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A NOTE ON THE RELATIONSHIP BETWEEN ACTION ACCESSIBLE AND WEAKLY ACTION REPRESENTABLE CATEGORIES

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ABSTRACT. The main purpose of this paper is to show that the converse of the known implication *weakly action representable implies action accessible* is false. In particular we show that both action accessibility, as well as the (at least formally stronger) condition requiring the existence of all normalizers do not imply weakly-action-representability even for varieties. In addition we show that in contrast to both action accessibility and the condition requiring the existence of all normalizers, weakly-action-representability is not necessarily inherited by Birkoff subcategories.

1. Introduction

Recall that for a pointed category $\mathbb{C}$, a split extension (of B with kernel X) is a diagram in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} \tag{1}$$

where (X, κ) is the kernel of α, and $\alpha\beta = 1_B$. A morphism of split extensions is a diagram in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, "u"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & B \arrow[l, shift left, "\beta"] \arrow[d, "w"] \\ X' \arrow[r, "\kappa'"] & A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"] \end{tikzcd} \tag{2}$$

where the top and bottom rows are split extensions (the domain and codomain respectively), and $v\kappa = \kappa' u$, $v\beta = \beta' w$ and $w\alpha = \alpha' v$. Let us denote by $\mathbf{SplExt}(\mathbb{C})$ the category of split extensions in $\mathbb{C}$, and by K and P the functors sending a split extension to its kernel and codomain, respectively. These data together form a span

$$\mathbb{C} \xleftarrow{P} \mathbf{SplExt}(\mathbb{C}) \xrightarrow{K} \mathbb{C}. \tag{3}$$

Recall also that when $\mathbb{C}$ is pointed protomodular [2], for each object X in $\mathbb{C}$, the assignment of each object B to the isomorphism class of split extensions Ξ with $K(\Xi) = X$ and $P(\Xi) = B$, determines a functor $\mathrm{SplExt}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ which assigns to each

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morphism $p : E \to B$ the morphism $\mathrm{SplExt}(p, X)$ defined by *pulling back along* p. The category $\mathbb{C}$ is action representable in the sense of [1] when each of these functors is representable and is weakly action representable [7] when for each X in $\mathbb{C}$ there exists a weak representation, that is there is a pair (M, μ) where M is an object in $\mathbb{C}$ and $\mu : \mathrm{SplExt}(-, X) \to \hom(-, M)$ is a monomorphism. Note that when the functor $\mathrm{SplExt}(-, X)$ is representable the representing object will be written $[\![X]\!]$.

Action representability can also be rephrased as requiring that for each X in $\mathbb{C}$ the fiber $K^{-1}(X)$ has a terminal object, and action accessibility [4] can be phrased, by the weakening of this, to instead require that for each X in $\mathbb{C}$ the fiber $K^{-1}(X)$ has *enough sub-terminal objects*, that is, each object admits a morphism into a sub-terminal object (= an object admitting at most one morphism into it from any object). The sub-terminal objects in $K^{-1}(X)$ are called faithful split extensions.

In [7], G. Janelidze proved for a semi-abelian category (in the sense of Janelidze, Marki, Tholen [8]) weakly-action-representability implies action accessibility (Theorem 4.6 of [7]). The main purpose of this paper is to show that the converse does not hold. We show that a Birkoff subcategory of a (weakly) action representable category is not necessarily weakly action representable. This should be contrasted with the fact that a Birkoff sub-category of an action accessible category is necessarily action accessible [4], and the immediate Proposition 2.1 below which shows if $\mathbb{C}$ is a category admitting all normalizers (in the sense of [5] or in the sense of [3]), then every full subcategory of $\mathbb{C}$ closed under sub-objects and finite limits, admits all normalizers. In particular we show that the category of n-solvable groups ($n \geq 3$) is action accessible and has normalizers, but is not weakly action representable.

2. The results

In this section we prove our main results.

Recall that the normalizer of a monomorphism $f : W \to X$ in [5] was defined to be the universal factorization of f as normal monomorphism followed by a monomorphism

$$\begin{tikzcd} W \arrow[r, "n"] \arrow[rr, bend right, "f"'] & N \arrow[r, "m"] & X. \end{tikzcd}$$

A different definition was given in [3], which in pointed, finitely complete context can be formulated as a commutative diagram

$$\begin{tikzcd} W \arrow[r, "\kappa"] \arrow[d, "n"] \arrow[dd, bend right, "f"'] & R \arrow[d, "\langle r_1, r_2 \rangle"] \\ N \arrow[r, "\langle 0,1 \rangle"] \arrow[d, "m"] & N \times N \\ X & \end{tikzcd}$$

where the upper square is a pullback and the morphisms $r_1, r_2 : R \to N$ are the projections of an equivalence relation, which is universal amongst such commutative diagrams. The two definitions coincide in the pointed exact protomodular context, where $r_1, r_2 : R \to N$ is necessarily the kernel pair of its coequalizer, which in turn is necessarily a normal epimorphism with kernel n.

2.1. Proposition. *Let $\mathbb{C}$ be a pointed category admitting normalizers (in either sense). If $\mathbb{X}$ is a full sub-category of $\mathbb{C}$ closed under subobjects and finite limits, then $\mathbb{X}$ admits normalizers (in the same sense).*

Proof. It is easy to check that under the conditions above the normalizer in $\mathbb{C}$ of a monomorphism f in $\mathbb{X}$ is also the normalizer of f in $\mathbb{X}$. ∎

Recall that a span of monomorphisms $m : S \to B$ and $m' : S \to B'$ in a category $\mathbb{C}$ can be amalgamated in $\mathbb{C}$ if there exist monomorphisms $u : B \to D$ and $u' : B' \to D$ in $\mathbb{C}$ such that $um = u'm'$.

2.2. Proposition. *Let $\mathbb{C}$ be an action representable category, and let $\mathbb{X}$ be a Birkoff subcategory of $\mathbb{C}$. The category $\mathbb{X}$ is not weakly action representable (and hence not action representable), if there exist monomorphisms $m : S \to B$ and $m' : S \to B'$ in $\mathbb{X}$, monomorphisms $u : B \to D$ and $u' : B' \to D$ in $\mathbb{C}$, and X in $\mathbb{X}$ with a monomorphism $v : D \to [\![X]\!]$ in $\mathbb{C}$ such that*

(i) $um = u'm'$;

(ii) m and m' cannot be amalgamated in $\mathbb{X}$;

(iii) the split extensions corresponding to vu and vu' in $\mathbb{C}$ are in $\mathbb{X}$.

Proof. The monomorphisms vu, vu', and $vum = vu'm'$ produce the span of faithful split extensions in $\mathbb{X}$.

$$\begin{tikzcd}
X \arrow[r, "\kappa"] \arrow[d, equal] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \\
X \arrow[r, "\lambda"] \arrow[d, equal] & R \arrow[u, "f"] \arrow[d, "g"'] \arrow[r, shift left, "\rho"] & S \arrow[l, shift left, "\sigma"] \arrow[u, "m"'] \arrow[d, "m'"] \\
X \arrow[r, "\kappa'"] & A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"]
\end{tikzcd}$$

If $\mathbb{X}$ were weakly action representable, then $\mathrm{SplExt}(-, X)$ would have weak representation M and there would (by Corollary 4.3 of [7]) be monomorphisms $i : B \to M$ and $i' : B' \to M$ in $\mathbb{X}$ such that $im = i'm'$. This is impossible since m and m' can't be amalgamated in $\mathbb{X}$. ∎

2.3. Example. Let $\mathbb{C}$ be the category of groups. Recall that: $\mathbb{C}$ is action representable [1] with $[\![X]\!] = \mathrm{Aut}(X)$ (the automorphism group of X), $\mathbb{C}$ admits normalizers (in the sense of [5] or equivalently – in this context – in the sense of [3]), and amalgamation holds in $\mathbb{C}$ (which according to [9] was first proved in [12]). It is well-known that every group can be embedded in the automorphism group of an abelian group (to prove this one can recall that every group can be embedded in the symmetric group on its underlying set, and the symmetric group on a set W can be embedded in $\mathrm{Aut}(\mathbb{Z}_2^W)$). Now let $\mathbb{X}$ be the sub-variety of n-solvable groups ($n \geq 3$). Neumann showed [11] that there exists an abelian group S, a 2-nilpontent group B and two monomorphisms $m : S \to B$ and $m' : S \to B$ which can't be amalgamated as a solvable group. Since n-nilpotent implies n-solvable, and split extensions with kernel abelian and codomain 2-solvable are at most 3-solvable it follows by the previous proposition that $\mathbb{X}$ is not weakly action representable (one may take D any group with $u : B \to D$ and $u' : B' \to D$ monomorphisms such that $um = u'm'$ and then take $X = \mathbb{Z}_2^D$). It seems worth pointing out that a finite such D does exist (see e.g. Corollary 15.2 of [10]). Action accessibility of $\mathbb{X}$ follows from Proposition 2.3 of [4] together with action accessibility of $\mathbb{C}$ (which in turn follows immediately from $\mathbb{C}$ being action representable). Note that $\mathbb{X}$ also has normalizers by Proposition 2.1. Action accessibility of $\mathbb{X}$ can then also be obtained from Proposition 4.5 of [3] (see also [6] where it is proved that action accessibility is equivalent to the existence of certain normalizers).

2.4. Remark. A further interesting application of Proposition 2.2 was suggested by the anonymous referee, and I hope that the referee will consider publishing this finding.

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