AN ELEMENTARY CHARACTERIZATION OF STABLY PRECOHESIVE GEOMETRIC MORPHISMS AS PARTICULAR PRECOHESIVE GEOMETRIC MORPHISMS

THOMAS STREICHER

Dedicated to the Memory of Bill Lawvere

ABSTRACT. We consider various characterizations of stably precohesive geometric morphisms which are elementary in the sense that they avoid reference to concepts of relative category theory.

1. Introduction

Bill Lawvere to whom we owe among many other things the quintessential notion of elementary topos in his last decades has spent quite some effort on investigating a notion of cohesive geometric morphism. The aim of this concept is to axiomatize what is a "topos of spaces" over some base topos, see e.g. [LM15]. Cohesive geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ are defined as precohesive geometric morphims satisfying the further requirement that the canonical morphism $L(A^{FI}) \to L(A)^{I}$ is an isomorphism for all $I \in \mathcal{S}$ and $A \in \mathcal{E}$ where $L \dashv F$.

A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is precohesive iff it is local and hyperconnected and F has a left adjoint L which preserves binary products. This is not verbatim Lawvere's original definition but equivalent to it as shown in [Jo11]. Recall that hyperconnected means that U preserves subobject classifiers and local means that F is full and faithful and U has a right adjoint R. As shown in [Jo11] a hyperconnected and local geometric morphism is precohesive iff F preserves exponentials (which as shown in Theorem 2 of [BP80] is equivalent to F having an \mathcal{S} -strong left adjoint L).

As shown in [LM15] a precohesive geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is stably precohesive, i.e. all its slices over some $I \in \mathcal{S}$ are precohesive, iff it is molecular (aka locally connected). This is obvious from the fact that a geometric morphism $F \dashv U$ is molecular iff one of the following equivalent conditions holds

(1) F has a left adjoint L fibered (or indexed) over \mathcal{S}

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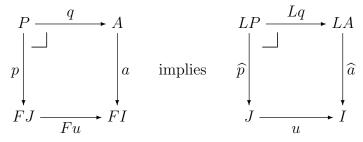
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(2) all slices of F preserve exponentials

(3) F preserves dependent products (i.e. right adjoints to pullback functors)

as shown in C3.3.1 of [Jo02]. Notice that condition (1) can be explicit ated as the requirement that



where we write \hat{a} for the upper transpose $\varepsilon_I \circ La : LA \to I$ of a (and similarly for p).

As shown by J.-L. Moens and M. Jibladze and explained in [Str23] geometric morphisms to S correspond to toposes fibered over S which have (internal) sums and are locally small from which it follows that the geometric morphism extends to a fibered one of the kind $\Delta \dashv \Gamma$. From this point of view it appears as most natural to require that the left adjoint of the inverse image part of the geometric morphism extends to a fibered left adjoint of Δ and, accordingly, stably precohesive appears as the more natural notion.

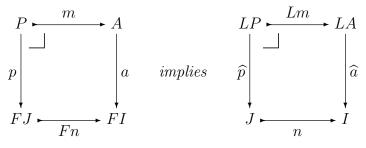
But in [LM15] the authors raise the question whether precohesive and stably precohesive coincide or there is a counterexample separating these two notions. Though we cannot answer this question in this paper we instead will provide alternative characterizations of stably precohesive geometric morphisms in terms of conditions which are "elementary" in the sense that they avoid any direct reference to concepts of the theory of fibered categories and thus might be easier to check.

2. The case of bireflective exponential ideals

Suppose $F \dashv U : \mathcal{E} \to \mathcal{S}$ is a geometric morphism which is connected, i.e. F is full and faithful, and which has a left adjoint L preserving binary products, i.e. \mathcal{S} is a bireflective exponential ideal within \mathcal{E} via F.

We will identify various conditions equivalent to $F \dashv U$ being molecular.

2.1. THEOREM. The geometric morphism $F \dashv U$ as above is molecular iff

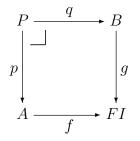


for all monos $n: J \rightarrow I$ in S.

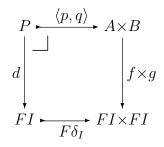
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PROOF. The condition given is an immediate consequence of the (explicitation of the) requirement that L is a fibered left adjoint of F as described in the introduction.

For the reverse direction we show that the condition implies that for all $I \in S$ the left adjoint to $F/I : S/I \to \mathcal{E}/FI$ preserves binary products. For this purpose suppose that



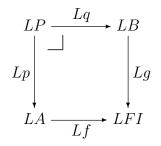
i.e.



where pf = d = qg. Then by the assumed condition it follows that

$$\begin{array}{c|c} LP & \stackrel{\langle Lp, Lq \rangle}{\longrightarrow} & LA \times LB \\ \widehat{d} & & & & \\ \widehat{d} & & & & \\ I & & & & \\ I & & & \delta_I \end{array} \xrightarrow{f \times \widehat{g}} I \times I \end{array}$$

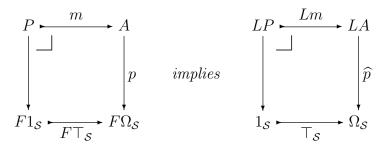
and thus



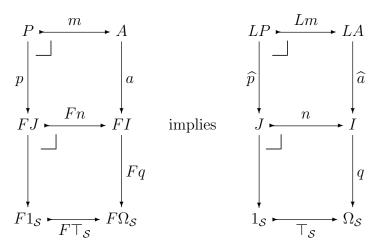
since $\widehat{f} = \varepsilon_I \circ Lf$ and $\widehat{g} = \varepsilon_I \circ Lg$ and the counit ε of $L \dashv F$ is a natural isomorphism because F is full and faithful as follows from the assumption that the geometric morphism $F \dashv U$ is connected.

Next we show that the previous theorem holds already when requiring the condition given there only for the cases where n is $\top_{\mathcal{S}}$.

2.2. THEOREM. The geometric morphism $F \dashv U$ is molecular iff



PROOF. The necessity of this condition follows from Theorem 2.1. The reverse direction follows since



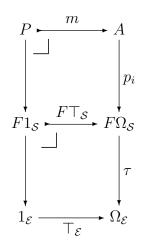
because our assumption guarantees the required implication for the outer rectangles, the lower squares are pullbacks anyway and thus the required implication holds for the upper squares by the usual pullback lemma.

Recall from [BP80] that a morphism in \mathcal{E} is called \mathcal{S} -definable if it can be obtained as pullback of Fu for some morphism u in \mathcal{S} . It is straightforward to check that a monomorphism $m: P \to A$ in \mathcal{E} is \mathcal{S} -definable iff for some mono $n: J \to I$ in \mathcal{S} it can be obtained as pullback of $F(n: J \to I)$ along some $a: A \to FI$. As already observed in the proof of the previous theorem m can be obtained as pullback of $F \top_{\mathcal{S}}$ along $Fq \circ a$ where $q: I \to \Omega_{\mathcal{S}}$ classifies n. Thus, a monomorphism in \mathcal{E} is \mathcal{S} -definable iff can be obtained as pullback of $F \top_{\mathcal{S}}$.

Recall from [Jo80] that $F \dashv U$ is called *subopen* iff the classifying map $\tau : F\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ for $F \top_{\mathcal{S}}$ is monic. Thus, if $F \dashv U$ is subopen then $F \top_{\mathcal{S}}$ classifies \mathcal{S} -definable monos in \mathcal{E} .

2.3. LEMMA. If $F \dashv U$ is molecular then it is subopen.

PROOF. Suppose $p_1, p_2 : A \to \Omega_S$ with $\tau \circ p_1 = \tau \circ p_2$. Let $m : P \to A$ be the mono classified by this morphism. Then for i = 1, 2 we have

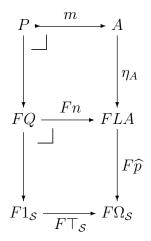


from which it follows by Theorem 2.2 that $\hat{p_1}$ and $\hat{p_2}$ both classify the mono Lm. Thus $\hat{p_1} = \hat{p_2}$ from which it follows that $p_1 = p_2$.

Next we give a further characterization of S-definable monos under the assumption that the geometric morphism under consideration is essential.

2.4. LEMMA. Suppose $F \dashv U : \mathcal{E} \to \mathcal{S}$ is an essential geometric morphism, i.e. F has a left adjoint L. Then a mono $m : P \to A$ in \mathcal{E} is \mathcal{S} -definable iff it appears as pullback along $\eta_A : A \to FLA$ of Fn for some mono $n : Q \to LA$ in \mathcal{S} .

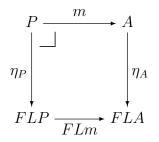
PROOF. Obviously, the condition is sufficient. For the reverse direction suppose that m appears as pullback of $F \top_{\mathcal{S}}$ along some $p : A \to F\Omega_{\mathcal{S}}$. Let $n : Q \to LA$ be the mono classified by $\hat{p} : LA \to \Omega_{\mathcal{S}}$. Then we have



where the lower square is a pullback since F preserves pullbacks and the outer rectangle is a pulback since m arises as pullback of $F \top_{\mathcal{S}}$ along $p = F \hat{p} \circ \eta_A$ and thus the upper square is a pullback as desired.

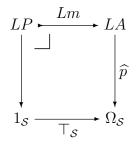
For the rest of this section we resume the assumption that $F \dashv U$ is connected and F has a left adjoint preserving binary products.

2.5. THEOREM. The geometric morphism $F \dashv U$ is molecular iff it is subopen and for all S-definable monos $m : P \rightarrow A$ in \mathcal{E} the map Lm is a mono in S and

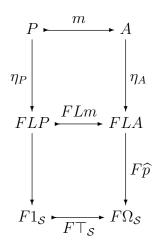


is a pullback in \mathcal{E} .

PROOF. For the forward direction suppose $F \dashv U$ is molecular. Then it is also subopen by Lemma 2.3. Suppose $m : P \to A$ is S-definable. Then m appears as pullback of $F \top_S$ along a unique map $p : A \to F\Omega_S$. Thus, by Theorem 2.2 it holds that



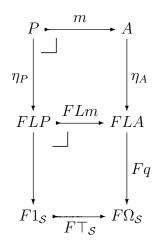
from which it follows that in



the lower square is a pullback since F preserves pullbacks and the outer rectangle is a pulback since $p = F\hat{p} \circ \eta_A$ classifies m. Accordingly, the upper square is a pullback square as well.

For the backwards direction suppose $F \dashv U$ is subopen and L sends S-definable monos in \mathcal{E} to monos in \mathcal{S} such that the naturality square for η is a pullback square for S-definable monos m. For showing that $F \dashv U$ is molecular we verify the criterion provided by Theorem 2.2.

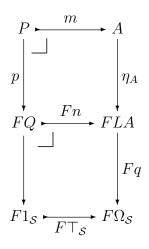
For this purpose suppose $m: P \rightarrow A$ appears as pullback of $F \top_{\mathcal{S}}$ along some (necessarily unique) map $p: A \rightarrow F\Omega_{\mathcal{S}}$. Then Lm is monic. Thus, there exists a unique map $q: LA \rightarrow \Omega_{\mathcal{S}}$ classifying Lm. Then we have



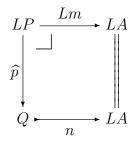
from which it follows that $p = Fq \circ \eta_A$ and thus $q = \hat{p}$. Thus Lm appears as pullback of $\top_{\mathcal{S}}$ along \hat{p} as desired.

2.6. LEMMA. If $F \dashv U$ is molecular then $m \mapsto Lm$ and $n \mapsto \eta_A^* Fn$ establishes a 1-1correspondence between S-definable subobjects m of A and arbitrary subobjects n of LA.

PROOF. Suppose $n: Q \to LA$ is classified by $q: LA \to \Omega_S$. We have to show that n and $L(\eta_A^*Fn)$ are isomorphic as subobjects of LA. For this purpose consider



from which it follows by Theorem 2.1 that



and thus Lm and n are isomorphic as subobjects of LA via \hat{p} , which is an isomorphism since it arises as pullback of id_{LA} .

Suppose $m: P \rightarrow A$ is S-definable. Then $m \cong \eta_A^* FLm$ follows from Theorem 2.5.

In [GS21] one can find examples of local geometric morphism having a further left adjoint preserving finite products but nevertheless are not molecular. However, none of these counterexamples is hyperconnected for which reason the question of Lawvere and Menni still remains open.

Nevertheless assuming $F \dashv U$ to be hyperconnected or even precohesive allows us to improve some of our results as we will see in the next section.

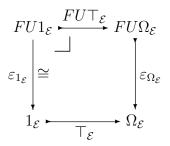
3. Exploiting the additional assumption that $F \dashv U$ is precohesive

From now on we assume that the geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ under consideration is not only connected with F having a finite product preserving left adjoint L but also that it is hyperconnected, i.e. fulfills one of the following equivalent conditions

- (1) U preserves subobject classifiers
- (2) for all $I \in S$ the functor $F/I : S/I \to \mathcal{E}/FI$ restricts to an equivalence between $\mathsf{Sub}_{\mathcal{S}}(I)$ and $\mathsf{Sub}_{\mathcal{E}}(FI)$
- (3) all units and counits of $F \dashv U$ are monic.

See A4.6.6 of [Jo02] for a proof and further characterizations. The intuition is that hyperconnected is the opposite of localic in the sense that these two classes form a factorization system on geometric morphisms in the appropriate 2-categorical sense.

Since $U \top_{\mathcal{E}} \cong \top_{\mathcal{S}}$ we have



with $\varepsilon_{\Omega_{\mathcal{E}}}$ being monic because the counit of a hyperconnected geometric morphism is monic. Since $FU \top_{\mathcal{E}} \cong F \top_{\mathcal{S}}$ it follows that $\varepsilon_{\Omega_{\mathcal{E}}} \cong \tau$ and thus τ is monic. Thus hyperconnected geometric morphisms are necessarily subopen. Actually, by Corollary C3.1.9(i) of [Jo02] they are even open.

Thus, under the current assumptions a mono $m: P \to A$ is S-definable iff its classifying map $\chi: A \to \Omega_{\mathcal{E}}$ factors through $\tau: F\Omega_{\mathcal{S}} \to \Omega_{\mathcal{E}}$ via a unique map $p: A \to F\Omega_{\mathcal{S}}$ whose transpose $\hat{p}: LA \to \Omega_{\mathcal{S}}$ classifies the unique mono $n: Q \to LA$ with $m \cong \eta_A^* Fn$. Without the assumption that $F \dashv U$ is molecular it is, however, not clear at all why this n should be isomorphic to Lm nor is it clear why the additional assumption of $F \dashv U$ being local should ensure this.

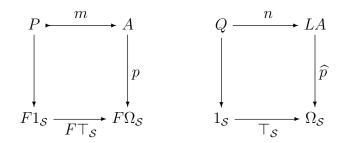
For the rest of this section we further assume that $F \dashv U$ is precohesive, i.e. also hyperconnected and local.

3.1. THEOREM. Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a precohesive geometric morphism. Thus F has a left adjoint L. Then $n \mapsto \eta_A^* Fn$ establishes a 1-1-correspondence between subobjects of LA in \mathcal{S} and \mathcal{S} -definable subobjects of A in \mathcal{E} . Its inverse is given by sending an \mathcal{S} -definable subobject m of A in \mathcal{E} to the image of Lm in \mathcal{S} .

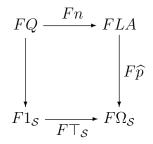
PROOF. Since $F \dashv U$ is precohesive it follows by Cor. 2.5 in [Jo11] that for all $A \in \mathcal{E}$ the unit map $\eta_A : A \to FLA$ is epic.

Since $F \dashv U$ is hyperconnected it is in particular subopen and thus a subobject of A is S-definable iff its characteristic map factors through the mono $\tau : F\Omega_S \to \Omega_{\mathcal{E}}$ as argued immediately before Lemma 2.3.

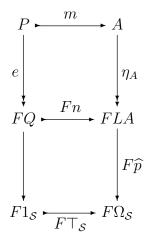
Suppose $p : A \to F\Omega_{\mathcal{S}}$. We write $n : Q \to LA$ for the mono classified by \hat{p} and $m : P \to A$ for the \mathcal{S} -definable mono classified by $p = F\hat{p} \circ \eta_A$. Thus, the two squares



are both pullbacks. Since F as a right adjoint preserves pullbacks the square



is a pullback, too. Let e be the unique arrow making the following diagram



commute. It exists since the lower square and the outer rectangle are pullbacks. Thus the upper square is a pullback as well. Thus e is epic since η_A is epic and epis are stable under pullbacks along arbitrary morphisms. Another consequence is that m is the pullback of Fn along η_A . Moreover, from $\eta_A \circ m = Fn \circ e$ it follows by taking the upper transpose of both sides w.r.t. $L \dashv F$ that $Lm = \operatorname{id}_{LA} \circ Lm = \widehat{\eta_A} \circ Lm = n \circ \widehat{e}$. Since left adjoints preserve epis Le is an epi and since ε_Q is an iso it follows that $\widehat{e} = \varepsilon_Q \circ Le$ is an epi. Thus, from $Lm = n \circ \widehat{e}$ it follows that n is the image of Lm.

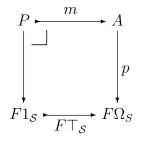
Based on this result we can give the following characterization of stably precohesive geometric morphism as precohesive ones validating a simple preservation property.

3.2. THEOREM. A precohesive geometric morphism $F \dashv U$ is stably precohesive, i.e. also molecular, iff the left adjoint L of F sends S-definable monos in \mathcal{E} to monos in \mathcal{S} .

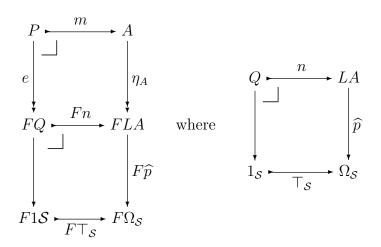
PROOF. Suppose $F \dashv U$ is precohesive.

If $F \dashv U$ is molecular then it follows from Theorem 2.1 that L sends S-definable monos in \mathcal{E} to monos in \mathcal{S} .

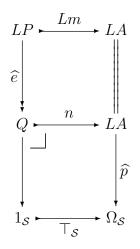
For the backward direction suppose F has a left adjoint L sending S-definable monos in \mathcal{E} to monos in \mathcal{S} . For showing that $F \dashv U$ is molecular we will verify the condition given in Thm 2.2. For this purpose suppose



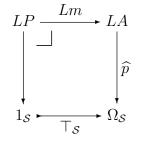
for some $p: A \to F\Omega_{\mathcal{S}}$. Then we have



since $p = F\hat{p} \circ \eta_A$ and F as a right adjoint preserves pullbacks. Transposing the left rectangle according to $L \dashv F$ we obtain



since $\hat{e} = \varepsilon_Q \circ Le$ is epic because Le is epic since L as a left adjoint preserves epis and ε_Q is an iso. But \hat{e} is also monic since $Lm = n \circ \hat{e}$ is monic due to the assumption that L sends \mathcal{S} -definable monos in \mathcal{E} to monos in \mathcal{S} . Thus \hat{e} is an isomorphism from which it follows by the previous diagram that



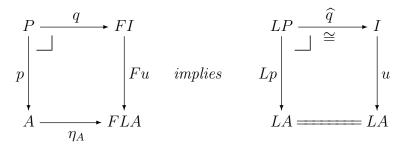
is a pullback as required by the condition given in Theorem 2.2.

One calls a geometric morphism stably precohesive iff it is precohesive and molecular. For precohesive geometric morphisms by Theorem 3.1 for S-definable subobjects m of Athe (unique up to iso) subobject n of LA with m isomorphic to η_A^*Fn is obtained as the image of Lm whereas for stably precohesive geometric morphism by Theorem 3.2 this n is isomorphic to Lm in S/LA. Thus, if one does not adopt the fibered point of view the requirement of being stably precohesive may appear as somewhat *ad hoc* and thus presumably will not hold automatically for all precohesive geometric morphisms!

4. Relation to an alternative characterization of being molecular

First we give an alternative characterization of being molecular.

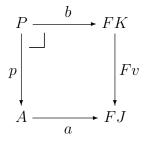
4.1. THEOREM. A geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ is molecular iff F has a left adjoint L such that



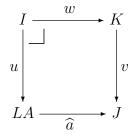
for all $u: I \to LA$ in \mathcal{S} .

PROOF. Obviously the condition is necessary.

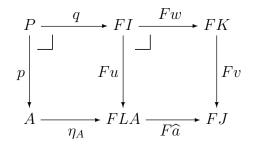
For showing it is also sufficient suppose



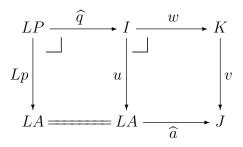
for some $v: K \to J$. There is a pullback



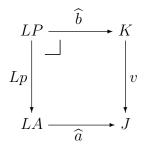
in \mathcal{S} . Since F is a right adjoint it sends this pullback to a pullback in \mathcal{E} . Let $q: P \to FI$ be the unique mediating arrow with $Fu \circ q = \eta_A \circ p$ and $Fw \circ q = b$. Then we have



since F preserves pullbacks and the rectangle is a pullback by assumption from which it follows by the usual pullback lemma that the left square is a pullback, too. Transposing the previous diagram w.r.t. the adjunction $L \dashv F$ we obtain



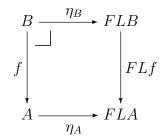
where $w \circ \hat{q} = \hat{b}$. Thus



as required.

Notice that for an essential geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ with $L \dashv F$ we have $L/A \dashv \eta_A^* \circ F/LA$ for all objects A in \mathcal{E} (see Theorem 16.4 of [Str23] for an explicitation of the units and counits of this adjunction which also works when the left adjoint is not required to preserve finite limits). The right adjoint is full and faithful iff the counit of this adjunction is an iso. That the latter holds for all objects A in \mathcal{E} is obviously equivalent to the condition given in Theorem 4.1 since the square on the right is the counit of $L/A \dashv \eta_A^* \circ F/LA$ at u. Thus $F \dashv U$ is molecular iff $\eta_A^* \circ F/LA$ is full and faithful for all A in \mathcal{E} . This condition is given in C3.3.5(ii) of [Jo02] but we think our proof is more transparent.

Thus, for molecular geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ the \mathcal{S} -definable maps to A form a full reflective subcategory of \mathcal{E}/A . Thus, a map $f : B \to A$ in \mathcal{E} is \mathcal{S} -definable iff the unit of $L/A \dashv \eta_A^* \circ F/LA$ at f is an isomorphism, i.e.



is a pullback. This reflection restricts to an equivalence between S-definable maps to A in \mathcal{E} and maps to LA in \mathcal{S} and thus, in particular, to an equivalence between S-definable subobjects of A and ordinary subobjects of LA.

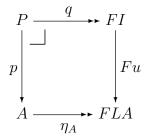
Accordingly, as suggested by M. Menni in private communication, showing that $F \dashv U$ is not molecular amounts to exhibiting an A in \mathcal{E} such that $\eta_A^* \circ F/LA$ is not full and faithful. However, so far we have not been inventive enough to find such a thing. The main reason for this failure is that we only know examples of precohesive geometric morphisms to boolean base toposes (or whose construction is analogous to such ones) and these are all necessarily stably precohesive as shown in [Men22].

5. For precohesive geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ the \mathcal{S} -definable subobjects of A are a full reflective subcategory of \mathcal{E}/A canonically equivalent to \mathcal{S}/LA

Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a precohesive geometric morphism and $L \dashv F$. For objects A of \mathcal{E} let $i_A : \mathsf{Sub}_{\mathcal{S}}(LA) \hookrightarrow \mathcal{S}/LA$ be the inclusion of subobjects of LA into the slice \mathcal{S}/LA . This functor has a left adjoint r_A sending maps to LA to their image in \mathcal{S} .

5.1. THEOREM. Let $F \dashv U : \mathcal{E} \to \mathcal{S}$ be a precohesive geometric morphism and L be a left adjoint of F. Then for every object A of \mathcal{E} the functor $\eta_A^* \circ F/LA \circ i_A : \mathsf{Sub}_{\mathcal{S}}(LA) \to \mathcal{E}/A$ is full and faithful with left adjoint $r_A \circ L/A$ and thus the \mathcal{S} -definable subobjects of A form a full reflective subcategory of \mathcal{E}/A .

PROOF. First recall that the counit of $L/A \dashv \eta_A^* \circ F/LA$ at $u : I \to LA$ is given by $\widehat{q} : Lp \to u$ where



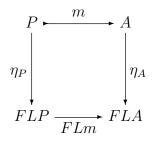
and $\hat{q} = \varepsilon_I \circ Lq$. As already observed at the beginning of the proof of Theorem 3.1 by Cor. 2.5 of [Jo11] the map η_A is epic since $F \dashv U$ is precohesive. Notice that \hat{q} is an epi since L as a left adjoint preserves epis and ε_I is an iso. Thus the functor r_A sends the counit \hat{q} at u in S/LA to an iso in $Sub_S(LA)$. Thus, since all counits of $r_A \dashv i_A$ are isos the counit of $r_A \circ L/A \dashv \eta_A^* \circ F/LA \circ i_A$ at u is an iso as desired.

6. Conclusion

We have shown in Theorem 3.1 that for a precohesive geometric morphism $F \dashv U : \mathcal{E} \to \mathcal{S}$ with $L \dashv F$ it holds that

- (1) $F^* \top_{\mathcal{S}}$ classifies \mathcal{S} -definable monos in \mathcal{E} and
- (2) for $p: A \to F\Omega_{\mathcal{S}}$ from the subobject n of LA classified by the upper transpose $\hat{p}: LA \to \Omega_{\mathcal{S}}$ the corresponding \mathcal{S} -definable subobject m of A classified by p may be constructed as $\eta_A^* Fn$ and
- (3) n is the image of Lm.

In the subsequent Theorem 3.2 we have shown that $F \dashv U$ is stably precohesive iff L sends S-definable monos in \mathcal{E} to monos in \mathcal{S} . For such geometric morphisms a mono $m: P \rightarrow A$ is S-definable iff



is a pullback square.

In section 4 following a suggestion of M. Menni we have observed that a precohesive geometric morphism $F \dashv U$ fails to be stably precohesive iff for some A in \mathcal{E} the functor $\eta_A^* \circ F/LA : \mathcal{S}/LA \to \mathcal{E}/A$ fails to be full and faithful.

Finally, in section 5 we have shown in Theorem 5.1 that for precohesive geometric morphisms $F \dashv U : \mathcal{E} \to \mathcal{S}$ the \mathcal{S} -definable subobjects of A form a full reflective subcategory of \mathcal{E}/A canonically equivalent to \mathcal{S}/LA .

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