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FACTORIZATION SYSTEMS FOR RESTRICTION CATEGORIES

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Abstract. This paper describes the notion of a (latent) factorization system for a restriction category. Analogous to factorization systems for ordinary categories, a description in terms of a Galois connection based on an orthogonality relation is given. The orthogonality relation involving a unique cross-map (or lifting), however, is between a pair, consisting of a map together with a restriction idempotent, and a map. This gives a significantly different character to the theory.

A restriction category in which the idempotents split is precisely a partial map category whose partiality is given by the system of monics which split the restriction idempotents. A restriction factorization on such a partial map category is completely characterized by a factorization on the total maps which is stable with respect to the system of monics; that is, pullbacks of both $\mathcal{E}$ and $\mathcal{M}$-maps along these special monics are (respectively) $\mathcal{E}$ and $\mathcal{M}$-maps.

Examples of latent factorization systems are discussed. In particular, one source of latent factorizations is provided by lifting latent factorizations from the base of a latent fibration into the total category: this is a generalization of the observation that one can lift a factorization from the base of an ordinary fibration into the total category.

1. Introduction

For the authors of this paper, the issue of what a factorization system looks like in a restriction category arose while studying latent fibrations in [5]. Latent fibrations are the analogue of fibrations for restriction categories. In an analogous manner to an ordinary fibration, we had reasoned, a latent fibration should induce a factorization system on the maps of the "total" category. The desire to understand the analogue of this observation for latent fibrations led to the development of the notion of factorization system for restriction categories, which we call a **latent factorization system**. As the name suggests – and, indeed, somewhat to our frustration – the notion of a latent factorization was not as completely obvious as we initially thought it might be. This ultimately led us to splitting the development of latent factorization systems away from the development of latent fibrations and to create, as far as is possible, a self-contained exposition. However, the development of latent fibrations and latent factorizations was synergistic, as the latter used an important bit of technology from

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the former: namely the notion of *precise* diagrams.

Although we did not realize it at the time, there was already a well-understood source of examples of a latent factorization system, for range categories [3] are, in fact, just restriction categories with a (special) latent factorization. In retrospect, this example – if we had but realized it was an example – would have already told us that the notion of factorization in a restriction category was going to have a rather different feel from a factorization in an ordinary category.

The paper [3] was, of course, developed with Pieter Hofstra. His remarkable gift for clarity of thought and careful exposition is all over that paper (and indeed its sequel [4]). In as much as this paper provides a new perspective on range categories, we feel it appropriate to dedicate this work to Pieter even if we cannot hope to match his clarity of exposition.

To see that range categories provide a source of examples of restriction categories with a latent factorization, recall that a restriction category, in which the idempotents split, is precisely a partial map category, [2], in which partiality is given by the system of monics which split the restriction idempotents. A latent factorization on such a partial map category should be – and we shall prove that it actually *is* – completely characterized by a factorization on the total maps which is stable with respect to pulling back along maps in the system of monics. That is, pullbacks of both $\mathcal{E}$ and $\mathcal{M}$-maps along these special monics exist and are (respectively) $\mathcal{E}$ and $\mathcal{M}$-maps. That a split range category provides an example of this was proven in [3, Theorem 4.7]. Notice, however, this is a rather special case of a latent factorization because the $\mathcal{M}$-maps for range categories also provide the system of monics – which is not necessarily the case for a general latent factorization.

Factorization systems in ordinary categories are often formulated as factorizations satisfying an orthogonality relation involving a unique cross-map (or lifting) [1]. One of our first objectives was to give an analogous description of latent factorizations. Indeed, obtaining this correspondence guided the manner in which a latent factorization was defined. As restriction categories are not self-dual, the notion of orthogonality for restriction categories takes a correspondingly non-self-dual character. In particular, in contrast to the orthogonality relation for mere categories, which is between maps, the orthogonality relation in restriction categories is between a pair, consisting of a map and a restriction idempotent, and a map. The theory then runs a parallel, albeit more complicated path to the corresponding theory for mere categories.

That the notion of a latent factorization system, which we introduce in Definition 3.8, is the correct one must be justified. A basic requirement is that when the restriction category is total, a latent factorization should reduce to the standard notion of a factorization: this is, indeed, the case. A more significant requirement is that in a *split* restriction category it should be synonymous with having a factorization system on the subcategory of total maps viewed as an M-category. In fact, this should be not merely any factorization system but rather an M-stable factorization system. It is this last requirement which has a far reaching effect on the form of a latent factorization system.

Proposition 4.1, below, shows that if one has a latent factorization on a restriction category $\mathbb{X}$ then this extends uniquely to a latent factorization on its category of split restriction idempotents $\mathsf{Split}_r(\mathbb{X})$. Proposition 4.3 shows that a latent factorization on a split restriction category induces an M-stable factorization on its category of total maps. Proposition 4.4 says that having an M-stable factorization on an M-category, $(\mathbb{X}, \mathsf{M})$, induces a latent factorization system on the partial map category, $\mathsf{Par}(\mathbb{X}, \mathsf{M})$, which is, of course, a split restriction category. Furthermore, these last two transitions are inverse to each other. These observations are collected into Theorem 4.5 which states that to have an M-category with an M-stable factorization system is precisely to have a split restriction category with a latent factorization.

As mentioned above, this work was originally motivated by the desire for an analogous result for latent fibrations to that of ordinary fibrations, which says that, for a fibration, a factorization on the base always lifts to a factorization on the total category. In the last section we turn to this issue and show in Proposition 5.3 that latent factorizations "fit" with the notion of a latent fibration in the sense that they lift along latent fibrations.

2. Restriction preliminaries

A **restriction category** (see [2] for details) is a category equipped with a **restriction** combinator which given a map $f : A \longrightarrow B$, returns an endomorphism on the domain $\overline{f} : A \longrightarrow A$ which satisfies just four identities[1]:

$$\text{[R.1]}\ \overline{f}f = f \qquad \text{[R.2]}\ \overline{f}\ \overline{g} = \overline{g}\ \overline{f} \qquad \text{[R.3]}\ \overline{f}\ \overline{g} = \overline{\overline{f}g} \qquad \text{[R.4]}\ f\overline{g} = \overline{fg}f$$

The prototypical restriction category is the category of sets and partial maps, Par. The restriction of a partial map in Par is the partial identity on the domain which is defined precisely when the partial map is defined.

In a restriction category $\overline{f}$ is always an idempotent: an idempotent, $e = ee$, in a restriction category is a **restriction idempotent** when $e = \overline{e}$. It is not the case that every idempotent need be a restriction idempotent. The restriction idempotents on an object A form a meet semilattice (with the meet given by composition) which we denote by $\mathcal{O}(A)$. The elements of the semilattice $\mathcal{O}(A)$ are the predicates of the object A which are determined by the restriction.

Restriction categories are always full subcategories of partial map categories (see [2, Proposition 3.3]) and this means that parallel maps in a restriction category can be partially ordered: this can be expressed directly using the restriction by setting $f \leq g$ if and only if $\overline{f}g = f$; this gives a partial order on parallel maps called the **restriction order** which is an enrichment. If parallel partial maps agree where they are both defined, they are said to be **compatible**; thus, f is compatible with g in a restriction category, written $f \smile g$, if and only if $\overline{f}g = \overline{g}f$. This is also an enrichment but into symmetric reflexive relations: in [6] the partial order and compatibility – there called "linking" relations – are combined into a single enrichment into "cohesive" sets.

[1] Note that we are writing composition in diagrammatic order in this paper, so that f, followed by g, is written as fg.

A map $f : A \longrightarrow B$ in a restriction category is said to be **total** in case $\overline{f} = 1_A$. Total maps compose and form a (non-full) subcategory, $\mathsf{Total}(\mathbb{X}) \subseteq \mathbb{X}$.

2.1. Partial isomorphisms and inverse categories. A map $s : A \longrightarrow B$ in a restriction category is a **partial isomorphism** if there is a map $s^{(-1)} : B \longrightarrow A$, the **partial inverse** of s, with $ss^{(-1)} = \overline{s}$ and $s^{(-1)}s = \overline{s^{(-1)}}$. We recall:

2.2. Lemma. *In any restriction category:*

(i) *Partial inverses are unique;*

(ii) *Partial isomorphisms are closed under composition;*

(iii) *Restriction idempotents are precisely partial isomorphisms which are idempotent;*

(iv) *If s is a partial isomorphism then $ss^{(-1)}s = s$ and $s^{(-1)}ss^{(-1)} = s^{(-1)}$;*

(v) *If $s_1, s_2 : A \longrightarrow B$ are partial isomorphisms then $s_1 \smile s_2$ if and only if $s_2^{(-1)}s_1$ is a restriction idempotent.*

Proof.

(i) Suppose both u and v are partial inverses to s then we have: $u = \overline{u}u = usu = u\overline{s} = usv = \overline{u}v$ so $u \leq v$ and similarly $v \leq u$ so they are equal.

(ii) The partial inverse of st is $(st)^{(-1)} := t^{(-1)}s^{(-1)}$ and for identity maps $1_A^{(-1)} = 1_A$.

(iii) It is clear that restriction idempotents are partial isomorphisms; conversely, if e is an idempotent which is a partial isomorphism then $\overline{e} = ee^{(-1)} = eee^{(-1)} = e\overline{e} = \overline{\overline{e}e} = \overline{e}e = e$.

(iv) $ss^{(-1)}s = \overline{s}s = s$ and similarly, $s^{(-1)}ss^{(-1)} = \overline{s^{(-1)}}s^{(-1)} = s^{(-1)}$.

(v) Assume $s_2^{(-1)}s_1 = \overline{s_2^{(-1)}s_1}$ then $\overline{s_2}s_1 = s_2s_2^{(-1)}s_1 = s_2\overline{s_2^{(-1)}s_1} = \overline{s_2s_2^{(-1)}s_1}s_2 = \overline{\overline{s_2}s_1}s_2 = \overline{s_1}s_2$ so that $s_1 \smile s_2$. Now assume $\overline{s_1}s_2 = \overline{s_2}s_1$ then $s_2^{(-1)}s_1 = s_2^{(-1)}\overline{s_2}s_1 = s_2^{(-1)}\overline{s_1}s_2 = \overline{s_2^{(-1)}s_1}s_2^{(-1)}s_2 = \overline{s_2^{(-1)}s_1}\ \overline{s_2^{(-1)}}$ and so $s_2^{(-1)}s_1$ is a restriction idempotent.

∎

A restriction category in which all maps are partial isomorphisms is called an **inverse category**. The partial isomorphisms of any restriction category form a subcategory (which includes all restriction idempotents) and is an inverse category. Inverse categories are to restriction categories as groupoids are to ordinary categories.

There are two additional characterizations of inverse categories:

2.3. Theorem. [Theorem 2.20 in [2]] *The following are equivalent:*

(i) $\mathbb{X}$ *is an inverse category (that is, a restriction category in which all maps are partial isomorphisms);*

(ii) $\mathbb{X}$ *is a category with an involution* $(\_)^* : \mathbb{X}^{\text{op}} \longrightarrow \mathbb{X}$ *which is stationary on objects so that* $(f^{(-1)})^{(-1)} = f$, $ff^{(-1)}f = f$, *and* $ff^{(-1)}gg^{(-1)} = gg^{(-1)}ff^{(-1)}$*;*

(iii) $\mathbb{X}$ *is a category in which each map* $f : A \longrightarrow B$ *has a unique map* $f^{(-1)} : B \longrightarrow A$ *satisfying* $ff^{(-1)}f = f$ *and* $f^{(-1)}ff^{(-1)} = f^{(-1)}$.

In a category with such an involution the restriction of a map $f : A \longrightarrow B$ may be defined by $\overline{f} := ff^{(-1)}$. The restriction of the partial inverse of f we shall often write as $\widehat{f} := \overline{f^{(-1)}} = f^{(-1)}f$: this we shall see below is an example of a range combinator. In an inverse category all idempotents are restriction idempotents by Lemma 2.2 (iii).

2.4. Restriction functors. There are various sorts of morphisms between restriction categories which can be considered: the most basic is that of a **restriction functor**, $F : \mathbb{X} \longrightarrow \mathbb{Y}$, which is a functor between restriction categories which in addition preserves the restriction structure, that is $F(\overline{f}) = \overline{F(f)}$. Restriction functors so defined preserve total maps, restriction idempotents, and partial isomorphisms.

In this paper we will also consider **restriction semifunctors**; a restriction semifunctor F from X to Y is a semifunctor $X \longrightarrow Y$ (that is, a map on objects and arrows which preserves domains, codomains, and composition, but not necessarily identities) which preserves restrictions. In this case, while $F(1_X)$ is not the identity, it is still a restriction idempotent since

$$\overline{F(1_X)} = F(\overline{1_X}) = F(1_X).$$

2.5. Open maps and range categories. A ready source of examples of restriction categories with a latent factorization is given by range restriction categories. This subsection recalls this idea.

A map $f : A \longrightarrow B$ in a restriction category is **open** in case there is a poset morphism $f_! : \mathcal{O}(A)/\overline{f} \longrightarrow \mathcal{O}(B)$ between restriction semilattices which is Frobenius left adjoint to the map induced by the restriction $f^*(\_) : \mathcal{O}(B) \longrightarrow \mathcal{O}(A)/\overline{f} : e \mapsto \overline{fe}$. Being a Frobenius left adjoint means that in addition to the usual conditions the adjunction satisfies the Frobenius identity (see [9, A.1.5]):

$$e' \wedge f_!(e) = f_!(f^*(e') \wedge e)$$

Recall that it is automatic that $f_!(f^*(e') \wedge e) \leq e' \wedge f_!(e)$, so the content of the identity is in the reverse inequality.

Explicitly this means, see [3, Lemma 2.6], that $f : A \longrightarrow B$ is an **open** map in case there is a map $\exists_f : \mathcal{O}(A) \longrightarrow \mathcal{O}(B); e \mapsto f_!(\overline{f}e)$ such that:

$$\text{[Open.1] } \exists_f(\overline{fe'}) \leq e' \quad \text{[Open.2] } e\overline{f} \leq \overline{f\exists_f(e)} \quad \text{[Open.3] } e'\exists_f(e) \leq \exists_f(\overline{fe'}e).$$

Here the first identity is the counit of the adjunction, the second is the unit, while the third is the direction of the Frobenius identity which is not automatic.

A category is a **range** (restriction) category in case it is a restriction category in which all maps are open. In this case one obtains a combinator, see [3], which given a map $f : A \longrightarrow B$ returns a restriction idempotent on the codomain $\hat{f} : B \longrightarrow B$, where $\hat{f} := \exists_f(1_A)$, which satisfies:

$$\text{[rR.1]}\ \overline{\hat{f}} = \hat{f} \quad \text{[rR.2]}\ f\hat{f} = f \quad \text{[rR.3]}\ \widehat{f\overline{g}} = \hat{f}\ \overline{g} \quad \text{[rR.4]}\ \widehat{\hat{f}g} = \widehat{fg}$$

We observe:

2.6. Lemma. *In a restriction category one may equivalently define an open map as a map $f : A \longrightarrow B$ with a combinator $\widehat{(\_)}$ on the arrows $\{f' | f' \leq f\}$ satisfying* [rR.1-3].

Proof. Let us start by showing that [rR.1-3] imply [Open.1-3]. We define $\exists_f(e) := \widehat{ef}$ and we must start by showing this defines a monotone map, that is if $e \leq e'$ then $\widehat{ef} \leq \widehat{e'f}$. But we have

$$\overline{\widehat{ef}}\ \widehat{e'f} = \widehat{ef}\ \widehat{e'f} = \widehat{ef\widehat{e'f}} = \widehat{ee'f\widehat{ef}} = \widehat{ee'f} = \widehat{ef}$$

establishing the inequality. For the remaining conditions:

$$\begin{aligned} \exists_f(\overline{fe'}) \leq e' &:= \widehat{\overline{fe'}f} = \widehat{fe'} = \hat{f}e' \leq e' \\ \overline{f\exists_f(e)} &:= \overline{f\widehat{ef}} \geq \overline{ef\widehat{ef}} = \overline{ef} = e\overline{f} \\ e'\exists_f(e) &:= e'\widehat{ef} = \widehat{ef}e' = \widehat{efe'} = \widehat{e\overline{fe'}f} =: \exists_f(\overline{fe'}e). \end{aligned}$$

Now let us show that [Open,1-3] imply [rR.1-3]:

$$\begin{aligned} \overline{\hat{f}} &:= \overline{\exists_f(1_B)} = \exists_f(1_B) =: \hat{f} \\ f\hat{f} &:= f\exists_f(1_B) = \overline{f\exists_f(1_B)}f \geq 1_B\overline{f}f = f \ \text{ but also } f\hat{f} \leq f \\ \widehat{f\overline{g}} &:= \exists_f(\overline{f\overline{g}}1_B) = \overline{g}\exists_f(1_B) =: \hat{f}\ \overline{g}. \end{aligned}$$

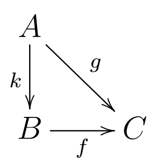
■

Notice that **[rR.4]** just defines the composite effect of an open map on idempotents.

An inverse category is self-dual, thus, it has in addition a *corestriction* combinator, $\hat{f} := f^*f : B \longrightarrow B$, introduced above. This one can easily check is a range combinator. It follows from this that partial isomorphisms in any restriction category are always open. In the category of sets and partial maps every map is open: so this is a range category. However, most restriction categories are not range categories and, thus, not every map will be open. A prototypical example in which not every map is open is the category of topological spaces with maps with domain of definition open sets. This example motivated the terminology $\mathcal{O}(X)$. However, from any restriction category one can always extract the subcategory of open maps: this subcategory will be a range category which, in particular, contains all the restriction idempotents.

2.7. M-categories and split restriction categories. A restriction category is **split**, if all its restriction idempotents split. Given an arbitrary restriction category, $\mathbb{X}$, one may always split its restriction idempotents to obtain, $\mathsf{Split}_r(\mathbb{X})$, a split restriction category. The 2-category of split restriction categories, restriction functors, and total natural transformations (i.e., natural transformations in which each component is a total map) is 2-equivalent to the 2-category of M-categories [2]. M-categories are categories with a system of monic maps M which is closed to composition and pullbacks along any map. Functors between M-categories must not only preserve the M-maps but also the pullbacks along M-maps. Natural transformations between M-functors are natural transformations which, in addition, are Cartesian (or tight) for transformations between M-maps – that is, the naturality squares for M-maps are pullbacks. The 2-equivalence is given on the one hand, by moving from an M-category $(\mathbb{X}, \mathsf{M}_{\mathbb{X}})$ to its partial map category $\mathsf{Par}(\mathbb{X}, \mathsf{M}_{\mathbb{X}})$ and on the other hand by moving to the M-category consisting of the total map category of the split restriction category, $\mathsf{Total}(\mathbb{E})$, with the restriction monics, $\mathsf{Monic}(\mathbb{E})$, $(\mathsf{Total}(\mathbb{E}), \mathsf{Monic}(\mathbb{E}))$. The restriction monics can be variously described as partial isomorphisms which are total, restriction sections, or, more interestingly, as left adjoints – "maps" – with respect to the partial order enrichment.

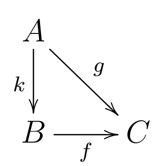
2.8. Precise diagrams. We shall call a commuting triangle of maps



precise in case $\overline{g} = \overline{k}$. More generally a commuting diagram in a restriction category is precise if the maps leaving the starting node have the same restriction as the map to the final node expressed by the whole diagram. We refer to k as the **left factor** (and f the right factor) of the triangle. Observe that:

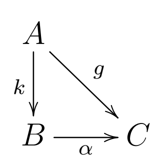
2.9. Lemma. *In any restriction category:*

(i) A commuting triangle



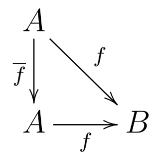
is precise if and only if $k\overline{f} = k$.

(ii) A commuting triangle with right factor a restriction idempotent or, more generally, a partial isomorphism



is precise if and only if $k = g\alpha^{(-1)}$ *(when* α *is a restriction idempotent this means* $k = g$*).*

(iii) The commuting triangle



is always precise.

PROOF.

(i) We have the calculations:

($\Rightarrow$) If $\overline{g} = \overline{k}$ then as $g = kf$ we have $k\overline{f} = \overline{kf}k = \overline{g}k = \overline{k}k = k$.

($\Leftarrow$) If $k\overline{f} = k$ then $\overline{g} = \overline{kf} = \overline{k\overline{f}} = \overline{k}$.

(ii) We use *(i)* above: when the triangle is precise we have $g\alpha^{(-1)} = k\alpha\alpha^{(-1)} = k\overline{\alpha} = k$. Conversely, if $k = g\alpha^{(-1)}$ we have

$$k\overline{\alpha} = \overline{k\alpha}k = \overline{g\alpha^{(-1)}\alpha}k = \overline{g\overline{\alpha^{(-1)}}}k = \overline{g\alpha^{(-1)}}k = \overline{k}k = k$$

showing the triangle is precise.

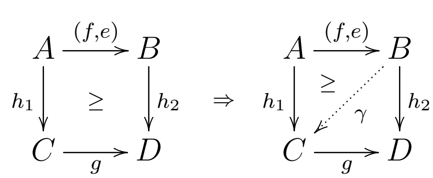
(iii) $\overline{f}f = f$ is precise with left factor $\overline{f}$ because $\overline{\overline{f}} = \overline{f}$.

■

3. Orthogonality and latent factorization systems

Fundamental to ordinary factorization systems is the orthogonality relation between $\mathcal{E}$- and $\mathcal{M}$-maps: in this section we develop the analogous story for restriction categories. Interestingly, the notion of an orthogonality relation for a restriction category is not a relation between maps but rather between a pair, consisting of a map and a restriction idempotent, and a map. While this difference is quite fundamental, a parallel theory to that of factorizations in mere categories will still emerge from the details.

3.1. DEFINITION. *In a restriction category* $\mathbb{X}$ *we say* $(f,e) \perp_r g$ *– pronounced* (f,e) *is* **(restriction) orthogonal** *to* g *– where* $f, e, g \in \mathbb{X}$*,* e *is a restriction idempotent on the codomain of* f *such that* $\overline{e} = e$ *and* $fe = f$*, when for all lax squares with* $eh_2 = h_2$*, there is a unique cross map* γ*:*



with $\gamma g = h_2$ and $\overline{\gamma} = \overline{h_2}$ (in other words, the lower triangle is precise).

We shall call a composite fg **split** at e when $fe = f$ and $eg = g$: this is clearly related to composition in the idempotent splitting. We shall indicate the split composition as $(f, e)g$; a composition fg which is split at e is **precise** when $\overline{g} = e$. Note that in the definition above, the composite fh_2 is required to split at e.

This relation gives a Galois connection between classes $\mathcal{E}$ of pairs of the form (f, e) above and classes of maps $\mathcal{M}$. We then make the following definitions:

3.2. DEFINITION. *In a restriction category:*

- *We write $\mathcal{E} \perp_r \mathcal{M}$ to indicate that every member of $(f, e) \in \mathcal{E}$ is restriction orthogonal to every member in $m \in \mathcal{M}$ (that is $\forall (f, e) \in \mathcal{E}, m \in \mathcal{M}, (f, e) \perp_r m$).*

- *Given any class of maps $\mathcal{M}$ we may form the class of pairs*

$$^{\perp}\mathcal{M} = \{(f, e) | \forall m \in \mathcal{M}, (f, e) \perp_r m\}$$

and similarly, given $\mathcal{E}$ a class of pairs we may form

$$\mathcal{E}^{\perp} = \{m | \forall (f, e) \in \mathcal{E}, (f, e) \perp_r m\}.$$

- *Two classes $\mathcal{E} \perp_r \mathcal{M}$ are* **maximally** *orthogonal in case $\mathcal{E} = {}^{\perp}\mathcal{M}$ and $\mathcal{E}^{\perp} = \mathcal{M}$.*

- *A class of pairs $\mathcal{E}$ is **maximal** if $\mathcal{E} = {}^{\perp}(\mathcal{E}^{\perp})$ and, similarly, a class of maps $\mathcal{M}$ is **maximal** if $\mathcal{M} = ({}^{\perp}\mathcal{M})^{\perp}$.*

It is standard from the theory of Galois connections that if $\mathcal{E} \perp_r \mathcal{M}$ is a possibly non-maximal pair of restriction orthogonal classes, then ${}^{\perp}\mathcal{M} \perp_r \mathcal{E}^{\perp}$ is the maximal pair of classes with $\mathcal{E} \subseteq {}^{\perp}\mathcal{M}$ and $\mathcal{M} \subseteq \mathcal{E}^{\perp}$.

3.3. LEMMA. *In any restriction category:*

(i) For any maximal $\mathcal{E}$, if α is a partial isomorphism then $(\alpha, \widehat{\alpha}) \in \mathcal{E}$;

(ii) If (f, e) is restriction orthogonal to all maps, then f is a partial isomorphism and $e = \widehat{f}$;

(iii) Any maximal $\mathcal{E}$ is closed to split composition; that is, if $(f, e), (f', e') \in \mathcal{E}$ and $ef' = f'$ then $(f, e)(f', e') := (ff', e') \in \mathcal{E}$;

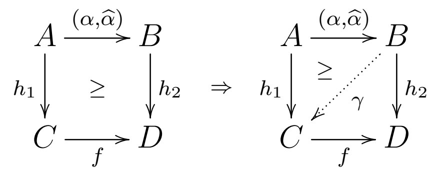
(iv) Any maximal $\mathcal{E}$ is closed to **right extension** *by partial isomorphisms; that is, if $(f, e) \in \mathcal{E}$ and α is a partial isomorphism then $(f\alpha, \widehat{e\alpha}) \in \mathcal{E}$;*

(v) Any maximal $\mathcal{M}$ is closed to composition and contains all isomorphisms;

(vi) Any maximal $\mathcal{M}$ is closed to **left extension** *by partial isomorphisms; that is, if $m \in \mathcal{M}$ and α is a partial isomorphism with $\widehat{\alpha} m = m$ then $\alpha m \in \mathcal{M}$.*

PROOF.

(i) We show that setting $\gamma := \overline{h_2}\alpha^{(-1)}h_1$ provides the desired cross map for any square so that



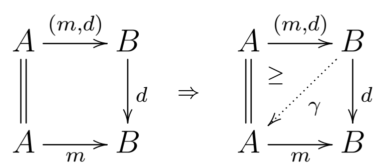
where we have:

$$\begin{array}{rcl} \alpha\gamma & := & \alpha\overline{h_2}\alpha^{(-1)}h_1 \leq \alpha\alpha^{(-1)}h_1 = \overline{\alpha}h_1 \leq h_1 \\ \gamma f & := & \overline{h_2}\alpha^{(-1)}h_1 f \geq \overline{h_2}\alpha^{(-1)}\alpha h_2 = \widehat{\alpha}h_2 = h_2 \text{ whence } \gamma f = h_2 \text{ since } \overline{\gamma} \leq \overline{h_2} \\ \overline{\gamma} & \geq & \overline{\gamma f} = \overline{h_2} \text{ but } \overline{\gamma} = \overline{\overline{h_2} f^{(-1)} h_1} \leq \overline{h_2} \text{ whence } \overline{\gamma} = \overline{h_2}. \end{array}$$

Uniqueness of this map is straightforward.

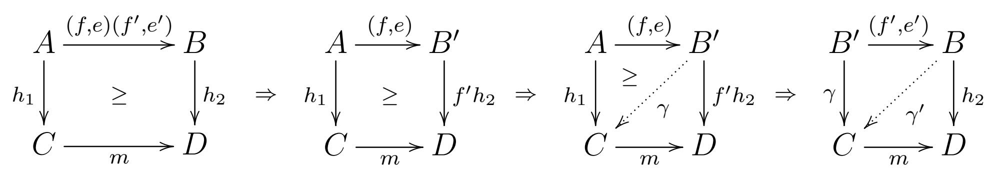
In the special case when f is a partial isomorphism, we note that there is an alternative simpler expression for γ as $h_2 f^{(-1)}$ because $\gamma = \gamma\overline{f} = \gamma f f^{(-1)} = h_2 f^{(-1)}$.

(ii) If $(m, d) \perp_r m$ then we have



with $\overline{\gamma} = \overline{d} = d$. However, then $\gamma m = d$ and $m\gamma \leq 1$ so $m\gamma = \overline{m\gamma} = \overline{m\overline{\gamma}} = \overline{md} = \overline{m}$. So m is a partial isomorphism with $\widehat{m} = d$.

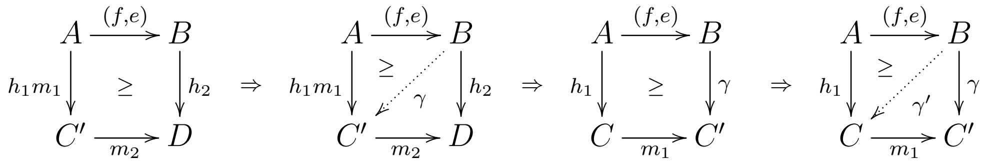
(iii) We have:



and it is easily shown that γ' supplies the required cross map.

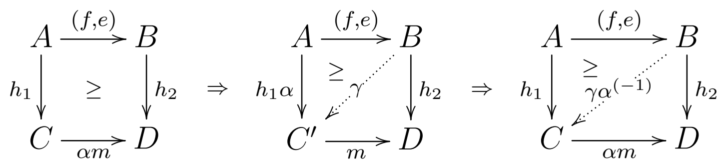
(iv) Note that $(f\alpha, \widehat{e\alpha}) = (f, e)(e\alpha, \widehat{e\alpha})$ where $(e\alpha, \widehat{e\alpha}) \in \mathcal{E}$ by *(i)*, above, and the composition is split at e (with $fe\alpha = f\alpha$) so, by *(iii)* above, this means that $(f\alpha, \widehat{e\alpha})$ is in $\mathcal{E}$.

(v) To show $\mathcal{M}$ is closed to composition we have:



That isomorphisms are in $\mathcal{M}$ is immediate.

(vi) We have:



In order to show the last diagram has the correct form we have (using $\widehat{\alpha}m = m$):

$$\begin{array}{rcl}\gamma\alpha^{(-1)}\alpha m & = & \gamma\widehat{\alpha}m = \gamma m = h_2 \\ f\gamma\alpha^{(-1)} & \leq & h_1\alpha\alpha^{(-1)} = h_1\overline{\alpha} \leq h_1 \\ \overline{\gamma\alpha^{(-1)}} & \leq & \overline{\gamma} = \overline{h_2} \\ \overline{h_2} & = & \overline{\gamma\alpha^{(-1)}\alpha m} \leq \overline{\gamma\alpha^{(-1)}}.\end{array}$$

■

Using Lemma 3.3 *(i)* and *(ii)* we observe:

3.4. Corollary. *In any restriction category, the class $\mathcal{E}_0 = \{(\alpha, \widehat{\alpha}) | \alpha$ is a partial iso$\}$ is maximally restriction orthogonal to all maps.*

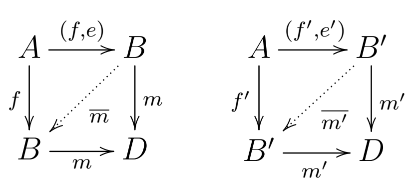
We can now define what it means to factor a map relative to an orthogonal pair $\mathcal{E} \perp_r \mathcal{M}$:

3.5. Definition. *Given two orthogonal such classes $\mathcal{E} \perp_r \mathcal{M}$, we say a map h has a* **precise $(\mathcal{E}, \mathcal{M})$-factorization** *in case there is an $(f, e) \in \mathcal{E}$ and an $m \in \mathcal{M}$ such that $e = \overline{m}$ and $fm = h$.*

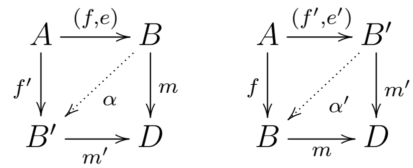
This has an expected uniqueness property:

3.6. Lemma. *Given two orthogonal classes, $\mathcal{E} \perp_r \mathcal{M}$, if $(f, e)m = h = (f', e')m'$ are two different precise $(\mathcal{E}, \mathcal{M})$-factorizations of h, then there is a unique partial isomorphism α, with $\overline{\alpha} = e$, $(f', e') = (f\alpha, \widehat{\alpha})$ and $\alpha m' = m$.*

Proof. For any precise factorizations of h we have the self cross maps given by:



However, given the two factorizations we also have the two cross maps



where we claim that the upper triangles commute on the nose. For this it suffices to show $f\alpha = f'$. We certainly have $f\alpha \leq f'$ which equivalently means $f\alpha = \overline{f\alpha}f'$. However, we now observe that $\overline{f\alpha} = \overline{f'}$: for this first observe $\overline{f'} = \overline{f'e'} = \overline{f'\overline{m'}} = \overline{f'm'} = \overline{h}$, so that $f\alpha = \overline{f\alpha}f' = \overline{f\overline{\alpha}}f' = \overline{f\overline{m}}f' = \overline{fm}f' = \overline{h}f' = f'$.

The maps α and α' now compose to give the self cross maps above and thus are partial inverses from which the required properties are immediate. ■

We may reverse this result to obtain a characterization of precise factorizations for a pair of maximally orthogonal classes:

3.7. Proposition. *If $\mathcal{E} \perp_r \mathcal{M}$ are maximally orthogonal and $h = (f,e)m$ is a precise $(\mathcal{E},\mathcal{M})$-factorization, then every precise $(\mathcal{E},\mathcal{M})$-factorization is of the form $h = (f\alpha, \widehat{\alpha})\alpha^{(-1)}m$ for some partial isomorphism α with $\overline{\alpha} = e$.*

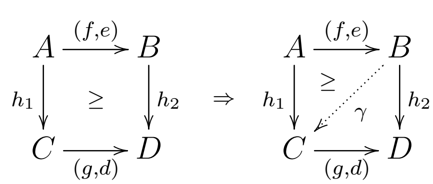
Proof. First note that $f\alpha\alpha^{(-1)}m = f\overline{\alpha}m = fem = fm = h$ and by Lemma 3.3 *(iii)* $(f\alpha, \widehat{\alpha})$ is in $\mathcal{E}$. By Lemma 3.3 *(v)*, as $\widehat{\alpha^{(-1)}} = \overline{\alpha} = \overline{m}$, $\alpha^{(-1)}m \in \mathcal{M}$, so this is a precise $(\mathcal{E},\mathcal{M})$-factorization. Conversely by Lemma 3.6 any precise $(\mathcal{E},\mathcal{M})$-factorization is of this form. ■

Thus these "precise factorizations" can pass through intermediate objects which are wildly different but, nonetheless, must have patches (determined by the partial isomorphism) which are isomorphic. In this sense they are precisely what we expect latent factorizations to be.

3.8. Definition. *A* **latent factorization system** *on a restriction category consists of two maximally orthogonal classes $\mathcal{E} \perp_r \mathcal{M}$ of maps such that every map admits a precise $(\mathcal{E},\mathcal{M})$-factorization.*

Before giving examples of latent factorization systems, it is useful to have different equivalent forms for this definition (as for factorization systems of mere categories); to establish these results we need to develop the theory further. To facilitate this development it is useful to introduce a slightly finer notion of orthogonality between map/restriction idempotent pairs:

3.9. Definition. *We shall say that $(f,e) \perp_r (g,d)$, where $e = \overline{e}$, $fe = f$, $d = \overline{d}$, and $gd = g$, if for every square on the left, in which $eh_2d = h_2$ we have a unique cross map γ, as indicated in the square on the right with $\overline{\gamma} = \overline{h_2}$:*



It is clear that if $(f, e) \perp_r g$ then certainly $(f, e) \perp_r (g, d)$. The converse may not hold as in the latter case h_2 satisfies an additional property.

This allows us to state the following somewhat technical lemma:

3.10. Lemma.

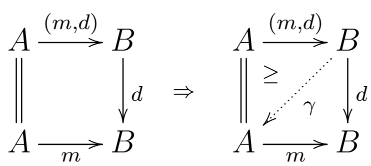
(i) Whenever $(m, d) \perp_r (m, d)$ then m is a partial isomorphism and $d = \widehat{m}$;

(ii) If $(f, e)(f', e') = (ff', e')$ is a precise composite with $(f, e) \perp_r m$ and $(ff', e') \perp_r m$ then $(f', e') \perp_r m$;

(iii) If for any h, $(g, d)h = gh$ is a precise composite, $(f, e) \perp_r h$, and $(f, e) \perp_r gh$ then $(f, e) \perp_r (g, d)$.

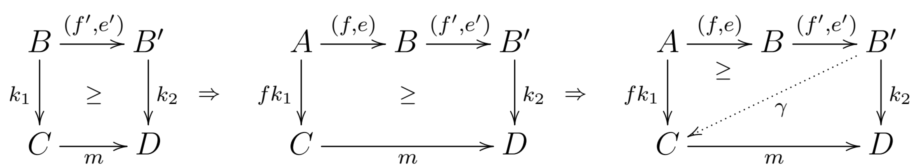
Proof.

(i) See Lemma 3.3 *(ii)*: the same proof works. If $(m, d) \perp_r (m, d)$ then we have

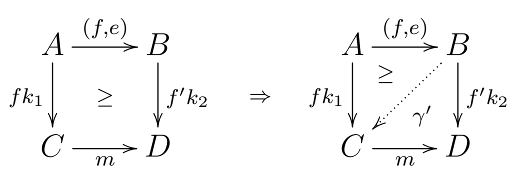


with $\overline{\gamma} = \overline{d} = d$. However, then $\gamma m = d$ and $m\gamma \le 1$ so $m\gamma = \overline{m\gamma} = \overline{m\overline{\gamma}} = \overline{md} = \overline{m}$. So m is a partial isomorphism with $\widehat{m} = d$.

(ii) By assumption the square on the right



has a unique cross map γ as does



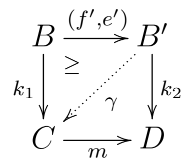
We claim that setting either $\gamma' := \overline{f'k_2}k_1\overline{m} = \overline{f'k_2}k_1$ or $\gamma' := \overline{f'k_2}f'\gamma = f'\gamma$ works to give cross maps. Starting with the first, $\gamma' := \overline{f'k_2}k_1$ we have

$$
\begin{aligned}
\gamma' m &:= \overline{f'k_2}k_1 m = f'k_2 \\
f\gamma' &:= f\overline{f'k_2}k_1 = \overline{ff'k_2}fk_1 \le fk_1 \\
\overline{\gamma'} &:= \overline{\overline{f'k_2}k_1} = \overline{\overline{f'k_2}\;\overline{k_1mk_1}} = \overline{\overline{f'k_2}k_1\overline{m}} = \overline{\overline{f'k_2}k_1m} = \overline{f'k_2}
\end{aligned}
$$

Now setting $\gamma' := f'\gamma$ we have:

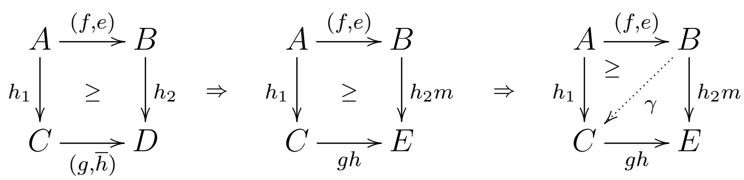
$$\begin{aligned} \gamma' m &:= f'\gamma m = f'k_2 \\ f\gamma' &:= ff'\gamma \leq fk_1 \\ \overline{\gamma'} &:= \overline{f'\gamma} = \overline{f'\overline{\gamma}} = \overline{f'\overline{k_2}} = \overline{f'k_2}. \end{aligned}$$

We therefore have $\overline{f'k_2}k_1 = f'\gamma$ and thus

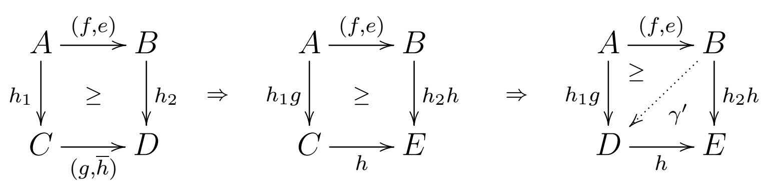


as required.

(iii) As $(g,d)h = gh$ is a precise composite $d = \overline{h}$. Consider the sequence of squares:



In the leftmost square $h_2 = eh_2\overline{h}$ and $fe = f$, $g\overline{h} = g$ and this gives the next square for which by assumption there is a cross map γ. We may also form the sequence of squares:



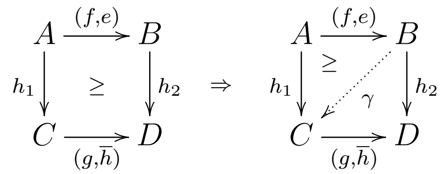
and the second by assumption has a cross map γ'. We claim that setting either $\gamma' := h_2\overline{h} = h_2$ or $\gamma' := \gamma g$ satisfy the requirements of a cross map as, for $\gamma' := h_2$ we have:

$$\begin{aligned} \gamma' h &:= h_2 h \\ f\gamma' &:= fh_2 \leq h_1 g \\ \overline{\gamma'} &:= \overline{h_2} = \overline{h_2\overline{h}} = \overline{h_2 h} \end{aligned}$$

and for $\gamma' := \gamma h$ we have:

$$\begin{aligned} \gamma' h &:= \gamma g h = h_2 h \\ f\gamma' &:= f\gamma g \leq h_1 g \\ \overline{\gamma'} &:= \overline{\gamma g} \leq \overline{\gamma} = \overline{h_2 h} \\ \overline{h_2 h} &= \overline{\gamma' g h} \leq \overline{\gamma'}. \end{aligned}$$

This means $h_2 = \gamma' = \gamma g$ and hence we have the required cross map



■

We are now ready to prove:

3.11. THEOREM. *For classes $\mathcal{E}, \mathcal{M}$ in a restriction category, the following are equivalent:*

(i) $(\mathcal{E}, \mathcal{M})$ *is a latent factorization system;*

(ii) $\mathcal{E} \perp_r \mathcal{M}$, $\mathcal{E}$ *is closed to right and* $\mathcal{M}$ *to left extensions by partial isomorphisms (in the sense of Lemma 3.3), and every map in* $\mathbb{X}$ *admits a precise* $(\mathcal{E}, \mathcal{M})$*-factorization;*

(iii) $\mathcal{E}$ *and* $\mathcal{M}$ *satisfy*

[ℓ**F.1**] $\mathcal{E}$ *is closed to split composition and right extension by partial isomorphisms;*

[ℓ**F.2**] $\mathcal{M}$ *is closed to composition and left extension by partial isomorphisms;*

[ℓ**F.3**] *Every map in* $\mathbb{X}$ *admits a precise factorization* $h = (f, e)m$ *and for any other precise factorization* $h = (f', e')m'$ *there is a unique partial isomorphism* α *such that* $\overline{\alpha} = e$, $\widehat{\alpha} = e'$ *and* $f\alpha = f'$, $\alpha m' = m$.

PROOF.

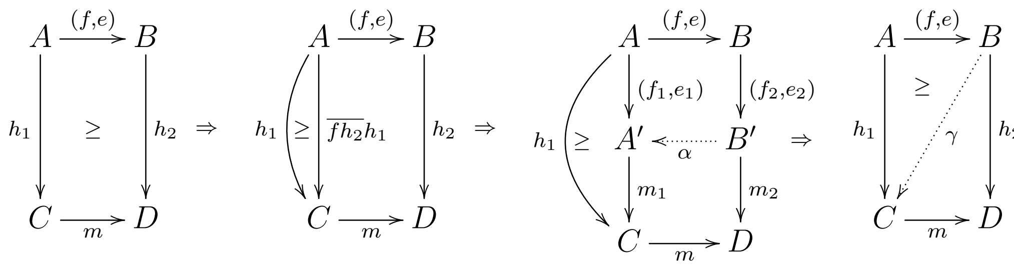
(ii)* ⇒ *(i) Note it is immediate that *(i)* ⇒ *(ii)* from Lemma 3.3; we want to establish the converse. For this it is necessary to show that $\mathcal{E} \perp_r \mathcal{M}$ maximally. We thus must show that if $(f, e) \perp_r \mathcal{M}$ then $(f, e) \in \mathcal{E}$ and, similarly, if $\mathcal{E} \perp_r m$ then $m \in \mathcal{M}$.

For the former: we know that we can precisely factorize $f = (f', e')m$ so that $(f, e) = (f', e')(m, e)$ is a split composition. By Lemma 3.10 *(ii)* it follows that $(m, e) \perp_r \mathcal{M}$. However, this means $(m, e) \perp_r (m, e)$ from which it follows, by Lemma 3.10 *(i)*, that m is a partial isomorphism and $e = \widehat{m}$. But this makes (f, e) a right extension of (f', e') by m and thus $(f, e) \in \mathcal{E}$.

For the latter suppose $\mathcal{E} \perp_r m$ then we may precisely factorize $m = (g, d)m'$. It follows by Lemma 3.10 *(iii)* that $\mathcal{E} \perp_r (g, d)$ but then $(g, d) \perp_r (g, d)$ so that, by Lemma 3.10 *(i)*, g is a partial isomorphism and $d = \widehat{g}$. However, this makes m a left extension of $m' \in \mathcal{M}$ by g; thus, $m \in \mathcal{E}$.

(i)* ⇒ *(iii) Is immediate from Lemma 3.3 and Proposition 3.7.

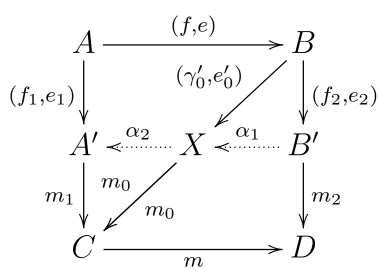
(iii) $\Rightarrow$ ***(ii)*** It remains to show that $\mathcal{E} \perp_r \mathcal{M}$. Suppose therefore we have $m \in \mathcal{M}$ and $(f, e) \in \mathcal{E}$ and the leftmost square below:



Next we may obtain a commuting square inscribed inside the first square and then we may precisely factorize the vertical maps ($h_1 = (f_1, e_1)m_1$ and $h_2 = (f_2, e_2)m_2$). This gives two different factorizations of fh_2 and so there is a partial isomorphism α which mediates them. This gives the cross arrow $\gamma := f_2\alpha m_1$. The only thing we need to check is that $\overline{\gamma} = \overline{h_2}$; for this we have:

$$\begin{aligned}
\overline{h_2} &= \overline{(f_2, e_2)m_2} = \overline{(f_2, e_2)\overline{m_2}} = \overline{(f_2, e_2)e_2} = \overline{(f_2, e_2)} \\
&= \overline{(f_2, e_2)\overline{\alpha}} = \overline{(f_2, e_2)\alpha\widehat{\alpha}} = \overline{(f_2, e_2)\alpha\overline{m_1}} \\
&= \overline{(f_2, e_2)\alpha m_1} = \overline{\gamma}.
\end{aligned}$$

Finally we must show this γ is unique: suppose therefore there is a γ' which also serves as a cross map. Then we may precisely factorize $\gamma' = (\gamma_0', e_0')m_0$, and this gives:



where $\alpha_1\alpha_2 = \alpha$ and so $\gamma' = f_2\alpha_2\alpha_1 m_1 = f_2\alpha m_1 = \gamma$.

■

We are now ready to consider various examples of latent factorization systems.

3.12. Example.

(1) In any trivial restriction category (that is, for every $f : A \longrightarrow B$, $\overline{f} = 1_A$) the notion of restriction orthogonality reduces to the usual notion of orthogonality for the mere category and the notion of a precise factorization reduces to a factorization. This means that a latent factorization on such a restriction category is precisely an (orthogonal) factorization system in the usual sense.

(2) Every restriction category has at least one latent factorization system: the *trivial* latent factorization system. This has $\mathcal{E}_0$ as the partial isomorphisms with their range, thus $\mathcal{E}_0 = \{(\alpha, \widehat{\alpha}) | \alpha \text{ is a partial isomorphism}\}$ and $\mathcal{M}$ consists of all maps. This is a maximal orthogonality by Corollary 3.4. A factorization of a map h is given whenever one has a partial isomorphism β with $\overline{\beta} = \overline{h}$; $(\beta^{(-1)}, \widehat{\beta})$ followed by βh is then a factorization (in particular, $\beta := \overline{h}$ works).

Note that as the notion of restriction category is not self-dual there may be no factorization, in general, in which the $\mathcal{M}$ maps are partial isomorphisms. The next example, in fact, shows what the latent factorization with the $\mathcal{M}$-maps partial isomorphism must be if it exists.

(3) The prototypical example of a latent factorization is given by a range structure, [3]. Here the class $\mathcal{E}$ consists of the pairs $(f, \widehat{f})$ while the class of $\mathcal{M}$-maps consists of all partial isomorphisms. A factorization of a map h is then a precise composite $h = (f, \widehat{f})m$. To see this is a latent factorization we shall use the characterization Theorem 3.11(iii). Clearly, partial isomorphisms are closed to composition and left extension by partial isomorphisms. We must show that $\mathcal{E}$ is closed to composition and right extension by partial isomorphisms; however, note that the latter is implied, in this case, by the former as for any partial isomorphism α, $(\alpha, \widehat{\alpha}) \in \mathcal{E}$. For the former we must show that the split composite $(f, \widehat{f})(g, \widehat{g}) = (fg, \widehat{g})$ has $(fg, \widehat{g}) = (fg, \widehat{fg})$ which follows as $\widehat{fg} = \widehat{\widehat{f}g} = \widehat{g}$. Clearly any h has a precise factorization as $h = (h, \widehat{h})\widehat{h}$ so all that remains is to show that any two precise factorizations of a map h are partially isomorphic. Suppose, therefore, that $h = (f_1, \widehat{f_1})\alpha_1 = (f_2, \widehat{f_2})\alpha_2$ are two precise factorizations of h then we claim $\alpha = \alpha_1 \alpha_2^{(-1)}$ makes them partially isomorphic. To see this we have:

$$\begin{aligned}
\alpha\alpha_2 &:= \alpha_1 \alpha_2^{(-1)} \alpha_2 = \alpha_1 \widehat{\alpha_2} = \alpha_1 \widehat{f_2 \alpha_2} = \alpha_1 \widehat{h} = \alpha_1 \widehat{f_1 \alpha_1} = \alpha_1 \widehat{\alpha_1} = \alpha_1 \\
f_1 \alpha &:= f_1 \alpha_1 \alpha_2^{(-1)} = h \alpha_2^{(-1)} = f_2 \alpha_2 \alpha_2^{(-1)} = f_2 \overline{\alpha_2} = f_2 \\
\overline{\alpha} &:= \overline{\alpha_1 \alpha_2^{(-1)}} = \overline{\alpha_1 \widehat{\alpha_2}} = \overline{\alpha_1 \widehat{\widehat{f_2} \alpha_2}} = \overline{\alpha_1 \widehat{h}} = \overline{\alpha_1} = \widehat{f_1} \\
\widehat{\alpha} &= \overline{\alpha^{(-1)}} := \overline{\alpha_2 \alpha_1^{*}} = \overline{\alpha_2 \overline{\alpha_1^{(-1)}}} = \overline{\alpha_2 \widehat{\alpha_1}} = \overline{\alpha_2 \widehat{h}} = \overline{\alpha_2} = \widehat{h_2}.
\end{aligned}$$

Thus this does indeed give a latent factorization.

Recall that every inverse category is a range category and so has the latent factorization as above. It is not unique but it is the smallest factorization system with respect to the order given by inclusion of $\mathcal{E}$-map pairs.

We will see more examples of latent factorization systems in the next section.

4. Latent factorizations in split restriction categories

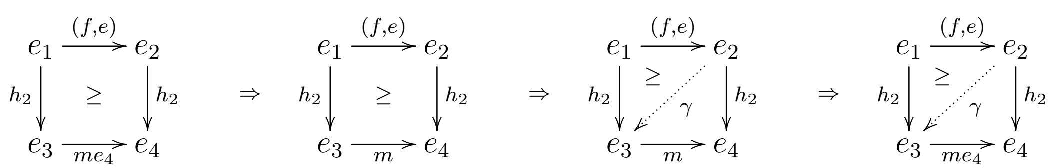
Given a restriction category $\mathbb{X}$ with a latent factorization system $(\mathcal{E}, \mathcal{M})$, we wish to show that splitting the restriction idempotents yields a split restriction category $\mathsf{Split}_r(\mathbb{X})$ which itself has a latent factorization system inherited from $\mathbb{X}$, defined as follows:

$$\mathcal{E}' = \{e_1 \xrightarrow{(f,e)} e_2 | (f,e) \in \mathcal{E}, e_1 f = f, e e_2 = e\} \text{ and } \mathcal{M}' = \{e_1 \xrightarrow{me_2} e_2 | m \in \mathcal{M}, e_1 m = m\}.$$

In particular, it is immediate from this definition that the embedding of $\mathbb{X}$ into $\mathsf{Split}_r(\mathbb{X})$ preserves both $\mathcal{E}$ and $\mathcal{M}$.

To show $(\mathcal{E}', \mathcal{M}')$ is a latent factorization system on $\mathsf{Split}_r(\mathbb{X})$ we will use the characterization given in Theorem 3.11(ii). It is easy to see that $\mathcal{M}'$ is closed to left extensions by partial isomorphisms and that $\mathcal{E}'$ is closed to right extensions by partial isomorphisms as this is so in $\mathbb{X}$. It is also immediate that we can precisely factorize any map as given an $h : e_1 \longrightarrow e_2$ we may factorize it in $\mathbb{X}$, so $h = (f,e)m$ with $e = \overline{m}$, $(f,e) \in \mathcal{E}$, and $m \in \mathcal{M}$, and then modify the factorization to $(f, \overline{me_2})me_2$. It is clear that this is precise and that $me_2 : \overline{me_2} \longrightarrow e_2 \in \mathcal{M}'$: we must show that $(f, \overline{me_2}) : e_1 \longrightarrow \overline{me_2} \in \mathcal{E}'$. First note that $f\overline{me_2} = \overline{fme_2}f = \overline{he_2}f = \overline{h}f = \overline{f}f = f$ where in the penultimate step we use $\overline{f} = \overline{fe} = \overline{f\overline{m}} = \overline{fm} = \overline{h}$. Next we note that $(f, \overline{me_2})$ is the right extension of (f,e) by $\overline{me_2}$ where it is useful to note $e\overline{me_2} = \overline{m}\ \overline{me_2} = \overline{me_2}$. This means $(f, \overline{me_2}) \in \mathcal{E}$.

It remains to prove that $\mathcal{E}' \perp_r \mathcal{M}'$ in $\mathsf{Split}_r(\mathbb{X})$. Toward this end consider the square in $\mathsf{Split}_r(\mathbb{X})$



where $(f,e) \in \mathcal{E}$ and $m \in \mathcal{M}$. In the third square we know $\overline{\gamma} = \overline{h_2}$ so that $e_2\gamma = \gamma$, and we need to show that $\gamma e_3 = \gamma$. Toward this end we show that γe_3 satisfies all the requirements of a cross map:

$$\begin{aligned} \gamma e_3 m &= \gamma m = h_2 \\ f\gamma e_3 &= h_2 e_3 = h_2 \\ \overline{\gamma e_3} &\leq \overline{\gamma} = \overline{h_2} \\ \overline{h_2} &= \overline{\gamma m} = \overline{\gamma e_3 m} \leq \overline{\gamma e_3}. \end{aligned}$$

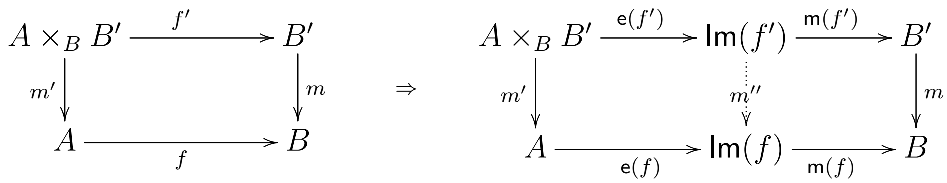
This shows that $\gamma e_3 = \gamma$ and that the maps in the third square are well-defined. For the final diagram we note that $\gamma m e_4 = h_2 e_4 = h_2$ and that this gives the desired cross map.

We have now shown:

4.1. PROPOSITION. *If $\mathbb{X}$ is a restriction category with a latent factorization system $(\mathcal{E}, \mathcal{M})$ then the latent factorization extends to a latent factorization system $(\mathcal{E}', \mathcal{M}')$, as defined above, on $\mathsf{Split}_r(\mathbb{X})$ so that the embedding $\mathbb{X} \longrightarrow \mathsf{Split}_r(\mathbb{X})$ preserves the factorization systems.*

Notice that the latent factorization system $(\mathcal{E}', \mathcal{M}')$ has the class $\mathcal{M}'$ defined in a slightly unexpected manner in order to ensure that every map admits a precise factorization. Also notice that the factorization that was used in the proof – and certainly it could have been chosen differently – makes the $\mathcal{M}'$-component a total map.

4.2. Definition. *An* M**-stable** $(\mathcal{E}, \mathcal{M})$*-factorization system on an* M*-category; that is, a category with a stable system of monics* M *in the sense of [2], consists of a normal factorization system which is, in addition,* **stable** *with respect to pulling back along* M*-maps. Thus, given an* M*-pullback square, as on the left below, it can be factorized, as on the right:*



and we require that whenever the square is a pullback both the squares on the right are pullbacks.

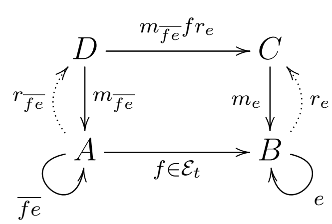
Note that because the pullback along any map of an $\mathcal{M}$-map, if it exists, is always an $\mathcal{M}$-map, it follows that it suffices to demand that pullbacks along M-maps of $\mathcal{E}$-maps are always $\mathcal{E}$-maps.

4.3. Proposition. *If* $\mathbb{X}$ *is a split restriction category with a latent factorization system* $(\mathcal{E}, \mathcal{M})$ *then this induces an* M*-stable factorization* $(\mathcal{E}_t, \mathcal{M}_t)$ *on* $\mathsf{Total}(\mathbb{X})$ *where*

$$\mathcal{E}_t = \{A \xrightarrow{f} B | (f, 1_B) \in \mathcal{E}, 1_A = \overline{f}\} \quad \textit{and} \quad \mathcal{M}_t = \{A \xrightarrow{m} B | m \in \mathcal{M}, \overline{m} = 1_A\}$$

Proof. First we note that a latent factorization when restricted to idempotents which are identities produces two classes of maps which are clearly orthogonal. The extension properties show that $\mathcal{M}_t$ is closed to composition on the left with isomorphisms and that $\mathcal{E}_t$ is closed to composition on the right with isomorphisms. Finally, we must show that we can factorize into total maps given a precise factorization $g = (f, e)m$. This is achieved by splitting $e : B \longrightarrow B$ into $r_e : B \longrightarrow B'$ and $s_e : B' \longrightarrow B$, with $e = r_e s_e$ and $s_e r_e = 1_{B'}$, and modifying $(f, e)m$ into the precise factorization $g = (fr_e, 1'_B)s_e m$, using Proposition 3.7. This factorization has $fr_e s_e m = fem = fm = g$ and then as g is total fr_e is total and, as the factorization, is precise $\overline{s_e m} = 1_{B'}$ and so $s_e m$ is a total map. This provides a factorization system on the total maps of $\mathbb{X}$.

It remains to show that this is an M-stable factorization. Consider, therefore, a pullback of an M-map. It has the form:



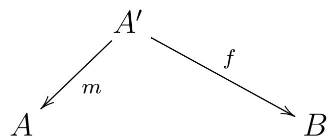
We have to show that $(m_{\overline{fe}} f r_e, 1_C)$ is in $\mathcal{E}$. Toward this end observe that $(m_{\overline{ef}}, \overline{ef})$, being a partial isomorphism with its range, is in $\mathcal{E}$. Similarly, $(r_e, 1_C) \in \mathcal{E}$ as r_e is the retraction of m_e. Split composing $(f, 1_B)(r_e, 1_C)$ gives $(fr, 1_C) \in \mathcal{E}$. Now note that $\overline{fr_e} = \overline{fe}$ and this means we can split compose $(m_{\overline{fe}}, \overline{fe})(fr_e, 1_C)$ to show that $(m_{\overline{fe}} f r_e, 1_C)$ is in $\mathcal{E}$ as required.

■

Next we observe that if we start with an M-category and an M-stable factorization system then this induces a latent factorization system on the partial map category, $\mathsf{Par}(\mathbb{X}, \mathsf{M})$:

4.4. Proposition. *If* $(\mathbb{X}, \mathsf{M})$ *is an* M*-category with an* M*-stable factorization system* $(\mathcal{E}, \mathcal{M})$*, then* $\mathsf{Par}(\mathbb{X}, \mathsf{M})$ *has a latent factorization system* $(\widehat{\mathcal{E}}, \widehat{\mathcal{M}})$*.*

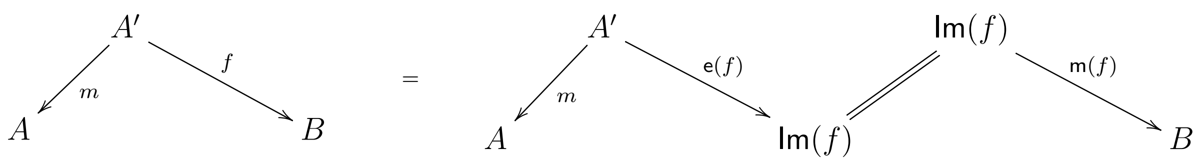
Proof. We shall indicate a map in $\mathsf{Par}(\mathbb{X}, \mathsf{M})$ by $(m, f) : A \longrightarrow B$ where



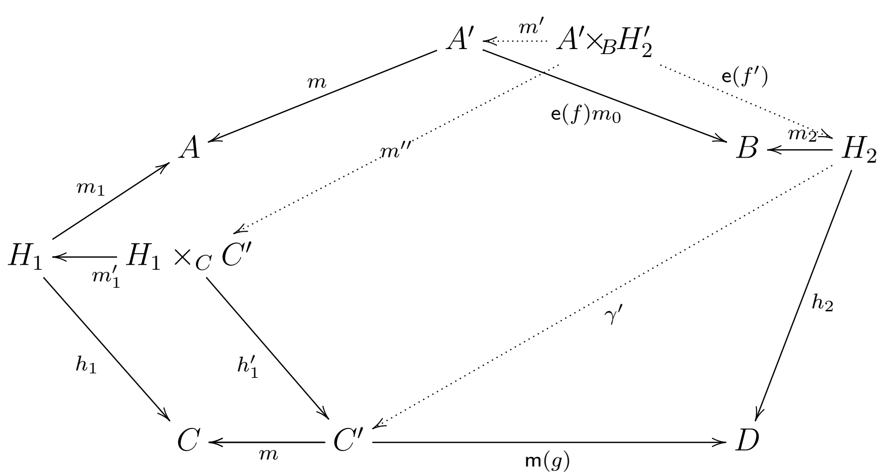
Define the latent factorization classes in $\mathsf{Par}(\mathbb{X}, \mathsf{M})$ based on the $(\mathcal{E}, \mathcal{M})$ of $\mathbb{X}$ by:

$$\begin{aligned} \widehat{\mathcal{E}} &= \{((m, gm'), (m'm'))|g \in \mathcal{E}, m, m' \in \mathsf{M}\} \\ \widehat{\mathcal{M}} &= \{(m, h)|h \in \mathcal{M}, m \in \mathsf{M}\} \end{aligned}$$

Given an arbitrary partial map (m, f) it can be latently factorized as $(m, \mathsf{e}(f), (1_{\mathsf{Im}(f)}, 1_{\mathsf{Im}(f)}))$ followed by $(1_{\mathsf{Im}(f)}, \mathsf{m}(f))$ where $\mathsf{e}(f)\mathsf{m}(f)$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f:



Thus letting the $\mathcal{E}$-maps of the partial map category be spans $(m, \mathsf{e}(f)m_0)$ we immediately satisfy all the conditions of Theorem 3.11(ii) except the orthogonality requirement. Thus, it remain to check orthogonality which is given by the following:



where m'' guarantees the laxness of the square of partial maps and the cross-map for the latent factorization is $\gamma = (m_2, \gamma' m)$ where γ' is the cross map for the ordinary factorization in $\mathbb{X}$. ∎

We can combine these propositions to prove:

4.5. THEOREM. *For a split restriction category,* $\mathbb{X} = \mathsf{Par}(\mathsf{Total}(\mathbb{X}), \mathsf{M})$*, to have a latent factorization is precisely to have an* M*-stable factorization on the total map category,* $\mathsf{Total}(\mathbb{X})$.

PROOF. It suffices to show for the $\mathcal{M}$-maps of a latent factorization that $\widehat{\mathcal{M}_t} = \mathcal{M}$ and that for M-stable $\mathcal{M}'$-maps that $\left(\widehat{\mathcal{M}'}\right)_t = \mathcal{M}'$. We have for the former:

$$\widehat{\mathcal{M}_t} = \{m \in \widehat{\mathcal{M}|\overline{m}} = 1\} = \{rm | rs = \overline{r}, sr = 1, m \in \mathcal{M}, \overline{m} = 1\}$$

This is contained in $\mathcal{M}$ as it is a left extension of elements in $\mathcal{M}$ by partial isomorphisms. But also for any $m \in \mathcal{M}$ we can split $\overline{m}$, so that $rs = \overline{m}$ and $sr = 1$, and then $m = rsm$ and sm is total and rsm is a left extension by a partial isomorphism. Thus $m \in \widehat{\mathcal{M}_t}$ so $\widehat{\mathcal{M}_t} = \mathcal{M}$.

Conversely given $\mathcal{M}'$ of a M-stable factorization on $\mathsf{Total}(\mathbb{X})$ then $\widehat{\mathcal{M}'} = \{rm' | r \in \mathsf{ParIso}(\mathbb{X}), m' \in \mathcal{M}', \widehat{r} = \overline{m'} = 1\}$ and $\widehat{\mathcal{M}'}_t = \{rm' \in \widehat{\mathcal{M}'} | rstr = \overline{rm'} = 1\}$. However, this makes the maps r used in $\widehat{\mathcal{M}'}_t$ isomorphisms. However, $\mathcal{M}'$ is part of an ordinary factorization system, but $\mathcal{M}'$ is certainly closed to composition on the left by isomorphisms. Thus, $\widehat{\mathcal{M}'}_t \subseteq \mathcal{M}'$. However, considering the case when $r = 1$ immediately gives the other containment, so $\widehat{\mathcal{M}'}_t = \mathcal{M}'$ as required. ∎

4.6. EXAMPLE.

(1) An example of an M-factorization (where M is chosen to be the class of all monics) is the usual epi-mono factorization on sets. This follows immediately from Proposition 4.4 as epic in the category of sets are closed to pulling back along any map. Thus, the category of sets and partial maps is a range category [3] and so, as explained in Example 3.12(3), this gives the latent factorization. The same argument, of course, works more generally for any regular category as the regular monic factorization is always monic-stable.

(2) A special case of an M-stable factorization $(\mathcal{E}, \mathcal{M})$ is the case when $\mathsf{M} = \mathcal{M}$. As explained in [3] this immediately makes the partial map category, $\mathsf{Par}(\mathbb{X}, \mathsf{M})$ a range category. Clearly regular categories provide a broad range of examples of this phenomenon (as mentioned above). However, there are many examples which do not arise in this way. A trivial example is given by the trivial restriction ($\overline{f} = 1_A$) of any ordinary category: this is always a range category. A further example, which is non-trivial arises from the factorization induced on (small) categories by taking the $\mathcal{M}$ maps to be full embeddings and $\mathcal{E}$ to be the maps that are "bijective on object" functors. The partial map category of partial functors with domains full embeddings is then a range category although it is far from regular.

(3) A well-known factorization on topological spaces is into "quotient maps" followed by "injective continuous maps" (see for example [7]). A quotient map is a surjection $f : X \to Y$ which has the codomain's topology determined by having $U \subseteq Y$ open if and only if

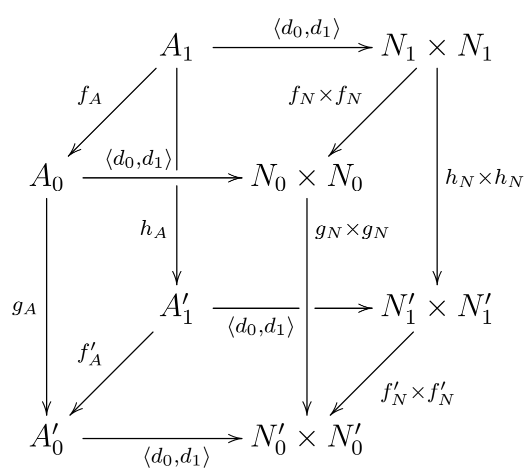
$f^{-1}(U)$ is open. An injective continuous map is just a continuous map whose underlying set map is injective. It is not hard to see that quotient maps, in this sense, are closed to pulling back along open maps, and thus to pulling back along open inclusions. This means that in $\mathsf{Top}_{\text{open}}$ (the partial map category of topological spaces whose domains are open inclusions) has a restriction factorization system which is non-trivial and not a range.

(4) In this example we describe explicitly a factorization system on directed graphs which is stable with respect to full subgraphs. This exhibits a non-trivial (non-range) restriction factorization on the full subgraph partial map category of directed graphs. (The factorization at the level of categories is the "bijective on object"–"full subcategory" factorization.)

Consider the category of directed graphs, DGraph, whose objects are diagrams $\mathcal{A} := A \underset{d_1}{\overset{d_0}{\rightrightarrows}} N$ in sets (in fact, in any category with finite limits), where A is the set of arrows and N the set of nodes. The maps $(f_A, f_N) : \mathcal{A} \longrightarrow \mathcal{A}'$ are pairs with $f_A : A \longrightarrow A'$ and $f_N : N \longrightarrow N'$ such that $f_A d_i = d_i f_N$ (for $i = 0, 1$). We let the M-maps be the **full subgraphs**: these are determined intuitively by choosing a subset of the nodes but leaving the arrows between those nodes unchanged. To describe this formally it is useful first to describe **full and faithful graph maps**: a graph map (f_A, f_N) is full and faithful in case

$$\begin{array}{ccc} A & \xrightarrow{\langle d_0, d_1\rangle} & N \times N \\ {\scriptstyle f_A}\downarrow & \lrcorner & \downarrow{\scriptstyle f_N \times f_N} \\ A' & \xrightarrow[\langle d_0, d_1\rangle]{} & N' \times N' \end{array}$$

is a pullback. (f_A, f_N) is a subgraph if f_N is monic. From this formulation it is clear that these maps are closed to composition. We must show that they are closed to pulling back along any map. To see this consider the cube:

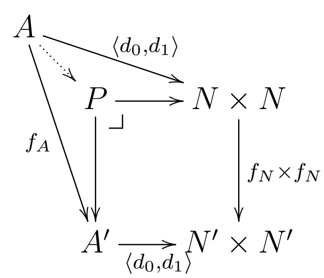


If this is a pullback in DGraph then the two side squares are pullbacks. If (g_A, g_N) is full and faithful, the front face is a pullback. This immediately means the back face (h_A, h_N) is a pullback and so pullbacks of full and faithful maps along any map are full and faithful. Finally it is clear that h_N must be monic if g_N is monic, as monics pullback. This show that pullbacks of full and faithful subgraphs along any map are full and faithful subgraphs as desired.

Next we describe a factorization system on DGraph. The $\mathcal{E}$-maps are those with their node component an isomorphism while the $\mathcal{M}$ maps are the full and faithful graph maps (that is those maps which make the first diagram above a pullback). Note that both classes are closed to composition and include isomorphisms, so we must show the classes are orthogonal and that every map can be factored. To show $\mathcal{E} \perp \mathcal{M}$ consider the square of maps in the cube above and assume this time that f_N is an isomorphism and (f'_A, f'_N) is full and faithful then

$$A_0 \xrightarrow{\langle d_0,d_1\rangle(f_N^{-1}\times f_N^{-1})(h_N\times h_N)} N_1\times N_1 \xrightarrow{f'_N\times f'_N} N'_0\times N'_0 = A_0 \xrightarrow{g_A} A'_0 \xrightarrow{\langle d_0,d_1\rangle} N'_0\times N'_0$$

Thus, as the bottom square is a pullback, there is a unique map $c : A_0 \longrightarrow A'_1$ providing the required cross map to establish orthogonality. Finally, to show there is a factorization we simply factor a graph map through the pullback:



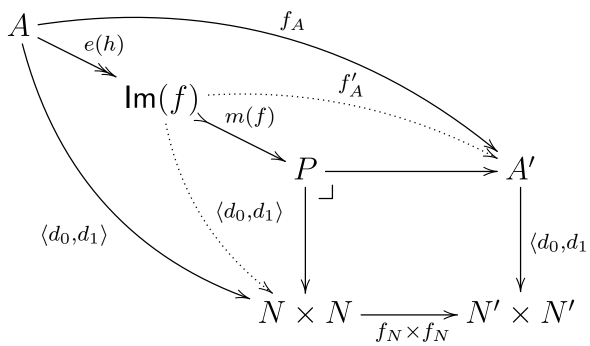
The upper triangle is then a $\mathcal{E}$-map while the pullback square is an $\mathcal{M}$-map, that is a full and faithful graph morphism.

To show this factorization is M-stable we must show that pullbacks of $\mathcal{E}$-maps along $\mathcal{M}$-maps are $\mathcal{E}$. However, as $\mathcal{E}$-maps are characterized by having f_M an isomorphism, the pullback of an $\mathcal{E}$-map along any map is an $\mathcal{E}$-map.

Thus, directed graphs with full subgraph partial maps, $\mathsf{DGraph}_{\text{full}}$, has a restriction factorization system determined by the factorization into bijective on nodes followed by full and faithful graph maps.

(5) The partial map category, $\mathsf{DGraph}_{\text{full}}$, has more than one non-trivial restriction factorization. A graph map may be factored into a (bijective on nodes) quotient followed by a faithful graph map. This factorization works for the directed graphs of any regular category which is stable with respect to full subgraphs. The factorization of an arbitrary

graph map is given by:



The lefthand lower triangle is then a quotient map (identity on objects) while (f'_A, f_N) is a faithful graph map. It is then straightforward to show this is M-stable with respect to full subgraph embeddings and, thus, gives another latent factorization system on the partial map category.

5. Factorizations induced on latent fibrations

An observation, which likely goes back to the early 1970's and is recorded, for example, in a slightly more general form in [8, Proposition II.5.7.1], is that factorization systems, $\mathcal{E}$-$\mathcal{M}$, on the base category of an ordinary fibration can be lifted to a factorization, $\mathcal{E}'$-$\mathcal{M}'$, on the total category. The $\mathcal{E}'$-maps consist of all maps which lie above an $\mathcal{E}$-map (this includes, in particular, all the vertical maps). The $\mathcal{M}'$-maps consists of those maps which are Cartesian above $\mathcal{M}$-maps:

5.1. Proposition. *If $p : \mathbb{E} \longrightarrow \mathbb{B}$ is a (standard) fibration, and $\mathbb{B}$ has an $\mathcal{E}$-$\mathcal{M}$ factorization then there is a unique induced factorization $\mathcal{E}'$-$\mathcal{M}'$ on $\mathbb{E}$, the total category, with $\mathcal{M}'$ the Cartesian maps above $\mathcal{M}$ and $\mathcal{E}'$ all maps above maps in $\mathcal{E}$.*

Proof. Set $\mathcal{M}' = \{f | p(f) \in \mathcal{M}$ and f is Cartesian$\}$ and $\mathcal{E}' = \{g | p(g) \in \mathcal{E}\}$. Clearly $\mathcal{E}'\ \mathsf{Iso} \subseteq \mathcal{E}'$ and $\mathsf{Iso}\ \mathcal{M}' \subseteq \mathcal{M}'$. Furthermore, that these two sets are orthogonal is immediate using the fact that $\mathcal{M}'$-maps are prone (the cross map is given by the lifting property of Cartesian arrows). Finally any map can be factorized by $f = \widetilde{p(\mathsf{e}(f))}p(m(f))^*$, where we note that $p(\widetilde{p(\mathsf{e}(f))}) = \mathsf{e}(f)$ and so $\widetilde{p(\mathsf{e}(f))} \in \mathcal{E}'$. This proves that $\mathcal{E}'$-$\mathcal{M}'$ is a factorization system with the required properties.

It is unique as $\mathcal{M}'$, which is completely determined as the Cartesian maps over $\mathcal{M}$, uniquely determines the factorization system. ∎

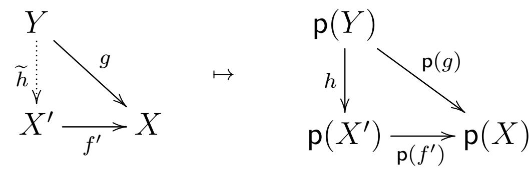
The usual manner in which this observation is deployed is to use the trivial factorization Iso-All on the base category in order to factorize maps in the total category into essentially vertical maps followed by prone maps.

Our objective in this section is to prove that a latent factorization on the base of a latent fibration induces a latent factorization on the total category in much the same manner.

Latent fibrations are explored in [5]. Here is the definition:

5.2. Definition. *Let $\mathbb{E}$ and $\mathbb{B}$ be restriction categories and $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ a restriction semi-functor.*

(i) An arrow $f' : X' \longrightarrow X$ in $\mathbb{E}$ is **p-prone** *in case whenever we have $g : Y \longrightarrow X$ in $\mathbb{E}$ and $h : \mathsf{p}(Y) \longrightarrow \mathsf{p}(X')$ in $\mathbb{X}$ such that $h\mathsf{p}(f') = \mathsf{p}(g)$ is a precise triangle, then there is a unique* **lifting** *of h to $\widetilde{h} : Y \longrightarrow X'$ so that $\mathsf{p}(\widetilde{h}) = h$ and $\widetilde{h}f' = g$ is a precise triangle:*



(ii) $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a **latent fibration** *if for each $X \in \mathbb{E}$ and each $f : A \longrightarrow \mathsf{p}(X)$ in $\mathbb{B}$ such that $f = f\mathsf{p}(1_X)$, there is a p-prone map $f' : X' \longrightarrow X$ sitting over f (that is, $\mathsf{p}(f') = f$).*

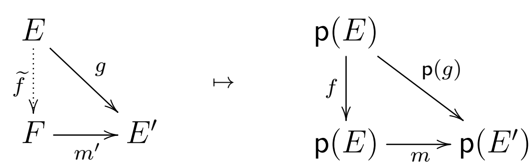
The term "prone" is used above – and in [5] – to emphasize that the analogue of a "Cartesian" map in a latent fibration is significantly more elaborate. However, for a trivial restriction category – one in which the restriction of all maps is the identity – the notion of prone coincides with that of Cartesian.

Note that in Definition 5.2, if p is a restriction functor (so that p preserves identities) then the condition $f = f\mathsf{p}(1_X)$ is vacuous. However, if p is a genuine semifunctor, then that condition is necessary, as if there is an arrow $f' : X' \longrightarrow X$ sitting over f, then since $f' = f'1_X$, $\mathsf{p}(f') = \mathsf{p}(f')\mathsf{p}(1_X)$; i.e., f must satisfy $f = f\mathsf{p}(1_X)$.

5.3. Proposition. *If $\mathsf{p} : \mathbb{E} \longrightarrow \mathbb{B}$ is a latent fibration, and $\mathbb{B}$ has a latent factorization system $(\mathcal{E}, \mathcal{M})$, then there is an induced latent factorization system on $\mathbb{E}$ with*

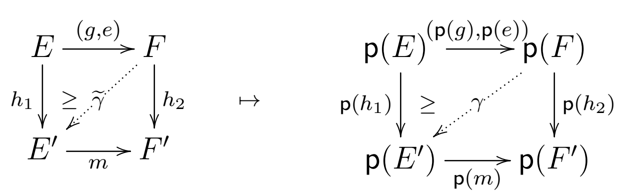
$$\mathcal{E}' = \{(g,e) | (\mathsf{p}(g), \mathsf{p}(e)) \in \mathcal{E}\} \qquad \textit{and} \qquad \mathcal{M}' = \{m | \mathsf{p}(f) \in \mathcal{M} \textit{ and } \mathsf{p}(f) \textit{ is prone}\}.$$

Proof. If $g : E \longrightarrow E'$ is a map in $\mathbb{E}$ then we may factorize it into $(\widetilde{f}, \overline{m'})$ followed by m' where m' is prone over m and $\widetilde{f}$ is the lift of f where $(f, \overline{m})m$ is the precise factorization of $\mathsf{p}(g)$:



Note that $\mathcal{E}'$ is closed to right extension as $\mathcal{E}$ is, and $\mathcal{M}'$ is closed to left extension as when α has $\widehat{\alpha}m' = m'$ then $\alpha\overline{m'}$ is a mediating morphism so that $\alpha\overline{m'}m' = \alpha m'$ is prone and over $\mathsf{p}(\alpha)m$ which is in $\mathcal{M}$ as it is closed to left extension.

It remains to show that $\mathcal{E}' \perp_r \mathcal{M}'$; for this consider



where note that the lift of γ certainly makes the lower triangle of the first square commute. By restricting to $\overline{\mathsf{p}(g)}\mathsf{p}(h_1)$ we can make the square precise and $\overline{g}h_1$ is thus its lifting. However, $g\widetilde{\gamma}$ is also a lifting and so $\overline{g}h_1 = g\widetilde{\gamma}$.

This proves the proposition. ∎

We may use this result to give an alternative proof of Proposition 4.1 by noticing that the underlying functor $\mathsf{U} : \mathsf{Split}_r(\mathbb{X}) \longrightarrow \mathbb{X}$ (see [5] 3.2.7) is an example of a latent fibration and, thus, by the above we can lift a latent fibration on $\mathbb{X}$ to one on $\mathsf{Split}_r(\mathbb{X})$. In this case the lifting gives a bijection between latent factorizations on $\mathbb{X}$ and $\mathsf{Split}_r(\mathbb{X})$ as there is clearly an inverse which takes the latent factorization induced on $\mathbb{X}$ from the embedding $I : \mathbb{X} \longrightarrow \mathsf{Split}_r(\mathbb{X})$.

6. Conclusion

The paper shows that – with some modifications – the theory of factorization systems has a natural analogue for restriction categories. Significantly the self-dual nature of ordinary factorization systems is lost in this analogy as restriction categories are not self-dual. This asymmetry becomes apparent almost immediately when one tries to develop the appropriate analogue of orthogonality. The notion of orthogonality presented here differs significantly from the usual notion of orthogonality: this departure may be useful for emboldening researchers who are studying factorization in other specialized contexts.

That range categories provide an important source of examples of latent factorization systems was largely for us a post-hoc realization: the development here was largely driven and motivated by the relation to latent fibrations. It is incumbent on us, therefore, to re-emphasize the significance of the fact – implicit here from the examples, see 3.12(3), 4.6(2), and, of course, [3] – that range restriction categories can be equivalently described as restriction categories with a latent factorization in which the $\mathcal{M}$-maps are exactly the partial isomorphisms. Having independent descriptions of a notion is always a good indicator of mathematical significance. In this manner, the view of range restriction categories offered by the existence of a special latent fibration, reinforces the mathematical importance of the theory of range restriction categories ... to which Pieter Hofstra contributed significantly.

While we have provided quite a few examples of latent factorizations, there is always room for more! We would always be happy to hear of new applications of this work.

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