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CAUCHY COMPLETENESS AND ADJOINTS IN DOUBLE CATEGORIES

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ABSTRACT. We consider Cauchy completeness in the double categories of toposes, topological spaces, locales, and other suplattice based settings. We also present a uniform approach to the relationship between adjoints and projectivity in double categories with applications to (not-necessarily commutative) rings, rigs, and quantales.

1. Introduction

This article is in memory of Bill Lawvere and a tribute to his insightful contributions to the study of adjoint functors, algebraic theories, elementary toposes, enriched categories, synthetic differential geometry, and much more. He will also be remembered for his generosity and willingness to share his ideas with other mathematicians at all levels. Although he has influenced my work in many ways, I have chosen a topic for this article which has roots in one of his many important ideas, namely the identification of metric spaces as enriched categories.

The observation that a metric space X can be viewed as an enriched category was made by Lawvere and presented in his 1973 paper [L73]. Writing the distance formula $d_X(x,x') = X(x,x')$, the triangle inequality and reflexivity

$$\begin{aligned} X(x',x'') + X(x,x') &\geq X(x,x'') \\ 0 &\geq X(x,x) \end{aligned}$$

give the composition and identity of X as a category enriched in the extended interval poset $([0,\infty],\geq)$ which is a symmetric monoidal closed category via $+$, 0, and truncated subtraction. Thus, we get the category of *generalized* or *Lawvere metric spaces* with morphisms $f\colon X \longrightarrow Y$ such that

$$X(x,x') \geq Y(fx,fx')$$

i.e., Lipschitz functions with constant 1. As Lawvere points out, the other metric space axioms can be added, as needed, but much can be done in this more general setting.

An important result in [L73] is the following characterization of Cauchy completeness. A bimodule from X to Y, denoted $m\colon X \to\!\!\!\!\bullet\; Y$, is a function $X \times Y \longrightarrow [0,\infty]$ satisfying

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$$\bigwedge_y Y(y,y') + m(x,y) \geq m(x,y') \quad \text{and} \quad \bigwedge_x m(x,y) + X(x',x) \geq m(x',y)$$

Composition with a bimodule $n\colon Y \to\!\!\bullet\!\!\to Z$ is defined by

$$(n\bullet m)(x,z) = \bigwedge_y n(y,z) + m(x,y)$$

and $\mathrm{id}_X = X(-,-)$ is the identity bimodule on X. As for any enriched category, a morphism $f\colon X \longrightarrow Y$ defines an adjoint pair of bimodules $f_*\colon X \to\!\!\bullet\!\!\to Y$ and $f^*\colon Y \to\!\!\bullet\!\!\to X$, i.e., $f_* f^* \geq \mathrm{id}_Y$ and $\mathrm{id}_X \geq f^* f_*$. Moreover, every left adjoint bimodule $X \to\!\!\bullet\!\!\to Y$ is induced by a morphism $X \longrightarrow Y$ if and only if Y is Cauchy complete (see [L73]; Page 163).

In the subsequent two decades, the notions of Cauchy completion and Cauchy complete objects in $\mathcal{V}$-Cat were developed by Kelly [K82], Walters [W81], Borceaux/ Dejean [BD86], and Carboni/Street [CS86], where $\mathcal{V}$ is a symmetric monoidal closed category. More recently, Paré [P21] considered Cauchy completeness for double categories with companions and conjoints, concentrating on the double category $\mathbb{R}$ing of commutative rings, homomorphisms and bimodules. Following this double category approach, we will see that Cauchy completeness arises in many other familiar cases, including locales, toposes, topological spaces, posets, suplattices, quantales, and more.

We begin in Section 2, with a review of double categories and Cauchy completeness for several examples from the 1980s. In the next two sections, we show that some familiar constructions can be viewed in this context. We conclude in Section 5, with a non-commutative generalization of Paré's result for rings, which we then apply to the double category of quantales, homomorphisms and bimodules.

2. Preliminaries for Cauchy Completeness

In this section, we recall the definition of a Cauchy complete object in a double category and recall some of the examples of interest in the 1980s.

2.1. Definition. *A double category is an internal pseudo category*

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\;\bullet\;} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \quad (\mathrm{id}^\bullet\colon \mathbb{D}_0 \to \mathbb{D}_1)$$

in the 2-category **CAT** *of locally small categories. It consists of objects (those of $\mathbb{D}_0$), two types of morphisms: horizontal (those of $\mathbb{D}_0$) and vertical (objects of $\mathbb{D}_1$ with domain and codomain given by s and t), and cells (morphisms of $\mathbb{D}_1$)*

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\bullet & \varphi & \bullet\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array} \qquad (\star)$$

Composition and identity morphisms are given horizontally in $\mathbb{D}_0$ *and vertically via* $\bullet$ *and* $\mathrm{id}^\bullet$*, respectively.*

2.2. Remark. A cell $(\star)$ is called special if f_s and f_t are identity morphisms. The vertical morphisms and special cells form a bicategory denoted by $\mathrm{Vert}(\mathbb{D})$.

2.3. Example. $\mathbb{R}\mathrm{el}$ has sets and functions as objects and horizontal morphisms. Vertical morphisms $v\colon X_s \mathrel{-\!\!\bullet\!\!\to} X_t$ are relations, i.e., subsets $v \subseteq X_s \times X_t$. Vertical identities and composition are given by the diagonal relations and the usual relation composition. There is a cell $(\star)$ if and only if $(x_s, x_t) \in v$ implies $(f_s(x_s), f_t(x_t)) \in w$. More generally, one can consider the double category $\mathbb{R}\mathrm{el}(\mathcal{E})$ of objects, morphism, and relations in any regular category $\mathcal{E}$, in the sense of Barr [B71].

2.4. Example. $\mathbb{P}\mathrm{os}$ has partially-ordered sets as objects and order-preserving maps as horizontal morphisms. Vertical morphisms $v\colon X_s \mathrel{-\!\!\bullet\!\!\to} X_t$ are order ideals $v \subseteq X_s^{\mathrm{op}} \times X_t$, with $\mathrm{id}^\bullet_X = \{(x, x') | x \leq x'\}$ and composition and cells as in $\mathbb{R}\mathrm{el}$.

2.5. Example. $\mathbb{C}\mathrm{at}$ has small categories as objects and functors as horizontal morphisms. Vertical morphisms $v\colon X_s \mathrel{-\!\!\bullet\!\!\to} X_t$ are profunctors (also called bimodules or distributors) $v\colon X_s^{\mathrm{op}} \times X_t \longrightarrow \mathrm{Sets}$, and cells $(\star)$ are natural transformations $v \longrightarrow w(f_s-, f_t-)$.

2.6. Definition. *A companion for* $f\colon X \longrightarrow Y$ *in* $\mathbb{D}$ *is a vertical morphism* $f_*\colon X \mathrel{-\!\!\bullet\!\!\to} Y$ *together with cells*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow & \alpha & \big\downarrow{\scriptstyle f_*} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f_*}\big\downarrow & \beta & \big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells. A conjoint for f *is a vertical morphism* $f^*\colon Y \mathrel{-\!\!\bullet\!\!\to} X$ *together with cells*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow & \rho & \big\downarrow{\scriptstyle f^*} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle f^*}\big\downarrow & \sigma & \big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells.

2.7. Remark. If f has a companion and a conjoint, then f_* is left adjoint to f^* in the vertical bicategory $\mathrm{Vert}(\mathbb{D})$.

2.8. Definition. *An object* Y *is called Cauchy complete if every left adjoint* $v\colon X \mathrel{-\!\!\bullet\!\!\to} Y$ *in* $\mathrm{Vert}(\mathbb{D})$ *is the companion of some horizontal morphism* $f\colon X \longrightarrow Y$ *of* $\mathbb{D}$*.*

As noted in the introduction, this agrees with the usual definition of a Cauchy complete metric space for the double category $\mathbb{M}et$ of Lawvere metric spaces, morphisms, and bimodules, (see [L73]). The following examples can also be found in the early literature.

2.9. Example. Every morphism $f\colon X \to Y$ of $\mathbb{R}\mathrm{el}$ has a companion $f_* = \{(x,y)|y = f(x)\}$ and conjoint $f^* = \{(y,x)|y = f(x)\}$, and every set is Cauchy complete in $\mathbb{R}\mathrm{el}$, as is every object of $\mathbb{R}\mathrm{el}(\mathcal{E})$, for every regular category $\mathcal{E}$ (see [FS90]).

2.10. Example. Every morphism $f\colon X \to Y$ of $\mathbb{P}\mathrm{os}$ has a companion $f_* = \{(x,y)|f(x) \leq y\}$ and conjoint $f^* = \{(y,x)|y \leq f(x)\}$, and every poset is Cauchy complete in $\mathbb{P}\mathrm{os}$ (see [CS86]).

2.11. Example. Every functor $f\colon X \to Y$ in $\mathbb{C}\mathrm{at}$ has a companion $f_* = Y(f(-),-)$ and a conjoint $f^* = Y(-,f(-))$. It is well known that a category Y is Cauchy complete if and only if idempotents split in Y (see [K82]). Moreover, this was generalized to $\mathcal{V}$-$\mathbb{C}\mathrm{at}$, for a suitable symmetric monoidal closed category $\mathcal{V}$ (see [LT22]).

3. Locales, Toposes, and Spaces

In this section, we show that every topos and locale is Cauchy complete, and it is precisely the sober spaces that are in an appropriate double category of topological spaces.

Our interest in companions and conjoints (as well as "cotabulators" in the sense of [GP99]) for toposes, locales, and topological spaces goes back to their implicit appearance in the construction of exponentials of locally closed inclusions [N81] using Artin-Wraith glueing [J77]. This was achieved in [N12a] by a construction for double categories with "glueing" which applied to the following three double categories.

3.1. Example. Objects and horizontal morphisms of $\mathbb{L}\mathrm{oc}$ are locales and locale homomorphisms, in the sense of [J82]. Vertical morphisms are left exact (i.e., finite meet preserving) functions, and there is a cell

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ v \big\downarrow & \geq & \big\downarrow w \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

if and only if $w(f_s)_* \leq (f_t)_* v$, or equivalently, $(f_t)^* w \leq v(f_s)^*$. Then $\mathbb{L}\mathrm{oc}$ has companions and conjoints by definition of a locale homomorphism as an order preserving map $f_*\colon X \to Y$ which has a left exact left adjoint f^*.

3.2. Proposition. *Every locale Y is Cauchy complete in $\mathbb{L}\mathrm{oc}$.*

Proof. Suppose $v\colon X \to Y$ is left adjoint to $w\colon Y \to X$ in $\mathbb{L}\mathrm{oc}$. Then $vw \geq \mathrm{id}^\bullet_Y$ and $\mathrm{id}^\bullet_X \geq wv$, and so v is right adjoint to w as poset maps. Since v is left exact, it follows that $f = v$ is a locale homomorphism such that $f_* = v$. ■

3.3. Example. Objects and horizontal morphisms of $\mathbb{T}$opos are elementary toposes, in the sense of Lawvere [L71] and Tierney [T73], and geometric morphisms. Vertical morphisms are left exact (i.e., finite limit preserving) functors, with cells

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \overset{\varphi}{\Leftarrow} & \bullet\!\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

where φ is a natural transformation $w(f_s)_* \longrightarrow (f_t)_* v$, or equivalently, $(f_t)^* w \longrightarrow v(f_s)^*$. Then $\mathbb{T}$opos has companions and conjoints by definition of a geometric morphism as a limit preserving functor $f_*: X \longrightarrow Y$ whose left adjoint f^* is left exact.

3.4. Proposition. *Every topos Y is Cauchy complete in* $\mathbb{T}$opos.

Proof. Suppose $v: X \longrightarrow\!\!\!\!\!\!\bullet\;\; Y$ is left adjoint to $w: Y \longrightarrow\!\!\!\!\!\!\bullet\;\; X$ in $\mathbb{T}$opos. Then we have cells

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle vw}\big\downarrow\!\bullet & \overset{\alpha}{\Leftarrow} & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow\!\bullet & \overset{\beta}{\Leftarrow} & \bullet\!\big\downarrow{\scriptstyle wv} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array}$$

such that $(v\beta)(\alpha v) = \mathrm{id}_v$ and $(\beta w)(w\alpha) = \mathrm{id}_w$, and so v is right adjoint to w as functors. Since v is left exact, it follows that $f = v$ is a geometric morphism such that $f_* = v$. ■

3.5. Example. Objects and horizontal morphisms of $\mathbb{T}$op are topological spaces and continuous maps. Vertical morphisms $X \longrightarrow\!\!\!\!\!\!\bullet\;\; Y$ are finite intersection preserving maps $\mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$. There is a cell

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \supseteq & \bullet\!\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

if and only if $f_t^{-1} w \subseteq v f_s^{-1}$ on the open set lattices. Given $f: X \longrightarrow Y$ continuous, we know $f^{-1}: \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ preserves finite intersections and arbitrary unions, and hence, has a finite intersection preserving right adjoint $f_*: \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$. Thus, we get a companion f_* and conjoint $f^* = f^{-1}$, for every $f: X \longrightarrow Y$.

Let $\mathcal{O}: \mathrm{Top} \longrightarrow \mathrm{Loc}$ denote the functor assigning the open set lattice $\mathcal{O}(X)$ to a space X and the locale homomorphism $f: \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ such that $f^* = f^{-1}$ to a continuous map $f: X \longrightarrow Y$. Then the functor $\mathcal{O}$ has a right adjoint taking a locale L to its space $\mathrm{pt}(L)$ of points of L. Moreover, Y is a sober space (i.e., every irreducible closed set is the closure of a unique point) if and only if the unit $\eta_Y: Y \longrightarrow \mathrm{pt}(\mathcal{O}(Y))$ is an isomorphism, or equivalently, every locale homomorphism $f: \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ corresponds to a continuous map $f: X \longrightarrow Y$ (see [J82] for details).

3.6. Proposition. *A space Y is Cauchy complete in* $\mathbb{T}\mathrm{op}$ *if and only if it is sober.*

Proof. A space Y is Cauchy complete in $\mathbb{T}\mathrm{op}$ if and only if every left adjoint vertical morphism $v\colon X \xrightarrow{\bullet} Y$ is a companion of a continuous map $f\colon X \longrightarrow Y$. Since every such map $v\colon \mathcal{O}(X) \xrightarrow{\bullet} \mathcal{O}(Y)$ is a locale homomorphism, this is equivalent to the sobriety of Y. ■

4. Suplattices, Quantales, and Modules

In this section, we show that there are familiar suplattice based double categories in which every object is Cauchy complete.

4.1. Example. $\mathbb{S}\mathrm{lat}$ denotes the double category whose objects are complete lattices, called suplattices. Horizontal and vertical morphisms are sup and order preserving maps, respectively, and cells are of the form

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ v\,\big\downarrow\bullet & \leq & \bullet\big\downarrow\,w \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

Since every sup-preserving map is order preserving and has an order preserving right adjoint, it follows that $\mathbb{S}\mathrm{lat}$ has companions and conjoints. Moreover, every left adjoint vertical morphism is easily seen to be the companion of a sup preserving map, and so every object is Cauchy complete.

Recall that a quantale X is a complete lattice together with a binary operation, with unit e, and which distributes over suprema on both sides. A quantale homomorphism $f\colon X \longrightarrow Y$ is a sup preserving map satisfying $f(x)f(x') = f(xx')$ and $f(e) = e$. Although the name "quantale" was introduced by Mulvey [M86], they were originally called "closed posets" in [NR85], following a suggestion of Lawvere.

4.2. Example. $\mathbb{Q}\mathrm{uant}$ is the double category whose objects and horizontal morphisms are quantales and their homomorphisms. Vertical morphisms $v\colon X \xrightarrow{\bullet} Y$ are order preserving maps satisfying $v(x)v(x') \leq v(xx')$ and $e \leq v(e)$, with cells as in $\mathbb{S}\mathrm{lat}$. Clearly, every horizontal morphism $f\colon X \longrightarrow Y$ is its own companion. Moreover, its right adjoint $g\colon Y \longrightarrow X$ satisfies $g(y)g(y') \leq g(yy')$ and $e \leq g(e)$, since $f(e) \leq e$ and

$$f(g(y)g(y')) = f(g(y))f(g(y')) \leq yy'$$

We claim that every quantale Y is Cauchy complete. Suppose $v\colon X \xrightarrow{\bullet} Y$ is a vertical morphism with right adjoint $w\colon Y \xrightarrow{\bullet} X$. Then v is clearly sup preserving, being a left adjoint, and $v(x)v(x') \leq v(xx')$ and $e \leq v(e)$. For the reverse inequalities, we know that $v(e) \leq e$, since $e \leq w(e)$, and $v(xx') \leq v(x)v(x')$, since

$$xx' \leq w(v(x))w(v(x')) \leq w(v(x)v(x'))$$

Therefore, v is a quantale homomorphism with companion $v_* = v$, and so Y is Cauchy complete.

Suppose Q is a quantale. Recall that a left Q-module is a suplattice X together with a sup preserving map $Q \otimes X \longrightarrow X$ satisfying $a(bx) = (ab)x$ and $ex = x$. A Q-module homomorphism $f: X \longrightarrow Y$ is a sup preserving map satisfying $af(x) = f(ax)$

4.3. Example. Q-$\mathbb{M}$od is the double category whose objects and horizontal morphisms are Q-modules and their homomorphisms. Vertical morphisms $v: X \mathrel{\bullet\!\!\to} Y$ are order preserving maps satisfying $av(x) \leq v(ax)$, with cells as in $\mathbb{S}$lat. These maps played a role in the characterization [N16] of projective Q-modules. Clearly, every horizontal morphism $f: X \longrightarrow Y$ is its own companion. Moreover, its right adjoint $g: Y \longrightarrow X$ satisfies $ag(y) \leq g(ay)$, since $f(ag(y)) = af(g(y)) \leq ay$. We claim that every Q-module Y is Cauchy complete. Suppose $v: X \mathrel{\bullet\!\!\to} Y$ is a vertical morphism with right adjoint $w: Y \mathrel{\bullet\!\!\to} X$. Then v is clearly sup preserving, being a left adjoint, and $av(x) \leq v(ax)$. For the reverse inequalities, we know that $v(ax) \leq av(x)$, since $ax \leq aw(v(x)) \leq w(av(x))$. Therefore, v is a Q-module homomorphism with companion $v_* = v$, and so Y is Cauchy complete.

5. Adjoints for Bimodules and Matrices

Recently, Paré [P21] showed that an (S, R)-bimodule $M: R \mathrel{\bullet\!\!\to} S$ has a right adjoint in $\mathbb{R}$ing if and only if M is finitely generated and projective as an S-module, where $\mathbb{R}$ing is the double category of commutative rings with unit, homomorphisms, and (S, R)-bimodules; and these bimodules correspond to non-unitary homomorphisms $R \longrightarrow \mathrm{Mat}_p(S)$, where $\mathrm{Mat}_p(S)$ is the ring of $p \times p$ matrices with coefficients in S. On the other hand, finitely generated projective modules M over a commutative ring S are those for which the functor $- \otimes_S M: S\mathrm{Mod} \longrightarrow S\mathrm{Mod}$ has a left adjoint. In [NW17], we presented a general proof characterizing the latter which we applied to commutative quantales, rings, and rigs.

In this section, we show that if $\mathcal{V}$ is a bicomplete symmetric monoidal closed category, then we can drop the commutativity assumption in the [NW17] characterization and add the condition that $M: R \mathrel{\bullet\!\!\to} S$ has a right adjoint in the double category $\mathbb{B}\mathrm{im}(\mathcal{V})$ whose objects are (not-necessarily commutative) monoids, horizontal morphisms are monoid homomorphisms, vertical morphisms $M: R \mathrel{\bullet\!\!\to} S$ are (S, R)-bimodules, and cells are bimodule homomorphisms in $\mathcal{V}$. Taking $\mathcal{V}$ to be the categories of suplattices, abelian groups, and commutative monoids, we see that we can directly relate the above mentioned projectivity condition to the existence of an adjoint for quantales, rings, and rigs; and extend this to the not-necessarily commutative case. Finally, we also obtain the above mentioned matrix representation for quantales and rigs.

Recall that given $K: Q \mathrel{\bullet\!\!\to} R$, $L: Q \mathrel{\bullet\!\!\to} S$, and $M: R \mathrel{\bullet\!\!\to} S$, one can define bimodules

$M \otimes_R K : Q \mathrel{-\!\!\bullet\!\!\!\to} S$ and $S\mathrm{Mod}(M, L) : Q \mathrel{-\!\!\bullet\!\!\!\to} R$ by the coequalizer

$$M \otimes R \otimes K \underset{\rho_R \otimes K}{\overset{M \otimes \lambda_R}{\rightrightarrows}} M \otimes K \twoheadrightarrow M \otimes_R K$$

where λ_R and ρ_R are the actions of R on K and M, respectively, and the equalizer

$$S\mathrm{Mod}(M, L) \rightarrowtail [M, L] \underset{f}{\overset{[\lambda_S, L]}{\rightrightarrows}} [S \otimes M, L]$$

where f is the adjoint transpose of

$$S \otimes M \otimes [M, L] \xrightarrow{S \otimes \varepsilon} S \otimes L \xrightarrow{\lambda_S} L$$

along the adjunction $(S \otimes M) \otimes - \dashv [S \otimes M, -]$ in $\mathcal{V}$. Thus, we get an adjunction between the bimodule categories

$$(R, Q)\text{-}\mathrm{Mod}(\mathcal{V}) \underset{S\mathrm{Mod}(M,-)}{\overset{M \otimes_R -}{\rightleftarrows}} (S, Q)\text{-}\mathrm{Mod}(\mathcal{V})$$

Given a set I and a monoid S, let $I \cdot S$ denote the coproduct S-modules

$$I \cdot S = \coprod_{i \in I} S$$

5.1. Theorem. *The following are equivalent for an (S, R)-bimodule M which admits an S-module presentation $I \cdot S \rightrightarrows J \cdot S \twoheadrightarrow M$.*

(a) $M : R \mathrel{-\!\!\bullet\!\!\!\to} S$ has a right adjoint in $\mathbb{B}\mathrm{im}(\mathcal{V})$.

(b) $- \otimes_S M : (Q, S)\text{-}\mathrm{Mod}(\mathcal{V}) \to (Q, R)\text{-}\mathrm{Mod}(\mathcal{V})$ has a left adjoint, for all Q.

(c) $- \otimes_S M : (Q, S)\text{-}\mathrm{Mod}(\mathcal{V}) \to (Q, R)\text{-}\mathrm{Mod}(\mathcal{V})$ preserves limits, for all Q.

(d) The canonical (S, R)-homomorphism $\theta_M : S\mathrm{Mod}(M, S) \otimes_S M \to S\mathrm{Mod}(M, M)$ is an isomorphism.

Proof. For $(a) \Rightarrow (b)$, suppose M is left adjoint to N. Then there are cells $R \xrightarrow{\eta} N \otimes_S M$ and $M \otimes_R N \xrightarrow{\varepsilon} S$ such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{M \otimes_R \eta} & M \otimes_R N \otimes_S M \\ & {\scriptstyle \mathrm{id}_M} \searrow & \downarrow {\scriptstyle \varepsilon \otimes_S M} \\ & & M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\eta \otimes_R N} & N \otimes_S M \otimes_R N \\ & {\scriptstyle \mathrm{id}_N} \searrow & \downarrow {\scriptstyle N \otimes_S \varepsilon} \\ & & N \end{array}$$

commute. To show that $- \otimes_R N$ is left adjoint to $- \otimes_S M$, let K be a (Q, R)-module, L be a (Q, S)-module, and take η_K and ε_L given by

$$K \xrightarrow{K \otimes_R \eta} K \otimes_R N \otimes_S M \quad \text{and} \qquad L \otimes_S M \otimes_R N \xrightarrow{L \otimes_S \varepsilon} L$$

Then we have commutative diagrams

$$\begin{array}{ccc} L \otimes_S M & \xrightarrow{L \otimes_S M \otimes_R \eta} & L \otimes_S M \otimes_R N \otimes_S M \\ & {\scriptstyle \mathrm{id}_{L \otimes_S M}} \searrow & \downarrow {\scriptstyle L \otimes_S \varepsilon \otimes_S M} \\ & & L \otimes_S M \end{array} \qquad \begin{array}{ccc} K \otimes_R N & \xrightarrow{K \otimes_R \eta \otimes_R N} & K \otimes_R N \otimes_S M \otimes_R N \\ & {\scriptstyle \mathrm{id}_{K \otimes_R N}} \searrow & \downarrow {\scriptstyle K \otimes_R N \otimes_S \varepsilon} \\ & & K \otimes_R N \end{array}$$

as desired.

Clearly, $(b) \Rightarrow (c)$ holds.

To show $(c) \Rightarrow (d)$, let $I \cdot S \rightrightarrows J \cdot S \longrightarrow M$ be a coequalizer, and consider the commutative diagram

$$\begin{array}{ccccc} S\mathrm{Mod}(M, S) \otimes_S M & \longrightarrow & S\mathrm{Mod}(J \cdot S, S) \otimes_S M & \rightrightarrows & S\mathrm{Mod}(I \cdot S, S) \otimes_S M \\ {\scriptstyle \theta_M} \downarrow & & \downarrow {\scriptstyle \theta_J} & & \downarrow {\scriptstyle \theta_I} \\ S\mathrm{Mod}(M, M) & \longrightarrow & S\mathrm{Mod}(J \cdot S, M) & \rightrightarrows & S\mathrm{Mod}(I \cdot S, M) \end{array}$$

where the rows are equalizers since $S\mathrm{Mod}(-, L)$ takes coequalizers to equalizers, for all L, and $- \otimes_S M$ preserves equalizers by assumption (c). Since $S\mathrm{Mod}(-, S)$ takes coproducts to products and $- \otimes_S M$ preserves products, again using (c), we know θ_I and θ_J are isomorphisms, and it follows that θ_M is as well.

For $(d) \Rightarrow (a)$, suppose θ is invertible. We will show that $M \dashv S\mathrm{Mod}(M, S)$. Consider

$$\eta\colon R \xrightarrow{f} S\mathrm{Mod}(M, M) \xrightarrow{\theta^{-1}} S\mathrm{Mod}(M, S) \otimes_S M$$

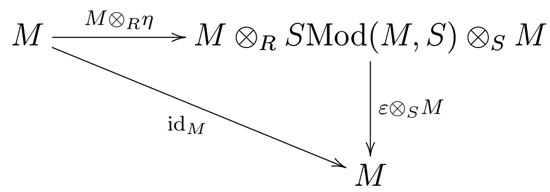
where f is the transpose of $\rho_R\colon M \otimes_R R \longrightarrow M$, and the evaluation map

$$\varepsilon\colon M \otimes_R S\mathrm{Mod}(M, S) \longrightarrow S$$

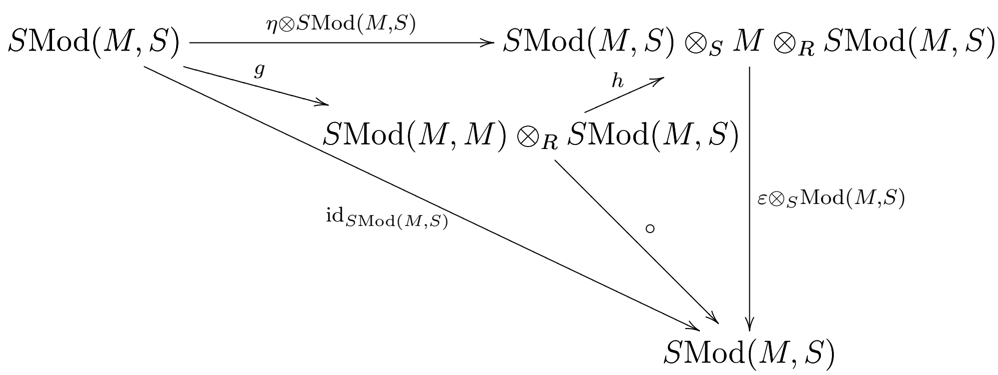
which is a left S-module homomorphism, since $S\mathrm{Mod}(M, S)$ is a right S-module and a right module homomorphism via the action of M on the left. Since

$$\begin{array}{ccc} M \otimes_R S\mathrm{Mod}(M, S) \otimes_S M & \xrightarrow{M \otimes_R \theta_M} & M \otimes_R S\mathrm{Mod}(M, M) \\ {\scriptstyle \varepsilon \otimes_S M} \downarrow & \swarrow {\scriptstyle \varepsilon} & \\ M & & \end{array}$$

commutes, by definition of θ_M, so does that diagram



Also, taking $g = f \otimes_R S\mathrm{Mod}(M,S)$ and $h = \theta^{-1} \otimes_R S\mathrm{Mod}(M,S)$, one shows that



commutes, to complete the proof. ■

Recall that when $\mathcal{V}$ is the the category $\mathbb{S}\mathrm{lat}$ of suplattices, then $\mathbb{B}\mathrm{im}(\mathcal{V})$ is a double category of quantales. Likewise, taking $\mathcal{V}$ to the category of abelian groups, respectively, commutative monoids, we get a double category of rings, respectively, rigs. Note the name "rig" was introduced by Lawvere [L92] and Schanuel [S91], to emphasize the lack of negatives in these semirings.

Recall that coproducts agree with products in the category of over a quantale, and are created by the underlying functor to the category of sets, we write the coproduct $I \cdot S$ as the product

$$S^I = \prod_{i \in I} S$$

5.2. Corollary. *The following are equivalent for an (S,R)-module M over quantales (respectively, rings or rigs).*

(a) $M\colon R \longrightarrow\!\!\bullet\!\!\rightarrow S$ has a right adjoint in $\mathbb{B}\mathrm{im}(\mathcal{V})$.

(b) $- \otimes_S M\colon (Q,S)\text{-}\mathrm{Mod}(\mathcal{V}) \longrightarrow (Q,R)\text{-}\mathrm{Mod}(\mathcal{V})$ has a left adjoint, for all Q.

(c) M is projective (respectively, and finitely generated) as an S-module.

Proof. Applying Theorem 5.1, we know that $(a) \Rightarrow (b)$ holds, and (b) implies that the canonical homomorphism $\theta\colon S\mathrm{Mod}(M,S) \otimes_S M \longrightarrow S\mathrm{Mod}(M,M)$ is an isomorphism. Consider

$$\theta^{-1}(\mathrm{id}_M) = \bigvee_{i \in I} \varphi_i \otimes m_i$$

Then $m = \bigvee_{i \in I} \varphi_i(m)m_i$, for all m, by definition of θ. Define $\tau\colon S^I \longrightarrow M$ and $\sigma\colon M \longrightarrow S^I$ by

$$\tau(\mathbf{s}) = \bigvee_{i \in I} s_i m_i \qquad \text{and} \qquad \sigma(m)_i = \varphi_i(m)$$

Then τ and σ are left S-homorphisms, and $\tau\sigma = \mathrm{id}_M$, since

$$\tau(\sigma(m)) = \bigvee_{i \in I} \varphi_i(m)m_i = m$$

and it follows that M is projective.

To prove $(c) \Rightarrow (a)$, suppose M is projective. Then $\tau\sigma = \mathrm{id}_M$, for some $\tau\colon S^I \longrightarrow M$ and $\sigma\colon M \longrightarrow S^I$. We will show that θ_M is an isomorphism and apply $(d) \Rightarrow (a)$ of Theorem 5.1. Consider the commutative diagram

$$\begin{array}{ccc}
S\mathrm{Mod}(M,S) \otimes_S M & \xrightarrow{\theta_M} & S\mathrm{Mod}(M,M) \\
\downarrow\uparrow & & \uparrow\downarrow \\
S\mathrm{Mod}(S^I,S) \otimes_S M & \xrightarrow[\theta_{S^I}]{} & S\mathrm{Mod}(S^I,M)
\end{array}$$

where the vertical morphisms are induced by σ and τ. Since coproducts agree with products and $- \otimes_S M$ preserves coproducts for modules over a quantale, it follows that θ_{S^I}, and hence, θ_M is an isomorphism.

Finally, to prove the corollary for rings and rigs, we replace the suprema by finite sums making the sets I finite, and proceed as above. ∎

We conclude with the analogue for quantales of Paré's matrix representation of projective modules. The result for (not-necessarily commutative) rings and rigs is similar.

Given a set I and a quantale S, let $\mathrm{Mat}_I(S)$ denote the quantale of $(I \times I)$-matrices with coefficients in S, i.e., the quantale of S-valued binary relation on I, in the sense of [HST14]. Now, every non-unitary homomorphism $f\colon R \longrightarrow \mathrm{Mat}_I(S)$ induces an (S,R)-bimodule defined by

$$M_f = \{\mathbf{s} \in S^I | \mathbf{s}f(e) = \mathbf{s}\}$$

with left action as in S^I and right action $\mathbf{s}\,r = \mathbf{s}f(r)$.

5.3. Theorem. *Suppose R and S are quantales and M is an (S,R)-bimodule. Then M is S-projective if and only if there is a set I and a non-unitary homomorphism $f\colon R \to \mathrm{Mat}_I(S)$ such that $M \cong M_f$ as an (S,R)-bimodule.*

Proof. Given $f\colon R \to \mathrm{Mat}_I(S)$, to show that M_f is S-projective, take σ to be the inclusion $\sigma\colon M_f \to S^I$, and define $\tau\colon S^I \to M_f$ by $\tau(\mathbf{s}) = \mathbf{s}f(e)$. Then $\tau(\mathbf{s}) \in M_f$, since $f(e) = f(e)f(e)$, and $\tau(\sigma(\mathbf{s})) = \mathbf{s}f(e) = \mathbf{s}$, since σ is the inclusion and $\mathbf{s} \in M_f$. Since $M \cong M_f$, it follows that M is S-projective.

Conversely, M is S-projective. Then $\tau\sigma = \mathrm{id}_M$, for some $\tau\colon S^I \to M$ and $\sigma\colon M \to S^I$, and so taking $m_i = \tau(\mathbf{e}_i)$, where $\mathbf{e}_i$ denotes the image of the unit e under the i^{th} coproduct injection $S \to S^I$, we see that

$$m = \bigvee_{j\in I} \sigma_j(m)m_j$$

In particular, substituting $m_i r$ and multiplying by r' on the right, we get

$$m_i r r' = \bigvee_{j\in I} \sigma_j(m_i r)m_j r'$$

Define $f\colon R \to \mathrm{Mat}_I(S)$ by $f(r)_{ij} = \sigma_j(m_i r)$. Then $f(rr')_{ik} = (f(r)f(r'))_{ik}$, since

$$\sigma_k(m_i r r') = \sigma_k\big(\bigvee_{j\in I} \sigma_j(m_i r)m_j r'\big) = \bigvee_{j\in I} \sigma_j(m_i r)\sigma_k(m_j r') = \bigvee_{j\in I} f(r)_{ij} f(r')_{jk}$$

and it follows that f is a (non-unitary) homomorphism.

To see $M \cong M_f$, we will show that $\sigma(M) = M_f$. Given $\mathbf{s} \in M_f$, we know that

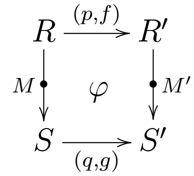
$$\mathbf{s} = \mathbf{s}f(e) = \big(\bigvee_{i\in I} s_i\sigma_j(m_i)\big)_j = \big(\sigma_j(\bigvee_{i\in I} s_i m_i)\big)_j = \sigma\big(\bigvee_{i\in I} s_i m_i\big)$$

and so $\mathbf{s} \in \sigma(M)$. Now, suppose $\mathbf{s} \in \sigma(M)$, say $s = \sigma(m)$, where $m \in M$. Then

$$\mathbf{s} = \sigma(m) = (\sigma_i(m))_i = \Big(\bigvee_{j\in I} \sigma_j(m)\sigma_i(m_j)\Big)_i = \sigma(m)f(e) = \mathbf{s}f(e)$$

as desired. ■

We conclude with a final remark. After showing that left adjoint bimodules are given by non-unitary homomorphisms, Paré [P21] introduced the following double category $\mathbb{A}$mpli of commutative rings in which every ring is Cauchy complete. Horizontal morphisms $R \to S$ are pairs (p,f), where $f\colon R \to \mathrm{Mat}_p(S)$ is a non-unitary homomorphism, vertical morphisms are the same as in $\mathbb{R}$ing, but cells



are of the form

$$\begin{array}{ccc} R & \xrightarrow{(p,f)} & \mathrm{Mat}_p(R') \\ {\scriptstyle M}\big\downarrow\!\bullet & \underset{\varphi}{\rightarrow} & \bullet\!\big\downarrow{\scriptstyle \mathrm{Mat}_{q,p}(M')} \\ S & \xrightarrow[(q,g)]{} & \mathrm{Mat}_q(S') \end{array}$$

where $\mathrm{Mat}_{q,p}(M')$ is the set of $q \times p$ matrices with entries in M' and φ is an appropriate additive map.

Although this construction of $\mathbb{A}$mpli works well for (non-commutative) rigs and quantales, we leave the latter for a future paper in which we develop the necessary properties of non-finite dimensional matrices over quantales and, more generally, suplattices.

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