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CYCLIC SETS FROM THE NERVE OF A GROUP AND THE BÖHM-ŞTEFAN CONSTRUCTION

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Abstract. Böhm and Ştefan developed a general method of construction of cyclic objects from (op)algebras over distributive laws of monads. The goal of this note is to show that all the cyclic sets resulting from the twisted nerve of a group G arise from the Böhm-Ştefan construction.

1. Introduction

Duplicial objects are a variation of the well-known simplicial objects, which mostly appear in homotopy theory and homological algebra. More specifically, the concept of duplicial objects due to Dwyer-Kan [Dwyer & Kan, 1985], generalises the concept of cyclic objects due to Connes[Connes, 1983]. Cyclic objects are central in the definition of cyclic homology, which is one of the building blocks of Connes' noncommutative geometry [Connes, 1994]. For some variants of cyclic homology like Hopf-cyclic homology and equivariant cyclic homology one must go beyond cyclic objects. It is in this context that duplicial objects come into play in a natural way. The discovery of Hopf-cyclic homology and twisted cyclic homology[Connes & Moscovici, 2001, Kustermans, Murphy & Tuset, 2003] led to the study of role of coefficients in cyclic homology. Since then, more and more general constructions of cyclic or more generally duplicial objects from algebraic data have been developed and studied, see e.g. [Hajac, Khalkhali, Rangipour & Sommerhäuser, 2004, Kaygun, 2005, Böhm & Ştefan, 2008, Shapiro, 2020] and the references therein.

Furthermore, Böhm and Ştefan [Böhm & Ştefan, 2008] gave a general method of construction of cyclic objects using distributive laws [Beck, 1969]. Their work mainly considers what additional data is needed in order to turn a simplicial object formed from a comonad into a duplicial one. We understand from their work that in the presence of a second comonad and a distributive law between them, together with two natural transformations which served as additional structures on the coefficients, one is able to define a duplicial structure on the simplicial object.

Our main motivation arises from this question:

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1.1. Question. *When does a simplicial object in a category that computes the simplicial homology of an algebraic structure carry a duplicial structure?*

The main example is the canonical simplicial $\mathbb{F}$-module $C(U, M, N)$ whose n-simplices are the elements of $M \otimes U^{\otimes n} \otimes N$ and whose homology is $\mathrm{Tor}^{U/\mathbb{F}}(M, N)$, where U is an associative unital $\mathbb{F}$-algebra and M and N are right resp. left U-modules [Weibel, 1994, Section 8.7.1]. In the literature, this has been studied extensively when U is a Hopf algebra (or more generally a Hopf algebroid). In [Kowalzig, Kraehmer, Slevin, 2015], Kowalzig, Kraehmer and Slevin studied the above Question for this example from the perspective of the results of Böhm and Ştefan in [Böhm & Ştefan, 2008]. This leads to a complete description of the additional structures on M and N that turn $C(U, M, N)$ into a duplicial $\mathbb{F}$-module. However, this description is relatively vague and it was only made explicit for N, which is one of the two modules by realising $C(U, M, N)$ in terms of the bar resolution of it. Our focus in this paper is different.

A Hopf algebra in the category **Set** of sets is just a group G, and for any pair M, N of a left and a right G-set, we obtain a simplicial set $C(G, M, N)$ with simplices

$$C_n(G, M, N) := M \times G^{\times n} \tilde{\times} N \tag{1}$$

which for the trivial (one-point) sets M, N is the nerve of the group G. Note the use of $\tilde{\times}$ in (1); this is to distinguish between two comonad structures on the category **G-Set** of G-sets (see Section 2).

Now the nerve of a group G can always be turned into a cyclic set, see [Weibel, 1994, Loday, 1998]. Loday in [Loday, 1998, Section Section 7.3.3] discusses how this cyclic set is a special case of what he termed as the twisted nerve of a group. In both examples of the nerve of a group and the twisted nerve of the group, the underlying simplicial set has as simplices $C_n(G, \{*\}, \{*\}) := \{*\} \times G^{\times n} \times \{*\}$, where $\{*\}$ is a trivial G-set. In this paper we consider a more general simplicial set with simplices $C_n(G, M, N) := M \times G^{\times n} \tilde{\times} N$ where M and N are right and left G-sets respectively and classify all the duplicial structures on it thereby generalising the twisted nerve of a group. In order to classify all the duplicial structures on this simplicial set, we found it convenient to work with the simplicial set $C(G, \{*\}, N)$, which is isomorphic to the one in (1) (see Proposition 3.3). This simplicial set has as simplices $C_n(G, \{*\}, N) := \{*\} \times G^{\times n} \tilde{\times} N$. Nonetheless, we classify all the duplicial structures on the twisted nerve of a group without loss of generality.

It is therefore natural to ask whether the duplicial structures on this simplicial set arise from the Böhm-Ştefan construction. The contents of our computations can be summarised as follows:

1.2. Theorem. *Let G be a group and N a G-set, $M = \{*\}$ the trivial G-set and $C(G, \{*\}, N)$ the simplicial set associated to this data and $\alpha : N \to G$ any set map. Then the map*

$$t_n : \{*\} \times G^{\times n} \tilde{\times} N \to \{*\} \times G^{\times n} \tilde{\times} N$$

$$(g_1, \ldots, g_n, y) \mapsto (\alpha(y)(g_1 \cdot \ldots \cdot g_n)^{-1}, g_1, \ldots, g_{n-1}, g_n y)$$

is a duplicial operator on $C(G,\{*\},N)$ and any duplicial operator on $C(G,\{*\},N)$ is of this form. Furthermore, this duplicial operator arises from the Böhm-Ştefan construction.

2. Preliminaries and notation

2.1. The category **G-Set**. Throughout this paper, G denotes a (not necessarily finite) group (in the category **Set** of sets) with unit element 1. By a G-set N, we mean a set with a left action $\theta\colon G \times N \to N$ that we usually just write as concatenation, $gn := \theta(g,n)$. If M is a set with a right action of G, we consider it also as a G-set by setting $gm := mg^{-1}$, $g \in G, m \in M$. The G-sets form a category **G-Set** (see e.g. [MacLane, 1988, Leinster, 2014] for the notions from category theory used) whose morphisms $L \to N$ are the G-equivariant maps

$$\mathrm{Hom}_G(L,N) := \{f\colon L \to N \mid \forall g \in G, x \in L : f(gx) = gf(x)\}.$$

2.2. **G-Set** is Cartesian. The category **G-Set** is Cartesian, i.e. is a symmetric monoidal category in which every object is in a unique way a cocommutative comonoid. Concretely this means that given any G-sets L, N, the Cartesian product $L \times N$ of sets becomes a G-set with respect to the diagonal action

$$g(l,n) := (gl, gn),$$

and for a fixed L this operation extends to an endofunctor $S := L \times -$ on **G-Set**; furthermore, this endofunctor carries a unique (up to isomorphism) comonad structure which is given by the natural maps

$$\Delta_N\colon L \times N \to L \times L \times N, \quad (l,n) \mapsto (l,l,n),$$

$$\varepsilon_N\colon L \times N \to N, \quad (l,n) \mapsto n.$$

Throughout, we denote this comonad by $\mathbb{S} = (S, \Delta, \varepsilon)$.

2.3. The comonad $\mathbb{T}$. Given any set X and any G-set L, we can construct the G-set $FX := L \tilde{\times} X$ with action given by $g(a,x) := (ga,x)$. This extends in the obvious way to a functor $F\colon \mathbf{Set} \to \mathbf{G\text{-}Set}$. When $L = G$, this is left adjoint to the forgetful functor $U\colon \mathbf{G\text{-}Set} \to \mathbf{Set}$ that forgets the action of G. Hence we have a natural bijection

$$\begin{gathered}\theta : \mathrm{Hom}_{\mathbf{G\text{-}Set}}(FUN, SN) \to \mathrm{Hom}_{\mathbf{Set}}(UN, USN) \\ \theta(\rho)(x) := \rho(1,x)\end{gathered} \tag{2}$$

which has as inverse the map

$$\begin{gathered}\xi : \mathrm{Hom}_{\mathbf{Set}}(UN, USN) \to \mathrm{Hom}_{\mathbf{G\text{-}Set}}(FUN, SN) \\ \xi(\bar{\rho})(a,x) := a\bar{\rho}(x).\end{gathered} \tag{3}$$

Now the composition $T := FU$ becomes (part of) a comonad $\mathbb{T} = (T, \tilde{\Delta}, \tilde{\varepsilon})$ on **G-Set** whose structure maps are given by

$$\tilde{\Delta}_N \colon FN \to FFN, \quad (g, n) \mapsto (g, 1, n),$$

$$\tilde{\varepsilon}_N \colon FN \to N, \quad (g, n) \mapsto gn.$$

3. Bar construction

3.1. The simplicial set $C(G, M, N)$. The bar construction produces simplicial objects out of comonads. Recall the free-forgetful adjunction in Section 2, from which we obtain the comonad $\mathbb{T}$ on **G-Set**. We use this comonad to obtain a simplicial set from the bar construction.

Let $N : \mathbf{1} \to \mathbf{G\text{-}Set}$ be a functor from the terminal category to the category **G-Set** of G-sets so that N can be regarded as a G-set. Next let $\tilde{M} : \mathbf{G\text{-}Set} \to \mathbf{Set}$ be a functor that sends any G-set X to $(M \times X)/G$, the orbit of the G-set $M \times G$, where M is any right G-set. Then the bar resolution $\tilde{M}B_*(\mathbb{T}, N) : \mathbf{1} \to \mathbf{Set}$ of the functor N is

$$\bar{C}_*(M, G, N) := (M \times G^{\times(*+1)} \tilde{\times} N)/G,$$

that is, the set of G-orbits in $M \times G^{\times(*+1)} \tilde{\times} N$. Then $\bar{C}(M, G, N)$ is a simplicial set with n-simplices as $\bar{C}_n(M, G, N) := (M \times G^{\times(n+1)} \tilde{\times} N)/G$, and face and degeneracy operators defined as follows:

$$d_i([x, g_1, \dots, g_{n+1}, y]) := \begin{cases} [g_1^{-1}x, g_2, \dots, g_{n+1}, y] & \text{if } i = 0 \\ [x, g_1, \dots, g_{i+1}g_{i+2}, \dots, g_{n+1}, y] & \text{if } 0 < i < n \\ [x, g_1, \dots, g_n, g_{n+1}y] & \text{if } i = n \end{cases}$$

$$s_i([x, g_1, \dots, g_{n+1}, y]) := [x, g_1, \dots, g_i, 1, g_{i+1}, \dots, g_{n+1}, y]$$

where $[x, g_1, \dots, g_{n+1}, y] \in (M \times G^{\times(n+1)} \tilde{\times} N)/G$.

Now identify $(M \times G^{\times(n+1)} \tilde{\times} N)/G \cong M \times G^{\times n} \tilde{\times} N$ via

$$[x, g_1, \dots, g_{n+1}, y] \mapsto (g_1^{-1}x, g_2, \dots, g_{n+1}, y). \tag{4}$$

It is straightforward to verify that this map is bijective with an inverse map given by

$$(x, g_1, \dots, g_n, y) \mapsto [x, 1, g_1, \dots, g_n, y].$$

Therefore under this identification the above face and degeneracy maps are written as follows:

$$d_i(x, g_1, \dots, g_n, y) := \begin{cases} (g_1^{-1}x, g_2, \dots, g_n, y) & \text{if } i = 0 \\ (x, g_1, \dots, g_i g_{i+1}, \dots, g_n, y) & \text{if } 0 < i < n \\ (x, g_1, \dots, g_{n-1}, g_n y) & \text{if } i = n \end{cases}$$
$$s_i(x, g_1, \dots, g_n, y) := (x, g_1, \dots, g_{i-1}, 1, g_i, \dots, g_n, y)$$

where $(x, g_1, \dots, g_n, y) \in (M \times G^{\times n} \tilde{\times} N)$.

3.2. Example. Let $M = N$ be the trivial G-set $\{*\}$ then by the identification in (4),

$$\begin{gathered} (\{*\} \times G \tilde{\times} \{*\})/G \cong \{*\} \times \{*\} \cong \{*\} \\ (\{*\} \times G^{\times 2} \tilde{\times} \{*\})/G \cong \{*\} \times G \tilde{\times} \{*\} \cong G \\ \vdots \\ (\{*\} \times G^{\times (n+1)} \tilde{\times} \{*\})/G \cong \{*\} \times G^{\times n} \tilde{\times} \{*\} \cong G^{\times n} \end{gathered}$$

thus the nerve of G [Weibel, 1994, Loday, 1998].

It turns out that without loss of generality we can focus on the case when M is the trivial G-set. This is justified by the following known result [Feng & Tsygan, 1991]:

3.3. Proposition. *The simplicial sets $C(G, M, N)$ and $C(G, \{*\}, \mathrm{ad}(N \times M))$ where $\mathrm{ad}(N \times M)$ is $N \times M$ as a set with conjugation action, are isomorphic as simplicial sets via the map*

$$\begin{gathered} \theta : C(G, M, N) \to C(G, \{*\}, \mathrm{ad}(N \times M)), \\ \theta_j(x, g_1, \dots, g_j, y) \mapsto (g_1, \dots, g_j, y, x(g_1 \cdot \ldots \cdot g_j)), \quad j \geq 0. \end{gathered}$$

4. Duplicial structures on $C(G, \{*\}, N)$

4.1. Duplicial sets. Our aim is to enrich the simplicial set $C(G, \{*\}, N)$ by defining an additional operator $t : C(G, \{*\}, N) \to C(G, \{*\}, N)$ which is compatible with the face and degeneracy maps as follows:

$$d_i t = \begin{cases} t d_{i-1} & 1 \leq i \leq n \\ d_n & i = 0 \end{cases} \tag{5}$$

$$s_i t = \begin{cases} t s_{i-1} & 1 \leq i \leq n \\ t^2 s_n & i = 0. \end{cases} \tag{6}$$

4.2. Definition. *In such a case one says that $(C(G, \{*\}, N), s_i, d_i, t)$ is a duplicial set. If in addition $t^{n+1} = \mathrm{id}$ holds, one says that $(C(G, \{*\}, N), s_i, d_i, t)$ is a cyclic set.*

4.3. Example. Let $C(G,\{*\},\{*\})$ be the classifying space of G and define the operator $t : C_n(G,\{*\},\{*\}) \to C_n(G,\{*\},\{*\})$ by

$$(g_1,\dots,g_n) \mapsto ((g_1\cdot\ldots\cdot g_n)^{-1}, g_1,\dots,g_{n-1}).$$

Then $(C(G,\{*\},\{*\}), s_i, d_i, t)$ is a cyclic set [Weibel, 1994, Example 9.6.2].

In what follows we give a classification of all the duplicial structures on $C(G,\{*\},N)$.

4.4. Theorem. *Consider the simplicial set $C(G,\{*\},N)$ and let $\alpha : N \to G$ be any set map. Then for any $n \in \mathbb{N}$, the map*

$$t : \{*\} \times G^n \tilde{\times} N \to \{*\} \times G^n \tilde{\times} N$$

$$(g_1,\dots,g_n,y) \mapsto (\alpha(y)(g_1\cdot\ldots\cdot g_n)^{-1}, g_1,\dots,g_{n-1}, g_n y)$$

is a duplicial operator on $C(G,\{*\},N)$ and any duplicial operator on $C(G,\{*\},N)$ is of this form.

Proof. It is straightforward to verify that t is a duplicial operator on $C(G,\{*\},N)$, i.e, that the duplicial relations (5), (6) are satisfied. Assume now that

$$t\colon \{*\} \times G^{\times n} \tilde{\times} N \to \{*\} \times G^{\times n} \tilde{\times} N$$

is any duplicial operator and define maps $f_i : \{*\} \times G^{\times n} \tilde{\times} N \to G, (1 \le i \le n)$ and $f_y : \{*\} \times G^{\times n} \tilde{\times} N \to N$ by

$$t_n(g_1,\dots,g_n,y) =: (f_1(g_1,\dots,g_n,y),\dots,f_n(g_1,\dots,g_n,y), f_y(g_1,\dots,g_n,y)).$$

In what follows we obtain conditions on the maps for which t satisfies the duplicial relations (5), (6). We proceed by induction on $n \in \mathbb{N} = \{0,1,\dots\}$. Take $n = 2$, then

$$\begin{aligned} d_1 t(g_1,g_2,y) &= t d_0(g_1,g_2,y) \\ &\Leftrightarrow \\ f_1(g_1,g_2,y) f_2(g_1,g_2,y) &= f_1(g_2,y) \\ f_y(g_1,g_2,y) &= f_y(g_2,y) \end{aligned} \tag{7}$$

$$\begin{aligned} d_0 t(g_1,g_2,y) &= d_2(g_1,g_2,y) \\ &\Leftrightarrow \\ f_2(g_1,g_2,y) &= g_1 \\ f_y(g_1,g_2,y) &= g_2 y \end{aligned} \tag{8}$$

From equations (7) and (8) we have that

$$f_1(g_1g_2, y) = f_1(g_2, y)g_1^{-1}.$$

In particular if $g_2 = 1$, then we obtain

$$f_1(g_1, y) = f_1(1, y)g_1^{-1}, \tag{9}$$

that is

$$f_1(g_1, y) = \alpha(y)g_1^{-1}$$

where $\alpha : N \to G$ is any set map. Thus

$$f_1(g_1, g_2, y) = f_1(g_1g_2, y) = \alpha(y)(g_1g_2)^{-1}. \tag{10}$$

So for $n = 2$,

$$\begin{aligned} t(g_1, g_2, y) &=: (f_1(g_1, g_2, y), f_2(g_1, g_2, y), f_y(g_1, g_2, g_3)) \\ &\Leftrightarrow \\ f_1(g_1, g_2, y) &= \alpha(y)(g_1g_2)^{-1} \\ f_2(g_1, g_2, y) &= g_1 \\ f_y(g_1, g_2, y) &= g_2y \end{aligned}$$

Now suppose that for $n = k$, the following holds:

$$\begin{aligned} t(g_1, \ldots, g_k, y) &=: (f_1(g_1, \ldots, g_k, y), \ldots, f_g(g_1, \ldots, g_k, y), f_y(g_1, \ldots, g_k, y)) \Leftrightarrow \\ f_1(g_1, \ldots, g_k, y) &= \alpha(y)(g_1g_2 \ldots g_g)^{-1} \\ f_2(g_1, \ldots, g_k, y) &= g_1 \\ f_3(g_1, \ldots, g_k, y) &= g_2 \\ &\vdots \\ f_g(g_1, \ldots, g_k, y) &= g_{k-1} \\ f_y(g_1, \ldots, g_k, y) &= g_ky. \end{aligned}$$

Let $n = k + 1$. Then

$$\begin{aligned} d_1t(g_1, g_2, \ldots, g_{k+1}, y) &= td_0(g_1, g_2, \ldots, g_{k+1}, y) \Leftrightarrow \\ f_1(g_1, \ldots, g_{k+1}, y)f_2(g_1, \ldots, g_{k+1}, y) &= f_1(g_2, \ldots, g_{k+1}, y) \\ f_3(g_1, \ldots, g_{k+1}, y) &= g_2 \\ f_4(g_1, \ldots, g_{k+1}, y) &= g_3 \\ &\vdots \\ f_{g+1}(g_1, \ldots, g_{k+1}, y) &= g \\ f_y(g_1, \ldots, g_{k+1}, y) &= g_{k+1}y \end{aligned} \tag{11}$$

$$\begin{aligned} d_2t(g_1, \ldots, g_{k+1}, y) &= td_1(g_1, \ldots, g_{k+1}, y) \\ &\Leftrightarrow \\ f_1(g_1, \ldots, g_{k+1}, y) &= \alpha(y)(g_1g_2 \ldots g_{k+1})^{-1} \\ f_2(g_1, \ldots, g_{k+1}, y)f_3(g_1, \ldots, g_{k+1}, y) &= f_2(g_1g_2, \ldots, g_{k+1}, y) = g_1g_2. \end{aligned} \tag{12}$$

Now from equations (11) and (12) we obtain $f_2(g_1, \ldots, g_{k+1}, y) = g_1$. Thus for all n in $\mathbb{N}$,

$$\begin{aligned} t(g_1, \ldots, g_n, y) &=: (f_1(g_1, \ldots, g_n, y), \ldots, f_n(g_1, \ldots, g_n, y), f_y(g_1, \ldots, g_n, y)) \Leftrightarrow \\ f_1(g_1, \ldots, g_n, y) &= \alpha(y)(g_1 \ldots g_n)^{-1} \\ f_2(g_1, \ldots, g_n, y) &= g_1 \\ &\vdots \\ f_n(g_1, \ldots, g_n, y) &= g_{n-1} \\ f_y(g_1, \ldots, g_n, y) &= g_ny. \end{aligned}$$

Therefore $t(g_1, \ldots, g_n, y) = (\alpha(y)(g_1 \ldots, g_n)^{-1}, g_1, g_2, \ldots, g_{n-1}, g_ny)$ is a duplicial operator on $C(G, \{*\}, N)$ for any set map $\alpha : N \to G$ and any duplicial operator on $C(G, \{*\}, N)$ is of this form. ■

4.5. Corollary. *Let N be a trivial G-set, then the duplicial operator*

$$t : C(G, \{*\}, N) \to C(G, \{*\}, N)$$

$$t_n(g_1, \ldots, g_n, y) = (\alpha(y)(g_1 \ldots g_n)^{-1}, g_1, \ldots, g_{n-1}, y)$$

is cyclic if and only if $\alpha(y)$ is in the center of the group G.

PROOF. Suppose that the duplicial operator t is cyclic, then for any $n \in \mathbb{N}$, $t_n^{n+1} = \mathrm{id}$. To be more specific, composing t_n with itself n times we obtain

$$t_n^n(g_1, g_2, \ldots, g_n, y) = (\alpha(y)g_2\alpha(y)^{-1}, \alpha(y)g_3\alpha(y)^{-1},$$

$$\ldots, \alpha(y)g_{n-1}\alpha(y)^{-1}, \alpha(y)g_n\alpha(y)^{-1}, \alpha(y)(g_1 \cdot \ldots \cdot g_n)^{-1}, y).$$

Setting $u_1 = \alpha(y)g_2\alpha(y)^{-1}, \ldots, u_{n-1} = \alpha(y)g_n\alpha(y)^{-1}, u_n = \alpha(y)(g_1 \cdot \ldots \cdot g_n)^{-1}$, we obtain

$$t_n(u_1, u_2, \ldots, u_n, y) = (\alpha(y)(u_1 \cdot \ldots \cdot u_n)^{-1}, u_1, u_2, \ldots, u_{n-1}, y).$$

Since t is cyclic, we obtain $(\alpha(y)(u_1 \cdot \ldots \cdot u_n)^{-1}, u_1, u_2, \ldots, u_{n-1}, y) = (g_1, g_2, \ldots, g_n, y)$. We thus see that $\alpha(y)g_1\alpha(y)^{-1} = g_1, \alpha(y)g_2\alpha(y)^{-1} = g_2, \ldots, \alpha(y)g_n\alpha(y)^{-1} = g_n$, hence $\alpha(y)$ is in the center of the group G. Conversly if $\alpha(y)$ belongs to the center of the group G, then

$$t_n^n(g_1, \ldots, g_n, y) = (g_2, g_3, \ldots, g_{n-1}, g_n, \alpha(y)(g_1 \cdot \ldots \cdot g_n)^{-1}, y).$$

Furthermore,

$$\begin{aligned} t_n(g_2, g_3, \ldots, g_{n-1}, g_n, \alpha(y)(g_1 \cdot \ldots \cdot g_n)^{-1}, y) &= (\alpha(y)g_1\alpha(y)^{-1}, g_2, \ldots, g_n, y) \\ &= (g_1, g_2, \ldots, g_n, y) \end{aligned}$$

thus making t cyclic. ∎

4.6. EXAMPLE. Let $C(G, \{*\}, N)$ be the simplicial set discussed above, let z be an element in the center of the group G. Then for any element y in N, the operator $t : C(G, \{*\}, N) \to C(G, \{*\}, N)$ defined by

$$(g_1, \ldots, g_n, y) \mapsto (z(g_1 \cdot \ldots \cdot g_n)^{-1}, g_1, \ldots, g_{n-1}, g_n y)$$

is a cyclic operator and this generalises the twisted nerve of a group as found in [Loday, 1998, Section 7.3.3]. To get the twisted nerve of G, we set $N = \{*\}$.

4.7. REMARK. If the map $\alpha : N \to G$ satisfies an additional property being that for any g in G and y in N, $\alpha(gy) = g\alpha(y)g^{-1}$,(i.e. that N is a crossed G-set [Freyd & Yetter, 1989]) then we do not require that N be a trivial G-set for t to be a cyclic operator as given in Corollary 4.5. The condition to be satisfied is that $\alpha(y) \in Stab(y)$ for any y in N.

5. Böhm-Ştefan Construction

In this section we provide the data that is needed to equip the simplicial set discussed in Section 3.1 with a duplicial structure. This requires that we give the additional structures on the functors $\tilde{M}$ and N discussed in the aforementioned Section.

5.1. Additional structures on the coefficients $\tilde{M}$ and N. Following the construction by Böhm and Ştefan[Böhm & Ştefan, 2008] the duplicial operator t can be written as a composite which involves additional structures on $\tilde{M}$ and N. Here we discuss these additional structures. However we begin by recalling the following:

5.2. Definition. [Distributive laws] *A distributive law between comonads* $\mathbb{A} = (A, \Delta^A, \varepsilon^A)$ *and* $\mathbb{B} = (B, \Delta^B, \varepsilon^B)$ *in a category* $\mathcal{C}$ *is a natural transformation* $\chi : \mathbb{AB} \to \mathbb{BA}$ *such that the following diagrams commute:*

$$\begin{tikzcd}
AB \arrow[r, "A\Delta^B"] \arrow[d, "\chi"'] & ABB \arrow[r, "\chi B"] & BAB \arrow[d, "B\chi"] \\
BA \arrow[rr, "\Delta^B A"'] & & BBA
\end{tikzcd}
\qquad
\begin{tikzcd}
AB \arrow[r, "\chi"] \arrow[dr, "A\varepsilon^B"'] & BA \arrow[d, "\varepsilon^B A"] \\
 & A
\end{tikzcd}$$

$$\begin{tikzcd}
AB \arrow[r, "\Delta^A B"] \arrow[d, "\chi"'] & AAB \arrow[r, "\chi B"] & ABA \arrow[d, "\chi A"] \\
BA \arrow[rr, "B\Delta^A"'] & & BAA
\end{tikzcd}
\qquad
\begin{tikzcd}
AB \arrow[r, "\chi"] \arrow[dr, "\varepsilon^A B"'] & BA \arrow[d, "B\varepsilon^A"] \\
 & B
\end{tikzcd}$$

Now recall the comonads $\mathbb{S}, \mathbb{T}$ discussed in Sections 2.2, 2.3 respectively. The following is a description of the distributive law that exists between these comonads:

5.3. Proposition. *The natural transformation* $\chi : TS \Rightarrow ST$ *given by*

$$\chi_N : K\tilde{\times}L \times N \to L \times K\tilde{\times}N \quad (k, l, n) \mapsto (kl, k, n)$$

is the unique distributive law between the comonads $\mathbb{T}$ *and* $\mathbb{S}$.

Proof. It is straightforward to verify that χ is a distributive law, i.e. that the following diagrams

$$\begin{tikzcd}
K\tilde{\times}L \times N \arrow[r, "K\tilde{\times}\Delta_N"] \arrow[d, "\chi_N"'] & K\tilde{\times}L \times L \times N \arrow[r, "\chi_{L\times N}"] & L \times K\tilde{\times}L \times N \arrow[d, "L\times\chi_N"] \\
L \times K\tilde{\times}N \arrow[rr, "\Delta_{K\tilde{\times}N}"'] & & L \times L \times K\tilde{\times}N
\end{tikzcd}
\qquad
\begin{tikzcd}
K\tilde{\times}L \times N \arrow[r, "\chi_N"] \arrow[dr, "K\tilde{\times}\varepsilon_N"'] & L \times K\tilde{\times}N \arrow[d, "\varepsilon_{K\tilde{\times}N}"] \\
 & K\tilde{\times}N
\end{tikzcd}$$

$$\begin{tikzcd}
K\tilde{\times}L \times N \arrow[r, "\tilde{\Delta}_{L\times N}"] \arrow[d, "\chi_N"'] & K\tilde{\times}K\tilde{\times}L \times N \arrow[r, "K\tilde{\times}\chi_N"] & K\tilde{\times}L \times K\tilde{\times}N \arrow[d, "\chi_{K\tilde{\times}N}"] \\
L \times K\tilde{\times}N \arrow[rr, "L\times\tilde{\Delta}_N"'] & & L \times K\tilde{\times}K\tilde{\times}N
\end{tikzcd}
\qquad
\begin{tikzcd}
K\tilde{\times}L \times N \arrow[r, "\chi_N"] \arrow[dr, "\tilde{\varepsilon}_{L\times N}"'] & L \times K\tilde{\times}N \arrow[d, "L\times\tilde{\varepsilon}_N"] \\
 & L \times N
\end{tikzcd}$$

commute.

Assume now that $\chi_N : K\tilde{\times}L \times N \to L \times K\tilde{\times}N$ is any distributive law and define

α, β, γ by $\chi_N(a,b,x) =: (\alpha(a,b,x), \beta(a,b,x), \gamma(a,b,x))$. From the first counit compatibility condition, we obtain

$$\varepsilon_{K\tilde{\times}N} \circ \chi_N(a,b,x) = \varepsilon_{K\tilde{\times}N}(\alpha(a,b,x), \beta(a,b,x), \gamma(a,b,x)) = (\beta(a,b,x), \gamma(a,b,x))$$

$$K\tilde{\times}\varepsilon_N(a,b,x) = (a,x).$$

Thus $\beta(a,b,x) = a$ and $\gamma(a,b,x) = x$. The second counit compatibility condition gives

$$L \times \tilde{\varepsilon}_N \circ \chi_N(a,b,x) = L \times \tilde{\varepsilon}_N(\alpha(a,b,x), \beta(a,b,x), \gamma(a,b,x)) = (\alpha(a,b,x), \beta(a,b,x)\gamma(a,b,x))$$

$$\tilde{\varepsilon}_{L\times N}(a,b,x) = a(b,x) = (ab, ax).$$

Thus $\alpha(a,b,x) = ab$ and $\beta(a,b,x)\gamma(a,b,x) = ax$. ∎

Let $L \times -$ be as in Section 2.2. In what follows, we give a classification of all the right χ-coalgebras (see [Kowalzig, Kraehmer, Slevin, 2015, Section 3.2] for definition). This can be understood as an instance of Proposition 3.5 in [Kowalzig, Kraehmer, Slevin, 2015].

5.4. Proposition. *Let N be a G-set and $f : N \to L$ be any map. Then the natural transformation*

$$\rho_N : G\tilde{\times}N \to L \times N \quad (a,n) \mapsto (af(n), an)$$

is a right χ-coalgebra and all right χ-coalgebras are of this form.

Proof. It is straightforward to show that ρ is a right χ-coalgebra, i.e that the following diagrams

$$\begin{tikzcd} G\tilde{\times}N \arrow[r, "\tilde{\Delta}_N"] \arrow[d, "\rho_N"'] & G\tilde{\times}G\tilde{\times}N \arrow[r, "G\tilde{\times}\rho_N"] & G\tilde{\times}L\times N \arrow[d, "\chi_N"] \\ L\times N \arrow[r, "\Delta_N"'] & L\times L\times N & L\times G\tilde{\times}N \arrow[l, "L\times\rho_N"] \end{tikzcd} \qquad \begin{tikzcd} G\tilde{\times}N \arrow[r, "\rho_N"] \arrow[dr, "\tilde{\varepsilon}_N"'] & L\times N \arrow[d, "\varepsilon_N"] \\ & N \end{tikzcd}$$

commute. Assume now that $\rho_N : G\tilde{\times}N \to L \times N$ is any right χ-coalgebra and define μ, ν by

$$\rho_N(a,x) := (\mu(a,x), \nu(a,x)).$$

Since ρ_N is G-equivariant we have

$$\rho_N(a,x) = \rho_N(a(1,x)) = a\rho_N(1,x) = a(\mu(1,x), \nu(1,x)) = (a\mu(1,x), a\nu(1,x))$$

This implies $\mu(a,x) = a\mu(1,x), \quad \nu(a,x) = a\nu(1,x)$. Now by the counit compatibility condition, we obtain

$$\tilde{\varepsilon}_N(a,x) = ax.$$

$$\varepsilon_N \circ \rho_N(a,x) = \varepsilon_N(\mu(a,x),\nu(a,x)) = \nu(a,x).$$

Thus $\nu(a,x) = ax$ and $\mu(a,x) = af(x)$ where $f : N \to L$ is any set map. Thus

$$\rho(a,x) = (af(x), ax).$$

■

In theory left χ-coalgebra structures can be characterised in an entirely dual way, see Remark 3.6 in [Kowalzig, Kraehmer, Slevin, 2015]. In what follows we classify all the left χ-coalgebras:

5.5. Proposition. *Let $\tilde{M}$: **G-Set** $\to$ **Set** be the functor which takes any G-set X to the set of orbits in X,*

$$\tilde{M}X := X/G.$$

1. *Let $h : L \to G$ be a G-equivariant map and let $\tilde{\lambda}_N : L \times N \to G\tilde{\times}N$ be a map defined by $\tilde{\lambda}_N(b,n) = (h(b), h(b)^{-1}n)$. Then $\tilde{\lambda}_N$ is a G-equivariant map.*
2. *The natural transformation*

$$\lambda_N : (L \times N)/G \to (G\tilde{\times}N)/G \quad ([b,n]) \mapsto [\tilde{\lambda}_N(b,n)]$$

is a left χ-coalgebra.

Proof.

1. We show that $\tilde{\lambda}_N$ is G-equivariant, i.e for any $a \in G$

$$\begin{aligned}\tilde{\lambda}(a(b,n)) &= \tilde{\lambda}(ab, an)\\ &= (h(ab), h(ab)^{-1}an)\\ &= (ah(b), h(b)^{-1}n) = a(h(b), h(b)^{-1}n) = a\tilde{\lambda}(b,n)\end{aligned}$$

So $\tilde{\lambda}$ is G-equivariant.

2. We verify that λ is a left χ-coalgebra, i.e, that the following diagrams

$$\begin{tikzcd}
(L\times N)/G \arrow[r, "\Delta_N"] \arrow[d, "\lambda_N"'] & (L\times L\times N)/G \arrow[r, "\lambda_{L\times N}"] & (G\tilde{\times}L\times N)/G \arrow[d, "\chi_N"] \\
(G\tilde{\times}N)/G \arrow[r, "\tilde{\Delta}_N"'] & (G\tilde{\times}G\tilde{\times}N)/G & (L\times G\tilde{\times}N)/G \arrow[l, "\lambda_{G\tilde{\times}N}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(L \times N)/G \arrow[r, "\lambda_N"] \arrow[dr, "\varepsilon_N"'] & (G\tilde{\times}N)/G \arrow[d, "\tilde{\varepsilon}_N"] \\
& N/G
\end{tikzcd}$$

commute. Let $[b, n] \in (L \times N)/G$. Then

$$\begin{aligned}
\lambda_{G\tilde{\times}N} \circ \chi_N \circ \lambda_{L\times N} \circ \Delta_N^L([b, n]) &= \lambda_{G\tilde{\times}N} \circ \chi_N \circ \lambda_{L\times N}([b, b, n]) \\
&= \lambda_{G\tilde{\times}N} \circ \chi_N([h(b), h(b)^{-1}(b, n)]) \\
&= \lambda_{G\tilde{\times}N} \circ \chi_N([h(b), h(b)^{-1}b, h(b)^{-1}n]) \\
&= \lambda_{G\tilde{\times}N}([b, h(b), h(b)^{-1}n]) \\
&= [h(b), h(b)^{-1}(h(b), h(b)^{-1}n)] \\
&= [h(b), 1, h(b)^{-1}n]
\end{aligned}$$

Further,

$$\tilde{\Delta}_N \circ \lambda_N([b, n]) = \tilde{\Delta}_N([h(b), h(b)^{-1}n]) = [h(b), 1, h(b)^{-1}n].$$

Lastly,

$$\tilde{\varepsilon}_N \circ \lambda_N([b, n]) = \tilde{\varepsilon}_N([h(b), h(b)^{-1}n]) = [n] \text{ and } \varepsilon_N([y, n]) = [n].$$

So the above diagrams commute and so λ_N is a left χ-coalgebra. ∎

5.6. The natural transformations ρ, χ, λ and operator t. Here we explicitly give the relationship between the duplicial operator t in Theorem 4.4 on $C(G, \{*\}, N)$ and the additional structures ρ and λ on the functors $\tilde{M}$ and N together with the distributive law described in Section 5. As discussed in Proposition 3.3, there is no loss of generality if we consider M in the definition of the simplicial set to be the terminal G-set. We use definitions of χ, ρ and λ as discussed in Proposition 5.3, Proposition 5.4 and Proposition 5.5 respectively.

5.7. Theorem. *Assume that L is a faithful G-set. Let t be the duplicial operator on $C(G, \{*\}, N)$ described in Theorem 4.4 and let ρ and λ be the natural transformations obtained in Propositions 5.4 and 5.5 respectively. Define $\alpha(y) = (h \circ f(y))^{-1}$. Then $t = \lambda \times \mathrm{Id}^n \circ \chi^n \circ \mathrm{Id}^n \times \rho$.*

Proof. Let $[*, g_1, \ldots, g_{n+1}, y] \in \bar{C}_n(G, \{*\}, N)$. Then

$$\begin{aligned}
\lambda \times \mathrm{Id}^n \circ \chi^n \circ \mathrm{Id}^n \times \rho([*, g_1, \ldots, g_{n+1}, y]) &= \lambda \times \mathrm{Id}^n \circ \chi^n([*, g_1, \ldots, g_{n+1}f(y), g_{n+1}y]) \\
&= \lambda \times \mathrm{Id}^n([*, u, g_1, \ldots, g_n, g_{n+1}y]) \\
&= [*, h(u), (h(u))^{-1}(g_1, \ldots, g_n), g_{n+1}y),]
\end{aligned}$$

where $u = g_1 \cdot \ldots \cdot g_{n+1} f(y)$. Now by the identification in (4), $[*, h(u), (h(u))^{-1}(g_1, \ldots, g_n), g_{n+1}y]$, an element in $\bar{C}_n(G, \{*\}, N)$ can be identified as $((h \circ f(y))^{-1}(g_2 \cdot \ldots \cdot g_{n+1})^{-1}, g_2, \ldots, g_n, g_{n+1}y)$, an element in $C_n(G, \{*\}, N)$. On the other hand, by using the identification (4), one can view $t_n([*, g_1, \ldots, g_{n+1}, y])$ as

$$t_n(g_2, \ldots, g_{n+1}, y) = (\alpha(y)(g_2 \cdot \ldots \cdot g_{n+1})^{-1}, g_2, \ldots, g_n, g_{n+1}y).$$

Thus $t = \lambda \times \mathrm{Id}^n \circ \chi^n \circ \mathrm{Id}^n \times \rho$. In other words the duplicial operator defined in Theorem 4.4 always arises from the Böhm-Ştefan construction [Böhm & Ştefan, 2008, Section 1, Theorem 1.23]. ∎

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