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STRICTIFICATION OF ∞-GROUPOIDS IS COMONADIC

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ABSTRACT. We investigate the universal strictification adjunction from weak infininity-groupoids (modeled as simplicial sets) to "strict infinity-groupoids", more commonly called "omega-groupoids". Modeling these with simplicial T-complexes, we prove that any simplicial set can be recovered up to weak homotopy equivalence as the totalization of its canonical cosimplicial resolution induced by this adjunction. We explain how this generalizes the fact due to Bousfield and Kan that the homotopy type of a simply connected space can be recovered as the totalization of its canonical cosimplicial resolution induced by the free simplicial abelian group adjunction. Furthermore, we leverage this result to show that this strictification adjunction induces a comonadic adjunction between the quasicategories of simplicial sets and omega-groupoids.

Contents

1. Introduction

A basic goal of classical algebraic topology is to construct algebraic invariants of homotopy types: functors $F : \mathrm{Ho}(\mathrm{Top}) \to \mathscr{A}$ where Ho(Top) is the category of topological spaces localized at the weak equivalences and $\mathscr{A}$ is some category of algebraic objects, e.g. groups or rings. Grothendieck's *Homotopy Hypothesis* is the statement that there is a *complete* such invariant generalizing the notion of groupoid, which he called an ∞-groupoid. By

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"complete" we mean that there is a notion of "weak equivalence" for ∞-groupoids, and that there is an equivalence of categories

$$\mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(\infty\text{-Groupoid})$$

The idea of an ∞-groupoid is based on axiomatizing the algebraic structure that the points, paths, homotopies, homotopies between homotopies, etc. in a topological space carry. Unfortunately, this information is too unwieldy to work with directly— Grothendieck provided a complete definition, but no one has yet been able to prove or disprove that it leads to the desired equivalence of categories. Various alternate definitions along the same lines have been proposed, see for instance [Cisinski, 2006; Henry, 2016].

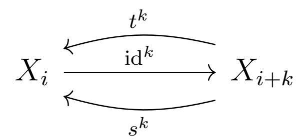
The enormous success of simplicial methods in homotopy theory led to a solution of a different sort: taking "∞-groupoid" to mean "Kan complex". Kan complexes certainly satisfy the condition of modeling the homotopy theory of spaces. If one is interested not necessarily in understanding Grothendieck's algebraic vision of "∞-groupoids" and is mainly looking for some convenient object with which to do homotopy theory, there is no reason to seek anything else. However, Kan complexes do not entirely fit the bill of axiomatizing the algebraic data in a space: they are not particularly algebraic (in the sense of being equipped with operations satisfying certain relations). One can define "algebraic Kan complexes", in which horn filling is an operation (rather than a property); in Nikolaus [2011] it is shown that these model spaces. This still falls somewhat short of the original vision— Kan complexes effectively hide much of the complicated nature of ∞-groupoids by relegating it to simplicial combinatorics. For instance: S^2 is the "free ∞-groupoid on a single 2-cell". Interpreting "free ∞-groupoid on a 2-cell" in Kan complexes gives a model for the 2-sphere, and while in principle you can compute anything you like from this, nothing is "intrinsically obvious". By contrast, in the weak 3-groupoid model (which one can use to model the homotopy 3-type of S^2), you can obtain that $\pi_3(S^2) \cong \mathbb{Z}$ directly from the axioms: the generator comes from a coherence 3-cell which expresses homotopy-commutativity of composition of 2-cells (this is an axiomatization of the Eckmann–Hilton argument). Essentially, more homotopical information is encoded "by hand" into the axioms of a weak 3-groupoid than in Kan complexes.

Unfortunately, encoding homotopical information directly into axioms for weak n-groupoids becomes infeasible as n gets larger. Fortunately, one can still retain a significant amount of information by working with "*strict*" ∞-groupoids, which we for clarity shall refer to exclusively as ω-groupoids, which are much easier to define and manipulate:

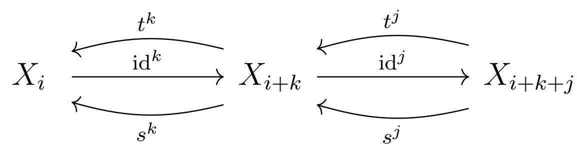
1.1. Definition. *An ω-**groupoid** is a sequence of sets*

$$X_0 \;\underset{s}{\overset{t}{\leftleftarrows}}\!\xrightarrow{\;\mathrm{id}\;}\; X_1 \;\underset{s}{\overset{t}{\leftleftarrows}}\!\xrightarrow{\;\mathrm{id}\;}\; X_2 \;\underset{s}{\overset{t}{\leftleftarrows}}\!\xrightarrow{\;\mathrm{id}\;}\; \cdots$$

such that each diagram



is equipped with the structure of a groupoid, and such that these are compatible in the sense that



is a strict 2*-groupoid. A map between* ω*-groupoids is a map of diagrams which preserves all the groupoidal structure. The resulting category we notate as* ωGpd*.*

One can define a functor Strict : sSet $\to$ ωGpd, but it is well known that ω-groupoids are not a complete invariant for homotopy types (see [Ara, 2013] for an overview of how much homotopical information they can model; essentially it is a mixture of the fundamental group and higher homological information). They nonetheless provide a useful concept while trying to understand the general problem of constructing algebraic invariants, in particular the problem of giving a convenient definition of ∞-groupoids. We think of the functor Strict : sSet $\to$ ωGpd as giving the "strictification" of an ∞-groupoid (presented up to homotopy equivalence by a simplicial set). It has a right adjoint $U : \omega\mathrm{Gpd} \to \mathrm{sSet}$.

The main goal of this paper is to examine this strictification functor, and in particular to prove that it induces a comonadic adjunction of quasicategories. One can think of these results as providing a complete algebraic model for homotopy types— coalgebras in ωGpd are a working model for ∞-groupoid. The weak (and somewhat imprecise) form of what we will prove is:

1.2. Theorem. *The homotopy type of a space* X *is determined by the homotopy type of its strictification Strict*(X)*, along with a natural coalgebra structure over the comonad induced by the adjunction Strict* $\dashv U$*.*

The stronger form is

1.3. Theorem. *The strictification functor is comonadic on the level of quasicategories: that is, it induces an equivalence of quasicategories between spaces and the quasicategory of coalgebras for the comonad Strict* $\circ\, U$ *on* ωGpd*.*

These appear more precisely as Theorems 5.1 and 6.1, respectively.

In relation to previous work, this is a generalization of the main result on homotopy categories in [Blomquist and Harper, 2019], in which the authors prove an analagous result for simply connected spaces and chain complexes. As simply connected chain complexes are equivalent to simply connected strict ∞-groupoids, restricting our result to the simply connected case recovers their main result.[1] Our Theorem 5.1 is a direct generalization of the fact that the $\mathbb{Z}$-completion (in the sense of Bousfield-Kan) of a nilpotent space is itself to the non-nilpotent case. Another closely related result is in [Rivera et al., 2022], where the authors succeed in giving an equivalence of homotopy theories between localizations of spaces at algebraically closed fields and simplicial coalgebras (with no simple connectedness assumptions).

In order to arrive at our result we will actually do very little directly with ω-groupoids, or with topological spaces. Instead, we will work with simplicial models: instead of topological spaces we will work with simplicial sets, and instead of ω-groupoids we will work with simplicial T-complexes, which are a certain sort of Kan complex which form a category equivalent to the category of ω-groupoids.

2. Technical background

2.1. Algebraic Kan complexes. The category of simplicial sets models the homotopy theory of ∞-groupoids under the model structure in which the fibrant-cofibrant objects are Kan complexes and weak equivalences are those maps which induce isomorphisms on all homotopy groups, for any choice of basepoint. As the composition operation in a Kan complex is a relation rather than a function (that is, compositions are asserted to exist rather than given as the output of a composition function), it is difficult to work directly with Kan complexes to provide a "strictification" of their ∞-groupoid structure. Therefore, we work with the notion of an **algebraic Kan complex,** due to [Nikolaus, 2011].

2.2. Definition. [[Nikolaus, 2011], definition 3.1] *The category AlgKan is defined as follows:*

- *The objects are simplicial sets X such that for every diagram*

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{h} & X \\ \downarrow{\scriptstyle \iota^n_k} & & \\ \Delta^n & & \end{array}$$

  there is a chosen *horn filler* $\mathrm{fill}_X(h) : \Delta^n \to X$ *which makes the diagram commute. In other words, objects are pairs* (X, fill_X) *where* fill_X *is a function from horns of* X *to horn fillers. For a horn* $h : \Lambda^n_k \to X$*, we call* $d_k(\mathrm{fill}_X(h))$ *the* composition *of the horn* h*. We call the simplices in the image of* fill_X distinguished fillers*.*

[1] One advantage of the approach by the authors there is that they work directly with a simplicially enriched category of coalgebras, whereas we utilize the Barr-Beck-Lurie theorem, which lets us avoid explicitly constructing a category of coalgebras.

- *The morphisms are morphisms of underlying simplicial sets which preserve the fillings: a morphism $f : (X, \mathrm{fill}_X) \to (Y, \mathrm{fill}_Y)$ is a map of simplicial sets $f : X \to Y$ such that for any horn $h : \Lambda^n_k \to X$,*

$$f(\mathrm{fill}_X(h)) = \mathrm{fill}_Y(f \circ h)$$

2.3. Example. Denote by $|-|$ and $S_\bullet(-)$ the functors for the geometric realization of a simplicial set and the singular simplicial set of a topological space, respectively. It is straightforward to functorially equip the singular simplicial set of a topological space with choices of horn fillers: for each of the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, fix a retract $R^n_k : |\Delta^n| \twoheadrightarrow |\Lambda^n_k|$. Then for any topological space X and any diagram

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & S_\bullet(X) \\ \Big\downarrow{\scriptstyle \iota^n_k} & & \\ \Delta^n & & \end{array}$$

We have an adjoint diagram

$$\begin{array}{ccc} |\Lambda^n_k| & \xrightarrow{|f|} & X \\ \Big\downarrow{\scriptstyle |\iota^n_k|} & & \\ |\Delta^n| & & \end{array}$$

And the adjoint of the map $|f| \circ R^n_k$ gives us the horn filler. This equips each horn with a choice of horn filler, defining a functor $AlgS_\bullet$, such that we have a factorization

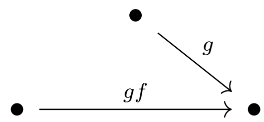
$$\begin{array}{ccccc} \mathrm{Top} & & \xrightarrow{S_\bullet} & & \mathrm{sSet} \\ & \searrow^{AlgS_\bullet} & & \nearrow^{U_A} & \\ & & \mathrm{AlgKan} & & \end{array}$$

Where U_A is the evident forgetful functor. In [Nikolaus, 2011], it is shown that the functor U_A is the right adjoint of a Quillen equivalence, and thus the category AlgKan models the homotopy theory of ∞-groupoids.

2.4. Simplicial T-complexes. Suppose that we have an algebraic Kan complex (X, fill_X), and two composable simplices $f, g \in X_1$. Then the diagram

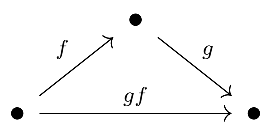
$$\begin{array}{ccccc} & & \bullet & & \\ & \nearrow^{f} & & \searrow^{g} & \\ \bullet & & & & \bullet \end{array}$$

Can be canonically filled to obtain the composition gf. We can then consider the diagram



And again use our choice of horn filler to obtain a composition $g^{-1}(gf)$. Now, if we working in a strict groupoid, we would have that this is equal to f. However, in the context of a weak ∞-groupoid we can conclude that $g^{-1}(gf)$ is homotopic to f, but not necessarily equal to it. If we wish to model strict groupoids, therefore, we must impose some sort of compatibility conditions on our fillers.

Suppose that we have a horn $h : \Lambda^n_k \to X$, and its composition filler is $\mathrm{fill}_X(h) : \Delta^n \to X$. Then, if we are trying to model strict higher groupoids, a reasonable thing to assert is that $\mathrm{fill}_X(h)$ is the composition filler all of its horns, not just the kth. In the above example, this would mean that since we have a filler



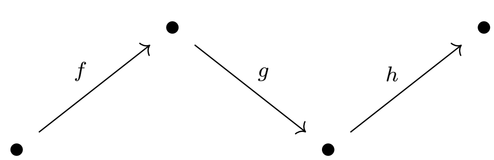
The composition filler for $g^{-1}(gf)$ must be the again the same 2-simplex, so that $f = g^{-1}(gf)$. This motivates the following definition, which appeared originally in [Dakin, 1977].

2.5. Definition. [[Dakin, 1977], Definition 1.1] *A* ***Simplicial T-Complex*** *is a simplicial set X equipped with a set of marked simplices, which we refer to as "thin." These are not required to form a subcomplex, but must satisfy the following axioms:*

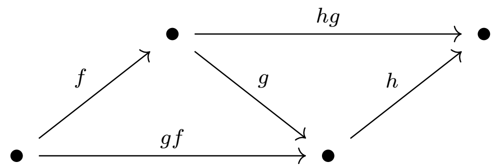
1. *Every degenerate simplex is thin.*
2. *For each horn $h : \Lambda^n_k \to X$, there is a* unique *filler $C : \Delta^n \to X$ such that the image of the nondegenerate n-simplex is thin in X. Given such an extension, we refer to the kth boundary of this filler as the* composition *of the horn h.*
3. *If $h : \Lambda^n_k \to X$ is a horn with all nondegenerate $(n-1)$ simplices thin, then the composition of h is thin as well.*

A map of simplicial T-complexes is a map on underlying simplicial sets which sends thin simplices to thin simplices. We denote the resulting category by sTCom.

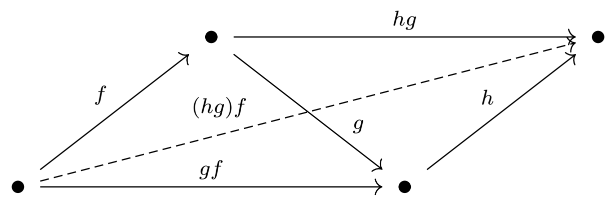
Condition 1 imposes that the composition of a simplex with degenerate simplices is the original simplex. Condition 2 is the compatibility condition we reasoned out above. Condition 3 is slightly more subtle, and is best illuminated by example: suppose we have a diagram of three composable 1-simplices:



Then we can fill in the horn defined by f and g, and the horn defined by g and h, giving us:



Then we can furthermore fill in the horn defined by f and hg, giving us



At this point, we have a horn Λ^3_1 where every face is thin. By axiom 3, the filler is thin, and therefore, looking at face 1, which we get by filling, we have $(hg)f = h(gf)$. So condition 3 is a uniqueness condition that ensures a composition is uniquely defined regardless of the order we choose to do compositions (that is, fill horns) in. Concretely, if we have already obtained every $n-1$-simplex of an n-horn via composition of the $n-2$ simplices, then the additional $n-1$ simplex we get should also express composition.

Another simpler motivation is this: suppose that we have a map $\partial\Delta^n \to X$ where each face is thin. Then we may look at any of the sub-horns, $\Lambda^n_k \hookrightarrow \partial\Delta^n \hookrightarrow X$, and obtain a composition of this sub-horn, giving us a thin filler $\Lambda^n_k \hookrightarrow \Delta^n \to X$. Imposing condition (2) tells us that the composition must be thin, and by uniqueness of thin fillers, we must have that it is equal to the kth face of our original map $\partial\Delta^n \to X$. This implies that our thin filler is in fact a filler for every sub-horn of this map $\partial\Delta^n \to X$. In particular, we get that the simplicial T complex formed by taking the n-skeleton of X and iteratively throwing in all thin fillers of horns is $(n+1)$-coskeletal, much in the way the nerve of an ordinary strict 1-category is 2-coskeletal.

Of course, the best motivation for simplicial T-complexes is simply that they are the correct simplicial analogue of strict ∞-groupoids, which is a consequence of [Ashley, 1978]:

2.6. Theorem. *There is an equivalence of categories* sTCom $\rightleftarrows$ ωGpd.

We will say more about this equivalence in Section 3.

2.7. Crossed complexes. We have already defined the category sTCom of simplicial T-complexes and the category ωGpd of ω-groupoids. We now define a third category, the category CrCom of crossed complexes, which are a sort of "nonabelian chain complex" in a way which we will clarify shortly.

2.8. Definition. [[Brown et al., 2011], Definition 7.1.9] *A* ***crossed complex*** C *is a sequence of sets*

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}}\!\!\xrightarrow{\text{id}} C_1 \xleftarrow[\delta_2]{} C_2 \xleftarrow[\delta_3]{} C_3 \longleftarrow \cdots$$

Such that:

1. *The diagram*

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}}\!\!\xrightarrow{\text{id}} C_1$$

forms a groupoid, which we will abuse notation by referring to simply as C_1.

2. *Each C_i for $i \geq 2$ is a skeletal module over the groupoid C_1: that is, a family of groups of the form*

$$C_i = \coprod_{c \in C_0} C_i(c)$$

where each $C_i(c)$ is a group, equipped with morphisms

$$\varphi_\ell : C_i(s(\ell)) \to C_i(t(\ell))$$

for each $\ell \in C_1$, satisfying that for composable ℓ and p in C_1,

$$\varphi_{\ell \circ p} = \varphi_\ell \circ \varphi_p$$

and that $\varphi_{\text{id}_x} = \text{id}_{C_i(x)}$. Further, each $C_i(c)$ is abelian for $i > 2$. From now on we shall generally suppress the $c \in C_0$ from our notation when our meaning is clear, saying for example "C_i is abelian for $i > 2$."

3. *For $i > 2$, the maps δ_i are families of maps of groups $\delta_i : C_i \to C_{i-1}$, satisfying $\delta_{i-1} \circ \delta_i = 0$.*

4. *δ_2 is a family of maps of groups $\delta_2(c) : C_2(c) \to \text{Aut}(c)$, where by $\text{Aut}(c)$ we mean the automorphism group of c in the groupoid C_1.*

5. *The action of C_1 on C_i is compatible with the δ_i in the sense that for $i > 2$, $\ell \in C_1$, and $a \in C_i(x)$*

$$\varphi_\ell \circ \delta_i = \delta_{i-1} \circ \varphi_\ell$$

6. *For any $a \in C_2$, $\delta_2(a)$ acts by conjugation by a on C_2 and trivially on C_i for $i > 2$.*

To a crossed complex C, we can associate homology groups:

2.9. Definition. *For* $C \in \mathrm{CrCom}$ *and* $c \in C_0$*, define* π_0*,* π_1*, and* H_n *for* $n \geq 2$ *by the following:*

- $\pi_0(C)$ *is* π_0 *of the groupoid* C_1*.*
- $\pi_1(C, c)$ *is* $Aut(C_1, c)/\delta_2(C_2(c))$
- *For* $n \geq 2$*,* $H_n(C, c)$ *is* $ker(\delta_n)(c)/im(\delta_{n+1})(c)$*.*

Note that H_2 *is always abelian although* C_2 *may not be, as the condition that* $\delta(C_2)$ *acts on itself by conjugation means that* $ker(\delta_2)$ *commutes with everything in* C_2*.*

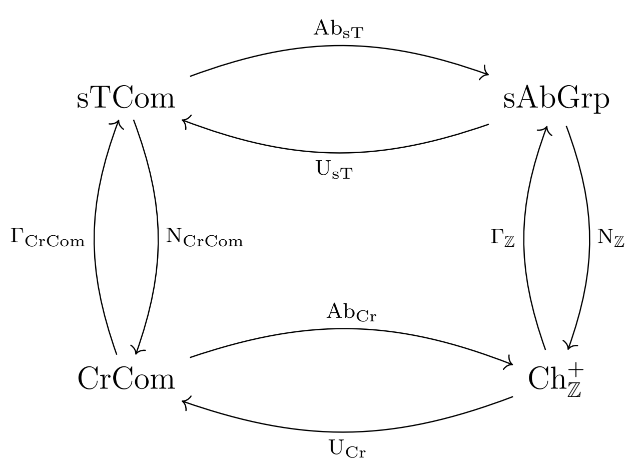
Our interest in CrCom is motivated by the following theorem, which is the primary content of [Brown, 1981]:

2.10. Theorem. *There is an equivalence of categories* $\mathrm{CrCom} \rightleftarrows \omega\mathrm{Gpd}$*.*

Thus, both sTCom and CrCom can be used to analyze ωGpd, and we will use both towards proving our main theorem.

3. The Dold-Kan correspondences: abelian and nonabelian

We use sAbGrp to denote the category of simplicial abelian groups and $\mathrm{Ch}_{\mathbb{Z}}^{+}$ to denote the category of nonnegatively graded chain complexes of abelian groups. In this section, we will recall the adjoint pairs in the following diagram:



(1)

Here the vertical edges are equivalences of categories, the left facing arrows are left adjoints, and the right facing arrows are the corresponding right adjoints. The right edge is the best known: it is the Dold-Kan correspondence. The left edge is a sort of nonabelian version, due to [Ashley, 1978]. The functors on the top and bottom edges we believe are known, with the right adjoint appearing in [Brown et al., 2011],[2] but lacking

[2]Where it is called Θ and defined more generally for chain complexes with a groupoid of operators.

a comprehensive reference we fully define them here. The top and bottom edges each form adjunctions, but not equivalences of categories or homotopy theories. However, they induce equivalences of homotopy theories of 1-connected objects, a fact which will be key to us. The main result of this section which we will use in proving our main theorem is the following.

3.1. THEOREM. *There are adjoint pairs of functors as in Diagram 1 above satisfying:*

1. *The left and right edges are equivalences.*
2. *For* $A \in \mathrm{sAbGrp}$*, the underlying simplicial set of* $U_{sT}(A)$ *is the same as the underlying simplicial set of* A*.*
3. *For* $C \in \mathrm{CrCom}$ *with* C *1-reduced (that is,* C_0 *and* C_1 *are singletons), we have that the natural map* $C \to U_{Cr}(\tilde{Ab}_{Cr}(C))$ *induces an isomorphism, where* $\tilde{Ab}_{Cr}(C)$ *is the reduced chain complex coming from* $Ab_{Cr}(C)$*.*
4. *The diagram is commutative in the sense that there is a natural isomorphism* $U_{sT} \circ \Gamma_{\mathbb{Z}} \cong \Gamma_{\mathrm{CrCom}} \circ U_{Cr}$*.*

The proof is spread throughout this section.

3.2. THE RIGHT EDGE: THE DOLD-KAN CORRESPONDENCE. For $A \in \mathrm{sAbGrp}$, denote by $D(A_n)$ the subgroup of A_n generated by the degenerate simplices.

3.3. DEFINITION. [The functor $\mathrm{N}_{\mathbb{Z}}$] *The functor* $N_{\mathbb{Z}} : \mathrm{sAbGrp} \to Ch^+_{\mathbb{Z}}$ *is defined by* $N_{\mathbb{Z}}(A)_n = A_n/D(A_n)$*, with differential the alternating sum of the face maps.*

The Dold-Kan correspondence is the following well-known result:

3.4. THEOREM. [The Dold-Kan Correspondence] *There is an equivalence of categories* $N_{\mathbb{Z}} : \mathrm{sAbGrp} \rightleftarrows Ch^+_{\mathbb{Z}} : \Gamma_{\mathbb{Z}}$*, where for* $C_\bullet \in Ch^+_{\mathbb{Z}}$*,*

$$\Gamma_{\mathbb{Z}}(C_\bullet)_n := \bigoplus_{[n]\twoheadrightarrow[k]} C_k$$

For a reference, see [Matthew, 2011]. We will need the following fact about this adjunction:

3.5. LEMMA. *For* $A_\bullet \in Ch^+_{\mathbb{Z}}$*, the* n*-simplices of* $\Gamma_{\mathbb{Z}}(A_\bullet)$ *are in bijection with chain complex maps* $N_{\mathbb{Z}}(\mathbb{Z}[\Delta^n]) \to A_\bullet$*. Under this identification, the subgroup generated by the degenerate simplices consists of those maps which send the unique nondegenerate* n*-simplex to* 0*.*

PROOF. The bijection follows from a general fact about adjunctions on sSet being defined by cosimplicial objects. The second fact is less trivial; see the above reference for details. ■

3.6. The left edge: the nonabelian Dold-Kan correspondence. In [Ashley, 1978], the following is proved:

3.7. Theorem. *There is an equivalence of categories* $N_{\mathrm{CrCom}} : \mathrm{CrCom} \rightleftarrows \mathrm{sTCom} : \Gamma_{\mathrm{CrCom}}$.

Proof. For a full proof, see [Ashley, 1978]. We describe here just the functor $\mathrm{N}_{\mathrm{CrCom}} : \mathrm{CrCom} \to \mathrm{sTCom}$, as this is the only particular we will make use of: for $C \in \mathrm{CrCom}$, the n-simplices of $\mathrm{N}_{\mathrm{CrCom}}(C)$ are defined inductively as follows:

- For $n = 0$, they are simply C_0.
- For $n = 1$, they are simply C_1, with boundary maps d_0 and d_1 defined as the source and target maps of the groupoid C_1.
- For $n \geq 2$ they are tuples $x = (x_0, ..., x_n; \alpha)$ where $x_i \in \mathrm{N}_{\mathrm{CrCom}}(C)_{n-1}$, $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$, and $\alpha \in C_n(d_1 \cdots d_{n-1}(x_n))$. Further, they must satisfy

$$\delta(\alpha) = \begin{cases} x_2 x_0 x_1^{-1} & \text{if } n = 2 \\ \varphi_{p^{-1}}(x_0) x_2 x_1^{-1} x_3^{-1} & \text{if } n = 3 \\ \varphi_{p^{-1}}(x_0) \sum_{i=1}^{n} (-1)^i x_i & \text{if } n \geq 4 \end{cases}$$

  Here $p = d_2 d_3 \cdots d_{n-1}(x_n)$, and φ is the action of C_1 on C_n for $n \geq 2$. The boundary maps are defined as $d_i(x) = x_i$. The degeneracy maps are defined inductively as follows: for $x = (x_0, \ldots, x_n; \alpha)$

$$s_i(x) = (s_{i-1}x_0, ..., s_{i-1}x_{i-1}, x, x, s_{i+1}x_i, ..., s_i x_n; e)$$

  Where e is the identity element of (the appropriate component of) C_{n+1}.

For a simplex $x = (x_0, ..., x_n; \alpha) \in \mathrm{N}_{\mathrm{CrCom}}(C)$, we will call α as the *automorphism element* of the simplex, and refer to it by $\mathrm{aut}(x)$. One can think of the simplices of $\mathrm{N}_{\mathrm{CrCom}}(C)$ as being inductively built out of compatible automorphism elements. A simplex is thin precisely when α is the identity element. ∎

We have now finished verifying item (1) of Theorem 3.1. This equivalence of categories respects the natural homotopical structures on the categories, in particular the fundamental algebraic invariants:

3.8. Lemma. *For* $C \in \mathrm{CrCom}$ *and* $x \in C_0$*, there are natural isomorphisms*

$$\pi_0(C) \cong \pi_0(N_{\mathrm{CrCom}}(C)) \quad \pi_1(C, x) \cong \pi_1(N_{\mathrm{CrCom}}(C), x) \quad H_n(C, x) \cong \pi_n(N_{\mathrm{CrCom}}(C), x)$$

For $n \geq 2$*.*

Proof. In [Brown and Higgins, 1991] they prove the analogous statement for the the classifying space of a crossed complex. Since the classifying space of a crossed complex C is defined to be the geometric realization of $\mathrm{N}_{\mathrm{CrCom}}(C)$, and geometric realization preserves homotopy groups, the result follows. ∎

3.9. The top edge: the adjunction $\mathrm{sTCom} \rightleftarrows \mathrm{sAbGrp}$. We first define the inclusion functor $\mathrm{U_{sT}} : \mathrm{sAbGrp} \to \mathrm{sTCom}$.

3.10. Definition. [The functor $\mathrm{U_{sT}}$] *For A a simplicial abelian group, $U_{sT}(A)$ is the simplicial T-complex whose underlying simplicial set is the underlying simplicial set of A, and whose thin simplices are the sums of degenerate simplices. In [Ashley, 1978], it is proven that this gives a simplicial T-complex structure. As maps of simplicial abelian groups preserve both sums and degenerate simplices, this is functorial.*

We now define the left adjoint $\mathrm{Ab_{sT}} : \mathrm{sTCom} \to \mathrm{sAbGrp}$. Recall $\mathbb{Z}[-]$, the free simplicial abelian group functor, given by applying the free abelian group functor to every level of a simplicial set.

3.11. Definition. [The functor $\mathrm{Ab_{sT}}$] *Let $X \in \mathrm{sTCom}$. Let $|X|$ be its underlying simplicial set, and η the natural map of simplicial sets $|X| \to \mathbb{Z}[|X|]$. We define*

$$Ab_{sT}(X) = \frac{\mathbb{Z}[|X|]}{\langle \mathrm{fill}(\eta \circ h) - \eta \circ \mathrm{fill}(h) | h : \Lambda^n_k \to |X| \rangle}$$

Where $\mathrm{fill}(h)$ denotes the unique thin filler of a horn in a simplicial T-complex.

3.12. Theorem. *Ab_{sT} is the left adjoint to U_{sT}.*

Proof. Let $f : X \to \mathrm{U_{sT}}(A)$ be a map of simplicial T-complexes. This induces a map $\mathbb{Z}[|X|] \to A$, which is a map of simplicial abelian groups and therefore a map of simplicial T-complexes. In particular, this map preserves thin fillers. It follows by the universal property of the quotient that this induces a map $\mathrm{Ab_{sT}}(X) \to A$ of simplicial abelian groups.

Conversely, let $g : \mathrm{Ab_{sT}}(X) \to A$ be a map of simplicial abelian groups. Then we get a composite map of simplicial T-complexes $X \to \mathrm{U_{sT}}(\mathrm{Ab_{sT}}(X)) \to \mathrm{U_{sT}}(A)$. ∎

We have now verified item (2) of Theorem 3.1.

3.13. The bottom edge: the adjunction $\mathrm{CrCom} \rightleftarrows \mathrm{Ch}^+_{\mathbb{Z}}$. The adjunction between crossed complexes and chain complexes will involve the most detail:

3.14. Definition. [The functor $\mathrm{U_{Cr}}$] *The right adjoint $U_{Cr} : Ch^+_{\mathbb{Z}} \to \mathrm{CrCom}$ is given by taking a chain complex $A_\bullet$ to the crossed complex*

$$|A_0| \underset{\pi_0}{\overset{\pi_0+d}{\leftleftarrows}} \xrightarrow{\mathrm{id}\times\mathbf{0}} |A_0| \times |A_1| \longleftarrow \coprod_{|A_0|} A_2 \longleftarrow \amalg_{|A_0|} A_3 \qquad \cdots$$

where $|-|$ denotes the underlying set of an abelian group. The composition operation on $|A_0| \times |A_1|$ is given by $(a_1, \ell_1) \circ (a_2, \ell_2) = (a_1, \ell_1 + \ell_2)$. The action of $A_0 \times A_1$ is trivial in the sense that each (p, ℓ) acts as the identity morphism from the a component to the $a + d(\ell)$ component of $\amalg_{a \in |A_0|} A_n$. In particular, if (a, ℓ) is an automorphism, its action is trivial.

3.15. Definition. [The functor $\mathrm{Ab}_{\mathrm{Cr}}$] *Let* $C =$

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1 \xleftarrow[\delta]{} C_2 \xleftarrow[\delta]{} C_3 \xleftarrow[\delta]{} \cdots$$

Be a crossed complex. The action of the elements C_1 *on* C_n *for* $n \geq 2$ *lets us define a certain sort of quotient of this action, which we write as* C_n/C_1*:*

$$C_n/C_1 := \frac{\oplus_{p \in C_0} C_n(p)}{\langle a - \varphi_\ell(a) | \ell \in C_1 \rangle}$$

Then we define the chain complex $Ab_{Cr}(C)$ *to be*

$$\mathbb{Z}[C_0] \xleftarrow[t-s]{} \mathbb{Z}[C_1]/\sim \xleftarrow[\delta]{} C_2/C_1 \xleftarrow[\delta]{} C_3/C_1 \xleftarrow[\delta]{} \cdots$$

Here the relation $\sim$ *on* $\mathbb{Z}[C_1]$ *is generated by* $g \circ f \sim g + f$ *for composable* g *and* f *in* C_1*. We note that while* C_2 *is not abelian,* C_2/C_1 *must be as* $\delta(C_2) \subset C_1$ *acts by conjugation on* C_2*.*

3.16. Theorem. *The functors* Ab_{Cr} *and* U_{Cr} *as described above form an adjoint pair.*

Proof. We define a natural bijection between hom sets: let $C \in \mathrm{CrCom}$, $A_\bullet \in \mathrm{Ch}^+_{\mathbb{Z}}$. Given $f : \mathrm{Ab}_{\mathrm{Cr}}(C) \to A_\bullet$, we define $\bar{f} : C \to \mathrm{U}_{\mathrm{Cr}}(A_\bullet)$ by:

- $\bar{f}_0 : C_0 \to |A_0|$ is the adjoint of $f_0 : \mathbb{Z}[C_0] \to A_0$.
- $\bar{f}_1 : C_1 \to |A_0| \times |A_1|$ is given on the first component by the composition $\bar{f}_0 \circ s$. On the second component, it is the adjoint of the composition $\mathbb{Z}[C_1] \to \mathbb{Z}[C_1]/\sim \to A_1$.
- For $n \geq 2$, recall that each C_n is of the form $\coprod_{p \in C_0} C_n(p)$. The map $\bar{f}_n : C_n \to \coprod_{|A_0|} A_n$ is simply given on each component $C_n(p)$ by the composition $C_n(p) \hookrightarrow C_n/C_1 \to A_n$.

Now, for the other direction: given $g : C \to \mathrm{U}_{\mathrm{Cr}}(A_\bullet)$, we define the adjoint $\bar{g} : \mathrm{Ab}_{\mathrm{Cr}}(C) \to A_\bullet$ as follows:

- $\bar{g}_0 : \mathbb{Z}[C_0] \to A_0$ is given by the adjoint to $g_0 : C_0 \to |A_0|$.
- Take the map $h : \mathbb{Z}[C_1] \to A_1$ adjoint to the second component of $C_1 \to |A_0| \times |A_1|$. Then for composable ℓ and k in C_1, we have by the definition of composition in $|A_0| \times |A_1|$ that $h(\ell \circ k) = h(\ell) + h(k)$. Hence, we can let $\bar{g}_1$ be the induced map $\mathbb{Z}[C_1]/\sim \to A_1$.
- For $n \geq 2$, we need to define a map $\bar{g}_n : C_n/C_1 \to A_n$, given a map $g_n : C_n \to \coprod_{|A_0|} A_n$. By definition of C_n/C_1, it suffices to argue that for $\ell \in C_1$ and $\alpha \in C_n$, $g_n(\alpha) = g_n(\varphi_\ell(\alpha))$. But this follows as we said the action is trivial.

■

We can now conclude the proof of 3.1:

Proof Proof of 3.1, item (3). Let C be a 1-reduced crossed complex. Then $\mathrm{Ab}_{\mathrm{Cr}}(C)$ is the chain complex where $\mathrm{Ab}_{\mathrm{Cr}}(C)_0 = \mathbb{Z}$, $\mathrm{Ab}_{\mathrm{Cr}}(C)_1 = 0$, $\mathrm{Ab}_{\mathrm{Cr}}(C)_n = C_n$ for $n > 1$. Then $\tilde{\mathrm{Ab}}_{\mathrm{Cr}}(C)$ is 0 in dimensions 0 and 1, and $\tilde{\mathrm{Ab}}_{\mathrm{Cr}}(C)_n = C_n$ for higher dimensions. Then the map is $C \to (\mathrm{U}_{\mathrm{Cr}} \circ \tilde{\mathrm{Ab}}_{\mathrm{Cr}})(C)$ is evidently the identity in dimensions 1 and higher, and in dimension 0 is the composition $\{\bullet\} \to |\mathbb{Z}| \to |\mathbb{Z}/\mathbb{Z}|$, and so also an isomorphism there. ■

Proof Proof of 3.1, item (4). We will construct a natural isomorphism $\eta : \mathrm{U}_{\mathrm{sT}} \circ \Gamma_{\mathbb{Z}} \Rightarrow \Gamma_{\mathrm{CrCom}} \circ \mathrm{U}_{\mathrm{Cr}}$. Let $A_\bullet \in \mathrm{Ch}^+_{\mathbb{Z}}$. We inductively define η as follows:

- Since the 0-simplices of $(\mathrm{U}_{\mathrm{sT}} \circ \Gamma_{\mathbb{Z}})(A_\bullet)$ and $(\Gamma_{\mathrm{CrCom}} \circ \mathrm{U}_{\mathrm{Cr}})(A_\bullet)$ are both A_0, η is the identity on 0-simplices.
- The 1-simplices of $(\mathrm{U}_{\mathrm{sT}} \circ \Gamma_{\mathbb{Z}})(A_\bullet)$ are $A_0 \oplus A_1$, with boundary maps $d_0(a_0, a_1) = a_0$ and $d_1(a_0, a_1) = a_0 + d(a_1)$. Since the 1-simplices of $(\Gamma_{\mathrm{CrCom}} \circ \mathrm{U}_{\mathrm{Cr}})(A_\bullet)$ are $|A_0| \times |A_1|$, we can define η as the identity, and the boundary maps agree.
- Let $x \in (\mathrm{U}_{\mathrm{sT}} \circ \Gamma_{\mathbb{Z}})(A_\bullet)_n$ be represented by a map $\mathrm{N}_{\mathbb{Z}}(\mathbb{Z}[\Delta^n]) \to A$, and let α be the image of the unique nondegenerate n-simplex. Then we let

$$\eta(x) = (\eta(d_0(x)), \ldots, \eta(d_n(x)); \alpha)$$

We must prove that this is a well-defined map of simplicial T-complexes, and that it is injective and surjective. This is easier to verify for the 0 and 1-simplices, so we focus on the inductive step:

- For η to be well-defined we must have that $d_i(\eta(d_j(x))) = d_j(\eta(d_i(x)))$ whenever $d^i d^j = d^j d^i$ in Δ, the simplex category. This follows from the inductive definition of boundaries and that $d_j(d_i(x)) = d_i(d_j(x))$. Furthermore, we must have that $d(\alpha) = \sum (-1)^i \mathrm{aut}(\eta(d_i(x)))$,[3] which follows from the definition of $\mathrm{N}_{\mathbb{Z}}(\mathbb{Z}[\Delta^n])$.
- For η to be a map of simplicial T-complexes, we must have that it commutes with the face and degeneracy maps, and that it preserves thin simplices. The former follows directly from the definitions. The latter follows from the identification of the thin simplices in Lemma 3.5.
- For injectivity: suppose $x, y \in (\mathrm{U}_{\mathrm{sT}} \circ \Gamma_{\mathbb{Z}})(A_\bullet)_n$ and $x \neq y$. We wish to show that $\eta(x) \neq \eta(y)$. Inductively assume that η is injective on the $n-1$ simplices. Since $x \neq y$, their representing maps $\mathrm{N}_{\mathbb{Z}}(\mathbb{Z}[\Delta^n]) \to A$ must differ, so they must differ either in the image of the nondegenerate n-simplex, or for some i we must have that $d_i(x) \neq d_i(y)$. If the former, they are different as their automorphism elements are different. If the latter, it follows by our inductive hypothesis.

[3]Recall that if $x = (x_0, ..., x_n; \alpha)$, then $\mathrm{aut}(x) = \alpha$

- For surjectivity: let $(x_0, ..., x_n; \alpha) \in (\Gamma_{\mathrm{CrCom}} \circ \mathrm{U}_{\mathrm{Cr}})(A)$. Inductively assume that η is surjective, then define a map $\mathrm{N}_{\mathbb{Z}}(\mathbb{Z}[\Delta^n]) \to A$ by sending the nondegenerate n-simplex to α, and sending the ith boundary to $\eta^{-1}(x_i)$.

■

4. The Quillen Adjunction $\mathrm{St}_{\mathrm{Alg}} \dashv \mathrm{U}_{\mathrm{Alg}}$

4.1. The functor AlgKan $\to$ sTCom. We are ready to define our strictification $\mathrm{St}_{\mathrm{Alg}}$: AlgKan $\to$ sTCom. We will construct this as an iterated quotient of algebraic Kan complexes, using the work in [Nikolaus, 2011] on the structure of the category AlgKan.

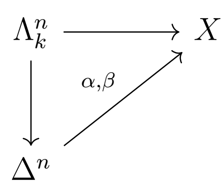
4.2. Definition. *Let X be an algebraic Kan complex. We inductively call a simplex $\alpha : \Delta^n \to X$* thin *if any of the following conditions hold:*

1. α is degenerate.

2. α is a distinguished filler of X.

3. α is a composition of thin simplices.

4.3. Definition. *Let $\alpha : \Delta^n \to X$ and $\beta : \Delta^n \to X$ be thin simplices of the algebraic Kan complex X. We call α and β co-thin if there is a map $\Lambda^n_k \to X$ such that α and β make the lifting diagram*



commute.

In other words, two thin simplices are co-thin if they share a horn.

4.4. Lemma. *Let $f : X \to Y$ be a map of algebraic Kan complexes. Then f preserves thin simplices. In particular, if α and β are co-thin in X, then $f\alpha$ and $f\beta$ are co-thin in Y.*

Proof. Since f is a map of simplicial sets, if α and β share a horn, then $f\alpha$ and $f\beta$ will share a horn, so it remains only to show that f preserves thin simplices. We do this inductively, checking that conditions (1)-(3) in the definition of thinness are preserved by f:

1. if α is degenerate, so is $f\alpha$ since f is a map of simplicial sets.

2. if α is a distinguished filler, then so is $f\alpha$, since f is a map of algebraic Kan complexes.

3. if α is a composition of the thin simplices $\alpha_0, ..., \alpha_n$, then $f\alpha$ is a composition of $f\alpha_0, ..., f\alpha_n$ (because f is a map of algebraic Kan complexes) and each of these is thin by the inductive hypothesis.

■

We can recast our definition of simplicial T-complexes in terms of thinness:

4.5. Lemma. *An algebraic Kan complex is a simplicial T-complex (with thin simplices as in definition 4.2) if and only if it has no distinct, co-thin simplices.*

Proof. This follows directly from the definitions. ■

4.6. Lemma. *The inclusion $U_{Alg} : \mathrm{sTCom} \to \mathrm{AlgKan}$ is fully faithful.*

Proof. For $X, Y \in \mathrm{sTCom}$, a map $X \to Y$ is a map of underlying simplicial sets which preserves the thin simplices. By 4.4, all maps $\mathrm{U_{Alg}}(X) \to \mathrm{U_{Alg}}(Y)$ are of this form. ■

Given this setup, we construct our left adjoint as follows: for $X \in \mathrm{AlgKan}$, let X_1 be the colimit of the diagram with X and an object Δ^n for every pair of co-thin maps α and β, with the two maps $\alpha, \beta : \Delta^n \to X$. More succinctly, let I be an indexing set for all pairs (α_i, β_i) of co-thin simplices in X. Then X_1 is the coequalizer (in AlgKan) of the diagram

$$\coprod_{i\in I} \Delta^{n_i} \underset{\coprod_I \beta_i}{\overset{\coprod_I \alpha_i}{\rightrightarrows}} X$$

This gives us an algebraic Kan complex X_1 equipped with a map $X \to X_1$, with the following properties:

4.7. Lemma. *The assignment $X \mapsto X_1$ is functorial. Let α and β be co-thin maps of X_1. If they factor through X, then they are equal.*

Proof. Functoriality follows from the fact that maps in AlgKan must preserve distinguished fillers by Lemma 4.4, so if we have a map $X \to Y$, this induces maps between the diagrams which define X_1 and Y_1. The second claim follows from the definition of the colimit. ■

Of course, we are not guaranteed that X_1 is a simplicial T-complex, as there may be many co-thin maps which do not factor through X. So, we simply repeat the process, and obtain X_2 with a map $X_1 \to X_2$, and a similar property. Continuing on gives us a sequence

$$X \to X_1 \to X_2 \to \cdots$$

And we define $\mathrm{St_{Alg}}(X)$ to be the colimit of this diagram in AlgKan. Overloading notation, we denote each of the natural maps $X_k \to \mathrm{St_{Alg}}(X)$ by ι. We then have the following:

4.8. Lemma. *A simplex $\alpha : \Delta^n \to St_{Alg}(X)$ is a distinguished filler iff there is some n and some distinguished filler $\tilde{\alpha} : \Delta^n \to X_k$ such that $\alpha = \iota \circ \tilde{\alpha}$.*

Proof. This follows from the construction of filtered colimits given in [Nikolaus, 2011]. ■

4.9. Lemma. *A simplex* $\alpha : \Delta^n \to St_{Alg}(X)$ *is thin iff there is an* $k \in \mathbb{N}$ *and a thin simplex* $\tilde{\alpha} : \Delta_n \to X_k$ *such that* $\alpha = \iota \circ \tilde{\alpha}$.

Proof. We proceed again by induction.

1. Suppose α is degenerate. Then it is degenerate in some X_k, so this follows.
2. Suppose that α is a distinguished filler. Then this follows from the previous lemma.
3. Suppose that α is a composition of thin simplices. Then we can pull the horn which α is a composition of back to some X_n in which each face is thin, and therefore the composition $\tilde{\alpha}$ is thin in X_n.

■

4.10. Lemma. *The mapping* $X \to St_{Alg}(X)$ *defines a functor AlgKan* $\to$ *AlgKan such that* $St_{Alg}(X)$ *has no distinct, co-thin simplices. Furthermore,* $St_{Alg}(X)$ *is initial among algebraic Kan complexes equipped with a map from X which have no distinct, co-thin simplices.*

Proof. Let α and β be co-thin simplices in $\mathrm{St}_{\mathrm{Alg}}(X)$. Then by the previous lemma they factor through some finite stage X_n in which they are co-thin, and by construction are therefore equal in X_{n+1}, and therefore $\mathrm{St}_{\mathrm{Alg}}(X)$. Functoriality follows from functoriality of the colimit and Lemma 4.7. ■

4.11. Theorem. *The mapping* $X \to St_{Alg}(X)$ *defines a functor* AlgKan $\to$ sTCom *(where the thin simplices of* $St_{Alg}(X)$ *are as in Definition 4.2) left adjoint to the forgetful functor* sTCom $\to$ AlgKan.

Proof. Any map $X \to Y$ where Y is a simplicial T-complex factors uniquely through $\mathrm{St}_{\mathrm{Alg}}(X)$, so this defines the left adjoint to the forgetful functor (as it is the inclusion of a full subcategory). ■

We would like to take this adjunction and bump it up to the level of Quillen adjunction. To do this we will need to understand the model structure on the the category of simplicial T-complexes.

4.12. Model Structure on Simplicial T-Complexes. In [Brown and Golasinski, 1989] it was shown that the category of crossed complexes admits a model structure with distinguished maps as follows:

- The weak equivalences are those which induce an isomorphism on π_0, π_1, and H_n for $n \geq 2$.
- The fibrations are those maps $f : C \to D$ such that f given a $p \in C_0$ and $y \in D_n$ with $\delta^n(y) = f(p)$, there exists a $z \in C_n$ with $f(z) = y$.
- The cofibrations are all those maps which lift on the left of the acyclic fibrations.

From this, we easily obtain

4.13. Theorem. *The category of simplicial T-complexes admits a model structure where*

- *The weak equivalences are the weak equivalences of underlying simplicial sets.*
- *The fibrations are the Kan fibrations of underlying simplicial sets.*

Proof. The model structure is the one transferred along the equivalence of categories with the category of crossed complexes. By Proposition 6.2 in [Brown and Higgins, 1991], the fibrations are the Kan fibrations on underlying simplicial sets. By 3.8, the weak equivalences are the weak equivalences on underlying simplicial sets. ∎

4.14. Corollary. *The adjunction $St_{Alg} \dashv U_{Alg}$ is a Quillen adjunction.*

We recall from earlier that the forgetful functor $U_A : \mathrm{AlgKan} \to \mathrm{sSet}$ is the right adjoint of a Quillen adjunction.

4.15. Definition. *The functor $F_A : \mathrm{sSet} \to \mathrm{AlgKan}$ is the left adjoint to the forgetful functor $U_A : \mathrm{AlgKan} \to \mathrm{sSet}$, as defined in [Nikolaus, 2011].*

4.16. Definition. *We define $St : \mathrm{sSet} \to \mathrm{sTCom}$ as the composition $St_{Alg} \circ F_A$ and $U_{St} : \mathrm{sTCom} \to \mathrm{sSet}$ as $U_A \circ U_{Alg}$.*

4.17. Corollary. *The adjunction $St \dashv U_{St}$ is a Quillen adjunction.*

Proof. In [Nikolaus, 2011] it is proven that $F_A \dashv U_A$ is a Quillen adjunction. As a composition of Quillen adjunctions is a Quillen adjunction, the result follows. ∎

The functor St therefore represents the universal "strictification" of the ∞-groupoid represented by an arbitrary simplicial set. It is natural to ask how lossy this functor is. An easy general example is the following:

4.18. Theorem. *Let X be an n-skeletal simplicial set. Then $St(X)$ is $(n+1)$-coskeletal.*

Proof. Note that by construction of F_A, every simplex of dimension $n+1$ or higher is degenerate, a distinguished filler, or a composition of simplices which are either degenerate or distinguished fillers. It follows then that in $\mathrm{St}(X)$ every simplex of dimension $n+1$ or higher is thin. Let $B : \partial\Delta^N \to X$ be the inclusion of a boundary, where $N \geq n+2$. Then note that a filler of any horn must be a filler for the boundary, by axiom (3). Of course, any filler for the boundary is also a filler for each horn. Hence, by existence and uniqueness of horn fillers, it must be the case that there is a unique extension of B to Δ^N. ∎

4.19. Remark. This generalizes that if X is n-skeletal, then $\mathbb{Z}[X]$ is $(n+1)$-coskeletal.

4.20. Corollary. *Let X be an n-skeletal simplicial set. The map $\pi_\bullet(X) \to \pi_\bullet(St(X))$ contains in its kernel $\pi_{\geq n+1}(X)$.*

This motivates the following definition:

4.21. DEFINITION. *Let X be a simplicial set. The* higher homotopy groups *of X are the graded pieces of the kernel of the map $\pi_\bullet(F_A(X)) \to \pi_\bullet(St(X))$.*

We call these groups the "higher" homotopy groups because their dimension may exceed the geometric dimension of X; they are generated from lower dimensional data.

4.22. EXAMPLE. Let X be $S^1 \vee S^2$. Then the higher homotopy groups of X are precisely $\pi_n(X)$ for $n \geq 3$.

4.23. REMARK. A natural expectation would be that "strictification" of ∞-groupoids should be idempotent - after all, if X is an ω-groupoid, sitting inside the category of weak ∞-groupoids (spaces), the universal ω-groupoid with a map from X is just X itself. And indeed, the adjunction $\mathrm{St}_{\mathrm{Alg}} \dashv \mathrm{U}_{\mathrm{Alg}}$ is idempotent. However, this adjunction is "homotopically incorrect:" $\mathrm{St}_{\mathrm{Alg}}$ does not preserve weak equivalences, and so it is necessary to take the derived functors in order to get the correct strictification adjunction. The derived adjunction is *not* idempotent: for instance, $K(\mathbb{Z}, 2)$ "is" a strict ∞-groupoid (that is, has a model as a simplicial T-complex). However, the strictification is homotopy equivalent to (the basepoint component of) $\mathbb{Z}[K(\mathbb{Z}, 2)]$, by Theorem 3.1.

5. The induced coalgebra St(X) detects the homotopy type of X

The adjunction $\mathrm{St} \dashv \mathrm{U}_{\mathrm{St}}$ induces a free functor $\mathscr{F}_{\mathrm{St}} : \mathrm{sSet} \to \mathrm{CoAlg}(\mathrm{St})$, where CoAlg(St) is the category of coalgebras for the comonad $\mathrm{St} \circ \mathrm{U}_{\mathrm{St}}$ induced by the adjunction $\mathrm{St} \dashv \mathrm{U}_{\mathrm{St}}$. The free functor $\mathscr{F}_{\mathrm{St}}$ is given on objects by

$$\mathscr{F}_{\mathrm{St}}(X) = (\mathrm{St}(X), \mathrm{St}(\eta_X) : \mathrm{St}(X) \to (\mathrm{St} \circ \mathrm{U}_{\mathrm{St}})(\mathrm{St}(X)))$$

where η is the unit of the adjunction $\mathrm{St} \dashv \mathrm{U}_{\mathrm{St}}$. We will show in this section that $\mathscr{F}_{\mathrm{St}}(X)$ determines the homotopy type of X. In the following section we will upgrade to an equivalence of quasicategories. The precise theorem in this section is the following:

5.1. THEOREM. *For $X \in$ sSet, denote by $St^\bullet(X)$ the cosimplicial object of sSet whose nth space is $(U_{St} \circ St)^{n+1}(X)$ and whose coface and codegeneracy maps are induced by the coalgebra structure of $St(X)$. The natural map*

$$X \to \mathrm{holim}\,(St^\bullet(X))$$

is a weak equivalence.

Bousfield and Kan proved a similar result for nilpotent spaces and simplicial abelian groups in [Bousfield and Kan, 1987], which we rephrase as follows:

5.2. THEOREM. [[Bousfield and Kan, 1987], III.5.4] *Let $X \in$ sSet be nilpotent. Denote by $\tilde{\mathbb{Z}}^\bullet[X]$ the cosimplicial simplicial set whose nth space is $(\tilde{\mathbb{Z}}[-])^n(X)$, with coface and codegeneracy maps given by the coalgebra structure on $\tilde{\mathbb{Z}}[X]$. Then there is a natural weak equivalence*

$$X \to \mathrm{holim}\,(\tilde{\mathbb{Z}}^\bullet[X])$$

Recall that $\mathbb{Z}[-]$ is the functor which takes a simplicial set and applies the free abelian group functor at each level to produce a simplicial abelian group, and $\tilde{\mathbb{Z}}[-]$ is the functor which takes a *pointed* simplicial set and applies $\mathbb{Z}[-]$, and then quotients out by the subgroup generate by the sub-simplicial set consisting of the basepoint (and all its degeneracies). Our proof will essentially be an extension of the Bousfield-Kan result to the general case, using the identification between simply connected simplicial T-complexes and simply connected chain complexes that we established in section 3.

5.3. Remark. While the main result and the majority of the work in this paper is done unpointed, we will need to use pointed spaces in order to relate $\mathrm{St}(X)$ to $\tilde{\mathbb{Z}}[X]$ and thereby leverage the Bousfield-Kan completion. The adjunctions $\mathrm{St} \dashv \mathrm{U}_{\mathrm{St}}$ and $\mathrm{N}_{\mathrm{CrCom}} \dashv \Gamma_{\mathrm{CrCom}}$ naturally extend to pointed versions.

5.4. Lemma. *Let X be a simply connected pointed simplicial set. There is a natural weak equivalence*

$$U_{St}(St(X)) \to |\tilde{\mathbb{Z}}[X]|$$

Proof. Let $A \to X$ be a weak equivalence, with A a 1-reduced pointed simplicial set. We then have the commutative square

$$\begin{array}{ccc} \Gamma_{\mathrm{CrCom}}(\mathrm{St}(A)) & \longrightarrow & \mathrm{U}_{\mathrm{Cr}}(\tilde{\mathrm{Ab}}_{\mathrm{Cr}}(\Gamma_{\mathrm{CrCom}}(\mathrm{St}(A)))) \\ \downarrow & & \downarrow \\ \Gamma_{\mathrm{CrCom}}(\mathrm{St}(X)) & \longrightarrow & \mathrm{U}_{\mathrm{Cr}}(\tilde{\mathrm{Ab}}_{\mathrm{Cr}}(\Gamma_{\mathrm{CrCom}}(\mathrm{St}(X)))) \end{array}$$

Where $\tilde{\mathrm{Ab}}_{\mathrm{Cr}}$ being the reduced version of $\mathrm{Ab}_{\mathrm{Cr}}$. The downwards arrows are weak equivalences by basic model category theory, using that all objects of $\mathrm{Ch}^{+}_{\mathbb{Z}}$ and CrCom are fibrant and all objects of sSet are cofibrant. The top arrow is an isomorphism by Theorem 3.1. Therefore, the bottom horizontal arrow is a weak equivalence. Finally, the equivalence between the simplicial and the chain complex models (item (4) of Theorem 3.1) completes the argument. ■

5.5. Lemma. *Let $X \in$ sSet be connected. For $n \geq 2$ and for any choice of basepoint there is a natural isomorphism of functors*

$$H_n(\widehat{X}) \cong \pi_n(U_{St}(St(X)))$$

Where $\widehat{X}$ is the universal cover of X. For $n = 0, 1$ and $x \in X$ there is a natural isomorphism

$$\pi_n(X) \cong \pi_n(U_{St}(St(X)))$$

Proof. The $n = 0$ case follows readily from the construction, as adding horn fillers and quotienting co-thin simplices do not change π_0. The $n = 1$ case follows from 3.8, and that $H_1(\Gamma_{\mathrm{CrCom}}(\mathrm{St}(X)), x)$ is by definition $\pi_1(X_1, x)$ modulo the boundaries of 2-simplices. For the $n \geq 2$ case, in ([Brown et al., 2011], Section 8.4) it is proven that $H_n(\widehat{X}) \cong H_n(\Gamma_{\mathrm{CrCom}}(\mathrm{St}(X)))$. The identification then follows from Lemma 3.8. ■

5.6. Lemma. *The map $X \to U_{St}(St(X))$ is 1-connected.*

Proof. Note that that we have a factorization $X \to \mathrm{U}_{\mathrm{St}}(\mathrm{St}(X)) \to |\mathbb{Z}[X]|$, which when we apply π_1 gives us $\pi_1(X) \to \pi_1(X) \to \pi_1(X)^{\mathrm{ab}}$ by 5.5, where the composite is the abelianization. Let $\alpha \in \pi_1(X)$. Considering a map $S^1 \to X$ representing α and applying functoriality of the sequence, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \pi_1(X) & \longrightarrow & \pi_1(\mathrm{U}_{\mathrm{St}}(\mathrm{St}(X))) \end{array}$$

Where the top arrow is an isomorphism, as if the composition $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, each map must be. We then see that $\pi_1(X) \to \pi_1(\mathrm{U}_{\mathrm{St}}(\mathrm{St}(X))$ must be the identity under the natural isomorphism $\pi_1(X) \cong \pi_1(\mathrm{U}_{\mathrm{St}}(\mathrm{St}(X)))$. ■

5.7. Lemma. *If G is a 1-type (that is, if $\pi_i(G) \cong 0$ for $i > 1$), then the unit $G \to U_{St}(St(G))$ is a weak equivalence.*

Proof. This follows from Lemmas 5.6 and 5.5. ■

5.8. Definition. *By a* homotopy fibre sequence *we shall mean a sequence of pointed simplicial sets $X \to Y \to Z$, such that*

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a homotopy pullback square.

For our purposes, it will really be sufficient just to know that the property of being a homotopy fibre sequence is invariant under weak equivalence of diagrams, and that for a pointed, connected simplicial set X the sequence

$$\widehat{X} \to X \to X_{\leq 1}$$

where again $\widehat{X}$ is the universal cover, and $X_{\leq 1}$ is the 2-coskeleton of X, is a homotopy fibre sequence.

5.9. Lemma. *If $Y \to X \to G$ is a fibre sequence of pointed, connected simplicial sets where G is a 1-type and $X \to G$ induces an isomorphism on π_1, then*

$$U_{St}(St(Y)) \to U_{St}(St(X)) \to U_{St}(St(G))$$

is a fibre sequence.

PROOF. The given hypotheses tell us that $Y \to X \to G$ is weak homotopy equivalent to the standard homotopy fibre sequence $\widehat{X} \to X \to X_{\leq 1}$. As $\mathrm{U_{St}}$ and St preserve weak equivalences, it suffices to show therefore that

$$\mathrm{U_{St}}(\mathrm{St}(\widehat{X})) \to \mathrm{U_{St}}(\mathrm{St}(X)) \to \mathrm{U_{St}}(\mathrm{St}(X_{\leq 1}))$$

is a homotopy fibre sequence, which we will do by showing it is equivalent to the standard homotopy fibre sequence

$$\mathrm{U_{St}}\widehat{(\mathrm{St}(X})) \to \mathrm{U_{St}}(\mathrm{St}(X)) \to \mathrm{U_{St}}(\mathrm{St}(X))_{\leq 1}$$

By 5.6 and 5.7, the map $\mathrm{U_{St}}(\mathrm{St}(X)) \to \mathrm{U_{St}}(\mathrm{St}(X_{\leq 1}))$ is 1-connected and $\mathrm{U_{St}}(\mathrm{St}(X_{\leq 1}))$ is a 1-type. Hence, it is weak equivalent to $\mathrm{U_{St}}(\mathrm{St}(X))_{\leq 1}$. Further, we can lift our map $\mathrm{U_{St}}(\mathrm{St}(\widehat{X})) \to \mathrm{U_{St}}(\mathrm{St}(X))$ to get the dashed arrow in

$$\begin{array}{ccc} & & \mathrm{U_{St}}\widehat{(\mathrm{St}(X})) \\ & \nearrow\!\!\!\text{- - -} & \downarrow \\ \mathrm{U_{St}}(\mathrm{St}(\widehat{X})) & \longrightarrow & \mathrm{U_{St}}(\mathrm{St}(X)) \end{array}$$

Invoking Lemma 5.5, we obtain that this map is an isomorphism on π_n for $n > 2$ and therefore is a weak equivalence. Putting this together, we have a weak equivalence of diagrams

$$\begin{array}{ccccc} \mathrm{U_{St}}(\mathrm{St}(\widehat{X})) & \longrightarrow & \mathrm{U_{St}}(\mathrm{St}(X)) & \longrightarrow & \mathrm{U_{St}}(\mathrm{St}(X_{\leq 1})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{U_{St}}\widehat{(\mathrm{St}(X})) & \longrightarrow & \mathrm{U_{St}}(\mathrm{St}(X)) & \longrightarrow & \mathrm{U_{St}}(\mathrm{St}(X))_{\leq 1} \end{array}$$

In which the left vertical map and the right vertical map are the two weak equivalences we just described, and the middle vertical map is the identity. We therefore have a weak equivalence of diagrams as desired. ∎

PROOF PROOF OF THEOREM 5.1. First consider the case where X is connected, and pick a basepoint for X. We have a fibre sequence $\widehat{X} \to X \to X_{\leq 1}$, where $X_{\leq 1}$ is the 2-coskeleton of X and $\widehat{X}$ is therefore (up to homotopy) a universal cover of X. We consider the following diagram:

$$\begin{array}{ccccccccc} \widehat{X} & \longrightarrow & \mathrm{holim}(\mathrm{St}^\bullet(\widehat{X})) & \longrightarrow & (\mathrm{U_{St}} \circ \mathrm{St})(\widehat{X}) & \rightrightarrows & (\mathrm{U_{St}} \circ \mathrm{St})^2(\widehat{X}) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X & \longrightarrow & \mathrm{holim}(\mathrm{St}^\bullet(X)) & \longrightarrow & (\mathrm{U_{St}} \circ \mathrm{St})(X) & \rightrightarrows & (\mathrm{U_{St}} \circ \mathrm{St})^2(X) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X_{\leq 1} & \longrightarrow & \mathrm{holim}(\mathrm{St}^\bullet(X_{\leq 1})) & \longrightarrow & (\mathrm{U_{St}} \circ \mathrm{St})(X_{\leq 1}) & \rightrightarrows & (\mathrm{U_{St}} \circ \mathrm{St})^2(X_{\leq 1}) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \cdots \end{array}$$

By Lemma 5.9, each of the sequences

$$(\mathrm{U}_{\mathrm{St}} \circ \mathrm{St})^n(\widehat{X}) \longrightarrow (\mathrm{U}_{\mathrm{St}} \circ \mathrm{St})^n(X) \longrightarrow (\mathrm{U}_{\mathrm{St}} \circ \mathrm{St})^n(X_{\leq 1})$$

is a fibre sequence. Since homotopy limits commute, it follows that

$$\mathrm{holim}(\mathrm{St}^\bullet(\widehat{X})) \longrightarrow \mathrm{holim}(\mathrm{St}^\bullet(X)) \longrightarrow \mathrm{holim}(\mathrm{St}^\bullet(X_{\leq 1}))$$

is a fibre sequence. By Lemma 5.4, we have a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \mathrm{holim}(\mathrm{St}^\bullet(\widehat{X})) \\ & \searrow & \downarrow \\ & & \mathrm{holim}(\tilde{\mathbb{Z}}^\bullet(\widehat{X})) \end{array}$$

Where the downwards arrows is a weak equivalence. By 5.2, $\widehat{X} \to \mathrm{holim}(\tilde{\mathbb{Z}}^\bullet(\widehat{X}))$ is a weak equivalence, so $\widehat{X} \to \mathrm{holim}((\mathrm{U}_{\mathrm{St}} \circ \mathrm{St})^\bullet(\widehat{X}))$ is as well. By Lemma 5.7, we get a commutative diagram

$$\begin{array}{ccc} X_{\leq 1} & \longrightarrow & \mathrm{holim}(\mathrm{St}^\bullet(X_{\leq 1})) \\ & \searrow & \downarrow \\ & & X_{\leq 1} \end{array}$$

Where the downwards arrow is a weak equivalence. Therefore, the map

$$X_{\leq 1} \to \mathrm{holim}(\mathrm{St}^\bullet(X_{\leq 1}))$$

is a weak equivalence. It follows by the five lemma that $X \to \mathrm{holim}(\mathrm{St}^\bullet(X))$ is a weak equivalence.

Finally, for the general case (where X may not be connected): Both the functors U_{St} and St commute with coproducts, and therefore the map $X \to \mathrm{holim}(\mathrm{St}^\bullet(X))$ is the coproduct of the map on each of the connected components, from which the result follows. ∎

6. The adjunction $\mathrm{St} \dashv \mathrm{U}_{\mathrm{St}}$ induces a comonadic adjunction of ∞-categories

In this section we will do a little more work to achieve our main goal: providing an equivalence of homotopy theories between spaces and coalgebras in sTCom for the comonad $\mathrm{St} \circ \mathrm{U}_{\mathrm{St}}$. We will do this via an application of the Barr–Beck–Lurie theorem. The main theorem of this section is

6.1. THEOREM. *Let $\mathcal{L} : \mathscr{sSet} \rightleftarrows \mathscr{sTCom} : \mathcal{R}$ be the adjunction of quasicategories induced by the Quillen adjunction $St : \mathrm{sSet} \rightleftarrows \mathrm{sTCom} : U_{St}$. Then $\mathcal{L} \dashv \mathcal{R}$ is comonadic: that is, it induces an equivalence of quasicategories between the quasicategory $\mathscr{sSet}$ and the quasicategory of coalgebras for the comonad $\mathcal{L} \circ \mathcal{R}$.*

6.2. The induced functor of quasicategories. In general, a Quillen adjunction of model categories induces an adjunction of quasicategories. Here, our categories are rather nice: they are both simplicially enriched. It is therefore reasonable to expect that we might be able to use this simplicial enrichment to define the ∞-categorical adjunction we seek. Unfortunately, the functor U_{St} is not simplicial (see [Tonks, 2003]), which complicates using these tools. Therefore, we ignore entirely the underlying simplicial enrichment and use basic model categorical tools.

In ([Mazel-Gee, 2016], A.3), a general technique for obtaining adjunctions of quasicategories from model categories with functorial (co)fibrant replacement is described. Fortunately, we are in the simplest possible case: sSet has as its cofibrant replacement functor the identity functor, and sTCom has as its fibrant replacement functor the identity functor. Hence, the argument there allows us to conclude the following:

6.3. Theorem. *The adjunction $St \dashv U_{St}$ induces an adjunction of quasicategories $\mathcal{L} : s\mathcal{S}et \rightleftarrows s\mathcal{T}\mathcal{C}om : \mathcal{R}$ with unit transformation induced directly by the unit transformation $\mathbf{1}_{\mathrm{sSet}} \to U_{St} \circ St$.*

To see this argument more fleshed out, see the appendix.

6.4. Comonadicity. In order to prove our main theorem, we will apply the Barr-Beck-Lurie Theorem:

6.5. Theorem. [[Lurie, 2009], Theorem 4.7.2.2] *A functor $F : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$ if and only if it admits a right adjoint, is conservative, and preserves all limits of F-split coaugmented cosimplicial objects.*

Our aim is to prove that the functor $\mathcal{L} : s\mathcal{S}et \to s\mathcal{T}\mathcal{C}om$ exhibits $s\mathcal{S}et$ as comonadic over $s\mathcal{T}\mathcal{C}om$. Of course, we already have a right adjoint. Conservativity of $\mathcal{L}$ is, as it turns out, a classical fact:

6.6. Lemma. *$\mathcal{L} : s\mathcal{S}et \to s\mathcal{T}\mathcal{C}om$ is conservative.*

Proof. We want to prove that for a morphism $f : X \to Y$ in $s\mathcal{S}et$, if $\mathcal{L}(f)$ is a weak equivalence, then so is f. If $\mathcal{L}(f)$ is a weak equivalence, then it induces isomorphisms on all the homotopy groups of the underlying simplicial sets of $\mathcal{L}(X)$ and $\mathcal{L}(Y)$. Equivalently by Lemma 5.5, it induces an isomorphism on all homology groups of the associated crossed complexes. By 5.5, this is equivalent to saying that f induces an isomorphisms on $\pi_0(X) \to \pi_0(Y)$, $\pi_1(X) \to \pi_1(Y)$ for any choice of basepoint, and $H_n(\widehat{X}) \to H_n(\widehat{Y})$ for $n \geq 2$. By ([Dieck, 2010], 20.1.8), f is a weak homotopy equivalence. ∎

It therefore remains to verify the last condition about F-split coaugmented cosimplicial objects. The following lemma simplifies the work to be done:

6.7. Lemma. [[Holmberg-Peroux, 2020], Prop 6.1.4] *Given a pair of adjoint functors $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ in quasicategories, such that L is conservative. Then L is comonadic if and only if the map $X \to \lim_{\Delta}^{\mathcal{C}}(RL^{\bullet+1}X)$ is an equivalence for all objects X in $\mathcal{C}$.*

PROOF PROOF OF 6.1. By Lemma 6.6 and the above lemma, we only need to prove that for any $X \in \mathcal{sSet}$, the map $X \to \lim_{\Delta}^{\mathcal{C}}(\mathcal{R}\mathcal{L}^{\bullet+1}X)$ is an equivalence. However, by our construction of the quasicategorical adjunction, the cosimplicial object $\mathcal{R}\mathcal{L}^{\bullet+1}X$ is the same as the diagram $\mathrm{St}^{\bullet}X$ as in Theorem 5.1. Since homotopy limits compute limits in the associated quasicategory, this then follows directly by Theorem 5.1. ∎

6.8. REMARK. We have phrased our main theorem as being about simplicial sets and simplicial T-complexes, but of course from the equivalence of categories $\mathrm{sTCom} \rightleftarrows \omega\mathrm{Gpd}$ the analagous statement for ω-groupoids immediately follows.

7. Future Work

- The most immediate line of future inquiry is to extend this work to categories, not just groupoids: that is, construct adjunctions between categories of strict and weak (∞, n) categories, and determine if these are comonadic. Perhaps the most full extension along these lines would be to construct for any $m \geq n$ a strictification functor from weak (m, n)-categories to strict (m, n)-categories, and of course show that the resulting diagram of homotopy theories indexed by $\mathbb{N}^2 \times [1]$ commutes.

- One consequence of this work is a globular model for homotopy types: the objects are strict ∞-groupoids equipped with a coalgebra map for the comonad $\mathrm{U}_{\mathrm{St}}\Gamma_{\mathrm{CrCom}} \circ \mathrm{N}_{\mathrm{CrCom}} \circ \mathrm{St}$. However,this description strongly relies on simplicial sets as an intermediary to describe the comonad. Thus, one goal is to construct a comonad of $\omega\mathrm{Gpd}$ which is equivalent to the comonad induced by the comonad on sTCom, but which is more "inherently globular."

- Although this work proves that ω-groupoids can be used to model weak ∞-groupoids, it does not provide a convenient model category of coalgebras. A strengthening of this result would be to construct a model structure on the category of coalgebras for the comonad, along with a Quillen equivalence to the model category of simplicial sets.

A. The passage to quasicategories

In this section we flesh out in greater detail the passage from the Quillen adjunction $\mathrm{St} : \mathrm{sSet} \rightleftarrows \mathrm{sTCom} : \mathrm{U}_{\mathrm{St}}$ to the adjunction of quasicategories $\mathcal{L} : \mathcal{sSet} \rightleftarrows \mathcal{sTCom} : \mathcal{R}$, in order to fully justify our results. There is nothing original in this section: we simply flesh out some of the argument in [Mazel-Gee, 2016] as it applies to our particular case of interest, in order to make it more accessible to the non-expert reader.

In order to obtain a quasicategory from a relative category (in particular, a model category), we shall pass through complete Segal spaces, which are the fibrant objects in a certain model structure on bisimplicial sets. We will not exposit much of the theory of complete Segal spaces, but direct the interested reader to [Rezk, 1998]. We begin with our model category, viewed as a relative category $(M, \mathcal{W})$. We then apply the following two functors:

$$\mathrm{RelCat} \xrightarrow{N_r} \mathrm{ssSet} \xrightarrow{i_1^*} \mathrm{sSet}$$

The first functor N_r is called the "Rezk nerve" (originally called the "classifying diagram" in [Rezk, 1998]). It is given as follows:

A.1. Definition. *Given a relative category $(M, \mathcal{W})$, the Rezk nerve $N_r(M)$ is given by $(n, m) \mapsto Fun([n], Core(Fun([m], M)))$.*

Here "Core" denotes the subcategory $\mathcal{W} \subset M$. Importantly, N_r preserves products, and so a natural transformation $M \times I \to M$ is sent to $N_r(M) \times N_r(I) \to N_r(M)$. Here I is the standard interval category, equipped with the relative category structure where only the isomorphisms (the two identity arrows) are weak eqivalences. We wish to obtain a quasicategory from this. First, we fibrantly replace $N_r(M)$ in the Reedy model structure to get a bisimplicial set denoted $N_R^f(M)$, which is a complete Segal space. We note that $N_R(I)$ is already a complete Segal space, as $N_r(C)$ is a complete Segal space whenever C is equipped with the minimal relative category structure. Applying this all to our starting situation of the unit natural transformation of the adjunction $\mathrm{St} \dashv \mathrm{U_{St}}$, we have a map

$$N_r^f(\mathrm{sTCom}) \times N_r(I) \to N_r(\mathrm{sTCom})$$

whose component at any object of $N_r^f(\mathrm{sTCom})$ is precisely the unit map $X \to \mathrm{U_{St}}(\mathrm{St}(X))$. We now apply the functor i_1^* of [Joyal and Tierney, 2006] to obtain a quasicategory. Conveniently, this functor merely restricts a bisimplicial set $X_{\star\star}$ to its first row $X_{\star 0}$, and so we have now a diagram

$$i_1^* \left(N_r^f(\mathrm{sSet})\right) \times N(I) \to i_1^* \left(N_r^f(\mathrm{sSet})\right)$$

Where $N(I)$ is the ordinary nerve of the category I. We take $i_1^* \left(N_r^f(\mathrm{sSet})\right)$ as our model for the quasicategory of simplicial sets $\mathcal{sSet}$ (and similarly $i_1^* \left(N_r^f(\mathrm{sTCom})\right)$ models the quasicategory of simplicial T-complexes $\mathcal{sTCom}$). This map is a candidate unit map for the quasicategorical adjunction $\mathcal{L} : \mathcal{sSet} \rightleftarrows \mathcal{sTCom} : \mathcal{R}$ in the sense of ([Lurie, 2009], Definition 5.2.2.7); to show that it truly provides a quasicategorical adjunction it must be shown that it induces weak equivalences

$$\underline{Map}_{\mathcal{sTCom}}(\mathcal{L}(x), y) \to \underline{Map}_{\mathcal{sSet}}(\mathcal{R}(\mathcal{L}(x)), \mathcal{R}(y)) \to \underline{Map}_{\mathcal{sSet}}(x, \mathcal{R}(y))$$

For any x and y objects in $\mathcal{sSet}$, where $\underline{Map}_{\mathcal{C}}$ is the mapping space between objects in a quasicategory. This follows from the fact that these mapping spaces are functorially equivalent to the mapping spaces given by hammock localization, which in turn are equivalent by [Dwyer and Kan, 1980] to mapping spaces calculated via cosimplicial resolutions. This reduces the problem to checking the weak equivalence on the level of model categories, where we can apply ([Dwyer and Kan, 1980], Proposition 5.4) to conclude that the induced map of mapping spaces is an isomorphism.

We now have that the unit transformation for the adjunction of quasicategories can be taken to be the unit transformation at the level of categories (more precisely, that the component at any object X is the arrow $X \to (\mathrm{U}_{\mathrm{St}} \circ \mathrm{St})(X))$. Hence the canonical cosimplicial resolution induced by the adjunction of quasicategories is equivalent to the canonical cosimplicial resolution induced by the adjunction of model categories, and therefore addressing the question of whether it is a limit diagram in the quasicategory $\mathcal{sSet}$ reduces to the question of whether it is a homotopy limit diagram in the simplicial model category sSet, justifying our usage of Theorem 5.1 to prove Theorem 6.1.

References

D. Ara. On homotopy types modeled by strict infinity-groupoids: In memory of jean-louis loday. *Theory and Applications of Categories*, 28:552–576, 01 2013.

N. Ashley. *T-complexes and crossed complexes.* PhD thesis, 1978.

Jacobson R. Blomquist and John E. Harper. Integral chains and bousfield–kan completion. *Homology, Homotopy and Applications*, 21(2):29–58, 2019. doi: 10.4310/hha.2019.v21.n2.a4.

Aldridge K. Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and Localizations.* Springer, 1987.

Higgins-Philip J. Brown, Ronald. The equivalence of ∞-groupoids and crossed complexes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 22(4):371–386, 1981. URL http://eudml.org/doc/91280.

Ronald Brown and Marek Golasinski. A model structure for the homotopy theory of crossed complexes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 30 (1):61–82, 1989. URL http://www.numdam.org/item/CTGDC_1989__30_1_61_0/.

Ronald Brown and Philip J. Higgins. The classifying space of a crossed complex. *Mathematical Proceedings of the Cambridge Philosophical Society*, 110(1):95–120, 1991. doi: 10.1017/s0305004100070158.

Ronald Brown, Philip J. Higgins, and Rafael Sivera. *Nonabelian algebraic topology: Filtered spaces, crossed complexes, cubical homotopy groupoids.* European Mathematical Society, 2011.

Denis-Charles Cisinski. Batanin higher groupoids and homotopy types, 2006.

Michael Keith Dakin. *Kan complexes and multiple groupoid structures.* PhD thesis, 1977.

Tammo tom Dieck. *Algebraic topology.* European Mathematical Society, 2010.

W.G. Dwyer and D.M. Kan. Function complexes in homotopical algebra. *Topology*, 19 (4):427–440, 1980. doi: 10.1016/0040-9383(80)90025-7.

Simon Henry. Algebraic models of homotopy types and the homotopy hypothesis, 2016.

Maximilien Thomas Holmberg-Peroux. *Highly Structured Coalgebras and Comodules*. PhD thesis, 2020. URL https://indigo.uic.edu/articles/thesis/Highly_Structured_Coalgebras_and_Comodules/13475667.

André Joyal and Myles Tierney. Quasi-categories vs segal spaces. *arXiv: Algebraic Topology*, 2006. URL https://api.semanticscholar.org/CorpusID:119749421.

Jacob Lurie. *Higher Topos theory*. Princeton University Press, 2009.

Akhil Matthew. The Dold-Kan Correspondence. 2011. http://math.uchicago.edu/~amathew/doldkan.pdf.

Aaron Mazel-Gee. Quillen adjunctions induce adjunctions of quasicategories. 2016.

Thomas Nikolaus. Algebraic models for higher categories. *Indagationes Mathematicae*, 21(1):52–75, 2011. ISSN 0019-3577. doi: https://doi.org/10.1016/j.indag.2010.12.004. URL https://www.sciencedirect.com/science/article/pii/S0019357710000133.

Charles Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353:973–1007, 1998. URL https://api.semanticscholar.org/CorpusID:1140520.

Manuel Rivera, Felix Wierstra, and Mahmoud Zeinalian. The simplicial coalgebra of chains determines homotopy types rationally and one prime at a time. *Transactions of the American Mathematical Society*, Feb 2022. doi: 10.1090/tran/8579.

A.P. Tonks. On the eilenberg–zilber theorem for crossed complexes. *Journal of Pure and Applied Algebra*, 179(1–2):199–220, Apr 2003. doi: 10.1016/s0022-4049(02)00160-3.

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