abstractly, the double category $\mathbb{T}$ with finite products is the *delooping* of the category T with finite products.

When the algebraic theory T is commutative, its category of models, $\mathsf{Mod}_{\mathsf{T}}$, has a symmetric monoidal closed structure [Borceux, 1994, Vol. 2, Theorem 3.10.3]. Models of the finite-product double theory $\mathbb{T}$ are then equivalent to categories enriched over $\mathsf{Mod}_{\mathsf{T}}$, similarly to Example 10.2. In fact, as Borceux remarks, the hypothesis of commutativity is not actually needed to construct the tensor product on $\mathsf{Mod}_{\mathsf{T}}$, only to make it well behaved; nor, as we have already noted, is the tensor product used at all in constructing the double theory.

The enrichment construction can be combined with other double theories from [Lambert and Patterson, 2024] to present finite-product double theories whose models are, say, monoidal categories or multicategories enriched over an algebraic category. We leave these examples to the interested reader and turn to a rather different application of finite products: using restrictions of products along structure arrows (Theorem 8.6) to axiomatize several flavors of generalized multicategory [Pisani, 2014, §4], [Shulman, 2016].

10.4. Example. [Cartesian and symmetric multicategories] The **theory of promonoids** is the finite-product double theory generated by one object x and two proarrows, $m : x^2 \nrightarrow x$ and $j : 1 \nrightarrow x$, subject to axioms of associativity and unitality,

$$\begin{tikzcd} x^3 \arrow[r, "(p\times \mathrm{id}_x)", "\shortmid"{marking}] \arrow[d, "(\mathrm{id}_x \times p)"', "\shortmid"{marking}] & x^2 \arrow[d, "p", "\shortmid"{marking}] \\ x^2 \arrow[r, "p"', "\shortmid"{marking}] & x \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 1 \times x \arrow[r, "j\times \mathrm{id}_x", "\shortmid"{marking}] \arrow[dr, "\cong"', "\shortmid"{marking}] & x^2 \arrow[d, "p"', "\shortmid"{marking}] & x \times 1 \arrow[l, "\mathrm{id}_x \times j"', "\shortmid"{marking}] \arrow[dl, "\cong", "\shortmid"{marking}] \\ & x & \end{tikzcd} .$$

The axioms ensure that, for each arity $n \geq 0$, there is a unique proarrow $p_n : x^n \nrightarrow x$ built out of the generators p and j. For example, $p_1 = \mathrm{id}_x$ and $p_4 = (p \times p) \odot p$. So far we have simply recalled the cartesian double theory of promonoids [Lambert and Patterson, 2024, Theory 6.9] but we now regard it as a finite-product double theory. Models of the theory are equivalent to multicategories.

Let F be the skeleton of FinSet spanned by the sets $[n] := \{1, \ldots, n\}$ for each $n \geq 0$. The **theory of cartesian promonoids** is the finite-product double theory presented as the theory of promonoids augmented with, for each function $\sigma : [m] \to [n]$ in F, a globular cell

$$\begin{tikzcd} x^n \arrow[r, "x^\sigma_!", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho(\sigma)"] & x^m \arrow[r, "p_m", "\shortmid"{marking}] & x \arrow[d, equal] \\ x^n \arrow[rr, "p_n"', "\shortmid"{marking}] & & x \end{tikzcd} ,$$

called the **action by** σ, where $x^\sigma := \Pi(\sigma) : x^n \to x^m$ is the structure arrow between products induced by σ and $x^\sigma_! = \Pi(\sigma)_! : x^n \nrightarrow x^m$ is its companion (Corollary 8.4). The composite proarrow $p^\sigma_m := x^\sigma_! \odot p_m$ is thus the restriction of p_m along the arrows x^σ and 1_x. The action cells obey the following relations.

- Functorality of action: for every pair of composable maps $[m] \xrightarrow{\sigma} [n] \xrightarrow{\tau} [q]$ in F, we have

$$\begin{tikzcd}
x^q \arrow[r, "x_!^\tau", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "1"] & x^n \arrow[r, "x_!^\sigma", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho(\sigma)"] & x^m \arrow[r, "p_m", "\shortmid"{marking}] & x \arrow[d, equal] \\
x^q \arrow[r, "x_!^\tau"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\rho(\tau)"] & x^n \arrow[rr, "p_n"', "\shortmid"{marking}] & & x \arrow[d, equal] \\
x^q \arrow[rrr, "p_q"', "\shortmid"{marking}] & & & x
\end{tikzcd}
=
\begin{tikzcd}
x^q \arrow[r, "x_!^\tau", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & x^n \arrow[r, "x_!^\sigma", "\shortmid"{marking}] & x^m \arrow[r, "p_m", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "1"] & x \arrow[d, equal] \\
x^q \arrow[rr, "x_!^{\sigma\cdot\tau}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\rho(\sigma\cdot\tau)"] & & x^m \arrow[r, "p_m"', "\shortmid"{marking}] & x \arrow[d, equal] \\
x^q \arrow[rrr, "p_q"', "\shortmid"{marking}] & & & x
\end{tikzcd},$$

and for every $n \geq 0$, the cell $\rho(1_{[n]})$ is the canonical isomorphism $x_!^{1_{[n]}} \odot p_n \cong p_n$.

- Naturality of action (i): for all $k \geq 0$ and maps $\sigma_i : [m_i] \to [n_i]$, $1 \leq i \leq k$, in F, we have

$$\begin{tikzcd}
x^n \arrow[r, "p_{m_1}^{\sigma_1}\times\cdots\times p_{m_k}^{\sigma_k}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\rho(\sigma_1)\times\cdots\times\rho(\sigma_k)"] & x^k \arrow[r, "p_k", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "1_{p_k}"] & x \arrow[d, equal] \\
x^n \arrow[r, "p_{n_1}\times\cdots\times p_{n_k}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "1"] & x^k \arrow[r, "p_k"', "\shortmid"{marking}] & x \arrow[d, equal] \\
x^n \arrow[rr, "p_n"', "\shortmid"{marking}] & & x
\end{tikzcd}
=
\begin{tikzcd}
x^n \arrow[r, "p_{m_1}^{\sigma_1}\times\cdots\times p_{m_k}^{\sigma_k}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & x^k \arrow[r, "p_k", "\shortmid"{marking}] & x \arrow[d, equal] \\
x^n \arrow[rr, "p_m^{\sigma_1+\cdots+\sigma_k}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho(\sigma_1+\cdots+\sigma_k)"] & & x \arrow[d, equal] \\
x^n \arrow[rr, "p_n"', "\shortmid"{marking}] & & x
\end{tikzcd},$$

where we put $m \coloneqq m_1 + \cdots + m_k$ and $n \coloneqq n_1 + \cdots + n_k$ and the isomorphism on the right-hand side is built out of the canonical isomorphisms (Proposition 8.2)

$$x_!^{\sigma_1} \times \cdots \times x_!^{\sigma_k} \cong (x^{\sigma_1} \times \cdots \times x^{\sigma_k})_! \cong x_!^{\sigma_1+\cdots+\sigma_k}.$$

- Naturality of action (ii): for every map $\sigma : [k] \to [\ell]$ in F and all $n_j \geq 0$, $1 \leq j \leq \ell$, we have

$$\begin{tikzcd}
x^n \arrow[r, "p_{n_1}\times\cdots\times p_{n_\ell}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "1"] & x^\ell \arrow[r, "p_k^\sigma", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\rho(\sigma)"] & x \arrow[d, equal] \\
x^n \arrow[r, "p_{n_1}\times\cdots\times p_{n_\ell}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "1"] & x^\ell \arrow[r, "p_\ell"', "\shortmid"{marking}] & x \arrow[d, equal] \\
x^n \arrow[rr, "p_n"', "\shortmid"{marking}] & & x
\end{tikzcd}
=
\begin{tikzcd}
x^n \arrow[r, "p_{n_1}\times\cdots\times p_{n_\ell}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & x^\ell \arrow[r, "p_k^\sigma", "\shortmid"{marking}] & x \arrow[d, equal] \\
x^n \arrow[rr, "p_m^{\sigma\wr(n_1,\dots,n_\ell)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho(\sigma\wr(n_1,\dots,n_\ell))"] & & x \arrow[d, equal] \\
x^n \arrow[rr, "p_n"', "\shortmid"{marking}] & & x
\end{tikzcd},$$

where we put $m \coloneqq n_{\sigma 1} + \cdots + n_{\sigma k}$ and $n \coloneqq n_1 + \cdots + n_\ell$, and the map $\sigma\wr(n_1, \dots, n_\ell) : [m] \to [n]$ applies σ blockwise and is the identity within each block of size $n_{\sigma i}$.

The isomorphism on the right-hand side uses a canonical isomorphism between products obtained as follows. The family of proarrows $p_{n_j} : x^{n_j} \nrightarrow x$, $1 \leq j \leq \ell$, indexed by the identity span $\mathrm{id}_{[\ell]}$, and the family of identity proarrows id_x, indexed by the span $\sigma^* = ([\ell] \xleftarrow{\sigma} [k] = [k])$, have as their composite the family $p_{n_{\sigma i}} : x^{n_{\sigma i}} \nrightarrow x$, $1 \leq i \leq k$, also indexed by the span σ^*. But the same is true for the family of identity proarrows $\mathrm{id}_{x^{n_{\sigma i}}} \cong \mathrm{id}_x^{n_{\sigma i}}$, $i \leq 1 \leq k$, indexed by σ^*, composed with the proarrow family $p_{n_{\sigma i}} : x^{n_{\sigma i}} \nrightarrow x$, $1 \leq i \leq k$, indexed by the identity span $\mathrm{id}_{[k]}$. Since a finite-product double theory has *strong* finite products, we obtain a canonical isomorphism

$$(p_{n_1} \times \cdots \times p_{n_\ell}) \odot x_!^{\sigma} \cong x_!^{\sigma\wr(n_1,\ldots,n_\ell)} \odot (p_{n_{\sigma 1}} \times \cdots \times p_{n_{\sigma k}}), \tag{13}$$

which, by composing with the identity 1_{p_k} on the right, yields the isomorphism used in the right-hand side.

A model of the theory of cartesian promonoids is equivalent to a cartesian multicategory. This follows by reasoning similar to, but simpler than, that in [Lambert and Patterson, 2024, Proposition 6.21], now appealing to Lemma 9.3 to control the model's laxators for companions.

The **theory of symmetric promonoids** is defined in the same way except that the skeleton of FinSet is replaced by its core, the skeleton of the groupoid of finite sets and bijections. Models of the theory of symmetric promonoids are equivalent to symmetric multicategories.

10.5. Remark. [Comparison with restriction sketches] Possessing restrictions of products along structure arrows, finite-product double theories subsume some, though not all, of the intended applications of "restriction sketches," a provisional notion introduced in [Lambert and Patterson, 2024, Definition 6.17]. While, unlike an equipment or a restriction sketch, finite-product double theories do not allow restriction along *arbitrary* arrows, they have the advantage over equipments of not producing uncontrolled laxators, as the additional laxators here are uniquely determined by Lemma 9.3. And they have the advantage over restriction sketches of not needing to manually impose extra equations that should follow from properties of products and restrictions; compare the *derived* Equation (13) above with the *postulated* [Lambert and Patterson, 2024, Equations 6.3 & 6.4].

11. Higher morphisms and double categories of models

If models of finite-product double theories are product-preserving lax functors, then morphisms between models, and cells between those, ought to be the higher morphisms between lax functors that also preserve finite products. In this final section, we determine the sense in which lax natural transformations and modules between lax functors, and modulations between squares of those, should or already do preserve products. The definitions and lemmas are all more or less obvious, following the pattern of Lemma 9.3 on lax functors and having counterparts for cartesian double categories in previous work [Lambert and Patterson, 2024]. Nevertheless, it is important to state and prove these results, as they ensure that the morphisms and cells between models of finite-product double theories behave as intended and recover the expected notions in examples of interest. We will conclude by showing that these higher morphisms assemble into a unital virtual double category, and in particular a 2-category, of models of a finite-product double theory.

Natural transformations between product-preserving lax double functors require no extra assumptions, just as ordinary natural transformations between product-preserving functors do not. However, *lax* natural transformations between double functors, introduced in [Lambert and Patterson, 2024, §7] to generalize lax natural transformations between 2-functors, need an extra axiom that make products be respected *strictly*.

11.1. Definition. [Product-preserving lax transformation] *Let $F, G : \mathbb{D} \to \mathbb{E}$ be lax functors between double categories with lax products. A lax natural transformation $\alpha : F \Rightarrow G$* ***preserves products*** *if it is* strictly *natural with respect to projections, meaning that for every family of objects $(I, \underline{x})$ in $\mathbb{D}$ and every $i \in I$, the square*

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "F\pi_i"] \arrow[d, "\alpha_{\Pi\underline{x}}"'] & Fx_i \arrow[d, "\alpha_{x_i}"] \\
G\Pi\underline{x} \arrow[r, "G\pi_i"'] & Gx_i
\end{tikzcd}$$

of arrows in $\mathbb{E}$ commutes, and the naturality comparison for the projection π_i has the form

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\alpha_{\Pi\underline{x}}"'] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
G\Pi\underline{x} \arrow[r, phantom, "\alpha_{\pi_i}"] \arrow[d, "G\pi_i"'] & Fx_i \arrow[d, "\alpha_{x_i}"] \\
Gx_i \arrow[r, "G\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Gx_i
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marking}] \arrow[d, "\alpha_{\Pi\underline{x}}"'] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
G\Pi\underline{x} \arrow[r, phantom, "\mathrm{id}"] \arrow[d, "G\pi_i"'] & Fx_i \arrow[d, "\alpha_{x_i}"] \\
Gx_i \arrow[r, "\mathrm{id}_{Gx_i}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "G_{x_i}"] & Gx_i \arrow[d, equal] \\
Gx_i \arrow[r, "G\,\mathrm{id}_{x_i}"', "\shortmid"{marking}] & Gx_i
\end{tikzcd} .$$

Lax natural transformation that restrict to strict natural transformations on product projections have appeared previously in the literature on two-dimensional categorical logic [Bourke, 2021, Example 9.1]. The $\mathcal{F}$-natural transformations from $\mathcal{F}$-category theory [Lack and Shulman, 2012, §4.1] are a general tool to handle such situations.

The next lemma generalizes [Lambert and Patterson, 2024, Lemma 7.11] on the naturality comparisons of a cartesian lax natural transformation.

11.2. Lemma. [Naturality comparisons for products] *Let $F, G : \mathbb{D} \to \mathbb{E}$ be lax functors between double categories with lax products, and let $\alpha : F \Rightarrow G$ be a product-preserving* *lax transformation. Then for every arrow $(f_0, f) : (I, \underline{x}) \to (J, \underline{y})$ in $\mathbb{F}\mathsf{am}^*(\mathbb{D})$, we have*

$$
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid" marking] \arrow[d, "\alpha_{\Pi\underline{x}}"'] & F\Pi\underline{x} \arrow[d, "F\Pi(f_0,f)"] \\
G\Pi\underline{x} \arrow[r, phantom, "\alpha_{\Pi(f_0,f)}"] \arrow[d, "G\Pi(f_0,f)"'] & F\Pi\underline{y} \arrow[d, "\alpha_{\Pi\underline{y}}"] \\
G\Pi\underline{y} \arrow[r, "G\,\mathrm{id}_{\Pi\underline{y}}", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "G\Pi_{\underline{y}}"] & G\Pi\underline{y} \arrow[d, equal] \\
G\Pi\underline{y} \arrow[r, "G\Pi\,\mathrm{id}_{\underline{y}}", "\shortmid" marking] \arrow[d, "\Psi_{\underline{y}}"'] \arrow[dr, phantom, "\Psi_{\mathrm{id}_{\underline{y}}}"] & G\Pi\underline{y} \arrow[d, "\Psi_{\underline{y}}"] \\
\Pi G\underline{y} \arrow[r, "\Pi G\,\mathrm{id}_{\underline{y}}"', "\shortmid" marking] & \Pi G\underline{y}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid" marking] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{id}_{\Phi_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\mathrm{id}_{\Pi F\underline{x}}"', "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Pi_{F\underline{x}}"] & \Pi F\underline{x} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[r, "\Pi\,\mathrm{id}_{F\underline{x}}"', "\shortmid" marking] \arrow[d, "\Pi\alpha_{\underline{x}}"'] & \Pi F\underline{x} \arrow[d, "\Pi(f_0,Ff)"] \\
\Pi G\underline{x} \arrow[r, phantom, "\Pi(f_0,\alpha_f)"] \arrow[d, "\Pi(f_0,Gf)"'] & \Pi F\underline{y} \arrow[d, "\Pi\alpha_{\underline{y}}"] \\
\Pi G\underline{y} \arrow[r, "\Pi G\,\mathrm{id}_{\underline{y}}"', "\shortmid" marking] & \Pi G\underline{y}
\end{tikzcd}
\quad , \tag{14}
$$

where α_f denotes the family of naturality comparisons α_{f_j} for the arrows $f_j : x_{f_0(j)} \to y_j$, $j \in J$.

In particular, when $\mathbb{D}$ has normal lax products and G preserves products of identities, the naturality comparison for the product arrow $\Pi(f_0, f) : \Pi\underline{x} \to \Pi\underline{y}$ is uniquely determined by the product $\Pi(f_0, \alpha_f)$ of naturality comparisons.

Proof. Fixing an element $j \in J$, the post-composite of the left-hand side of Equation (14) with the projection π_j in $\mathbb{E}$ is

$$
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\mathrm{LHS}"] & F\Pi\underline{x} \arrow[d] \\
\Pi G\underline{y} \arrow[r, "\Pi G\,\mathrm{id}_{\underline{y}}"', "\shortmid" marking] \arrow[d, "\pi_j"'] \arrow[dr, phantom, "\pi_j"] & \Pi G\underline{y} \arrow[d, "\pi_j"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid" marking] & Gy_j
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha_{\Pi(f_0,f)}"] & F\Pi\underline{x} \arrow[d] \\
G\Pi\underline{y} \arrow[r, "G\,\mathrm{id}_{\Pi\underline{y}}", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "G\Pi_{\underline{y}}"] & G\Pi\underline{y} \arrow[d, equal] \\
G\Pi\underline{y} \arrow[r, "G\Pi\,\mathrm{id}_{\underline{y}}", "\shortmid" marking] \arrow[d, "G\pi_j"'] \arrow[dr, phantom, "G\pi_j"] & G\Pi\underline{y} \arrow[d, "G\pi_j"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid" marking] & Gy_j
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid" marking] \arrow[d, "\alpha_{\Pi\underline{x}}"'] & F\Pi\underline{x} \arrow[d, "F\Pi(f_0,f)"] \\
G\Pi\underline{x} \arrow[r, phantom, "\alpha_{\Pi(f_0,f)}"] \arrow[d, "G\Pi(f_0,f)"'] & F\Pi\underline{y} \arrow[d, "\alpha_{\Pi\underline{y}}"] \\
G\Pi\underline{y} \arrow[r, "G\,\mathrm{id}_{\Pi\underline{y}}", "\shortmid" marking] \arrow[d, "G\pi_j"'] \arrow[dr, phantom, "G\,\mathrm{id}_{\pi_j}"] & G\Pi\underline{y} \arrow[d, "G\pi_j"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid" marking] & Gy_j
\end{tikzcd}
\quad .
$$

Setting $i \coloneqq f_0(j) \in I$ and using the equations

$$
\begin{tikzcd}
\Pi\underline{x} \arrow[r, "\Pi(f_0,f)"] \arrow[d, "\pi_i=\pi_{f_0(j)}"'] & \Pi\underline{y} \arrow[d, "\pi_j"] \\
x_{f_0(j)} \arrow[r, "f_j"'] & y_j
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Pi\,\mathrm{id}_{F\underline{x}} \arrow[r, "\Pi(f_0,\alpha_f)"] \arrow[d, "\pi_i=\pi_{f_0(j)}"'] & \Pi G\,\mathrm{id}_{\underline{y}} \arrow[d, "\pi_j"] \\
\mathrm{id}_{Fx_i} \arrow[r, "\alpha_{f_j}"'] & G\,\mathrm{id}_{y_j}
\end{tikzcd}
\tag{15}
$$

in $\mathbb{E}_0$ and $\mathbb{E}_1$, the post-composite of the right-hand side of Equation (14) with π_j is

$$
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marker}] \arrow[d] \arrow[dr, phantom, "\text{RHS}"] & F\Pi\underline{x} \arrow[d] \\
\Pi G\underline{y} \arrow[r, "\Pi G\,\mathrm{id}_{\underline{y}}"', "\shortmid"{marker}] \arrow[d, "\pi_j"'] \arrow[dr, phantom, "\pi_j"] & \Pi G\underline{y} \arrow[d, "\pi_j"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid"{marker}] & Gy_j
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marker}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{id}_{\Phi_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\mathrm{id}_{\Pi F\underline{x}}"', "\shortmid"{marker}] \arrow[d, equal] \arrow[dr, phantom, "\Pi_{F\underline{x}}"] & \Pi F\underline{x} \arrow[d, equal] \\
\Pi F\underline{x} \arrow[r, "\Pi\,\mathrm{id}_{F\underline{x}}"', "\shortmid"{marker}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_i"] & \Pi F\underline{x} \arrow[d, "\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marker}] \arrow[d] \arrow[dr, phantom, "\alpha_{f_j}"] & Fx_i \arrow[d] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid"{marker}] & Gy_j
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marker}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{id}_{\Phi_{\underline{x}}}"] & F\Pi\underline{x} \arrow[d, "\Phi_{\underline{x}}"] \\
\Pi F\underline{x} \arrow[r, "\mathrm{id}_{\Pi F\underline{x}}"', "\shortmid"{marker}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\mathrm{id}_{\pi_i}"] & \Pi F\underline{x} \arrow[d, "\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marker}] \arrow[d] \arrow[dr, phantom, "\alpha_{f_j}"] & Fx_i \arrow[d] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid"{marker}] & Gy_j
\end{tikzcd}
=
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marker}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "\mathrm{id}_{F\pi_i}"] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marker}] \arrow[d, "\alpha_{x_i}"'] & Fx_i \arrow[d, "Ff_j"] \\
Gx_i \arrow[r, phantom, "\alpha_{f_j}"] \arrow[d, "Gf_j"'] & Fy_j \arrow[d, "\alpha_{y_j}"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid"{marker}] & Gy_j
\end{tikzcd}
.
$$

But, applying the functorality of the naturality comparisons to the first square in Equation (15) and using the assumption that α is strictly natural with respect to projections, we also have

$$
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marker}] \arrow[d, "\alpha_{\Pi\underline{x}}"'] & F\Pi\underline{x} \arrow[d, "F\Pi(f_0,f)"] \\
G\Pi\underline{x} \arrow[r, phantom, "\alpha_{\Pi(f_0,f)}"] \arrow[d, "G\Pi(f_0,f)"'] & F\Pi\underline{y} \arrow[d, "\alpha_{\Pi\underline{y}}"] \\
G\Pi\underline{y} \arrow[r, "G\,\mathrm{id}_{\Pi\underline{y}}", "\shortmid"{marker}] \arrow[d, "G\pi_j"'] \arrow[dr, phantom, "G\,\mathrm{id}_{\pi_j}"] & G\Pi\underline{y} \arrow[d, "G\pi_j"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid"{marker}] & Gy_j
\end{tikzcd}
= \alpha_{\pi_j \circ \Pi(f_0,f)} = \alpha_{f_j \circ \pi_i} =
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "\mathrm{id}_{F\Pi\underline{x}}", "\shortmid"{marker}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "\mathrm{id}_{F\pi_i}"] & F\Pi\underline{x} \arrow[d, "F\pi_i"] \\
Fx_i \arrow[r, "\mathrm{id}_{Fx_i}"', "\shortmid"{marker}] \arrow[d, "\alpha_{x_i}"'] & Fx_i \arrow[d, "Ff_j"] \\
Gx_i \arrow[r, phantom, "\alpha_{f_j}"] \arrow[d, "Gf_j"'] & Fy_j \arrow[d, "\alpha_{y_j}"] \\
Gy_j \arrow[r, "G\,\mathrm{id}_{y_j}"', "\shortmid"{marker}] & Gy_j
\end{tikzcd}
.
$$

We have proved that the post-composite of Equation (14) with any projection π_j is true, hence the equation itself is true by the universal property of the product. ∎

We now turn to modules between lax double functors, introduced by Paré while developing the Yoneda theory of double categories [Paré, 2011]. A **module** between lax functors $F, G : \mathbb{D} \to \mathbb{E}$ can be succinctly defined as a lax functor $M : \mathbb{D} \times \mathbb{I} \to \mathbb{E}$, where $\mathbb{I} := \{0 \nrightarrow 1\}$ is the walking proarrow, such that $M(-,0) = F$ and $M(-,1) = G$. The definition is fully articulated in [Paré, 2011, §3.1] or [Lambert and Patterson, 2024, Definition 9.1]. Since a module is a kind of lax functor, a product-preserving module can be defined very similarly to a product-preserving lax functor (Definition 9.1), extending our previous definition of a cartesian module [Lambert and Patterson, 2024, Definition 9.4].

11.3. Definition. [Product-preserving module] *Let $F, G : \mathbb{D} \to \mathbb{E}$ be lax functors between double categories with lax products. A module $M : F \Rrightarrow G$* ***preserves products*** *if, for* *every family of proarrows* $(A, \underline{m}) : (I, \underline{x}) \mathrel{\mapstochar\rightarrow} (J, \underline{y})$ *in* $\mathbb{D}$*, the canonical comparison cell*

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{M}_{\underline{m}}"] & G\Pi\underline{y} \arrow[d, "\Psi_{\underline{y}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi M\underline{m}"', "\shortmid"{marking}] & \Pi G\underline{y}
\end{tikzcd},$$

defined by the equations

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{M}_{\underline{m}}"] & G\Pi\underline{y} \arrow[d, "\Psi_{\underline{y}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi M\underline{m}"', "\shortmid"{marking}] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\pi_a"] & \Pi G\underline{y} \arrow[d, "\pi_j"] \\
Fx_i \arrow[r, "Mm_a"', "\shortmid"{marking}] & Gy_j
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "M\pi_a"] & G\Pi\underline{y} \arrow[d, "G\pi_j"] \\
Fx_i \arrow[r, "Mm_a"', "\shortmid"{marking}] & Gy_j
\end{tikzcd}, \qquad (i \overset{a}{\mathrel{\mapstochar\rightarrow}} j) : (I \overset{A}{\mathrel{\mapstochar\rightarrow}} J), \tag{16}$$

is an isomorphism in $\mathbb{E}_1$.

For a module $M : F \mathrel{\mapstochar\Rightarrow} G$ between lax double functors to preserve products, it is clearly necessary that the underlying functors $F_0, G_0 : \mathbb{D}_0 \to \mathbb{E}_0$ preserve products. One typically assumes that the double functors $F, G : \mathbb{D} \to \mathbb{E}$ in their entirety preserve products.

A square of compatible transformations and modules are bound together by a *modulation*, defined for strict transformations in [Paré, 2011, §3.3] and for lax transformations in [Lambert and Patterson, 2024, Definition 9.7]. As a top-dimensional cell, modulations automatically preserve products.

11.4. Lemma. [Modulation components for products] *Let* $\mathbb{D}$ *and* $\mathbb{E}$ *be double categories with lax products, let* $F, G, H, K : \mathbb{D} \to \mathbb{E}$ *be lax functors, let* $\alpha : F \Rightarrow H$ *and* $\beta : G \Rightarrow K$ *be product-preserving lax transformations, let* $M : F \mathrel{\mapstochar\Rightarrow} G$ *and* $N : H \mathrel{\mapstochar\Rightarrow} K$ *be modules, and finally let*

$$\begin{tikzcd}
F \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\mu"] & G \arrow[d, "\beta"] \\
H \arrow[r, "N"', "\shortmid"{marking}] & K
\end{tikzcd}$$

be a modulation. Then, for any family of proarrows $(A, \underline{m}) : (I, \underline{x}) \mathrel{\mapstochar\rightarrow} (J, \underline{y})$ *in* $\mathbb{D}$*, we have*

$$\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\alpha_{\Pi\underline{x}}"'] \arrow[dr, phantom, "\mu_{\Pi\underline{m}}"] & G\Pi\underline{y} \arrow[d, "\beta_{\Pi\underline{y}}"] \\
H\Pi\underline{x} \arrow[r, "N\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\mathrm{H}_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{N}_{\underline{m}}"] & K\Pi\underline{y} \arrow[d, "\mathrm{K}_{\underline{y}}"] \\
\Pi H\underline{x} \arrow[r, "\Pi N\underline{m}"', "\shortmid"{marking}] & \Pi K\underline{y}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\Phi_{\underline{x}}"'] \arrow[dr, phantom, "\mathrm{M}_{\underline{m}}"] & G\Pi\underline{y} \arrow[d, "\Psi_{\underline{y}}"] \\
\Pi F\underline{x} \arrow[r, "\Pi M\underline{m}", "\shortmid"{marking}] \arrow[d, "\Pi\alpha_{\underline{x}}"'] \arrow[dr, phantom, "\Pi\mu_{\underline{m}}"] & \Pi G\underline{y} \arrow[d, "\Pi\beta_{\underline{y}}"] \\
\Pi H\underline{x} \arrow[r, "\Pi N\underline{m}"', "\shortmid"{marking}] & \Pi K\underline{y}
\end{tikzcd}, \tag{17}$$

where $(1_A, \mu_{\underline{m}})$ *is the cell in* $\mathbb{F}\mathsf{am}^*(\mathbb{E})$ *with components*

$$\begin{tikzcd} Fx_i \arrow[r, "Mm_a", "\shortmid"{marking}] \arrow[d, "\alpha_{x_i}"'] \arrow[dr, phantom, "\mu_{m_a}"] & Gy_j \arrow[d, "\beta_{y_j}"] \\ Hx_i \arrow[r, "Nm_a"', "\shortmid"{marking}] & Ky_j \end{tikzcd}, \qquad (i \overset{a}{\nrightarrow} j) : (I \overset{A}{\nrightarrow} J).$$

In particular, when the module $N : H \Rightarrow K$ *preserves products, the component of the modulation* μ *at the product* $\Pi\underline{m}$ *is uniquely determined by the product of the components at* m_a *for each* $a \in A$.

Proof. Fixing an element $a : i \nrightarrow j$ of the indexing span, the naturality of the modulation μ applied to the projection cell $\pi_a : \Pi\underline{m} \to m_a$ in $\mathbb{D}$ is the equation

$$\begin{tikzcd} F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "\alpha_{\Pi\underline{x}}"'] \arrow[dr, phantom, "\mu_{\Pi\underline{m}}"] & G\Pi\underline{y} \arrow[d, "\beta_{\Pi\underline{y}}"] \\ H\Pi\underline{x} \arrow[r, "N\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "H\pi_i"'] \arrow[dr, phantom, "N\pi_a"] & K\Pi\underline{y} \arrow[d, "K\pi_j"] \\ Hx_i \arrow[r, "Nm_a"', "\shortmid"{marking}] & Ky_j \end{tikzcd} = \begin{tikzcd} F\Pi\underline{x} \arrow[r, "M\Pi\underline{m}", "\shortmid"{marking}] \arrow[d, "F\pi_i"'] \arrow[dr, phantom, "M\pi_a"] & G\Pi\underline{y} \arrow[d, "G\pi_j"] \\ Fx_i \arrow[r, "Mm_a", "\shortmid"{marking}] \arrow[d, "\alpha_{x_i}"'] \arrow[dr, phantom, "\mu_{m_a}"] & Gy_j \arrow[d, "\beta_{y_j}"] \\ Hx_i \arrow[r, "Nm_a"', "\shortmid"{marking}] & Ky_j \end{tikzcd},$$

where we have also used the assumption that the lax transformations α and β preserve products. But, by Equation (16), the equation above is also the post-composite of Equation (17) with the projection cell $\pi_a : \Pi N\underline{m} \to Nm_a$ in $\mathbb{E}$. So, by the universal property of the product $\Pi N\underline{m}$, the claimed Equation (17) holds. ■

Since lax double functors are, in general, the objects of a merely virtual double category, we must also consider the multi-input generalization of a modulation. *Multimodulations* were defined for strict transformations in [Paré, 2011, §4.1] and for lax transformations in [Lambert and Patterson, 2024, Definition 10.1]. The following lemma on multimodulations is a straightforward generalization of Lemma 11.4 on modulations. We include it not just for completeness but to emphasize the role by played by strong products in the domain, irrelevant in the unary case.

11.5. Lemma. [Multimodulation components for products] *Let* $\mathbb{D}$ *and* $\mathbb{E}$ *be double categories with lax products and, for any* $k \geq 0$, *let* $F_0, \dots, F_k$, G, *and* H *be lax functors from* $\mathbb{D}$ *to* $\mathbb{E}$. *Suppose given a multimodulation*

$$\begin{tikzcd} F_0 \arrow[r, "M_1", "\shortmid"{marking}] \arrow[d, "\alpha"'] & \cdots \arrow[r, "M_k", "\shortmid"{marking}] \arrow[d, phantom, "\mu"] & F_k \arrow[d, "\beta"] \\ G \arrow[rr, "N"', "\shortmid"{marking}] & & H \end{tikzcd},$$

where α and β are product-preserving lax natural transformations. Then, for any length-k sequence of proarrow families $(I_0, \underline{x}_0) \overset{(A_1,\underline{m}_1)}{\nrightarrow} \cdots \overset{(A_k,\underline{m}_k)}{\nrightarrow} (I_k, \underline{x}_k)$ in $\mathbb{D}$, we have

$$
\begin{tikzcd}
F_0\Pi\underline{x}_0 \arrow[r, "M_1\Pi\underline{m}_1", "\shortmid"{marking}] \arrow[d, "\alpha_{\Pi\underline{x}_0}"'] & \cdots \arrow[r, "M_k\Pi\underline{m}_k", "\shortmid"{marking}] \arrow[d, phantom, "\mu_{\Pi\underline{m}_1,\ldots,\Pi\underline{m}_k}"] & F_k\Pi\underline{x}_k \arrow[d, "\beta_{\Pi\underline{x}_k}"] \\
G\Pi\underline{x}_0 \arrow[rr, "N(\Pi\underline{m}_1\odot\cdots\odot\Pi\underline{m}_k)"', "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, phantom, "N\Pi_{\underline{m}_1,\ldots,\underline{m}_k}"] & H\Pi\underline{x}_k \arrow[d, equal] \\
G\Pi\underline{x}_0 \arrow[rr, "N\Pi(\underline{m}_1\odot\cdots\odot\underline{m}_k)"', "\shortmid"{marking}] \arrow[d, "\Psi_{\underline{x}_0}"'] & {} \arrow[d, phantom, "\mathrm{N}_{\underline{m}_1\odot\cdots\odot\underline{m}_k}"] & H\Pi\underline{x}_k \arrow[d, "\mathrm{H}_{\underline{x}_k}"] \\
\Pi G\underline{x}_0 \arrow[rr, "\Pi N(\underline{m}_1\odot\cdots\odot\underline{m}_k)"', "\shortmid"{marking}] & & \Pi H\underline{x}_k
\end{tikzcd}
=
\begin{tikzcd}
F_0\Pi\underline{x}_0 \arrow[r, "M_1\Pi\underline{m}_1", "\shortmid"{marking}] \arrow[d, "(\Phi_0)_{\underline{x}_0}"'] \arrow[dr, phantom, "(\mathrm{M}_1)_{\underline{m}_1}"] & \cdots \arrow[r, "M_k\Pi\underline{m}_k", "\shortmid"{marking}] \arrow[dr, phantom, "(\mathrm{M}_k)_{\underline{m}_k}"] & F_k\Pi\underline{x}_k \arrow[d, "(\Phi_k)_{\underline{x}_k}"] \\
\Pi F_0\underline{x}_0 \arrow[r, "\Pi M_1\underline{m}_1"', "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "\Pi M_k\underline{m}_k"', "\shortmid"{marking}] \arrow[d, phantom, "\Pi_{M_1\underline{m}_1,\ldots,M_k\underline{m}_k}"] & \Pi F_k\underline{x}_k \arrow[d, equal] \\
\Pi F_0\underline{x}_0 \arrow[rr, "\Pi(M_1\underline{m}_1\odot\cdots\odot M_k\underline{m}_k)"', "\shortmid"{marking}] \arrow[d, "\Pi\alpha_{\underline{x}_0}"'] & {} \arrow[d, phantom, "\Pi\mu_{\underline{m}_1,\ldots,\underline{m}_k}"] & \Pi F_k\underline{x}_k \arrow[d, "\Pi\beta_{\underline{x}_k}"] \\
\Pi G\underline{x}_0 \arrow[rr, "\Pi N(\underline{m}_1\odot\cdots\odot\underline{m}_k)"', "\shortmid"{marking}] & & \Pi H\underline{x}_0
\end{tikzcd}
. \tag{18}
$$

In particular, when the domain double category $\mathbb{D}$ has strong products and the module N preserves products, the component of the multimodulation μ at the products $\Pi\underline{m}_1, \ldots, \Pi\underline{m}_k$ is uniquely determined by the product of the components at $m_{1,a_1}, \ldots, m_{k,a_k}$ indexed by $i_0 \overset{a_0}{\nrightarrow} \cdots \overset{a_k}{\nrightarrow} i_k$.

Proof. Fixing elements $i_0 \overset{a_0}{\nrightarrow} \cdots \overset{a_k}{\nrightarrow} i_k$ of the indexing spans $I_0 \overset{A_0}{\nrightarrow} \cdots \overset{A_k}{\nrightarrow} I_k$, the naturality of the multimodulation μ at the projection cells $\pi_{a_1} : \Pi\underline{m}_1 \to m_{1,a_1}$ through $\pi_{a_k} : \Pi\underline{m}_k \to m_{k,a_k}$ in $\mathbb{D}$ is the equation

$$
\begin{tikzcd}
F_0\Pi\underline{x}_0 \arrow[r, "M_1\Pi\underline{m}_1", "\shortmid"{marking}] \arrow[d, "\alpha_{\Pi\underline{x}_0}"'] & \cdots \arrow[r, "M_k\Pi\underline{m}_k", "\shortmid"{marking}] \arrow[d, phantom, "\mu_{\Pi\underline{m}_1,\ldots,\Pi\underline{m}_k}"] & F_k\Pi\underline{x}_k \arrow[d, "\beta_{\Pi\underline{x}_k}"] \\
G\Pi\underline{x}_0 \arrow[rr, "N(\Pi\underline{m}_1\odot\cdots\odot\Pi\underline{m}_k)"', "\shortmid"{marking}] \arrow[d, "G\pi_{i_0}"'] & {} \arrow[d, phantom, "N(\pi_{a_1}\odot\cdots\odot\pi_{a_k})"] & H\Pi\underline{x}_k \arrow[d, "H\pi_{i_k}"] \\
Gx_{0,i_0} \arrow[rr, "N(m_{1,a_1}\odot\cdots\odot m_{k,a_k})"', "\shortmid"{marking}] & & Hx_{k,i_k}
\end{tikzcd}
=
\begin{tikzcd}
F_0\Pi\underline{x}_0 \arrow[r, "M_1\Pi\underline{m}_1", "\shortmid"{marking}] \arrow[d, "F_0\pi_{i_0}"'] \arrow[dr, phantom, "M_1\pi_{a_1}"] & \cdots \arrow[r, "M_k\Pi\underline{m}_k", "\shortmid"{marking}] \arrow[dr, phantom, "M_k\pi_{a_k}"] & F_k\Pi\underline{x}_k \arrow[d, "F_k\pi_{i_k}"] \\
F_0x_{0,i_0} \arrow[r, "M_1m_{1,a_1}"', "\shortmid"{marking}] \arrow[d, "\alpha_{x_{0,i_0}}"'] & \cdots \arrow[r, "M_km_{k,a_k}"', "\shortmid"{marking}] \arrow[d, phantom, "\mu_{m_{1,a_1},\ldots,m_{k,a_k}}"] & F_kx_{k,i_k} \arrow[d, "\beta_{x_{k,a_k}}"] \\
Gx_{0,i_0} \arrow[rr, "N(m_{1,a_1}\odot\cdots\odot m_{k,a_k})"', "\shortmid"{marking}] & & Hx_{k,i_k}
\end{tikzcd}
.
$$

But this equation is the post-composite of Equation (18) with the projection cell

$$\pi_{(a_1,\ldots,a_k)} : \Pi N(\underline{m}_1 \odot \cdots \odot \underline{m}_k) \to N(m_{1,a_1} \odot \cdots \odot m_{k,a_k})$$

in $\mathbb{E}$. By the universal property of the product, Equation (18) holds. ∎

As anticipated, for two fixed double categories with lax products, the product-preserving lax functors and higher morphisms between them assemble into a unital virtual double category. The preceding lemmas are not needed to prove this, only to ensure that the construction is well behaved. For the definitions of virtual double categories and units in them, see [Cruttwell and Shulman, 2010].

11.6. THEOREM. [Virtual double category of product-preserving lax functors] *Let $\mathbb{D}$ and $\mathbb{E}$ be double categories with lax products. There is a virtual double category with units whose*

- *objects are product-preserving lax functors from $\mathbb{D}$ to $\mathbb{E}$,*
- *arrows are either (strict) natural transformations or product-preserving pseudo, lax, or oplax natural transformations,*
- *proarrows are product-preserving modules, and*
- *multicells are multimodulations.*

PROOF. For fixed double categories $\mathbb{D}$ and $\mathbb{E}$, there is a unital virtual double category of lax functors $\mathbb{D} \to \mathbb{E}$, natural transformations, modules, and multimodulations, for any choice of laxness in the natural transformations. This result was stated for strict transformations in [Paré, 2011, Theorem 4.3] and proved for pseudo or (op)lax transformations in [Lambert and Patterson, 2024, Theorem 10.3]. So we need only check that the product-preserving arrows and proarrows are closed under the operations of a unital virtual double category. Product-preserving lax transformations are closed under composition, by pasting the naturality squares for projections, and the identity lax transformation arises from the identity strict transformation, so certainly preserves products. Finally, the identity module id_F on a lax functor F preserves products if and only if F does. ■

11.7. COROLLARY. [Virtual double category of models] *Every finite-product double theory has a unital virtual double category of models, for any choice of strict, pseudo, or (op)lax maps of models.*

For instance, the theory of local commutative monoids (Example 10.2) gives rise to the unital virtual double category of categories, functors, profunctors, and natural transformations, all enriched in commutative monoids. In this case, the strict and lax maps of models coincide since the only arrows in the theory are the structure arrows between products. The verification of these claims uses the lemmas proved in this section. More generally, for any category $\mathcal{V}$ of models of a commutative algebraic theory, Construction 10.3 gives the unital virtual double category of $\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-profunctors, and $\mathcal{V}$-natural transformations.

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Email: evan@epatters.org

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