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THE SNAIL LEMMA AND THE LONG HOMOLOGY SEQUENCE

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Abstract. In the first part of the paper, we establish a version of the snail lemma (which is a generalization of the classical snake lemma) for categories with a structure of nullhomotopies. This lemma allows us to construct a six-term exact sequence in a (sufficiently nice) category with a structure of nullhomotopies associated to a morphism in the category. In the second part, we introduce the category with nullhomotopies $\mathbf{Seq}(\mathcal{A})$ of sequentiable families of arrows in a category $\mathcal{A}$ and we apply the homotopy snail lemma to a morphism in $\mathbf{Seq}(\mathcal{A})$ obtaining first a six-term exact sequence in $\mathbf{Seq}(\mathcal{A})$ and then, unrolling the sequence in $\mathbf{Seq}(\mathcal{A})$, a long exact sequence in $\mathcal{A}$. We then compare the category $\mathbf{Seq}(\mathcal{A})$ of sequentiable families with the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ and prove that, when $\mathcal{A}$ is abelian, the long exact sequence built using the snail lemma subsumes the usual long exact sequence of homology obtained from an extension of chain complexes. This result suggests that the category $\mathbf{Seq}(\mathcal{A})$, which has a nicer structure of nullhomotopies than that of $\mathbf{Ch}(\mathcal{A})$, could provide a useful alternative to chain complexes for the study of homological algebra.

Contents

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1. Introduction

A cornerstone in homological algebra is the fact that, starting from a short exact sequence of chain complexes in an abelian category $\mathcal{A}$, one can construct a long exact sequence relating the homology objects of the original complexes. A classical strategy to prove such a theorem is to prove first the snake lemma and then construct the long exact sequence in homology pasting together the infinitely many six-term exact sequences coming from the snake lemma (see for example [28]).

The snake lemma is a special case of a more general result, the snail lemma, introduced in [25, 19] and exploited in [17, 23] in order to unify some higher dimensional exact sequences appearing in homotopy theory, see [12], and in the study of groupoids and crossed modules, see [8, 10]. The difference between the snake and the snail lemmas lies in the fact that the snake requires as starting point a short exact sequence in $\mathbf{Arr}(\mathcal{A})$, the category of arrows in $\mathcal{A}$, whereas the snail works starting from any morphism in $\mathbf{Arr}(\mathcal{A})$.

In this paper, we first prove a version of the snail lemma for categories equipped with a structure of nullhomotopies. More concretely, given a category $\mathcal{B}$ with a structure of nullhomotopies Θ satisfying some nice properties, we build a six-term sequence of "homotopy objects" starting from a morphism in $\mathcal{B}$ (Construction 3.0.2) and we show that under some conditions, this sequence is exact with respect to a sufficiently nice class of "surjections" (Proposition 4.0.6, Proposition 4.0.9). The prime example of a category where this can be done is the category $\mathbf{Arr}(\mathcal{A})$ of arrows on an abelian category $\mathcal{A}$ equipped with the usual structure of nullhomotopies Θ_Δ (Subsection 2.5). Then, in the second part of the paper we show that it is possible to use the snail lemma instead of the snake lemma in order to construct a long exact sequence in homology starting from any morphism of chain complexes, and not necessarily from a short exact sequence of complexes (Corollary 6.2.2). Even if the idea is quite simple, to state and prove it properly we have to introduce a new concept, that we call a sequentiable family of arrows (Definition 5.1.1). The idea behind a sequentiable family is to focus our attention not on the homology objects associated with a chain complex, but on the homology arrows, that is, those arrows whose kernel and cokernel are the homology objects associated with a complex (see Section 6, and especially Subsection 6.1, for a more precise explanation on how sequentiable families of arrows naturally arise from chain complexes). This allows us to formulate the snail lemma inside the category of sequentiable families, which is equipped with a structure of nullhomotopies more convenient than the one usually considered in the category of chain complexes. This produces a single six-term exact sequence of sequentiable families, sequence which provides a compact presentation of a long exact sequence in the base category $\mathcal{A}$ (Proposition 5.3.2) and, as a special case, the long homology sequence (Proposition 6.2.3).

This last result, together with the fact that the structure of nullhomotopies in sequentiable families has more desirable properties than the usual structure of nullhomotopies in chain complexes, suggests that the category of sequentiable families has the potential to be a useful substitute of the category of chain complexes in the study of homological algebra. In future work we would like to explore this direction further by analyzing, for example, the use of sequentiable families to construct the derived category of an abelian category.

The layout of the paper is as follows:

In the first part of the paper (Sections 2, 3 and 4) we investigate a general version of the snail lemma in a pointed category $\mathcal{B}$ equipped with a structure of nullhomotopies Θ, the guiding example being $\mathbf{Arr}(\mathcal{A})$ for $\mathcal{A}$ an abelian category. First, in Section 2, we review the notion of a category with a structure of nullhomotopies and recall some properties and constructions in this framework that will be relevant for the rest of the paper. In particular, after reviewing the definition of structure of nullhomotopies, the condition of reduced interchange and the notion of homotopy-strong zero object in Subsection 2.1, we then go over homotopy (co)kernels (Subsection 2.2), the π_0 of an object (Subsection 2.3) and (co)discrete objects (Subsection 2.4), to finish the section by analyzing these concepts in the particular case of the category of arrows (Subsection 2.5). Then, in Section 3, we show that given a morphism in a category with nullhomotopies $(\mathcal{B}, \Theta)$, we can build a six-term exact sequence in $\mathcal{B}$, as long as $(\mathcal{B}, \Theta)$ satisfies some nice properties also satisfied by the category of arrows of an abelian category (the reduced interchange condition holds, $\mathcal{B}$ has a strong zero object, all homotopy kernels and all homotopy cokernels of terminal arrows). In Section 4, we show that the six-term sequence is exact relatively to a good class $\mathcal{S}$ of "surjections". For this result to hold, the homotopy kernel, the domain and the codomain of the starting morphism need to have a good behaviour with respect of the class $\mathcal{S}$: the homotopy kernel and the domain of the morphism need to be $\mathcal{S}$-proper and $\mathcal{S}$-global (Definition 4.0.4) and the codomain needs to be $\mathcal{S}$-proper. In the particular case of the category of arrows of an abelian category, all objects are proper and global and hence the snail sequence is always exact. It is worth noticing that, while the proof of the exactness of the snail sequence in this homotopy context is similar to that in the context of pointed regular protomodular categories from [25], they are not completely parallel (see Remark 4.0.11).

In the second part of the paper we introduce the new category $\mathbf{Seq}(\mathcal{A})$ of sequentiable families and we compare it with the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes. First, in Section 5, we build the category $\mathbf{Seq}(\mathcal{A})$ and equip it with a natural structure of nullhomotopies (Subsection 5.1). We then apply the homotopy snail lemma in $\mathbf{Seq}(\mathcal{A})$ in order to get a six-term exact sequence in $\mathbf{Seq}(\mathcal{A})$ (Subsection 5.2), which we then unrole to obtain a long exact sequence in $\mathcal{A}$ (Subsection 5.3). Then, in Section 6, we build a functor $\mathcal{F} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Seq}(\mathcal{A})$ from chain complexes to sequentiable families (Subsection 6.1) and we then proceed to prove the announced result: when $\mathcal{A}$ is pointed, regular and protomodular, if the morphism in $\mathbf{Seq}(\mathcal{A})$ that we use to construct the long exact sequence comes from an extension of complexes through the functor $\mathcal{F}$, the new sequence coincides with the classical long homology sequence (Subsection 6.2). To finish the paper, we prove in Subsection 6.3 that the functor $\mathcal{F}$ can be upgraded to a morphism of categories with nullhomotopies when $\mathcal{A}$ is preadditive and $\mathbf{Ch}(\mathcal{A})$ is endowed with the classical structure of nullhomotopies.

Starting from Section 4, regular, protomodular, preadditive and abelian categories will appear so to make the base category $\mathcal{A}$ rich enough. Basic references for these kinds of categories are [2, 3, 7].

N.B.: Given two arrows $A \xrightarrow{f} B \xrightarrow{g} C$, the composite arrow will be written as $f \cdot g$.

2. Preliminaries on nullhomotopies

Categories with a structure of nullhomotopies have been introduced by Grandis in [14]. In this section, we review their basic theory and provide several preliminary results that we will require later on.

2.1. Categories with a structure of nullhomotopies. Let us first introduce the definition of nullhomotopy structure. We follow [22, 26].

2.1.1. Definition. A *structure of nullhomotopies* Θ on a category $\mathcal{B}$ is given by:

1) For every arrow g in $\mathcal{B}$, a set $\Theta(g)$ whose elements are called nullhomotopies on g.

2) For every triple of composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

in such a way that, for every $\varphi \in \Theta(g)$, one has

(a) $(f' \cdot f) \circ \varphi \circ (h \cdot h') = f' \circ (f \circ \varphi \circ h) \circ h'$ whenever the compositions $f' \cdot f$ and $h \cdot h'$ are defined,

(b) $\mathrm{id}_B \circ \varphi \circ \mathrm{id}_C = \varphi$.

Equivalently, Θ is a functor from the twisted arrow category to the category of sets. See Remark 2.1.10 in [27] and the references therein for more details.

2.1.2. Notation. When $f = \mathrm{id}_B$ or $h = \mathrm{id}_C$, we write $\varphi \circ h$ and $f \circ \varphi$ instead of $\mathrm{id}_B \circ \varphi \circ h$ and $f \circ \varphi \circ \mathrm{id}_C$.

2.1.3. Notation. We sometimes depict a nullhomotopy $\lambda \in \Theta(g)$ as

$$\begin{tikzcd} A \arrow[r, bend left=40, "g"] \arrow[r, bend right=40, dash, dashed] \arrow[r, phantom, "\lambda \Uparrow"] & B \end{tikzcd}$$

In order to build the homotopy snail sequence in Section 3, we will need to impose some extra conditions on the categories with nullhomotopies with which we work. One condition we will need to impose is that the structure of nullhomotopies satisfies *reduced interchange*, which was introduced in [14]:

2.1.4. Definition. Let $(\mathcal{B}, \Theta)$ be a category with nullhomotopies. The structure Θ satisfies *reduced interchange* if, for any pair of composable arrows $f\colon A \to B$ and $g\colon B \to C$ and for any pair of nullhomotopies $\alpha \in \Theta(f)$ and $\beta \in \Theta(g)$, one has $\alpha \circ g = f \circ \beta$.

Another condition we will need to impose is the existence of a zero object 0 in $\mathcal{B}$ which is, moreover, Θ-*strong*:

2.1.5. Definition. Let $(\mathcal{B}, \Theta)$ be a category with nullhomotopies and assume that $\mathcal{B}$ has a zero object 0. We say that the zero object 0 is Θ-*strong* if for every object $X \in \mathcal{B}$, the set of nullhomotopies $\Theta(0^X\colon X \to 0)$ of the terminal arrow is reduced to a single element, denoted by $*^X$, and the set of nullhomotopies $\Theta(0_X\colon 0 \to X)$ of the initial arrow is reduced to a single element, denoted by $*_X$.

2.1.6. Remark. If $(\mathcal{B}, \Theta)$ satisfies the reduced interchange (Definition 2.1.4) and has a Θ-strong zero object (Definition 2.1.5), we get a canonical nullhomotopy

$$*^X_Y \in \Theta(0^X_Y\colon X \to 0 \to Y)$$

given by $*^X \circ 0_Y$ or, equivalently, by $0^X \circ *_Y$. Observe that in general $\Theta(0^X_Y)$ is not reduced to the element $*^X_Y$. Nevertheless, for all arrows $f\colon W \to X, g\colon X \to Y, h\colon Y \to Z$ and for all nullhomotopies $\varphi \in \Theta(g)$ we have:

$$f \circ *^X_Y \circ h = *^W_Z \quad \text{and} \quad 0^W_X \circ \varphi = *^W_Y \quad \text{and} \quad \varphi \circ 0^Y_Z = *^X_Z$$

2.1.7. Remark. Although these two conditions (the reduced interchange and the existence of a Θ-strong zero object) will be neccessary from Section 3 on and until the end of the paper, we will not impose them yet in this section, where we aim to provide the results in their most general version.

2.2. Homotopy (co)kernels. Homotopy (co)kernels are the only homotopy (co)limits we need in this paper. In this subsection, we introduce their definition following [22, 26], and provide two examples that are essential for our purposes. To conclude, we prove a general result that will be needed in Section 3 to construct the snail sequence.

2.2.1. Definition. Let $g\colon B \to C$ be an arrow in a category with nullhomotopies $(\mathcal{B}, \Theta)$.

1. A *homotopy kernel* of g with respect to Θ (or Θ-*kernel*) is a universal triple

$$\begin{tikzcd} \mathcal{N}(g) \arrow[r, "n_g"'] \arrow[rr, dashed, bend left, "\Downarrow \nu_g"] & B \arrow[r, "g"'] & C \end{tikzcd}$$

   This means that, for any other triple $(A \in \mathcal{B}, f\colon A \to B, \varphi \in \Theta(f \cdot g))$, there exists a unique arrow $f'\colon A \to \mathcal{N}(g)$ such that $f' \cdot n_g = f$ and $f' \circ \nu_g = \varphi$.

2. A Θ-kernel $(\mathcal{N}(g), n_g, \nu_g)$ is *strong* if, for any triple $(A, f\colon A \to \mathcal{N}(g), \varphi \in \Theta(f \cdot n_g))$ such that $\varphi \circ g = f \circ \nu_g$, there exists a unique nullhomotopy $\varphi' \in \Theta(f)$ such that $\varphi' \circ n_g = \varphi$.

3. The notion of (*strong*) Θ-*cokernel* is dual of the notion of (strong) Θ-kernel. The notation is:
$$\mathcal{C}(g) \in \mathcal{B}, c_g\colon C \to \mathcal{C}(g), \gamma_g \in \Theta(g \cdot c_g)$$

Let us recall from [22, 26] the cancellation properties satisfied by a Θ-kernel.

2.2.2. Proposition. *Let $g\colon B \to C$ be an arrow in a category with nullhomotopies $(\mathcal{B}, \Theta)$ and let $(\mathcal{N}(g), n_g, \nu_g)$ be a Θ-kernel of g. Then, the following cancellation property holds:*

(1) Given $f, h\colon A \to \mathcal{N}(g)$, if $f \cdot n_g = h \cdot n_g$ and $f \circ \nu_g = h \circ \nu_g$, then $f = h$.

In addition, if the reduced interchange is satisfied, then the following cancellation property holds:

(2) Given an arrow $f\colon A \to \mathcal{N}(g)$ and nullhomotopies $\varphi, \psi \in \Theta(f)$ such that $\varphi \circ n_g = \psi \circ n_g$, if the Θ-kernel of g is strong, then $\varphi = \psi$.

2.2.3. Remark. The dual cancellation properties hold for the Θ-cokernel $(\mathcal{C}(g), c_g, \gamma_g)$ of an arrow g.

Let us introduce two examples of Θ-kernels that play an important role in this paper.

2.2.4. Example. [The Θ-kernel of an initial arrow]The first relevant special case of Θ-kernel is the one of the initial arrow 0_Y of an object $Y \in \mathcal{B}$. Its universal property can be restated as follows:
For any object $X \in \mathcal{B}$ and for any nullhomotopy $\varphi \in \Theta(0_Y^X)$, there exists a unique arrow $g\colon X \to \mathcal{N}(0_Y)$ such that $g \circ \nu_{0_Y} = \varphi$

$$\begin{tikzcd} \mathcal{N}(0_Y) \arrow[dr, "n_{0_Y}"] \arrow[ddr, dashed, bend right, "\overset{\nu_{0_Y}}{\Rightarrow}"] & & X \arrow[ll, "g"'] \arrow[ddl, "0_Y^X"'] \arrow[ddl, dashed, bend left, "\overset{\varphi}{\Leftarrow}"'] \\ & 0 \arrow[d, "0_Y"] & \\ & Y & \end{tikzcd}$$

2.2.5. Remark. The universal property of the Θ-cokernel of a terminal arrow can be formulated dually.

2.2.6. Example. [The Θ-kernel of an identity arrow] The second relevant special case of Θ-kernel is the one of the identity arrow id_Y of an object $Y \in \mathcal{B}$. Its universal property can be restated as follows:
For every arrow $g\colon X \to Y$ and for every nullhomotopy $\varphi \in \Theta(g)$, there exists a unique arrow $\overline{g}\colon X \to \mathcal{N}(\mathrm{id}_Y)$ such that $\overline{g} \cdot n_{\mathrm{id}_Y} = g$ and $\overline{g} \circ \nu_{\mathrm{id}_Y} = \varphi$

$$\begin{tikzcd}
\mathcal{N}(\mathrm{id}_Y) \arrow[rd, "n_{\mathrm{id}_Y}"] \arrow[rd, dashed, no head, bend right=60, "\overset{\nu_{\mathrm{id}_Y}}{\Rightarrow}"] & & X \arrow[ll, "\overline{g}"'] \arrow[ld, "g"'] \arrow[ld, dashed, no head, bend left=60, "\overset{\varphi}{\Leftarrow}"'] \\
& Y &
\end{tikzcd}$$

Moreover, if the Θ-kernel $\mathcal{N}(\mathrm{id}_Y)$ is strong, there exists a unique nullhomotopy $\overline{\varphi} \in \Theta(\overline{g})$ such that $\overline{\varphi} \circ n_{\mathrm{id}_Y} = \varphi$.

2.2.7. Remark. If it exists, the Θ-kernel of the identity arrow of an object Y has a special role: if $\mathcal{B}$ has pullbacks, then the Θ-kernel of any arrow $g\colon X \to Y$ exists and is given by

$$\begin{tikzcd}
\mathcal{N}(g) \arrow[r, "n_g"'] \arrow[rr, dashed, no head, bend left=40, "\Downarrow g' \circ \nu_{\mathrm{id}_Y}"] & X \arrow[r, "g"'] & Y
\end{tikzcd}$$

where the diagram

$$\begin{tikzcd}
\mathcal{N}(g) \arrow[r, "g'"] \arrow[d, "n_g"'] & \mathcal{N}(\mathrm{id}_Y) \arrow[d, "n_{\mathrm{id}_Y}"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

is a pullback (see Proposition 5.3 in [22]).

We now prove the following general lemma, which is a variant of Remark 3.2 in [26]. Its dual also holds and both are needed in Section 3 to construct the snail sequence.

2.2.8. Lemma. *Consider the solid part of the following commutative diagram (the one on the left) in a category with nullhomotopies* $(\mathcal{B}, \Theta)$

$$\begin{tikzcd}
\mathcal{N}(a) \arrow[r, dotted, "n(g{,}g')"] \arrow[d, "n_a"] \arrow[dd, dashed, no head, bend right=70, "\nu_a \Rightarrow"'] & \mathcal{N}(b) \arrow[d, "n_b"'] \arrow[dd, dashed, no head, bend left=70, "\Leftarrow \nu_b"] \\
A \arrow[r, "g'"] \arrow[d, "a"] & B \arrow[d, "b"'] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{N}(a) \arrow[r, dotted, "n(g{,}g')"'] \arrow[r, dashed, no head, bend left=40, "n(\psi) \Downarrow"] \arrow[d, "n_a"] \arrow[d, dashed, no head, bend right=70, "\psi \Rightarrow"'] & \mathcal{N}(b) \arrow[d, "n_b"'] \\
A \arrow[r, "g'"] \arrow[d, "a"] & B \arrow[d, "b"'] \\
X \arrow[r, "g"] \arrow[r, dashed, no head, bend right=40, "\varphi \Uparrow"'] & Y
\end{tikzcd}$$

1. *There exists a unique arrow* $n(g,g')\colon \mathcal{N}(a) \to \mathcal{N}(b)$ *such that* $n(g,g') \cdot n_b = n_a \cdot g'$ *and* $n(g,g') \circ \nu_b = \nu_a \circ g$.

2. *Consider the diagram on the right and assume that* Θ *satisfies the reduced interchange. Given nullhomotopies* $\varphi \in \Theta(g)$ *and* $\psi \in \Theta(n_a)$, *if the* Θ*-kernel of* b *is strong, then there exists a unique nullhomotopy* $n(\psi) \in \Theta(n(g,g'))$ *such that* $n(\psi) \circ n_b = \psi \circ g'$.

Proof. 1. Just apply the universal property of $\mathcal{N}(b)$ to $\nu_a \circ g \in \Theta(n_a \cdot g' \cdot b)$.
2. Consider $\psi \circ g' \in \Theta(n(g,g') \cdot n_b)$. Since

$$\psi \circ g' \cdot b = \psi \circ a \cdot g = n_a \cdot a \circ \varphi = \nu_a \circ g = n(g,g') \circ \nu_b$$

we can use that $\mathcal{N}(b)$ is strong to get a unique nullhomotopy $n(\psi) \in \Theta(n(g,g'))$ such that $n(\psi) \circ n_b = \psi \circ g'$. ∎

2.3. The π_0 of an object. In this section we introduce the π_0 of an object, which is a special type of Θ-kernel that will be appearing in the snail homotopy sequence we will build in Section 3.

2.3.1. Definition. Let $(\mathcal{B}, \Theta)$ be a category with nullhomotopies and assume that $\mathcal{B}$ has a 0 object. The π_0 *of an object* Y is the Θ-kernel of the arrow part of the Θ-cokernel of the terminal arrow 0^Y. Diagrammatically:

$$\begin{tikzcd}
Y \arrow[rr, "0^Y"'] \arrow[dr, "\eta_Y"'] \arrow[rrr, dash, dashed, bend left, "\Downarrow \gamma_{0^Y}"] & & 0 \arrow[r, "c_{0^Y}"'] & \mathcal{C}(0^Y) \\
& \pi_0(Y) \arrow[ur, "n_{c_{0^Y}}"'] \arrow[urr, dash, dashed, bend right, "\Uparrow \nu_{c_{0^Y}}"] & &
\end{tikzcd}$$

where η_Y is the unique arrow such that $\eta_Y \circ \nu_{c_{0^Y}} = \gamma_{0^Y}$. Dually, the π_1 *of an object* Y is the Θ-cokernel of the arrow part of the Θ-kernel of the terminal arrow 0_Y.

2.3.2. Remark. The chosen terminology of π_0 and π_1 is not arbitrary, since this definition generalizes that of π_0 and π_1 of a groupoid. We explain this in more detail in Example 2.5.2. The meaning of the π_0-construction (and, dually, of the π_1-construction) in the category $\mathbf{Arr}(\mathcal{A})$ is explained in Subsection 2.5.

2.3.3. Remark. Observe that the construction of $\pi_0(Y)$ is a special case of a Θ-kernel of an initial arrow (see Example 2.2.4), since $\pi_0(Y) = \mathcal{N}(c_{0^Y}) = \mathcal{N}(0_{\mathcal{C}(0^Y)})$. Analogously, $\pi_1(Y)$ is a special case of a Θ-cokernel of a terminal arrow since $\pi_1(Y) = \mathcal{C}(n_{0_Y}) = \mathcal{C}(0^{\mathcal{N}(0_Y)})$.

2.4. (Co)discrete objects. We now proceed to introduce the general definition of (co)discrete object. For our purposes, the relevance of discrete objects lies in the fact that, in the categories with nullhomotopies of our interest, the categorical kernel and the Θ-kernel of any arrow with codomain a discrete object coincide (see Lemma 2.4.3 below). This property will be very useful later on, especially since objects of the form $\mathcal{N}(0_Y)$ (Example 2.2.4) and, in particular, of the form $\pi_0(Y)$ (Definition 2.3.1), are discrete under suitable assumptions (see Lemma 2.4.4 below).

2.4.1. Definition. Consider a category with nullhomotopies $(\mathcal{B}, \Theta)$. Assume that Θ satisfies the reduced interchange and that $\mathcal{B}$ has a Θ-strong zero object. An object $Y \in \mathcal{B}$ is *discrete* if, for any arrow $g\colon X \to Y$, the following conditions hold:

- if $g \neq 0^X_Y$, then $\Theta(g) = \emptyset$,
- if $g = 0^X_Y$, then $\Theta(g) = \{*^X_Y\}$.

Codiscrete objects are defined dually.

2.4.2. Remark. The terminology "discrete" will be motivated at the end of Section 2.5 (see Remark 2.5.1).

We now show that, under suitable assumptions, the categorical kernel and the Θ-kernel of a morphism with codomain a discrete object coincide.

2.4.3. Lemma. *Consider a category with nullhomotopies $(\mathcal{B}, \Theta)$. Assume that Θ satisfies the reduced interchange and that $\mathcal{B}$ has a Θ-strong zero object and Θ-kernels. If an object $Y \in \mathcal{B}$ is discrete, then for every arrow $g\colon X \to Y$ the Θ-kernel of g coincides with the usual categorical kernel of g.*

Proof. Since Y is discrete, $n_g \cdot g = 0^{\mathcal{N}(g)}_Y$ and $\nu_g = *^{\mathcal{N}(g)}_Y$. Consider an arrow $f\colon W \to X$ such that $f \cdot g = 0^W_Y$. This implies that $*^W_Y \in \Theta(f \cdot g)$, so that there exists a unique arrow $f'\colon W \to \mathcal{N}(g)$ such that $f' \cdot n_g = f$ and $f' \circ \nu_g = *^W_Y$. If $f''\colon W \to \mathcal{N}(g)$ is another arrow such that $f'' \cdot n_g = f$, then $f'' \circ \nu_g = f'' \circ *^{\mathcal{N}(g)}_Y = *^W_Y$ by Definition 2.1.5, so that $f'' = f'$. This proves that the Θ-kernel satisfies the universal property of the usual kernel.
Conversely, let $k_g\colon \mathrm{Ker}(g) \to X$ be the usual kernel of g. Since $k_g \cdot g = 0^{\mathrm{Ker}(g)}_Y$, we can take $\nu_g = *^{\mathrm{Ker}(g)}_Y \in \Theta(k_g \cdot g)$. Given now $f\colon W \to X$ and $\varphi \in \Theta(f \cdot g)$, since Y is discrete we get $f \cdot g = 0^W_Y$ and $\varphi = *^W_Y$. From $f \cdot g = 0^W_Y$, we get a unique $f'\colon W \to \mathrm{Ker}(g)$ such that $f' \cdot k_g = f$. Moreover, $f' \circ \nu_g = f' \circ *^{\mathrm{Ker}(g)}_Y = *^W_Y = \varphi$ by Definition 2.1.5, and we are done. ∎

In particular, we will be able to apply Lemma 2.4.3 to morphisms whose codomain is of the form $\mathcal{N}(0_Y)$ and, in particular, of the form $\pi_0(Y)$, on account of the following result.

2.4.4. Lemma. *Consider a category with nullhomotopies $(\mathcal{B}, \Theta)$. Assume that Θ satisfies the reduced interchange and that $\mathcal{B}$ has a Θ-strong zero object. Then, if $(\mathcal{B}, \Theta)$ has strong Θ-kernels, for any object $Y \in \mathcal{B}$, the object $\mathcal{N}(0_Y)$ is discrete. In particular, $\pi_0(Y)$ is discrete. Dually, if $(\mathcal{B}, \Theta)$ has strong Θ-cokernels, for any object $Y \in \mathcal{B}$, the object $\mathcal{C}(0^Y)$ is codiscrete.*

Proof. Consider an arrow $g \colon X \to \mathcal{N}(0_Y)$ and a nullhomotopy $\varphi \in \Theta(g)$. We have to prove that $g = 0^X_{\mathcal{N}(0_Y)}$ and $\varphi = *^X_{\mathcal{N}(0_Y)}$. Using the universal property of the Θ-kernel, the first condition follows from the equations:

$$g \cdot n_{0_Y} = 0^X = 0^X_{\mathcal{N}(0_Y)} \cdot n_{0_Y}$$

and

$$g \circ \nu_{0_Y} = \varphi \circ 0_Y^{\mathcal{N}(0_Y)} = *^X_Y = *^X_{\mathcal{N}(0_Y)} \circ 0_Y^{\mathcal{N}(0_Y)} = 0^X_{\mathcal{N}(0_Y)} \circ \nu_{0_Y}$$

(in the second one we use Definition 2.1.5). Using that the Θ-kernel is strong, the second condition follows form the equation

$$\varphi \circ n_{0_Y} = *^X = *^X_{\mathcal{N}(0_Y)} \circ n_{0_Y}$$

where we have used once again Definition 2.1.5. The proof that $\mathcal{C}(0^Y)$ is codiscrete is dual. ■

2.5. The classical example: The category of arrows and nullhomotopies.

To help the reader with the various constructions introduced so far, let us look at the case where $(\mathcal{B}, \Theta) = (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$:

- Objects, arrows and nullhomotopies for this example can be depicted as follows

  $$\begin{tikzcd}
  W \arrow[r, "f"] \arrow[d, "w"'] & X \arrow[r, "g"] \arrow[d, "x"'] & Y \arrow[r, "h"] \arrow[d, "y"] & Z \arrow[d, "z"] \\
  W_0 \arrow[r, "f_0"'] & X_0 \arrow[r, "g_0"'] \arrow[ur, dashed, "\varphi"] & Y_0 \arrow[r, "h_0"'] & Z_0
  \end{tikzcd}$$

  In other words,

  $$\Theta_\Delta((g, g_0) \colon (X, x, X_0) \to (Y, y, Y_0)) = \{\varphi \colon X_0 \to Y \mid x \cdot \varphi = g, \varphi \cdot y = g_0\}$$

  and $(f, f_0) \circ \varphi \circ (h, h_0) = f_0 \cdot \varphi \cdot h$.

- The structure Θ_Δ satisfies the reduced interchange.

- If $\mathcal{A}$ has pullbacks, then $\mathbf{Arr}(\mathcal{A})$ has strong Θ_Δ-kernels constructed as in the following diagram on the left (the dashed arrow is the nullhomotopy), the one on the right being a pullback

  $$\begin{tikzcd}
  X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\langle x{,}g \rangle"'] & X \arrow[r, "g"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
  X_0 \times_{g_0,y} Y \arrow[r, "y'"'] \arrow[urr, dashed, "g_0'"] & X_0 \arrow[r, "g_0"'] & Y_0
  \end{tikzcd}
  \qquad
  \begin{tikzcd}
  X_0 \times_{g_0,y} Y \arrow[r, "g_0'"] \arrow[d, "y'"'] & Y \arrow[d, "y"] \\
  X_0 \arrow[r, "g_0"'] & Y_0
  \end{tikzcd}$$

Dually, if $\mathcal{A}$ has pushouts, then $\mathbf{Arr}(\mathcal{A})$ has strong Θ_Δ-cokernels.

- If $\mathcal{A}$ has a zero object 0, then the object $(0, \mathrm{id}_0, 0)$ is a Θ_Δ-strong zero object in $\mathbf{Arr}(\mathcal{A})$. More details can be found in [22, 26].

Assume now that $\mathcal{A}$ has a zero object, kernels and cokernels. For an object (Y, y, Y_0) in $\mathbf{Arr}(\mathcal{A})$ we then have the following:

- The Θ_Δ-kernel $\mathcal{N}(0_{(Y,y,Y_0)})$ with its structural nullhomotopy is given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y \arrow[d, "y"] \\
\mathrm{Ker}(y) \arrow[r, "0"'] \arrow[ur, dashed, "k_y"] & Y_0
\end{tikzcd}$$

- Dually, the Θ_Δ-cokernel $\mathcal{C}(0^{(Y,y,Y_0)})$ with its structural nullhomotopy is given by

$$\begin{tikzcd}
Y \arrow[r, "0"] \arrow[d, "y"'] & \mathrm{Cok}(y) \arrow[d] \\
Y_0 \arrow[r] \arrow[ur, dashed, "c_y"] & 0
\end{tikzcd}$$

- Consequently, the canonical arrow $\eta_{(Y,y,Y_0)} \colon (Y, y, Y_0) \to \pi_0(Y, y, Y_0)$ is given by

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "y"'] & 0 \arrow[d] \\
Y_0 \arrow[r, "c_y"'] & \mathrm{Cok}(y)
\end{tikzcd}$$

- Finally, the Θ_Δ-kernel $\mathcal{N}(\mathrm{id}_{(Y,y,Y_0)})$ with its structural nullhomotopy is given by

$$\begin{tikzcd}
Y \arrow[r, "\mathrm{id}_Y"] \arrow[d, "\mathrm{id}_Y"'] & Y \arrow[d, "y"] \\
Y \arrow[r, "y"'] \arrow[ur, dashed, "\mathrm{id}_Y"] & Y_0
\end{tikzcd}$$

The following remark motivates the choice of the terminology "discrete".

2.5.1. Remark. Assume that $\mathcal{A}$ has a zero object, kernels and cokernels. Then, an object $(X, x, X_0) \in \mathbf{Arr}(\mathcal{A})$ is discrete if and only if $X = 0$ (and then, necessarily, $x = 0_{X_0}$). For the only if part, just consider the Θ_Δ-kernel of $\mathrm{id}_{(X,x,X_0)}$. Dually, an object $(X, x, X_0) \in \mathbf{Arr}(\mathcal{A})$ is codiscrete if and only if $X_0 = 0$.

The following example motivates the choice of the terminology π_0 and π_1.

2.5.2. Example. Let $\mathcal{A}$ be an abelian category. In particular, we have an equivalence of categories $\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$, where $\mathbf{Grpd}(\mathcal{A})$ denotes the category of internal groupoids in $\mathcal{A}$ (see [3]). This equivalence sends an object (Y, y, Y_0) in $\mathbf{Arr}(\mathcal{A})$ to the groupoid $\mathcal{G}(Y, y, Y_0)$ with underlying reflexive graph

$$\begin{tikzcd}
Y_0 \oplus Y \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\mathrm{id}\oplus y"'] & Y_0 \arrow[l, "i_1" description]
\end{tikzcd}$$

(every reflexive graph in $\mathcal{A}$ can be extended uniquely to a groupoid structure, see [9, 20]). Then:

- The π_0-construction in $\mathbf{Arr}(\mathcal{A})$ corresponds via the equivalence $\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$ to the usual π_0-construction in $\mathbf{Grpd}(\mathcal{A})$:

  The object $\pi_0(Y, y, Y_0) = (0, 0, \mathrm{Cok}(y))$ in $\mathbf{Arr}(\mathcal{A})$ gets sent via the equivalence to the groupoid $\pi_0(\mathcal{G}(Y, y, Y_0))$, i.e. the discrete groupoid of connected components of $\mathcal{G}(Y, y, Y_0)$, whose underlying reflexive graph is

$$\begin{tikzcd}
\mathrm{Cok}(y) \arrow[r, shift left=2, "\mathrm{id}"] \arrow[r, shift right=2, "\mathrm{id}"'] & \mathrm{Cok}(y) \arrow[l, "\mathrm{id}" description]
\end{tikzcd}$$

- The π_1-construction in $\mathbf{Arr}(\mathcal{A})$ corresponds via the equivalence $\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$ to the usual π_1-construction in $\mathbf{Grpd}(\mathcal{A})$:

  The object $\pi_1(Y, y, Y_0) = (\mathrm{Ker}(y), 0, 0)$ in $\mathbf{Arr}(\mathcal{A})$ gets sent via the equivalence to the groupoid $\pi_1(\mathcal{G}(Y, y, Y_0))$, i.e. the codiscrete groupoid of automorphisms of the 0-object of $\mathcal{G}(Y, y, Y_0)$, whose underlying reflexive graph is

$$\begin{tikzcd}
\mathrm{Ker}(y) \arrow[r, shift left=2, "0"] \arrow[r, shift right=2, "0"'] & 0 \arrow[l, "0" description]
\end{tikzcd}$$

3. The homotopy snail sequence

In this section, we work in a category with nullhomotopies $(\mathcal{B}, \Theta)$ under the following assumptions:

- $(\mathcal{B}, \Theta)$ satisfies the reduced interchange condition 2.1.4;
- There exists a Θ-strong zero object as in Definition 2.1.5;
- The strong Θ-kernel of any arrow exists;
- The strong Θ-cokernel of any terminal arrow exists.

Given an arrow g in $\mathcal{B}$ together with its Θ-kernel

$$\begin{tikzcd}
\mathcal{N}(g) \arrow[r, "n_g"'] \arrow[rr, dashed, bend left=40, ""{name=U, below}] & X \arrow[r, "g"'] & Y
\arrow[phantom, from=U, to=1-2, "\Downarrow \nu_g"]
\end{tikzcd}$$

we are going to construct the "snail" sequence connecting six discrete objects

$$\mathcal{N}(0_{\mathcal{N}(g)}) \xrightarrow{n(n_g)} \mathcal{N}(0_X) \xrightarrow{n(g)} \mathcal{N}(0_Y) \xrightarrow{\delta} \pi_0(\mathcal{N}(g)) \xrightarrow{\pi_0(n_g)} \pi_0(X) \xrightarrow{\pi_0(g)} \pi_0(Y)$$

where each pair of consecutive arrows gives, by composition, the zero arrow. The exactness of this sequence will be discussed in Section 4.

3.0.1. Remark. The reader may have expected an exact sequence in which the first three objects would have been of type π_1 (more precisely, $\pi_1(\mathcal{N}(g)), \pi_1(X)$ and $\pi_1(Y)$) as it happens, for example, in [17, 23, 25] (see also [21] and [24] for other instances of such a situation). Notice, however, that between a codiscrete object and a discrete object the only possible arrow is the zero arrow. On the contrary, the objects $\mathcal{N}(0_{\mathcal{N}(g)}), \mathcal{N}(0_X)$ and $\mathcal{N}(0_Y)$ are the discrete version of $\pi_1(\mathcal{N}(g)), \pi_1(X)$ and $\pi_1(Y)$ and therefore, they can participate in a non-trivial sequence with the objects of type π_0. What makes the present situation different from that in the above mentioned references is that in [17, 23, 25], $\pi_0(X)$ and $\pi_1(X)$ do not live in the same category where X lives, whereas in our context $X, \pi_0(X)$ and $\pi_1(X)$ all live in the same category $\mathcal{B}$ ($\pi_0(X)$ as a discrete object and $\pi_1(X)$ as an codiscrete one). We will go back to this problem in Remark 3.0.4.

3.0.2. Construction. We divide the construction of the "snail" sequence into five steps:
Step 1: By applying the first part of Lemma 2.2.8 to the situations

$$\begin{tikzcd}
\mathcal{N}(0_{\mathcal{N}(g)}) \arrow[r, dotted, "n(n_g)"] \arrow[d, "n_{0_{\mathcal{N}(g)}}"] \arrow[dd, dashed, dash, bend right=70, ""{name=L}] & \mathcal{N}(0_X) \arrow[d, "n_{0_X}"'] \arrow[dd, dashed, dash, bend left=70, ""{name=R}] \\
0 \arrow[r, "\mathrm{id}_0"] \arrow[d, "0_{\mathcal{N}(g)}"] & 0 \arrow[d, "0_X"'] \\
\mathcal{N}(g) \arrow[r, "n_g"'] & X
\arrow[phantom, from=L, to=2-1, "\nu_{0_{\mathcal{N}(g)}} \Rightarrow"]
\arrow[phantom, from=R, to=2-2, "\Leftarrow \nu_{0_X}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{N}(0_X) \arrow[r, dotted, "n(g)"] \arrow[d, "n_{0_X}"] \arrow[dd, dashed, dash, bend right=70, ""{name=L}] & \mathcal{N}(0_Y) \arrow[d, "n_{0_Y}"'] \arrow[dd, dashed, dash, bend left=70, ""{name=R}] \\
0 \arrow[r, "\mathrm{id}_0"] \arrow[d, "0_X"] & 0 \arrow[d, "0_Y"'] \\
X \arrow[r, "g"'] & Y
\arrow[phantom, from=L, to=2-1, "\nu_{0_X} \Rightarrow"]
\arrow[phantom, from=R, to=2-2, "\Leftarrow \nu_{0_Y}"']
\end{tikzcd}$$

we get the dotted arrows $n(n_g)$ and $n(g)$, which are unique with $n(n_g) \circ \nu_{0_X} = \nu_{0_{\mathcal{N}(g)}} \circ n_g$ and $n(g) \circ \nu_{0_Y} = \nu_{0_X} \circ g$. If we apply now the second part of Lemma 2.2.8 to

$$\begin{tikzcd}
\mathcal{N}(0_{\mathcal{N}(g)}) \arrow[r, "n(n_g)"] \arrow[d, "n_{0_{\mathcal{N}(g)}}"] \arrow[d, dashed, dash, bend right=70, ""{name=L}] & \mathcal{N}(0_X) \arrow[r, "n(g)"] & \mathcal{N}(0_Y) \arrow[d, "n_{0_Y}"] \\
0 \arrow[rr, "\mathrm{id}_0"] \arrow[d, "0_{\mathcal{N}(g)}"] & & 0 \arrow[d, "0_Y"] \\
\mathcal{N}(g) \arrow[r, "n_g"] \arrow[rr, dashed, bend right=30, ""{name=B, above}] & X \arrow[r, "g"] & Y
\arrow[phantom, from=L, to=1-1, "*^{\mathcal{N}(0_{\mathcal{N}(g)})} \Rightarrow"]
\arrow[phantom, from=B, to=3-2, "\Uparrow \nu_g"]
\end{tikzcd}$$

we get a nullhomotopy in $\Theta(n(n_g) \cdot n(g))$. Since $\mathcal{N}(0_Y)$ is discrete (see Lemma 2.4.4), we can conclude that $n(n_g) \cdot n(g) = 0$.

Step 2: Consider now the diagram

$$\begin{tikzcd}
X \arrow[rrr, "g"] \arrow[d, "\eta_X"] \arrow[dd, bend right=50, "0^X"'] \arrow[ddd, dash, dashed, bend right=90, "\gamma_{0^X}\Rightarrow"] & & & Y \arrow[d, "\eta_Y"'] \arrow[dd, bend left=50, "0^Y"] \arrow[ddd, dash, dashed, bend left=90, "\Leftarrow\gamma_{0^Y}"'] \\
\pi_0(X) \arrow[rrr, dotted, "\pi_0(g)"] \arrow[d, "n_{c_{0^X}}"] \arrow[dd, dash, dashed, bend left=70, "\Leftarrow\nu_{c_{0^X}}"] & & & \pi_0(Y) \arrow[d, "n_{c_{0^Y}}"'] \arrow[dd, dash, dashed, bend right=70, "\nu_{c_{0^Y}}\Rightarrow"'] \\
0 \arrow[d, "c_{0^X}"] & & & 0 \arrow[d, "c_{0^Y}"'] \\
\mathcal{C}(0^X) \arrow[rrr, dotted, "c(g)"'] & & & \mathcal{C}(0^Y)
\end{tikzcd}$$

Starting from $g\colon X \to Y$ and applying the dual of Lemma 2.2.8, we get a unique arrow $c(g)$ such that $\gamma_{0^X} \circ c(g) = g \circ \gamma_{0^Y}$. If we start now from the arrow $c(g)\colon \mathcal{C}(0^X) \to \mathcal{C}(0^Y)$ and we work as in Step 1, we get a unique arrow $\pi_0(g)$ such that $\pi_0(g) \circ \nu_{c_{0^Y}} = \nu_{c_{0^X}} \circ c(g)$. Moreover, $g \cdot \eta_Y = \eta_X \cdot \pi_0(g)$. To see this, we compose with $\nu_{c_{0^Y}}$

$$\eta_X \cdot \pi_0(g) \circ \nu_{c_{0^Y}} = \eta_X \circ \nu_{c_{0^X}} \circ c(g) = \gamma_{0^X} \circ c(g) = g \circ \gamma_{0^Y} = g \cdot \eta_Y \circ \nu_{c_{0^Y}}$$

and we can conclude using the first part of Proposition 2.2.2.

If we repeat the same argument starting from $n_g\colon \mathcal{N}(g) \to X$ instead of $g\colon X \to Y$, we get arrows

$$c(n_g)\colon \mathcal{C}(0^{\mathcal{N}(g)}) \to \mathcal{C}(0^X) \quad \text{and} \quad \pi_0(n_g)\colon \pi_0(\mathcal{N}(g)) \to \pi_0(X)$$

unique with $\gamma_{0^{\mathcal{N}(g)}} \circ c(n_g) = n_g \circ \gamma_{0^X}$ and $\pi_0(n_g) \circ \nu_{c_{0^X}} = \nu_{c_{0^{\mathcal{N}(g)}}} \circ c(n_g)$. Moreover, by Proposition 2.2.2 we get $n_g \cdot \eta_X = \eta_{\mathcal{N}(g)} \cdot \pi_0(n_g)$ as above.

Finally, the nullhomotopy $\nu_g \in \Theta(n_g \cdot g)$ and the dual of the second part of Lemma 2.2.8 allow us to prove that the composite arrow

$$\mathcal{C}(0^{\mathcal{N}(g)}) \xrightarrow{c(n_g)} \mathcal{C}(0^X) \xrightarrow{c(g)} \mathcal{C}(0^Y)$$

is the zero arrow (because $\mathcal{C}(0^Y)$ is codiscrete). By the second part of Lemma 2.2.8 again, we can conclude that also the composite arrow

$$\pi_0(\mathcal{N}(g)) \xrightarrow{\pi_0(n_g)} \pi_0(X) \xrightarrow{\pi_0(g)} \pi_0(Y)$$

is the zero arrow, as needed.

Step 3: In order to construct the connecting arrow $\delta\colon \mathcal{N}(0_Y) \to \pi_0(\mathcal{N}(g))$, consider the following diagram

$$\begin{tikzcd}
& & \mathcal{N}(0_Y) \arrow[dl, "0_X^{\mathcal{N}(0_Y)}"'] \arrow[d, "n_{0_Y}"] \arrow[dd, dashed, bend left=70, "\Leftarrow \nu_{0_Y}"'] \arrow[ddll, dotted, bend right, "\Delta"'] \\
& X \arrow[r, phantom] & 0 \arrow[d, "0_Y"] \\
\mathcal{N}(g) \arrow[ur, phantom] \arrow[r, "n_g"] \arrow[d, "\eta_{\mathcal{N}(g)}"'] \arrow[rr, dashed, bend right, "\nu_g \Uparrow"] & {} & Y \\
\pi_0(\mathcal{N}(g))
\end{tikzcd}$$

Since $\nu_{0_Y} \in \Theta(n_{0_Y} \cdot 0_Y) = \Theta(0_X^{\mathcal{N}(0_Y)} \cdot g)$, the universal property of the Θ-kernel of g gives a unique arrow $\Delta\colon \mathcal{N}(0_Y) \to \mathcal{N}(g)$ such that $\Delta \cdot n_g = 0_X^{\mathcal{N}(0_Y)}$ and $\Delta \circ \nu_g = \nu_{0_Y}$. We put

$$\delta = \Delta \cdot \eta_{\mathcal{N}(g)}\colon \mathcal{N}(0_Y) \to \mathcal{N}(g) \to \pi_0(\mathcal{N}(g))$$

Step 4: To check that the composite arrow

$$\mathcal{N}(0_X) \xrightarrow{n(g)} \mathcal{N}(0_Y) \xrightarrow{\delta} \pi_0(\mathcal{N}(g))$$

is the zero arrow, consider the diagram

$$\begin{tikzcd}
\mathcal{N}(\mathrm{id}_{\mathcal{N}(g)}) \arrow[dd, "n_{\mathrm{id}_{\mathcal{N}(g)}}"] \arrow[dd, dashed, bend right=60, "\nu_{\mathrm{id}_{\mathcal{N}(g)}} \Rightarrow"'] & & \mathcal{N}(\mathrm{id}_X) \arrow[ll, dotted, "r_X"'] \arrow[ddll, dotted, "s_X"'] \arrow[ddll, dashed, bend left, "\nu_{X,g} \Leftarrow"] \arrow[dd, "n_{\mathrm{id}_X}"] \arrow[dd, dashed, bend right, "\nu_{\mathrm{id}_X} \Rightarrow"'] & & \mathcal{N}(0_X) \arrow[ll, dotted, "t_X"'] \arrow[r, "n(g)"] \arrow[dl, "n_{0_X}"'] \arrow[ddll, dashed, bend left, "\Leftarrow \nu_{0_X}"] & \mathcal{N}(0_Y) \arrow[d, "n_{0_Y}"] \arrow[dd, dashed, bend right=60, "\nu_{0_Y} \Rightarrow"'] \\
& & & 0 \arrow[dl, "0_X"'] & & 0 \arrow[d, "0_Y"] \\
\mathcal{N}(g) \arrow[rr, "n_g"] \arrow[d, "\eta_{\mathcal{N}(g)}"'] \arrow[rrrrr, dashed, bend right=20, "\Uparrow \nu_g"'] & & X \arrow[rrr, "g"] & & & Y \\
\pi_0(\mathcal{N}(g))
\end{tikzcd}$$

By the universal property of $\mathcal{N}(\mathrm{id}_X)$, we get a unique arrow t_X such that $t_X \cdot n_{\mathrm{id}_X} = 0_X^{\mathcal{N}(0_X)}$ and $t_X \circ \nu_{\mathrm{id}_X} = \nu_{0_X}$. By the universal property of $\mathcal{N}(g)$, we get a unique arrow s_X such that $s_X \cdot n_g = n_{\mathrm{id}_X}$ and $s_X \circ \nu_g = \nu_{\mathrm{id}_X} \circ g$. Since $\nu_{\mathrm{id}_X} \in \Theta(n_{\mathrm{id}_X}) = \Theta(s_X \cdot n_g)$ and $\nu_{\mathrm{id}_X} \circ g = s_X \circ \nu_g$, the fact that $\mathcal{N}(g)$ is strong gives us a unique nullhomotopy $\nu_{X,g} \in \Theta(s_X)$ such that $\nu_{X,g} \circ n_g = \nu_{\mathrm{id}_X}$. Finally, since $\nu_{X,g} \in \Theta(s_X)$, by the universal property of $\mathcal{N}(\mathrm{id}_{\mathcal{N}(g)})$ we get a unique arrow r_X such that $r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} = s_X$ and $r_X \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} = \nu_{X,g}$. Now we can check that $n(g) \cdot \Delta = t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}}$ using the first part of Proposition 2.2.2:

- $n(g) \cdot \Delta \cdot n_g = n(g) \cdot 0_X^{\mathcal{N}(0_Y)} = 0_X^{\mathcal{N}(0_X)} = t_X \cdot n_{\mathrm{id}_X} = t_X \cdot s_X \cdot n_g = t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g$

- $n(g)\cdot\Delta\circ\nu_g = n(g)\circ\nu_{0_Y} = \nu_{0_X}\circ g = t_X\circ\nu_{\mathrm{id}_X}\circ g = t_X\cdot s_X\circ\nu_g = t_X\cdot r_X\cdot n_{\mathrm{id}_{\mathcal{N}(g)}}\circ\nu_g$

Thanks to the previous equation, we have that

$$t_X\cdot r_X\circ\nu_{\mathrm{id}_{\mathcal{N}(g)}}\circ\eta_{\mathcal{N}(g)}\in\Theta(t_X\cdot r_X\cdot n_{\mathrm{id}_{\mathcal{N}(g)}}\cdot\eta_{\mathcal{N}(g)})=\Theta(n(g)\cdot\Delta\cdot\eta_{\mathcal{N}(g)})=\Theta(n(g)\cdot\delta)$$

and, since $\pi_0(\mathcal{N}(g))$ is discrete (see Lemma 2.4.4), we can conclude that $n(g)\cdot\delta$ is the zero arrow.

Step 5: The fact that the composite arrow

$$\mathcal{N}(0_Y)\xrightarrow{\ \delta\ }\pi_0(\mathcal{N}(g))\xrightarrow{\pi_0(n_g)}\pi_0(X)$$

is the zero arrow is obvious:

$$\delta\cdot\pi_0(n_g)=\Delta\cdot\eta_{\mathcal{N}(g)}\cdot\pi_0(n_g)=\Delta\cdot n_g\cdot\eta_X=0_X^{\mathcal{N}(0_Y)}\cdot\eta_X=0_{\pi_0(X)}^{\mathcal{N}(0_Y)}$$

3.0.3. Example. We go back to the classical example treated in Section 2.5, where $(\mathcal{B},\Theta)=(\mathbf{Arr}(\mathcal{A}),\Theta_\Delta)$. We assume that $\mathcal{A}$ has a zero object, pullbacks and cokernels. Starting from a Θ_Δ-kernel

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\langle x{,}g\rangle"'] & X \arrow[r, "g"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
X_0\times_{g_0,y}Y \arrow[r, "y'"'] \arrow[urr, dashed, "g'_0"] & X_0 \arrow[r, "g_0"'] & Y_0
\end{tikzcd}$$

the associated snail sequence is

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}\langle x,g\rangle \arrow[r] & \mathrm{Ker}(x) \arrow[r] & \mathrm{Ker}(y) \arrow[r, "\delta_0"] & \mathrm{Cok}\langle x,g\rangle \arrow[r] & \mathrm{Cok}(x) \arrow[r] & \mathrm{Cok}(y)
\end{tikzcd}$$

where the unlabelled arrows are the obvious ones and δ_0 is given by

$$\delta_0=\langle 0,k_y\rangle\cdot c_{\langle x,g\rangle}\colon\mathrm{Ker}(y)\longrightarrow X_0\times_{g_0,y}Y\longrightarrow\mathrm{Cok}\langle x,g\rangle$$

so that the bottom line precisely is the snail sequence appearing in [19, 25].

3.0.4. Remark. Example 3.0.3 gives us the occasion to clarify the problem discussed in Remark 3.0.1. If, instead of the sequence just constructed, we try to get a sequence of the form π_1-π_0, in the case of $(\mathbf{Arr}(\mathcal{A}),\Theta_\Delta)$ we would get

$$\begin{tikzcd}
\mathrm{Ker}\langle x,g\rangle \arrow[r] \arrow[d] & \mathrm{Ker}(x) \arrow[r] \arrow[d] & \mathrm{Ker}(y) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathrm{Cok}\langle x,g\rangle \arrow[r] & \mathrm{Cok}(x) \arrow[r] & \mathrm{Cok}(y)
\end{tikzcd}$$

so that the only possible connecting homomorphism would be the zero morphism.

We end this section with a lemma about the arrow $\Delta\colon\mathcal{N}(0_Y)\to\mathcal{N}(g)$ defined in Step 3 in Construction 3.0.2 above. This lemma, needed in the next section, is a generalization of Lemma 3.2 in [23].

3.0.5. Lemma. *Consider the notation introduced in Construction 3.0.2 above. The diagram*

$$\begin{tikzcd}\mathcal{N}(0_Y) \arrow[r, "\Delta"] & \mathcal{N}(g) \arrow[r, "n_g"] & X\end{tikzcd}$$

is a kernel (in the usual categorical sense).

Proof. The fact that $\Delta \cdot n_g = 0_X^{\mathcal{N}(0_Y)}$ is part of the definition of Δ (see Step 3 in Construction 3.0.2). Consider now an arrow $a \colon A \to \mathcal{N}(g)$ such that $a \cdot n_g = 0_X^A$. This implies that $a \cdot n_g \cdot g = 0_Y^A$ and then $a \circ \nu_g \in \Theta(0_Y^A)$. By the universal property of $\mathcal{N}(0_Y)$, we get a unique arrow $b \colon A \to \mathcal{N}(0_Y)$ such that $b \circ \nu_{0_Y} = a \circ \nu_g$. To check that $b \cdot \Delta = a$ we use part 1 of Proposition 2.2.2:

- $b \cdot \Delta \cdot n_g = b \cdot 0_X^{\mathcal{N}(0_Y)} = 0_X^A = a \cdot n_g$
- $b \cdot \Delta \circ \nu_g = b \circ \nu_{0_Y} = a \circ \nu_g$

As far as the uniqueness of b is concerned, let $b' \colon A \to \mathcal{N}(0_Y)$ be such that $b' \cdot \Delta = a$. It follows that $b' \circ \nu_{0_Y} = b' \cdot \Delta \circ \nu_g = a \circ \nu_g$, so that $b' = b$ by definition of b. ■

4. Exactness of the snail sequence

We work under the same assumptions as in Section 3: the nullhomotopy structure Θ satisfies the reduced interchange condition 2.1.4, the category $\mathcal{B}$ is equipped with a Θ-strong zero object and has all the needed strong Θ-kernels and strong Θ-cokernels. We moreover assume that $\mathcal{B}$ has pullbacks.

In this section we are going to study the $\mathcal{S}$-exactness of the snail sequence built in Section 3, where $\mathcal{S}$ is a suitable class of morphisms in $\mathcal{B}$. The notion of $\mathcal{S}$-exactness (Definition 4.0.3) is a generalization to categories with nullhomotopies of the notion of exactness in the context of regular pointed categories, where the role of $\mathcal{S}$ is played by the class of regular epimorphisms (see, for example, [6]). This notion, in turn also generalizes the classical notion of exactness in abelian categories.

4.0.1. Condition. We fix a class of arrows $\mathcal{S}$ in $\mathcal{B}$ satisfying the following conditions:

1. $\mathcal{S}$ is closed under composition,
2. $\mathcal{S}$ is stable under pullbacks,
3. $\mathcal{S}$ contains all the identities,
4. $\mathcal{S}$ has the left cancellation property: if a composite $f \cdot g$ is in $\mathcal{S}$, then g is in $\mathcal{S}$.

4.0.2. Remark. Note that such a class $\mathcal{S}$ contains all the isomorphisms. Note also that, if we ask that all monomorphisms in $\mathcal{S}$ are isomorphisms, we get the notion of surjection-like class of arrows discussed in [16] (where condition 4 is called strong right cancellation property).

4.0.3. Definition. Consider the diagram in $(\mathcal{B}, \Theta)$

$$\begin{tikzcd}
W \arrow[r, "f"'] \arrow[rr, bend left, dashed, no head, "\Downarrow \varphi"] & X \arrow[r, "g"'] & Y
\end{tikzcd}$$

We say that (f, φ, g) is $\mathcal{S}$-exact if the unique factorization of (f, φ) through the Θ-kernel (n_g, ν_g) of g is in $\mathcal{S}$.
Note that if Y is discrete, to be $\mathcal{S}$-exact means that the unique factorization of f through the categorical kernel of g is in $\mathcal{S}$.

We now proceed to introduce the notions of $\mathcal{S}$-proper and $\mathcal{S}$-global objects generalizing the corresponding notions of proper morphism and object with a global support. Recall that a proper morphism in a pointed regular and protomodular category is a morphism such that (the first morphism in) its factorization through the kernel of its cokernel is a regular epimorphism (see [6], [11], [25]), while an object with a global support is an object such that the terminal morphism with the object as domain is a regular epimorphism (see [4], [5], [15]). As already mentioned in the Introduction, in an abelian category every morphism is proper and every object is global. This is the reason why these notions disappear in the abelian context, whereas they are used in more general contexts to develop basic homological algebra (see [3]).

4.0.4. Definition. Let Y be an object in $(\mathcal{B}, \Theta)$.

1. Y is $\mathcal{S}$*-proper* if $\overline{y} \colon \mathcal{N}(\mathrm{id}_Y) \to \mathcal{N}(\eta_Y)$ is in $\mathcal{S}$, where $\overline{y}$ is the unique arrow such that $\overline{y} \cdot n_{\eta_Y} = n_{\mathrm{id}_Y}$ and $\overline{y} \circ \nu_{\eta_Y} = \nu_{\mathrm{id}_Y} \circ \eta_Y$

$$\begin{tikzcd}
\mathcal{N}(\mathrm{id}_Y) \arrow[rr, "n_{\mathrm{id}_Y}"] \arrow[rr, bend left, dashed, no head, "\Downarrow \nu_{\mathrm{id}_Y}"] \arrow[dr, "\overline{y}"'] & & Y \arrow[r, "\eta_Y"] & \pi_0(Y) \\
& \mathcal{N}(\eta_Y) \arrow[ur, "n_{\eta_Y}"] \arrow[urr, bend right, dashed, no head, "\Uparrow \nu_{\eta_Y}"'] & &
\end{tikzcd}$$

2. Y is $\mathcal{S}$*-global* if $\eta_Y \colon Y \to \pi_0(Y)$ is in $\mathcal{S}$.

4.0.5. Example. Let us consider once again $(\mathcal{B}, \Theta) = (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$ and take as $\mathcal{S}$ the class of arrows

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
X_0 \arrow[r, "g_0"'] & Y_0
\end{tikzcd}$$

such that both g and g_0 are regular epimorphisms in $\mathcal{A}$. Observe that in $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, the factorization $\overline{y}$ of Definition 4.0.4 is essentially the factorization of the arrow part y

of an object (Y, y, Y_0) through the kernel of its cokernel

$$\begin{tikzcd}
Y \arrow[rr, "\mathrm{id}_Y"] \arrow[d, "\mathrm{id}_Y"'] \arrow[ddr, "\mathrm{id}_Y"] & & Y \arrow[r] \arrow[d, "y"] & 0 \arrow[d] \\
Y \arrow[rr, "y"] \arrow[ddr, "\overline{y}"'] & & Y_0 \arrow[r, "c_y"'] & \mathrm{Cok}(y) \\
 & Y \arrow[uur, "\mathrm{id}_Y"] \arrow[d, "\overline{y}"'] & & \\
 & \mathrm{Ker}(c_y) \arrow[uur, "k_{c_y}"'] & &
\end{tikzcd}$$

To start, assume just that $\mathcal{A}$ has a zero object, kernels and cokernels. Under these conditions, $\mathcal{S}$ contains isomorphisms, each object is $\mathcal{S}$-global and an object (Y, y, Y_0) is $\mathcal{S}$-proper precisely when the factorization of y through the kernel of its cokernel is a regular epimorphism. This is the definition of proper arrow used in [6, 25]. Now keep in mind that limits and colimits in $\mathbf{Arr}(\mathcal{A})$ are computed level-wise in $\mathcal{A}$. If we add the assumption that $\mathcal{A}$ is regular, then $\mathcal{S}$ satisfies Condition 4.0.1.

4.0.6. Proposition. *Consider an arrow g in $(\mathcal{B}, \Theta)$ together with its Θ-kernel*

$$\begin{tikzcd}
\mathcal{N}(g) \arrow[r, "n_g"'] \arrow[rr, dashed, bend left, "\Downarrow \nu_g"] & X \arrow[r, "g"'] & Y
\end{tikzcd}$$

If Y and $\mathcal{N}(g)$ are $\mathcal{S}$-proper and X is $\mathcal{S}$-global where $\mathcal{S}$ is a class of morphisms in $\mathcal{B}$ satisfying Condition 4.0.1, then the associated snail sequence

$$\mathcal{N}(0_{\mathcal{N}(g)}) \xrightarrow{n(n_g)} \mathcal{N}(0_X) \xrightarrow{n(g)} \mathcal{N}(0_Y) \xrightarrow{\delta} \pi_0(\mathcal{N}(g)) \xrightarrow{\pi_0(n_g)} \pi_0(X) \xrightarrow{\pi_0(g)} \pi_0(Y)$$

is $\mathcal{S}$-exact in $\mathcal{N}(0_X), \mathcal{N}(0_Y)$ and $\pi_0(X)$.

Proof. We use the constructions and the notations introduced in Section 3. In particular, for the arrows t_X, r_X and s_X and for the nullhomotopy $\nu_{X,g}$, see Step 4 of Section 3.

Exactness in $\mathcal{N}(0_X)$: we are going to prove that the diagram

$$\mathcal{N}(0_{\mathcal{N}(g)}) \xrightarrow{n(n_g)} \mathcal{N}(0_X) \xrightarrow{n(g)} \mathcal{N}(0_Y)$$

constructed in Step 1 of Section 3, is a kernel. This implies the $\mathcal{S}$-exactness because $\mathcal{S}$ contains the isomorphisms. Consider an arrow $a \colon A \to \mathcal{N}(0_X)$ such that $a \cdot n(g) = 0^A_{\mathcal{N}(0_Y)}$. As a preliminary step, we check that $a \cdot t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} = 0^A_{\mathcal{N}(g)}$ using the cancellation property provided by Proposition 2.2.2:

- $t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g = t_X \cdot s_X \cdot n_g = t_X \cdot n_{\mathrm{id}_X} = 0^{\mathcal{N}(0_X)}_X = 0^{\mathcal{N}(0_X)}_{\mathcal{N}(g)} \cdot n_g$, and then $a \cdot t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g = a \cdot 0^{\mathcal{N}(0_X)}_{\mathcal{N}(g)} \cdot n_g = 0^A_{\mathcal{N}(g)} \cdot n_g$

- $a \cdot t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \circ \nu_g = a \cdot t_X \cdot s_X \circ \nu_g = a \cdot t_X \circ \nu_{\mathrm{id}_X} \circ g = a \circ \nu_{0_X} \circ g$
  $= a \cdot n(g) \circ \nu_{0_Y} = *^A_{\mathcal{N}(0_Y)} \circ 0_Y^{\mathcal{N}(0_Y)} = *^A_{\mathcal{N}(0_Y)} \circ 0_{\mathcal{N}(g)}^{\mathcal{N}(0_Y)} \cdot n_g \cdot g$
  $= 0^A_{\mathcal{N}(g)} \circ \nu_g$

By the universal property of $\mathcal{N}(0_{\mathcal{N}(g)})$, there exists a unique arrow $a' \colon A \to \mathcal{N}(0_{\mathcal{N}(g)})$ such that $a' \circ \nu_{0_{\mathcal{N}(g)}} = a \cdot t_X \cdot r_X \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}}$. Now we show that $a' \cdot n(n_g) = a$ using once again Proposition 2.2.2:

- $a' \cdot n(n_g) \cdot n_{0_X} = 0^A = a \cdot n_{0_X}$
- $a' \cdot n(n_g) \circ \nu_{0_X} = a' \circ \nu_{0_{\mathcal{N}(g)}} \circ n_g = a \cdot t_X \cdot r_X \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g = a \cdot t_X \circ \nu_{X,g} \circ n_g$
  $= a \cdot t_X \circ \nu_{\mathrm{id}_X} = a \circ \nu_{0_X}$

As far as the uniqueness of the factorization is concerned, let $a'' \colon A \to \mathcal{N}(0_{\mathcal{N}(g)})$ be such that $a'' \cdot n(n_g) = a$. Since clearly $a' \cdot n_{0_{\mathcal{N}(g)}} = 0^A = a'' \cdot n_{0_{\mathcal{N}(g)}}$, to have $a'' = a'$ it remains to show that $a' \circ \nu_{0_{\mathcal{N}(g)}} = a'' \circ \nu_{0_{\mathcal{N}(g)}}$. Since the Θ-kernel $\mathcal{N}(g)$ is strong, it is enough to compose with n_g:

$$a' \circ \nu_{0_{\mathcal{N}(g)}} \circ n_g = a' \cdot n(n_g) \circ \nu_{0_X} = a'' \cdot n(n_g) \circ \nu_{0_X} = a'' \circ \nu_{0_{\mathcal{N}(g)}} \circ n_g$$

Exactness in $\mathcal{N}(0_Y)$: We are going to prove that the unique arrow σ making commutative the following diagram is in $\mathcal{S}$:

$$\begin{tikzcd}
\mathcal{N}(\delta) \arrow[r, "n_\delta"] & \mathcal{N}(0_Y) \arrow[r, "\delta"] & \pi_0(\mathcal{N}(g)) \\
& \mathcal{N}(0_X) \arrow[u, "n(g)"'] \arrow[ul, "\sigma"] &
\end{tikzcd}$$

For this, consider the factorization Δ' of $n_\delta \cdot \Delta$ through the (homotopy) kernel of $\eta_{\mathcal{N}(g)}$:

$$\begin{tikzcd}
\mathcal{N}(\delta) \arrow[r, "n_\delta"] \arrow[drr, "\Delta'"'] & \mathcal{N}(0_Y) \arrow[r, "\Delta"] & \mathcal{N}(g) \arrow[r, "\eta_{\mathcal{N}(g)}"] & \pi_0(\mathcal{N}(g)) \\
& & \mathcal{N}(\eta_{\mathcal{N}(g)}) \arrow[u, "n_{\eta_{\mathcal{N}(g)}}"'] &
\end{tikzcd}$$

Consider also the factorization $\overline{z}$ obtained when, in Definition 4.0.4.1, we start from the object $\mathcal{N}(g)$:

$$\begin{tikzcd}
\mathcal{N}(\mathrm{id}_{\mathcal{N}(g)}) \arrow[rr, "n_{\mathrm{id}_{\mathcal{N}(g)}}"] \arrow[dr, "\overline{z}"'] & & \mathcal{N}(g) \\
& \mathcal{N}(\eta_{\mathcal{N}(g)}) \arrow[ur, "n_{\eta_{\mathcal{N}(g)}}"'] &
\end{tikzcd}$$

Using the arrows σ, Δ' and $\overline{z}$, we can build up the diagram

$$\begin{tikzcd}
\mathcal{N}(\delta) \arrow[rr, "\Delta'"] & & \mathcal{N}(\eta_{\mathcal{N}(g)}) \\
\mathcal{N}(0_X) \arrow[u, "\sigma"] \arrow[r, "t_X"'] & \mathcal{N}(\mathrm{id}_X) \arrow[r, "r_X"'] & \mathcal{N}(\mathrm{id}_{\mathcal{N}(g)}) \arrow[u, "\overline{z}"']
\end{tikzcd}$$

To check its commutativity, we compose with $n_{\eta_{\mathcal{N}(g)}}$, which is a monomorphism because $\pi_0(\mathcal{N}(g))$ is discrete:

$$\sigma \cdot \Delta' \cdot n_{\eta_{\mathcal{N}(g)}} = \sigma \cdot n_\delta \cdot \Delta = n(g) \cdot \Delta = t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} = t_X \cdot r_X \cdot \overline{z} \cdot n_{\eta_{\mathcal{N}(g)}}$$

By assumption, $\mathcal{N}(g)$ is $\mathcal{S}$-proper, that is, $\overline{z} \in \mathcal{S}$. Since $\mathcal{S}$ is stable under pullbacks, to prove that $\sigma \in \mathcal{S}$ it remains to show that the previous square is a pullback. For this, consider two arrows

$$a\colon A \to \mathcal{N}(\mathrm{id}_{\mathcal{N}(g)}) \quad \text{and} \quad b\colon A \to \mathcal{N}(\delta)$$

such that $a \cdot \overline{z} = b \cdot \Delta'$. This equality implies that

$$a \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g = a \cdot \overline{z} \cdot n_{\eta_{\mathcal{N}(g)}} \cdot n_g = b \cdot \Delta' \cdot n_{\eta_{\mathcal{N}(g)}} \cdot n_g = b \cdot n_\delta \cdot \Delta \cdot n_g = b \cdot n_\delta \cdot 0_X^{\mathcal{N}(0_Y)} = 0_X^A$$

so that we can consider the nullhomotopy $a \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g \in \Theta(a \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g) = \Theta(0_X^A)$. By the universal property of $\mathcal{N}(0_X)$, we get a unique arrow $c\colon A \to \mathcal{N}(0_X)$ such that $c \circ \nu_{0_X} = a \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g$. We have to verify that c is a factorization of a and b. To check that $c \cdot \sigma = b$, we compose with n_δ, which is a monomorphism because $\pi_0(\mathcal{N}(g))$ is discrete, and then we use the universal property of $\mathcal{N}(0_Y)$:

$$c \cdot \sigma \cdot n_\delta \circ \nu_{0_Y} = c \cdot n(g) \circ \nu_{0_Y} = c \circ \nu_{0_X} \circ g = a \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g \cdot g = a \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \circ \nu_g = b \cdot n_\delta \cdot \Delta \circ \nu_g = b \cdot n_\delta \circ \nu_{0_Y}$$

where the fourth equality follows from the reduced interchange. To check that $c \cdot t_X \cdot r_X = a$, we use Proposition 2.2.2 and we compose a first time with $n_{\mathrm{id}_{\mathcal{N}(g)}}$ and a second time with $\nu_{\mathrm{id}_{\mathcal{N}(g)}}$. In the first case we use Proposition 2.2.2 again, and in the second case we use that the Θ-kernel $\mathcal{N}(g)$ is strong:

- $c \cdot t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g = c \cdot t_X \cdot s_X \cdot n_g = c \cdot t_X \cdot n_{\mathrm{id}_X} = c \cdot 0_X^{\mathcal{N}(0_X)} = 0_X^A = a \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g$
- $c \cdot t_X \cdot r_X \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \circ \nu_g = c \cdot t_X \cdot s_X \circ \nu_g = c \cdot t_X \circ \nu_{\mathrm{id}_X} \circ g = c \circ \nu_{0_X} \circ g$
  $= a \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g \cdot g = a \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \circ \nu_g$
- $c \cdot t_X \cdot r_X \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g = c \cdot t_X \circ \nu_{X,g} \circ n_g = c \cdot t_X \circ \nu_{\mathrm{id}_X} = c \circ \nu_{0_X} = a \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g$

As far as the uniqueness of the factorization c is concerned, let $c'\colon A \to \mathcal{N}(0_X)$ be such that $c' \cdot \sigma = b$ and $c' \cdot t_X \cdot r_X = a$. To verify that $c' = c$, we go back to the definition of c:

$$c' \circ \nu_{0_X} = c' \cdot t_X \circ \nu_{\mathrm{id}_X} = c' \cdot t_X \circ \nu_{X,g} \circ n_g = c' \cdot t_X \cdot r_X \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g = a \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g$$

Exactness in $\pi_0(X)$: We are going to prove that the unique arrow σ making commutative the following diagram is in $\mathcal{S}$:

$$\begin{tikzcd}
\mathcal{N}(\pi_0(g)) \arrow[r, "n_{\pi_0(g)}"] & \pi_0(X) \arrow[r, "\pi_0(g)"] & \pi_0(Y) \\
& \mathcal{N}(0_X) \arrow[lu, "\sigma"] \arrow[u, "\pi_0(n_g)"'] &
\end{tikzcd}$$

We can split the pullback describing the Θ-kernel $\mathcal{N}(g)$ (see 2.2.7) in two steps:

$$\begin{tikzcd}
\mathcal{N}(g) \arrow[rr, "g'"] \arrow[rd, "a"] \arrow[dd, "n_g"'] & & \mathcal{N}(\mathrm{id}_Y) \arrow[rd, "\overline{y}"] \arrow[dd, "n_{\mathrm{id}_Y}"'] & \\
& P \arrow[rr, "c"] \arrow[ld, "b"] & & \mathcal{N}(\eta_Y) \arrow[ld, "n_{\eta_Y}"] \\
X \arrow[rr, "g"'] & & Y &
\end{tikzcd}$$

Now observe that

$$b \cdot \eta_X \cdot \pi_0(g) = b \cdot g \cdot \eta_Y = c \cdot n_{\eta_Y} \cdot \eta_Y = c \cdot 0_{\pi_0(Y)}^{\mathcal{N}(\eta_Y)} = 0_{\pi_0(Y)}^{P}$$

so that there exists a unique arrow $t \colon P \to \mathcal{N}(\pi_0(g))$ such that $t \cdot n_{\pi_0(g)} = b \cdot \eta_X$. In fact, more is true: the square

$$\begin{tikzcd}
P \arrow[r, "t"] \arrow[d, "b"'] & \mathcal{N}(\pi_0(g)) \arrow[d, "n_{\pi_0(g)}"] \\
X \arrow[r, "\eta_X"'] & \pi_0(X)
\end{tikzcd}$$

is a pullback. This can be proved using the following commutative diagrams:

$$\begin{tikzcd}
P \arrow[r, "c"] \arrow[d, "b"'] \arrow[rd, phantom, "(1)"] & \mathcal{N}(\eta_Y) \arrow[r] \arrow[d, "n_{\eta_Y}"] \arrow[rd, phantom, "(2)"] & 0 \arrow[d] \\
X \arrow[r, "g"'] & Y \arrow[r, "\eta_Y"'] & \pi_0(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
P \arrow[r, "t"] \arrow[d, "b"'] \arrow[rd, phantom, "(3)"] & \mathcal{N}(\pi_0(g)) \arrow[r] \arrow[d, "n_{\pi_0(g)}"] \arrow[rd, phantom, "(4)"] & 0 \arrow[d] \\
X \arrow[r, "\eta_X"'] & \pi_0(X) \arrow[r, "\pi_0(g)"'] & \pi_0(Y)
\end{tikzcd}$$

Since (1) and (2) are pullbacks, then (1)+(2) is a pullback. This implies that (3)+(4) is a pullback because $\eta_X \cdot \pi_0(g) = g \cdot \eta_Y$. Since (4) also is a pullback, we can conclude that

(3) is a pullback. Finally, observe that $a \cdot t = \eta_{\mathcal{N}(g)} \cdot \sigma$. To check this equality, we can compose with $n_{\pi_0(g)}$, which is a monomorphism because $\pi_0(Y)$ is discrete:

$$a \cdot t \cdot n_{\pi_0(g)} = a \cdot b \cdot \eta_X = n_g \cdot \eta_X = \eta_{\mathcal{N}(g)} \cdot \pi_0(n_g) = \eta_{\mathcal{N}(g)} \cdot \sigma \cdot n_{\pi_0(g)}$$

Now we can conclude as follows: since Y is $\mathcal{S}$-proper, $\overline{y} \in \mathcal{S}$ and then $a \in \mathcal{S}$ by stability under pullbacks. Since X is $\mathcal{S}$-global, $\eta_X \in \mathcal{S}$ and then $t \in \mathcal{S}$ once again by stability under pullbacks. Since $\mathcal{S}$ is closed under composition, the equality $a \cdot t = \eta_{\mathcal{N}(g)} \cdot \sigma$ and the left cancellation property imply that $\sigma \in \mathcal{S}$. ∎

The exactness of the snail sequence in $\pi_0(\mathcal{N}(g))$ requires one more assumption on the class $\mathcal{S}$.

4.0.7. Condition. Let $\mathcal{S}$ be a class of arrows in $\mathcal{B}$ as in Condition 4.0.1. We say that $\mathcal{S}$ satisfies condition (Sub) if, in the following commutative diagram, if $K(x, y)$ and g are in $\mathcal{S}$, then $K(g, g_0)$ also is in $\mathcal{S}$.

$$\begin{tikzcd}
 & \mathrm{Ker}(x) \arrow[r, "K(g{,}g_0)"] \arrow[d, swap, "k_x"] & \mathrm{Ker}(y) \arrow[d, "k_y"] \\
\mathrm{Ker}(g) \arrow[r, "k_g"] \arrow[d, swap, "K(x{,}y)"] & X \arrow[r, "g"] \arrow[d, swap, "x"] & Y \arrow[d, "y"] \\
\mathrm{Ker}(g_0) \arrow[r, swap, "k_{g_0}"] & X_0 \arrow[r, swap, "g_0"] & Y_0
\end{tikzcd}$$

4.0.8. Remark. Condition (Sub), with $\mathcal{S} = \{$regular epimorphisms$\}$, has been isolated by Bourn in [6] (see also [3]) and is a special case of the snake lemma. It also appears in [13] under the name of *symmetric saturation property*. It has been used as well in [25] to prove the basic version of the snail lemma which, in the context of pointed protomodular regular categories, subsumes the snake lemma. The precise situation has been explained to us by Zurab Janelidze in a private communication and we report it here for the sake of completeness. Assume that the category $\mathcal{B}$ is pointed and regular. Then condition (Sub) is equivalent to the subtractivity of $\mathcal{B}$. First, by Theorem 3 in [19], the subtractivity of $\mathcal{B}$ is equivalent to the fact that the incomplete snail lemma holds in $\mathcal{B}$. Now, the incomplete snail lemma is equivalent to condition (Sub). Indeed, if in the first diagram of Section 3 in [19] we assume that the arrows $Y_1 \to X$ and $W_1 \to X$ are monos, then the incomplete snail lemma precisely gives condition (Sub). For the converse implication, one has to consider the (regular epi, mono) factorization of the same two arrows. The fact that subtractivity implies condition (Sub) can also be checked using the pointed subobject functor introduced in [18].

4.0.9. Proposition. *Consider an arrow g in $(\mathcal{B}, \Theta)$ together with its Θ-kernel*

$$\begin{tikzcd}
\mathcal{N}(g) \arrow[r, swap, "n_g"] \arrow[rr, dashed, bend left, "\Downarrow \nu_g"] & X \arrow[r, swap, "g"] & Y
\end{tikzcd}$$

If X is $\mathcal{S}$-proper and $\mathcal{N}(g)$ is $\mathcal{S}$-global and if $\mathcal{S}$ satisfies Conditions 4.0.1 and 4.0.7, then the associated snail sequence

$$\mathcal{N}(0_{\mathcal{N}(g)}) \xrightarrow{n(n_g)} \mathcal{N}(0_X) \xrightarrow{n(g)} \mathcal{N}(0_Y) \xrightarrow{\delta} \pi_0(\mathcal{N}(g)) \xrightarrow{\pi_0(n_g)} \pi_0(X) \xrightarrow{\pi_0(g)} \pi_0(Y)$$

is $\mathcal{S}$-exact in $\pi_0(\mathcal{N}(g))$.

Proof. We are going to prove that the unique arrow σ making commutative the following diagram is in $\mathcal{S}$:

$$\begin{tikzcd}
\mathcal{N}(\pi_0(n_g)) \arrow[r, "n_{\pi_0(n_g)}"] & \pi_0(\mathcal{N}(g)) \arrow[r, "\pi_0(n_g)"] & \pi_0(X) \\
 & \mathcal{N}(0_Y) \arrow[ul, "\sigma"] \arrow[u, "\delta"'] &
\end{tikzcd}$$

Consider the following diagram, where the square is the pullback describing the Θ-kernel of g (see 2.2.7)

$$\begin{tikzcd}
\mathcal{N}(\mathrm{id}_X) \arrow[r, dotted, "v"'] \arrow[r, dashed, no head, bend left=40, "\Downarrow \tau"] \arrow[rr, dotted, bend left=50, "\overline{g}"] \arrow[dr, "n_{\mathrm{id}_X}"] \arrow[dr, dashed, no head, bend right=50, "\Uparrow \nu_{\mathrm{id}_X}"] & \mathcal{N}(g) \arrow[r, "g'"] \arrow[d, "n_g"'] & \mathcal{N}(\mathrm{id}_Y) \arrow[d, "n_{\mathrm{id}_Y}"'] \arrow[d, dashed, no head, bend left=70, "\Leftarrow \nu_{\mathrm{id}_Y}"'] \\
 & X \arrow[r, "g"'] & Y
\end{tikzcd}$$

and recall that $\nu_g = g' \circ \nu_{\mathrm{id}_Y}$. By the universal property of $\mathcal{N}(\mathrm{id}_Y)$, we get a unique arrow $\overline{g}$ such that $\overline{g} \cdot n_{\mathrm{id}_Y} = n_{\mathrm{id}_X} \cdot g$ and $\overline{g} \circ \nu_{\mathrm{id}_Y} = \nu_{\mathrm{id}_X} \circ g$. Because of the first condition on $\overline{g}$, we can use the universal property of the pullback $\mathcal{N}(g)$ and we get a unique arrow v such that $v \cdot g' = \overline{g}$ and $v \cdot n_g = n_{\mathrm{id}_X}$. Since

$$v \circ \nu_g = v \cdot g' \circ \nu_{\mathrm{id}_Y} = \overline{g} \circ \nu_{\mathrm{id}_Y} = \nu_{\mathrm{id}_X} \circ g$$

the fact that the Θ-kernel $\mathcal{N}(g)$ is strong gives us a unique nullhomotopy $\tau \in \Theta(v)$ such that $\tau \circ n_g = \nu_{\mathrm{id}_X}$. Consider now the following diagram, where the solid part is commutative and where $\overline{z}$ (respectively, $\overline{x}$) is the factorization obtained as in Definition 4.0.4.1 when we start from the object $\mathcal{N}(g)$ (respectively, X), as in the second part of the proof of Proposition 4.0.6

$$\begin{tikzcd}
\mathcal{N}(\mathrm{id}_{\mathcal{N}(g)}) \arrow[dr, "\overline{z}"'] \arrow[drr, "n_{\mathrm{id}_{\mathcal{N}(g)}}"] \arrow[drr, dashed, no head, bend left=40, "\Downarrow \nu_{\mathrm{id}_{\mathcal{N}(g)}}"] \arrow[ddd, dotted, shift left, "\overline{n_g}"] & & \mathcal{N}(0_Y) \arrow[r, "\sigma"] \arrow[d, "\Delta"] & \mathcal{N}(\pi_0(n_g)) \arrow[d, "n_{\pi_0(n_g)}"] \\
& \mathcal{N}(\eta_{\mathcal{N}(g)}) \arrow[r, "n_{\eta_{\mathcal{N}(g)}}"'] \arrow[d, "n(n_g)"'] & \mathcal{N}(g) \arrow[r, "\eta_{\mathcal{N}(g)}"'] \arrow[d, "n_g"] & \pi_0(\mathcal{N}(g)) \arrow[d, "\pi_0(n_g)"] \\
& \mathcal{N}(\eta_X) \arrow[r, "n_{\eta_X}"] & X \arrow[r, "\eta_X"] \arrow[d, "g"] & \pi_0(X) \\
\mathcal{N}(\mathrm{id}_X) \arrow[uuu, dotted, shift left, "\overline{v}"] \arrow[ur, "\overline{x}"] \arrow[urr, "n_{\mathrm{id}_X}"] \arrow[urr, dashed, no head, bend right=40, "\Uparrow \nu_{\mathrm{id}_X}"'] & & Y &
\end{tikzcd}$$

By the universal property of $\mathcal{N}(\mathrm{id}_{\mathcal{N}(g)})$, we get a unique arrow $\overline{v}$ such that $\overline{v} \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} = v$ and $\overline{v} \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} = \tau$. Moreover, the universal property of $\mathcal{N}(\mathrm{id}_X)$ gives a unique arrow $\overline{n_g}$ such that $\overline{n_g} \cdot n_{\mathrm{id}_X} = n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g$ and $\overline{n_g} \circ \nu_{\mathrm{id}_X} = \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g$. Now we check that $\overline{v}$ and $\overline{n_g}$ realize an isomorphism. We will use three times the cancellation property from Proposition 2.2.2. First, we check that $\overline{n_g} \cdot v = n_{\mathrm{id}_{\mathcal{N}(g)}}$:

- $\overline{n_g} \cdot v \cdot n_g = \overline{n_g} \cdot n_{\mathrm{id}_X} = n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g$
- $\overline{n_g} \cdot v \circ \nu_g = \overline{n_g} \circ \nu_{\mathrm{id}_X} \circ g = \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g \cdot g = n_{\mathrm{id}_{\mathcal{N}(g)}} \circ \nu_g$, the last equality coming from the reduced interchange

Second, we check that $\overline{n_g} \cdot \overline{v} = \mathrm{id}$:

- $\overline{n_g} \cdot \overline{v} \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} = \overline{n_g} \cdot v = n_{\mathrm{id}_{\mathcal{N}(g)}}$
- $\overline{n_g} \cdot \overline{v} \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} = \overline{n_g} \circ \tau = \nu_{\mathrm{id}_{\mathcal{N}(g)}}$ where, for the last equality, we compose with n_g:
- $\overline{n_g} \circ \tau \circ n_g = \overline{n_g} \circ \nu_{\mathrm{id}_X} = \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g$

Third, we check that $\overline{v} \cdot \overline{n_g} = \mathrm{id}$:

- $\overline{v} \cdot \overline{n_g} \cdot n_{\mathrm{id}_X} = \overline{v} \cdot n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g = v \cdot n_g = n_{\mathrm{id}_X}$
- $\overline{v} \cdot \overline{n_g} \circ \nu_{\mathrm{id}_X} = \overline{v} \circ \nu_{\mathrm{id}_{\mathcal{N}(g)}} \circ n_g = \tau \circ n_g = \nu_{\mathrm{id}_X}$

It remains to check that $\overline{z} \cdot n(n_g) = \overline{n_g} \cdot \overline{x}$. For this, it is enough to compose with n_{η_X} which is a monomorphism because $\pi_0(X)$ is discrete:

$$\overline{z} \cdot n(n_g) \cdot n_{\eta_X} = \overline{z} \cdot n_{\eta_{\mathcal{N}(g)}} \cdot n_g = n_{\mathrm{id}_{\mathcal{N}(g)}} \cdot n_g = \overline{n_g} \cdot n_{\mathrm{id}_X} = \overline{n_g} \cdot \overline{x} \cdot n_{\eta_X}$$

We can conclude as follows: since X is $\mathcal{S}$-proper and $\overline{n_g}$ is an isomorphism, the equality $\overline{z} \cdot n(n_g) = \overline{n_g} \cdot \overline{x}$ implies that $n(n_g)$ is in $\mathcal{S}$ by Condition 4.0.1. Since $\mathcal{N}(g)$ is $\mathcal{S}$-global and $\Delta\colon \mathcal{N}(0_Y) \to \mathcal{N}(g)$ is the kernel of n_g (see Lemma 3.0.5), we can use Condition 4.0.7 and σ is in $\mathcal{S}$. ■

4.0.10. Example. Let us consider once again $(\mathcal{B}, \Theta) = (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$ with $\mathcal{S}$ the class of arrows introduced in Example 4.0.5 and $\mathcal{A}$ a pointed regular category with kernels and cokernels. We saw in Example 4.0.5 that $\mathcal{S}$ satisfies Condition 4.0.1. If we further assume that $\mathcal{A}$ is protomodular, then $\mathcal{S}$ satisfies also Condition 4.0.7, as proved in [6]. In particular, if $\mathcal{A}$ is abelian, then $\mathcal{S}$ satisfies Conditions 4.0.1 and 4.0.7 and each object in $\mathbf{Arr}(\mathcal{A})$ is $\mathcal{S}$-global and $\mathcal{S}$-proper. In conclusion, the exactness of the snail sequence appearing in [19, 25] (see Example 3.0.3) is the special case of Propositions 4.0.6 and 4.0.9 when $(\mathcal{B}, \Theta) = (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$ and $\mathcal{A}$ is pointed, regular and protomodular.

4.0.11. Remark. The proof of the exactness of the snail sequence (Propositions 4.0.6 and 4.0.9) is more elaborate but similar to the proof of the exactness of the snail sequence in the context of pointed regular protomodular categories done in [25]. Strangely enough, Lemma 3.0.5, which is essential in the present proof, does not appear in [25] but comes from the snail lemma for internal groupoids established in [23].

5. Sequentiable families of arrows

In this section, we fix a category $\mathcal{A}$ with a zero object 0, kernels and cokernels and we introduce the category with nullhomotopies $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ of *sequentiable families of arrows in* $\mathcal{A}$. Then, under the extra assumption of $\mathcal{A}$ having pullbacks, we build the homotopy snail sequence associated to a morphism in $\mathbf{Seq}(\mathcal{A})$ and analyze its $\mathcal{S}$-exactness where $\mathcal{S}$ is a class of arrows in $\mathbf{Seq}(\mathcal{A})$ inherited from a class of arrows, also denoted by $\mathcal{S}$, in $\mathcal{A}$. To conclude, we show how the snail homotopy sequence associated to a morphism in $\mathbf{Seq}(\mathcal{A})$ allows to construct a long sequence in $\mathcal{A}$, which, under suitable conditions, will be $\mathcal{S}$-exact.

5.1. The category with nullhomotopies $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ of sequentiable families of arrows in $\mathcal{A}$.

5.1.1. Definition. A *sequentiable family of arrows* $h_\bullet$ is a family of pairs of arrows

$$h_\bullet = \{h_n, i_n\}_{n \in \mathbb{Z}}$$

in $\mathcal{A}$ with i_n connecting the cokernel of h_{n+1} with the kernel of h_n:

$$\begin{tikzcd} \ldots\ \mathrm{Cod}(h_{n+1}) \arrow[r, "q_{n+1}"] & \mathrm{Cok}(h_{n+1}) \arrow[r, "i_n"] & \mathrm{Ker}(h_n) \arrow[r, "k_n"] & \mathrm{Dom}(h_n) \arrow[r, "h_n"] & \mathrm{Cod}(h_n)\ \ldots \end{tikzcd}$$

A *morphism of sequentiable families* $f_\bullet \colon h_\bullet \to h'_\bullet$ is a family of pairs of arrows

$$f_\bullet = \{\overline{f}_n \colon \mathrm{Dom}(h_n) \to \mathrm{Dom}(h'_n), \underline{f}_n \colon \mathrm{Cod}(h_n) \to \mathrm{Cod}(h'_n)\}_{n \in \mathbb{Z}}$$

such that for all $n \in \mathbb{Z}$

$$\overline{f}_n \cdot h'_n = h_n \cdot \underline{f}_n \quad \text{and} \quad \underline{f}_{n+1} \cdot q'_{n+1} \cdot i'_n \cdot k'_n = q_{n+1} \cdot i_n \cdot k_n \cdot \overline{f}_n$$

or, equivalently, such that

$$\overline{f}_n \cdot h'_n = h_n \cdot \underline{f}_n \quad \text{and} \quad C(f)_{n+1} \cdot i'_n = i_n \cdot K(f)_n$$

where $C(f)_{n+1}$ is the unique arrow such that $\underline{f}_{n+1} \cdot q'_{n+1} = q_{n+1} \cdot C(f)_{n+1}$ and $K(f)_n$ is the unique arrow such that $K(f)_n \cdot k'_n = k_n \cdot \overline{f}_n$, as in the diagram

$$\begin{tikzcd}
\vdots & \vdots \\
\mathrm{Dom}(h_{n+1}) \arrow[r, "\overline{f}_{n+1}"] \arrow[d, "h_{n+1}"'] & \mathrm{Dom}(h'_{n+1}) \arrow[d, "h'_{n+1}"] \\
\mathrm{Cod}(h_{n+1}) \arrow[r, "\underline{f}_{n+1}"] \arrow[d, "q_{n+1}"'] & \mathrm{Cod}(h'_{n+1}) \arrow[d, "q'_{n+1}"] \\
\mathrm{Cok}(h_{n+1}) \arrow[r, dotted, "C(f)_{n+1}"] \arrow[d, "i_n"'] & \mathrm{Cok}(h'_{n+1}) \arrow[d, "i'_n"] \\
\mathrm{Ker}(h_n) \arrow[r, dotted, "K(f)_n"] \arrow[d, "k_n"'] & \mathrm{Ker}(h'_n) \arrow[d, "k'_n"] \\
\mathrm{Dom}(h_n) \arrow[r, "\overline{f}_n"] \arrow[d, "h_n"'] & \mathrm{Dom}(h'_n) \arrow[d, "h'_n"] \\
\mathrm{Cod}(h_n) \arrow[r, "\underline{f}_n"] & \mathrm{Cod}(h'_n) \\
\vdots & \vdots
\end{tikzcd}$$

With the obvious identity morphisms and composition of morphisms, sequentiable families and their morphisms give rise to a category denoted $\mathbf{Seq}(\mathcal{A})$.

5.1.2. REMARK. Observe that the category $\mathbf{Seq}(\mathcal{A})$ has kernels and cokernels constructed level-wise in $\mathcal{A}$. The connecting arrows are obtained by an easy diagram chasing left to the reader. More generally, $\mathbf{Seq}(\mathcal{A})$ inherits level-wise all limits and colimits which eventually exist in $\mathcal{A}$.

The category $\mathbf{Seq}(\mathcal{A})$ of sequentiable families is equipped with a structure of nullhomotopies Θ_Δ which extends the one we considered in $\mathbf{Arr}(\mathcal{A})$, see Section 2.5.

5.1.3. DEFINITION. Let $f_\bullet \colon h_\bullet \to h'_\bullet$ be a morphism of sequentiable families. A *nullhomotopy* $\lambda_\bullet \in \Theta_\Delta(f_\bullet)$ is a family of arrows

$$\lambda_\bullet = \{\lambda_n \colon \mathrm{Cod}(h_n) \to \mathrm{Dom}(h'_n)\}_{n\in\mathbb{Z}}$$

such that $h_n \cdot \lambda_n = \overline{f}_n$ and $\lambda_n \cdot h'_n = \underline{f}_n$ for all $n \in \mathbb{Z}$, as depicted in the diagram

$$\begin{tikzcd}
\mathrm{Dom}(h_n) \arrow[r, "\overline{f}_n"] \arrow[d, "h_n"'] & \mathrm{Dom}(h'_n) \arrow[d, "h'_n"] \\
\mathrm{Cod}(h_n) \arrow[r, "\underline{f}_n"'] \arrow[ur, "\lambda_n"] & \mathrm{Cod}(h'_n)
\end{tikzcd}$$

The composition of a nullhomotopy with morphisms on the left and on the right is defined level-wise as in $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$.

The following proposition shows that the category of sequentiable families of arrows with this structure of nullhomotopies satisfies some nice properties.

5.1.4. Proposition. *Consider the category with nullhomotopies* $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ *as in Definitions 5.1.1 and 5.1.3.*

1. *For a morphism $f_\bullet \in \mathbf{Seq}(\mathcal{A})$, if $\Theta_\Delta(f_\bullet) \neq \emptyset$ then for all $n \in \mathbb{Z}$ the arrows $K(f)_n$ and $C(f)_n$ are zero arrows.*
2. *The structure Θ_Δ satisfies the reduced interchange condition 2.1.4.*
3. *The sequentiable family $0_\bullet = \{\mathrm{id}_0, \mathrm{id}_0\}_{n\in\mathbb{Z}}$ is a Θ_Δ-strong zero object of $\mathbf{Seq}(\mathcal{A})$.*
4. *If $\mathcal{A}$ has pullbacks, then $\mathbf{Seq}(\mathcal{A})$ has Θ_Δ-kernels constructed level-wise as in $\mathbf{Arr}(\mathcal{A})$, and they are strong.*
5. *Dually, if $\mathcal{A}$ has pushouts, then $\mathbf{Seq}(\mathcal{A})$ has Θ_Δ-cokernels constructed level-wise as in $\mathbf{Arr}(\mathcal{A})$, and they are strong.*

Proof. The proof of points 1, 2 and 3 is straightforward. We proceed to show point 4, the proof of 5 is dual. Given a morphism $f_\bullet : h_\bullet \to h'_\bullet$ in $\mathbf{Seq}(\mathcal{A})$, we can construct for each level

$$\begin{tikzcd}
\mathrm{Dom}(h_n) \arrow[r, "\overline{f}_n"] \arrow[d, "h_n"'] & \mathrm{Dom}(h'_n) \arrow[d, "h'_n"] \\
\mathrm{Cod}(h_n) \arrow[r, "\underline{f}_n"'] & \mathrm{Cod}(h'_n)
\end{tikzcd}$$

the correspoding Θ_Δ-kernel in $\mathbf{Arr}(\mathcal{A})$. Adopting the notation depicted in the following diagram (where the region marked as p.b. is a pullback),

$$\begin{tikzcd}
\mathrm{Dom}(h_n) \arrow[rr, "\overline{f}_n"] \arrow[dd, "h_n"'] \arrow[dr, "h_n^P"] & & \mathrm{Dom}(h'_n) \arrow[dd, "h'_n"] \\
& P_n \arrow[ur, "\pi'_n"'] \arrow[dl, "\pi_n"] \arrow[dr, phantom, "\text{p.b.}"] & \\
\mathrm{Cod}(h_n) \arrow[rr, "\underline{f}_n"'] & & \mathrm{Cod}(h'_n)
\end{tikzcd}$$

we have that this Θ_Δ-kernel in $\mathbf{Arr}(\mathcal{A})$ is given by the triple $(h_n^P, (\mathrm{id}_{\mathrm{Dom}(h_n)}, \pi_n), \pi_n')$ (see Section 2.5). Consider now the following commutative diagram, where the upper-most horizontal pair of rectangles depicts the Θ_Δ-kernel in $\mathbf{Arr}(\mathcal{A})$ at level $n+1$ and the lower-most horizontal pair of rectangles depicts the Θ_Δ-kernel in $\mathbf{Arr}(\mathcal{A})$ at level n:

$$\begin{tikzcd}
\mathrm{Dom}(h_{n+1}) \arrow[r, "\mathrm{id}"] \arrow[d, "h^P_{n+1}"'] & \mathrm{Dom}(h_{n+1}) \arrow[r, "\overline{f}_{n+1}"] \arrow[d, "h_{n+1}"] & \mathrm{Dom}(h'_{n+1}) \arrow[d, "h'_{n+1}"] \\
P_{n+1} \arrow[r, "\pi_{n+1}"'] \arrow[d, "q^P_{n+1}"'] \arrow[urr, dashed, "\pi'_{n+1}"] & \mathrm{Cod}(h_{n+1}) \arrow[r, "\underline{f}_{n+1}"'] \arrow[d, "q_{n+1}"] & \mathrm{Cod}(h'_{n+1}) \arrow[d, "q'_{n+1}"] \\
\mathrm{Cok}(h^P_{n+1}) \arrow[r, "C(\pi)_{n+1}"] \arrow[d, dotted, "i^P_n"'] & \mathrm{Cok}(h_{n+1}) \arrow[r, "C(f)_{n+1}"] \arrow[d, "i_n"] & \mathrm{Cok}(h'_{n+1}) \arrow[d, "i'_n"] \\
\mathrm{Ker}(h^P_n) \arrow[r, "K(\pi)_n"'] \arrow[d, "k^P_n"'] & \mathrm{Ker}(h_n) \arrow[r, "K(f)_n"'] \arrow[d, "k_n"] & \mathrm{Ker}(h'_n) \arrow[d, "k'_n"] \\
\mathrm{Dom}(h_n) \arrow[r, "\mathrm{id}"] \arrow[d, "h^P_n"'] & \mathrm{Dom}(h_n) \arrow[r, "\overline{f}_n"] \arrow[d, "h_n"] & \mathrm{Dom}(h'_n) \arrow[d, "h'_n"] \\
P_n \arrow[r, "\pi_n"'] \arrow[urr, dashed, "\pi'_n"] & \mathrm{Cod}(h_n) \arrow[r, "\underline{f}_n"'] & \mathrm{Cod}(h'_n)
\end{tikzcd}$$

We are going to prove that, for all $n \in \mathbb{Z}$, there exists a unique arrow

$$i_n^P : \mathrm{Cok}(h_{n+1}^P) \to \mathrm{Ker}(h_n^P)$$

such that $i_n^P \cdot K(\pi)_n = C(\pi)_{n+1} \cdot i_n$. This makes $\mathcal{N}(f_\bullet) = \{h_n^P, i_n^P\}_{n \in \mathbb{Z}}$ an object of $\mathbf{Seq}(\mathcal{A})$.

- Existence of i_n^P: first, we check that $C(\pi)_{n+1} \cdot i_n \cdot k_n \cdot h_n^P = 0$. For this, we compose with the projections of the pullback P_n:

  ∗ $C(\pi)_{n+1} \cdot i_n \cdot k_n \cdot h_n^P \cdot \pi_n = C(\pi)_{n+1} \cdot i_n \cdot k_n \cdot h_n = C(\pi)_{n+1} \cdot i_n \cdot 0 = 0$

  ∗ $C(\pi)_{n+1} \cdot i_n \cdot k_n \cdot h_n^P \cdot \pi_n' = C(\pi)_{n+1} \cdot i_n \cdot k_n \cdot \overline{f}_n = C(\pi)_{n+1} \cdot i_n \cdot K(f)_n \cdot k_n'$
  $= C(\pi)_{n+1} \cdot C(f)_{n+1} \cdot i_n' \cdot k_n' = 0$
  where, to justify the last equality, we precompose with the epimorphism q_{n+1}^P:
  $q_{n+1}^P \cdot C(\pi)_{n+1} \cdot C(f)_{n+1} \cdot i_n' \cdot k_n' = \pi_{n+1} \cdot \underline{f}_{n+1} \cdot q_{n+1}' \cdot i_n' \cdot k_n' =$
  $= \pi_{n+1}' \cdot h_{n+1}' \cdot q_{n+1}' \cdot i_n' \cdot k_n' = \pi_{n+1}' \cdot 0 \cdot i_n' \cdot k_n' = 0$

  Now, from $C(\pi)_{n+1} \cdot i_n \cdot k_n \cdot h_n^P = 0$, by virtue of the universal property of $\mathrm{Ker}(h_n^P)$ we get a unique arrow i_n^P such that $i_n^P \cdot k_n^P = C(\pi)_{n+1} \cdot i_n \cdot k_n$. This can be rewritten as $i_n^P \cdot K(\pi)_n \cdot k_n = C(\pi)_{n+1} \cdot i_n \cdot k_n$. Since k_n is a monomorphism, we can conclude that $i_n^P \cdot K(\pi)_n = C(\pi)_{n+1} \cdot i_n$.

- Uniqueness of i_n^P: this follows easily by composing with the monomorphism $k_n^P = K(\pi)_n \cdot k_n$.

We thus have that $\mathcal{N}(f_\bullet) = \{h_n^P, i_n^P\}_{n\in\mathbb{Z}}$ is an object of $\mathbf{Seq}(\mathcal{A})$. Now, the fact that $(\mathcal{N}(f_\bullet), \{\mathrm{id}, \pi_n\}_{n\in\mathbb{Z}}, \pi'_\bullet)$ is the Θ_Δ-kernel of $f_\bullet$ in $\mathbf{Seq}(\mathcal{A})$ follows easily from the fact that level-wise it is the Θ_Δ-kernel in $\mathbf{Arr}(\mathcal{A})$. One just needs to check that the unique morphisms obtained level-wise by the universal property of the Θ_Δ-kernel in $\mathbf{Arr}(\mathcal{A})$ do assemble into a morphism in $\mathbf{Seq}(\mathcal{A})$. This follows readily by a simple diagram chasing, we leave it to the reader as an exercise. Finally, the fact that $(\mathcal{N}(f_\bullet), \{\mathrm{id}, \pi_n\}_{n\in\mathbb{Z}}, \pi'_\bullet)$ is Θ_Δ-strong follows immediately from the fact that it is level-wise Θ_Δ-strong in $\mathbf{Arr}(\mathcal{A})$. ■

5.1.5. Examples. Based on the description of the Θ_Δ-kernel in $\mathbf{Seq}(\mathcal{A})$ contained in the proof of Proposition 5.1.4, we list here some of the special cases of Θ_Δ-kernels involved in the snail sequence:

1. The n-th level of the Θ_Δ-kernel of $\mathrm{id}_{h_\bullet}$ is given by

$$\begin{tikzcd}
\mathrm{Dom}(h_n) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathrm{Dom}(h_n) \arrow[r, "\mathrm{id}"] \arrow[d, "h_n"] & \mathrm{Dom}(h_n) \arrow[d, "h_n"] \\
\mathrm{Dom}(h_n) \arrow[r, "h_n"'] \arrow[urr, dashed, "\mathrm{id}"] & \mathrm{Cod}(h_n) \arrow[r, "\mathrm{id}"'] & \mathrm{Cod}(h_n)
\end{tikzcd}$$

   and the n-th connecting arrow of $\mathcal{N}(\mathrm{id}_{h_\bullet})$ is $\mathrm{id}\colon 0 \to 0$.

2. The n-th level of the Θ_Δ-kernel of $0_{h_\bullet}$ is given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{Dom}(h_n) \arrow[d, "h_n"] \\
\mathrm{Ker}(h_n) \arrow[r] \arrow[urr, dashed, "k_{h_n}"] & 0 \arrow[r] & \mathrm{Cod}(h_n)
\end{tikzcd}$$

   and the n-th connecting arrow of $\mathcal{N}(0_{h_\bullet})$ is $0\colon \mathrm{Ker}(h_{n+1}) \to 0$.

3. The n-th level of the morphism $\eta_{h_\bullet}\colon h_\bullet \to \pi_0(h_\bullet)$ is given by

$$\begin{tikzcd}
\mathrm{Dom}(h_n) \arrow[r] \arrow[d, "h_n"'] & 0 \arrow[d] \\
\mathrm{Cod}(h_n) \arrow[r, "c_{h_n}"'] & \mathrm{Cok}(h_n)
\end{tikzcd}$$

   and the n-th connecting arrow of $\pi_0(h_\bullet)$ is $0\colon \mathrm{Cok}(h_{n+1}) \to 0$.

4. The n-th level of the factorization $\overline{h}_\bullet\colon \mathcal{N}(\mathrm{id}_{h_\bullet}) \to \mathcal{N}(\eta_{h_\bullet})$ of $n_{\mathrm{id}_{h_\bullet}}\colon \mathcal{N}(\mathrm{id}_{h_\bullet}) \to h_\bullet$ through $n_{\eta_{h_\bullet}}\colon \mathcal{N}(\eta_{h_\bullet}) \to h_\bullet$ (see Definition 4.0.4) is given by

$$\begin{tikzcd}
\mathrm{Dom}(h_n) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathrm{Dom}(h_n) \arrow[d, "\overline{h}_n"] \\
\mathrm{Dom}(h_n) \arrow[r, "\overline{h}_n"'] & \mathrm{Ker}(c_{h_n})
\end{tikzcd}$$

where $\overline{h}_n$ is the factorization of h_n through the kernel of its cokernel.

5.2. THE HOMOTOPY SNAIL SEQUENCE ASSOCIATED TO AN ARROW IN $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$.

Assume now that the category $\mathcal{A}$ has pullbacks and let $f_\bullet \colon h_\bullet \to h'_\bullet$ be a morphism in $\mathbf{Seq}(\mathcal{A})$.

5.2.1. THE CONSTRUCTION OF THE SEQUENCE. By virtue of Proposition 5.1.4, the category with nullhomotopies $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ satisfies the required conditions to peform the construction of the homotopy snail sequence built in Section 3. Applying the construction to $f_\bullet \colon h_\bullet \to h'_\bullet$, we obtain a sequence

$$\mathcal{N}(0_{\mathcal{N}(f_\bullet)}) \longrightarrow \mathcal{N}(0_{h_\bullet}) \longrightarrow \mathcal{N}(0_{h'_\bullet}) \longrightarrow \pi_0(\mathcal{N}(f_\bullet)) \longrightarrow \pi_0(h_\bullet) \longrightarrow \pi_0(h'_\bullet). \qquad (1)$$

Keeping in mind the special cases of Θ_Δ-kernels described in Examples 5.1.5, we can make the previous sequence explicit. Here it is from level $n+1$ to level n:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}(h^P_{n+1}) \arrow[r, "K(\pi)_{n+1}"] \arrow[d, "\mathrm{id}"'] & \mathrm{Ker}(h_{n+1}) \arrow[r, "K(f)_{n+1}"] \arrow[d, "\mathrm{id}"'] & \mathrm{Ker}(h'_{n+1}) \arrow[r, "\langle 0,k'_{n+1}\rangle\cdot q'_{n+1}"] \arrow[d, "\mathrm{id}"'] & \mathrm{Cok}(h^P_{n+1}) \arrow[r, "C(\pi)_{n+1}"] \arrow[d, "\mathrm{id}"] & \mathrm{Cok}(h_{n+1}) \arrow[r, "C(f)_{n+1}"] \arrow[d, "\mathrm{id}"] & \mathrm{Cok}(h'_{n+1}) \arrow[d, "\mathrm{id}"] \\
\mathrm{Ker}(h^P_{n+1}) \arrow[r, "K(\pi)_{n+1}"] \arrow[d] & \mathrm{Ker}(h_{n+1}) \arrow[r, "K(f)_{n+1}"] \arrow[d] & \mathrm{Ker}(h'_{n+1}) \arrow[r, "\langle 0,k'_{n+1}\rangle\cdot q'_{n+1}"] \arrow[d] & \mathrm{Cok}(h^P_{n+1}) \arrow[r, "C(\pi)_{n+1}"] \arrow[d] & \mathrm{Cok}(h_{n+1}) \arrow[r, "C(f)_{n+1}"] \arrow[d] & \mathrm{Cok}(h'_{n+1}) \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}(h^P_n) \arrow[r, "K(\pi)_n"] & \mathrm{Ker}(h_n) \arrow[r, "K(f)_n"] & \mathrm{Ker}(h'_n) \arrow[r, "\langle 0,k'_n\rangle\cdot q'_n"] & \mathrm{Cok}(h^P_n) \arrow[r, "C(\pi)_n"] & \mathrm{Cok}(h_n) \arrow[r, "C(f)_n"] & \mathrm{Cok}(h'_n).
\end{tikzcd}$$

5.2.2. THE $\mathcal{S}$-EXACTNESS OF THE SEQUENCE. Consider a class $\mathcal{S}$ of arrows in $\mathcal{A}$.

5.2.3. CONSTRUCTION. The class $\mathcal{S}$ of arrows in $\mathcal{A}$ can be extended to a class, that we will denote once more by $\mathcal{S}$, of morphisms in $\mathbf{Seq}(\mathcal{A})$ consisting of those morphisms $f_\bullet \colon h_\bullet \to h'_\bullet$ such that both $\overline{f}_n$ and $\underline{f}_n$ are in $\mathcal{S}$ for all $n \in \mathbb{Z}$.

5.2.4. PROPOSITION. *The homotopy snail sequence (1) is $\mathcal{S}$-exact in $\mathbf{Seq}(\mathcal{A})$ if and only if the sequence*

$$\mathrm{Ker}(h^P_n) \xrightarrow{K(\pi)_n} \mathrm{Ker}(h_n) \xrightarrow{K(f)_n} \mathrm{Ker}(h'_n) \xrightarrow{\langle 0,k'_n\rangle\cdot q'_n} \mathrm{Cok}(h^P_n) \xrightarrow{C(\pi)_n} \mathrm{Cok}(h_n) \xrightarrow{C(f)_n} \mathrm{Cok}(h'_n)$$

is $\mathcal{S}$-exact in $\mathcal{A}$ for all $n \in \mathbb{Z}$.

PROOF. From Lemma 2.4.4, we know that all the objects involved in the homotopy snail sequence (1) are discrete. The result then follows directly from Lemma 2.4.3 together with Remark 5.1.2. ■

5.3. The long exact homotopy snail sequence in $\mathcal{A}$. Let us observe that the connecting arrows of the sequentiable families $\mathcal{N}(f_\bullet)$, $h_\bullet$ and $h'_\bullet$ allow us to paste together all the six-term sequences in $\mathcal{A}$, getting a long sequence as follows:

$$\begin{tikzcd}
\mathrm{Ker}(h^P_{n+1}) \arrow[r, "K(\pi)_{n+1}"] & \mathrm{Ker}(h_{n+1}) \arrow[r, "K(f)_{n+1}"] & \mathrm{Ker}(h'_{n+1}) \arrow[r, "\langle 0,k'_{n+1}\rangle\cdot q'_{n+1}"] & \mathrm{Cok}(h^P_{n+1}) \arrow[r, "C(\pi)_{n+1}"] \arrow[dlll, "i^P_n"'] & \mathrm{Cok}(h_{n+1}) \arrow[r, "C(f)_{n+1}"] \arrow[dlll, "i_n"'] & \mathrm{Cok}(h'_{n+1}) \arrow[dlll, "i'_n"'] \\
\mathrm{Ker}(h^P_n) \arrow[r, "K(\pi)_n"'] & \mathrm{Ker}(h_n) \arrow[r, "K(f)_n"'] & \mathrm{Ker}(h'_n) \arrow[r, "\langle 0,k'_n\rangle\cdot q'_n"'] & \mathrm{Cok}(h^P_n) \arrow[r, "C(\pi)_n"'] & \mathrm{Cok}(h_n) \arrow[r, "C(f)_n"'] & \mathrm{Cok}(h'_n)
\end{tikzcd}$$

Here is an ad hoc condition which ensures the preservation of exactness of the six-term sequences in $\mathcal{A}$ when we paste them all together.

5.3.1. Definition. We denote by $\mathbf{IsoSeq}(\mathcal{A})$ the full subcategory of $\mathbf{Seq}(\mathcal{A})$ of the isosequentiable families, that is, those families $h_\bullet = \{h_n, i_n\}_{n\in\mathbb{Z}}$ such that i_n is an isomorphism for all $n \in \mathbb{Z}$.

5.3.2. Proposition. *Let $f_\bullet\colon h_\bullet \to h'_\bullet$ be a morphism in $\mathbf{Seq}(\mathcal{A})$. Take as $\mathcal{S}$ the class of regular epimorphisms in $\mathcal{A}$ and extend $\mathcal{S}$ to $\mathbf{Seq}(\mathcal{A})$ as in 5.2.3. Consider the following sequences, the first one being in $\mathbf{Seq}(\mathcal{A})$ and the second one being in $\mathcal{A}$:*

$$\mathcal{N}(0_{\mathcal{N}(f_\bullet)}) \longrightarrow \mathcal{N}(0_{h_\bullet}) \longrightarrow \mathcal{N}(0_{h'_\bullet}) \longrightarrow \pi_0(\mathcal{N}(f_\bullet)) \longrightarrow \pi_0(h_\bullet) \longrightarrow \pi_0(h'_\bullet)$$

$$\begin{tikzcd}
\mathrm{Ker}(h^P_{n+1}) \arrow[r, "K(\pi)_{n+1}"] & \mathrm{Ker}(h_{n+1}) \arrow[r, "K(f)_{n+1}"] & \mathrm{Ker}(h'_{n+1}) \arrow[r, "\langle 0,k'_{n+1}\rangle\cdot q'_{n+1}"] & \mathrm{Cok}(h^P_{n+1}) \arrow[r, "C(\pi)_{n+1}"] & \mathrm{Cok}(h_{n+1}) \arrow[r, "C(f)_{n+1}"] \arrow[dll, "C(f)_{n+1}\cdot i'_n = i_n\cdot K(f)_n"'] & \mathrm{Cok}(h'_{n+1}) \\
\mathrm{Ker}(h^P_n) \arrow[r, "K(\pi)_n"'] & \mathrm{Ker}(h_n) \arrow[r, "K(f)_n"'] & \mathrm{Ker}(h'_n) \arrow[r, "\langle 0,k'_n\rangle\cdot q'_n"'] & \mathrm{Cok}(h^P_n) \arrow[r, "C(\pi)_n"] & \mathrm{Cok}(h_n) \arrow[r, "C(f)_n"] \arrow[dll, "C(f)_n\cdot i'_{n-1} = i_{n-1}\cdot K(f)_{n-1}"'] & \mathrm{Cok}(h'_n) \\
\mathrm{Ker}(h^P_{n-1}) \arrow[r, "K(\pi)_{n-1}"'] & \mathrm{Ker}(h_{n-1}) \arrow[r, "K(f)_{n-1}"'] & \mathrm{Ker}(h'_{n-1}) \arrow[r, "\langle 0,k'_{n-1}\rangle\cdot q'_{n-1}"'] & \mathrm{Cok}(h^P_{n-1}) \arrow[r, "C(\pi)_{n-1}"'] & \mathrm{Cok}(h_{n-1}) \arrow[r, "C(f)_{n-1}"'] & \mathrm{Cok}(h'_{n-1})
\end{tikzcd}$$

1. *If $\mathcal{A}$ is regular and protomodular and if $\mathcal{N}(f_\bullet)$, $h_\bullet$ and $h'_\bullet$ are $\mathcal{S}$-proper, then the six-term sequence in $\mathbf{Seq}(\mathcal{A})$ is $\mathcal{S}$-exact and each row of the long sequence in $\mathcal{A}$ is $\mathcal{S}$-exact.*

2. *If moreover $h_\bullet$ and $h'_\bullet$ are isosequentiable, then the long sequence in $\mathcal{A}$ is $\mathcal{S}$-exact at each point.*

Proof. 1. Observe that every object in $\mathbf{Seq}(\mathcal{A})$ is $\mathcal{S}$-global; this can be shown as it was done for $\mathbf{Arr}(\mathcal{A})$ (see Example 4.0.5). Moreover, note that the class $\mathcal{S}$ in $\mathbf{Seq}(\mathcal{A})$ satisfies Conditions 4.0.1 and 4.0.7; the argument for $\mathbf{Seq}(\mathcal{A})$ is again the same as for $\mathbf{Arr}(\mathcal{A})$ (see Example 4.0.10). Consequently, by Propositions 4.0.6 and 4.0.9, the six-term sequence in $\mathbf{Seq}(\mathcal{A})$ is $\mathcal{S}$-exact. Therefore, by Proposition 5.2.4, each horizontal row of the long sequence is $\mathcal{S}$-exact. 2. By assumption, the connecting arrows i_n and i'_n are now isomorphisms. Therefore, we can paste vertically the different rows and we are done. ∎

5.3.3. Remark. As already observed in Example 4.0.10 in the case $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, not only each object of $\mathbf{Seq}(\mathcal{A})$ is $\mathcal{S}$-global and, but if $\mathcal{A}$ is abelian, it is also $\mathcal{S}$-proper. Therefore, in the abelian case, the assumptions in Proposition 5.3.2 that $\mathcal{N}(h_\bullet), h_\bullet$ and $h'_\bullet$ are $\mathcal{S}$-proper are redundant.

5.3.4. Remark. In general, the Θ_Δ-kernel of a morphism between isosequentiable families is not isosequentiable. The second example in 5.1.5 provides a counterexample: the Θ_Δ-kernel of the initial morphism $0 \to h_\bullet$ is isosequentaible precisely when each h_n has trivial kernel.

Let $\mathcal{A}$ be a pointed regular and protomodular category with cokernels. In this section we have built a category with nullhomotopies $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ from $\mathcal{A}$, and, associated to each morphism between suitable objects in $\mathbf{Seq}(\mathcal{A})$, we have constructed a long exact sequence in $\mathcal{A}$ by glueing the level-wise homotopy snail sequences. This construction may remind the reader of the construction of the long exact sequence of homology associated to a short exact sequence of chain complexes, which is performed by pasting the exact sequences obtained by applying the snake lemma level-wise. The precise relation between both constructions will be explored in the next section.

6. From chain complexes to sequentiable families

As in Section 5, we fix a category $\mathcal{A}$ with a zero object 0, kernels and cokernels. We fix as class $\mathcal{S}$ of arrows in $\mathcal{A}$ the class of regular epimorphisms.

This section is divided in three parts. In Subsection 6.1 we build a functor $\mathcal{F} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Seq}(\mathcal{A})$, from the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ to the category $\mathbf{Seq}(\mathcal{A})$ of sequentiable families of arrows in $\mathcal{A}$. Then, in Subsection 6.2, we prove that, when $\mathcal{A}$ is regular and protomodular, the long exact sequence of homology associated to an extension of proper chain complexes is isomorphic to the long exact sequence constructed in Section 5.2 associated to the image via $\mathcal{F}$ of the second morphism of this extension. To conclude, in Subsection 6.3, we show that when $\mathcal{A}$ is preadditive, the functor $\mathcal{F}$ can be upgraded to a functor between categories with nullhomotopies, where $\mathbf{Ch}(\mathcal{A})$ is equipped with the structure of nullhomotopies $\Theta_{\mathbf{Ch}}$ provided by the classical chain homotopies, while $\mathbf{Seq}(\mathcal{A})$ is equipped with the structure of nullhomotopies Θ_Δ described in Definition 5.1.3.

6.0.1. Notation. A typical object in the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ will be depicted as

$$\begin{tikzcd}
C_\bullet\colon \ \dots\ C_{n+1} \arrow[r, "d^C_{n+1}"] & C_n \arrow[r, "d^C_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] & C_{n-2}\ \dots
\end{tikzcd}$$

6.0.2. Notation. Along the section, we write proper instead of $\mathcal{S}$-proper. By Example 4.0.10, we already know that an arrow in $\mathcal{A}$ is proper (in the sense that its factorization through the kernel of its cokernel is a regular epimorphism) iff it is proper as an object of $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$. Similarly, for a sequentiable family $h_\bullet = \{h_n, i_n\}_{n\in\mathbb{Z}}$, to be proper as an object of $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ amounts to the fact that each h_n is a proper arrow. By analogy, we will say that a complex $C_\bullet \in \mathbf{Ch}(\mathcal{A})$ is proper if each d_n^C is a proper arrow. In this way, our terminology for complexes agrees with the terminology introduced in [11].

6.1. Construction of the functor $\mathcal{F} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Seq}(\mathcal{A})$. In this subsection we are going to construct a functor $\mathcal{F} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Seq}(\mathcal{A})$.

Let us start by looking more carefully at the classical link between the snake lemma and the long homology sequence in the abelian case (see for example [28]). Given an extension of complexes

$$\begin{tikzcd} A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] & C_\bullet \end{tikzcd}$$

we get a family of extensions in $\mathbf{Arr}(\mathcal{A})$

$$\begin{tikzcd}
A_n \arrow[r, "f_n"] \arrow[d, "d_n^A"'] & B_n \arrow[r, "g_n"] \arrow[d, "d_n^B"'] & C_n \arrow[d, "d_n^C"] \\
A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1} \arrow[r, "g_{n-1}"'] & C_{n-1}
\end{tikzcd}$$

that is, g_n is the cokernel of f_n and f_n is the kernel of g_n, and this for each $n \in \mathbb{Z}$. Nevertheless, in order to get the homology sequence, we do not apply the snake lemma directly to these extensions. Instead, we factorize each one of them and we get a new family of dotted commutative diagrams (notation $\mathcal{F}(-)$ is explained in Proposition 6.1.1)

$$\begin{tikzcd}
A_n \arrow[rr, "f_n"] \arrow[dr, "q_n^A"] \arrow[ddd, "d_n^A"'] & & B_n \arrow[rr, "g_n"] \arrow[dr, "q_n^B"] \arrow[ddd, "d_n^B"'] & & C_n \arrow[dr, "q_n^C"] \arrow[ddd, "d_n^C"'] & \\
& \mathrm{Cok}(d_{n+1}^A) \arrow[rr, dotted, "\overline{\mathcal{F}f}_n"] \arrow[d, dotted, "h_n^{\mathcal{F}A}"'] & & \mathrm{Cok}(d_{n+1}^B) \arrow[rr, dotted, "\overline{\mathcal{F}g}_n"] \arrow[d, dotted, "h_n^{\mathcal{F}B}"'] & & \mathrm{Cok}(d_{n+1}^C) \arrow[d, dotted, "h_n^{\mathcal{F}C}"'] \\
& \mathrm{Ker}(d_{n-1}^A) \arrow[rr, dotted, "\underline{\mathcal{F}f}_n"'] \arrow[dl, "k_n^A"] & & \mathrm{Ker}(d_{n-1}^B) \arrow[rr, dotted, "\underline{\mathcal{F}g}_n"'] \arrow[dl, "k_n^B"] & & \mathrm{Ker}(d_{n-1}^C) \arrow[dl, "k_n^C"] \\
A_{n-1} \arrow[rr, "f_{n-1}"'] & & B_{n-1} \arrow[rr, "g_{n-1}"'] & & C_{n-1} &
\end{tikzcd} \tag{2}$$

Finally we apply the snake lemma to this second family of diagrams (this is possible because $\overline{\mathcal{F}g}_n$ is still the cokernel of $\overline{\mathcal{F}f}_n$ and $\underline{\mathcal{F}f}_n$ is still the kernel of $\underline{\mathcal{F}g}_n$) and we get the long exact sequence in homology. Indeed, the kernel of $h_n^{\mathcal{F}A}$ is the homology object $H_n(A_\bullet)$ and the cokernel of $h_n^{\mathcal{F}A}$ is the homology object $H_{n-1}(A_\bullet)$, and the same holds for the complexes $B_\bullet$ and $C_\bullet$. The sequentiable families of arrows in $\mathcal{A}$ arise precisely from this intermediate construction, as formalized in the next proposition.

6.1.1. Proposition. *From any complex $C_\bullet \in \mathbf{Ch}(\mathcal{A})$, we obtain a sequentiable family of arrows $\mathcal{F}(C_\bullet) = \{h_n^{\mathcal{F}C}, i_n^{\mathcal{F}C}\}_{n\in\mathbb{Z}} \in \mathbf{Seq}(\mathcal{A})$ depicted hereunder (from level n+1 to level n)*

$$\begin{tikzcd}
C_{n+1} \arrow[r, "q_{n+1}^C"] \arrow[dr, "k(d_{n+1}^C)"'] \arrow[ddd, "d_{n+1}^C"'] & \mathrm{Cok}(d_{n+2}^C) \arrow[d, "h_{n+1}^{\mathcal{F}C}"] \arrow[dddl, "q(d_{n+1}^C)"'] \\
 & \mathrm{Ker}(d_n^C) \arrow[d, "q_{n+1}^{\mathcal{F}C}"] \arrow[ddl, "k_n^C"] \\
 & \mathrm{Cok}(h_{n+1}^{\mathcal{F}C}) \arrow[dd, "i_n^{\mathcal{F}C}"] \\
C_n \arrow[ddr, "q_n^C"] \arrow[dddr, "k(d_n^C)"'] \arrow[ddd, "d_n^C"'] & \\
 & \mathrm{Ker}(h_n^{\mathcal{F}C}) \arrow[d, "k_n^{\mathcal{F}C}"] \\
 & \mathrm{Cok}(d_{n+1}^C) \arrow[d, "h_n^{\mathcal{F}C}"] \arrow[dl, "q(d_n^C)"'] \\
C_{n-1} & \mathrm{Ker}(d_{n-1}^C) \arrow[l, "k_{n-1}^C"]
\end{tikzcd}$$

where $k(d_n^C)$ is the unique arrow such that $k(d_n^C) \cdot k_{n-1}^C = d_n^C$ and $q(d_n^C)$ is the unique arrow such that $q_n^C \cdot q(d_n^C) = d_n^C$. The components of the family $\mathcal{F}(C_\bullet)$ are as follows:

- *$h_n^{\mathcal{F}C}$ is the unique arrow such that $q_n^C \cdot h_n^{\mathcal{F}C} = k(d_n^C)$ or, equivalently, the unique arrow such that $h_n^{\mathcal{F}C} \cdot k_{n-1}^C = q(d_n^C)$.*
- *$i_n^{\mathcal{F}C}$ is the unique arrow such that $q_{n+1}^{\mathcal{F}C} \cdot i_n^{\mathcal{F}C} \cdot k_n^{\mathcal{F}C} = k_n^C \cdot q_n^C$.*

This construction extends to a functor

$$\mathcal{F}\colon \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathbf{Seq}(\mathcal{A})$$

Moreover, if $\mathcal{A}$ is regular and protomodular, then the family $\mathcal{F}(C_\bullet)$ is isosequentiable provided that the complex $C_\bullet$ is proper.
In particular, if $\mathcal{A}$ is abelian, we get a functor $\mathcal{F}\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{IsoSeq}(\mathcal{A})$.

Proof. The equivalence between the two possible definitions of $h_n^{\mathcal{F}C}$ comes from the fact that q_n^C is an epimorphism and k_{n-1}^C is a monomorphism. As far as $i_n^{\mathcal{F}C}$ is concerned, observe that, since q_{n+1}^C is an epimorphism, $\mathrm{Cok}(h_{n+1}^{\mathcal{F}C}) = \mathrm{Cok}(k(d_{n+1}^C))$. Similarly, since k_{n-1}^C is a monomoprphism, $\mathrm{Ker}(h_n^{\mathcal{F}C}) = \mathrm{Ker}(q(d_n^C))$. Now the argument runs as usual: since $k(d_{n+1}^C) \cdot k_n^C \cdot q_n^C = 0$, there exists a unique $j_n\colon \mathrm{Cok}(k(d_{n+1}^C)) \to \mathrm{Cok}(d_{n+1}^C)$ such that

$q^{\mathcal{F}C}_{n+1} \cdot j_n = k^C_n \cdot q^C_n$. Since moreover $j_n \cdot q(d^C_n) = 0$ (precompose with $q^{\mathcal{F}C}_{n+1}$), there exists a unique $i^{\mathcal{F}C}_n$ such that $i^{\mathcal{F}C}_n \cdot k^{\mathcal{F}C}_n = j_n$.
The construction of a morphism $\mathcal{F}(g_\bullet) = \{\overline{\mathcal{F}g}_n, \underline{\mathcal{F}g}_n\}_{n\in\mathbb{Z}} \colon \mathcal{F}(B_\bullet) \to \mathcal{F}(C_\bullet)$ from a morphism $g_\bullet \colon B_\bullet \to C_\bullet$ in $\mathbf{Ch}(\mathcal{A})$ has been already depicted in diagram (2) and the functoriality of the construction is obvious. Let us check just that $\mathcal{F}(g_\bullet)$ is indeed a morphism in $\mathbf{Seq}(\mathcal{A})$:

- $\underline{\mathcal{F}g}_{n+1}\cdot q^{\mathcal{F}C}_{n+1}\cdot i^{\mathcal{F}C}_n\cdot k^{\mathcal{F}C}_n = \underline{\mathcal{F}g}_{n+1}\cdot k^C_n\cdot q^C_n = k^B_n\cdot g_n\cdot q^C_n = k^B_n\cdot q^B_n\cdot\overline{\mathcal{F}g}_n = q^{\mathcal{F}B}_{n+1}\cdot i^{\mathcal{F}B}_n\cdot k^{\mathcal{F}B}_n\cdot\overline{\mathcal{F}g}_n$

Finally, keeping in mind what we have observed above about $\mathrm{Cok}(h^{\mathcal{F}C}_{n+1})$ and $\mathrm{Ker}(h^{\mathcal{F}C}_n)$, the fact that $\mathcal{F}(C_\bullet)$ is isosequentiable if $C_\bullet$ is proper and $\mathcal{A}$ is regular and protomodular follows from Lemma 4.5.1 in [3]. The special case follows once again because in an abelian category all arrows are proper. ■

6.1.2. Remark. Let us point out explicitly from the proof of Proposition 6.1.1 that the homology objects of the proper complex $C_\bullet$ can be recovered, for all $n \in \mathbb{Z}$, as $H_n(C_\bullet) = \mathrm{Cok}(h^{\mathcal{F}C}_{n+1})$ or, equivalently, as $H_n(C_\bullet) = \mathrm{Ker}(h^{\mathcal{F}C}_n)$. In other words, for an isosequentiable family $h_\bullet$, we can put $H_n(h_\bullet) = \mathrm{Cok}(h_{n+1})$ or $H_n(h_\bullet) = \mathrm{Ker}(h_n)$. In this way, if $\mathcal{A}$ is regular and protomodular, the diagram

$$\begin{tikzcd}
\mathbf{Ch}_{\mathrm{prop}}(\mathcal{A}) \arrow[rr, "\mathcal{F}"] \arrow[dr, "H_n"'] & & \mathbf{IsoSeq}(\mathcal{A}) \arrow[dl, "H_n"] \\
& \mathcal{A} &
\end{tikzcd}$$

obtained by restricting $\mathcal{F}$ to the full subcategory $\mathbf{Ch}_{\mathrm{prop}}(\mathcal{A}) \subseteq \mathbf{Ch}(\mathcal{A})$ of proper complexes, commutes (up to isomorphism) for all $n \in \mathbb{Z}$. In particular, if $\mathcal{A}$ abelian, the diagram

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}) \arrow[rr, "\mathcal{F}"] \arrow[dr, "H_n"'] & & \mathbf{IsoSeq}(\mathcal{A}) \arrow[dl, "H_n"] \\
& \mathcal{A} &
\end{tikzcd}$$

is commutative, since every complex is proper. We can relate this fact to Remark 5.3.4: for a complex $C_\bullet$, the Θ_Δ-kernel of the initial morphism $0 \to \mathcal{F}(C_\bullet)$ is isosequentiable precisely when $C_\bullet$ is acyclic.

6.2. The long exact sequence of homology and the long exact homotopy snail sequence. In this subsection we show that, when $\mathcal{A}$ is regular and protomodular, given a morphism $g_\bullet : B_\bullet \to C_\bullet$ between proper chain complexes such that $\mathrm{Ker}(g_\bullet)$ is a proper chain complex and $\mathcal{N}(\mathcal{F}(g_\bullet))$ is proper in $\mathbf{Seq}(\mathcal{A})$, the long exact sequence in $\mathcal{A}$ obtained by applying Proposition 5.3.2 to $\mathcal{F}(g_\bullet)$ is precisely the classical long exact sequence of homology associated to the extension of chain complexes $\mathrm{Ker}(g_\bullet) \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet$.

We first show that, when $\mathcal{A}$ is regular, proper chain complexes are sent to proper objects in $\mathbf{Seq}(\mathcal{A})$ under the functor $\mathcal{A}$.

6.2.1. Lemma. *Let $\mathcal{A}$ be a regular category.*

1. *Consider the following situation*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & D
\end{tikzcd}$$

If $f \cdot g \cdot h$ is proper, f is an epimorphism and h is a monomorphism, then g is proper.

2. *If a complex $C_\bullet$ is proper, then the associated sequentiable family $\mathcal{F}(C_\bullet)$ is proper.*

Proof. 1. We split the proof in two parts.
(a) We assume that $a = g \cdot h$ is proper and h is a monomorphism and we show that g is proper. Consider the following commutative diagram

$$\begin{tikzcd}
 & B \arrow[ld, "\overline{a}"'] \arrow[rd, "\overline{g}"] \arrow[ldd, "a"] \arrow[rdd, "g"'] & \\
\mathrm{Ker}(c_a) \arrow[rr, dotted, shift left, "s"] \arrow[d, "k_{c_a}"'] & & \mathrm{Ker}(c_g) \arrow[ll, dotted, shift left, "t"] \arrow[d, "k_{c_g}"] \\
D \arrow[d, "c_a"'] & & C \arrow[ll, "h"'] \arrow[d, "c_g"] \\
\mathrm{Cok}(a) & & \mathrm{Cok}(g) \arrow[ll, "h'"]
\end{tikzcd}$$

Since $k_{c_g} \cdot h \cdot c_a = 0$, there exists a unique arrow t such that $t \cdot k_{c_a} = k_{c_g} \cdot h$. Since $\overline{a} \cdot k_{c_a} = \overline{g} \cdot k_{c_g} \cdot h$, with $\overline{a}$ a regular epimorphism (because a is proper) and $k_{c_g} \cdot h$ a monomorphism, there exists a unique arrow s such that $\overline{a} \cdot s = \overline{g}$ and $s \cdot k_{c_g} \cdot h = k_{c_a}$. Composing on the left with $\overline{a}$ and on the right with k_{c_a}, one checks that $s \cdot t = \mathrm{id}$. Composing on the right with $k_{c_g} \cdot h$, one checks that $t \cdot s = \mathrm{id}$. Therefore, $\overline{a}$ and $\overline{g}$ are equal up to an isomorphism, so that $\overline{g}$ is a regular epimorphism, that is, g is proper.
(b) We assume that $b = f \cdot g$ is proper and f is an epimorphism and we show that g is proper. Consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "\overline{b}"'] \arrow[rdd, "b"] & & B \arrow[d, "\overline{g}"] \arrow[ldd, "g"'] \\
\mathrm{Ker}(c_b) \arrow[rr, dotted, "t"'] \arrow[rd, "k_{c_b}"'] & & \mathrm{Ker}(c_g) \arrow[ld, "k_{c_g}"] \\
 & C \arrow[ld, "c_b"'] \arrow[rd, "c_g"] & \\
\mathrm{Cok}(b) \arrow[rr, dotted, "s"'] & & \mathrm{Cok}(g)
\end{tikzcd}$$

Since f is an epimorphism, the unique arrow s such that $c_b \cdot s = c_g$ is an isomorphism. Therefore, the unique arrow t such that $t \cdot k_{c_g} = k_{c_b}$ also is an isomorphism. Since $\overline{b} \cdot t = f \cdot \overline{g}$ (compose with k_{c_g} to check this) and since $\overline{b}$ is a regular epimorphism (because b is proper), we conclude that $\overline{g}$ is a regular epimorphism, that is, g is proper.
2. This follows from 1. because, for all $n \in \mathbb{Z}$, we have $q_n^C \cdot h_n^{\mathcal{F}C} \cdot k_{n-1}^C = d_n^C$, which is proper (notation as in Proposition 6.1.1). ■

6.2.2. Corollary. *Let $\mathcal{A}$ be regular and protomodular and let $g_\bullet \colon B_\bullet \to C_\bullet$ be a morphism of proper complexes. If $\mathcal{N}(\mathcal{F}(g_\bullet))$ is proper, then we get a long exact sequence in homology*

$$\begin{tikzcd} \ldots\ H_{n+1}(B_\bullet) \arrow[r] & H_{n+1}(C_\bullet) \arrow[r] & \mathrm{Cok}(h^P_{n+1}) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet)\ \ldots \end{tikzcd}$$

In particular, if $\mathcal{A}$ is abelian this holds with no assumption on $B_\bullet, C_\bullet$ and $\mathcal{N}(\mathcal{F}(g_\bullet))$.

Proof. This is the long exact sequence of Proposition 5.3.2 applied to $\mathcal{F}(g_\bullet) \colon \mathcal{F}(B_\bullet) \to \mathcal{F}(C_\bullet)$ (we need Lemma 6.2.1 to use Proposition 5.3.2). Therefore, $\{h_n^P\}_{n\in\mathbb{Z}}$ is the first component of the Θ_Δ-kernel $\mathcal{N}(\mathcal{F}(g_\bullet))$ in $\mathbf{Seq}(\mathcal{A})$ and the homology objects of $B_\bullet$ and $C_\bullet$ appear in the sequence as explained in Remark 6.1.2. ■

It is time to compare the long sequence of Corollary 6.2.2 with the usual one obtained from an extension of complexes, but this is an easy task.

6.2.3. Proposition. *Let $\mathcal{A}$ be a regular and protomodular category and consider an extension of proper complexes*

$$\begin{tikzcd} A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] & C_\bullet \end{tikzcd}$$

Assume also that the sequentiable family $\mathcal{N}(\mathcal{F}(g_\bullet))$ is proper. Then, the long exact sequence of Corollary 6.2.2 is nothing but the usual one

$$\begin{tikzcd} \ldots\ H_{n+1}(B_\bullet) \arrow[r] & H_{n+1}(C_\bullet) \arrow[r] & H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet)\ \ldots \end{tikzcd}$$

Proof. The long exact sequence of Corollary 6.2.2 is obtained pasting together infinitely many copies of the six-term snail sequence coming from

$$\begin{tikzcd} \mathrm{Cok}(d^B_{n+1}) \arrow[r, "\mathrm{id}"] \arrow[d, "h^P_n"'] & \mathrm{Cok}(d^B_{n+1}) \arrow[r, "\overline{\mathcal{F}g}_n"] \arrow[d, "h^{\mathcal{F}B}_n"] & \mathrm{Cok}(d^C_{n+1}) \arrow[d, "h^{\mathcal{F}C}_n"] \\ P_n \arrow[r, "\pi_n"'] & \mathrm{Ker}(d^B_{n-1}) \arrow[r, "\underline{\mathcal{F}g}_n"'] & \mathrm{Ker}(d^C_{n-1}) \end{tikzcd}$$

The usual long exact sequence in homology is obtained pasting together infinitely many copies of the six-term snake sequence coming from

$$\begin{tikzcd} \mathrm{Cok}(d^A_{n+1}) \arrow[r, "\overline{\mathcal{F}f}_n"] \arrow[d, "h^{\mathcal{F}A}_n"'] & \mathrm{Cok}(d^B_{n+1}) \arrow[r, "\overline{\mathcal{F}g}_n"] \arrow[d, "h^{\mathcal{F}B}_n"] & \mathrm{Cok}(d^C_{n+1}) \arrow[d, "h^{\mathcal{F}C}_n"] \\ \mathrm{Ker}(d^A_{n-1}) \arrow[r, "\underline{\mathcal{F}f}_n"'] & \mathrm{Ker}(d^B_{n-1}) \arrow[r, "\underline{\mathcal{F}g}_n"'] & \mathrm{Ker}(d^C_{n-1}) \end{tikzcd}$$

Since, for all $n \in \mathbb{Z}$, the arrow $\overline{\mathcal{F}g_n}$ is a regular epimorphism (because $g_n \cdot q_n^C = q_n^B \cdot \overline{\mathcal{F}g_n}$, see (2), and g_n is a regular epimorphism), we can apply Proposition 4.3 in [25] and we have that the two six-term exact sequences are isomorphic. ■

6.2.4. Remark. Notice that if $\mathcal{A}$ is abelian, then every complex is proper and every sequentiable family is also proper (see Example 4.0.10). Therefore, in this case, the assumptions in Proposition 6.2.3 that the complexes $A_\bullet, B_\bullet$ and $C_\bullet$ are proper and the assumption that $\mathcal{N}(\mathcal{F}(g_\bullet))$ is proper are all redundant.

6.2.5. Remark. Under the same assumptions, we can reformulate the proof of Proposition 6.2.3 in a more concise way. First, observe that the functor $\mathcal{F}\colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Seq}(\mathcal{A})$ preserves kernels and regular epimorphisms (use Remark 5.1.2). Second, compare the kernel and the Θ_Δ-kernel of $\mathcal{F}(g_\bullet)$ in $\mathbf{Seq}(\mathcal{A})$:

$$\begin{tikzcd}
\mathcal{F}(A_\bullet) \arrow[r, "\mathcal{F}(f_\bullet)"] \arrow[dr, "\sigma"'] & \mathcal{F}(B_\bullet) \arrow[r, "\mathcal{F}(g_\bullet)"] & \mathcal{F}(C_{\bullet)} \\
& \mathcal{N}(\mathcal{F}(g_\bullet)) \arrow[u, "n_{\mathcal{F}(g_\bullet)}"'] &
\end{tikzcd}$$

Proposition 4.3 in [25] provides the non-trivial part of the proof that the comparison σ is a *quasi-isomorphism*, that is, all the induced arrows $C(\sigma)_n$ and $K(\sigma)_n$ are isomorphisms. Keeping in mind Remark 6.1.2, this means that $H_n(A_\bullet) \simeq H_n(\mathcal{F}(A_\bullet)) \simeq \mathrm{Cok}(h_{n+1}^P)$ and we can conclude that the long exact sequence of Corollary 6.2.2 is the usual long homology sequence.

6.3. Nullhomotopies in $\mathbf{Ch}(\mathcal{A})$ and nullhomotopies in $\mathbf{Seq}(\mathcal{A})$. To conclude, assume that $\mathcal{A}$ is preadditive (= enriched in abelian groups) and let us make more precise the comparison between chain complexes and sequentiable families in this situation.

Recall that, when $\mathcal{A}$ is preadditive, the category $\mathbf{Ch}(\mathcal{A})$ is equipped with a structure of nullhomotopies $\Theta_{\mathbf{Ch}}$ defined as follows. For a morphism $g_\bullet\colon B_\bullet \to C_\bullet$, a nullhomotopy $\varphi_\bullet \in \Theta_{\mathbf{Ch}}(g_\bullet)$ is a family $\{\varphi_n\colon B_n \to C_{n+1}\}_{n\in\mathbb{Z}}$ of arrows in $\mathcal{A}$ such that, for all $n \in \mathbb{Z}$, $\varphi_n \cdot d_{n+1}^C + d_n^B \cdot \varphi_{n-1} = g_n$, as in the following diagram:

$$\begin{tikzcd}
B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d_{n+1}^B"'] & C_{n+1} \arrow[d, "d_{n+1}^C"] \\
B_n \arrow[ur, "\varphi_n"] \arrow[r, "g_n"] \arrow[d, "d_n^B"'] & C_n \arrow[d, "d_n^C"] \\
B_{n-1} \arrow[ur, "\varphi_{n-1}"] \arrow[r, "g_{n-1}"'] & C_{n-1}
\end{tikzcd}$$

For morphisms $f_\bullet\colon A_\bullet \to B_\bullet$ and $h_\bullet\colon C_\bullet \to D_\bullet$, the nullhomotopy $f_\bullet \circ \varphi_\bullet \circ h_\bullet$ is defined by $\{f_n \cdot \varphi_n \cdot h_{n+1}\}_{n\in\mathbb{Z}}$.

6.3.1. Remark. The structure $\Theta_{\mathbf{Ch}}$ in $\mathbf{Ch}(\mathcal{A})$ is not so good as the structure Θ_{Δ} in $\mathbf{Seq}(\mathcal{A})$ is. Here is why:

1. The main problem with $\Theta_{\mathbf{Ch}}$ is that the reduced interchange condition 2.1.4 is not satisfied. To see this, start with any morphism of complexes $g_\bullet \colon B_\bullet \to C_\bullet$ and any nullhomotopy $\varphi_\bullet \in \Theta_{\mathbf{Ch}}(g_\bullet)$. Then, construct the following diagram, where $C_\bullet[-1]$ is the (-1)-translate of $C_\bullet$,

$$C_\bullet \longrightarrow 0 \longrightarrow C_\bullet[-1]$$

equipped with the nullhomotopy given by the family of identity arrows. Explicitly:

$$\begin{tikzcd}
B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^B_{n+1}"'] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \arrow[r] \arrow[d] & C_n \arrow[d, "-d^C_n"] \\
B_n \arrow[r, "g_n"'] \arrow[ur, "\varphi_n"] & C_n \arrow[r] \arrow[urr, "\mathrm{id}"] & 0 \arrow[r] & C_{n-1}
\end{tikzcd}$$

The reduced interchange condition between the two depicted nullhomotopies gives $\varphi_n \cdot 0 = g_n \cdot \mathrm{id}_{C_n}$, so that one would have $g_n = 0$ for all $n \in \mathbb{Z}$.

2. Another problem is that, for any complex $C_\bullet$, the canonical arrow $\eta_{C_\bullet} \colon C_\bullet \to \pi_0(C_\bullet)$ is the identity arrow. This comes from the fact that in the diagram

$$C_\bullet \longrightarrow 0 \longrightarrow C_\bullet[-1]$$

equipped with the already mentioned nullhomotopy, the left part is the $\Theta_{\mathbf{Ch}}$-kernel of the right part and the right part is the $\Theta_{\mathbf{Ch}}$-cokernel of the left part. As a consequence, in $(\mathbf{Ch}(\mathcal{A}), \Theta_{\mathbf{Ch}})$ each object is $\mathcal{S}$-global and $\mathcal{S}$-proper (in the sense of Definition 4.0.3), and this whatever the class S is.

Recall the following definition from [26].

6.3.2. Definition. A *morphism* $\mathcal{F} \colon (\mathcal{A}, \Theta_{\mathcal{A}}) \to (\mathcal{B}, \Theta_{\mathcal{B}})$ *of categories with nullhomotopies* is a functor $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ equipped, for every arrow $g \colon B \to C$ in $\mathcal{A}$, with a map

$$\mathcal{F}_g \colon \Theta_{\mathcal{A}}(g) \longrightarrow \Theta_{\mathcal{B}}(\mathcal{F}(g))$$

such that $\mathcal{F}_{f \cdot g \cdot h}(f \circ \varphi \circ h) = \mathcal{F}(f) \circ \mathcal{F}_g(\varphi) \circ \mathcal{F}(h)$ for all $f \colon A \to B$ and $h \colon C \to D$.

6.3.3. Proposition. *If $\mathcal{A}$ is preadditive, the functor $\mathcal{F}$ of Proposition 6.1.1 extends to a morphism of categories with nullhomotopies*

$$\mathcal{F} \colon (\mathbf{Ch}(\mathcal{A}), \Theta_{\mathbf{Ch}}) \longrightarrow (\mathbf{Seq}(\mathcal{A}), \Theta_{\Delta})$$

Proof. To start, consider a morphism of complexes $g_\bullet \colon B_\bullet \to C_\bullet$ and a nullhomotopy $\varphi_\bullet \in \Theta_{\mathbf{Ch}}(g_\bullet)$. The situation is depicted by the following diagram

$$\begin{tikzcd}
 & B_n \arrow[r, "g_n"] \arrow[dl, "q_n^B"'] \arrow[ddd, "d_n^B"'] & C_n \arrow[dr, "q_n^C"] \arrow[ddd, "d_n^C"'] & \\
\mathrm{Cok}(d_{n+1}^B) \arrow[rrr, "\overline{\mathcal{F}g_n}"] \arrow[d, "h_n^{\mathcal{F}B}"'] & & & \mathrm{Cok}(d_{n+1}^C) \arrow[d, "h_n^{\mathcal{F}C}"] \\
\mathrm{Ker}(d_{n-1}^B) \arrow[rrr, "\underline{\mathcal{F}g}_n"'] \arrow[dr, "k_{n-1}^B"'] & & & \mathrm{Ker}(d_{n-1}^C) \arrow[dl, "k_{n-1}^C"] \\
 & B_{n-1} \arrow[r, "g_{n-1}"'] \arrow[uuur, "\varphi_{n-1}"] & C_{n-1} &
\end{tikzcd}$$

The nullhomotopy $\mathcal{F}(\varphi_\bullet) \in \Theta_\Delta(\mathcal{F}(g_\bullet))$ is defined, in degree n, by the formula

$$\mathcal{F}(\varphi)_n = k_{n-1}^B \cdot \varphi_{n-1} \cdot q_n^C$$

To check the condition $h_n^{\mathcal{F}B} \cdot \mathcal{F}(\varphi)_n = \overline{\mathcal{F}g_n}$, compose on the left with the epimorphism q_n^B. To check the condition $\mathcal{F}(\varphi)_n \cdot h_n^{\mathcal{F}C} = \underline{\mathcal{F}g}_n$, compose on the right with the monomorphism k_{n-1}^C. The rest of the proof is straightforward. ■

6.3.4. Remark. Proposition 6.3.3 can actually be slightly improved: Under suitable assumptions, the categories with nullhomotopies $(\mathbf{Ch}(\mathcal{A}), \Theta_{\mathbf{Ch}})$ and $(\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ can be upgraded to 2-categories and the functor $\mathcal{F} \colon (\mathbf{Ch}(\mathcal{A}), \Theta_{\mathbf{Ch}}) \to (\mathbf{Seq}(\mathcal{A}), \Theta_\Delta)$ can be upgraded to a 2-functor:

1. When $\mathcal{A}$ is preadditive, the category $\mathbf{Ch}(\mathcal{A})$ has a 2-categorical structure: an homotopy $\varphi_\bullet \colon f_\bullet \Rightarrow g_\bullet \colon B_\bullet \rightrightarrows C_\bullet$ is an element in $\Theta_{\mathbf{Ch}(\mathcal{A})}(g_\bullet - f_\bullet)$. Explicitly, $\varphi_\bullet$ is a family $\{\varphi_n \colon B_n \to C_{n+1}\}_{n \in \mathbb{Z}}$ such that $g_n = f_n + \varphi_n \cdot d_{n+1}^C + d_n^B \cdot \varphi_{n-1}$ for all $n \in \mathbb{Z}$. A 2-cell $[\varphi_\bullet] \colon f_\bullet \Rightarrow g_\bullet$ is a class of homotopies, where two homotopies $\varphi_\bullet, \psi_\bullet \colon f_\bullet \Rightarrow g_\bullet$ are equivalent if there exists a family $\{\alpha_n \colon B_n \to C_{n+2}\}_{n \in \mathbb{Z}}$ such that $\varphi_n = \psi_n + \alpha_n \cdot d_{n+2}^C - d_n^B \cdot \alpha_{n-1}$ for all $n \in \mathbb{Z}$.

2. When $\mathcal{A}$ is additive and has finite limits, there is an equivalence of categories $\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$, where the latter is the category of internal groupoids and internal functors in $\mathcal{A}$. Since $\mathbf{Grpd}(\mathcal{A})$ is a 2-category (2-cells are the internal natural transformations in the sense of [1], Chapter 8), using this equivalence we get a 2-categorical structure on $\mathbf{Arr}(\mathcal{A})$. A 2-cell $\varphi \colon (f, f_0) \Rightarrow (g, g_0) \colon (B, b, B_0) \rightrightarrows (C, c, C_0)$ is an arrow $\varphi \colon B_0 \to C$ such that $g = f + b \cdot \varphi$ and $g_0 = f_0 + \varphi \cdot c$. Clearly, the 2-categorical structure of $\mathbf{Arr}(\mathcal{A})$ can be extended level-wise to a 2-categorical structure for $\mathbf{Seq}(\mathcal{A})$.

3. It is then easy to see that, when $\mathcal{A}$ is additive and has finite limits, the functor $\mathcal{F} \colon \mathbf{Ch}(\mathcal{A}) \to \mathbf{Seq}(\mathcal{A})$ of Proposition 6.1.1 is a 2-functor. The argument is essentially

> the same as in the proof of Proposition 6.3.3, one has just to check that the definition of $\mathcal{F}$ on homotopies is compatible with the equivalence relation used in $\mathbf{Ch}(\mathcal{A})$ to define 2-cells.

Let us end this remark with a comment about the terminology introduced in Remark 6.2.5. On one hand, the use of the name quasi-isomorphism in the category $\mathbf{Seq}(\mathcal{A})$ is justified by the fact that, by Remark 6.1.2, a morphism in $\mathbf{Ch}(\mathcal{A})$ is a quasi-isomorphism if and only if its image under the functor $\mathcal{F}$ is a quasi-isomorphism in $\mathbf{IsoSeq}(\mathcal{A})$. On the other hand, by analogy with the simpler situation of $\mathbf{Arr}(\mathcal{A})$, quasi-isomoprhisms in $\mathbf{Seq}(\mathcal{A})$ could be called weak equivalences. Indeed, under the biequivalence $\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$, morphisms $(g, g_0) \colon (B, b, B_0) \to (C, c, C_0)$ such that $K(g)$ and $C(g_0)$ are isomorphisms correspond to weak equivalences.

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