WHAT IS THE UNIVERSAL PROPERTY OF THE 2-CATEGORY OF MONADS?

In memory of our colleague Pieter Hofstra

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ABSTRACT. For a 2-category $\mathcal{K}$, we consider Street's 2-category $\mathrm{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$, along with Lack and Street's 2-category $\mathrm{EM}(\mathcal{K})$ and the identity-on-objects-and-1-cells 2-functor $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ between them. We show that this 2-functor can be obtained as a "free completion" of the 2-functor $1\colon \mathcal{K} \to \mathcal{K}$. We do this by regarding 2-functors which act as the identity on both objects and 1-cells as categories enriched a cartesian closed category **BO** whose objects are identity-on-objects functors. We also develop some of the theory of **BO**-enriched categories.

1. Introduction

In [11], Street introduced, for a given 2-category $\mathcal{K}$, a 2-category $\mathrm{Mnd}(\mathcal{K})$ whose objects are the monads in $\mathcal{K}$, and showed how various aspects of the theory of monads can be understood in terms of this construction. For example, there is a 2-functor $\mathrm{Id}\colon \mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ sending each object of $\mathcal{K}$ to the identity monad on that object, and this Id has a right adjoint just when $\mathcal{K}$ admits what later came to be known as Eilenberg-Moore objects (which in the classical case, $\mathcal{K} = \mathbf{Cat}$, correspond to the category of algebras for the monad, introduced by Eilenberg and Moore [4]).

Some thirty years later, in [9], a variant $\mathrm{EM}(\mathcal{K})$ of $\mathrm{Mnd}(\mathcal{K})$ was introduced, having the same objects and 1-cells, but a different notion of 2-cell. There is a 2-functor $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ between them, which acts as the identity on objects and 1-cells. As shown in [9], it is still the case that the composite 2-functor $\mathcal{K} \to \mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ has a right adjoint just when $\mathcal{K}$ admits Eilenberg-Moore objects, but now there is a conceptual reason: $\mathrm{EM}(\mathcal{K})$ is the free completion of $\mathcal{K}$ under Eilenberg-Moore objects.

One might then ask whether the original $\mathrm{Mnd}(\mathcal{K})$ itself has a universal property, and lo these twenty years after [9] we offer our response to this question.

In fact, as often happens in mathematics, rather than answer this question as is, we first reformulate it, and then respond to the modified question. Thus rather than seek to characterize $\mathrm{Mnd}(\mathcal{K})$ alone in terms of some universal property, we consider $\mathrm{Mnd}(\mathcal{K})$ and $\mathrm{EM}(\mathcal{K})$ together, along with the 2-functor $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ which connects them, which as mentioned above *acts as the identity on both objects and 1-cells*. We consider 2-functors with this latter property as a structure in its own right, and describe a universal property of $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ relative to the identity 2-functor $\mathcal{K} \to \mathcal{K}$.

The way we do this is to consider such 2-functors as a certain sort of enriched category. In more detail, there is a cartesian closed category **BO** whose objects are identity-on-objects functors and whose morphisms are commutative squares of functors. A **BO**-enriched category is essentially the same as a 2-functor which acts as the identity on objects and on 1-cells.

We then develop a little of the theory of **BO**-enriched category theory, including in particular weighted limits and colimits, and free completions under these. We also show how every **Cat**-enriched weight gives rise to a corresponding **BO**-enriched weight (actually in two different ways), and so in particular we have a **BO**-enriched notion of Eilenberg-Moore object.

Our answer to the (reformulated) problem, then, is that $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ is the free completion of $1\colon \mathcal{K} \to \mathcal{K}$ under these **BO**-enriched Eilenberg-Moore objects.

As was the case in [9], it is technically more convenient to deal with colimits rather than limits when it comes to free completions, so we actually work with the colimit notion corresponding to Eilenberg-Moore objects, namely Kleisli objects.

We begin, in Section 2, by introducing our cartesian closed category **BO**, and describing various relationships it has to **Cat**. Then in Section 3 we begin our study of **BO**-enriched categories, including enriched presheaf categories, as well as various "change-of-base" constructions linking **BO**-categories with other sorts of enriched categories. In Section 4, we study various examples of **BO**-enriched colimits, leading up to our main result, Theorem 4.18, characterizing $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$ as a free completion.
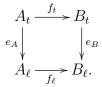
2. The cartesian closed category **BO**

We write $\mathbf{Cat}_1$ for the cartesian closed category of (small) categories and functors, where the subscript 1 is to emphasize that we are thinking of this as a mere category rather than a 2-category. The arrow category $\mathbf{Cat}_1^{\mathbf{2}}$ is also cartesian closed. We write **BO** for the full subcategory of $\mathbf{Cat}_1^{\mathbf{2}}$ consisting of the identity-on-object functors.[1] Our standard notation for objects of **BO** is a letter like A, then we write $e_A\colon A_t \to A_\ell$ for the corresponding functor[2], sometimes dropping the subscript A in e_A. A typical morphism $f\colon A \to B$ has

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[1]Very little would change if we were to work with functors which are merely bijective on objects, and indeed these are the source of our notation **BO**. Of course every bijective-on-object functor is isomorphic in $\mathbf{Cat}_1^{\mathbf{2}}$ to an object of **BO**.

[2]The subscripts t and ℓ stand for tight and loose, as in [8]; see also Section 3.1 below.

the form of a commutative square

$$
\begin{array}{ccc}
A_t & \xrightarrow{f_t} & B_t \\
e_A \downarrow & & \downarrow e_B \\
A_\ell & \xrightarrow{f_\ell} & B_\ell.
\end{array}
$$

Given any functor $g\colon X \to Y$ there is an essentially unique factorization (often called the *bo-ff factorization*)
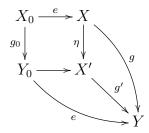
$$
X \xrightarrow{\;e\;} Z \xrightarrow{\;j\;} Y
$$

with e the identity on objects and j fully faithful.

2.1. PROPOSITION. *The full subcategory* **BO** *of* $\mathbf{Cat}_1^2$ *is both reflective and coreflective.*

PROOF. The *coreflection* of an object $g\colon X \to Y$ is given by the identity-on-objects part $e\colon X \to Z$ of the bo-ff factorization; indeed, if $(\mathcal{E}, \mathcal{M})$ is a factorization system on a category $\mathcal{C}$, then $\mathcal{E}$ determines a full coreflective subcategory of $\mathcal{C}^2$ and the coreflection sends a morphism to the $\mathcal{E}$-part of the factorization.

The *reflection* of $g\colon X \to Y$ is given by the induced map $g'\colon X \to Y$ in the diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\;e\;} & X \\
g_0 \downarrow & & \eta \downarrow \qquad g \\
Y_0 & \longrightarrow & X' \\
& e & \qquad g' \\
& & Y
\end{array}
$$

in which the arrows labelled e are identity-on-object inclusions of discrete subcategories, and the square is a pushout. ∎

2.2. REMARK. Of course $\mathbf{Cat}_1^2$ can be made into a 2-category $\mathbf{Cat}^2$, and so **BO** becomes a full sub-2-category. As such, it is still coreflective, but not reflective; indeed it is not closed under powers.

It follows from the coreflectivity and the fact that **BO** is closed in $\mathbf{Cat}_1^2$ under finite products (which in turn follows from reflectivity or can easily be checked) that **BO** is, like $\mathbf{Cat}_1^2$, cartesian closed, with the internal hom in **BO** formed as the coreflection of the internal hom in $\mathbf{Cat}^2$.

Explicitly, if $e_A\colon A_t \to A_\ell$ and $e_B\colon B_t \to B_\ell$ are in **BO**, the internal hom $[A, B]$ has:

- objects are commutative squares (in other words, morphisms $f = (f_t, f_\ell)\colon A \to B$ in **BO**);

- $[A, B]_t(f, g)$ consists of natural transformations $f_t \to g_t$ and $f_\ell \to g_\ell$ subject to the obvious compatibility condition (these are also the 2-cells if **BO** is made into a 2-category, as in Remark 2.2).

- $[A, B]_\ell(f, g)$ consists just of natural transformations $f_\ell \to g_\ell$.

As well as the adjunctions between $\mathbf{BO}$ and $\mathbf{Cat}_1^2$, there are also adjunctions between $\mathbf{BO}$ and $\mathbf{Cat}_1$. In particular, the codomain functor cod: $\mathbf{BO} \to \mathbf{Cat}_1$ has a left adjoint disc: $\mathbf{Cat}_1 \to \mathbf{BO}$ sending a category C to the inclusion $C_0 \to C$ of the discrete category with the same objects as C. Once again, this is just an adjunction of ordinary categories, not of 2-categories. On the other hand, both adjoints preserve finite products and so this is a monoidal adjunction, and induces a 2-adjunction $\mathrm{disc}_* \dashv \mathrm{cod}_*$ between $\mathbf{BO}$-categories and 2-categories.

In fact there is a chain of monoidal adjunctions[3]

$$\pi \dashv \mathrm{disc} \dashv \mathrm{cod} \dashv \mathrm{id} \dashv \mathrm{dom} \dashv \mathrm{ch}$$

between $\mathbf{BO}$ and $\mathbf{Cat}_1$. It is disc $\dashv$ cod and id $\dashv$ dom which will be particularly important in what follows. Here dom: $\mathbf{BO} \to \mathbf{Cat}_1$ sends an identity-on-objects functor to its domain, while id: $\mathbf{Cat}_1 \to \mathbf{BO}$ sends a category to the corresponding identity functor.

2.3. REMARK. The adjunction $\pi \dashv \mathrm{disc}$ will not be significant in what follows, so we shall not describe it in detail. It sends an identity-on-objects map $A \to A'$ to its pushout with the canonical map $A \to \pi_0 A$. In order to see that this π preserves finite products, and so that $\pi \dashv \mathrm{disc}$ is monoidal, it is perhaps easiest to use the Day reflection theorem [3, Theorem 1.2 and Corollary 2.1], according to which it will suffice to check that if $A \in \mathbf{BO}$ and $C \in \mathbf{Cat}_1$, then the internal hom $[A, \mathrm{disc}\, C]$ is in the image of disc. To do this, one observes that an object B of $\mathbf{BO}$ is in the image of disc if and only if B_t is discrete, and then uses the explicit description of $[A, \mathrm{disc}\, C]_t$ given above.

2.4. REMARK. We have directly described a cartesian closed structure on $\mathbf{BO}$ and in the following section we shall consider enrichment over this structure. But it is also worth pointing out that $\mathbf{BO}$ can itself be understood in terms of enrichment. The category $\mathbf{Set^2}$ is also cartesian closed, and the category of $\mathbf{Set^2}$-enriched categories and $\mathbf{Set^2}$-enriched functors is isomorphic to $\mathbf{BO}$: see [10, Example 1]. This immediately implies that $\mathbf{BO}$ is cartesian closed; furthermore, the adjunctions cod $\dashv$ id $\dashv$ dom $\dashv$ ch between $\mathbf{BO}$ and $\mathbf{Cat}_1$ can themselves be seen as arising via change of base-of-enrichment from monoidal adjunctions between $\mathbf{Set^2}$ and $\mathbf{Set}$.

2.5. REMARK. Another point of view on $\mathbf{BO}$ relates it to the cartesian closed category $\mathbf{DblCat}$ of double categories. There is a functor $\mathbf{BO} \to \mathbf{DblCat}$ sending an identity on objects functor to its "higher kernel"; this functor is fully faithful, with image given by the cateads, and moreover has a finite-product-preserving reflection. See [2] for the notion of cateads and their equivalence to bijective on objects functors in a more general context, or [1] for further details, including a simpler treatment for the specific case at hand. The fact that the reflection preserves finite products can be found in [1, Proposition 8.30].

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[3] An adjunction between cartesian closed categories is monoidal just when the left adjoint preserves finite products.

3. $\mathbf{BO}$-enriched categories

Since $\mathbf{BO}$ is cartesian closed, we can consider enrichment over it. In this section we investigate what some of the basic theory of enriched categories looks like in the case of enrichment over $\mathbf{BO}$.

3.1. $\mathbf{BO}$-ENRICHED CATEGORIES. We generally use blackboard bold for the names of $\mathbf{BO}$-categories. A $\mathbf{BO}$-category $\mathbb{A}$ has objects A, B, C, and so on; and, for objects A and B, a $\mathbf{BO}$-valued hom $\mathbb{A}(A, B)$. This consists of an identity-on-objects functor, which we write as $E_{\mathbb{A}}\colon \mathcal{A}_t(A, B) \to \mathcal{A}_\ell(A, B)$. The composition and identities make $\mathcal{A}_t(A, B)$ and $\mathcal{A}_\ell(A, B)$ into the hom-categories of 2-categories $\mathcal{A}_t$ and $\mathcal{A}_\ell$, in such a way that the $E_{\mathbb{A}}$ define a 2-functor $E_{\mathbb{A}}\colon \mathcal{A}_t \to \mathcal{A}_\ell$ which acts as the identity on objects and on 1-cells. We often drop the subscript $\mathbb{A}$ and write simply E. Conversely, any 2-functor which acts as the identity on objects and on 1-cells arises in this way from a unique $\mathbf{BO}$-category. We shall routinely identify $\mathbf{BO}$-categories with the corresponding 2-functors.

Following the naming convention of [8], 2-cells of $\mathcal{A}_t$ are referred to as *tight*, and 2-cells of $\mathcal{A}_\ell$ as *loose*; this is also the origin of the subscripts t and ℓ.

Given $\mathbf{BO}$-categories $\mathbb{A}$ and $\mathbb{B}$, seen as 2-functors $E_{\mathbb{A}}\colon \mathcal{A}_t \to \mathcal{A}_\ell$ and $E_{\mathbb{B}}\colon \mathcal{B}_t \to \mathcal{B}_\ell$, a $\mathbf{BO}$-functor $F\colon \mathbb{A} \to \mathbb{B}$ consists of 2-functors $F_t\colon \mathcal{A}_t \to \mathcal{B}_t$ and $F_\ell\colon \mathcal{A}_\ell \to \mathcal{B}_\ell$ making the square

$$
\begin{array}{ccc}
\mathcal{A}_t & \xrightarrow{\;F_t\;} & \mathcal{B}_t \\
{\scriptstyle E_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle E_{\mathbb{B}}} \\
\mathcal{A}_\ell & \xrightarrow[\;F_\ell\;]{} & \mathcal{B}_\ell
\end{array}
$$

commute.

Similarly, if $G\colon \mathbb{A} \to \mathbb{B}$ is also a $\mathbf{BO}$-functor, a $\mathbf{BO}$-natural transformation $F \to G$ consists of a pair of 2-natural transformations $\varphi_t\colon F_t \to G_t$ and $\varphi_\ell\colon F_\ell \to G_\ell$ satisfying the evident compatibility condition $E_{\mathbb{B}}\varphi_t = \varphi_\ell E_{\mathbb{A}}$.

We can summarize the results of this analysis using the 2-category 2-$\mathbf{Cat}$ of 2-categories, 2-functors, and 2-natural transformations, and the 2-category 2-$\mathbf{Cat}^{\mathbf{2}}$ of arrows in 2-$\mathbf{Cat}$.

3.2. PROPOSITION. *The 2-category* $\mathbf{BO}$-$\mathbf{Cat}$ *of* $\mathbf{BO}$-*categories,* $\mathbf{BO}$-*functors, and* $\mathbf{BO}$-*natural transformations is isomorphic to the full sub-2-category of* 2-$\mathbf{Cat}^{\mathbf{2}}$ *consisting of those 2-functors which act as the identity on objects and on 1-cells.*
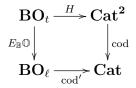
Every enriched category has an underlying ordinary category; the underlying ordinary category of a $\mathbf{BO}$-category $\mathbb{A}$ is the underlying ordinary category of the 2-category $\mathcal{A}_t$, which is of course the same as the underlying ordinary category of the 2-category $\mathcal{A}_\ell$.

3.3. THE $\mathbf{BO}$-CATEGORY $\mathbb{BO}$. We write $\mathbb{BO}$ for the $\mathbf{BO}$-category coming from the cartesian closed structure of $\mathbf{BO}$ itself. The corresponding 2-functor $\mathbf{BO}_t \to \mathbf{BO}_\ell$ can be described as follows.

The objects of $\mathbf{BO}_t$, $\mathbf{BO}_\ell$, $\mathbb{BO}$, and $\mathbf{BO}$ all coincide, and are just the identity-on-objects functors. Once again, the 1-cells of $\mathbf{BO}_t$, $\mathbf{BO}_\ell$, and $\mathbf{BO}$ all coincide (and we

could add $\mathbb{BO}$ to this list if we understand the 1-cells of a $\mathbf{BO}$-category to be the 1-cells of the underlying ordinary category): these are all just the commutative squares of functors, with vertical maps acting as the identity on objects.
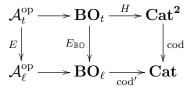
Given 1-cells $F = (F_t, F_\ell)$ and $G = (G_t, G_\ell)$ from $E_{\mathbb{A}} \colon \mathcal{A}_t \to \mathcal{A}_\ell$ to $E_{\mathbb{B}} \colon \mathcal{B}_t \to \mathcal{B}_\ell$, a 2-cell $F \to G$ in $\mathbf{BO}_t$ consists of natural transformations $\varphi_t \colon F_t \to G_t$ and $\varphi_\ell \colon F_\ell \to G_\ell$ satisfying the evident compatibility condition $\varphi_\ell E_{\mathbb{A}} = E_{\mathbb{B}} \varphi_t$. Thus in fact $\mathbf{BO}_t$ can be identified with the full sub-2-category of $\mathbf{Cat^2}$ consisting of those functors which act as the identity on objects.

A 2-cell $F \to G$ in $\mathbf{BO}_\ell$ consists of just the single natural transformation $\varphi_\ell \colon F_\ell \to G_\ell$, and $E_{\mathbb{BO}}$ sends a 2-cell $(\varphi_t, \varphi_\ell)$ to φ_ℓ. Thus in fact we have a commutative square

$$
\begin{array}{ccc}
\mathbf{BO}_t & \xrightarrow{\;H\;} & \mathbf{Cat^2} \\
{\scriptstyle E_{\mathbb{BO}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{cod}} \\
\mathbf{BO}_\ell & \xrightarrow[\mathrm{cod}']{} & \mathbf{Cat}
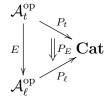\end{array}
$$

where the upper horizontal H is the fully faithful inclusion mentioned above, the left vertical acts as the identity on objects and 1-cells, and the lower horizontal cod' is fully faithful on 2-cells. This is enough to determine $\mathbf{BO}_\ell$.

3.4. PRESHEAVES. A presheaf on $\mathbb{A}$ consists of a $\mathbf{BO}$-functor $\mathbb{A}^{\mathrm{op}} \to \mathbb{BO}$, but we can also analyze this in terms of the 2-functor $E_{\mathbb{A}} \colon \mathcal{A}_t \to \mathcal{A}_\ell$. Given such a presheaf, we obtain a commutative diagram
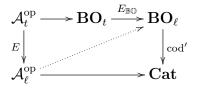
$$
\begin{array}{ccccc}
\mathcal{A}_t^{\mathrm{op}} & \longrightarrow & \mathbf{BO}_t & \xrightarrow{\;H\;} & \mathbf{Cat^2} \\
{\scriptstyle E}\big\downarrow & & {\scriptstyle E_{\mathbb{BO}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{cod}} \\
\mathcal{A}_\ell^{\mathrm{op}} & \longrightarrow & \mathbf{BO}_\ell & \xrightarrow[\mathrm{cod}']{} & \mathbf{Cat}
\end{array}
$$

of 2-functors, and by the universal property of the power $\mathbf{Cat^2}$, this in turn determines a diagram

$$
\begin{array}{c}
\mathcal{A}_t^{\mathrm{op}} \\
{\scriptstyle E}\big\downarrow \quad \Downarrow{\scriptstyle P_E} \quad \searrow{\scriptstyle P_t} \\
\quad\quad\quad \mathbf{Cat} \\
\mathcal{A}_\ell^{\mathrm{op}} \quad \nearrow{\scriptstyle P_\ell}
\end{array}
$$

where P_ℓ is the composite lower horizontal in the previous diagram, while the composite upper horizontal $\mathcal{A}_t^{\mathrm{op}} \to \mathbf{Cat^2}$ corresponds to the 2-natural map $P_E \colon P_t \to P_\ell E$.

When does a 2-natural $P_E \colon P_t \to P_\ell E$ arise in this way from some $\mathbf{BO}$-presheaf? Such a P_E does determine a unique map $\overline{P} \colon \mathcal{A}_t^{\mathrm{op}} \to \mathbf{Cat^2}$ with $\mathrm{cod}\, \overline{P} = P_\ell E$. This $\overline{P}$ will land in $\mathbf{BO}_t$ just when the components of P_E act as the identity on objects. And indeed when

this happens, we have the solid part of the diagram

$$
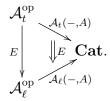\begin{array}{ccc}
\mathcal{A}_t^{\mathrm{op}} \longrightarrow \mathbf{BO}_t \xrightarrow{E_{\mathbb{BO}}} \mathbf{BO}_\ell \\
{\scriptstyle E}\big\downarrow \qquad\qquad\qquad \big\downarrow {\scriptstyle \mathrm{cod}'} \\
\mathcal{A}_\ell^{\mathrm{op}} \longrightarrow \mathbf{Cat}
\end{array}
$$

in which the left vertical is the identity on objects and 1-cells, and the right vertical is fully faithful on 2-cells. It follows that there is a unique induced diagonal filler.
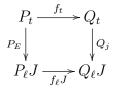
This proves:

3.5. PROPOSITION. *Let $\mathbb{A}$ be a $\mathbf{BO}$-category and $E\colon \mathcal{A}_t \to \mathcal{A}_\ell$ the corresponding 2-functor. To give a presheaf on $\mathbb{A}$ is equivalently to give $\mathbf{Cat}$-presheaves P_t and P_ℓ on $\mathcal{A}_t$ and $\mathcal{A}_\ell$ respectively, along with a 2-natural transformation $P_E\colon P_t \to P_t E$ whose components are identity-on-objects functors.* ∎

From this point of view, a representable presheaf $\mathbb{A}(-, A)$ has the form

$$
\begin{array}{c}
\mathcal{A}_t^{\mathrm{op}} \\
\quad\searrow {\scriptstyle \mathcal{A}_t(-,A)} \\
{\scriptstyle E}\big\downarrow \quad \Downarrow E \quad \mathbf{Cat}. \\
\quad\nearrow {\scriptstyle \mathcal{A}_\ell(-,A)} \\
\mathcal{A}_\ell^{\mathrm{op}}
\end{array}
$$

3.6. ENRICHED PRESHEAF CATEGORIES. Since the cartesian closed category $\mathbf{BO}$ is complete, we can construct enriched presheaf categories as in [6, Section 2.2].

Given two presheaves P and Q on the $\mathbf{BO}$-category $\mathbb{A}$, there is a $\mathbf{BO}$-valued hom $[\mathbb{A}^{\mathrm{op}}, \mathbb{BO}](P, Q)$. An object is a $\mathbf{BO}$-natural $f\colon P \to Q$: this amounts to maps $f_t\colon P_t \to Q_t$ and $f_\ell\colon P_\ell \to Q_\ell$ making the square

$$
\begin{array}{ccc}
P_t & \xrightarrow{f_t} & Q_t \\
{\scriptstyle P_E}\big\downarrow & & \big\downarrow {\scriptstyle Q_j} \\
P_\ell J & \xrightarrow[f_\ell J]{} & Q_\ell J
\end{array}
$$

commute. A tight map $f \to g$ consists of compatible 2-cells $f_t \to g_t$ and $f_\ell \to g_\ell$, while a loose map consists of just a 2-cell $f_\ell \to g_\ell$.

A more abstract way to summarize this is as follows. First form the pullback of 2-categories

$$
\begin{array}{ccc}
[\mathbb{A}^{\mathrm{op}}, \mathbb{BO}]_t & \longrightarrow & [\mathcal{A}_\ell^{\mathrm{op}}, \mathbf{Cat}] \\
\big\downarrow & & \big\downarrow {\scriptstyle P_E^*} \\
[\mathcal{A}_t^{\mathrm{op}}, \mathbf{BO}_t] & \xrightarrow[{[\mathcal{A}_t^{\mathrm{op}}, \mathrm{cod}]}]{} & [\mathcal{A}_t^{\mathrm{op}}, \mathbf{Cat}]
\end{array}
$$

and now factorize the upper horizontal as

$$[\mathbb{A}^{\mathrm{op}}, \mathbb{BO}]_t \xrightarrow{E_{[\mathbb{A}^{\mathrm{op}},\mathbb{BO}]}} [\mathbb{A}^{\mathrm{op}}, \mathbb{BO}]_\ell \longrightarrow [\mathcal{A}_\ell^{\mathrm{op}}, \mathbf{Cat}]$$

where $E_{[\mathbb{A}^{\mathrm{op}},\mathbb{BO}]}$ acts as the identity on objects and on 1-cells, and the other map is fully faithful on 2-cells.

3.7. CHANGE OF BASE. As observed in Section 2 above, there is a chain of monoidal adjunctions disc $\dashv$ cod $\dashv$ id $\dashv$ dom between **BO** and $\mathbf{Cat}_1$. These induce 2-adjunctions $\mathrm{disc}_* \dashv \mathrm{cod}_* \dashv \mathrm{id}_* \dashv \mathrm{dom}_*$ between **BO**-**Cat** and 2-**Cat**.

Given a **BO**-category $\mathbb{A}$, the 2-category $\mathrm{dom}_* \mathbb{A}$ is $\mathcal{A}_t$, while $\mathrm{cod}_* \mathbb{A}$ is $\mathcal{A}_\ell$.

Given a 2-category $\mathcal{D}$, the **BO**-category $\mathrm{id}_* \mathcal{D}$ is the one corresponding to the identity 2-functor $1: \mathcal{D} \to \mathcal{D}$. We sometimes identify a 2-category $\mathcal{D}$ with the corresponding **BO**-category $\mathrm{id}_* \mathcal{D}$, and call a **BO**-category of this form *tight*, since it has the property that every loose 2-cell has a unique tight structure (thus we might "loosely" say that all 2-cells are tight!)

On the other hand, $\mathrm{disc}_* \mathcal{D}$ is the **BO**-category corresponding to the 2-functor $\mathcal{D}_1 \to \mathcal{D}$, where $\mathcal{D}_1$ is the underlying ordinary category of $\mathcal{D}$, seen as a locally discrete 2-category (no non-identity 2-cells). We call a **BO**-category of this form *loose*, since there are no non-identity tight 2-cells.

The tight and loose **BO**-categories are analogous, respectively, to the *chordate* and *inchordate* $\mathcal{F}$-categories of [8]; see also [5, Example 16.2].

3.8. CHANGE OF BASE AND TIGHT PRESHEAVES. As well as these 2-adjunctions between **BO**-**Cat** and 2-**Cat**, there are also various connections between the enriched presheaf categories.

First observe that the monoidal adjunction id $\dashv$ dom also induces a 2-adjunction between $\mathbf{BO}_t = \mathrm{dom}_* \mathbb{BO}$ and $\mathbf{Cat}$: recall that $\mathbf{BO}_t$ is just the full sub-2-category of $\mathbf{Cat}^{\mathbf{2}}$ consisting of the identity-on-objects functors. For any 2-category $\mathcal{D}$, we have

$$\mathrm{dom}_*[\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}] \cong [\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}]_t \cong [\mathcal{D}^{\mathrm{op}}, \mathbf{BO}_t]$$

and so id $\dashv$ dom induces a 2-adjunction between $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}]$ and this $\mathrm{dom}_*[\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}]$. In particular, a 2-functor $F: \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$ will be sent to the **BO**-functor $F^{\mathrm{id}}: \mathrm{id}_* \mathcal{D}^{\mathrm{op}} \to \mathbb{BO}$ with $F_t^{\mathrm{id}} = F_\ell^{\mathrm{id}} = F$ and $F_E: F_t \to F_\ell$ equal to the identity.

3.9. DEFINITION. *A weight of the form* $F^{\mathrm{id}}: \mathrm{id}_* \mathcal{D}^{\mathrm{op}} \to \mathbb{BO}$ *for a* **Cat**-*weight* $F: \mathcal{D}^{\mathrm{op}} \to$ **Cat** *will be called a* tight weight.

3.10. REMARK. In fact these tight weights arise by general change of base arguments. If $\varphi: \mathcal{V} \to \mathcal{W}$ is a monoidal functor, then any $\mathcal{V}$-weight $F: \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ determines a $\mathcal{W}$-weight as follows. First apply change of base along φ to obtain a $\mathcal{W}$-functor $\varphi_* F: \varphi_* \mathcal{D}^{\mathrm{op}} \to \varphi_* \mathcal{V}$; now compose with the $\mathcal{W}$-functor $\varphi: \varphi_* \mathcal{V} \to \mathcal{W}$. If moreover φ has a monoidal right adjoint ψ, then this process defines a $\mathcal{V}$-functor $[\mathcal{D}^{\mathrm{op}}, \mathcal{V}] \to \psi_*[\varphi_* \mathcal{D}^{\mathrm{op}}, \mathcal{W}]$ which in turn has a right $\mathcal{V}$-functor induced by ψ. Applying this in the case of the monoidal adjunction id $\dashv$ dom sends a **Cat**-weight F to the corresponding tight **BO**-weight F^{id}.

3.11. Change of base and loose presheaves. This time we consider the monoidal adjunction disc ⊣ cod. Once again, we start with a **Cat**-weight $F\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$. This time we apply Remark 3.10 using disc ⊣ cod to obtain a **BO**-weight $\mathbb{F}\colon \mathrm{disc}_* \mathcal{D}^{\mathrm{op}} \to \mathbb{BO}$.
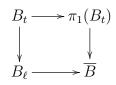
Recall that $\mathrm{disc}_* \mathcal{D}$ corresponds to the inclusion 2-functor $E\colon \mathcal{D}_1 \to \mathcal{D}$, where $\mathcal{D}_1$ is obtained from $\mathcal{D}$ by discarding all non-identity 2-cells. Then $F_t\colon \mathcal{D}_1^{\mathrm{op}} \to \mathbf{Cat}$ is defined to send $D \in \mathcal{D}_1$ to the discrete category with the same set of objects as FD. The identity-on-objects inclusions $F_t D \to FD$ are the components of a 2-natural transformation $F_E\colon F_t \to FE$ whose components act as the identity on objects. This in turn determines the presheaf $\mathbb{F}$.

3.12. Definition. *A weight of the form* $\mathbb{F}\colon \mathrm{disc}_* \mathcal{D}^{\mathrm{op}} \to \mathbb{BO}$ *for a* **Cat***-weight* $F\colon \mathcal{D}^{\mathrm{op}} \to$ **Cat** *will be called a* loose weight.

3.13. Another example of change-of-base. In this section we include, for interest's sake, a further example of change-of-base. It will be not be used in the remainder of the paper.

If $\mathcal{K}$ is a 2-category then we obtain another 2-category $\mathcal{K}_g$ with the same objects and morphisms by discarding all non-invertible 2-cells. Thus $\mathcal{K}_g \to \mathcal{K}$ can be seen as a **BO**-category.

In fact this also arises through a change-of-base process. There is a functor core$\colon \mathbf{Cat}_1 \to$ **BO** sending a category A to the inclusion $A_g \to A$, where A_g is the subcategory of A consisting of the isomorphisms. This has a left adjoint π sending $B_t \to B_\ell$ to the pushout

$$
\begin{array}{ccc}
B_t & \longrightarrow & \pi_1(B_t) \\
\downarrow & & \downarrow \\
B_\ell & \longrightarrow & \overline{B}
\end{array}
$$

where $B_t \to \pi_1(B_t)$ is the map which universally inverts all morphisms in B_t. The counit of $\pi \dashv$ core is invertible.

Much as in Remark 2.3, the Day reflection theorem [3, Theorem 1.2 and Corollary 2.1] can be used to see that π preserves finite products and so $\pi \dashv$ core is monoidal, by observing that $[B, \mathrm{core}\, A]$ is in the image of core for any $B \in \mathbf{BO}$ and $A \in \mathbf{Cat}_1$.

4. **BO**-enriched colimits

Let $M\colon \mathbb{A}^{\mathrm{op}} \to \mathbb{BO}$ be a presheaf, corresponding as in Proposition 3.5 to $M_t\colon \mathcal{A}_t^{\mathrm{op}} \to \mathbf{Cat}$, $M_\ell\colon \mathcal{A}_\ell^{\mathrm{op}} \to \mathbf{Cat}$, and $M_E\colon M_t \to M_\ell E_\mathbb{A}$. By the Yoneda lemma, for an object $A \in \mathbb{A}$ there is a bijection between **BO**-natural $\widehat{a}\colon \mathbb{A}(-, A) \to M$ and maps $a\colon 1 \to MA$ in $\mathbb{BO}$, or equivalently of $a \in M_t A$ (or $a \in M_\ell A$). Then M is representable when there is an $A \in \mathbb{A}$ and $a \in M_t A$ as above, for which the induced $\widehat{a}$ is invertible. This in turn is equivalent to invertibility of the 2-natural transformations $\widehat{a}_t\colon \mathcal{A}_t(-, A) \to M_t$ and $\widehat{a}_\ell\colon \mathcal{A}_\ell(-, A) \to M_\ell$. We call the invertibility of $\widehat{a}_t$ and $\widehat{a}_\ell$, respectively, *the tight and the loose aspects of the universal property.*

Recall [6, Section 3.1] that if $\mathbb{F}\colon \mathbb{D} \to \mathbb{BO}$ and $\mathbb{S}\colon \mathbb{D} \to \mathbb{K}$ are **BO**-functors, the weighted limit $\{\mathbb{F}, \mathbb{S}\}$ in $\mathbb{K}$ exists just when the presheaf $[\mathbb{D}, \mathbb{BO}](\mathbb{F}, \mathbb{K}(-, \mathbb{S}))\colon \mathbb{D}^{\mathrm{op}} \to \mathbb{BO}$ is representable.

A weighted colimit in $\mathbb{K}$ is a weighted limit in $\mathbb{K}^{\mathrm{op}}$. As usual, however, rather than consider a weight $\mathbb{F}\colon \mathbb{D} \to \mathbb{BO}$ and diagram $\mathbb{S}\colon \mathbb{D} \to \mathbb{K}^{\mathrm{op}}$, we think of the diagram as having the form $\mathbb{S}\colon \mathbb{D}^{\mathrm{op}} \to \mathbb{K}$. In fact we more often (as is also usual) replace $\mathbb{D}$ by its opposite, so that we have a presheaf $\mathbb{F}\colon \mathbb{D}^{\mathrm{op}} \to \mathbb{BO}$ and diagram $\mathbb{S}\colon \mathbb{D} \to \mathbb{K}$.

In what follows, we work through what this means in various cases, concentrating on the colimits, and leading up to the notion of Kleisli object which appears in our main theorem.

4.1. Tight colimits. A tight colimit is a weighted colimit for which the weight is tight, in the sense of Definition 3.9.

Consider a 2-category $\mathcal{D}$, a presheaf $F\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$, and the corresponding tight weight $F^{\mathrm{id}}\colon \mathrm{id}_* \mathcal{D}^{\mathrm{op}} \to \mathbb{BO}$ of Section 3.8. By adjointness, a diagram $\mathbb{S}\colon \mathrm{id}_* \mathcal{D} \to \mathbb{K}$ corresponds to a 2-functor $S\colon \mathcal{D} \to \mathcal{K}_t$ (its tight part); the loose part is necessarily $E_{\mathbb{K}}S\colon \mathcal{D} \to \mathcal{K}_\ell$.

4.2. Proposition. *To give a **BO**-enriched weighted colimit $F^{\mathrm{id}} * \mathbb{S}$ in the **BO**-category $\mathbb{K}$ is equivalently to give a **Cat**-enriched weighted colimit $F * S$ in $\mathcal{K}_t$, with the further property that the induced functor*

$$\mathcal{K}_\ell(E_{\mathbb{K}}(F * S), B) \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}_\ell(E_{\mathbb{K}}S, B))$$

is fully faithful.

Proof. Since $[\mathcal{D}^{\mathrm{op}}, \mathbb{BO}](F^{\mathrm{id}}, \mathbb{K}(\mathbb{S}, B))_t \cong [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}(S, B))$, the tight aspect of the universal property says precisely that the **BO**-colimit $F^{\mathrm{id}} * \mathbb{S}$ in $\mathbb{K}$ should be the **Cat**-colimit $F * S$ in $\mathcal{K}$.

To understand the loose part of the universal property, suppose that $F * S$ exists in $\mathcal{K}_t$, and consider the following diagram.

$$
\begin{array}{ccc}
\mathcal{K}_t(F * S, B) & \xrightarrow{\quad E_{\mathbb{K}} \quad} & \mathcal{K}_\ell(E_{\mathbb{K}}(F * S), B) \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \theta} \\
[\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}](F^{\mathrm{id}}, \mathbb{K}(\mathbb{S}, B))_t & \xrightarrow{E_{[\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}]}} & [\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}](F^{\mathrm{id}}, \mathbb{K}(\mathbb{S}, B))_\ell \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \psi} \\
[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}_t(S, B)) & \xrightarrow[{[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, E_{\mathbb{K}})}]{} & [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}_\ell(E_{\mathbb{K}}S, B))
\end{array}
\tag{1}
$$

In the lower square, the left vertical is part of the change-of-base isomorphism of Section 3.8, while $E_{[\mathrm{id}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}]}$ is the identity on objects and the right vertical is fully faithful by Section 3.6. In the upper square, the left vertical is the isomorphism expressing the universal property of the colimit $F * S$ (that is, the tight aspect of the universal property

of $F^{\mathrm{id}} * \mathbb{S}$), while the loose aspect says that θ is invertible. Since the other three maps in the top square are bijective on objects, so is θ. On the other hand, since ψ is fully faithful, θ will be fully faithful (and so invertible) if and only if $\psi\theta$ is fully faithful. ∎

4.3. PROPOSITION. *In the case where* $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, -)\colon [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Cat}$ *sends pointwise bijective-on-objects maps to bijective-on-objects maps,* $F^{\mathrm{id}} * \mathbb{S}$ *exists in* $\mathbb{K}$ *just when* $F * S$ *exists in* $\mathcal{K}_t$ *and is preserved by* $E\colon \mathcal{K}_t \to \mathcal{K}_\ell$.

PROOF. Suppose that $F*S$ exists in $\mathcal{K}_t$. It will be preserved by $E_{\mathbb{K}}$ just when the displayed map in Proposition 4.2 is invertible. This would be equivalent to it being fully faithful, as in the condition in Proposition 4.2, just when it is already known to be bijective on objects. But it is the image under $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, -)$ of the pointwise identity-on-objects map $E_{\mathbb{K}}\colon \mathcal{K}_t(S, B) \to \mathcal{K}_\ell(E_{\mathbb{K}}S, B)$, thus will indeed be bijective on objects under the hypotheses of the proposition. ∎

4.4. EXAMPLE. Suppose that $\mathcal{D}$ is a locally discrete 2-category — it has no non-identity 2-cells — and that $F = \Delta 1\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$. Then $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](\Delta 1, -)\colon [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{Cat}$ does send pointwise bijective-on-object maps to bijective-on-object ones.

Thus $\mathbb{K}$ has conical colimits indexed by ordinary categories if and only if $\mathcal{K}_t$ has them and they are preserved by $\mathcal{K}_t \to \mathcal{K}_\ell$.

Dually, $\mathbb{K}$ has conical limits indexed by ordinary categories if and only if $\mathcal{K}_t$ has them and they are preserved by $\mathcal{K}_t \to \mathcal{K}_\ell$.

4.5. EXAMPLE. On the other hand, consider the case of copowers by categories. Then $\mathcal{D} = 1$ and $F\colon 1 \to \mathbf{Cat}$ picks out a category X. It is not the case that $\mathbf{Cat}(X, -)$ preserves bijective-on-object maps unless X is discrete. A diagram $1 \to \mathbb{K}$ corresponds to an object $A \in \mathcal{K}$. The copower by X will be a copower $X \cdot A$ in $\mathcal{K}_t$ with the property that the induced

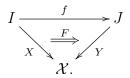$$\mathcal{K}_\ell(X \cdot A, B) \to \mathbf{Cat}(X, \mathcal{K}_\ell(A, B))$$

is fully faithful.

Thus we have bijections between maps $f\colon X \cdot A \to B$ in $\mathcal{K}_t$ and functors $f'\colon X \to \mathcal{K}_t(A, B)$; and furthermore, given $f, g\colon X \cdot A \to B$ and the corresponding $f', g'\colon X \to \mathcal{K}_t(A, B)$ there are bijections between items in the left and right columns of the following table.

tight 2-cells $f \to g$	natural transformations $f' \to g'$
loose 2-cells $f \to g$	natural transformations $E_{\mathbb{K}}f' \to E_{\mathbb{K}}g'$

4.6. COPRODUCT COMPLETIONS. Our main result will concern free completions under a certain sort of **BO**-enriched Kleisli object. But perhaps it is also worth discussing briefly the free completions under *coproducts*, seen as tight colimits as in Example 4.4.

Recall that the free completion of an ordinary category $\mathcal{X}$ under coproducts is given by $\mathrm{Fam}(\mathcal{X})$: an object is a "family" $X\colon I \to \mathcal{X}$ of objects of $\mathcal{X}$, where I is an indexing

set, seen as a discrete category, and X a functor; while a morphism has the form

$$
\begin{array}{ccc}
I & \xrightarrow{\quad f \quad} & J \\
 & \overset{F}{\Longrightarrow} & \\
X \searrow & & \swarrow Y \\
 & \mathcal{X}. &
\end{array}
$$

If $\mathcal{X}$ is not just a category but a 2-category, then there is a natural way to make $\mathrm{Fam}(\mathcal{X})$ into a 2-category: a 2-cell $(f, F) \to (g, G)$ can exist only if $f = g$, in which case it consists of a modification $F \to G$. The resulting 2-category is the free completion of $\mathcal{X}$ under **Cat**-enriched coproducts.

Since this **Cat**-enriched completion agrees with the ordinary one at the level of underlying ordinary categories, if $E_{\mathbb{A}} \colon \mathcal{A}_t \to \mathcal{A}_\ell$ acts as the identity on objects and on 1-cells, then the same is true of $\mathrm{Fam}(E) \colon \mathrm{Fam}(\mathcal{A}_t) \to \mathrm{Fam}(\mathcal{A}_\ell)$. This defines a **BO**-category $\mathrm{Fam}(\mathbb{A})$ for each **BO**-category $\mathbb{A}$.

By the universal property of the free completion, $\mathrm{Fam}(\mathcal{A}_t)$ has coproducts, and these are preserved by $\mathrm{Fam}(E_{\mathbb{A}}) \colon \mathrm{Fam}(\mathcal{A}_t) \to \mathrm{Fam}(\mathcal{A}_\ell)$. Thus the **BO**-category $\mathrm{Fam}(\mathbb{A})$ has tight coproducts.

It follows easily from the universal property of the **Cat**-enriched Fam construction that $\mathrm{Fam}(\mathbb{A})$ is in fact the **BO**-enriched free completion under tight coproducts.

Of course one can also modify this so as to deal with finite coproducts (or indeed κ-small coproducts for some regular cardinal κ) by limiting the size of the indexing sets I.

4.7. LOOSE COLIMITS. Once again, a loose colimit is a weighted colimit for which the weight is loose, in the sense of Definition 3.12.

Consider a 2-presheaf $F \colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$ and the corresponding $\mathbb{F} \colon \mathrm{disc}_* \mathcal{D}^{\mathrm{op}} \to \mathbb{BO}$ as in Section 3.11 above. By adjointness again, a diagram $\mathbb{S} \colon \mathrm{disc}_* \mathcal{D} \to \mathbb{K}$ consists of just a 2-functor $S \colon \mathcal{D} \to \mathcal{K}_\ell$ — we write $S_t \colon \mathcal{D}_1 \to \mathcal{K}_t$ for the uniquely determined tight part of $\mathbb{S}$, which satisfies $E_{\mathbb{K}} S_t = S E_{\mathrm{disc}_* \mathcal{D}}$ — and a map $\mathbb{F} \to \mathbb{K}(S, C)$ corresponds to a map $F \to \mathcal{K}_\ell(S, C)$.

4.8. PROPOSITION. *A* **BO***-enriched colimit* $\mathbb{F} * \mathbb{S}$ *in* $\mathbb{K}$ *is a* **Cat***-enriched colimit* $F * S$ *in* $\mathcal{K}_\ell$ *with the property that, for maps* $f, g \colon F * S \to B$ *and a loose 2-cell* $\varphi \colon f \to g$, *to give a tight structure to* φ *is equivalently to give a lifting* $\varphi' \colon f' \to g'$ *as in*

$$
\begin{array}{ccc}
 & \overset{f'}{\cdots\cdots\cdots\cdots\rightarrow} & \\
F_1 & \Downarrow\varphi' & \mathcal{K}_t(S_t-, B) \\
 & \underset{g'}{\cdots\cdots\cdots\cdots\rightarrow} & \\
F_E \downarrow & & \downarrow E_{\mathbb{K}} \\
FE & & \\
\downarrow & & \\
 & \overset{\mathcal{K}_\ell(SE-, f)}{\longrightarrow} & \\
\mathcal{K}_\ell(SE-, F * S) & \Downarrow\mathcal{K}_\ell(SE-, \varphi) & \mathcal{K}_\ell(SE-, B). \\
 & \underset{\mathcal{K}_\ell(SE-, g)}{\longrightarrow} &
\end{array}
$$

PROOF. First note that the existence and uniqueness of f' and g' are automatic, since F_1 is pointwise discrete and $E_{\mathbb{K}} \colon \mathcal{K}_t(S_t-, B) \to \mathcal{K}_\ell(SE-, B)$ is pointwise the identity on objects.

By virtue of the isomorphism

$$[\mathrm{disc}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}](\mathbb{F}, \mathbb{K}(\mathbb{S}, B))_\ell \cong [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}_\ell(S, B))$$

the loose aspect of the universal property says precisely that $\mathbb{F} * \mathbb{S}$ should be given by a **Cat**-enriched limit $F * S$ in $\mathcal{K}_\ell$. As for the tight aspect, since

$$
\begin{array}{ccc}
[\mathrm{disc}_* \mathcal{D}^{\mathrm{op}}, \mathbb{BO}](\mathbb{F}, \mathbb{K}(\mathbb{S}, B))_t & \longrightarrow & [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}_\ell(S, B)) \\
\downarrow & & \downarrow \\
& & [\mathcal{D}_1^{\mathrm{op}}, \mathbf{Cat}](FE, \mathcal{K}_\ell(SE, B)) \\
& & \downarrow \\
[\mathcal{D}_1^{\mathrm{op}}, \mathbf{Cat}](F_1, \mathcal{K}_t(S_t, B)) & \longrightarrow & [\mathcal{D}_1^{\mathrm{op}}, \mathbf{Cat}](F_1, \mathcal{K}_\ell(SE, B))
\end{array}
$$

is a pullback as in Section 3.6, it follows that

$$
\begin{array}{ccc}
\mathcal{K}_t(F * S, B) & \longrightarrow & \mathcal{K}_\ell(F * S, B) \\
\downarrow & & \downarrow \cong \\
& & [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](F, \mathcal{K}_\ell(S, B)) \\
& & \downarrow \\
[\mathcal{D}_1^{\mathrm{op}}, \mathbf{Cat}](F_1, \mathcal{K}_t(S_t, B)) & \longrightarrow & [\mathcal{D}_1^{\mathrm{op}}, \mathbf{Cat}](F_1, \mathcal{K}_\ell(SE, B))
\end{array}
$$

will need to be a pullback of categories. Since the two horizontals are both bijective on objects, this reduces to the condition in the proposition. ∎

4.9. REMARK. If $\mathcal{D}$ is locally discrete, and F takes values in discrete categories, then in fact $\mathbb{F} * \mathbb{S}$ is also a tight colimit, and just amounts to a colimit $F * S_t$ in $\mathcal{K}_t$ which is preserved by $E_{\mathbb{K}} \colon \mathcal{K}_t \to \mathcal{K}_\ell$.

4.10. REMARK. If on the other hand $\mathbb{K}$ is tight, so that $\mathcal{K}_t = \mathcal{K}_\ell$ and $E_{\mathbb{K}}$ is the identity, then a loose colimit in $\mathbb{K}$ is just a (2-categorical) colimit in $\mathcal{K}_\ell$.

4.11. KLEISLI OBJECTS. Our main example involves the special case of the previous section arising from 2-categorical Kleisli objects. Let $\mathcal{M}$ be the universal 2-category containing a monad, so that for any 2-category $\mathcal{K}$, there is a natural bijection between 2-functors from $\mathcal{M}$ to $\mathcal{K}$ and monads in $\mathcal{K}$. Now let $F \colon \mathcal{M}^{\mathrm{op}} \to \mathbf{Cat}$ be the weight for Kleisli objects, so that for a 2-functor $T \colon \mathcal{M} \to \mathcal{K}$, a colimit $F * T$ is exactly a Kleisli object for the monad corresponding to T. For details concerning $\mathcal{M}$ and F, see for example [12, Section 5] or [7, Section 8.2].

Then $\mathcal{M}_1$ is the universal category containing an endomorphism (in other words, the one-object category whose morphisms are the natural numbers with composition given by addition). And $F_1 \colon \mathcal{M}_1^{\mathrm{op}} \to \mathbf{Cat}$ picks out the discrete category $\mathbb{Z}_{>0}$ of positive integers together with the successor endomorphism.

A diagram $\mathrm{disc}_* \mathcal{M} \to \mathbb{K}$ consists of a monad (A, t) in $\mathcal{K}_\ell$; we might call this a *loose monad in* $\mathbb{K}$. An $\mathbb{F}$-weighted colimit of it is a Kleisli object $e \colon A \to A_t$ for the monad in $\mathcal{K}_\ell$, with the additional property that if $f, g \colon A_t \to B$ and $\varphi \colon f \to g$ is loose, then restriction along e defines a bijection between liftings of φ to a tight 2-cell and liftings of $\varphi e \colon ge \to he$ to a tight 2-cell.

4.12. REMARK. Of course this extra tightness condition is automatic if $\mathbb{K} = \mathrm{id}_* \mathcal{K}$, so that all 2-cells are tight.

4.13. PROPOSITION. *Let* $e \colon A \to A'$ *have a right adjoint* $e \dashv r$ *in* $\mathcal{K}_\ell$, *and let* t *be the induced monad in* $\mathcal{K}_\ell$; *that is, the induced loose monad in* $\mathbb{K}$. *Then* e *exhibits* A' *as the Kleisli object in* $\mathbb{K}$ *if and only if*

(i) e *exhibits* A' *as the Kleisli object in* $\mathcal{K}_\ell$, *and*

(ii) *the square*

$$
\begin{array}{ccc}
\mathcal{K}_t(A_t, B) & \xrightarrow{\mathcal{K}_t(e,B)} & \mathcal{K}_t(A, B) \\
{\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\
\mathcal{K}_\ell(A_t, B) & \xrightarrow[\mathcal{K}_\ell(e,B)]{} & \mathcal{K}_\ell(A, B)
\end{array}
$$

is a pullback in $\mathbf{Cat}$, *for each* $B \in \mathbb{K}$.

PROOF. Here (i) expresses the loose aspect of the universal property. In the square in (ii), the vertical maps labelled E are the identity on objects, so the object part of the pullback property is always true. What is left is the tight aspect of the universal property of the Kleisli object. ∎

4.14. DEFINITION. *In this case we say that* $e \dashv r$ *is* an adjunction of enhanced Kleisli type in $\mathbb{K}$. *We call the colimit the* enhanced Kleisli object, *or just the Kleisli object if the context makes clear we are dealing with the* $\mathbf{BO}$-*enriched notion.*

Adjunctions of enhanced Kleisli type clearly compose, since Kleisli adjunctions compose in $\mathcal{K}_\ell$ and pullback squares can be pasted to give pullback squares.

4.15. PROPOSITION. *Let* $P = (P_t, P_\ell, P_E)$ *be a presheaf on the* $\mathbf{BO}$-*category* $\mathbb{K} = (\mathcal{K}_t, \mathcal{K}_\ell, E_\mathbb{K})$. *A loose monad on* P *consists of a monad* (S, m, i) *on* P_ℓ *in* $[\mathcal{K}_\ell^{\mathrm{op}}, \mathbf{Cat}]$, *together with a lifting of the endomorphism* $S \colon P_\ell \to P_\ell$ *to a map* $\overline{S} \colon P_t \to P_t$ *in* $[\mathcal{K}_t^{\mathrm{op}}, \mathbf{Cat}]$ *making the diagram*

$$
\begin{array}{ccc}
P_t & \xrightarrow{\overline{S}} & P_t \\
{\scriptstyle P_E}\downarrow & & \downarrow{\scriptstyle P_E} \\
P_\ell E_\mathbb{K} & \xrightarrow[SE_\mathbb{K}]{} & P_\ell E_\mathbb{K}
\end{array}
$$

commute.

The enhanced Kleisli object $P' = (P'_t, P'_\ell, P'_E)$ has $P'_t = P_t$, P'_ℓ the Kleisli object in $[\mathcal{K}^{\mathrm{op}}_\ell, \mathbf{Cat}]$ of S, and $P'_E = e_S E_{\mathbb{K}}.P_E$, where $e_S\colon P_\ell \to P'_\ell$ is the Kleisli map for S. The identity $1\colon P_t \to P'_t$ and $e_S\colon P_\ell \to P'_\ell$ define the Kleisli map $P \to P'$.

PROOF. The description of loose monads on P follows immediately from the definitions. We derive the description of the enhanced Kleisli object using the characterization of these given in Proposition 4.13.

First we construct the loose adjunction $e \dashv r$. The right adjoint $r\colon P' \to P$ is defined by $S\colon P'_t = P_t \to P_t$ and $r_S\colon P'_\ell \to P_\ell$, where r_S is the right adjoint of e_S. The unit and counit of $e \dashv r$ are just the unit and counit of the adjunction $e_S \dashv r_S$ in $[\mathcal{K}^{\mathrm{op}}_\ell, \mathbf{Cat}]$. This is clearly a loose adjunction and induces the original loose monad. We have to show that it is of enhanced Kleisli type.

First suppose given a map $g\colon P \to Q$ with a loose opaction $gS \to g$; that is, an opaction $g_\ell S \to g_\ell$. Then g_ℓ induces a unique $g'_\ell\colon P'_\ell \to Q_\ell$, which together with $g'_t := g_t$ determines a map $g'\colon P' \to Q$. This defines a bijection between the objects of $[\mathbb{K}^{\mathrm{op}}, \mathbb{BO}]_\ell(P', Q)$ and the objects of $[\mathbb{K}^{\mathrm{op}}, \mathbb{BO}]_\ell(P, Q)$ equipped with an opaction of S, and this bijection extends to an isomorphism of categories exhibiting P' as the Kleisli object in $[\mathbb{K}^{\mathrm{op}}, \mathbb{BO}]_\ell$.

It remains to check the tight aspect. This says that if $x, y\colon P' \to Q$ and $\xi\colon x_\ell \to y_\ell$, then to give a lifting of ξ to some $\bar{\xi}\colon x_t \to y_t$ is equivalent to giving a lifting $\xi e_\ell\colon x_\ell e_\ell \to y_\ell e_\ell$ to some $x_t \to y_t$, which is trivially true.   ■

For a 2-category $\mathcal{K}$, let $\mathrm{KL}(\mathcal{K})$ be the free completion of $\mathcal{K}$ under (**Cat**-enriched) Kleisli objects. As described in [9, Section 1], this is equipped with a 2-functor $\mathrm{Mnd}_*(\mathcal{K}) \to \mathrm{KL}(\mathcal{K})$ which acts as the identity on objects and on 1-cells. Here $\mathrm{Mnd}_*(\mathcal{K}) = \mathrm{Mnd}(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$, and similarly $\mathrm{KL}(\mathcal{K}) = \mathrm{EM}(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$.

Thus $\mathrm{Mnd}_*(\mathcal{K}) \to \mathrm{KL}(\mathcal{K})$ can be seen as a **BO**-category.

4.16. REMARK. In particular, if we take $\mathcal{K} = \mathbf{Cat}$, then the **BO**-category corresponding to $\mathrm{Mnd}_*(\mathbf{Cat}) \to \mathrm{KL}(\mathbf{Cat})$ is isomorphic to the full sub-2-category of **BO** consisting of those identity-on-objects functors which are left adjoints.

4.17. THEOREM. *The free completion of* $\mathrm{id}_* \mathcal{K}$ *under enhanced Kleisli objects for loose monads is the* **BO***-category* $\mathrm{Mnd}_*(\mathcal{K}) \to \mathrm{KL}(\mathcal{K})$.

PROOF. The free completion $\widehat{\mathcal{K}}$ will be given by the closure of the representables in $[\mathrm{id}_* \mathcal{K}, \mathbb{BO}]$ under enhanced Kleisli objects.

As observed above, the enhanced Kleisli-type adjunctions in a **BO**-category are closed under composition, and so this closure under Kleisli objects will be of the "one-step" variety, and we can simply consider (the full subcategory consisting of) those objects which are Kleisli objects of loose monads on representables. Thus we can take the objects of $\widehat{\mathcal{K}}$ to be the loose monads on representables in $[\mathrm{id}_* \mathcal{K}, \mathbb{BO}]$, or equivalently the loose monads in $\mathrm{id}_* \mathcal{K}$, or equivalently the monads in $\mathcal{K}$.

Given monads (A, t) and (B, s), the hom $\widehat{\mathcal{K}}((A, t), (B, s))$ will be the hom in $[\mathrm{id}_* \mathcal{K}, \mathbb{BO}]$ between the Kleisli objects.

By Proposition 4.15 and the fact that $\mathcal{K}_\ell = \mathcal{K}_t$, the Kleisli object of (A, t) is just the Kleisli object and Kleisli morphism $\mathcal{K}_\ell(-, A) \to \mathcal{K}_\ell(-, A)_{\mathcal{K}_\ell(-, t)}$ in $[\mathcal{K}_\ell^{\mathrm{op}}, \mathbf{Cat}]$. We can now read off the various morphisms and 2-cells. In particular, the loose 2-category $\widehat{\mathcal{K}}_\ell$ is precisely $\mathrm{KL}(\mathcal{K})$, by exactly the calculation that was given in [9]. A morphism $(A, t) \to (B, s)$ is a pair (f, φ), where $f\colon A \to B$ and $\varphi\colon ft \to sf$, subject to two equations. Given another morphism $(g, \psi)\colon (A, t) \to (B, s)$, a loose 2-cell $(f, \varphi) \to (g, \psi)$ consists of a 2-cell $\rho\colon f \to sg$, subject to a single equation. To make this into a tight 2-cell is to give $\overline{\rho}\colon f \to g$ with $\eta g.\overline{\rho} = \rho$. But $\eta g.\overline{\rho}$ is a 2-cell in $\mathrm{KL}(\mathcal{K})$ just when $\overline{\rho}\colon f \to g$ defines a 2-cell $(f, \varphi) \to (g, \psi)$ in $\mathrm{Mnd}_*(\mathcal{K})$. ∎

Dually, we have the notion of *enhanced Eilenberg-Moore object*, and the corresponding theorem.

4.18. THEOREM. *The free completion of* $\mathrm{id}_* \mathcal{K}$ *under enhanced Eilenberg-Moore objects for loose monads is the* $\mathbf{BO}$*-category* $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$.

This is the promised universal property of $\mathrm{Mnd}(\mathcal{K}) \to \mathrm{EM}(\mathcal{K})$. Having fulfilled our promise, we conclude with a few observations about Eilenberg-Moore objects as adjoints to the inclusion $\mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$.

Since $\mathrm{EM}(\mathcal{K})$ is the free completion of $\mathcal{K}$ under Eilenberg-Moore objects, $\mathcal{K}$ will have Eilenberg-Moore objects just when the inclusion $\mathcal{K} \to \mathrm{EM}(\mathcal{K})$ has a right adjoint. But why should $\mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ also have a right adjoint, as observed already in [11]?

By Remark 4.10, the tight $\mathbf{BO}$-category $\mathrm{id}_* \mathcal{K}$ has enhanced Eilenberg-Moore objects for loose monads if and only if $\mathcal{K}$ has ordinary Eilenberg-Moore objects for monads. By Theorem 4.18, this will be the case if and only if the inclusion

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathrm{Mnd}(\mathcal{K}) \\ \downarrow & & \downarrow \\ \mathcal{K} & \longrightarrow & \mathrm{EM}(\mathcal{K}) \end{array}$$

has a right $\mathbf{BO}$-adjoint. Such an adjoint will imply in particular that the horizontal components each have right $\mathbf{Cat}$-adjoints. Thus we recover the result of [11] that if $\mathcal{K}$ has Eilenberg-Moore objects then $\mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ has a right adjoint.

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