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THE GROUP LAW OF PICARD STACKS VIA MATRICES

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Abstract. An algebraic theory is a category whose objects are finite products of a distinguished object with itself. In this framework, abelian groups can be described as finite product-preserving functors from an appropriate algebraic theory to the category of sets.

The purpose of this paper is to extend this categorical description of abelian groups to the setting of strictly commutative Picard stacks over a site **S**. More precisely, we show that such stacks can be characterized as morphisms of 2-stacks from a suitable algebraic 2-stack theory to the 2-stack of stacks over **S**.

Our results provide a new formulation of the group law for strictly commutative Picard stacks, which may help clarify the notion of a group structure in higher stacks. We expect that this perspective will contribute further to the study of torsors, extensions, and biextensions in the higher categorical context.

1. Introduction

In his doctoral thesis [Lawvere, 1963], F. W. Lawvere defined an algebraic theory as a category whose objects are finite products of a given object with itself [1]. In this framework, abelian groups can be described as functors from an appropriate algebraic theory to the category of sets, which preserve finite products (see [Adamek,Rosicky, Vitale, 2010, Example 1.6]). The aim of this paper is to generalize this description of groups to strictly commutative Picard stacks over a site **S**.

We start by recalling the general properties of stacks and strictly commutative Picard stacks (see Section 2). We define the notions of algebraic 2-stack theory over a site **S** and of 2-algebra for an algebraic 2-stack theory, which generalize to 2-stacks the notions of algebraic theory and of algebra for an algebraic theory respectively (see Definition 3.1). This leads us to the definition of an algebraic 2-stack, that is a 2-stack 2-equivalent to the 2-stack of 2-algebras for an appropriate algebraic 2-stack theory (see Definition 3.2). Finally in Theorem 4.1 we prove that the 2-stack of strictly commutative Picard stacks over a site **S** is algebraic.

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[1] The notion of algebraic theory in nowadays literature is weaker than Lawvere's one. In this paper we always refer to Lawvere's original notion of algebraic theory, that is also an algebraic theory in the modern sense of the term.

In order to define an appropriate group law for stacks we need a morphism of stacks, six natural isomorphisms and five compatibility conditions expressed via diagrams (see [Bertolin, 2013, §1]), and in the case of 2-stacks this definition becomes even more complicated: it involves a morphism of 2-stacks, two natural 2-transformations, five modifications and five compatibility conditions expressed via diagrams (see [Bertolin, Tatar 2, 2018, §1]). A group law for higher stacks gets more and more difficult to describe. The results of this paper provide a new description of the group law of a strictly commutative Picard stack in terms of a 2-algebra $A : \mathbb{T} \to \mathbb{S}\mathrm{tack}$ of an appropriate algebraic 2-stack theory $\mathbb{T}$ and we hope that these results will shed some light on the notion of group law for higher stacks [2]. We expect that our new description of group law for stacks will contribute further to the study of torsors, extensions and biextensions for higher stacks, which has been done so far only for strictly commutative Picard stacks (see [Bertolin, 2011, Bertolin, 2013]) and strictly commutative Picard 2-stacks (see [Bertolin, Tatar, 2014, Bertolin, Tatar 2, 2018, §1]).

An antecedent of the present work is [Power, 1999], where Power introduces the notion of a Lawvere V-theory, which specializes to the classical concept when V is the category Set. In the case $V = Cat$, a Lawvere V-theory is precisely a Lawvere 2-theory $\mathcal{T}$, that is, a Cat-enriched Lawvere theory. Furthermore, in [Power, 1999, Theorem 5.3], Power establishes that if T is a 2-monad, then the 2-category of T-algebras is biequivalent to the 2-category of models of the Lawvere 2-theory $\mathcal{T}$. This result provides the conceptual foundation for our definition of 2-algebra given in Definition 3.1.

Via the dictionary between Picard stacks over $\mathbf{S}$ and length one complexes of abelian sheaves on $\mathbf{S}$ given in Proposition 2.1, 1-motives can be seen as Picard stacks. This implies that Picard stacks have an important role in the theory of motives (see [Bertolin, 2009, Bertolin2, 2009, Bertolin, 2012, Bertolin, 2019, Bertolin, Philippon, Saha,Saha, 2022, Bertolin, Philippon, 2024, Bertolin,Mazza, 2009, Bertolin, Brochard, 2019] and [Bertolin,Galluzzi, 2020, Bertolin, Galluzzi, 2021]).

2. Preliminaries

Let $\mathbf{S}$ be a site. In this paper U denotes an object of $\mathbf{S}$. A *stack* over $\mathbf{S}$ is a fibered category $\mathcal{X}$ over $\mathbf{S}$ such that

- (*Gluing condition on objects*) descent is effective for objects of $\mathcal{X}$, and
- (*Gluing condition on arrows*) for any U and for every pair of objects X, Y of the category $\mathcal{X}(U)$, the presheaf of arrows $\mathrm{Hom}_{\mathcal{X}(U)}(X, Y)$ of $\mathcal{X}(U)$ is a sheaf over U

(see [Breen, 2010, Definition 2.2] for a more explicit description of stacks). For the notions of morphism of stacks (i.e. cartesian functor), and morphism of cartesian

[2] At the level of stacks, one can already distinguish several notions of abelian group, according to the degree of coherence required for associativity, commutativity, the unit, and the inverse. In the case of strictly commutative Picard stacks, all the structural laws are required to hold strictly, whereas for a pseudo-group in the 2-category of stacks these laws hold only up to coherent 2-isomorphisms. In higher-stack contexts, the situation becomes considerably more intricate.

functors, we refer to [Giraud, 1971, Chp II 1.2]. An *equivalence of stacks* $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism of stacks such that any object Y of $\mathcal{X}_2(U)$ is isomorphic to an object of the form $F(U)(X)$ with X an object of $\mathcal{X}_1(U)$, and for any pair of objects X, Y of $\mathcal{X}_1(U)$, the map $\mathrm{Hom}_{\mathcal{X}_1(U)}(X,Y) \to \mathrm{Hom}_{\mathcal{X}_2(U)}(F(U)(X), F(U)(Y))$ is bijective. Two stacks are *equivalent* if there exists an equivalence of stacks between them. A *stack in groupoids* over $\mathbf{S}$ is a stack $\mathcal{X}$ over $\mathbf{S}$ such that the category $\mathcal{X}(U)$ is a groupoid, i.e. a category whose arrows are invertible. *From now on, all stacks will be stacks in groupoids.*

A *strictly commutative Picard stack* over the site $\mathbf{S}$ (just called a Picard stack) is a stack $\mathcal{P}$ over $\mathbf{S}$ endowed

- with a morphism of stacks $+_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, called *the group law* of $\mathcal{P}$, which assigns to each object (X_1, X_2) of $(\mathcal{P} \times_\mathbf{S} \mathcal{P})(U)$ the object $X_1 +_\mathcal{P} X_2$ of $\mathcal{P}(U)$, and which assigns to each arrow $(f,g) : (X_1, X_2) \to (X_1', X_2')$ of $(\mathcal{P} \times_\mathbf{S} \mathcal{P})(U)$ the arrow $f +_\mathcal{P} g : X_1 +_\mathcal{P} X_2 \to X_1' +_\mathcal{P} X_2'$ of $\mathcal{P}(U)$, and
- two natural isomorphisms, called respectively the *associativity* and the *commutativity*

$$\mathsf{a} : +_\mathcal{P} \circ (+_\mathcal{P} \times \mathrm{id}_\mathcal{P}) \Rightarrow +_\mathcal{P} \circ (\mathrm{id}_\mathcal{P} \times +_\mathcal{P}),$$
$$\mathsf{c} : +_\mathcal{P} \circ \mathsf{s} \Rightarrow +_\mathcal{P},$$

  with $\mathsf{s}(X,Y) = (Y,X)$ for all $X, Y \in \mathcal{P}(U)$, which express respectively the associativity and the commutativity constraints of the group law $+_\mathcal{P}$ of $\mathcal{P}$,

such that $\mathcal{P}(U)$ is a strictly commutative Picard category (i.e. it is possible to make the sum of two objects of $\mathcal{P}(U)$, this sum is associative and commutative, and any object of $\mathcal{P}(U)$ has an inverse with respect to this sum, see [Bertolin, 2013, §1] or [Deligne, 1973, 1.4, page 39] for more details). The word "strictly" means that $\mathsf{c}_{X,X} : X +_\mathcal{P} X \to X +_\mathcal{P} X$ is the identity for all $X \in \mathcal{P}(U)$.

An *additive functor* $(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2$ between two Picard stacks is a morphism of stacks $F : \mathcal{P}_1 \to \mathcal{P}_2$ endowed with a natural isomorphism $\sum_{X,Y} : F(X +_{\mathcal{P}_1} Y) \cong F(X) +_{\mathcal{P}_2} F(Y)$ (for all $X, Y \in \mathcal{P}_1(U)$) which is compatible with the natural isomorphisms a and c underlying $\mathcal{P}_1$ and $\mathcal{P}_2$. A *morphism of additive functors* $u : (F, \sum) \Rightarrow (F', \sum')$ is a morphism of cartesian functors which is compatible with the natural isomorphisms $\sum$ and $\sum'$ of F and F' respectively. The stack whose objects are additive functors from $\mathcal{P}$ to $\mathcal{Q}$ and whose arrows are morphisms of additive functors, is a stack in groupoid: any morphism of additive functors is invertible (i.e. it is an isomorphism of additive functors).

Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site $\mathbf{S}$: all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K = (K^i)_i$ such that $K^i = 0$ for $i \neq -1$ or 0. In [Deligne, 1973, 1.4.11 page 45] Deligne proves the following links between Picard $\mathbf{S}$-stacks and complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, between additive functors and morphisms of complexes, and between morphisms of additive functors and homotopies of complexes:

- to any complex $K = [K^{-1} \stackrel{d}{\to} K^0]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is associated a Picard stack $\mathrm{PicSt}(K)$ which is the stack generated by the following pre-stack $\mathrm{PreSt}(K)$: for any object U of $\mathbf{S}$, the objects of $\mathrm{PreSt}(K)(U)$ are the elements of $K^0(U)$, and if x and y are two objects of $\mathrm{PreSt}(K)(U)$ (i.e. $x, y \in K^0(U)$), an arrow of $\mathrm{PreSt}(K)(U)$ from x to y is an element f of $K^{-1}(U)$ such that $df = y - x$;

- a morphism of complexes $g : K \to L$ induces an additive functor $\mathrm{PicSt}(g) : \mathrm{PicSt}(K) \to \mathrm{PicSt}(L)$ between the Picard stacks associated to the complexes K and L;

- for any Picard $\mathbf{S}$-stack $\mathcal{P}$, there exists a complex K of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ such that $\mathcal{P} = \mathrm{PicSt}(K)$;

- if K, L are two complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then for any additive functor $F : \mathrm{PicSt}(K) \to \mathrm{PicSt}(L)$ there exists a quasi-isomorphism $k : K' \to K$ and a morphism of complexes $l : K' \to L$ such that F is isomorphic as additive functor to $st(l) \circ st(k)^{-1}$;

- if $f, g : K \to L$ are two morphisms of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then the morphisms of additive functors $\mathrm{PicSt}(f) \Rightarrow \mathrm{PicSt}(g)$ are in bijection with the homotopies $H : K \to L$ such that $g - f = dH + Hd$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $\mathrm{H}^i(K) = 0$ for $i \neq -1$ or 0. Let $\mathrm{Picard}(\mathbf{S})$ be the category whose objects are small Picard stacks over $\mathbf{S}$ and whose arrows are isomorphism classes of additive functors. According to [Deligne, 1973, Lemma 1.4.13 page 46], we have

2.1. Proposition. *The functor*

$$\begin{array}{rcl} \mathrm{PicSt} : \mathcal{D}^{[-1,0]}(\boldsymbol{S}) & \longrightarrow & \mathrm{Picard}(\boldsymbol{S}) \\ K & \mapsto & \mathrm{PicSt}(K) \\ K \stackrel{f}{\to} L & \mapsto & \mathrm{PicSt}(K) \stackrel{\mathrm{PicSt}(f)}{\to} \mathrm{PicSt}(L), \end{array}$$

is an equivalence of categories.

We denote by $[\quad]$ the inverse equivalence of PicSt.

The *product of two stacks* $\mathcal{P}$ *and* $\mathcal{Q}$ *over* $\boldsymbol{S}$ is simply the categorical product over each fibre. More precisely the stack $\mathcal{P} \times \mathcal{Q}$ over $\mathbf{S}$ is defined as follows:

- an object of the category $\mathcal{P} \times \mathcal{Q}(U)$ is a pair (X, Y) of objects with X an object of $\mathcal{P}(U)$ and Y an object of $\mathcal{Q}(U)$;

- if (X, Y) and (X', Y') are two objects of $\mathcal{P} \times \mathcal{Q}(U)$, an arrow of $\mathcal{P} \times \mathcal{Q}(U)$ from (X, Y) to (X', Y') is a pair (f, g) of arrows with $f : X \to X'$ an arrow of $\mathcal{P}(U)$ and $g : Y \to Y'$ an arrow of $\mathcal{Q}(U)$;

For the notion of product of two Picard stacks see [Bertolin, 2011, Definition 2.1] or [Bertolin, Tatar, 2014, Definition 4.1]. If $\mathcal{P}$ and $\mathcal{Q}$ are Picard stacks, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$[\mathcal{P} \times \mathcal{Q}] = [\mathcal{P}] + [\mathcal{Q}]. \tag{2.1}$$

2.2. Remark. For any integer $n > 0$, $\mathcal{P}^n$ denotes the product of the stack $\mathcal{P}$ with itself n times, i.e. $\mathcal{P}^n := \mathcal{P} \times \cdots \times \mathcal{P}$. Clearly $\mathcal{P}^n \times \mathcal{P}^m = \mathcal{P}^{n+m}$. We view an object of $\mathcal{P}^n$ as a column vector $(X_1, \ldots, X_n)^t$ of length n (here t is the transpose) and an arrow of $\mathcal{P}^n$ as a row vector $(f_1, \ldots, f_n)$ of length n.

Since written references in English on 2-categories and 2-fibered categories are difficult to find, we recall here the main definitions (for references in French see [Hakim, 1972, Chp I]). A *2-category* $\mathbb{X}$ is given by the following data:

1. a class of objects;
2. for every pair of objects (X, Y) of $\mathbb{X}$, a category $\mathrm{Hom}_{\mathbb{X}}(X, Y)$, whose objects are called the *1-arrows* from X to Y and whose arrows are called the *2-arrows*;
3. for every triple of objects (X, Y, Z) of $\mathbb{X}$, a composition functor

$$\mu_{X,Y,Z} : \mathrm{Hom}_{\mathbb{X}}(X, Y) \times \mathrm{Hom}_{\mathbb{X}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathbb{X}}(X, Z),$$

and a 1-arrow $\mathrm{id}_X \in \mathrm{Hom}_{\mathbb{X}}(X, X)$ such that the following conditions are satisfied:

(i) for every triple of objects (X, Y, Z) of $\mathbb{X}$,

$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}_{\mathbb{X}}(X,Y)}.$$

(ii) for any quadruple of object (X, Y, Z, T) of $\mathbb{X}$,

$$\mu_{X,Z,T} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}_{\mathbb{X}}(Z,T)}) = \mu_{X,Y,T} \circ (\mathrm{id}_{\mathrm{Hom}_{\mathbb{X}}(X,Y)} \times \mu_{Y,Z,T})$$

Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two 2-categories. A *2-functor* $F : \mathbb{X}_1 \to \mathbb{X}_2$ is the collection of the following data:

(i) a map $F : \mathrm{Ob}\,\mathbb{X}_1 \to \mathrm{Ob}\,\mathbb{X}_2$,

(ii) a functor $F_{X,Y} : \mathrm{Hom}_{\mathbb{X}_1}(X, Y) \to \mathrm{Hom}_{\mathbb{X}_2}(F(X), F(Y))$ for every pair (X, Y) of objects of $\mathbb{X}_1$,

such that the following conditions are satisfied:

(i)′ for every object $X \in \mathrm{Ob}\,\mathbb{X}_1$, $F_{X,X}(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(ii)′ for every triple (X, Y, Z) of objects of $\mathbb{X}_1$, the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{X}_1}(X, Y) \times \mathrm{Hom}_{\mathbb{X}_1}(Y, Z) & \xrightarrow{\mu} & \mathrm{Hom}_{\mathbb{X}_1}(X, Z) \\
\downarrow{\scriptstyle F_{X,Y} \times F_{Y,Z}} & & \downarrow{\scriptstyle F_{X,Z}} \\
\mathrm{Hom}_{\mathbb{X}_2}(F(X), F(Y)) \times \mathrm{Hom}_{\mathbb{X}_2}(F(Y), F(Z)) & \xrightarrow[\mu']{} & \mathrm{Hom}_{\mathbb{X}_2}(F(X), F(Z))
\end{array}$$

Let $F : \mathbb{X}_1 \to \mathbb{X}_2$ be a 2-functor. A 1-arrow $f : X \to Y$ of $\mathbb{X}_1$ is called *cartesian* when it has the following two properties:

1. for all $h : Z \to Y$ and $u : F(Z) \to F(X)$ with $F(h) = F(f) \circ u$, there is a unique $\hat{u} : Z \to X$ with $F(\hat{u}) = u$ and $h = f \circ \hat{u}$;
2. for all $\sigma : h \Rightarrow k, \tau : u \Rightarrow v$ with $F(\sigma) = F(f) \cdot \tau$ and lift $\hat{u}, \hat{v}$ of u, v, there is a unique $\hat{\tau} : \hat{u} \Rightarrow \hat{v}$ with $F(\hat{\tau}) = \tau$ and $\sigma = f \cdot \hat{\tau}$.

A 2-arrow $\alpha : f \Rightarrow g : X \to Y$ of $\mathbb{X}_1$ is cartesian if it is cartesian as 1-arrow for the functor $F_{X,Y} : \mathrm{Hom}_{\mathbb{X}_1}(X, Y) \to \mathrm{Hom}_{\mathbb{X}_2}(F(X), F(Y))$.

Let $F : \mathbb{X}_1 \to \mathbb{X}_2$ be a 2-functor. $\mathbb{X}_1$ is a *fibered 2-category over* $\mathbb{X}_2$ via F if the following conditions are satisfied:

(i) For every object $X \in \mathrm{Ob}\,\mathbb{X}_1(U)$ and every 1-arrow $u : V \to U$ of $\mathbb{X}_2$, there exists a cartesian 1-arrow $v : Y \to X$ lying over u.

(ii) The functor $F_{X,Y} : \mathrm{Hom}_{\mathbb{X}_1}(X, Y) \to \mathrm{Hom}_{\mathbb{X}_2}(F(X), F(Y))$ is a fibration of categories for every pair (X, Y) of objects of $\mathbb{X}_1$.

(iii) The composite of any two composable cartesian 2-arrows is cartesian.

According to [Buckley, 2014, Remark 2.1.7] condition (ii) is equivalent to the following condition: For every arrow g of $\mathbb{X}_1$ and every 2-arrow $\alpha : f \Rightarrow F(g)$ of $\mathbb{X}_2$, there exists a cartesian 2-arrow $\sigma : f \Rightarrow g$ lying over α. Hence our definition coincide with Buckley's one ([Buckley, 2014, Definition 2.1.6]).

A *2-stack* over a site $\mathbf{S}$ is a fibered 2-category $\mathbb{X} \to \mathbf{S}$ such that

- 2-descent is effective for objects, and
- for every object U of $\mathbf{S}$ and every pair of objects $X, Y \in \mathbb{X}(U)$, the fibered category $(V \to U) \mapsto \mathrm{Hom}_{\mathbb{X}(V)}(X|_V, Y|_V)$ is a stack.

Concretely, 2-descent for objects means that given a covering family $(U_\alpha \to U)_{\alpha \in I}$, we are given objects $X_\alpha \in \mathbb{X}(U_\alpha)$, 1-arrows $\phi_{\alpha\beta} \colon X_\alpha|_{U_{\alpha\beta}} \longrightarrow X_\beta|_{U_{\alpha\beta}}$, and invertible 2-arrows $\psi_{\alpha\beta\gamma} \colon \phi_{\beta\gamma} \circ \phi_{\alpha\beta} \Rightarrow \phi_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$, such that the tetrahedral coherence condition is satisfied: in the following diagram of 2-arrows

$$\begin{array}{ccccc}
\phi_{\gamma\delta} \circ (\phi_{\beta\gamma} \circ \phi_{\alpha\beta}) & \xRightarrow{\mathrm{id} * \psi_{\alpha\beta\gamma}} & \phi_{\gamma\delta} \circ \phi_{\alpha\gamma} & & \\
\Downarrow \mathsf{a} & & & \searrow^{\psi_{\alpha\gamma\delta}} & \\
(\phi_{\gamma\delta} \circ \phi_{\beta\gamma}) \circ \phi_{\alpha\beta} & \xRightarrow[\psi_{\beta\gamma\delta} * \mathrm{id}]{} & \phi_{\beta\delta} \circ \phi_{\alpha\beta} & \xRightarrow[\psi_{\alpha\beta\delta}]{} & \phi_{\alpha\delta}
\end{array}$$

the two composites from $\phi_{\gamma\delta}\phi_{\beta\gamma}\phi_{\alpha\beta}$ to $\phi_{\alpha\delta}$ coincide. The 2-descent condition $(X_\alpha, \phi_{\alpha\beta}, \psi_{\alpha\beta\gamma})$ is effective if there exists an object $X \in \mathbb{X}(U)$, together with 1-arrows $X|_{U_\alpha} \cong X_\alpha$ in $\mathbb{X}(U_\alpha)$, compatible with the given 1- and 2-arrows $\phi_{\alpha\beta}$ and $\psi_{\alpha\beta\gamma}$.

Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be 2-stacks over a site $\mathbf{S}$. A *morphism of* 2*-stacks* $F\colon \mathbb{X}_1 \longrightarrow \mathbb{X}_2$ is a cartesian 2-functor over $\mathbf{S}$, that is, a 2-functor such that the diagram

$$\begin{array}{ccc} \mathbb{X}_1 & \xrightarrow{F} & \mathbb{X}_2 \\ & \searrow \quad \swarrow & \\ & \mathbf{S} & \end{array}$$

commutes and F sends cartesian i-arrows of $\mathbb{X}_1$ to cartesian i-arrows of $\mathbb{X}_2$ for $i = 1, 2$.

Let $F, G\colon \mathbb{X}_1 \longrightarrow \mathbb{X}_2$ be cartesian 2-functors over a site $\mathbf{S}$. A *morphism of cartesian* 2*-functors* $\eta\colon F \Rightarrow G$ consists of:

- for every object X of $\mathbb{X}_1$, there is a 1-arrow $\eta_X\colon F(X) \longrightarrow G(X)$ lying over the identity arrow of the image of X in $\mathbf{S}$;
- for every 1-arrow $f\colon X \to Y$ in $\mathbb{X}_1$, an invertible 2-arrow $\eta_f\colon G(f) \circ \eta_X \Rightarrow \eta_Y \circ F(f)$, such that the following coherence conditions hold: for every object X, $\eta_{\mathrm{id}_X} = \mathrm{id}_{\eta_X}$, and for every pair of composable 1-arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, the naturality constraints are compatible with composition;
- the components η_X preserve cartesian 1-arrows, i.e. if f is cartesian in $\mathbb{X}_1$, then the naturality constraint η_f is compatible with the cartesian structure.

Let $F, G\colon \mathbb{X}_1 \to \mathbb{X}_2$ be cartesian 2-functors and let $\eta, \theta\colon F \Rightarrow G$ be morphisms of cartesian 2-functors. A *modification* of morphisms of cartesian 2-functors $\Gamma\colon \eta \Rrightarrow \theta$ is given by a family of 2-arrows $\Gamma_X\colon \eta_X \Rightarrow \theta_X$ in $\mathbb{X}_2$, one for each object X of $\mathbb{X}_1$, such that for every 1-arrow $f\colon X \to Y$ in $\mathbb{X}_1$ the following diagram of 2-arrows commutes:

$$\begin{array}{ccc} G(f) \circ \eta_X & \xRightarrow{\eta_f} & \eta_Y \circ F(f) \\ {\scriptstyle \mathrm{id}*\Gamma_X}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \Gamma_Y*\mathrm{id}} \\ G(f) \circ \theta_X & \xRightarrow[\theta_f]{} & \theta_Y \circ F(f). \end{array}$$

A *2-equivalence of 2-stacks* $F : \mathbb{X}_1 \to \mathbb{X}_2$ is a morphism of 2-stacks such that any object Y of $\mathbb{X}_2(U)$ is equivalent to an object of the form $F(U)(X)$ with X an object of $\mathbb{X}_1(U)$, and for any pair of objects X, Y of $\mathbb{X}_1(U)$, the morphism of stacks $\mathrm{Hom}_{\mathbb{X}_1(U)}(X, Y) \to \mathrm{Hom}_{\mathbb{X}_2(U)}(F(U)(X), F(U)(Y))$ is an equivalence of stacks. Two 2-stacks are *2-equivalent* if there exists a 2-equivalence of 2-stacks between them. A *2-stack in 2-groupoids* over $\mathbf{S}$ is a 2-stack $\mathbb{X}$ over $\mathbf{S}$ such that the 2-category $\mathbb{X}(U)$ is a 2-groupoid, i.e. a 2-category whose 1-arrows are invertible up to a 2-arrow and whose 2-arrows are strictly invertible. *From now on, all 2-stacks will be 2-stacks in 2-groupoids.*

Stacks over $\mathbf{S}$ form a 2-stack over $\mathbf{S}$ (see [Breen, 1994, Examples 1.11 i) page 32] and [Stack Project, 2005–2026, Chp 8, Definitions 8.4.5 and 8.5.5 and 112.4 Related references on foundations of stacks]), denoted

$$\mathbb{S}\mathrm{tack}$$

whose objects are stacks and whose hom-stack consists of morphisms of stacks and morphisms of cartesian functors. Picard stacks over $\mathbf{S}$ form a 2-stack over $\mathbf{S}$, denoted

$$\mathbb{P}\text{icard}$$

whose objects are Picard stacks and whose hom-stack consists of additive functors and morphisms of additive functors.

3. An algebraic 2-stack theory

An *algebraic theory* is a small category T with finite products. An *algebra of the algebraic theory* T is a functor $A : \mathrm{T} \to \mathrm{Set}$ preserving finite products (here Set is the category of small sets). Denote by AlgT the category whose objects are algebras of T and whose arrows are natural transformations. A category is *algebraic* if it is equivalent to AlgT for some algebraic theory T ([Adamek,Rosicky, Vitale, 2010, Chp 1, Definition 1.1]). Generalizing from categories to 2-stacks these definitions we propose

3.1. Definition. *An algebraic 2-stack theory over* $\boldsymbol{S}$ *is a 2-stack* $\mathbb{T}$ *over* $\boldsymbol{S}$ *with finite products, that is each fibre is a 2-category with finite products and base change preserves them up to equivalence. A 2-algebra of the algebraic 2-stack theory* $\mathbb{T}$ *is a morphism of 2-stacks*

$$A : \mathbb{T} \longrightarrow \mathbb{S}\text{tack}$$

preserving finite products.

Denote by Alg$\mathbb{T}$ the 2-stack of 2-algebras for $\mathbb{T}$, whose hom-stack consists of morphisms of cartesian 2-functors and modification of morphisms of cartesian 2-functors.

3.2. Definition. *A 2-stack* $\mathbb{X}$ *over* $\boldsymbol{S}$ *is algebraic if it is 2-equivalent to* Alg$\mathbb{T}$ *for some algebraic 2-stack theory* $\mathbb{T}$ *over* $\boldsymbol{S}$*.*

In algebraic geometry, the term "algebraic" 2-stack is used to denote a 2-stack admitting a suitable 2-dimensional atlas. Our Definition 3.2 does not follow this convention.

Denote by $\mathrm{Sh}(\mathbf{S})$ (resp. $\mathrm{Ab}(\mathbf{S})$) the category of sheaves (resp. of abelian sheaves) over $\mathbf{S}$. Let $\underline{\mathbb{Z}}$ be the constant sheaf over $\mathbf{S}$ obtained as the sheafification of the constant pre-sheaf $\mathbb{Z}$. The forgetful functor $\iota : \mathrm{Ab}(\mathbf{S}) \to \mathrm{Sh}(\mathbf{S})$ has a left adjoint

$$\underline{\mathbb{Z}}[-] : \mathrm{Sh}(\mathbf{S}) \to \mathrm{Ab}(\mathbf{S}),$$

called the free abelian sheaf functor (see [Grothendieck, Verdier, 1972, 11.1.5, page 198]).

Let pt be the terminal sheaf of the category $\mathrm{Sh}(\mathbf{S})$ of sheaves (i.e. the sheafification of the constant pre-sheaf with value the terminal object in the category of sets, i.e. any one-element set). For any abelian sheaf F on $\mathbf{S}$, the Yoneda Lemma and the adjointness of the functors ι and $\underline{\mathbb{Z}}[-]$ imply the isomorphisms

$$F(U) \cong {}^{3}\mathrm{Hom}_{\mathrm{Sh}(\mathbf{S})}(\mathrm{pt}_{|U}, \iota(F)) \cong \mathrm{Hom}_{\mathrm{Ab}(\mathbf{S})}(\underline{\mathbb{Z}}[\mathrm{pt}_{|U}], F). \tag{3.1}$$

[3]This isomorphism if true for any sheaf.

Consider the disjoint union $\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt}$ of n copies of the terminal sheaf. The free abelian sheaf $\underline{\mathbb{Z}}[\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt}]$ is isomorphic to the direct sum $\underline{\mathbb{Z}}[\mathrm{pt}] \oplus \cdots \oplus \underline{\mathbb{Z}}[\mathrm{pt}]$ of n copies of the sheaf $\underline{\mathbb{Z}}[\mathrm{pt}]$, that we will denote $\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]$.

3.3. Definition. *A locally constant matrix* $P = (p_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,n}}$ *over* $\boldsymbol{S}$ *is a matrix whose entries* p_{ij} *are locally constant functions, i.e. sections of a constant sheaf over* $\boldsymbol{S}$.

With the notations introduced until now, we can prove that

3.4. Lemma. *The category* $\mathrm{Ab}(\boldsymbol{S})$ *of abelian sheaves over* $\boldsymbol{S}$ *is algebraic.*

Proof. Denote by T the category whose

- objects are the abelian sheaves $\mathrm{T}^n := \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]$ for any integer n, $n \geq 1$, and
- arrows from T^n to T^k are morphisms of sheaves defined by locally constant matrices $P = (p_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,n}}$ with k rows, n columns and entries in the constant sheaf $\underline{\mathbb{Z}}$. More explicitly, viewing elements of $\mathrm{T}^n(U)$ as column vectors $(z_1, \ldots, z_n)^t$, the matrix $P_U = (p_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,n}}$, with $p_{ij} \in \underline{\mathbb{Z}}(U)$, sends an element $(z_1, \ldots, z_n)^t$ of $\mathcal{T}^n(U)$ to the element $(\sum_{j=1}^n p_{1j} z_j, \ldots, \sum_{j=1}^n p_{kj} z_j)^t$ of $\mathcal{T}^k(U)$ (the products $p_{ij} z_j$ are given by the natural morphism of sheaves $\underline{\mathbb{Z}} \times \underline{\mathbb{Z}}[\mathrm{pt}] \to \underline{\mathbb{Z}}[\mathrm{pt}]$).

The category $\mathrm{Ab}(\mathbf{S})$ of abelian sheaves is equivalent to the category AlgT, whose objects are algebras of T. In fact, an algebra $A : \mathrm{T} \to \mathrm{Sh}(\mathbf{S})$ defines an abelian sheaf F in the following way: the image $A(\mathrm{T}^1)$ of T^1 gives the sheaf F and the image $A((1_{|U}, 1_{|U}))$ of the arrow $(1_{|U}, 1_{|U}) : \mathrm{T}^2(U) \to \mathrm{T}^1(U)$ provides the group law of F (here $1_{|U} \in \underline{\mathbb{Z}}(U)$ is the locally constant function 1). Conversely, via the Yoneda Lemma (3.1), any abelian sheaf F determines an algebra $A : \mathrm{T} \to \mathrm{Sh}(\mathbf{S}), \mathrm{T}^1 \mapsto F$, of T. ■

Consider the disjoint union $\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt}$ of n copies the terminal sheaf pt and denote by e_i (for $i = 1, \ldots, n$) the image via the canonical morphism of sheaves $\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt} \to \underline{\mathbb{Z}}[\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt}] \cong \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]$ of the one-element set (seen as global constant function) defining the i-th terminal sheaf pt in this disjoint union. We need the following stack version of [Deligne, 1973, Lemme 1.4.3, page 41], proved in [Tatar, 2011, Proposition 6.2]:

3.5. Lemma. *Let* $\mathcal{P}$ *be a Picard stack. If* $\{X_1, \ldots, X_n\}$ *is a family of* n *objects of the category* $\mathcal{P}(U)$*, then it exists*

- *a map* $\Sigma : \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}](U) \to \mathrm{Objects}(\mathcal{P}(U))$,
- *isomorphisms* $a_i : \Sigma(\mathrm{e}_i) \to X_i$ *in the category* $\mathcal{P}(U)$ *for* $i = 1, \ldots, n$,
- *isomorphisms* $a_{s,t} : \Sigma(s +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} t) \to \Sigma(s) +_{\mathcal{P}} \Sigma(t)$ *in the category* $\mathcal{P}(U)$ *for any* $s, t \in \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}](U)$,

such that the diagrams

$$\begin{array}{ccccc}
\Sigma\big(s +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} (t +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} u)\big) & \xrightarrow{a_{s,t+u}} & \Sigma(s) +_{\mathcal{P}} \Sigma(t +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} u) & \xrightarrow{a_{t,u}} & \Sigma(s) +_{\mathcal{P}} \big(\Sigma(t) +_{\mathcal{P}} \Sigma(u)\big) \\
\| & & & & \uparrow \mathsf{a} \\
\Sigma\big((s +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} t) +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} u\big) & \xrightarrow[a_{s+t,u}]{} & \Sigma(s +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} t) +_{\mathcal{P}} \Sigma(u) & \xrightarrow[a_{s,t}]{} & \big(\Sigma(s) +_{\mathcal{P}} \Sigma(t)\big) +_{\mathcal{P}} \Sigma(u),
\end{array} \tag{3.2}$$

$$\begin{array}{ccc}
\Sigma(s +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} t) & \xrightarrow{a_{s,t}} & \Sigma(s) +_{\mathcal{P}} \Sigma(t) \\
\| & & \uparrow \mathsf{c} \\
\Sigma(t +_{\oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]} s) & \xrightarrow[a_{t,s}]{} & \Sigma(t) +_{\mathcal{P}} \Sigma(s)
\end{array} \tag{3.3}$$

commute. The set of data $(\Sigma, a_i, a_{s,t})_{\substack{i=1,\dots,n \\ s,t \in \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}](U)}}$ *is unique modulo a unique isomorphism and it is functorial in the family* $\{X_1, \dots, X_n\}$.

For any integer n, $n \geq 1$, define the following Picard stack

$$\mathcal{T}^n := \mathrm{PicSt}\big([0 \to \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]]\big) = \underbrace{\mathcal{T}^1 \times \cdots \times \mathcal{T}^1}_{n}$$

where the last equality is due to (2.1). Remark that for any $n \geq 1$ and any U, the category $\mathcal{T}^n(U)$ is discrete.

3.6. Definition. *Denote by* $\mathbb{T}$ *the 2-stack in which*

- *the objects are the Picard stacks* $\mathcal{T}^n$ *for* $n \geq 1$,
- *a 1-arrow from* $\mathcal{T}^n$ *to* $\mathcal{T}^k$

$$P : \mathcal{T}^n \to \mathcal{T}^k$$

  is a morphism of stacks defined by a locally constant matrix $P = (p_{ij})_{\substack{i=1,\dots,k \\ j=1,\dots,n}}$, *with* k *rows,* n *columns and entries in the constant sheaf* $\underline{\mathbb{Z}}$, *in the following way:*

  – *the object function:* $P_U = (p_{ij})_{\substack{i=1,\dots,k \\ j=1,\dots,n}}$ *with* $p_{ij} \in \underline{\mathbb{Z}}(U)$ *sends an object* $(Z_1, \dots, Z_n)^t$ *of* $\mathcal{T}^n(U)$ *to the following* k*-uplet of* $\mathcal{T}^k(U)$:

$$P_U \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n p_{1j} Z_j \\ \vdots \\ \sum_{j=1}^n p_{kj} Z_j \end{pmatrix}.$$

  Remark that the products $p_{ij}Z_j$ *are defined by the group law of the Picard category* $\mathcal{T}^1(U)$[4],

  – *the arrow function:* P_U *sends identities of* $\mathcal{T}^n(U)$ *to identities of* $\mathcal{T}^k(U)$.

[4] By Lemma 3.5, if $(\Sigma, a_j, a_{s,t})_{\substack{j=1,\dots,n \\ s,t \in \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}](U)}}$ denotes the set of data associated to the family of objects $\{Z_1, \dots, Z_n\}$ of $\mathcal{T}^1(U)$, Z_j is the image $\Sigma(e_j)$ of e_j via Σ and $p_{ij}Z_j$ is the image $\Sigma(p_{ij}\mathrm{e}_j)$ of $p_{ij}\mathrm{e}_j$ via Σ (recall that $\mathcal{T}^1(U)$ is discrete and so the isomorphisms a_j and $a_{s,t}$ are identities).

The composite of two morphisms of stacks $P : \mathcal{T}^n \to \mathcal{T}^k$ and $Q : \mathcal{T}^k \to \mathcal{T}^m$ is the morphism of stacks $QP : \mathcal{T}^n \to \mathcal{T}^m$ defined by the product QP of the two matrices Q and P.

- *a 2-arrow $u : P \Rightarrow Q$ between two morphisms of stacks $Q, P : \mathcal{T}^n \to \mathcal{T}^k$ is the morphism of cartesian functors which assigns to the object $(Z_1, \dots, Z_n)^t$ of $\mathcal{T}^n(U)$ the identity of $\mathcal{T}^k(U)$ if $P_U(Z_1, \dots, Z_n)^t = Q_U(Z_1, \dots, Z_n)^t$ and nothing otherwise:*

$$u_{(Z_1,\dots,Z_n)^t} = \mathrm{id}_{P_U(Z_1,\dots,Z_n)^t} : P_U \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \longrightarrow Q_U \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}.$$

By construction, we have that

3.7. Lemma. *The 2-stack $\mathbb{T}$ over $\boldsymbol{S}$ is an algebraic 2-stack theory over $\boldsymbol{S}$.*

Let $\mathcal{P}$ be a Picard stack. By the 2-Yoneda Lemma [Stack Project, 2005–2026, 4.41] and by [Deligne, 1973, 1.4.24, page 51] we have the equivalences of categories

$$\mathcal{P}(U) \cong {}^5\mathrm{Hom}_{\mathbb{S}\mathrm{tack}}\big(\mathrm{st}(\mathrm{pt}_{|U}), \mathcal{P}\big) \cong \mathrm{Hom}_{\mathbb{P}\mathrm{icard}}\big(\mathrm{PicSt}\big([0 \to \mathbb{Z}[\mathrm{pt}_{|U}]]\big), \mathcal{P}\big), \tag{3.4}$$

where $\mathrm{st}(\mathrm{pt}_{|U})$ is the stack associated to the sheaf pt (see [Stack Project, 2005–2026, Example 4.38.5 and 95.10]), $\mathrm{PicSt}\big([0 \to \mathbb{Z}[\mathrm{pt}_{|U}]]\big)$ is the strictly commutatine Picard stack associated to the complex $[0 \to \mathbb{Z}[\mathrm{pt}_{|U}]]$ via Proposition 2.1, and $\mathrm{Hom}_{\mathbb{S}\mathrm{tack}}$ (resp. $\mathrm{Hom}_{\mathbb{P}\mathrm{icard}}$) denotes the hom-category whose objects are morphisms of stacks and whose arrows are morphisms of cartesian functors (resp. whose objects are additive functors and whose arrows are morphisms of additive functors).

4. Algebraicity of the 2-stack of Picard stacks

4.1. Theorem. *The 2-stack* $\mathbb{P}$icard *is 2-equivalent to the 2-stack* Alg$\mathbb{T}$.

Proof. We construct a 2-equivalence of 2-stacks

$$\widehat{(\ )} : \mathbb{P}\mathrm{icard} \longrightarrow \mathrm{Alg}\mathbb{T}$$

from the 2-stack of Picard stacks to the 2-stack of 2-algebras for the algebraic 2-stack theory $\mathbb{T}$. We proceed in several steps.

1. Via the 2-Yoneda Lemma (3.4), any Picard stack $\mathcal{P}$ defines a 2-algebra of $\mathbb{T}$

$$\widehat{\mathcal{P}} : \mathbb{T} \longrightarrow \mathbb{S}\mathrm{tack}, \quad \mathcal{T}^1 \longmapsto \widehat{\mathcal{P}}(\mathcal{T}^1) := \mathcal{P}.$$

   Taking the product of stacks, we get that $\widehat{\mathcal{P}}(\mathcal{T}^n) = \mathcal{P}^n$, i.e. $\widehat{\mathcal{P}}$ preserves finite products.

[5] This equivalence is true for any stacks.

2. Any additive functor $F : \mathcal{P}_1 \to \mathcal{P}_2$ defines a morphism of cartesian 2-functors $\widehat{F} : \widehat{\mathcal{P}_1} \Rightarrow \widehat{\mathcal{P}_2}$: for any object $\mathcal{T}^n$ of $\mathbb{T}$, we set

$$\widehat{F}_{\mathcal{T}^n} := F^n : \widehat{\mathcal{P}_1}(\mathcal{T}^n) = \mathcal{P}_1^n \xrightarrow{F^n} \mathcal{P}_2^n = \widehat{\mathcal{P}_2}(\mathcal{T}^n), \tag{4.1}$$

and for any 1-arrow $P : \mathcal{T}^n \to \mathcal{T}^k$ of $\mathbb{T}$ we have the following commutative diagram in $\mathbb{S}$tack which involves stacks and morphisms of stacks

$$\begin{array}{ccc} \mathcal{P}_1^n & \xrightarrow{\widehat{F}_{\mathcal{T}^n}} & \mathcal{P}_2^n \\ {\scriptstyle \widehat{\mathcal{P}_1}(P)}\downarrow & & \downarrow{\scriptstyle \widehat{\mathcal{P}_2}(P)} \\ \mathcal{P}_1^k & \xrightarrow[\widehat{F}_{\mathcal{T}^k}]{} & \mathcal{P}_2^k. \end{array}$$

3. Any morphism $u : F \Rightarrow G$ between two additive functors $F, G : \mathcal{P}_1 \to \mathcal{P}_2$ defines a modification of morphisms of cartesian 2-functors $\widehat{u} : \widehat{F} \Rrightarrow \widehat{G}$: for any object $\mathcal{T}^n$ of $\mathbb{T}$, we set

$$\widehat{u}_{\mathcal{T}^n} := u^n = (\widehat{F}_{\mathcal{T}^n} : \mathcal{P}_1^n \overset{F^n}{\to} \mathcal{P}_2^n) \Longrightarrow (\widehat{G}_{\mathcal{T}^n} : \mathcal{P}_1^n \overset{G^n}{\to} \mathcal{P}_2^n).$$

4. Consider an algebra $A : \mathbb{T} \to \mathbb{S}$tack for the algebraic 2-stack theory $\mathbb{T}$, i.e. a morphism of 2-stacks

$$A : \mathbb{T} \longrightarrow \mathbb{S}\text{tack}$$

preserving finite products. Now we show that A is isomorphic (as morphism of 2-stacks) to $\widehat{\mathcal{P}}$ for some Picard stack $\mathcal{P}$.

Denote by $\mathcal{P}$ the stack $A(\mathcal{T}^1)$. Since A preserves finite products, we have that

$$A(\mathcal{T}^n) = A(\mathcal{T}^1 \times \cdots \times \mathcal{T}^1) = A(\mathcal{T}^1) \times \cdots \times A(\mathcal{T}^1) = \mathcal{P}^n.$$

We define a group law on the stack $\mathcal{P}$ as follows. Consider the morphism of stacks $(1_{|U}, 1_{|U}) : \mathcal{T}^2(U) \to \mathcal{T}^1(U)$ given by the locally constant matrix $(1_{|U}, 1_{|U})$ (here $1_{|U} \in \underline{\mathbb{Z}}(U)$ is the locally constant function 1). The image $A((1_{|U}, 1_{|U}))$ of this 1-arrow via the morphism of 2-stacks A is a morphism of stacks

$$\begin{aligned} A((1_{|U}, 1_{|U})) : \mathcal{P}^2(U) &\longrightarrow \mathcal{P}(U) \\ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &\longmapsto (1_{|U}, 1_{|U}) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = X_1 +_{\mathcal{P}} X_2 \end{aligned} \tag{4.2}$$

Consider the disjoint union $\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt}$ of n copies of the terminal sheaf pt and denote by e_i (for $i = 1, \ldots, n$) the image via the canonical morphism of sheaves $\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt} \to \underline{\mathbb{Z}}[\mathrm{pt} \sqcup \cdots \sqcup \mathrm{pt}] \cong \oplus_n \underline{\mathbb{Z}}[\mathrm{pt}]$ of the one-element set (seen as global constant function) defining the i-th terminal sheaf pt in this disjoint union. Let $\{X_1, \ldots, X_n\}$ be a family of objects of $\mathcal{P}(U)$. Since $A(\mathcal{T}^1) = \mathcal{P}$, there exists a family of objects $\{Z_1, \ldots, Z_n\}$ of $\mathcal{T}^1(U)$ such that $A_U(Z_i) = X_i$

for $i = 1, \dots, n$. Denote by $(\Sigma^Z, a_i^Z, a_{s,t}^Z)_{\substack{i=1,\dots,n \\ s,t\in\oplus_n\underline{\mathbb{Z}}[\mathrm{pt}](U)}}$ the set of data associated to the family of objects $\{Z_1, \dots, Z_n\}$ of $\mathcal{T}^1(U)$ given by Lemma 3.5. Since the category $\mathcal{T}^1(U)$ is discrete, the isomorphisms $a_i^Z, a_{s,t}^Z$ are identities, $Z_i = \Sigma^Z(\mathrm{e}_i)$ and $\Sigma^Z(s) +_{\mathcal{T}^1} \Sigma^Z(t) = \Sigma^Z(s +_{\oplus_n\underline{\mathbb{Z}}[\mathrm{pt}](U)} t)$ for any $s, t \in \oplus_n\underline{\mathbb{Z}}[\mathrm{pt}](U)$. Let $\Sigma^X : \oplus_n\underline{\mathbb{Z}}[\mathrm{pt}](U) \to \mathcal{P}(U)$ be the composite $A_U \circ \Sigma^Z$:

$$\begin{array}{ccccc} & & & & \oplus_n\underline{\mathbb{Z}}[\mathrm{pt}](U) \\ & & & \swarrow^{\Sigma^Z} & \\ \mathcal{T}^2(U) & \xrightarrow{(1_{|U},1_{|U})} & \mathcal{T}^1(U) & & \Sigma^X \\ \downarrow{\scriptstyle A_U\times A_U} & & \downarrow{\scriptstyle A_U} & \swarrow & \\ \mathcal{P}^2(U) & \xrightarrow[A((1_{|U},1_{|U}))]{} & \mathcal{P}(U). & & \end{array} \tag{4.3}$$

For $i = 1, \dots, n$, denote by $a_i^X : \Sigma^{\mathcal{P}}(\mathrm{e}_i) \to X_i$ the isomorphisms in the category $\mathcal{P}(U)$ which are the images via A_U of the identities $a_i^Z : \Sigma^Z(\mathrm{e}_i) \to Z_i$. For any $s, t \in \oplus_n\underline{\mathbb{Z}}[\mathrm{pt}](U)$, define an isomorphism $a_{s,t} : \Sigma^X(s +_{\oplus_n\underline{\mathbb{Z}}[\mathrm{pt}]} t) \to \Sigma^X(s) +_{\mathcal{P}} \Sigma^X(t)$ in the category $\mathcal{P}(U)$ as the composite

$$\begin{aligned}\Sigma^X(s +_{\oplus_n\underline{\mathbb{Z}}[\mathrm{pt}]} t) &= A_U\big(\Sigma^Z(s +_{\oplus_n\underline{\mathbb{Z}}[\mathrm{pt}]} t)\big) \cong A_U\big(\Sigma^Z(s) +_{\mathcal{T}^1} \Sigma^Z(t)\big) \\ &= A_U\big(\Sigma^Z(s)\big) +_{\mathcal{P}} A_U\big(\Sigma^Z(t)\big) = \Sigma^X(s) +_{\mathcal{P}} \Sigma^X(t),\end{aligned}$$

where the isomorphism is given by the identity $a_{s,t}^Z$ and the second equality holds because A preserves finite products, i.e. the square diagram in (4.3) commutes. The image of the 2-arrow of $\mathbb{T}$

$$(1,1) \circ (\mathrm{id}_{\mathcal{T}^1}, (1,1)) \Rightarrow (1,1) \circ ((1,1), \mathrm{id}_{\mathcal{T}^1})$$

via the 2-algebra A is the natural isomorphism

$$\mathsf{a} : +_{\mathcal{P}} \circ (\mathrm{id}_{\mathcal{P}} \times +_{\mathcal{P}}) \Rightarrow +_{\mathcal{P}} \circ (+_{\mathcal{P}} \times \mathrm{id}_{\mathcal{P}}),$$

and by construction the set of data $(\Sigma^X, a_i^X, a_{s,t}^X)_{\substack{i=1,\dots,n \\ s,t\in\oplus_n\mathbb{Z}[\mathrm{pt}](U)}}$ associated to the family of objects $\{X_1, \dots, X_n\}$ of $\mathcal{P}(U)$ is such that the diagram (3.2) applied to the morphism of stacks (4.2), the natural isomorphism a and Σ^X commutes. Hence (4.2) is an associative functor according to Deligne's definition in [Deligne, 1973, 1.4, page 39]. In an analogous way, the image of the 2-arrow of $\mathbb{T}$

$$(1,1) \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow (1,1)$$

via A is the natural isomorphism

$$\mathsf{c} : +_{\mathcal{P}} \circ \mathsf{s} \Rightarrow +_{\mathcal{P}},$$

where $\mathsf{s}(X,Y) = (Y,X)$ for all $X, Y \in \mathcal{P}(U)$, and by construction the set of data $(\Sigma^X, a_i^X, a_{s,t}^X)_{\substack{i=1,\dots,n \\ s,t\in\oplus_n\mathbb{Z}[\mathrm{pt}](U)}}$ is such that the diagram (3.3) applied to the morphism of stacks (4.2), the natural isomorphism c and Σ^X commutes. Hence (4.2) is a strictly commutative functor according to Deligne's definition in [Deligne, 1973, 1.4, page 39]. We can conclude that the morphism of stack (4.2) defines a group law on the stack $\mathcal{P}$ such that $\mathcal{P}(U)$ is a strictly commutative Picard category, i.e. $\mathcal{P}$ is a Picard stack.

Observe that for any $n \geq 1$, $A(\mathcal{T}^n) = A(\mathcal{T}^1)^n = \mathcal{P}^n = \widehat{\mathcal{P}}(\mathcal{T}^n)$, that is the two 2-algebras A and $\widehat{\mathcal{P}}$ take the same values on objects.

For any 1-arrow $P : \mathcal{T}^n \to \mathcal{T}^k$ of $\mathbb{T}$, the two morphisms of stacks

$$\widehat{\mathcal{P}}(P), A(P) : \mathcal{P}^n \longrightarrow \mathcal{P}^k$$

coincide on objects $(X_1, \ldots, X_n)^t$ and on arrows $(f_1, \ldots, f_n) : (X_1, \ldots, X_n)^t \to (Y_1, \ldots, Y_n)^t$ of $\mathcal{P}^n(U)$:

$$A(P)(X_1, \ldots, X_n)^t = P(X_1, \ldots, X_n)^t = \widehat{\mathcal{P}}(P)(X_1, \ldots, X_n)^t,$$

$$A(P)(f_1, \ldots, f_n) = \widehat{\mathcal{P}}(P)(f_1, \ldots, f_n) : P(X_1, \ldots, X_n)^t \to P((f_1, \ldots, f_n)(X_1, \ldots, X_n)^t).$$

Finally for any 2-arrow $u : P \Rightarrow Q$ of $\mathbb{T}$ between two morphisms of stacks $Q, P : \mathcal{T}^n \to \mathcal{T}^k$, the two morphisms of cartesian functors

$$\widehat{\mathcal{P}}(u), A(u) : \widehat{\mathcal{P}}(P) \Longrightarrow \widehat{\mathcal{P}}(Q)$$

also coincide. Hence the 2-algebras A and $\widehat{\mathcal{P}}$ are isomorphic as morphism of 2-stacks.

5. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two Picard stacks, the morphism of stacks

$$\mathrm{Hom}_{\mathbb{P}\mathrm{icard}}(\mathcal{P}_1, \mathcal{P}_2) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}\mathbb{T}}(\widehat{\mathcal{P}_1}, \widehat{\mathcal{P}_2}), \quad F \longmapsto \widehat{F}$$

is an equivalence of stacks, since by (4.1) $F = \widehat{F}_{\mathcal{T}^1}$.

We can conclude that $\widehat{(\ )} : \mathbb{P}\mathrm{icard} \to \mathrm{Alg}\mathbb{T}$ is a 2-equivalence of 2-stacks. ∎

4.2. Corollary. *The 2-stack* $\mathbb{P}\mathrm{icard}$ *of Picard stacks over* $\boldsymbol{S}$ *is algebraic.*

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