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QUOTIENTS, PURE EXISTENTIAL COMPLETIONS AND ARITHMETIC UNIVERSES

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ABSTRACT. We provide a new description of Joyal's arithmetic universes through a characterization of the exact and regular completions of pure existential completions.

We show that the regular and exact completions of the pure existential completion of an elementary doctrine P are equivalent to the reg/lex and ex/lex-completions, respectively, of the category of predicates of P.

This result generalizes a previous one by the first author with F. Pasquali and G. Rosolini about doctrines equipped with Hilbert's ϵ-operators.

Thanks to this characterization, each arithmetic universe in the sense of Joyal can be seen as the exact completion of the pure existential completion of the doctrine of predicates of its Skolem theory.

In particular, the initial arithmetic universe in the standard category of ZFC-sets turns out to be the completion with exact quotients of the doctrine of recursively enumerable predicates.

Dedicated to Pieter Hofstra for his inspiring creative work.

1. Introduction

This paper provides a new contribution to the description of Joyal's arithmetic universes thought the application of a new characterization of free completions of elementary Lawvere doctrines.

Free completions of categories with quotients are ubiquitous in category theory. In particular those leading to exact and regular categories in [1, 3] have been widely studied in the literature of category theory, with applications both to mathematics and computer science, see [32, 27, 28, 23, 37, 36, 20].

Such free completions are also involved in the construction of arithmetic universes introduced by A. Joyal to prove Gödel incompleteness theorems in some lectures (still unpublished) in the seventies and recalled in [14] (see [19, 38] for more information). A more general abstract definition of arithmetic universe as list-arithmetic pretopos has been proposed in [19].

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In more detail, an arithmetic universe in the sense of Joyal can be described as the exact completion – in the sense of [1] – of the lex category of predicates of a given Skolem theory. In particular, the initial arithmetic universe within the standard category of ZFC-sets turns out to be the exact completion of the lex category of primitive recursive predicates. In recent years, the regular and exact completions of a lex category have been proved to be instances of more general completions of certain Lawvere doctrines in [23, 22, 24].

More precisely, the exact completion of a lex category [1, 3], also known as ex/lex-completion, and the exact completion of a regular category, referred to as ex/reg-completion, have been proven to be an instance of a more general exact completion $\mathsf{Ex}(P)$ relative to an elementary, pure existential doctrine P.

The construction of the exact category $\mathsf{Ex}(P)$ is, essentially, achieved through the tripos-to-topos construction developed by J.M.E. Hyland, P.T. Johnstone and A.M. Pitts in [11, 30], applied to an elementary existential doctrine. Similarly, the regular completion of a lex category, also called reg/lex-completion, introduced in [2, 3] has been proved to be a specific instance of a more general construction for doctrines in [21], which is the regular completion $\mathsf{Reg}(P)$ of an elementary, pure existential doctrine P. The regular and exact completions of doctrines are related via the ex/reg-completion, namely we have that $\mathsf{Ex}(P) \equiv (\mathsf{Reg}(P))_{\mathsf{ex/reg}}$ [24].

In this paper we provide a characterization which further relates the exact and regular completions of a lex category with the notion of exact completion of an elementary existential doctrine as presented in [23, 22, 24] by involving a third kind of free completion, namely the *pure existential completion* of an elementary doctrine.

The notion of *existential completion* was introduced by the second author in his PhD thesis and later published in [33], and it is a construction that freely adds existential quantifiers to a given primary doctrine, along with a class Λ of base morphisms that is closed under compositions, pullbacks and isomorphisms. To distinguish some particularly interesting instances, this free construction has been renamed as *generalized existential completion* in [26]. Following the terminology in [26], the *pure existential completion* is the instance of the generalized existential completion where the class Λ consists of product projections, while we call *full existential completion* the instance where Λ is the class of all the base morphisms. Furthermore, the pure existential completion also coincides with the restriction to faithful fibrations of the *simple coproduct (or sum) completion* of a fibration employed by P. Hofstra in [9], while the full existential completion coincides with the $\exists$-completion introduced by J. Frey in [6, 5] and a particular case of it has also played a significant role in the works by P. Hofstra [8, 7].

In this work, we show that the regular and exact completions of the pure existential completion of an elementary doctrine P are equivalent to the reg/lex and ex/lex-completions, respectively, of the category of predicates of P.

In detail, we show that for an elementary, pure existential doctrine P and an elementary subdoctrine P' of P on the same base category, the regular completion $\mathsf{Reg}(P)$ of P corresponds to the reg/lex-completion $(\mathcal{P}rd_{P'})_{\mathsf{reg/lex}}$ of the category of predicates $\mathcal{P}rd_{P'}$ of

P' (via an equivalence induced by the canonical embedding of $\mathcal{Prd}_{P'}$ into $\mathsf{Reg}(P)$) if and only if P is the pure existential completion of P' (Theorem 5.8).

Then, by combining this result with the aforementioned decomposition of exact completions, we immediately deduce that the exact completion $\mathsf{Ex}(P)$ of P corresponds to the ex/lex-completion $(\mathcal{Prd}_{P'})_{\mathsf{ex/lex}}$ of the category of predicates $\mathcal{Prd}_{P'}$ of P' (again via an equivalence induced by the canonical embedding of $\mathcal{Prd}_{P'}$ into $\mathsf{Ex}(P)$) if and only if P is the pure existential completion of P' (Corollary 5.9).

The crucial intuition is that there is a tight connection between *regular projective objects* of $\mathsf{Reg}(P)$ and *pure existential free elements* of P. In particular, we show that whenever P is the pure existential completion of P' (and hence the elements of P' are pure existential free elements according to [26]), we can use P' to define a projective cover for $\mathsf{Reg}(P)$ which in addition satisfies the property that every object of $\mathsf{Reg}(P)$ can be embedded into a projective of this cover.

Our characterization generalizes a previous one in [21] by the first author with F. Pasquali and G. Rosolini about doctrines equipped with Hilbert's ϵ-operators for the fact that in [26] we showed that a doctrine is equipped with Hilbert's ϵ-operators if and only if it is equivalent to the pure existential completion of itself.

Then, we apply our characterization to deduce that each arithmetic universe turns out to be the exact completion – in the sense of [24] – of the pure existential completion of the doctrine of predicates of a given Skolem theory. As a consequence we deduce that the initial arithmetic universe in the standard category of ZFC-sets is the completion with exact quotients of the doctrine of recursively enumerable predicates.

Another notable application of our characterization, already published in [35] by employing our main theorem first presented in [25] (with a different proof), regards the exact completion of Gödel hyperdoctrines introduced in [35, 34] as an equivalent presentation of the restriction to faithful fibrations of Hofstra's Dialectica fibrations [9].

2. Elementary and existential doctrines

The term *doctrine*, when accompanied by certain adjectives, is often associated with a generalization of the concept of *hyperdoctrine* introduced by F.W. Lawvere in a series of seminal papers [15, 17, 16]. We recall from *loc. cit.* some definitions which will be useful in the following. The reader can find all the details about the theory of elementary and pure existential doctrines also in [22, 23, 24, 21], and an algebraic analysis of the elementary structure of a doctrine in [4].

2.1. Definition. *A* ***primary doctrine*** *is a functor* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *from the opposite of a category* $\mathcal{C}$ *with finite products to the category* InfSl *of inf-semilattices.*

We will use the notation $\alpha \wedge \beta$ to denote the binary inf of α and β in $P(A)$ and $\top_A$ to denote the top element of $P(A)$.

2.2. Definition. *A primary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *is* ***elementary*** *if for every* A *in* $\mathcal{C}$ *there exists an object* δ_A *in* $P(A \times A)$*, called* ***fibered equality****, such that*

1. *the assignment*

$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\pi_1}(\alpha) \wedge \delta_A$$

for an element α of $P(A)$ determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle} \colon P(A \times A) \to P(A)$;

2. *for every morphism e of the form $\langle \pi_1, \pi_2, \pi_2 \rangle \colon X \times A \to X \times A \times A$ in $\mathcal{C}$, the assignment*

$$\exists_e(\alpha) := P_{\langle \pi_1, \pi_2 \rangle}(\alpha) \wedge P_{\langle \pi_2, \pi_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to $P_e \colon P(X \times A \times A) \to P(X \times A)$.

2.3. Example. Let $\mathcal{L}_{=}$ be the $(\top, \wedge, =)$-fragment of Intuitionistic Logic, i.e. the fragment with true constant, conjunctions and equality, called Horn-fragment in [13, Sec. D.1.1, p. 810]. Let $\mathbb{T}$ be a theory in such a fragment. Let us denote by $\mathcal{V}$ the syntactic category whose objects are contexts (up to α-equivalence), and arrows are term substitutions. Consider the functor

$$\mathsf{LT}^{\mathbb{T}}_{=} \colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$$

defined on a given context Γ of $\mathcal{V}$ by taking $\mathsf{LT}^{\mathbb{T}}_{=}(\Gamma)$ as the Lindenbaum-Tarski algebra of well-formed formulas of $\mathcal{L}_{=}$ with free variables in Γ and on a substitution morphism between contexts by taking the substitution homomorphism between formulas of the Lindenbaum-Tarski algebras. The functor $\mathsf{LT}^{\mathbb{T}}_{=}$ is an elementary doctrine.

2.4. Definition. *A primary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is **pure existential** if, for every object A and B in $\mathcal{C}$ for any product projection $\pi_A \colon A \times B \to A$, the functor*

$$P_{\pi_A} \colon P(A) \to P(A \times B)$$

has a left adjoint $\exists_{\pi_A}$, and these satisfy:

(BCC) ***Beck-Chevalley condition:*** *for any pullback diagram*

$$\begin{array}{ccc} C \times B & \xrightarrow{\pi_C} & C \\ {\scriptstyle f \times \mathrm{id}_B}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ A \times B & \xrightarrow[\pi_A]{} & A \end{array}$$

the canonical arrow

$$\exists_{\pi_C} P_{f \times \mathrm{id}_B}(\alpha) \leq P_f \exists_{\pi_A}(\alpha)$$

is an isomorphism for every element α of the fibre $P(A \times B)$;

(FR) ***Frobenius reciprocity:*** *for any projection $\pi_A \colon A \times B \to A$, for any object α in $P(A)$ and β in $P(X \times A)$, the canonical arrow*

$$\exists_{\pi_A}(P_{\pi_A}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{\pi_A}(\beta)$$

in $P(A)$ is an isomorphism.

Notation: in this work, given two primary doctrines P and P' on the same base category $\mathcal{C}$, we will say that P' is a **subdoctrine** of P if $P'(X)$ is a sub-inf-semilattice of $P(X)$ for every object X of $\mathcal{C}$, and if, for every arrow $f\colon Y \to X$, the action of P_f and P'_f is the same (on the object of $P'(X)$). Moreover, we say that P' is an **elementary subdoctrine** of P whenever both doctrines are elementary and also the fibred equality of P' coincides with that of P.

2.5. Example. Let $\mathcal{L}_{=,\exists}$ be the $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic Logic (also called *regular* in [13, Sec. D1.3], see remark 5.12), i.e. the fragment with the true constant, conjunction, equality and existential quantifiers. Let $\mathbb{T}$ be a theory of such a fragment. Consider the syntactic doctrine

$$\mathsf{LT}^{\mathbb{T}}_{=,\exists}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$$

where $\mathcal{V}$ is the category of contexts and substitutions and $\mathsf{LT}^{\mathbb{T}}_{=,\exists}(\Gamma)$ is given by the Lindenbaum-Tarski algebra of well-formed formulas of $\mathcal{L}_{=,\exists}$ with free variables in Γ as in Example 2.3. The doctrine $\mathsf{LT}^{\mathbb{T}}_{=,\exists}$ is elementary and pure existential.

2.6. Remark. In a pure existential elementary doctrine, for every arrow $f\colon A \to B$ of $\mathcal{C}$ the functor P_f has a left adjoint $\exists_f$ that can be computed as

$$\exists_{\pi_2}(P_{f\times \mathrm{id}_B}(\delta_B) \wedge P_{\pi_1}(\alpha))$$

for α in $P(A)$, where π_1 and π_2 are the projections from $A \times B$. However, observe that such a definition guarantees only the validity of the corresponding Frobenius reciprocity condition for $\exists_f$, but it does not guarantee the validity of the Beck-Chevalley condition with respect to pullbacks along f (see the counterexample in [26, Rem. 6.4]). In particular, primary doctrines, whose base category has finite limits, having left adjoints along every morphisms satisfying BCC and FR are called *full existential* in [26].

The following examples are discussed in [15, 10].

2.7. Example. Let $\mathcal{C}$ be a category with finite limits. The subobject functor $\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is an elementary doctrine. Moreover, it is pure existential if and only if the category $\mathcal{C}$ is regular.

2.8. Example. Let $\mathcal{D}$ be a category with finite products and weak pullbacks. The weak subobjects (or variations) functor $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, assigning to an object A the poset reflection of the slice category $\mathcal{D}/A$, is an elementary and pure existential doctrine (left adjoints are given by the post-composition). Moreover, we know from [26] that every weak subobject doctrines is a full existential completion (and that every element of the fibre can be written as an existential quantifier of a top element).

The category of primary doctrines PD is a 2-category, and we refer to [23, 22] for a complete description of the 1-cells and 2-cells of this 2-category. We denote by ED the 2-full subcategory of PD whose objects are pure existential doctrines, and whose 1-cells are those 1-cells of PD which preserve the pure existential structure. Similarly, we denote by EED the 2-full subcategory of PD whose objects are elementary and pure existential doctrines, and whose 1-cells preserve both the pure existential and the elementary structure.

We conclude this section recalling from [17, 22, 23] the **Grothendieck category** and the **category of predicates** of an elementary doctrine.

2.9. Definition. *Given a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, its **Grothendieck category** $\mathcal{G}_P$ is defined as follows:*

- *an object of $\mathcal{G}_P$ is a pair (A, α) where A is a set and $\alpha \in P(A)$;*
- *an arrow $f\colon (A, \alpha) \to (B, \beta)$ is an arrow $f\colon A \to B$ of $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$.*

We just remind that the Grothendieck category $\mathcal{G}_P$ of P is the base of the free completion adding *comprehensions* to P as shown in [22, 24].

2.10. Definition. *Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, its **category of predicates** $\mathcal{P}\!rd_P$ is defined as follows:*

- *an object of $\mathcal{P}\!rd_P$ is a pair (A, α) where A is a set and $\alpha \in P(A)$;*
- *an arrow $[f]\colon (A, \alpha) \to (B, \beta)$ of $\mathcal{P}\!rd_P$ is an equivalence class of arrows $(A, \alpha) \to (B, \beta)$ of $\mathcal{G}_P$ with respect to the following equivalence relation: $f \sim g$ when $\alpha \leq P_{\langle f,g\rangle}(\delta_B)$.*

We just remind that the category of predicates $\mathcal{P}\!rd_P$ of P is the base of the free completion adding an extensional equality, formally named *comprehensive diagonals* (see [22, Def. 5.2]), to the free completion of P with comprehensions, as shown in [22].

2.11. Remark. Notice that the category of predicates $\mathcal{P}\!rd_P$ of an elementary doctrine has always finite limits. We refer to [23, Prop.4.15] or [21, Rem. 2.14] for the explicit description of the pullbacks in $\mathcal{P}\!rd_P$, being the base of an elementary doctrine with full comprehensions and comprehensive diagonals. The name *category of predicates* in [21] was inspired by Joyal's category of predicates in [19] (see [21, Ex. 2.18]) which we will recall in the last section.

3. The pure existential completion

In [33] the second author introduced a free construction, called *existential completion*, that freely adds left adjoints along a given class of morphisms Λ (closed under pullbacks, compositions and isomorphisms) to a given primary doctrine. Such a notion has been renamed *generalized existential completion* in [26] to distinguish some of its relevant instances. Following the terminology in [26], the *pure existential completion* is the instance of the generalized existential completion where the class Λ consists of product projections.

In [26] we provided an intrinsic characterization of generalized existential completions through the notion of *existential free elements* with respect to Λ. Here, we present a further version of this characterization (Theorem 3.14) only for the pure existential completion (but it can be smoothly extended to all generalized existential completions) by introducing the notion of *pure existential free elements of an existential doctrine P relative to a subdoctrine P'* which slightly generalizes the notion of *pure existential free objects of a doctrine P*. This result will be relevant in the proof of Theorem 5.8.

Since we will mainly perform our calculations on pure existential completions by just referring to their intrinsic characterizations, here we do not recall the original construction from [33]. We just remind from [33] that the pure existential completion $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of a given primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ provides a 2-adjunction

$$\mathsf{PD} \underset{}{\overset{(-)^{\exists}}{\rightleftarrows}} \mathsf{ED} \quad (\perp)$$

from the 2-category PD of primary doctrines into the 2-category ED of pure existential doctrines [33] and this justifies its name. Moreover, the pure existential completion preserves the elementary structure as shown in [33] and this is necessarily so as shown in [26, Thm. 6.1].

Now, we start by giving the main definitions necessary for the intrinsic characterization of the pure existential completion.

For the rest of this section, let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be a fixed pure existential doctrine, and let $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be a fixed subdoctrine of P.

3.1. Definition. *An element α of the fibre $P(A)$ is said to be a* ***pure existential splitting*** *if for every projection $\pi_A\colon A\times B \to A$ and for every element β of the fibre $P(A\times B)$, whenever $\alpha = \exists_{\pi_A}(\beta)$ holds then there exists an arrow $h\colon A \to B$ such that $\alpha = P_{\langle \mathrm{id}_A, h\rangle}(\beta)$. Moreover, α is said to be* ***pure existential free*** *if for every morphism $f\colon B \to A$, $P_f(\alpha)$ is a pure existential splitting.*

In the following proposition we recall from [26, Prop. 4.4] a useful equivalent characterization of pure existential splitting elements that is a strengthening of the usual Existence Property:

3.2. Proposition. *Let α be an element of the fibre $P(A)$. Then α is pure existential splitting if and only if for every projection $\pi_A\colon A\times B \to A$ and for every element β of the fibre $P(A\times B)$, whenever $\alpha \leq \exists_{\pi_A}(\beta)$ holds then there exists an arrow $h\colon A \to B$ such that $\alpha \leq P_{\langle \mathrm{id}_A, h\rangle}(\beta)$.*

3.3. Definition. *P satisfies the* ***Rule of Choice****, for short* **(RC)***, if whenever $\top_A \leq \exists_{\pi_A}(\beta)$ there exists an arrow $h\colon A \to B$ such that $\top_A \leq P_{\langle \mathrm{id}_A, h\rangle}(\beta)$.*

3.4. Remark. Observe that P satisfies (RC) if and only if for every object A of $\mathcal{C}$, the element $\top_A \in P(A)$ is a pure existential splitting.

3.5. Definition. *We say that an element α of the fibre $P(A)$ is* ***covered*** *by an element $\beta \in P(A \times B)$ if $\alpha = \exists_{\pi_A}(\beta)$.*

3.6. Definition. *We say that P has* ***enough pure existential free elements*** *if for every object A of $\mathcal{C}$, any element $\alpha \in P(A)$ is covered by some pure existential free element $\beta \in P(A \times B)$ for some object B of $\mathcal{C}$.*

3.7. Remark. Observe that if all elements of a subdoctrine P' of a pure existential doctrine P are pure existential splitting for P then they are also pure existential free for P, being the doctrine P' closed under re-indexing. It also holds that, if a doctrine has enough pure existential free elements then every pure existential splitting is pure existential free. We refer to [26, Lem. 4.11] for a proof of this fact.

3.8. Definition. *We say that P' is a* ***pure existential cover*** *of P if for any object A of $\mathcal{C}$, every element α' of $P'(A)$ is a pure existential splitting for P (and hence pure existential free) and every element α of $P(A)$ is covered by an element of P'.*

We summarize in the following proposition some useful properties, and refer to [26] for all the details:

3.9. Lemma. *The following two results hold:*

- *if P is the pure existential completion of a primary doctrine P' then P' is a pure existential cover of P;*
- *if P' is a pure existential cover of P, then the existential free elements of P coincides exactly with the elements of P'. Hence, if a pure existential cover exists, it is unique.*

The previous notions can be generalized by relativizing each concept to a given subdoctrine:

3.10. Definition. *An object α of the fibre $P'(A)$ is said to be a* ***pure existential splitting of*** *P* ***relative to*** *P' if for every projection $\pi_A \colon A \times B \to A$ and for every element β of the fibre $P'(A \times B)$, whenever $\alpha = \exists_{\pi_A}(\beta)$ holds in $P(A)$ then there exists an arrow $h \colon A \to B$ such that $\alpha = P_{\langle \mathrm{id}_A, h\rangle}(\beta)$. Moreover, α is said to be* ***pure existential free of*** *P* ***relative to*** *P' if $P_f(\alpha)$ is a pure existential splitting of P relative to P' for every morphism $f \colon B \to A$.*

3.11. Definition. *We say that P' is a* ***pure existential relative cover of*** *P if for any object A, every element α' of $P'(A)$ is a pure existential splitting element of P relative to P' and every element α of $P(A)$ is covered by an element of P'.*

3.12. Lemma. *If every element of P is covered by an element of P', then every pure existential splitting of P relative to P' is a pure existential splitting.*

PROOF. Let α be an element of $P'(A)$, and let us suppose that it is a pure existential splitting of P relative to P'. Now suppose that $\alpha \leq \exists_{\pi_A}(\beta)$, with β element of $P(A \times B)$. By assumption, β can be written as $\beta = \exists_{\pi_{A\times B}}(\gamma)$ with γ element of $P'(A \times B \times C)$ and $\pi_{A\times B}\colon A \times B \times C \to A \times B$. Hence, since left adjoints compose, we have that

$$\alpha \leq \exists_{\pi'_A}(\ \exists_{\pi_{A\times B}}(\gamma)\ ) = \exists_{\pi_A}(\gamma)$$

with $\pi'_A\colon A \times B \to A$ and $\pi_A\colon A \times B \times C \to A$. Since α is a pure existential splitting of P relative to P', then there exists an arrow $\langle f, g\rangle\colon A \to B \times C$ such that

$$\alpha \leq P_{\langle \mathrm{id}_A, f, g\rangle}(\gamma).$$

Then, since $\beta = \exists_{\pi_{A\times B}}(\gamma)$, and hence, $\gamma \leq P_{\pi_{A\times B}}(\beta)$, we deduce that

$$\alpha \leq P_{\langle \mathrm{id}_A, f, g\rangle}(\ P_{\pi_{A\times B}}(\beta)\ ) = P_{\langle \mathrm{id}_A, f\rangle}(\beta).$$

Therefore, by Proposition 3.2, we can conclude that α is a pure existential splitting. ∎

Combining this lemma with the definition of pure existential cover, we obtain the following corollary:

3.13. COROLLARY. *P' is a pure existential cover of P if and only if P' is a pure existential relative cover of P.*

PROOF. If P' is a pure existential cover of P, then it is in particular a pure existential relative cover of P since every pure existential splitting element of P is obviously a pure existential splitting element of P relative to P'. The converse follows from Lemma 3.12 since every element of P' which is a pure existential free element of P relative to P' is also a pure existential splitting element for P. ∎

Now we are ready to recall the main result from [26]. Notice that, with respect to the original result, here we present an extra equivalent condition, based on Corollary 3.13:

3.14. THEOREM. *The following are equivalent:*

1. *P is isomorphic to the pure existential completion $(P')^{\exists}$ of a primary doctrine P';*
2. *P satisfies the following points:*
   - *(a) P satisfies the rule of choice RC;*
   - *(b) for every pure existential free element α and β of $P(A)$, then $\alpha \wedge \beta$ is a pure existential free.*
   - *(c) P has enough pure existential free elements;*
3. *P has a (unique) pure existential cover;*
4. *P has a (unique) pure existential relative cover.*

A relevant application of the previous characterization is given in the context of doctrines with *Hilbert's ϵ-operators*. We recall from [21] the following definitions:

3.15. Definition. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *be an elementary pure existential doctrine. An object* B *of* $\mathcal{C}$ *is equipped with Hilbert's* ϵ-***operator*** *if, for any object* A *in* $\mathcal{C}$ *and any* α *in* $P(A \times B)$ *there exists an arrow* $\epsilon_\alpha\colon A \to B$ *such that* $\exists_{\pi_A}(\alpha) = P_{\langle \mathrm{id}_A, \epsilon_\alpha\rangle}(\alpha)$ *holds in* $P(A)$.

3.16. Definition. *We say that an elementary pure existential doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *is* ***equipped with*** *Hilbert's* ϵ-***operators*** *if every object in* $\mathcal{C}$ *is equipped with an* ϵ*-operator.*

Doctrines equipped with Hilbert's ϵ-operators have been characterized in [26] in terms of pure existential completions as follows:

3.17. Theorem. *Every elementary pure existential doctrine is equipped with Hilbert's* ϵ*-operators if and only if it is (equivalent to) the pure existential completion of itself.*

We conclude this section by recalling the following example from [26]:

3.18. Example. Let $\mathbb{T}_0$ be the fragment $\mathcal{L}_{=,\exists}$ of first-order Intuitionistic Logic, defined in Example 2.5, with no extra-logical axioms on an arbitrary signature and let H_0 be the Horn theory given the Horn fragment $\mathcal{L}_{=}$ of first-order Intuitionistic Logic, defined in Example 2.3, with no extra-logical axioms on the same signature. Then, the elementary pure existential doctrine $\mathsf{LT}^{\mathbb{T}_0}_{=,\exists}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is the pure existential completion of the syntactic elementary doctrine $\mathsf{LT}^{\mathsf{H}_0}_{=}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, namely $(\mathsf{LT}^{\mathsf{H}_0}_{=})^{\exists} \equiv \mathsf{LT}^{\mathbb{T}_0}_{=,\exists}$.

4. Regular and Exact completions of elementary pure existential doctrines

In this section we first recall well-known characterizations of the reg/lex and ex/lex-completions in [1, 2, 3] and then we pass to remind notions and results related to the regular and exact completions of elementary, pure existential doctrines in [23, 22, 24, 21].

Remember from [1, Lem. 5.1] the following characterization of the reg/lex completion (we refer also [12, Sec. A.1.3] for a detailed analysis of this completion):

4.1. Theorem. [1] *Any regular category* $\mathcal{A}$ *is the regular completion of the full subcategory* $\mathcal{P}_{\mathcal{A}}$ *of its regular projectives if and only if* $\mathcal{P}_{\mathcal{A}}$ *is closed under finite limits in* $\mathcal{A}$ *and* $\mathcal{A}$ *has enough regular projectives, and in addition every object of* $\mathcal{A}$ *can be embedded in a regular projective.*

Then, recall the following decomposition of ex/lex-completion shown by A. Carboni and E. Vitale [3]:

4.2. Theorem. [3] *For any category* $\mathcal{C}$ *with finite limits, the* ex/lex*-completion of* $\mathcal{C}$ *is equivalent to the* ex/reg*-completion of the* reg/lex*-completion of* $\mathcal{C}$*, namely*

$$(\mathcal{C})_{\mathsf{ex/lex}} \equiv ((\mathcal{C})_{\mathsf{reg/lex}})_{\mathsf{ex/reg}}$$

4.3. Regular and exact completions of doctrines. We recall from [21] the *regular completion* of an elementary and pure existential doctrine. We provide a direct explicit description of this construction, while we refer to [21] for its equivalent presentation in terms of the category of *entire and functional relations* of the completion with comprehensions and comprehensive diagonals of an elementary existential doctrine.

4.4. Definition. *Let P be an elementary pure existential doctrine. The* ***regular completion*** $\mathsf{Reg}(P)$ *of P is the category defined as follows:*

- *an object is a pair (A, α) where A is an object of $\mathcal{C}$ and $\alpha \in P(A)$;*
- *an arrow from (A, α) to (B, β) is given by an element ϕ of $P(A \times B)$ such that:*
  1. *$\phi \leq P_{\pi_1}(\alpha) \wedge P_{\pi_2}(\beta)$;* ***(well-defined)***
  2. *$\alpha \leq \exists_{\pi_1}(\phi)$;* ***(entire)***
  3. *$P_{\langle\pi_1,\pi_2\rangle}(\phi) \wedge P_{\langle\pi_1,\pi_3\rangle}(\phi) \leq P_{\langle\pi_2,\pi_3\rangle}(\delta_B)$.* ***(functional)***

4.5. Remark. Observe that, if ψ and φ are morphisms of $\mathsf{Reg}(P)$ from (A, α) to (B, β) and $\psi \leq \varphi$ then $\psi = \varphi$. We refer to [29] for this remark.

The universal properties of the regular completion of a doctrine are studied in [21, Thm. 3.3]. We recall here the main result:

4.6. Theorem. *Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. Then the category $\mathsf{Reg}(P)$ is regular, and the assignment $P \mapsto \mathsf{Reg}(P)$ extends to a 2-functor*

$$\mathsf{Reg}(-)\colon \mathsf{EED} \to \mathsf{RegCat}$$

which is a left biadjoint to the inclusion of the 2-category RegCat of regular categories in the 2-category EED of elementary and pure existential doctrines given by the assignment $\mathcal{C} \mapsto \mathsf{Sub}_{\mathcal{C}}$.

4.7. Remark. Recall from [21] that given an elementary and pure existential doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ and an arrow $f \colon A \to B$ of $\mathcal{C}$, we have that the *graph* $P_{f\times \mathrm{id}_B}(\delta_B)$ of f is an entire and functional relation from A to B and this defines the *graph functor* $G \colon \mathcal{C} \to \mathsf{Reg}(P)$ which preserves finite products.

4.8. Remark. Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. Notice that the categories $\mathcal{P}rd_P$, $\mathcal{G}_P$ and $\mathsf{Reg}(P)$ have the same objects, but increasingly general morphisms.

Observe that the graph functor $G \colon \mathcal{C} \to \mathsf{Reg}(P)$ defined in Remark 4.7 extends to a functor from $\mathcal{P}rd_P$ (and also from $\mathcal{G}_P$) to $\mathsf{Reg}(P)$ and, more generally, to a functor from $\mathcal{P}rd_{P'}$ (and also from $\mathcal{G}_{P'}$) to $\mathsf{Reg}(P)$ for a given subdoctrine P'.

4.9. Definition. *Given an elementary pure existential doctrine P and an elementary subdoctrine P' we can define an embedding, called* ***graph functor***

$$G_{|_{P'}} \colon \mathcal{P}rd_{P'} \to \mathsf{Reg}(P)$$

by mapping (A, α) of $\mathcal{P}rd_{P'}$ into (A, α) of $\mathsf{Reg}(P)$ and an arrow $[f] \colon (A, \alpha) \to (B, \beta)$ of $\mathcal{P}rd_{P'}$ into the arrow $G_{|_{P'}}([f]) = P_{f \times \mathrm{id}_B}(\delta_B) \wedge \ (P_{\pi_1}(\alpha) \ \wedge \ P_{\pi_2}(\beta)\ )$ from (A, α) to (B, β) of $\mathsf{Reg}(P)$.

4.10. Remark. The graph functor of an elementary pure existential doctrine P with an elementary subdoctrine P' can be also defined as $G_{|_{P'}}([f]) = \exists_{\langle \mathrm{id}_A, f\rangle}(\alpha)$ because P has left adjoints along arbitrary arrows, see Remark 2.6.

We refer to [21] for the following result:

4.11. Proposition. *The previous assignments provide a well-defined functor $G_{|_{P'}} \colon \mathcal{P}rd_{P'} \to \mathsf{Reg}(P)$, and it preserves finite limits. Moreover, it is faithful, and it induces a regular functor $G^{\mathsf{reg}}_{|_{P'}}$:*

$$\begin{array}{ccc} & & (\mathcal{P}rd_{P'})_{\mathsf{reg/lex}} \\ & \nearrow & \downarrow {\scriptstyle G^{\mathsf{reg}}_{|_{P'}}} \\ \mathcal{P}rd_{P'} & \xrightarrow[G_{|_{P'}}]{} & \mathsf{Reg}(P) \end{array}$$

Proof. The first part follows by [21, Thm. 3.2] (and from the fact that $\mathcal{P}rd_{P'}$ is lex, see Remark 2.11), while the existence of the regular functor $G^{\mathsf{reg}}_{|_{P'}}$ follows by the universal property of the $\mathsf{reg/lex}$-completion in [1]. ■

4.12. Example. The regular completion $\mathsf{Reg}(\mathsf{Sub}_{\mathcal{C}})$ of the doctrine $\mathsf{Sub}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of subobjects of a regular category is equivalent to $\mathcal{C}$ [21, Cor. 5.4].

4.13. Example. The regular completion $\mathsf{Reg}(\Psi_{\mathcal{D}})$ of the doctrine $\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of weak subobjects presented in Example 2.8 coincides with the regular completion $(\mathcal{D})_{\mathsf{reg/lex}}$ of the lex category $\mathcal{D}$, in the sense of [3]. We refer to [21] for more details.

4.14. Example. The regular completion $\mathsf{Reg}(\mathsf{LT}^{\mathbb{T}}_{=,\exists})$ performed on the syntactic doctrine $\mathsf{LT}^{\mathbb{T}}_{=,\exists} \colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ defined in Example 2.5 provides exactly the syntactic category denoted $\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}}$ associated with the theory $\mathbb{T}$ in [13, Sec. D1.4].

Combining the regular completion of an elementary pure existential doctrine with the exact completion $(-)_{\mathsf{ex/reg}}$ of a regular category [1, 3], one can define the so-called *exact completion of an elementary and pure existential doctrine* as pointed out in [24, Sec. 3]:

4.15. Definition. *Let P be an elementary pure existential doctrine. We call the category* $\mathsf{Ex}(P) := (\mathsf{Reg}(P))_{\mathsf{ex/reg}}$ the **exact completion** of P.

The universal properties of the exact completion of an elementary and pure existential doctrine can be deduced by combining the universal properties of the regular completion of a doctrine (see Theorem 4.6) with the universal properties of the ex/reg-completion, see [24, Cor. 3.4]. We recall here the main result:

4.16. Theorem. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. The category $\mathsf{Ex}(P)$ is exact, and the assignment $P \mapsto \mathsf{Ex}(P)$ extends to a 2-functor*

$$\mathsf{Ex}(-)\colon \mathsf{EED} \to \mathsf{ExCat}$$

which is a left biadjoint to the inclusion of the 2-category ExCat of exact categories in the 2-category EED of elementary and pure existential doctrines given by the assignment $\mathcal{C} \mapsto \mathsf{Sub}_{\mathcal{C}}$.

4.17. Example. The exact completion $\mathsf{Ex}(\Psi_{\mathcal{D}})$ of the weak subobjects doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ presented in Example 2.8 coincides with the exact completion $(\mathcal{D})_{\mathsf{ex/lex}}$ of the lex category $\mathcal{D}$, in the sense of [3]. We refer to [24, Ex. 4.4] for more details.

4.18. Example. The exact completion $\mathsf{Ex}(\mathsf{LT}^{\mathbb{T}}_{=,\exists}) = (\mathsf{Reg}(\mathsf{LT}^{\mathbb{T}}_{=,\exists}))_{\mathsf{ex/reg}}$ of the syntactic doctrine $\mathsf{LT}^{\mathbb{T}}_{=,\exists}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ coincides with the *effectivization* $\mathcal{E}_{\mathbb{T}} := \mathbf{Eff}(\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}})$ of the syntactic category $\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}}$ in [13, pp. 849-850].

4.19. Example. The exact completion of the subobject doctrine $\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of a regular category $\mathcal{C}$ coincides with the well known construction of the exact completion of a regular category $(\mathcal{C})_{\mathsf{ex/reg}}$ as observed in [24].

4.20. Remark. We recall from [21, Ex. 5.6] that the subobject doctrine $\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ on a regular category satisfies the rule of choice (RC) if and only if every regular epi of $\mathcal{C}$ splits. Therefore, when $\mathcal{C}$ is a regular category whose regular epimorphisms split, we have that $\mathsf{Sub}_{\mathcal{C}}$ happens to be equivalent to the weak-subobject doctrine by [21, Thm. 5.9], and, by combining this fact with previous examples, we obtain an abstract proof that $(\mathcal{C})_{\mathsf{reg/lex}} \equiv \mathcal{C}$ (by Examples 4.12 and 4.13) and $(\mathcal{C})_{\mathsf{ex/reg}} \equiv (\mathcal{C})_{\mathsf{ex/lex}}$ (by Examples 4.19 and 4.17). In [12, Rem. 1.3.10(c)], when $\mathcal{C}$ is a regular category whose regular epimorphisms split, there is a direct proof of the fact that $(\mathcal{C})_{\mathsf{reg/lex}} \equiv \mathcal{C}$, and from this we can alternatively deduce that $(\mathcal{C})_{\mathsf{ex/reg}} \equiv (\mathcal{C})_{\mathsf{ex/lex}}$ directly by Theorem 4.2.

5. Characterization of regular and exact completions of pure existential completions

In this section we present our main results characterizing the regular and exact completions of pure existential completions of elementary doctrines (Theorem 5.8 and Corollary 5.9).

In particular, we first show that an elementary and pure existential doctrine P is the pure existential completion of an elementary doctrine P' if and only if the canonical embedding of $\mathcal{P}\!rd_{P'}$ into $\mathsf{Reg}(P)$ gives rise to an equivalence $\mathsf{Reg}(P) \equiv (\mathcal{P}\!rd_{P'})_{\mathsf{reg/lex}}$. To this aim, we show that, inside the regular completion $\mathsf{Reg}(P)$ of the pure existential completion P of an elementary subdoctrine P', the comprehension (A,α) of a pure existential free element α of $P(A)$ is a regular projective. To this purpose, we recall a standard, but useful lemma holding in every regular category. We refer to [39, Sec. 4.3].

5.1. Lemma. *In a regular category $\mathcal{C}$, an arrow $f\colon A \to B$ is a regular epi if and only if the subobject doctrine $\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ satisfies $\top_B \leq \exists_{\pi_B}\mathsf{Sub}_{f\times \mathrm{id}_B}(\delta_B)$.*

This lemma, combined with the definition of arrows in $\mathsf{Reg}(P)$, allows us to provide a simple description of the regular epimorphisms of the regular completion $\mathsf{Reg}(P)$ of an elementary pure existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$. We refer to [29, Sec. 2.5] and [39] for more details.

5.2. Lemma. *A morphism $\phi\colon (A,\alpha) \to (B,\beta)$ of $\mathsf{Reg}(P)$ is a regular epimorphism if and only if $\beta = \exists_{\pi_B}(\phi)$ in $P(B)$.*

Now, we are going to show that pure existential splitting elements of a pure existential doctrine P single out regular projective objects in the regular completion $\mathsf{Reg}(P)$.

To this purpose, we first prove the following useful lemma:

5.3. Lemma. *If $\phi\colon (A,\alpha) \to (B,\beta)$ is an arrow of $\mathsf{Reg}(P)$ and α is a pure existential splitting element of P, then there exists an arrow $f\colon A \to B$ such that $\alpha = P_{\langle \mathrm{id}_A, f\rangle}(\phi)$, with $\alpha \leq P_f(\beta)$. Moreover, for every arrow $g\colon A \to B$ with such a property, we have that $\alpha \leq P_{\langle f,g\rangle}(\delta_B)$.*

Proof. Let $\phi\colon (A,\alpha) \to (B,\beta)$ be an arrow of in $\mathsf{Reg}(P)$. By definition of arrows in $\mathsf{Reg}(P)$, we have that $\alpha = \exists_{\pi_1}(\phi)$, and then, by the universal property of pure existential splittings, we can conclude that there exists an arrow $f\colon A \to B$ such that $\alpha = P_{\langle \mathrm{id}_A, f\rangle}(\phi)$. Moreover, since $\phi \leq P_{\pi_2}(\beta)$, we can conclude that $\alpha \leq P_f(\beta)$.

Now let us consider another arrow $g\colon A \to B$ such that $\alpha = P_{\langle \mathrm{id}_A, g\rangle}(\phi)$. By definition, we have that ϕ is functional in P, namely

$$P_{\langle \pi_1,\pi_2\rangle}(\phi) \wedge P_{\langle \pi_1,\pi_3\rangle}(\phi) \leq P_{\langle \pi_2,\pi_3\rangle}(\delta_B).$$

Then we can apply $P_{\langle \mathrm{id}_A, f, g\rangle}$ to both sides of this inequality, obtaining

$$P_{\langle \mathrm{id}_A, f\rangle}(\phi) \wedge P_{\langle \mathrm{id}_A, g\rangle}(\phi) \leq P_{\langle f,g\rangle}(\delta_B)$$

that is

$$\alpha \leq P_{\langle f,g\rangle}(\delta_B).$$

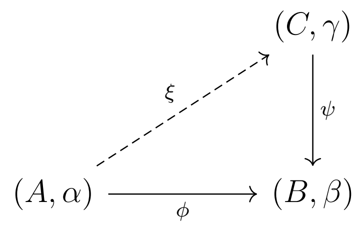
■

5.4. Remark. Notice that, by Lemma 5.3, we have that every arrow $\phi\colon (A,\alpha) \to (B,\beta)$ of $\mathsf{Reg}(P)$ with α pure existential splitting induces a unique arrow in $\mathcal{Prd}_P$.

Furthermore, pure existential splitting elements single out regular projectives as follows:

5.5. Proposition. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. Then every object (A,α) where α is pure existential splitting is a regular projective in $\mathsf{Reg}(P)$.*

Proof. Let us consider the following diagram



(1)

with ψ regular epi in $\mathsf{Reg}(P)$. By Lemma 5.2 we know that $\beta = \exists_{\pi_B}(\psi)$, with $\pi_2\colon C\times B \to B$. Hence, we are going to use the fact that α is pure existential splitting to show that there exists a morphism ξ of $\mathsf{Reg}(P)$ such that the diagram (1) commutes.

From $\beta = \exists_{\pi_2}(\psi)$ and $\alpha \le \exists_{\pi_1}(\phi) \le \exists_{\pi_1}(\phi \wedge P_{\pi_2}(\beta))$ (since ϕ is entire and well-defined) we can deduce (combining these with BCC and FR) that

$$\alpha \le \exists_{\pi_1}(P_{\langle \pi_1,\pi_3\rangle}(\phi) \wedge P_{\langle \pi_2,\pi_3\rangle}(\psi)).$$

where here projections have domain $A \times C \times B$.

Therefore, since α is existential splitting, there exists a morphism $\langle \mathrm{id}_A, h_1, h_2\rangle\colon A \to A \times C \times B$ such that

$$\alpha \le P_{\langle \mathrm{id}_A, h_2\rangle}(\phi) \wedge P_{\langle h_1,h_2\rangle}(\psi). \tag{2}$$

Then, we can define

$$\xi := G([h_1]) = P_{h_1\times \mathrm{id}_C}(\delta_C) \wedge (P_{\pi_1}(\alpha) \wedge P_{\pi_2}(\gamma))$$

after noting that $[h_1]\colon (A,\alpha) \to (C,\gamma)$ is an arrow of $\mathcal{Prd}_P$ since $\alpha \le P_{\langle h_1,h_2\rangle}(\psi) \le P_{h_1}(\gamma)$ being ψ an arrow of $\mathsf{Reg}(P)$.

Now we show that $\psi \circ \xi = \phi$ (where $\psi\circ\xi$ denotes the composition of morphisms in $\mathsf{Reg}(P)$). The fact that predicates are descent objects for the equality allows us to deduce that

$$\psi \circ \xi = \exists_{\langle \pi_1,\pi_3\rangle}(P_{\langle \pi_1,\pi_2\rangle}(G([h_1])) \wedge P_{\langle \pi_2,\pi_3\rangle}(\psi)) \le P_{\pi_1}(\alpha) \wedge P_{h_1\times \mathrm{id}_B}(\psi)$$

and by functionality of ψ, together with $\alpha \le P_{\langle h_1,h_2\rangle}(\psi)$ by (2), we can deduce

$$P_{\pi_1}(\alpha) \wedge P_{h_1\times\mathrm{id}_B}(\psi) \le P_{h_2\times \mathrm{id}_B}(\delta_B).$$

Therefore, since $\alpha \leq P_{\langle \mathrm{id}_A, h_2\rangle}(\phi)$ by (2), and hence $P_{\pi_1}(\alpha) \leq P_{\langle \pi_1, h_2\pi_1\rangle}(\phi)$, we get that

$$\psi \circ \xi \leq \phi.$$

Hence, by Remark 4.5, we can conclude that $\phi = \psi \circ \xi$, i.e. the diagram (1) commutes in $\mathsf{Reg}(P)$. This concludes the proof that (A, α) is a regular projective.

■

Recall that in the context of regular categories, we say that an object A is *covered* by a regular projective B if there exists a regular epi $e\colon B \to A$.

5.6. Lemma. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine with enough pure existential free elements. Then every object (B, β) of $\mathsf{Reg}(P)$ is covered by a regular projective object $(A \times B, \alpha)$, with α pure existential splitting.*

Proof. By definition of doctrine with enough pure existential free elements, for any element β of $P(B)$ there exists a pure existential free element α of $P(A \times B)$ such that $\beta = \exists_{\pi_2}(\alpha)$ (hence, $[\pi_2]\colon (A \times B, \alpha) \to (B, \beta)$ is a well-defined arrow of $\mathcal{Prd}_P$). Thus, for every object (B, β) of $\mathsf{Reg}(P)$, we can define in $\mathsf{Reg}(P)$ the arrow

$$G([\pi_2])\colon (A \times B, \alpha) \to (B, \beta)$$

where $G\colon \mathcal{Prd}_P \to \mathsf{Reg}(P)$ is the graph functor defined in Definition 4.9 (with respect to P itself), and it is a regular epi of $\mathsf{Reg}(P)$ since by Remark 4.10 $G([\pi_2]) = \exists_{\langle \mathrm{id}_{A\times B}, \pi_2\rangle}(\alpha)$, which implies $\exists_{\pi_3}(G([\pi_2])) = \exists_{\pi_3}((\exists_{\langle \mathrm{id}_{A\times B}, \pi_2\rangle}(\alpha)) = \exists_{\pi_2}(\alpha) = \beta$.

Finally, since every pure existential free element is in particular a pure existential splitting element, by Proposition 5.5 we conclude that $(A \times B, \alpha)$ is a regular projective object of $\mathsf{Reg}(P)$ and it covers (B, β). ■

5.7. Lemma. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. If P satisfies the rule of choice then every object of $\mathsf{Reg}(P)$ is a subobject of a regular projective.*

Proof. If P satisfies the rule of choice, then we have that every top element $\top_A$ is a pure existential splitting element (see Remark 3.4), and hence $(A, \top_A)$ is a regular projective of $\mathsf{Reg}(P)$ by Proposition 5.5. Therefore, every object (A, α) is a subobject of $(A, \top)$ in $\mathsf{Reg}(P)$ via $G([\mathrm{id}_A])$ with $[\mathrm{id}_A]\colon (A, \alpha) \to (A, \top_A)$, i.e. $\exists_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha)$.

■

By employing the previous results we can prove our main theorem:

5.8. Theorem. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. Then P is the pure existential completion of an elementary subdoctrine P' if and only if the functor $G^{\mathsf{reg}}_{|P'}\colon (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}} \to \mathsf{Reg}(P)$ provides an equivalence $\mathsf{Reg}(P) \equiv (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}}$.*

Proof. ($\Rightarrow$) Suppose that P is the pure existential completion of an elementary subdoctrine P'. To prove that $G^{\mathsf{reg}}_{|_{P'}}\colon (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}} \to \mathsf{Reg}(P)$ provides an equivalence $\mathsf{Reg}(P) \equiv (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}}$, we will employ the characterization of the regular completion of a lex category as presented in [1, Lem. 5.1] and recalled in Theorem 4.1.

In particular, we are going to show that the image via $G_{|_{P'}}$ of $\mathcal{Prd}_{P'}$ into $\mathsf{Reg}(P)$ is a full subcategory of regular projectives, and that every object of $\mathsf{Reg}(P)$ is covered (via a regular epi) by an object lying in the image of $G_{|_{P'}}$, and that every object of $\mathsf{Reg}(P)$ is a subobject of an object lying in the image of $G_{|_{P'}}$.

Now, by Theorem 3.14 and Lemma 3.9, we have that the pure existential free elements of P are precisely the elements of P' and of course, we have that every pure existential free is pure existential splitting.

Therefore, by Proposition 5.5, we have that every object (A,α) with α element of $P'(A)$ is a regular projective and, by Lemma 5.6, we can conclude that every object (B,β) of $\mathsf{Reg}(P)$ is covered by a regular projective object of the form $(A\times B,\alpha)$ with α pure existential free. Then, combining Lemma 5.7 with the fact that every pure existential completion satisfies the rule of choice, we can conclude that every object (A,α) of $\mathsf{Reg}(P)$ is a subobject of a regular projective $(A,\top_A)$ (which is in the image of $G_{|_{P'}}$). Finally, the image of $\mathcal{Prd}_{P'}$ via $G_{|_{P'}}$ into $\mathsf{Reg}(P)$ is a lex full subcategory of $\mathsf{Reg}(P)$ by Proposition 4.11 and Remark 5.4. By the characterization of the regular completion in Theorem 4.1, we conclude that $G^{\mathsf{reg}}_{|_{P'}}\colon (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}} \to \mathsf{Reg}(P)$ is an equivalence of categories.

($\Leftarrow$) Let $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary subdoctrine of P, and let us suppose the functor $G^{\mathsf{reg}}_{|_{P'}}\colon (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}} \to \mathsf{Reg}(P)$ provides an equivalence $\mathsf{Reg}(P) \equiv (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}}$.

It is immediate to observe that the fibre $\mathsf{Sub}_{\mathsf{Reg}(P)}(A,\top_A)$ of the subobject doctrine of $\mathsf{Reg}(P)$ is equivalent to $P(A)$ and that the fibre $\mathsf{Sub}_{\mathsf{Reg}(\Psi_{\mathcal{Prd}_{P'}})}((A,\top_A),\mathrm{id}_A)$ of the subobject doctrine of $\mathsf{Reg}(\Psi_{\mathcal{Prd}_{P'}})$ is equivalent to $\Psi_{\mathcal{Prd}_{P'}}(A,\top_A)$. Hence, we conclude that $P(A)$ is equivalent to $\Psi_{\mathcal{Prd}_{P'}}(A,\top_A)$ by the equivalence $\mathsf{Reg}(P) \equiv (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}}$ induced by $G^{\mathsf{reg}}_{|_{P'}}$ (and by Example 4.13).

Therefore, since from Example 2.8 we know that every weak subobject doctrine is a full existential completion (and that every element of the fibre can be written as an existential quantifier of a top element), any γ in $P(B)$ can be written as $\gamma = \exists_f(\alpha) = \exists_{\pi_B}(P_{f\times \mathrm{id}_B}(\delta_B)\wedge P_{\pi_A}(\alpha))$ for some α in $P'(A\times B)$ and some arrow $f\colon A\to B$.

Since P' is elementary, then $P_{f\times \mathrm{id}_B}(\delta_B)\wedge P_{\pi_A}(\alpha)$ is an object of P' and hence we conclude that every γ can be written as $\gamma = \exists_{\pi_B}(\sigma)$ for some object σ of $P'(A\times B)$.

Now we show that every element α of $P'(A)$ is a pure existential splitting element of P relative to P' (see Definition 3.10).

Suppose that $\alpha = \exists_{\pi_A}(\beta)$ in $P(A)$ with α element of $P'(A)$ and β element of $P'(A\times B)$. Then, observe that $G_{|_{P'}}([\pi_A])\colon (A\times B,\beta)\to(A,\alpha)$ is a well defined arrow in $\mathcal{Prd}_{P'}$ and that the arrow $G_{|_{P'}}([\pi_A])\colon (A\times B,\beta)\to(A,\alpha)$ is a surjective epimorphisms and hence a regular epimorphism in $\mathsf{Reg}(P)$ (as in the proof of Lemma 5.6).

Since (A,α) is a regular projective, being in the image of $G^{\mathsf{reg}}_{|_{P'}}$, there exists an arrow ϕ such that the diagram

$$\begin{array}{ccc} & & (A\times B,\beta) \\ & \overset{\phi}{\nearrow} & \downarrow G_{|_{P'}}([\pi_A]) \\ (A,\alpha) & \xrightarrow[G_{|_{P'}}([\mathrm{id}_A])]{} & (A,\exists_{\pi_A}(\beta)) \end{array} \tag{3}$$

commutes in $\mathsf{Reg}(P)$.

Then, by fullness of $G^{\mathsf{reg}}_{|_{P'}}$ since β is an element of P', there exists a unique arrow $[\langle f_1,f_2\rangle]:(A,\alpha)\to(A\times B,\beta)$ of $\mathcal{Prd}_{P'}$ such that $\phi=G_{|_{P'}}([\langle f_1,f_2\rangle])$ and also

$$\alpha\le P_{\langle f_1,f_2\rangle}(\beta)$$

Hence we have that

$$\begin{array}{ccc} & & (A\times B,\beta) \\ & \overset{G_{|_{P'}}([\langle f_1,f_2\rangle])}{\nearrow} & \downarrow G_{|_{P'}}([\pi_1]) \\ (A,\alpha) & \xrightarrow[G_{|_{P'}}([\mathrm{id}_A])]{} & (A,\exists_{\pi_A}(\beta)) \end{array} \tag{4}$$

commutes in $\mathsf{Reg}(P)$ and by faithfulness of $G_{|_{P'}}$ we conclude

$$\alpha\le P_{\langle f_1,\mathrm{id}_A\rangle}(\delta_A).$$

Combining this with $\alpha\le P_{\langle f_1,f_2\rangle}(\beta)$ by the properties of equality we conclude

$$\alpha\le P_{\langle \mathrm{id}_A,f_2\rangle}(\beta).$$

This ends the proof that any α of P' is a pure existential splitting element of P relative to P'. Furthermore, since any object of P is (existentially) covered by an element of P', we get that P' is a pure existential relative cover for P and by Theorem 3.14 we finally conclude that P is the pure existential completion of P'.

■

Note that, by the universal property of the ex/lex-completion, the graph functor $G^{\mathsf{reg}}_{|_{P'}}\colon(\mathcal{Prd}_{P'})_{\mathsf{reg/lex}}\to\mathsf{Reg}(P)$ extends to a functor $G^{\mathsf{ex}}_{|_{P'}}:(\mathcal{Prd}_{P'})_{\mathsf{ex/lex}}\to\mathsf{Ex}(P)$:

$$\begin{array}{ccccc} \mathcal{Prd}_{P'} & \hookrightarrow & (\mathcal{Prd}_{P'})_{\mathsf{reg/lex}} & \hookrightarrow & (\mathcal{Prd}_{P'})_{\mathsf{ex/lex}} \\ & & \downarrow G^{\mathsf{reg}}_{|_{P'}} & & \vdots\, G^{\mathsf{ex}}_{|_{P'}} \\ & & \mathsf{Reg}(P) & \hookrightarrow & \mathsf{Ex}(P). \end{array}$$

Thus, we can extend our previous characterization to the exact completion of elementary and pure existential doctrines as follows:

5.9. Corollary. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. Then, P is the pure existential completion of an elementary sudoctrine P' if and only if the functor $G^{\mathsf{ex}}_{|_{P'}}\colon (\mathcal{P}rd_{P'})_{\mathsf{ex/lex}} \to \mathsf{Ex}(P)$ provides an equivalence $\mathsf{Ex}(P) \equiv (\mathcal{P}rd_{P'})_{\mathsf{ex/lex}}$.*

Proof. First observe that $G^{\mathsf{ex}}_{|_{P'}}$ is an equivalence if and only if $G^{\mathsf{reg}}_{|_{P'}}$ is an equivalence. This because, by definition, the restriction of $G^{\mathsf{ex}}_{|_{P'}}$ to $(\mathcal{P}rd_{P'})_{\mathsf{reg/lex}}$ is precisely $G^{\mathsf{reg}}_{|_{P'}}$. Then, the result follows by combining this fact with the definition of $\mathsf{Ex}(P)$ (see Definition 4.15), the decomposition of the ex/lex-completion (see Theorem 4.2) and Theorem 5.8. ∎

The characterization of the regular and exact completions of doctrines equipped with Hilbert's ϵ-operators presented in[21, Thm. 6.2 (ii)] can be seen now as a particular case of Theorem 5.8 and Corollary 5.9. In fact, combining Theorem 3.17 with these results we obtain the following corollary:

5.10. Corollary. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be an elementary and pure existential doctrine. Then the following are equivalent:*

- *P is equipped with Hilbert's ϵ-operators;*
- *the functor $G^{\mathsf{reg}}\colon (\mathcal{P}rd_P)_{\mathsf{reg/lex}} \to \mathsf{Reg}(P)$ provides an equivalence $\mathsf{Reg}(P) \equiv (\mathcal{P}rd_P)_{\mathsf{reg/lex}}$;*
- *the functor $G^{\mathsf{ex}}\colon (\mathcal{P}rd_P)_{\mathsf{ex/lex}} \to \mathsf{Ex}(P)$ provides an equivalence $\mathsf{Ex}(P) \equiv (\mathcal{P}rd_P)_{\mathsf{ex/lex}}$.*

Proof. It follows from Theorem 5.8 and Corollary 5.9, since P is isomorphic to the pure existential completion of itself $P^{\exists}$ by Theorem 3.17.

∎

Another corollary of our main results regards the presentation of the syntactic category $\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}_0}$ and of its effectivization $\mathcal{E}_{\mathbb{T}_0}$ associated to a theory $\mathbb{T}_0$ of the fragment with true constant, binary conjunctions, equality and existential quantifiers of first-order Intuitionistic Logic with no extra-logical axioms, as defined in [13] (see Examples 4.14 and 4.18).

5.11. Corollary. *Let $\mathbb{T}_0$ be a regular theory in the sense of [13, Sec. D1.3], i.e. a theory of the fragment $\mathcal{L}_{=,\exists}$ of first-order Intuitionistic Logic and no extra-logical axioms on a generic signature. Let H_0 be the Horn theory given by the corresponding fragment $\mathcal{L}_{=}$ with no extra-logical axioms on the same signature. The syntactic category $\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}_0}$ of $\mathbb{T}_0$ is equivalent to the reg/lex-completion $(\mathcal{P}rd_{\mathsf{LT}^{\mathsf{H}_0}_{=}})_{\mathsf{reg/lex}}$ of the category of predicates of the syntactic doctrine $\mathsf{LT}^{\mathsf{H}_0}_{=}$ of H_0. Hence, also its effectivization $\mathcal{E}_{\mathbb{T}_0}$ is the ex/lex-completion $(\mathcal{P}rd_{\mathsf{LT}^{\mathsf{H}_0}_{=}})_{\mathsf{ex/lex}}$ of the category of predicates of $\mathsf{LT}^{\mathsf{H}_0}_{=}$.*

Proof. This follows from Theorem 5.8 and Corollary 5.9 after recalling from Examples 4.14 and 4.18 that $\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}_0} = \mathsf{Reg}(\mathsf{LT}^{\mathbb{T}_0}_{=,\exists})$ and that $\mathcal{E}_{\mathbb{T}_0} = \mathsf{Ex}(\mathsf{LT}^{\mathbb{T}_0}_{=,\exists})$ and that $\mathsf{LT}^{\mathsf{T}_0}_{=,\exists}$ is the pure existential completion of $\mathsf{LT}^{\mathsf{H}_0}_{=}$ as observed in Example 3.18. ∎

5.12. Remark. Despite the name *regular theory* for a theory of the fragment $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic Logic in [13, Sec. D1.3], the syntactic doctrine of such a theory presented in Example 2.5 does not coincide with the subobject doctrine of a regular category. Indeed, the fragment $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic Logic does not provide the internal language of regular categories, which can be instead described by adopting a dependent type theory as that in [18] (a similar internal language is introduced there also for lex categories). The regular category $\mathcal{C}_{\mathbb{T}}^{\mathsf{reg}}$ presented in [13, pp. 849-850] associated to such a dubbed regular theory (see also Example 4.14) is instead the regular completion $\mathsf{Reg}(\mathsf{LT}^{\mathbb{T}}_{=,\exists})$ of the syntactic doctrine $\mathsf{LT}^{\mathbb{T}}_{=,\exists}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$.

Finally, a last relevant example of doctrines arising as pure existential completions is that of the so-called Gödel hyperdoctrines presented in [35], arising in context of Dialectica interpretation. The original observation, for the more general case of fibrations, that a Dialectica fibration can be obtained combining the simple product and simple coproduct completions (i.e. the pure universal and pure existential completions in the case of doctrines) is due to P. Hofstra [9].

By Corollary 5.9, we have that also every exact completion of a Gödel hyperdoctrine is an instance of the $\mathsf{ex/lex}$-completion a category of predicates:

5.13. Corollary. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be a Gödel hyperdoctrine (as defined in [35]). Then we have the equivalences*

- $\mathsf{Reg}(P) \equiv (\mathcal{P}rd_{P'})_{\mathsf{reg/lex}}$;
- $\mathsf{Ex}(P) \equiv (\mathcal{P}rd_{P'})_{\mathsf{ex/lex}}$;

where P' is the elementary subdoctrine of P given by the pure existential free elements of P.

6. A new description of Joyal's arithmetic universes

Now, we apply our main results to the categorical setting of *Joyal's arithmetic universes* reported in [19, 38]. In [19] a more general abstract notion of arithmetic universes in terms of list-arithmetic pretoposes is introduced. Here, we provide a new description only for arithmetic universes in the sense of Joyal. In the following, we refer to [19] for the definition of *predicates* on a *Skolem theory* and of Joyal's arithmetic universes.

6.1. Definition. *Let $\mathcal{S}$ be a Skolem theory. The* elementary doctrine of **$\mathcal{S}$-predicates** *is the functor $\mathsf{R}_{\mathcal{S}}\colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ sending an object Nat^n into the poset $\mathsf{R}_{\mathcal{S}}(\mathsf{Nat}^n)$ of predicates over Nat^n, namely the arrows $P : \mathsf{Nat}^n \to \mathsf{Nat}$ of the Skolem theory such that $P \cdot P = P$ where $\cdot$ is the multiplication of predicates (defined point-wise with the multiplication of natural numbers), and where $P \leq Q$ is the point-wise order induced by natural numbers. The fibered equality δ_N^n is defined via the equality of the Skolem theory, see [19, Def.4.1].*

6.2. Remark. The category denoted by $\mathcal{P}rd_{\mathcal{S}}$ in [19] built by Joyal is a key inspiring example of the category of predicates of an elementary doctrine introduced in [21]. It was described in terms of free completions already in [24, Ex. 4.5]. Using the language of doctrines, Joyal's category $\mathcal{P}rd_{\mathcal{S}}$ in [19] is exactly the category of predicates $\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}}$ associated with the elementary doctrine $\mathsf{R}_{\mathcal{S}} \colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$.

6.3. Remark. Let us call $\mathcal{S}_{in}^{\mathsf{Set}}$ the embedding of the initial Skolem theory described in [19] within Set. Hence, in this category every object is isomorphic to a finite product of Nat and the arrows of $\mathcal{S}_{in}^{\mathsf{Set}}$ are precisely *the primitive recursive functions*. In this case, the fibres of the elementary doctrine $\mathsf{R}_{\mathcal{S}_{in}^{\mathsf{Set}}} \colon (\mathcal{S}_{in}^{\mathsf{Set}})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of $\mathcal{S}_{in}^{\mathsf{Set}}$-predicates can be equivalently be presented as follows:

$$\mathsf{R}_{\mathcal{S}_{in}^{\mathsf{Set}}}(\mathsf{Nat}^n) \simeq \{f \colon \mathsf{Nat}^n \to \mathsf{Nat} \mid f \in \mathcal{S}_{in}^{\mathsf{Set}}(\mathsf{Nat}^n, \mathsf{Nat}) \text{ and } \forall m \in \mathsf{Nat}^n,\ f(m) = 0 \text{ or } 1\}$$

In the following proposition we summarize some useful properties of the category $\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}}$ associated with the doctrine $\mathsf{R}_{\mathcal{S}} \colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$. We refer to [19, Prop. 4.7] for more details.

6.4. Proposition. *Given a Skolem theory $\mathcal{S}$ and its elementary doctrine $\mathsf{R}_{\mathcal{S}} \colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of $\mathcal{S}$-predicates, the category $\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}}$ is regular and every regular epi splits.*

Now recall the construction of Joyal's arithmetic universes from [19, Def. 4.8]:

6.5. Definition. *Given a Skolem theory $\mathcal{S}$, an **arithmetic universe** in the sense of **Joyal** is the category $(\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}})_{\mathsf{ex/reg}}$.*

Then, combining Corollary 5.9 with Proposition 6.4 we obtain the following result:

6.6. Corollary. *Every arithmetic universe $(\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}})_{\mathsf{ex/reg}}$ in the sense of Joyal on a Skolem theory $\mathcal{S}$ is equivalent to the exact completion $\mathsf{Ex}(\mathsf{R}^{\exists}_{\mathcal{S}})$ of the pure existential completion $\mathsf{R}^{\exists}_{\mathcal{S}}$ of the elementary doctrine $\mathsf{R}_{\mathcal{S}} \colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of $\mathcal{S}$-predicates.*

Proof. In the category $\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}}$ we have that regular epimorphisms split by Proposition 6.4 and hence from Remark 4.20 we derive that $(\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}})_{\mathsf{ex/reg}} \equiv (\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}})_{\mathsf{ex/lex}}$. Finally, by applying Corollary 5.9 we conclude that the arithmetic universe $(\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}})_{\mathsf{ex/reg}}$ is equivalent to the exact completion of the pure existential completion $\mathsf{R}^{\exists}_{\mathcal{S}}$ of the elementary doctrine $\mathsf{R}_{\mathcal{S}}$:

$$(\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}}})_{\mathsf{ex/reg}} \equiv \mathsf{Ex}(\mathsf{R}^{\exists}_{\mathcal{S}}).$$

∎

6.7. Corollary. *The initial arithmetic universe $(\mathcal{P}rd_{\mathsf{R}_{\mathcal{S}_{in}^{\mathsf{Set}}}})_{\mathsf{ex/reg}}$ on the initial Skolem theory embedded in Set is equivalent to the exact completion $\mathsf{Ex}(\mathsf{R}^{\exists}_{\mathcal{S}_{in}^{\mathsf{Set}}})$ where the elements of the fibre $\mathsf{R}^{\exists}_{\mathcal{S}_{in}^{\mathsf{Set}}}(\mathsf{Nat})$ are exactly the recursive enumerable subsets of Nat in Set.*

Proof. By Corollary 6.6 we have that

$$(\mathcal{Prd}_{\mathsf{R}_{\mathcal{S}_{in}^{\mathsf{Set}}}})_{\mathsf{ex/reg}} \equiv \mathsf{Ex}(\mathsf{R}^{\exists}_{\mathcal{S}_{in}^{\mathsf{Set}}}).$$

Then, observe that the fibres of the pure existential completion $\mathsf{R}^{\exists}_{\mathcal{S}_{in}^{\mathsf{Set}}} \colon (\mathcal{S}_{in}^{\mathsf{Set}})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ of the elementary doctrine $\mathsf{R}_{\mathcal{S}_{in}^{\mathsf{Set}}} \colon (\mathcal{S}_{in}^{\mathsf{Set}})^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ are exactly the *recursively enumerable predicate* because, by Theorem 3.14, every element of $\mathsf{R}^{\exists}_{\mathcal{S}_{in}^{\mathsf{Set}}}(\mathsf{Nat})$ can be written as an existential quantifier of a primitive recursive predicate, and it is well-known that every recursively enumerable predicate can be proved to be presented as an existential quantifier of a primitive recursively enumerable predicate, for example, from [31, Thm. II.1.8, Thm. I.3.3, Ex I.2.8]. ∎

7. Conclusion

We have provided a new description of Joyal's arithmetic universes [19] as an application of a characterization of regular and exact completions of pure existential completions of elementary doctrines. This characterization extends a previous one proved in [21] for doctrines equipped with Hilbert's ϵ-operators.

In particular, we have proved that for elementary doctrines arising as pure existential completions, their regular and exact completions happen to be equivalent to the $\mathsf{reg/ex}$ and $\mathsf{ex/lex}$-completions, respectively, of the category of predicates associated with the subdoctrine of their pure existential free elements. To reach this goal, we took advantage of the intrinsic characterization of doctrines arising as pure existential completions presented in [26], slightly extended with another equivalent presentation here.

Using these results we have deduced that an arithmetic universe in the sense of Joyal can be seen as the exact completion of the pure existential completion of the doctrine of predicates of its Skolem theory. In particular, the initial arithmetic universe in the standard category of ZFC-sets turns out to be the completion with exact quotients of the doctrine of recursively enumerable predicates.

Other examples of application of our characterization include the so called syntactic category in [13] associated to the so called regular fragment of first-order logic in [13] (and its effectivization) and the regular and exact completion of a Gödel hyperdoctrine [35, 34].

As future work, we aim to extend our results to regular and exact completions of other classes of doctrines obtained as generalized existential completions, including the case of the *full existential completion* of primary doctrines in the sense of [26], as initiated in [25], with applications to sheaf theory.

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