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EXTENSION, DEFORMATION AND CATEGORIFICATION OF ASSDER PAIRS

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Abstract. In this paper, we consider associative algebras equipped with derivations. A pair consisting of an associative algebra and a distinguished derivation is called an AssDer pair. We study central extensions and formal one-parameter deformations of AssDer pairs in terms of cohomology. Finally, we define 2-derivations on associative 2-algebras and show that the category of associative 2-algebras with 2-derivations is equivalent to the category of 2-term A_∞-algebras with homotopy derivations.

Contents

1. Introduction

Algebraic structures, such as Lie algebras and associative algebras are important in various areas of mathematics and physics. Algebras are also useful via their derivations. If the algebra A is a polynomial algebra in n variables, some special types of derivations (locally nilpotent, locally finite, etc.) are studied extensively in the literature. One can construct a homotopy Lie algebra out of a graded Lie algebra with a special derivation [26]. In [7], the authors use noncommuting derivations in an associative algebra to construct deformation formulas. Derivations are useful in control theory and gauge theories in QFT [1, 2]. Algebras with derivations are also studied from an operadic point of view [20, 12]. Recently, the authors in [25] considered Lie algebras equipped with derivations (also called LieDer pairs). More precisely, they study central extensions and deformations of LieDer pairs from a cohomological point of view.

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In this paper, we consider a pair of an associative algebra A with a distinguished derivation ϕ_A. Such a pair (A, ϕ_A) of an associative algebra with a derivation is called an AssDer pair. It is also known as a differential algebra in the literature and is itself a significant object of study in areas such as differential Galois theory [21]. Derivations on unital, commutative associative algebras are also related to Lie-Rinehart algebras (cf. Proposition 2.2).

Here we construct a cohomology for an AssDer pair, study their central extensions, and formal one-parameter deformations. We relate the cohomology of an AssDer pair with the cohomology of the corresponding commutator LieDer pair. This cohomology might be a starting point to study cyclic theory for AssDer pairs. Additionally, we define and study 2-derivations on associative 2-algebras and relate them with homotopy derivations of 2-term A_∞-algebras.

In section 2, we study representations and cohomology of AssDer pairs. Let (A, ϕ_A) be an AssDer pair. A representation of it consists of an A-bimodule M together with a linear map ϕ_M which is compatible with the left and right actions of A on M. It turns out that any AssDer pair is a representation of itself. Given a representation (M, ϕ_M), the pair $(M^*, -\phi_M^*)$ is also a representation, where M^* is equipped with the A-bimodule structure dual to M (cf. Proposition 2.8). Given a representation of an AssDer pair, one can construct a semidirect product AssDer pair (cf. Proposition 2.9). Next, we study the cohomology of an AssDer pair with coefficients in a representation. This cohomology is a follow-up to the Hochschild cohomology of the associative structure and a factor modified by the fixed derivation. Like Hochschild cohomology, we show that the cohomology of an AssDer pair with coefficients in itself carries a degree -1 graded Lie bracket (cf. Proposition 2.12). Next, we construct a functor $U : \mathbf{LieDer} \rightarrow \mathbf{AssDer}$ from the category of LieDer pairs to the category of AssDer pairs.

In Sections 3, we study extensions of an AssDer pair by a trivial AssDer pair, called central extensions. We show that isomorphism classes of central extensions are classified by the second cohomology of the AssDer pair with coefficients in the trivial representation (cf. Theorem 3.4). Next, we study extensions of a pair of derivations in a central extension of associative algebras. Given a central extension of associative algebras $0 \rightarrow M \xrightarrow{i} \hat{A} \xrightarrow{p} A \rightarrow 0$ and a pair of derivations $(\phi_A, \phi_M) \in \mathrm{Der}(A) \times \mathrm{Der}(M)$, we associate a second cohomology class in the Hochschild cohomology of A with trivial representation M (cf. Proposition 3.7), called the obstruction class. When this cohomology class is null, the pair of derivations (ϕ_A, ϕ_M) is extensible to a derivation $\phi_{\hat{A}} \in \mathrm{Der}(\hat{A})$ which makes the above sequence into an exact sequence of AssDer pairs (cf. Theorem 3.8).

In Section 4, we study formal one-parameter deformations of AssDer pairs following the classical approach of Gerstenhaber for associative algebras [16] and Nijenhuis-Richardson for Lie algebras [22]. For this, we deform both the associative product as well as the given derivation. Here we will show that the vanishing of the second cohomology of an AssDer pair with coefficients in itself implies that the AssDer pair is rigid (cf. Theorem 4.7, Remark 4.8). Given a finite order deformation of an AssDer pair, we associate a third cohomology class, called the obstruction class of the deformation (cf. Proposition 4.10). If the class is trivial, then the deformation extends to a deformation of the next order (cf. Theorem 4.11). We also consider automorphisms of the deformed AssDer pair and their extensions (cf. Theorem 4.15).

Strongly homotopy associative algebras (A_∞-algebras) were introduced by Stasheff to recognize loop spaces [24]. In Section 5, we consider homotopy derivations on A_∞-algebras whose underlying graded vector space is concentrated in degrees 0 and 1 [20, 12]. We denote the category of 2-term A_∞-algebras with homotopy derivations by $2\mathbf{AssDer}_\infty$. Homotopy derivations on skeletal A_∞-algebras are characterized by third cocycles of AssDer pairs (cf. Proposition 5.4) and 'strict' homotopy derivations on strict A_∞-algebras are characterized by crossed modules of AssDer pairs (cf. Proposition 5.8).

In [5], Baez and Crans introduced Lie 2-algebras as the categorification of Lie algebras. They also showed that the category of 2-term L_∞-algebras and the category of Lie 2-algebras are equivalent. This result has been extended to various other algebraic structures, including groups, Leibniz algebras and twisted associative algebras [4, 3, 23, 10]. In section 6, we introduce the categorification of AssDer pairs. More precisely, we study AssDer pair structures on a 2-vector space. We call such an object an AssDer 2-pair. The category of AssDer 2-pairs and morphisms between them is denoted by **AssDer2**. Finally, we show that the categories $2\mathbf{AssDer}_\infty$ and **AssDer2** are equivalent (cf. Theorem 6.5).

In the whole paper, we assume that $\mathbb{K}$ is a fixed field of characteristic zero, all the vector spaces are over the field $\mathbb{K}$ and maps are $\mathbb{K}$-linear maps unless otherwise stated. Let XAlg be any given type of algebraic structure. Then by $\mathbf{2XAlg}_\infty$, we denote the category of 2-term XAlg_∞-algebras and by **XAlg2**, we denote the category of XAlg 2-algebras.

2. AssDer pairs

Let A be an associative algebra. An A-bimodule (also called a representation of A) is a vector space M together with two linear maps $l : A \otimes M \to M, (a, m) \mapsto am$ and $r : M \otimes A \to M, (m, a) \mapsto ma$ satisfying

$$(ab)m = a(bm), \quad (am)b = a(mb) \quad \text{and} \quad (ma)b = m(ab),$$

for all $a, b \in A$ and $m \in M$. It follows that A is a bimodule over the associative algebra A itself with the left and right actions given by the algebra multiplication. We call this the adjoint bimodule (representation).

A derivation on A with values in the A-bimodule M is given by a linear map $\phi : A \to M$ that satisfies

$$\phi(ab) = \phi(a)b + a\phi(b), \qquad \text{for } a, b \in A.$$

Our main object in this paper is a pair (A, ϕ_A) in which A is an associative algebra and $\phi_A : A \to A$ is a derivation on A with values in the adjoint representation. Thus, ϕ_A

satisfies

$$\phi_A(ab) = \phi_A(a)b + a\phi_A(b), \quad \text{for } a, b \in A.$$

Such a pair (A, ϕ_A) is called an AssDer pair. Here we give a few examples of AssDer pairs.

2.1. Example. *(i) The notion of derivation is a generalization of the usual derivative of functions. For instance, if $A = \mathbb{K}[x_1, \ldots, x_n]$ is the polynomial algebra in n variables, then for $1 \leq i \leq n$, the partial derivatives $\phi_i = \frac{\partial}{\partial x_i}$ are derivations on A. In fact, the space of derivations on A is linearly spanned by $\{\phi_i\}_{1 \leq i \leq n}$.*

(ii) Any derivation on the space $C^\infty(M)$ of smooth functions on a manifold M is given by a vector field. Therefore, $(C^\infty(M), X)$ is an AssDer pair, for any vector field X on M.

(iii) A (non-commutative) Poisson algebra is an associative algebra P together with a Lie bracket $\{~,~\}$ on it which is a derivation on each entry for the associative product. It follows that if $(P, \{~,~\})$ is a (noncommutative) Poisson algebra, then for any $a \in P$, the linear map $\phi_a = \{a,~\}$ is a derivation for the associative product. Hence (P, ϕ_a) is an AssDer pair.

(iv) Let V be a vector space and $d : V \to V$ be a linear map. Consider the reduced tensor algebra $\overline{T}(V) = \oplus_{n \geq 1} V^{\otimes n}$ with the concatenation product. The linear map d induces a linear map $\overline{d} : \overline{T}(V) \to \overline{T}(V)$ by

$$\overline{d}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes dv_i \otimes \cdots \otimes v_n.$$

It is easy to verify that $\overline{d}$ is a derivation on $\overline{T}(V)$. Hence $(\overline{T}(V), \overline{d})$ is an AssDer pair. Any derivation on $\overline{T}(V)$ arises in this way.

A *Lie-Rinehart algebra* [18] is a triple (A, L, ρ) where A is a commutative associative algebra, $(L, [~,~]_L)$ is a Lie algebra with L a left A-module and an A-module map $\rho : L \to \mathrm{Der}(A)$ which is a morphism of Lie algebras satisfying the following Leibniz rule

$$[X, aY]_L = a[X, Y]_L + \rho(X)(a)Y, \text{ for } X, Y \in L, a \in A.$$

Let A be a commutative associative algebra and ϕ_A be a derivation on A. Then ϕ_A induces a Lie bracket on A given by

$$[a, b]_{\phi_A} := a\phi_A(b) - \phi_A(a)b, \quad \text{for } a, b \in A. \tag{1}$$

We denote this Lie algebra by A_{ϕ_A}. This Lie bracket additionally satisfies the Leibniz rule

$$[a, fb]_{\phi_A} = f[a, b]_{\phi_A} + \rho(a)(f)b, \quad \text{for } a, b, f \in A, \tag{2}$$

where $\rho : (A, [~,~]_{\phi_A}) \to (\mathrm{Der}(A), [~,~])$ is the A-linear Lie algebra morphism given by $\rho(a) = a\phi_A$. In other words, the triple (A, A_{ϕ_A}, ρ) is a Lie-Rinehart algebra. The next proposition says that any Lie-Rinehart algebra structure on (A, A) arises by a derivation in this way when A is unital. Thus, to better understand Lie-Rinehart algebra structures on (A, A), one needs to know derivations on A.

2.2. Proposition. *Let A be an unital, commutative associative algebra. There is a one-to-one correspondence between derivations on A and Lie-Rinehart algebra structures on (A, A).*

Proof. Given a derivation ϕ_A, we have constructed a Lie-Rinehart algebra (A, A_{ϕ_A}, ρ). Conversely, let $(A, A_{\text{Lie}}, \rho)$ be a Lie-Rinehart algebra. Define $\phi_A : A \to A$ by $\phi_A = \rho(1)$, where 1 is the unit of A. By definition, ϕ_A is a derivation on A. Moreover, we have $\rho(a) = a\rho(1) = a\phi_A$. Hence, for $a, b \in A$,

$$[a, b]_{\text{Lie}} = b[a, 1]_{\text{Lie}} + \rho(a)(b) = -\rho_A(a) + a\phi_A(b) = [a, b]_{\phi_A}.$$

Therefore, the Lie-Rinehart algebra $(A, A_{\text{Lie}}, \rho)$ is obtained from the derivation ϕ_A. Finally, the above two correspondences are inverses of each other. ∎

2.3. Remark. In this remark, we will show that the Witt algebra is obtained from a derivation in Laurent polynomial algebra. Let $A = \mathbb{K}[x, x^{-1}]$ be the Laurent polynomial algebra. Consider the derivation $\phi_A : A \to A$ given by $\phi_A(x^n) = -nx^{n-1}$, for $n \in \mathbb{Z}$. It follows from (1) that $A = \mathbb{K}[x, x^{-1}]$ carries a Lie algebra structure given by

$$[x^m, x^n] = -x^m(nx^{n-1}) + (mx^{m-1})x^n = (m-n)x^{m+n-1}, \text{ for } x^m, x^n \in A, m, n \in \mathbb{Z}. \quad (3)$$

Consider the basis $\{l_n\}_{n\in\mathbb{Z}}$ for A, where $l_n = x^{n+1}$, $n \in \mathbb{Z}$. Then the Lie bracket (3) reads as $[l_m, l_n] = (m-n)\ l_{m+n}$, for $m, n \in \mathbb{Z}$. This is precisely the Witt algebra structure on $A =$ span $\{l_n\}_{n\in\mathbb{Z}}$.

2.4. Definition. *Let (A, ϕ_A) and (B, ϕ_B) be two AssDer pairs. A morphism between them consists of an algebra map $f : A \to B$ that commutes with derivations, i.e. $f \circ \phi_A = \phi_B \circ f$.*

We denote the category of AssDer pairs together with morphisms between them by **AssDer**.

Let (V, d) be a vector space together with a linear map. The free AssDer pair over (V, d) is an AssDer pair $(\mathcal{F}(V), \phi_{\mathcal{F}(V)})$ equipped with a linear map $i : V \to \mathcal{F}(V)$ that satisfies $\phi_{\mathcal{F}(V)} \circ i = i \circ d$ and the following universal condition holds: for any AssDer pair (A, ϕ_A) and a linear map $f : V \to A$ satisfying $\phi_A \circ f = f \circ d$, there exists a unique AssDer pair morphism $\widetilde{f} : (\mathcal{F}(V), \phi_{\mathcal{F}(V)}) \to (A, \phi_A)$ such that $\widetilde{f} \circ i = f$.

It follows that the free AssDer pair over (V, d) is well-defined up to a unique isomorphism.

2.5. Proposition. *Let (V, d) be a vector space with a linear map. Then $(T(V), \overline{d})$ (resp.$(\overline{T}(V), \overline{d})$) equipped with the inclusion map i is free unital (resp. nonunital) AssDer pair over (V, d).*

2.6. Representations and cohomology of AssDer pairs.

2.7. Definition. *Let (A, ϕ_A) be an AssDer pair. A left module over it consists of a pair (M, ϕ_M) in which M is a left A-module and $\phi_M : M \to M$ is a linear map satisfying*

$$\phi_M(am) = \phi_A(a)m + a\phi_M(m), \quad \textit{for } a \in A, m \in M. \tag{4}$$

Similarly, a right module over (A, ϕ_A) is a pair (M, ϕ_M) in which M is a right A-module and $\phi_M : M \to M$ is a linear map satisfying

$$\phi_M(ma) = \phi_M(m)a + m\phi_A(a), \quad \text{for } a \in A, m \in M. \tag{5}$$

A bimodule (representation) over (A, ϕ_A) is a pair (M, ϕ_M) which is both a left module and a right module over (A, ϕ_A) and M is an A-bimodule, i.e. $(am)b = a(mb)$, for all $a, b \in A$ and $m \in M$. It follows that the AssDer pair (A, ϕ_A) is a representation of itself for the adjoint bimodule structure on A.

2.8. Proposition. *Let (M, ϕ_M) be a representation of the AssDer pair (A, ϕ_A). Then $(M^*, -\phi_M^*)$ is also a representation of (A, ϕ_A) where the A-bimodule structure on M^* is given by*

$$\begin{array}{ll} A \otimes M^* \to M^* & M^* \otimes A \to M^* \\ (af)(m) = f(ma) & (fa)(m) = f(am), \end{array}$$

for $a \in A, f \in M^*$ and $m \in M$.

Proof. The fact that M^* is an A-bimodule is standard [19]. To verify that $(M^*, -\phi_M^*)$ is a representation of the AssDer pair, we observe that

$$\begin{aligned} \langle -\phi_M^*(af), m\rangle = \langle af, -\phi_M(m)\rangle = f(-\phi_M(m)a) &= f(m\phi_A(a)) - f(\phi_M(ma)) \quad (\text{by } (5)) \\ &= \langle \phi_A(a)f, m\rangle - \langle a\phi_M^*(f), m\rangle. \end{aligned}$$

This shows that $-\phi_M^*(af) = \phi_A(a)f + a(-\phi_M^*(f))$. Similarly, we have

$$\begin{aligned} \langle -\phi_M^*(fa), m\rangle = \langle fa, -\phi_M(m)\rangle = f(-a\phi_M(m)) &= f(\phi_A(a)m) - f(\phi_M(am)) \quad (\text{by } (4)) \\ &= \langle f\phi_A(a), m\rangle - \langle \phi_M^*(f)a, m\rangle. \end{aligned}$$

This shows that $-\phi_M^*(fa) = -\phi_M^*(f)a + f\phi_A(a)$. ■

It follows from the above Proposition, $(A^*, -\phi_A^*)$ is a representation of the AssDer pair (A, ϕ_A). This is called the coadjoint representation of the AssDer pair (A, ϕ_A).

Given an associative algebra and a bimodule over it, one can construct a semidirect product associative algebra [19]. The following result generalizes it to AssDer pairs. The proof is straightforward.

2.9. Proposition. *Let (A,ϕ_A) be an AssDer pair and (M,ϕ_M) be a representation of it. Then $(A\oplus M,\phi_A\oplus\phi_M)$ is an AssDer pair where the associative structure on $A\oplus M$ is given by the semi-direct product*

$$(a\oplus m)\cdot(b\oplus n)=(ab\oplus an+mb).$$

Let A be an associative algebra and M be an A-bimodule. Then the Hochschild cohomology of A with coefficients in M is given as follows [17]. The n-th cochain group $C^n(A,M)$ is given by $C^n(A,M):=\mathrm{Hom}(A^{\otimes n},M)$ for $n\geq 0$ and the coboundary map $\delta_{\mathrm{Hoch}}:C^n(A,M)\to C^{n+1}(A,M)$ is given by

$$\begin{aligned}\delta_{\mathrm{Hoch}}(f)(a_1,\ldots,a_{n+1})=a_1f(a_2,\ldots,a_{n+1})+\sum_{i=1}^{n}(-1)^i\ f(a_1,\ldots,a_{i-1},a_ia_{i+1},\ldots,a_{n+1})\\ +(-1)^{n+1}f(a_1,\ldots,a_n)a_{n+1}.\end{aligned}\tag{6}$$

The corresponding cohomology groups are called the Hochschild cohomology groups of A with coefficients in M. It has been observed by Gerstenhaber [15] that the graded vector space of Hochschild cochains $C^\star(A,A)=\oplus_nC^n(A,A)$ carries a degree -1 graded Lie bracket (called the Gerstenhaber bracket) given by

$$[f,g]:=f\circ g-(-1)^{(m-1)(n-1)}g\circ f,\quad\text{for } f\in C^m(A,A),\ g\in C^n(A,A),\tag{7}$$

where $(f\circ g)(a_1,\ldots,a_{m+n-1})$ is defined as

$$\sum_{i=1}^{m}(-1)^{(i-1)(n-1)}f(a_1,\ldots,a_{i-1},g(a_i,\ldots,a_{i+n-1}),\ldots,a_{m+n-1}).$$

Let $\mu:A^{\otimes 2}\to A$ denote the associative multiplication on A. With the above notations, the Hochschild coboundary map δ_{Hoch} with coefficients in itself is given by $\delta_{\mathrm{Hoch}}f=(-1)^{n-1}[\mu,f],\qquad\text{for } f\in C^n(A,A)$.

Let (A,ϕ_A) be an AssDer pair and (M,ϕ_M) be a representation of it. We aim to define the cohomology of the AssDer pair (A,ϕ_A) with coefficients in (M,ϕ_M). Define the space $C^0_{\mathrm{AssDer}}(A,M)$ of 0-cochains to be 0 and the space $C^1_{\mathrm{AssDer}}(A,M)$ of 1-cochains to be $\mathrm{Hom}(A,M)$. For $n\geq 2$, the space $C^n_{\mathrm{AssDer}}(A,M)$ of n-cochains to be defined by

$$C^n_{\mathrm{AssDer}}(A,M)=C^n(A,M)\oplus C^{n-1}(A,M).$$

Before we introduce the coboundary operator, we define a new map $\delta:C^n(A,M)\to C^n(A,M)$ by

$$\delta f=\sum_{i=1}^{n}f\circ(\mathrm{id}\otimes\cdots\otimes\phi_A\otimes\cdots\otimes\mathrm{id})-\phi_M\circ f.$$

When $(M, \phi_M) = (A, \phi_A)$, the map δ can be seen in terms of Gerstenhaber bracket as $\delta f = -[\phi_A, f]$.

Finally, we define the coboundary map $\partial : C^n_{\text{AssDer}}(A, M) \to C^{n+1}_{\text{AssDer}}(A, M)$ by

$$\begin{cases} \partial f = (\delta_{\text{Hoch}} f, -\delta f), & \text{for } f \in C^1_{\text{AssDer}}(A, M) = \text{Hom}(A, M), \\ \partial (f, g) = (\delta_{\text{Hoch}} f,\ \delta_{\text{Hoch}} g + (-1)^n \delta f), & \text{for } (f, g) \in C^n_{\text{AssDer}}(A, M). \end{cases} \tag{8}$$

To prove that $\partial^2 = 0$ we use the following lemma whose proof will be given after Proposition 2.12.

2.10. Lemma. $\delta_{\text{Hoch}} \circ \delta = \delta \circ \delta_{\text{Hoch}}$.

2.11. Proposition. *The map ∂ is a coboundary map, i.e. $\partial^2 = 0$.*

Proof. For $f \in C^1_{\text{AssDer}}(A, M)$, we have

$$\partial^2 f = \partial(\delta_{\text{Hoch}} f, -\delta f) = (\delta^2_{\text{Hoch}} f,\ -\delta_{\text{Hoch}} \delta f + \delta \delta_{\text{Hoch}} f) = 0.$$

For $(f, g) \in C^n_{\text{AssDer}}(A, M)$, we have

$$\partial^2 (f, g) = (\delta^2_{\text{Hoch}} f,\ \delta^2_{\text{Hoch}} g + (-1)^n \delta_{\text{Hoch}} \delta f + (-1)^{n+1} \delta \delta_{\text{Hoch}} f) = 0.$$

■

We denote the corresponding cohomology groups by $H^n_{\text{AssDer}}(A, M)$, for $n \geq 1$. Next, we will show that the cohomology of an AssDer pair with coefficients in itself carries a degree -1 graded Lie bracket.

2.12. Proposition. *The bracket $[\![\ ,\ ]\!] : C^m_{\text{AssDer}}(A, A) \times C^n_{\text{AssDer}}(A, A) \to C^{m+n-1}_{\text{AssDer}}(A, A)$ given by*

$$[\![(f, g), (f', g')]\!] := ([f, f'], (-1)^{m+1} [f, g'] + [g.f'])$$

is a degree -1 graded Lie bracket on $\bigoplus_n C^n_{\text{AssDer}}(A, A)$.

Proof. First note that, since $[\ ,\ ]$ is a degree -1 graded Lie bracket (Gerstenhaber bracket), we have

$$[f, [f', f'']] = [[f, f'], f''] + (-1)^{(m-1)(n-1)} [f', [f, f'']], \tag{9}$$

for $f \in \text{Hom}(A^{\otimes m}, A)$, $f' \in \text{Hom}(A^{\otimes n}, A)$ and $f'' \in \text{Hom}(A^{\otimes p}, A)$. Next for any $(f, g) \in C^m_{\text{AssDer}}(A, A)$, $(f', g') \in C^n_{\text{AssDer}}(A, A)$ and $(f'', g'') \in C^p_{\text{AssDer}}(A, A)$, we have

$$\begin{aligned} &[\![(f, g), [\![(f', g'), (f'', g'')]\!]]\!] \\ &= [\![(f, g), ([f', f''], (-1)^{n+1} [f', g''] + [g', f''])]\!] \\ &= \big([f, [f', f'']], (-1)^{m+n+2} [f, [f', g'']] + (-1)^{m+1} [f, [g', f'']] + [g, [f', f'']]\big). \end{aligned}$$

On the other hand,

$$\begin{aligned}
&[\![[\![(f,g),(f',g')]\!],(f'',g'')]\!] + (-1)^{(m-1)(n-1)}[\![(f',g'),[\![(f,g),(f'',g'')]\!]]\!] \\
&= [\![([f,f'],\ (-1)^{m+1}[f,g']+[g,f']),\ (f'',g'')]\!] \\
&+ (-1)^{(m-1)(n-1)}[\![(f',g'),\ ([f,f''],(-1)^{m+1}[f,g'']+[g,f''])]\!] \\
&= \big([[f,f'],f''],\ (-1)^{m+n}[[f,f'],g'']+(-1)^{m+1}[[f,g'],f'']+[[g,f'],f'']\big) \\
&+ (-1)^{(m-1)(n-1)}\big([f',[f,f'']],(-1)^{n+1+m+1}[f',[f,g'']]+(-1)^{n+1}[f',[g,f'']]+[g',[f,f'']]\big).
\end{aligned}$$

Hence, the result follows from (9). ■

It follows from the above proposition that the shifted graded space $\bigoplus_n C^{n+1}_{\mathrm{AssDer}}(A,A)$ carries a graded Lie bracket. This result is true for an arbitrary vector space A (not necessarily an associative algebra) and $C^n_{\mathrm{AssDer}}(A,A) = \mathrm{Hom}(A^{\otimes n},A) \oplus \mathrm{Hom}(A^{\otimes n-1},A)$, for all n. Let A be a vector space, $\mu : A^{\otimes 2} \to A$ and $\phi_A : A \to A$ be two linear maps. Consider the pair $(\mu,\phi_A) \in C^2_{\mathrm{AssDer}}(A,A)$. Then μ defines an associative product on A and ϕ_A is a derivation for the associative product if and only if $(\mu,\phi_A) \in C^2_{\mathrm{AssDer}}(A,A)$ is a Maurer-Cartan element in the graded Lie algebra $(\bigoplus_n C^{n+1}_{\mathrm{AssDer}}(A,A), [\![\ ,\ ]\!])$, i.e. $[\![(\mu,\phi_A),(\mu,\phi_A)]\!] = 0$. With these notations, the differential (8) of the AssDer pair (A,ϕ_A) with coefficients in itself is given by

$$\partial(f,g) = (-1)^{n-1}[\![(\mu,\phi_A),(f,g)]\!], \text{ for } (f,g) \in C^n_{\mathrm{AssDer}}(A,A).$$

As a consequence, we get that the graded space of cohomology $\bigoplus_n H^{n+1}_{\mathrm{AssDer}}(A,A)$ of the AssDer pair (A,ϕ_A) with coefficients in itself carries a graded Lie bracket.

Proof. (*of Lemma* 2.10) We first prove this result when the coefficient is the AssDer pair itself. Then, using the semidirect product AssDer pair, we conclude the same for any coefficients.

When $(M,\phi_M) = (A,\phi_A)$, we have $\delta_{\mathrm{Hoch}}(f) = (-1)^{n-1}[\mu,f]$ and $\delta f = -[\phi_A,f]$, for any $f \in C^n(A,A)$. Hence

$$\begin{aligned}
\delta_{\mathrm{Hoch}} \circ \delta(f) &= -\delta_{\mathrm{Hoch}}[\phi_A,f] = (-1)^n[\mu,[\phi_A,f]] \\
&= (-1)^n\,[[\mu,\phi_A],f] + (-1)^n[\phi_A[\mu,f]] = \delta \circ \delta_{\mathrm{Hoch}}(f).
\end{aligned}$$

For any coefficient (M,ϕ_M), we consider the semidirect product $(A \oplus M, \phi_A \oplus \phi_M)$ given in Proposition 2.9. We use the same notation δ_{Hoch} to denote the Hochschild cohomology of A with coefficients in M, as well as the Hochschild cohomology of the semi-direct product algebra $A \oplus M$. Similarly, we use the same notation for the operator δ. Note that, for any $f \in C^n(A,M)$ can be extended to a map $\widetilde{f} \in C^n(A \oplus M, A \oplus M)$ by

$$\widetilde{f}((a_1,m_1),\ldots,(a_n,m_n)) = (0, f(a_1,\ldots,a_n)).$$

The map f can be obtained from $\widetilde{f}$ just by restricting it to $A^{\otimes n}$. Moreover, $\widetilde{f} = 0$ implies that $f = 0$. Observe that $\widetilde{\delta_{\mathrm{Hoch}}(f)} = \delta_{\mathrm{Hoch}}(\widetilde{f})$ and $\widetilde{\delta f} = \delta\widetilde{f}$. Hence we have

$$\widetilde{\delta_{\mathrm{Hoch}} \circ \delta(f)} = \delta_{\mathrm{Hoch}}(\delta\widetilde{f}) = \delta_{\mathrm{Hoch}} \circ \delta(\widetilde{f}) = \delta \circ \delta_{\mathrm{Hoch}}(\widetilde{f}) = \delta \circ \widetilde{\delta_{\mathrm{Hoch}}(f)}.$$

This implies that $\delta_{\mathrm{Hoch}} \circ \delta = \delta \circ \delta_{\mathrm{Hoch}}$. ■

2.13. Relation with LieDer pairs. Let (A, ϕ_A) be an AssDer pair. Consider the Lie algebra structure on A with the commutator bracket $[a, b]_c = ab - ba$. We denote this Lie algebra by A_c. Then it is easy to see that ϕ_A is a derivation for the Lie algebra A_c. Thus we get a functor $(\ )_c : \mathbf{AssDer} \rightarrow \mathbf{LieDer}$. In the following, we construct a functor left adjoint to $(\ )_c$ using the universal enveloping algebra of a Lie algebra.

Let $(\mathfrak{g}, \phi_\mathfrak{g})$ be a LieDer pair. Consider the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$. The universal enveloping algebra $U(\mathfrak{g})$ is an associative algebra which is the quotient of $T(\mathfrak{g})$ by the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$, for $x, y \in \mathfrak{g}$. Note that the linear map $\overline{\phi_\mathfrak{g}} : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ (see Example 2.1 (iv)) induces a map $\phi_{U(\mathfrak{g})} : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ as

$$\begin{aligned}
&\overline{\phi_\mathfrak{g}}(x \otimes y - y \otimes x - [x, y]) \\
&\quad = \overline{\phi_\mathfrak{g}}(x) \otimes y - y \otimes \overline{\phi_\mathfrak{g}}(x) - [\overline{\phi_\mathfrak{g}}(x), y] + x \otimes \overline{\phi_\mathfrak{g}}(y) - \overline{\phi_\mathfrak{g}}(y) \otimes x - [x, \overline{\phi_\mathfrak{g}}(y)].
\end{aligned}$$

Moreover, $\phi_{U(\mathfrak{g})}$ is a derivation on $U(\mathfrak{g})$. Thus, a LieDer pair $(\mathfrak{g}, \phi_\mathfrak{g})$ induces an AssDer pair $(U(\mathfrak{g}), \phi_{U(\mathfrak{g})})$ on the universal enveloping algebra $U(\mathfrak{g})$.

2.14. Proposition. *The functor* $U : \mathbf{LieDer} \rightarrow \mathbf{AssDer}$ *is left adjoint to the functor* $(\ )_c : \mathbf{AssDer} \rightarrow \mathbf{LieDer}$. *In other words, there is an isomorphism*

$$\mathrm{Hom}_{\mathbf{AssDer}}(U(\mathfrak{g}), A) \cong \mathrm{Hom}_{\mathbf{LieDer}}(\mathfrak{g}, A_c),$$

for any AssDer pair (A, ϕ_A) *and any LieDer pair* $(\mathfrak{g}, \phi_\mathfrak{g})$.

Proof. For any AssDer pair morphism $f : U(\mathfrak{g}) \rightarrow A$, we consider its restriction to $\mathfrak{g}$, which is a Lie algebra morphism $\mathfrak{g} \rightarrow A_c$ and commutes with derivations. Hence, it is a morphism of LieDer pairs. Conversely, for any LieDer pair morphism $h : \mathfrak{g} \rightarrow A_c$, we consider the unique extension of h as an associative algebra morphism $\widetilde{h} : T(\mathfrak{g}) \rightarrow A$. This is indeed a morphism of AssDer pairs. It induces a map of AssDer pairs $\widetilde{\widetilde{h}} : U(\mathfrak{g}) \rightarrow A$ as h is a LieDer pair morphism. Finally, the above two correspondences are inverses to each other. ■

Let $(\mathfrak{g}, \phi_\mathfrak{g})$ be a LieDer pair. A module over it [25] consists of a $\mathfrak{g}$-module M together with a linear map $\phi_M : M \rightarrow M$ satisfying

$$\phi_M[x, m] = [\phi_\mathfrak{g}(x), m] + [x, \phi_M(m)], \quad \text{for all } x \in \mathfrak{g}, m \in M.$$

2.15. Proposition. *Let* $(\mathfrak{g}, \phi_\mathfrak{g})$ *be a LieDer pair. A* $(\mathfrak{g}, \phi_\mathfrak{g})$*-module is equivalent to a left* $(U(\mathfrak{g}), \phi_{U(\mathfrak{g})})$*-module.*

Proof. It is known that any left $U(\mathfrak{g})$-module is equivalent to a $\mathfrak{g}$-module [19]. More precisely, let M be a left $U(\mathfrak{g})$-module, then the $\mathfrak{g}$-module structure on M is given by $[x, m] = xm$, for $x \in \mathfrak{g}$ and $m \in M$.

Next, take (M, ϕ_M) be a left module over the AssDer pair $(U(\mathfrak{g}), \phi_{U(\mathfrak{g})})$. Then the condition $\phi_M(xm) = \phi_{U(\mathfrak{g})}(x)m + x\phi_M(m)$ is equivalent to $\phi_M[x, m] = [\phi_\mathfrak{g}(x), m] + [x, \phi_M(m)]$, for all $x \in \mathfrak{g}$ and $m \in M$. ■

Let (M, ϕ_M) be a representation of the AssDer pair (A, ϕ_A). Then M can be considered as an A_c-module via $[~,~] : A_c \times M \to M,~ [a,m] = am - ma$. Further (M, ϕ_M) is a representation of the LieDer pair (A_c, ϕ_A). Before we relate the cohomology of an AssDer pair with that of the corresponding commutator LieDer pair, we recall the following standard result [19].

2.16. Proposition. *The collection of maps* $T_n : \mathrm{Hom}(A^{\otimes n}, M) \to \mathrm{Hom}(\wedge^n A_c, M),~ n \geq 0$, *defined by*

$$T_n(f)(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^\sigma f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

is a morphism from the Hochschild complex of A with coefficients in the A-bimodule M to the Chevalley-Eilenberg cohomology of the commutator Lie algebra A_c with coefficients in the module M.

In [25], the authors introduced a cohomology for a LieDer pair with coefficients in a representation. Let $(\mathfrak{g}, \phi_\mathfrak{g})$ be a LieDer pair and (M, ϕ_M) be a representation of it. We denote by $\delta_{\mathrm{CE}} : \mathrm{Hom}(\wedge^n \mathfrak{g}, M) \to \mathrm{Hom}(\wedge^{n+1}\mathfrak{g}, M)$ the coboundary operator for the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in M. Define the 0-th cochain group of the LieDer pair $(\mathfrak{g}, \phi_\mathfrak{g})$ with coefficients in (M, ϕ_M) to be 0, and the higher cochain groups are defined by $C^1_{\mathrm{LieDer}}(\mathfrak{g}, M) = \mathrm{Hom}(\mathfrak{g}, M)$ and $C^n_{\mathrm{LieDer}}(\mathfrak{g}, M) = \mathrm{Hom}(\wedge^n \mathfrak{g}, M) \oplus \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, M)$, for $n \geq 2$. The coboundary map $\partial : C^n_{\mathrm{LieDer}}(\mathfrak{g}, M) \to C^{n+1}_{\mathrm{LieDer}}(\mathfrak{g}, M)$ is given by

$$\partial f = (\delta_{\mathrm{CE}} f, -\delta f) \text{ and } \partial(f,g) = (\delta_{\mathrm{CE}} f,~ \delta_{\mathrm{CE}} g + (-1)^n \delta f),$$

where $\delta : \mathrm{Hom}(\wedge^n \mathfrak{g}, M) \to \mathrm{Hom}(\wedge^n \mathfrak{g}, M)$ is the map

$$\delta f = \sum_{i=1}^{n} f \circ (\mathrm{id} \otimes \cdots \otimes \phi_\mathfrak{g} \otimes \cdots \otimes \mathrm{id}) - \phi_M \circ f$$

for $f \in C^1_{\mathrm{LieDer}}(\mathrm{g}, M)$ and $(f,g) \in C^n_{\mathrm{LieDer}}(\mathfrak{g}, M)$. When one consider the cohomology of the LieDer pair $(\mathfrak{g}, \phi_\mathfrak{g})$ with coefficients in itself, the cochain groups $\bigoplus_n C^n_{\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g})$ carries a degree -1 graded Lie bracket $[\![~,~]\!]$ given by

$$[\![(f,g),(f',g')]\!] = ([f,f'], (-1)^{m+1}[f,g'] + [g,f']),$$

for $(f,g) \in C^m_{\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g}),~ (f',g') \in C^n_{\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g})$, where $[~,~]$ is the Nijenhuis-Richardson bracket on the space of skew-symmetric multilinear maps on $\mathfrak{g}$ given by

$$\begin{aligned}
[f,f']~(x_1, \ldots, x_{m+n-1}) = &\sum_{\sigma \in \mathrm{Sh}(n,m-1)} (-1)^\sigma f(f'(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), x_{\sigma(n+1)}, \ldots, x_{\sigma(m+n-1)}) \\
&- (-1)^{(m-1)(n-1)} \sum_{\sigma \in \mathrm{Sh}(m,n-1)} (-1)^\sigma f'(f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n-1)}).
\end{aligned}$$

If the Lie bracket on $\mathfrak{g}$ is given by a map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, then $(\omega, \phi_\mathfrak{g}) \in C^2_{\mathrm{LieDer}}(\mathfrak{g}, \mathfrak{g})$ satisfies $[\![(\omega, \phi_\mathfrak{g}), (\omega, \phi_\mathfrak{g})]\!] = 0$ and the differential if given by

$$\partial(f,g) = (-1)^{n-1}[\![(\omega, \phi_\mathfrak{g}), (f,g)]\!].$$

2.17. Proposition. *Let (A, ϕ_A) be an AssDer pair and (M, ϕ_M) be a representation over it. Then the maps*

$$(T_n, T_{n-1}) : \mathrm{Hom}(A^{\otimes n}, M) \oplus \mathrm{Hom}(A^{\otimes n-1}, M) \to \mathrm{Hom}(\wedge^n A_c, M) \oplus \mathrm{Hom}(\wedge^{n-1} A_c, M)$$

defines a morphism from the cohomology of the AssDer pair (A, ϕ_A) to the cohomology of the corresponding LieDer pair (A_c, ϕ_A).

3. Central extensions of AssDer pairs

Let (A, ϕ_A) be an AssDer pair and (M, ϕ_M) a trivial AssDer pair. That is, the associative structure on M is trivial.

3.1. Definition. *A central extension of (A, ϕ_A) by (M, ϕ_M) consists of an exact sequence of AssDer pairs*

$$0 \to (M, \phi_M) \xrightarrow{i} (\hat{A}, \phi_{\hat{A}}) \xrightarrow{p} (A, \phi_A) \to 0 \qquad (10)$$

satisfying $i(m) \cdot \hat{a} = 0 = \hat{a} \cdot i(m)$, for all $m \in M$ and $\hat{a} \in \hat{A}$.

We identify M with the corresponding subalgebra of $\hat{A}$ and with this identification $\phi_M = \phi_{\hat{A}}|_M$.

3.2. Definition. *Let $(\hat{A}_1, \phi_{\hat{A}_1})$ and $(\hat{A}_2, \phi_{\hat{A}_2})$ be two central extensions of the AssDer pair (A, ϕ_A) by (M, ϕ_M). These two central extensions are said to be isomorphic if there exists an AssDer pair isomorphism $\eta : (\hat{A}_1, \phi_{\hat{A}_1}) \to (\hat{A}_2, \phi_{\hat{A}_2})$ such that the following diagram commute*

$$\begin{array}{ccccccc} & & & (\hat{A}_1, \phi_{\hat{A}_1}) & & & \\ & & \overset{i_1}{\nearrow} & & \overset{p_1}{\searrow} & & \\ 0 \longrightarrow & (M, \phi_M) & & \downarrow \eta & & (A, \phi_A) & \longrightarrow 0. \\ & & \underset{i_2}{\searrow} & & \underset{p_2}{\nearrow} & & \\ & & & (\hat{A}_2, \phi_{\hat{A}_2}) & & & \end{array} \qquad (11)$$

Let $0 \to (M, \phi_M) \xrightarrow{i} (\hat{A}, \phi_{\hat{A}}) \xrightarrow{p} (A, \phi_A) \to 0$ be a central extension of the AssDer pair (A, ϕ_A) by (M, ϕ_M). A section of it is given by a linear map $s : A \to \hat{A}$ such that $p \circ s = \mathrm{id}_A$.

Let s be a section. Define two maps $\psi : A^{\otimes 2} \to M$ and $\chi : A \to M$ by

$$\psi(a, b) = s(a) \cdot s(b) - s(ab), \qquad \chi(a) = \phi_{\hat{A}}(s(a)) - s(\phi_A(a)), \quad \text{for } a, b \in A.$$

Since the vector space $\hat{A}$ is isomorphic to $A \oplus M$ (via the section s), we may transfer the AssDer structure of $\hat{A}$ to $A \oplus M$. This will certainly depend on the section s. The product and the linear map on $A \oplus M$ are respectively given by

$$(a \oplus m) \cdot (b \oplus n) = ab \oplus \psi(a,b) \quad \text{and} \quad \phi_{A \oplus M}(a \oplus m) = \phi_A(a) \oplus \phi_M(m) + \chi(a).$$

With these notations, we have the following.

3.3. Proposition. *The pair $(A \oplus M, \phi_{A\oplus M})$ is an AssDer pair if and only if (ψ, χ) is a 2-cocycle in the cohomology of the AssDer pair (A, ϕ_A) with coefficients in the trivial representation $(M = (M, l = 0, r = 0), \phi_M)$.*

3.4. Theorem. *Let (A, ϕ_A) be an AssDer pair and (M, ϕ_M) be a trivial AssDer pair. Then the isomorphism classes of central extensions of (A, ϕ_A) by (M, ϕ_M) are classified by the second cohomology group $H^2_{\mathrm{AssDer}}(A, M)$ of the AssDer pair with coefficients in the trivial representation $(M = (M, l = 0, r = 0), \phi_M)$.*

Proof. First, we show that the cohomology class of the 2-cocycle (ψ, χ) does not depend on the choice of s. Let s_1 and s_2 be two sections of (10). Define a map $\phi : A \to M$ by $\phi(a) = s_1(a) - s_2(a)$. Then we have

$$\begin{aligned}\psi_1(a,b) = s_1(a) \cdot s_1(b) - s_1(ab) &= (s_2(a) + \phi(a)) \cdot (s_2(b) + \phi(b)) - s_2(ab) - \phi(ab)\\ &= \psi_2(a,b) - \phi(ab) \qquad (\text{as} \quad \phi(a),\ \phi(b) \in M),\end{aligned}$$

$$\begin{aligned}\chi_1(a) = \phi_{\hat{A}}(s_1(a)) - s_1(\phi_A(a)) &= \phi_{\hat{A}}(s_2(a) + \phi(a)) - s_2(\phi_A(a)) - \phi(\phi_A(a))\\ &= \chi_2(a) + \phi_M(\phi(a)) - \phi(\phi_A(a)).\end{aligned}$$

This shows that $(\psi_1, \chi_1) = (\psi_2, \chi_2) + \partial\phi$. Hence (ψ_1, χ_1) and (ψ_2, χ_2) are representative of the same cohomology class.

Let $(\hat{A}_1, \phi_{\hat{A}_1})$ and $(\hat{A}_2, \phi_{\hat{A}_2})$ be two isomorphic central extensions as of Definition 3.2, and the isomorphism is given by η. Let $s_1 : A \to \hat{A}_1$ be a section of the first extension. Then we have $p_2 \circ (\eta \circ s_1) = p_1 \circ s_1 = \mathrm{id}_A$. This shows that $s_2 := \eta \circ s_1$ is a section for the second central extension. Since η is a morphism of AssDer pairs, we have $\eta|_M = \mathrm{id}_M$. Thus, we have

$$\psi_2(a,b) = s_2(a) \cdot s_2(b) - s_2(ab) = \eta(s_1(a) \cdot s_1(b) - s_1(ab)) = \psi_1(a,b),$$

$$\chi_2(a) = \phi_{\hat{A}_2}(s_2(a)) - s_2(\phi_A(a)) = \eta\big(\phi_{\hat{A}_1}(s_1(a)) - s_1(\phi_A(a))\big) = \chi_1(a).$$

This shows that isomorphic central extensions give rise to same 2-cocycle, hence, correspond to same element in $H^2_{\mathrm{AssDer}}(A, M)$.

Conversely, let (ψ_1, χ_1) and (ψ_2, χ_2) be two cohomologous 2-cocycles. Therefore, there exists a linear map $\phi : A \to M$ such that $(\psi_1, \chi_1) - (\psi_2, \chi_2) = \partial\phi$. Consider the corresponding AssDer pairs $(A \oplus M, \phi^1_{A\oplus M})$ and $(A \oplus M, \phi^2_{A\oplus M})$ given in Proposition 3.3. They are isomorphic as AssDer pairs via the map $\eta : A \oplus M \to A \oplus M$ given by $\eta(a \oplus m) = a \oplus m + \phi(a)$. In fact, η defines an isomorphism between central extensions. ■

3.5. Extensions of a pair of derivations. In this subsection, we study extensions of a pair of derivations in a central extension of associative algebras.

Let

$$0 \longrightarrow M \xrightarrow{i} \hat{A} \xrightarrow{p} A \longrightarrow 0 \tag{12}$$

be a fixed central extension of associative algebras.

3.6. Definition. *A pair of derivations* $(\phi_A, \phi_M) \in \mathrm{Der}(A) \times \mathrm{Der}(M)$ *is said to be extensible if there exists a derivation* $\phi_{\hat{A}} \in \mathrm{Der}(\hat{A})$ *such that*

$$0 \longrightarrow (M, \phi_M) \longrightarrow (\hat{A}, \phi_{\hat{A}}) \longrightarrow (A, \phi_A) \longrightarrow 0 \tag{13}$$

is an exact sequence of AssDer pairs. In other words, $(\hat{A}, \phi_{\hat{A}})$ *is a central extension of* (A, ϕ_A) *by* (M, ϕ_M)*.*

Let $s : A \to \hat{A}$ be a section of the central extension (12). Define a map $\psi : A^{\otimes 2} \to M$ by

$$\psi(a, b) := s(a) \cdot s(b) - s(ab).$$

For any pair of derivations $(\phi_A, \phi_M) \in \mathrm{Der}(A) \times \mathrm{Der}(M)$, we define a map $\mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)} : A^{\otimes 2} \to M$ by

$$\mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)}(a, b) := \phi_M(\psi(a, b)) - \psi(\phi_A(a), b) - \psi(a, \phi_A(b)).$$

3.7. Proposition. *The map* $\mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)} : A^{\otimes 2} \to M$ *is a 2-cocycle in the Hochschild cohomology of* A *with coefficients in the trivial representation* $M = (M, l = 0, r = 0)$*. Moreover, the cohomology class* $[\mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)}] \in H^2_{\mathrm{Hoch}}(A, M)$ *does not depend on the choice of sections.*

Proof. Note that ψ is a 2-cocycle on A with coefficients in the trivial A-bimodule M, i.e.

$$\psi(ab, c) - \psi(a, bc) = 0. \tag{14}$$

Observe that

$$\begin{aligned}
(\delta_{\mathrm{Hoch}} \mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)})(a, b, c) &= -\mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)}(ab, c) + \mathrm{Ob}^{\hat{A}}_{(\phi_A, \phi_M)}(a, bc) \\
&= -\phi_M(\psi(ab, c)) + \psi(\phi_A(ab), c) + \psi(ab, \phi_A(c)) \\
&\quad + \phi_M(\psi(a, bc)) - \psi(\phi_A(a), bc) - \phi(a, \phi_A(bc)) \\
&= \psi(a\phi_A(b), c) + \psi(\phi_A(a)b, c) + \psi(ab, \phi_A(c)) \\
&\quad - \psi(\phi_A(a), bc) - \psi(a, b\phi_A(c)) - \psi(a, \phi_A(b)c) = 0 \quad (\text{by } (14)).
\end{aligned}$$

This proves the first part. To prove the second part, take s_1, s_2 to be two sections of (12). Define a map $\phi : A \to M$ by $\phi = s_1 - s_2$. Then we get

$$\begin{aligned}\psi_1(a,b) = s_1(a)\cdot s_1(b) - s_1(ab) &= (s_2(a)+\phi(a))\cdot(s_2(b)+\phi(b)) - s_2(ab) - \phi(ab)\\ &= s_2(a)\cdot s_2(b) - s_2(ab) - \phi(ab) = \psi_2(a,b) - \phi(ab).\end{aligned}$$

If the 2-cocycles corresponding to s_1 and s_2 are respectively denoted by ${}^1\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}$ and ${}^2\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}$, then

$$\begin{aligned}&{}^1\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}(a,b) = \ \phi_M(\psi_1(a,b)) - \psi_1(\phi_A(a),b) - \psi_1(a,\phi_A(b))\\ &= \phi_M(\psi_2(a,b)) - \phi_M(\phi(ab)) - \psi_2(\phi_A(a),b) + \phi(\phi_A(a)b) - \psi_2(a,\phi_A(b)) + \phi(a\phi_A(b))\\ &= {}^2\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}(a,b) + \delta_{\mathrm{Hoch}}(\phi_M\circ\phi - \phi\circ\phi_A)(a,b).\end{aligned}$$

This shows that the 2-cocycles ${}^1\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}$ and ${}^2\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}$ are cohomologous, hence, they correspond to same cohomology class in $H^2_{\mathrm{Hoch}}(A,M)$. ∎

The cohomology class $[\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}] \in H^2_{\mathrm{Hoch}}(A,M)$ considered above is called the *obstruction class* to extend the pair of derivations (ϕ_A,ϕ_M).

3.8. THEOREM. *A pair of derivations $(\phi_A,\phi_M) \in \mathrm{Der}(A)\times\mathrm{Der}(M)$ is extensible if and only if the obstruction class $[\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}] \in H^2_{\mathrm{Hoch}}(A,M)$ is trivial.*

PROOF. Suppose that the pair (ϕ_A,ϕ_M) is extensible. That is, there exists a derivation $\phi_{\hat{A}} \in \mathrm{Der}(\hat{A})$ such that (13) is an exact sequence of AssDer pairs. Define a map $\lambda : A \to M$ by $\lambda(a) = \phi_{\hat{A}}(s(a)) - s(\phi_A(a))$. Note that the image of λ lies in M as $p\big(\phi_{\hat{A}}(s(a)) - s(\phi_A(a))\big) = 0$, which implies that $\phi_{\hat{A}}(s(a)) - s(\phi_A(a)) \in \ker(p) = \mathrm{im}(i)$.

For any $s(a)+m \in \hat{A}$, we observe that

$$\begin{aligned}\phi_{\hat{A}}(s(a)+m) = \phi_{\hat{A}}(s(a)) + \phi_M(m) &= \phi_{\hat{A}}(s(a)) - s(\phi_A(a)) + s(\phi_A(a)) + \phi_M(m)\\ &= s(\phi_A(a)) + \lambda(a) + \phi_M(m).\end{aligned}$$

Hence, for any $s(a)+m,\ s(b)+n \in \hat{A}$, we have

$$\begin{aligned}\phi_{\hat{A}}((s(a)+m)\cdot(s(b)+n)) = \phi_{\hat{A}}(s(a)\cdot s(b)) &= \phi_{\hat{A}}(s(ab)+\psi(a,b))\\ &= s(\phi_A(ab)) + \lambda(ab) + \phi_M(\psi(a,b)). \quad (15)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\phi_{\hat{A}}(s(a)+m)\cdot(s(b)+n) + (s(a)+m)\cdot\phi_{\hat{A}}(s(b)+n)\\ &= (s(\phi_A(a)) + \lambda(a) + \phi_M(m))\cdot(s(b)+n) + (s(a)+m)\cdot(s(\phi_A(b)) + \lambda(b) + \phi_M(n))\\ &= s(\phi_A(a))\cdot s(b) + s(a)\cdot s(\phi_A(b))\\ &= s(\phi_A(a)\cdot b) + \psi(\phi_A(a),b) + s(a\cdot\phi_A(b)) + \psi(a,\phi_A(b)). \quad (16)\end{aligned}$$

Since $\phi_{\hat{A}}$ is a derivation, it follows from (15) and (16) that

$$\phi_M(\psi(a,b)) - \psi(\phi_A(a),b) - \psi(a,\phi_A(b)) = -\lambda(ab). \tag{17}$$

This implies that $\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)} = \partial\lambda$ is given by a coboundary. Hence $[\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}]$ is trivial.

Conversely, suppose that the obstruction cocycle is a coboundary, say $\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)} = \partial\lambda$, for some $\lambda : A \to M$. We define a map $\phi_{\hat{A}}$ on $\hat{A}$ by

$$\phi_{\hat{A}}(s(a)+m) = s(\phi_A(a)) + \lambda(a) + \phi_M(m).$$

Using (17), we can show that (13) is an exact sequence of AssDer pairs. Hence (ϕ_A,ϕ_M) is extensible. ■

As a consequence, we get the following.

3.9. Corollary. *If* $H^2_{\mathrm{Hoch}}(A,M) = 0$ *then any pair of derivations* $(\phi_A,\phi_M) \in \mathrm{Der}(A) \times \mathrm{Der}(M)$ *is extensible.*

Let A be an associative algebra and $M = (M, l = 0, r = 0)$ be a trivial bimodule. In the following, we give conditions on a pair of derivations $(\phi_A,\phi_M) \in \mathrm{Der}(A) \times \mathrm{Der}(M)$ such that it is extensible in every central extension of associative algebras.

Define a map $\Theta : \mathrm{Der}(A) \times \mathrm{Der}(M) \to \mathfrak{gl}(H^2_{\mathrm{Hoch}}(A,M))$ by

$$\Theta(\phi_A,\phi_M)([\psi]) := [\phi_M \circ \psi - \psi \circ (\phi_A \otimes \mathrm{id}) - \psi \circ (\mathrm{id} \otimes \phi_A)].$$

3.10. Theorem. *A pair of derivations* $(\phi_A,\phi_M) \in \mathrm{Der}(A) \times \mathrm{Der}(M)$ *is extensible in every central extensions of* A *by* M *if and only if* $\Theta(\phi_A,\phi_M) = 0$.

Proof. Let $0 \to M \xrightarrow{i} \hat{A} \xrightarrow{p} A \to 0$ be any central extension of A by M. For any section $s : A \to \hat{A}$, the map $\psi : A^{\otimes 2} \to M$, $\psi(a,b) = s(a)\cdot s(b) - s(ab)$ is a 2-cocycle in the cohomology of A with coefficients in M. If $\Theta(\phi_A,\phi_M) = 0$ then we have

$$[\mathrm{Ob}^{\hat{A}}_{(\phi_A,\phi_M)}] = [\phi_M \circ \psi - \psi \circ (\phi_A \otimes \mathrm{id}) - \psi \circ (\mathrm{id} \otimes \phi_A)] = \Theta(\phi_A,\phi_M)([\psi]) = 0.$$

Hence by Theorem 3.8, the pair (ϕ_A,ϕ_M) is extensible.

Conversely, suppose that (ϕ_A,ϕ_M) is extensible in every central extensions of A by M. Take any class $[\psi] \in H^2_{\mathrm{Hoch}}(A,M)$. This induces a central extension of A by M:

$$0 \to M \xrightarrow{i} A \oplus M \xrightarrow{p} A \to 0, \tag{18}$$

where the associative product on $A \oplus M$ is given by $(a \oplus m)\cdot(b \oplus n) = ab \oplus \psi(a,b)$. Since (ϕ_A,ϕ_M) is extensible in the central extension (18), by Theorem 3.8 we have

$$\Theta(\phi_A,\phi_M)([\psi]) = [\phi_M \circ \psi - \psi \circ (\phi_A \otimes \mathrm{id}) - \psi \circ (\mathrm{id} \otimes \phi_A)] = [\mathrm{Ob}^{A\oplus M}_{(\phi_A,\phi_M)}] = 0.$$

This shows that $\Theta(\phi_A,\phi_M) = 0$. ■

4. Deformations

Let (A, ϕ_A) be an AssDer pair. We denote the associative multiplication on A by μ. Consider the space $A[[t]]$ of formal power series in t with coefficients from A. Then $A[[t]]$ is a $\mathbb{K}[[t]]$-module.

A formal (one-parameter) deformation of the AssDer pair (A, ϕ_A) consists of two formal power series

$$\mu_t = \sum_{i\geq 0} t^i \mu_i, \qquad \mu_i \in \mathrm{Hom}(A^{\otimes 2}, A) \text{ with } \mu_0 = \mu,$$
$$\phi_t = \sum_{i\geq 0} t^i \phi_i, \qquad \phi_i \in \mathrm{Hom}(A, A) \quad \text{with } \phi_0 = \phi_A$$

such that the $\mathbb{K}[[t]]$-module $A[[t]]$ together with the multiplication μ_t forms an associative algebra and $\phi_t : A[[t]] \to A[[t]]$ is a derivation on it. In other words, $A[[t]]$ with the associative multiplication μ_t and the derivation ϕ_t forms an AssDer pair over $\mathbb{K}[[t]]$. It is clear from the definition that $\mu_t = \sum_{i\geq 0} t^i \mu_i$ defines a deformation of the associative structure on A in the sense of Gerstenhaber [16].

Let (μ_t, ϕ_t) defines a deformation of the AssDer pair (A, ϕ_A). Then we have

$$\mu_t(\mu_t(a, b), c) = \mu_t(a, \mu_t(b, c)) \quad \text{and} \quad \phi_t(\mu_t(a, b)) = \mu_t(\phi_t(a), b) + \mu_t(a, \phi_t(b)),$$

for all $a, b, c \in A$. Expanding both the equations as power series in t and equating coefficients of t^n in both the equations, we get for $n \geq 0$,

$$\sum_{i+j=n} \mu_i(\mu_j(a, b), c) = \sum_{i+j=n} \mu_i(a, \mu_j(b, c)), \tag{19}$$
$$\sum_{i+j=n} \phi_i(\mu_j(a, b)) = \sum_{i+j=n} \mu_i(\phi_j(a), b) + \mu_i(a, \phi_j(b)). \tag{20}$$

For $n = 0$, the identity (19) and (20) both holds automatically. However, for $n = 1$, we obtain

$$\mu_1(ab, c) + \mu_1(a, b)c = a\mu_1(b, c) + \mu_1(a, bc), \tag{21}$$
$$\phi(\mu_1(a, b)) + \phi_1(ab) = \phi_1(a)b + \mu_1(\phi(a), b) + a\phi_1(b) + \mu_1(a, \phi(b)). \tag{22}$$

The identity (21) is equivalent $\delta_{\mathrm{Hoch}}(\mu_1) = 0$ while the identity (22) is equivalent to $\delta_{\mathrm{Hoch}}(\phi_1) + \delta\mu_1 = 0$. It follows from (8) that $\partial(\mu_1, \phi_1) = 0$. Hence we get the following.

4.1. Proposition. *Let (μ_t, ϕ_t) be a formal deformation of an AssDer pair (A, ϕ_A). Then the linear term (μ_1, ϕ_1) is a 2-cocycle in the cohomology of the AssDer pair (A, ϕ_A) with coefficients in itself.*

The 2-cocycle (μ_1, ϕ_1) is called the infinitesimal of the formal deformation (μ_t, ϕ_t).

4.2. Definition. *Let (μ_t, ϕ_t) and (μ'_t, ϕ'_t) be two formal deformations of an AssDer pair (A, ϕ_A). They are said to be equivalent if there exists a formal isomorphism $\Phi_t = \sum_{i \geq 0} t^i \Phi_i : A[[t]] \to A[[t]]$ with $\Phi_0 = \mathrm{id}_A$, such that*

$$\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t) \quad \text{and} \quad \Phi_t \circ \phi_t = \phi'_t \circ \Phi_t.$$

It follows that the following identities must hold (by equating coefficients of t^n from both sides)

$$\sum_{i+j=n} \Phi_i \circ \mu_j = \sum_{i+j+k=n} \mu'_i \circ (\Phi_j \otimes \Phi_k) \quad \text{and} \quad \sum_{i+j=n} \Phi_i \circ \phi_j = \sum_{i+j=n} \phi'_i \circ \Phi_j.$$

For $n = 0$, both the identities hold as $\Phi_0 = \mathrm{id}_A$. For $n = 1$, we obtain

$$\mu_1 + \Phi_1 \circ \mu = \mu'_1 + \mu \circ (\Phi_1 \otimes \mathrm{id}) + \mu \circ (\mathrm{id} \otimes \Phi_1) \quad \text{and} \quad \phi_1 + \Phi_1 \circ \phi_A = \phi'_1 + \phi_A \circ \Phi.$$

This implies that $(\mu_1, \phi_1) - (\mu'_1, \phi'_1) = \partial(\Phi_1)$. Thus, we have the following.

4.3. Theorem. *The infinitesimals corresponding to equivalent deformations of an AssDer pair (A, ϕ_A) are cohomologous. Therefore, they correspond to the same cohomology class.*

To obtain a one-to-one correspondence between the cohomology group $H^2_{\mathrm{AssDer}}(A, A)$ and equivalence classes of certain type deformations, we use the truncated version of formal deformations.

4.4. Definition. *An infinitesimal deformation of an AssDer pair (A, ϕ_A) is a deformation of (A, ϕ_A) over $\mathbb{K}[[t]]/(t^2)$ (the local Artinian ring of dual numbers).*

Thus, an infinitesimal deformation of (A, ϕ_A) consists of a pair (μ_t, ϕ_t) in which $\mu_t = \mu + t\mu_1$ and $\phi_t = \phi_A + t\phi_1$ such that (μ_1, ϕ_1) is a 2-cocycle in the cohomology of the AssDer pair (A, ϕ_A).

4.5. Theorem. *There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of the AssDer pair (A, ϕ_A) and the second cohomology group $H^2_{\mathrm{AssDer}}(A, A)$.*

Proof. It is already shown that the map

$$\text{infinitesimal deformations}/\sim \;\longrightarrow\; H^2_{\mathrm{AssDer}}(A, A) \;\text{ given by }\; [(\mu_t, \phi_t)] \mapsto [(\mu_1, \phi_1)]$$

is well-defined. This map is bijective with the inverse given as follows. For any 2-cocycle $(\mu_1, \phi_1) \in C^2_{\mathrm{AssDer}}(A, A)$, the pair $(\mu_t = \mu + t\mu_1, \phi_t = \phi_A + t\phi_1)$ defines an infinitesimal deformation of (A, ϕ_A). If $(\mu'_1, \phi'_1) \in C^2_{\mathrm{AssDer}}(A, A)$ is another 2-cocycle cohomologous to (μ_1, ϕ_1), then we have $(\mu_1, \phi_1) - (\mu'_1, \phi'_1) = \partial h$, for some $h \in \mathrm{Hom}(A, A) = C^1_{\mathrm{AssDer}}(A, A)$. In such a case, $\Phi_t = \mathrm{id}_A + th$ defines an equivalence between the infinitesimal deformations $(\mu_t = \mu + t\mu_1, \phi_t = \phi_A + t\phi_1)$ and $(\mu'_t = \mu + t\mu'_1, \phi'_t = \phi_A + t\phi'_1)$. Therefore, the inverse map is also well-defined. ∎

4.6. Definition. *A formal deformation* (μ_t, ϕ_t) *of an* AssDer *pair* (A, ϕ_A) *is said to be trivial if it is equivalent to* $(\mu'_t = \mu, \phi'_t = \phi_A)$.

4.7. Theorem. *If* $H^2_{\text{AssDer}}(A, A) = 0$ *then every formal deformation of the AssDer pair* (A, ϕ_A) *is trivial.*

Proof. Let (μ_t, ϕ_t) be any formal one-parameter deformation of (A, ϕ_A). It follows from Proposition 4.1 that the linear term (μ_1, ϕ_1) is a 2-cocycle. From the given hypothesis, there exists a 1-cochain $\Phi_1 \in C^1_{\text{AssDer}}(A, A) = \text{Hom}(A, A)$ such that $(\mu_1, \phi_1) = \partial\Phi_1$. Setting $\Phi_t = \text{id}_A + \Phi_1 t : A[[t]] \to A[[t]]$ and define

$$\mu'_t = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t), \qquad \phi'_t = \Phi_t^{-1} \circ \phi_t \circ \Phi_t. \tag{23}$$

Then (μ'_t, ϕ'_t) is equivalent to (μ_t, ϕ_t). Moreover, it follows from (23) that μ'_t and ϕ'_t are of the form $\mu'_t = \mu + t^2\mu'_2 + \cdots$ and $\phi'_t = \phi_A + t^2\phi'_2 + \cdots$. In other words, the linear terms of μ'_t and ϕ'_t vanish. By repeating this argument, one can show that (μ_t, ϕ_t) is equivalent to (μ, ϕ_A). ■

4.8. Remark. An AssDer pair (A, ϕ_A) is said to be rigid if every formal deformation is equivalent to (μ, ϕ_A). It follows that the vanishing of the second cohomology is a sufficient condition for the rigidity.

4.9. Extensions of finite order deformation. Let (A, ϕ_A) be an AssDer pair. Consider the $\mathbb{K}[[t]]/(t^{n+1})$-module $A[[t]]/(t^{n+1})$. A deformation of order n of the AssDer (A, ϕ_A) consists of a pair (μ_t, ϕ_t) where $\mu_t = \sum_{i=0}^n t^i\mu_i$ and $\phi_t = \sum_{i=0}^n t^i\phi_i$ such that μ_t defines an associative product on $A[[t]]/(t^{n+1})$ and ϕ_t defines a derivation on it.

Thus, in a deformation of order n, the following identities must hold:

$$\sum_{i+j=k} \mu_i(\mu_j(a, b), c) = \sum_{i+j=k} \mu_i(a, \mu_j(b, c)),$$
$$\sum_{i+j=k} \phi_i(\mu_j(a, b)) = \sum_{i+j=k} \mu_i(\phi_j(a), b) + \mu_i(a, \phi_j(b)),$$

for $k = 0, 1, \ldots, n$. In other words,

$$\delta_{\text{Hoch}}(\mu_k) = \frac{1}{2} \sum_{i+j=k, i,j>0} [\mu_i, \mu_j], \quad \text{and} \tag{24}$$

$$\delta_{\text{Hoch}}(\phi_k) + \delta(\mu_k) = \sum_{i+j=k, i,j>0} [\phi_i, \mu_j]. \tag{25}$$

Let $(\mu_{n+1}, \phi_{n+1}) \in C^2_{\text{AssDer}}(A, A)$ be such that $(\mu'_t = \sum_{i=0}^n t^i\mu_i + t^{n+1}\mu_{n+1},\ \phi'_t = \sum_{i=0}^n t^i\phi_i + t^{n+1}\phi_{n+1})$ defines a deformation of order $n+1$. Then the deformation $(\mu_t = \sum_{i=0}^n t^i\mu_i, \phi_t = \sum_{i=0}^n t^i\phi_i)$ is said to be extensible. In such a case, two more equations need to be satisfied, namely,

$$\delta_{\text{Hoch}}(\mu_{n+1}) = \frac{1}{2} \sum_{i+j=n+1, i,j>0} [\mu_i, \mu_j] \quad (= \text{Ob}^3(a,b,c) \text{ say}),$$

$$\delta_{\text{Hoch}}(\phi_{n+1}) + \delta(\mu_{n+1}) = \sum_{i+j=n+1, i,j>0} [\phi_i, \mu_j] \quad (= \text{Ob}^2(a,b) \text{ say}).$$

4.10. Proposition. *The pair* $(\text{Ob}^3, \text{Ob}^2)$ *is a* 3*-cocycle in the cohomology of the AssDer pair* (A, ϕ_A) *with coefficients in itself.*

Proof. It is known from the finite order deformations of associative algebras [16] that the obstruction Ob^3 is a Hochschild 3-cocycle in the cohomology of A, i.e. $\delta_{\text{Hoch}}(\text{Ob}^3) = 0$. Moreover, we have

$$\begin{aligned}
&\delta_{\text{Hoch}}(\text{Ob}^2) + (-1)^3 \ \delta(\text{Ob}^3) = -[\mu, \text{Ob}^2] + [\phi_A, \text{Ob}^3] \\
&= - \sum_{\substack{i+j=n+1 \\ i,j>0}} \big([[\mu, \phi_i], \mu_j] + [\phi_i, [\mu, \mu_j]]\big) + \frac{1}{2} \sum_{\substack{i+j=n+1 \\ i,j>0}} \big([[\phi_A, \mu_i], \mu_j] + [\mu_i, [\phi_A, \mu_j]]\big) \\
&= - \sum_{\substack{i+j=n+1 \\ i,j>0}} \big([[\mu, \phi_i], \mu_j] + [\phi_i, [\mu, \mu_j]]\big) + \sum_{\substack{i+j=n+1 \\ i,j>0}} [[\phi_A, \mu_i], \mu_j] \\
&= - \sum_{\substack{i+j=n+1 \\ i,j>0}} \big([[\mu, \phi_i], \mu_j] - [[\phi_A, \mu_i], \mu_j]\big) + \frac{1}{2} \sum_{\substack{i+j'+j''=n+1 \\ i,j',j''>0}} [\phi_i, [\mu'_j, \mu''_j]] \quad (\text{by } (24)) \\
&= - \sum_{\substack{i'++i''+j=n+1 \\ i',i'',j>0}} [[\phi_{i'}, \mu_{i''}], \mu_j] - \sum_{\substack{i+j=n+1 \\ i,j>0}} ([[\phi_A, \mu_i], \mu_j] - [[\phi_A, \mu_i], \mu_j]) \\
&\quad + \frac{1}{2} \sum_{\substack{i+j'+j''=n+1 \\ i,j',j''>0}} ([[\phi_i, \mu_{j'}], \mu_{j''}] + [\mu_{j'}, [\phi_i, \mu_{j''}]]) \quad (\text{by } (25)) \\
&= - \sum_{\substack{i'+i''+j=n+1 \\ i',i'',j>0}} [[\phi_{i'}, \mu_{i''}], \mu_j] + \sum_{\substack{i+j'+j''=n+1 \\ i,j',j''>0}} [[\phi_i, \mu_{j'}], \mu_{j''}] = 0.
\end{aligned}$$

Thus, $\partial(\text{Ob}^3, \text{Ob}^2) = (\delta_{\text{Hoch}}(\text{Ob}^3), \delta_{\text{Hoch}}(\text{Ob}^2) + (-1)^3 \ \delta(\text{Ob}^3)) = 0$. ■

Therefore, $(\text{Ob}^3, \text{Ob}^2)$ defines a cohomology class in $H^3_{\text{AssDer}}(A, A)$. If this cohomology class vanishes, i.e. $(\text{Ob}^3, \text{Ob}^2)$ is a coboundary, then we have $\partial(\mu_{n+1}, \phi_{n+1}) = (\text{Ob}^3, \text{Ob}^2)$, for some $(\mu_{n+1}, \phi_{n+1}) \in C^2_{\text{AssDer}}(A, A)$. In such a case $(\mu'_t = \mu_t + t^{n+1}\mu_{n+1}, \phi'_t = \phi_t + t^{n+1}\phi_{n+1})$ defines a deformation of order $n+1$. Therefore, the deformation (μ_t, ϕ_t) becomes extensible. On the other hand, if (μ_t, ϕ_t) is extensible, there exists $(\mu_{n+1}, \phi_{n+1}) \in C^2_{\text{AssDer}}(A, A)$ such that $(\mu'_t = \mu_t + t^{n+1}\mu_{n+1}, \phi'_t = \phi_t + t^{n+1}\phi_{n+1})$ is a deformation of order $n+1$. Hence the obstruction $(\text{Ob}^3, \text{Ob}^2)$ is given by the coboundary $\partial(\mu_{n+1}, \phi_{n+1})$. Thus the corresponding cohomology class is null. Therefore, we obtain the following.

4.11. Theorem. *Let* (μ_t, ϕ_t) *be a deformation of order* n *of the AssDer pair* (A, ϕ_A)*. It is extensible if and only if the obstruction class* $[(\mathrm{Ob}^3, \mathrm{Ob}^2)]$ *vanishes.*

4.12. Theorem. *If* $H^3_{\mathrm{AssDer}}(A, A) = 0$ *then every finite order deformation of the AssDer pair* (A, ϕ_A) *extends to a deformation of the next order.*

4.13. Corollary. *If* $H^3_{\mathrm{AssDer}}(A, A) = 0$ *then every 2-cocycle is the infinitesimal of a formal deformation of* (A, ϕ_A)*.*

4.14. Automorphisms of the deformed AssDer pair. Let (A, ϕ_A) be an AssDer pair and (μ_t, ϕ_t) be a deformation of it. Suppose $\Phi_t = \sum_{i \geq 0} t^i \Phi_i : A[[t]] \to A[[t]]$ is an automorphism of the deformed AssDer pair $(A[[t]], \mu_t, \phi_t)$. Then we have

$$\Phi_t \circ \mu_t = \mu_t \circ (\Phi_t \otimes \Phi_t) \quad \text{ and } \quad \Phi_t \circ \phi_t = \phi_t \circ \Phi_t.$$

This, in particular, implies that

$$\Phi_1(ab) = \Phi_1(a)b + a\Phi_1(b) \quad \text{ and } \quad \Phi_1 \circ \phi_A = \phi_A \circ \Phi_1.$$

Therefore, the linear term Φ_1 of the automorphism Φ_t is a derivation on A commuting with ϕ_A. Thus, one may now ask when a derivation on A which commutes with ϕ_A can be extended to an automorphism of the deformed AssDer pair $(A[[t]], \mu_t, \phi_t)$. We will consider a more general situation about extensions of a finite order automorphism of the deformed AssDer pair.

Let $\Phi_t = \sum_{i=1}^{N} t^i \Phi_i$ be an automorphism of order N. It is said to be extensible if there exists a map $\Phi_{N+1} : A \to A$ such that $\Phi'_t = \sum_{i=1}^{N+1} t^i \Phi_i$ is an automorphism of order $N+1$. In other words, the following additional identities must hold:

$$\begin{aligned} & a\Phi_{N+1}(b) + \Phi_{N+1}(a)b - \Phi_{N+1}(ab) \\ & \quad = \sum_{i+j=N+1, i \neq N+1} \Phi(\mu_j(a,b)) - \sum_{i+j+k=N+1, j,k \neq N+1} \mu_i(\Phi_j(a), \Phi_k(b)) \quad (= \mathrm{Ob}^1_{\Phi_t} \text{ say}), \end{aligned} \tag{26}$$

$$\begin{aligned} & - \Phi_{N+1} \circ \phi_A + \phi_A \circ \Phi_{N+1} \\ & \quad = \sum_{i+j=N+1, i \neq N+1} \Phi_i \circ \phi_j - \sum_{i+j=N+1, j \neq N+1} \phi_i \circ \Phi_j \quad (= \mathrm{Ob}^2_{\Phi_t} \text{ say}). \end{aligned} \tag{27}$$

The pair $(\mathrm{Ob}^1_{\Phi_t}, \mathrm{Ob}^2_{\Phi_t})$ is called the obstruction to extending the order N automorphism Φ_t. It has been shown in [14] that $\mathrm{Ob}^1_{\Phi_t}$ is a Hochschild 2-cocycle. It is also not difficult to show that (similar to Proposition 4.10)

$$\delta_{\mathrm{Hoch}}(\mathrm{Ob}^2_{\Phi_t}) + \delta(\mathrm{Ob}^1_{\Phi_t}) = 0.$$

In other words, $(\mathrm{Ob}^1_{\Phi_t}, \mathrm{Ob}^2_{\Phi_t})$ is a 2-cocycle in the cohomology of the AssDer pair (A, ϕ_A).

Hence, from (26) and (27), we get the following.

4.15. Theorem. *An order N automorphism $\Phi_t = \sum_{i=1}^{N} t^i \Phi_i$ of the deformed AssDer pair is extensible if and only if the obstruction class $[(\mathrm{Ob}^1_{\Phi_t}, \mathrm{Ob}^2_{\Phi_t})] \in H^2_{\mathrm{AssDer}}(A, A)$ vanishes.*

5. Homotopy derivations on 2-term A_∞-algebras

In this section, we are interested in 2-term A_∞-algebras [24] with homotopy derivations. Note that homotopy derivation on A_∞-algebras was studied by Loday [20] and further developed by Doubek-Lada [12]. We classify homotopy derivations on skeletal and strict A_∞-algebras.

5.1. Definition. *A 2-term A_∞-algebra consists of a chain complex $A := (A_1 \xrightarrow{d} A_0)$ together with maps $\mu_2 : A_i \otimes A_j \rightarrow A_{i+j}$, for $0 \leq i, j, i+j \leq 1$ and a map $\mu_3 : A_0 \otimes A_0 \otimes A_0 \rightarrow A_1$ satisfying the followings: for any $a, b, c, e \in A_0$ and $m, n \in A_1$,*

(a) $d\mu_2(a, m) = \mu_2(a, dm)$,

(b) $d\mu_2(m, a) = \mu_2(dm, a)$,

(c) $\mu_2(dm, n) = \mu_2(m, dn)$,

(d) $d\mu_3(a, b, c) = \mu_2\big(\mu_2(a, b), c\big) - \mu_2\big(a, \mu_2(b, c)\big)$,

(e1) $\mu_3(a, b, dm) = \mu_2\big(\mu_2(a, b), m\big) - \mu_2\big(a, \mu_2(b, m)\big)$,

(e2) $\mu_3(a, dm, c) = \mu_2\big(\mu_2(a, m), c\big) - \mu_2\big(a, \mu_2(m, c)\big)$,

(e3) $\mu_3(dm, b, c) = \mu_2\big(\mu_2(m, b), c\big) - \mu_2\big(m, \mu_2(b, c)\big)$,

(f) $\mu_3\big(\mu_2(a, b), c, e\big) - \mu_3\big(a, \mu_2(b, c), e\big) + \mu_3\big(a, b, \mu_2(c, e)\big) = \mu_2\big(\mu_3(a, b, c), e\big) + \mu_2\big(a, \mu_3(b, c, e)\big)$.

A 2-term A_∞-algebra as above may be denoted by $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3)$.

5.2. Definition. *Let $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3)$ and $(A'_1 \xrightarrow{d'} A'_0, \mu'_2, \mu'_3)$ be 2-term A_∞-algebras. A morphism between them consists of a chain map $f : A \rightarrow A'$ (which consists of linear maps $f_0 : A_0 \rightarrow A'_0$ and $f_1 : A_1 \rightarrow A'_1$ with $f_0 \circ d = d' \circ f_1$) and a bilinear map $f_2 : A_0 \otimes A_0 \rightarrow A'_1$ such that for any $a, b, c \in A_0$ and $m \in A_1$, the following conditions hold*

(a) $d' f_2(a, b) = f_0(\mu_2(a, b)) - \mu'_2(f_0(a), f_0(b))$,

(b) $f_2(a, dm) = f_1(\mu_2(a, m)) - \mu'_2(f_0(a), f_1(m))$,

(c) $f_2(dm, a) = f_1(\mu_2(m, a)) - \mu'_2(f_1(m), f_0(a))$,

(d) $f_2(\mu_2(a, b), c) - f_2(a, \mu_2(b, c)) - \mu'_2(f_2(a, b), f_0(c)) + \mu'_2(f_0(a), f_2(b, c))$
$= f_1(\mu_3(a, b, c)) - \mu'_3(f_0(a), f_0(b), f_0(c))$.

We denote the category of 2-term A_∞-algebras and morphisms between them by $2\mathbf{A}_\infty$.

5.3. Definition. *Let* $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3)$ *be a* 2*-term* A_∞*-algebra. A homotopy derivation of degree* 0 *on it consists of a chain map* $\theta : A \to A$ *(which consists of linear maps* $\theta_i : A_i \to A_i$*, for* $i = 0, 1$ *satisfying* $\theta_0 \circ d = d \circ \theta_1$*) and* $\theta_2 : A_0 \otimes A_0 \to A_1$ *satisfying the followings: for any* $a, b, c \in A_0$ *and* $m \in A_1$*,*

(a) $d(\theta_2(a, b)) = \mu_2(\theta_0 a, b) + \mu_2(a, \theta_0 b) - \theta_0(\mu_2(a, b))$,

(b) $\theta_2(a, dm) = \mu_2(\theta_0 a, m) + \mu_2(a, \theta_1 m) - \theta_1(\mu_2(a, m))$,

(c) $\theta_2(dm, a) = \mu_2(\theta_1 m, a) + \mu_2(m, \theta_0 a) - \theta_1(\mu_2(m, a))$,

(d) $\theta_1(\mu_3(a, b, c)) = \theta_2(a, \mu_2(b, c)) - \theta_2(\mu_2(a, b), c) + \mu_2(a, \theta_2(b, c)) - \mu_2(\theta_2(a, b), c)$
$+ \mu_3(\theta_0 a, b, c) + \mu_3(a, \theta_0 b, c) + \mu_3(a, b, \theta_0 c)$.

We call a 2-term A_∞-algebra with a homotopy derivation a 2-term AssDer_∞-pair. We denote such a pair by $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$. An AssDer_∞-pair is said to be skeletal if the underlying 2-term A_∞-algebra is skeletal, i.e. $d = 0$.

5.4. Proposition. *There is a one-to-one correspondence between the set of all skeletal* AssDer_∞*-pairs and the set of all triples* $((A, \phi_A), (M, \phi_M), (\theta, \psi))$*, where* (A, ϕ_A) *is an AssDer pair,* (M, ϕ_M) *is a representation and* $(\theta, \psi) \in C^3_{\mathrm{AssDer}}(A, M)$ *is a* 3*-cocycle in the cohomology of the AssDer pair with coefficients in* (M, ϕ_M)*.*

Proof. Let $(A_1 \xrightarrow{0} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$ be a skeletal AssDer_∞-pair. Then it follows from Definition 5.3(a) that θ_0 is a derivation for the associative algebra (A_0, μ_2). Moreover, the conditions (b) and (c) say that (A_1, θ_1) is a representation of the AssDer pair (A_0, θ_0). Finally, the condition (f) of Definition 5.1 implies that $\delta_{\mathrm{Hoch}}(\mu_3) = 0$ and condition (d) of Definition 5.3 implies that $\delta_{\mathrm{Hoch}}\theta_2 + \delta\mu_3 = 0$. Therefore, $(\mu_3, -\theta_2) \in C^3_{\mathrm{AssDer}}(A_0, A_1)$ is a 3-cocycle in the cohomology of the AssDer pair with coefficients in (A_1, θ_1).

Conversely, let $((A, \phi_A), (M, \phi_M), (\theta, \psi))$ be such a triple. Then it can be easily verify that $(M \xrightarrow{0} A, \mu_2 = (\mu_A, l, r), \theta, \phi_A, \phi_M, -\psi)$ is a skeletal AssDer_∞-pair. The above correspondences are inverses of each other. ■

5.5. Definition. *A* 2*-term* AssDer_∞*-pair* $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$ *is called strict if* $\mu_3 = 0$ *and* $\theta_2 = 0$*.*

5.6. Example. *Let* (A, ϕ_A) *be an AssDer pair. Take* $A_0 = A_1 = A$*,* $d = \mathrm{id}$*,* $\mu_2 = \mu$ *(the associative multiplication on* A*),* $\theta_0 = \theta_1 = \phi_A$*. Then* $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3 = 0, \theta_0, \theta_1, \theta_2 = 0)$ *is strict* AssDer_∞*-pair.*

Next, we introduce crossed modules of AssDer pairs and show that strict AssDer_∞-pairs correspond to crossed modules of AssDer pairs.

5.7. Definition. *A crossed module of AssDer pairs is a tuple* $((A, \phi_A), (B, \phi_B), dt, \phi)$ *in which* $(A, \phi_A), (B, \phi_B)$ *are both AssDer pairs,* $dt : A \to B$ *is a morphism of AssDer pairs and*

$$\phi : B \otimes A \to A \qquad \phi : A \otimes B \to A$$

defines an AssDer pair bimodule on (A, ϕ_A) *satisfying the following conditions: for all* $b \in B$ *and* $m, n \in A$,

(i) $dt(\phi(b, m)) = \mu_B(b, dt(m)),$
$dt(\phi(m, b)) = \mu_B(dt(m), b),$

(ii) $\phi(dt(m), n) = \mu_A(m, n),$
$\phi(m, dt(n)) = \mu_A(m, n),$

(iii) $\mu_A(\phi(b, m), n) = \phi(b, \mu_A(m, n)),$
$\mu_A(\phi(m, b), n) = \mu_A(m, \phi(b, n)),$
$\phi(\mu_A(m, n), b) = \mu_A(m, \phi(n, b)),$

(iv) $\phi_A(\phi(b, m)) = \phi(\phi_B(b), m) + \phi(b, \phi_A(m)),$
$\phi_A(\phi(m, b)) = \phi(\phi_A(m), b) + \phi(m, \phi_B(b)).$

5.8. Proposition. *There is a one-to-one correspondence between strict AssDer*$_\infty$*-pairs and the crossed module of AssDer pairs.*

Proof. It is already known that strict A_∞-algebras are in one-to-one correspondence with crossed modules of associative algebras [10]. More precisely, $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3 = 0)$ is a strict A_∞-algebra if and only if (A_1, A_0, d, μ_2) is a crossed module of associative algebras. Note that the associative products on A_1 and A_0 are respectively given by $\mu_{A_1}(m, n) := \mu_2(dm, n) = \mu_2(m, dn)$ and $\mu_{A_0}(a, b) = \mu_2(a, b)$, for $m, n \in A_1$ and $a, b \in A_0$. It follows from (a) and (b) of Definition 5.3 that θ_1 is a derivation on A_1 and θ_0 is a derivation on A_0. Hence (A_1, θ_1) and (A_0, θ_0) are AssDer pairs. Since $\theta_0 \circ d = d \circ \theta_1$, we have $dt = d : A_1 \to A_0$ is a morphism of AssDer pairs. The conditions (i), (ii), (iii) of Definition 5.7 are also held. Finally, the conditions (b) and (c) of Definition 5.3 are equivalent to the last condition of Definition 5.7. ■

The crossed module corresponding to the strict AssDer$_\infty$-pair of Example 5.6 is given by the tuple $((A, \phi_A), (A, \phi_A), \mathrm{id}, \mu_A)$.

5.9. Definition. *Let* $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$ *and* $(A_1' \xrightarrow{d'} A_0', \mu_2', \mu_3', \theta_0', \theta_1', \theta_2')$ *be two* 2*-term* AssDer$_\infty$*-pair. A morphism between them consists of a morphism* (f_0, f_1, f_2) *between the underlying* 2*-term* A_∞*-algebras together with a map* $\mathcal{B} : A_0 \to A_1'$ *such that the following conditions hold:*

(i) $\theta_0'(f_0(a)) - f_0(\theta_0(a)) = d'(\mathcal{B}(a)),$

(ii) $\theta_1'(f_1(m)) - f_1(\theta_1(m)) = \mathcal{B}(dm),$

(iii) $f_1(\theta_2(a,b)) + f_2(\theta_0 a, b) + f_2(a, \theta_0 b) - \theta_1'(f_2(a,b)) - \theta_2'(f_0(a), f_0(b))$
$= \mu_2'(\mathcal{B}a, f_0(b)) + \mu_2'(f_0(a), \mathcal{B}b) - \mathcal{B}(l_2(a,b))$.

Let $A = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$ and $A' = (A_1' \xrightarrow{d'} A_0', \mu_2', \mu_3', \theta_0', \theta_1', \theta_2')$ be two 2-term AssDer$_\infty$-pairs and $f = (f_0, f_1, f_2, \mathcal{B})$ be a morphism between them. Let $A'' = (A_1'' \xrightarrow{d''} A_0'', \mu_2'', \mu_3'', \theta_0'', \theta_1'', \theta_2'')$ be another 2-term AssDer$_\infty$-pair and $g = (g_0, g_1, g_2, \mathcal{C})$ be a morphism from A' to A''. Their composition is a morphism $g \circ f : A \to A''$ of AssDer$_\infty$-pairs whose components are given by $(g \circ f)_0 = g_0 \circ f_0$, $(g \circ f)_1 = g_1 \circ f_1$, and

$$(g \circ f)_2(a,b) = g_2(f_0(a), f_0(b)) + g_1(f_2(a,b)), \qquad \mathcal{D} = g_1 \circ \mathcal{B} + \mathcal{C} \circ f_0 : A_0 \to A_1''.$$

For any 2-term AssDer$_\infty$-pair A, the identity morphism id_A is given by the identity chain map $A \to A$ together with $(\mathrm{id}_A)_2 = 0$ and $\mathcal{B} = 0$. The collection of 2-term AssDer$_\infty$-pairs and morphisms between them forms a category. We denote this category by $2\mathbf{AssDer}_\infty$.

6. Categorification of AssDer pairs

In this section, we study the categorification of AssDer pair, which we call AssDer 2-pair. We show that the category of AssDer 2-pairs and the category $2\mathbf{AssDer}_\infty$ are equivalent.

A 2-vector space C is a category with vector space of objects C_0 and the vector space of arrows C_1 such that all structure maps in the category C are linear. A morphism of 2-vector spaces is a functor $F = (F_0, F_1)$ which is linear in the space of objects and arrows. We denote the category of 2-vector spaces by 2**Vect**. Given a 2-vector space $C = (C_1 \rightrightarrows C_0)$, we have a 2-term complex $\ker(s) \xrightarrow{t} C_0$. A morphism between 2-vector spaces induces a morphism between 2-term complexes. Conversely, any 2-term complex $A_1 \xrightarrow{d} A_0$ gives rise to a 2-vector space $\mathbb{V} = (A_0 \oplus A_1 \rightrightarrows A_0)$ in which the set of objects is A_0 and the set of morphisms is $A_0 \oplus A_1$. The structure maps are given by $s(a,m) = a$, $t(a,m) = a + dm$ and $i(a) = (a, 0)$. A morphism between 2-term chain complexes induces a morphism between the corresponding 2-vector spaces. We denote the category of 2-term complexes by 2**TermCom**. There is an equivalence of categories 2**TermCom** $\simeq$ 2**Vect**.

6.1. Definition. *An associative 2-algebra is a 2-vector space C equipped with a bilinear functor $\mu : C \otimes C \to C$ and a trilinear natural isomorphism called the associator*

$$\mathcal{A}_{\xi,\eta,\zeta} : \mu(\mu(\xi,\eta),\zeta) \to \mu(\xi, \mu(\eta,\zeta))$$

satisfying the following identity represented by the pentagon

$$\begin{array}{ccc}
 & \mu\big(\mu(\mu(\xi,\eta),\zeta),\lambda\big) & \\
\swarrow^{\mathcal{A}_{\xi,\eta,\zeta}} & & \searrow^{\mathcal{A}_{\mu(\xi,\eta),\zeta,\lambda}} \\
\mu\big(\mu(\xi,\mu(\eta,\zeta)),\lambda\big) & & \mu\big(\mu(\xi,\eta),\mu(\zeta,\lambda)\big) \\
\searrow^{\mathcal{A}_{\xi,\mu(\eta,\zeta),\lambda}} & & \swarrow^{\mathcal{A}_{\xi,\eta,\mu(\zeta,\lambda)}} \\
\mu\big(\xi,\mu(\mu(\eta,\zeta),\lambda)\big) & \xrightarrow[\mathcal{A}_{\eta,\zeta,\lambda}]{} & \mu\big(\xi,\mu(\eta,\mu(\zeta,\lambda))\big).
\end{array}$$

6.2. Definition. *A morphism between associative 2-algebras $(C,\mu,\mathcal{A})$ and $(C',\mu',\mathcal{A}')$ consists of a functor $F=(F_0,F_1)$ from the underlying 2-vector space C to C' and a bilinear natural transformation*

$$F_2(\xi,\eta):\mu'(F_0(\xi),F_0(\eta))\to F_0(\mu(\xi,\eta))$$

such that the following diagram commutes

$$\begin{array}{ccc}
 & \mu'\big(\mu'(F_0(\xi),F_0(\eta)),F_0(\zeta)\big) & \\
\swarrow^{\mathcal{A}'_{F_0(\xi),F_0(\eta),F_0(\zeta)}} & & \searrow^{F_2(\xi,\eta)} \\
\mu'\big(F_0(\xi),\mu'(F_0(\eta),F_0(\zeta))\big) & & \mu'\big(F_0(\mu(\xi,\eta)),F_0(\zeta)\big) \\
\downarrow{\scriptstyle F_2(\eta,\zeta)} & & \downarrow{\scriptstyle F_2\left(\mu(\xi,\eta),\Phi(\zeta)\right)} \\
\mu'\big(F_0(\xi),F_0(\mu(\eta,\zeta))\big) & & F_0\big(\mu(\mu(\xi,\eta),\zeta)\big) \\
\searrow^{F_2\left(\xi,\mu(\eta,\zeta)\right)} & & \swarrow^{\mathcal{A}_{\xi,\eta,\zeta}} \\
 & F_0\big(\mu(\xi,\mu(\eta,\zeta))\big). &
\end{array}$$

The composition of two associative 2-algebra morphisms is again an associative 2-algebra morphism. More precisely, let C,C' and C'' be three associative 2-algebras and $F:C\to C'$, $G:C'\to C''$ be associative 2-algebra morphisms. Then their composition $G\circ F:C\to C''$ is an associative 2-algebra morphism given by $(G\circ F)_0=G_0\circ F_0$, $(G\circ F)_1=G_1\circ F_1$ and $(G\circ F)_2$ is given by the composition

$$\mu''(G_0\circ F_0(\xi),G_0\circ F_0(\eta))\xrightarrow{G_2(F_0(\xi),F_0(\eta))}G_0\big(\mu'(F_0(\xi),F_0(\eta))\big)\xrightarrow{G_0(F_2(\xi,\eta))}(G_0\circ F_0)(\mu(\xi,\eta)).$$

Finally, for any associative 2-algebra C, the identity morphism $\mathrm{id}_C:C\to C$ is given by the identity functor as its linear functor and the identity natural transformation as $(\mathrm{id}_C)_2$. We denote the category of associative 2-algebras and morphisms between them by **Ass**2.

6.3. Definition. *Let* $(C, \mu, \mathcal{A})$ *be an associative* 2*-algebra. A* 2*-derivation on it consists of a linear functor* $D : C \to C$ *and a bilinear natural isomorphism, called the derivator*

$$\mathcal{D}_{a,b} : D(\mu(a,b)) \rightsquigarrow \mu(Da,b) + \mu(a,Db)$$

satisfying the following

$$\begin{array}{ccc}
 & D((a\cdot b)\cdot c) & \\
{}^{\mathcal{D}_{a\cdot b,c}}\swarrow & & \searrow^{\mathcal{A}_{a,b,c}} \\
(a\cdot b)\cdot D(c) + D(a\cdot b)\cdot c & & D(a\cdot(b\cdot c)) \\
\downarrow \mathcal{D}_{a,b} & & \downarrow \mathcal{D}_{a,b\cdot c} \\
(a\cdot b)\cdot D(c) + (a\cdot D(b) + D(a)\cdot b)\cdot c & & a\cdot D(b\cdot c) + D(a)\cdot(b\cdot c) \\
{}_{\mathcal{A}_{a,b,Dc}+\mathcal{A}_{a,Db,c}+\mathcal{A}_{Da,b,c}}\searrow & & \swarrow_{\mathcal{D}_{b,c}} \\
 & a\cdot(b\cdot D(c)) + a\cdot(D(b)\cdot c) + D(a)\cdot(b\cdot c). &
\end{array}$$

In the above diagram, we use the notation $\mu(a,b) = a\cdot b$. We call an associative 2-algebra together with a 2-derivation by an AssDer2-pair.

6.4. Definition. *Let* $(C, \mu, \mathcal{A}, D, \mathcal{D})$ *and* $(C', \mu', \mathcal{A}', D', \mathcal{D}')$ *be two* AssDer2*-pairs. A morphism between them consists of an associative* 2*-algebra morphism* $(F = (F_0, F_1), F_2)$ *and a natural isomorphism* $\Phi_a : D' \circ F_0(a) \to F_0 \circ D(a)$ *such that the following diagram commutes*

$$\begin{array}{ccc}
 & D'(F_0(a)\cdot' F_0(b)) & \\
{}^{\mathcal{D}'}\swarrow & & \searrow^{\mathcal{A}_{a,b,c}} \\
D'(F_0(a))\cdot' F_0(b) + F_0(a)\cdot' D'(F_0(b)) & & D'(F_0(a\cdot b)) \\
\downarrow \Phi & & \downarrow \Phi \\
F_0(D(a)\cdot' F_0(b) + F_0(a)\cdot' F_0(D(a)) & & F_0(D(a\cdot b)) \\
{}_{F_2}\searrow & & \swarrow_{\mathcal{D}} \\
 & F_0(D(a)\cdot b + a\cdot D(b)). &
\end{array}$$

Here we use the bifunctors μ and μ' as $\cdot$ and $\cdot'$ respectively. We denote the category of AssDer2-pairs together with morphisms between them by **AssDer**2.

It is known that the categories $2\mathbf{A}_\infty$ and $\mathbf{Ass}2$ are equivalent. See, for example [10]. A functor $T : 2\mathbf{A}_\infty \to \mathbf{Ass}2$ is given as follows. Let $A = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3)$ be a 2-term A_∞-algebra. The corresponding associative 2-algebra is defined on the 2-vector space $A_0 \oplus A_1 \rightrightarrows A_0$. The bifunctor μ and the associator $\mathcal{A}$ is given by

$$\mu((a,m),(b,n)) = (\mu_2(a,b), \mu_2(a,n) + \mu_2(m,b) + \mu_2(dm,n)),$$
$$\mathcal{A}_{a,b,c} = \ ((ab)c, \mu_3(a,b,c)).$$

For any A_∞-algebra morphism (f_0, f_1, f_2) from A to A', the associative 2-algebra morphism from $T(A)$ to $T(A')$ is given by

$$F_0 = f_0, \qquad F_1 = f_1 \quad \text{and} \quad F_2(a,b) = (\mu'(f_0(a), f_0(b)), f_2(a,b)).$$

On the other hand, a functor $S : \mathbf{Ass}2 \to 2\mathbf{A}_\infty$ is given as follows. Given an associative 2-algebra $C = (C_1 \rightrightarrows C_0, \mu, \mathcal{A})$, the corresponding 2-term A_∞-algebra is defined on the complex $A_1 = \ker s \xrightarrow{d = t|_{\ker s}} C_0 = A_0$. Define $\mu_2 : A_i \otimes A_j \to A_{i+j}$ and $\mu_3 : A_0 \otimes A_0 \otimes A_0 \to A_1$ by

$$\mu_2(a,b) = \mu(a,b),\ \mu_2(a,m) = \mu(i(a),m),\ \mu_2(m,a) = \mu_2(m,i(a)),\ \mu_2(m,n) = 0,$$
$$\text{and} \ \ \mu_3(a,b,c) = \mathcal{A}_{a,b,c} - i(s(\mathcal{A}_{a,b,c})).$$

For any associative 2-algebra morphism $(F_0, F_1, F_2) : C \to C'$, the corresponding A_∞-algebra morphism from $S(C)$ to $S(C')$ is given by

$$f_0 = F_0, \qquad f_1 = F_1|_{\ker s}, \qquad f_2(a,b) = F_2(a,b) - i(sF_2(a,b)).$$

It is well-known that the above two functors provide the equivalence between $2\mathbf{A}_\infty$ and $\mathbf{Ass}2$ (see for instance, [10]). This equivalence can be extended to respective categories equipped with derivations. More precisely, we have the following.

6.5. Theorem. *The categories* $2\mathbf{AssDer}_\infty$ *and* $\mathbf{AssDer}2$ *are equivalent.*

Proof. Given a 2-term AssDer$_\infty$-pair $(A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$, consider the corresponding associative 2-algebra $T(A)$. A functor $D : T(A) \to T(A)$ and the derivator $\mathcal{D}$ is given by

$$D((a,m)) = (\theta_0(a), \theta_1(m)), \qquad \mathcal{D}_{a,b} = (ab, \theta_2(a,b)).$$

It can be checked that if $(f_0, f_1, f_2, \mathcal{B})$ is a morphism of AssDer$_\infty$-pairs, then (F_0, F_1, F_2, Φ) is a morphism of corresponding AssDer2-pairs, where $\Phi(a) = \mathcal{B}(a)$.

Conversely, for any AssDer2-pair $(C, \mu, \mathcal{A}, D, \mathcal{D})$, consider the 2-term A_∞-algebra $S(C)$. A homotopy derivation on $S(C)$ is given by $\theta_0 = D(i(a))$, $\theta_1(m) = D|_{\ker s}(m)$ and $\theta_2(a,b) = \mathcal{D}_{a,b} - i(s\mu(a,b))$. If (F_0, F_1, F_2, Φ) is a morphism of AssDer2-pairs, then $(f_0, f_1, f_2, \mathcal{B})$ is a morphism between corresponding 2-term AssDer$_\infty$-pairs, where $\mathcal{B}(a) = \Phi(a)$.

Thus, it remains to prove that the composition $T \circ S$ is naturally isomorphic to the identity functor 1_{AssDer2} and $S \circ T$ is naturally isomorphic to $1_{\mathbf{2AssDer_\infty}}$. For any AssDer2-pair $(C, \mu, \mathcal{A}, D, \mathcal{D})$, the AssDer2-pair structure on $(T \circ S)(C)$ is defined on the 2-vector space $A_0 \oplus A_1 \rightrightarrows A_0$, where $A_0 = C_0$ and $A_1 = \ker s$. Define $\theta : T \circ S \to 1_{\mathbf{AssDer2}}$ by $\theta_C : (T \circ S)(C) \to 1_{\mathbf{AssDer2}}(C)$ with $(\theta_C)_0(a) = a,\ (\theta_C)_1(a, m) = i(a) + m$. Then θ_C is an isomorphism of AssDer2-pairs. It is also a natural isomorphism.

For any 2-term AssDer$_\infty$-pair $A = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$, the AssDer$_\infty$-pair structure on $(S \circ T)(A)$ is defined on the same complex $A_1 \xrightarrow{d} A_0$. In fact, we get back the same 2-term AssDer$_\infty$-pair. Therefore, the natural isomorphism $\vartheta : S \circ T \to 1_{\mathbf{2AssDer_\infty}}$ is given by the identity. ∎

6.6. Example. *Let (A, ϕ_A) be an AssDer pair. Then $(A \xrightarrow{\text{id}} A, \mu_2 = \mu, \mu_3 = 0, \theta_0 = \theta_1 = \phi_A, \theta_2 = 0)$ is a strict* AssDer$_\infty$*-pair. The corresponding* AssDer2*-pair is also strict in the sense that the associator and the derivator are both trivial.*

Conclusions.

In this paper, we mainly concentrate on a pair of an associative algebra and a derivation on it. We call such a pair an AssDer pair. Among other things, we study central extensions and deformations of an AssDer pair by extending the classical extensions and deformations of associative algebras. For this, we define a cohomology theory for AssDer pairs generalizing the cohomology of LieDer pairs introduced in [25].

In [11], the author studies deformations of multiplications in a nonsymmetric operad which generalizes the deformation of associative algebras. As applications, the author formulates deformation of various Loday-type (e.g. dendriform, tridendriform, dialgebra, quadri) algebras. See [11] for explicit cohomology of Loday-type algebras. Given a nonsymmetric operad $\mathcal{O}$ with a multiplication $m \in \mathcal{O}(2)$ (i.e. m satisfies $m \circ_1 m = m \circ_2 m$), an element $\phi \in \mathcal{O}(1)$ is called a derivation for m if ϕ satisfies

$$\phi \circ m = m \circ_1 \phi + m \circ_2 \phi.$$

By the method of the present paper, one may study deformations of a pair (m, ϕ), where m is a multiplication on $\mathcal{O}$ and ϕ is a derivation for m. Therefore, one may deduce deformations of Loday-type algebras equipped with derivations. On the other hand, Balavoine [6] studied deformations of algebras over quadratic operads. Derivations on an algebra over an operad are studied in [20, 12]. In a subsequent paper, we aim to construct an explicit cohomology and deformation theory for $\mathcal{P}$-algebras equipped with derivations, where $\mathcal{P}$ is a quadratic operad. The results of the present paper can be dualized to study deformations of coalgebras with coderivations. Since an A_∞-algebra can be described by a square-zero coderivation on the tensor coalgebra of a graded vector space, one can explore formal deformations of A_∞-algebras and compare with the results of [13].

In [8] the authors considered deformations of a Lie algebroid A. Such deformations are governed by a (shifted) graded Lie algebra on the space of multiderivations on A. A

1-cocycle of the corresponding complex is given by a Lie algebroid derivation on A. A derivation of the underlying vector bundle A is a Lie algebroid derivation if it is also a derivation for the Lie bracket on ΓA. Lie algebroid derivations are worth interesting as their flows give rise to Lie algebroid automorphisms. It would be interesting to study deformations of Lie algebroids equipped with Lie algebroid derivation.

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