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FROM ABELIAN CATEGORIES TO 2-ABELIAN BICATEGORIES

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ABSTRACT. We show that, if $\mathcal{A}$ is an abelian category, then a certain bicategory of fractions $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$ of the 2-category $\mathbf{Arr}(\mathcal{A})$ of arrows in $\mathcal{A}$ is 2-abelian. On the way, we study homotopy kernels and homotopy cokernels, their relationship with 2-limits and bilimits, and how they pass through the general construction of the bicategory of fractions. We also introduce two new factorization systems in $\mathbf{Arr}(\mathcal{A})$ and we use them to describe the class Σ of "weak equivalences".

Contents

I would be happy if this paper was a jointly work with Mathieu Dupont. Since this is not the case, I can only express my deep gratitude to Mathieu for the numerous instructive exchanges on 2-abelian bicategories. I also thank Julia Ramos Gonzalez and Dorette Pronk for some help with bicategories of fractions. Finally, I thank the referee for some intriguing questions (and for pointing out to me reference [4]). I tried to formulate an answer to some of them (Remarks 2.1.10 and 3.1.2) and I keep in mind some others for future work.

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1. Introduction

Based on an extensive literature on symmetric categorical groups (see, for example, [14, 46, 47, 49, 33, 9, 8, 25, 45, 3, 12, 13, 20]), three solutions to the equation

$$\frac{\text{abelian categories}}{\text{abelian groups}} = \frac{???}{\text{symmetric categorical groups}}$$

have been proposed by Mathieu Dupont in his Ph.D. Thesis [15] and by Hiroyuki Nakaoka in [38]. We will recall in Section 7 the definition of 2-abelian bicategory from [15]. (A comparison among the three proposed solutions has been established in [39].) The main example of 2-abelian bicategory clearly is the 2-category **SCG** of symmetric categorical groups. The other examples discussed in [15, 38] (see also [29]) are all related to **SCG**.

In search of new examples of 2-abelian bicategories, the question has been posed in [15] to construct a 2-abelian bicategory from any abelian category $\mathcal{A}$ in such a way that the sub-bicategories of discrete or connected objects are equivalent to $\mathcal{A}$. A partial answer to this question has been proposed by Teimuraz Pirashvili in [42]: if the abelian category $\mathcal{A}$ has enough projective objects, then the 2-category $\mathbf{Arr}(\mathcal{A})$ of arrows in $\mathcal{A}$ is 2-abelian. A more general solution has been proposed by Mathieu Dupont in an unpublished manuscript [16]: for any abelian category $\mathcal{A}$, the bicategory whose objects are arrows in $\mathcal{A}$ and whose arrows are butterflies (in the sense of Berang Noohi, see [40]) is 2-abelian.

Recall now that, for any semi-abelian category $\mathcal{A}$, the bicategory of butterflies between crossed modules in $\mathcal{A}$ provides the bicategory of fractions of crossed modules with respect to weak equivalences (see [41, 2, 16, 44, 1] for this result and for some related results in terms of anafunctors). Moreover, when $\mathcal{A}$ is abelian, crossed modules in $\mathcal{A}$ reduces to arrows. Therefore, it turns out that the bicategory of arrows and butterflies in $\mathcal{A}$ used in [16] is the bicategory of fractions of arrows with respect to weak equivalences. This suggests the idea to revisit one of the main results of [16] in terms of bicategories of fractions, which is the aim of the present paper. In addition, since in general $\mathbf{Arr}(\mathcal{A})$ is not a 2-category but is always equipped with a structure of nullhomotopies, we work as far as possible with kernels and cokernels relative to nullhomotpies. This is relevant especially for the construction of three-step factorizations obtained in Subsection 4.2. Under adequate assumptions (see Section 5, which is devoted to bilimits in $\mathbf{Arr}(\mathcal{A})$), kernels and cokernels in $\mathbf{Arr}(\mathcal{A})$ relative to nullhomotopies become bikernels and bicokernels, which are needed to express the notion of 2-abelian bicategory.

To achieve our goal, we use seven different kinds of limits: Θ-kernels, strong Θ-kernels and Θ-strong limits (where Θ denotes a structure of nullhomotopies), 2-limits, homotopy limits and strong homotopy limits, and finally bilimits. This is why, in order to make the paper as self-contained as possible, we start with a long review section where the various limits involved are recalled and compared. (In fact, I adopted this self-contained philosophy throughout all the paper, and especially in Sections 3.1, 5.2 and 6.2. I hope that the reader will forgive me for the length of the paper and appreciate to have at hand all the needed material.) In the third section, after recalling some general facts on bicategories of fractions, we complete the study of bilimits in bicategories of fractions initiated in [27]. In the fourth section, we observe that, if $\mathcal{A}$ is a category with pullbacks and pushouts, then the category $\mathbf{Arr}(\mathcal{A})$ is equipped with two factorization systems, say $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$, related to Θ-kernels and Θ-cokernels. The interest of these factorization systems is that, if $\mathcal{A}$ has also a zero object, merging $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ we can produce two different three-step factorizations of an arrow of $\mathbf{Arr}(\mathcal{A})$. Moreover, in Section 6 we prove that, if $\mathcal{A}$ is abelian, then the middle step of both three-step factorizations of an arrow lies in $\mathcal{E}_1 \cap \mathcal{M}_2$. Since the definition of 2-abelian bicategory is in Puppe-exact style, this fact is a prelude to prove, in the final section, that the bicategory of fractions $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$, where $\Sigma = \mathcal{E}_1 \cap \mathcal{M}_2$, is 2-abelian if $\mathcal{A}$ is abelian.

N.B.: The composition of two arrows $A \xrightarrow{f} B \xrightarrow{g} C$ will be written as $f \cdot g$.

2. From homotopy kernels to bilimits

2.1. Categories with nullhomotopies, Θ-kernels and Θ-strong limits. We recall from [22, 52, 26] the notion of category with a structure of nullhomotopies. Examples and applications of categories with nullhomotopies are discussed in [37, 53]. Our object of study, the category with nullhomotopies $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, is introduced in Example 2.1.6. Other examples of interest for the present work will be introduced in Subsections 2.2 and 2.3.

2.1.1. Definition. A structure of nullhomotopies Θ on a category $\mathcal{B}$ is given by:

1) For every arrow g in $\mathcal{B}$, a set $\Theta(g)$ whose elements are called nullhomotopies on g.

2) For every triple of composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

in such a way that, for every $\varphi \in \Theta(g)$, one has

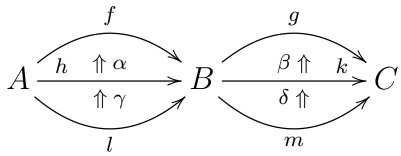
(a) $(f' \cdot f) \circ \varphi \circ (h \cdot h') = f' \circ (f \circ \varphi \circ h) \circ h'$ whenever the compositions $f' \cdot f$ and $h \cdot h'$ are defined,

(b) $\mathrm{id}_B \circ \varphi \circ \mathrm{id}_C = \varphi$.

When $f = \mathrm{id}_B$ or $h = \mathrm{id}_C$, we write $\varphi \circ h$ and $f \circ \varphi$ instead of $\mathrm{id}_B \circ \varphi \circ h$ and $f \circ \varphi \circ \mathrm{id}_C$.

2.1.2. Condition. (From [23]) Let $(\mathcal{B}, \Theta)$ be a category with nullhomotopies. The structure Θ satisfies the reduced interchange if, for any pair of composable arrows $f \colon A \to B$ and $g \colon B \to C$ and for nullhomotopies $\alpha \in \Theta(f)$ and $\beta \in \Theta(g)$, one has $\alpha \circ g = f \circ \beta$.

2.1.3. . The name of reduced interchange can be justified by the following analysis (see also Remark 2.2.6). In any 2-category $\mathcal{B}$, the usual interchange condition for 2-cells



$$(\gamma \circ \delta) \cdot (\alpha \circ \beta) = (\gamma \cdot \alpha) \circ (\delta \cdot \beta)$$

implies the following conditions, where only reduced horizontal composition (that is, horizontal composition between a 2-cell and an identity 2-cell) is involved:

1. reduced interchange : $(\alpha \circ k) \cdot (f \circ \beta) = (h \circ \beta) \cdot (\alpha \circ g)$
2. distributivities : $(h \circ \delta) \cdot (h \circ \beta) = h \circ (\delta \cdot \beta)$, $(\gamma \circ m) \cdot (\alpha \circ m) = (\gamma \cdot \alpha) \circ m$

Conversely, if only reduced horizontal composition is allowed, one can define

$$\alpha \circ \beta = (\alpha \circ k) \cdot (f \circ \beta)$$

and the interchange condition follows from reduced interchange and distributivities. Assume now that $\mathcal{B}$ has zero object 0 and write $0^A_B = 0^A \cdot 0_B \colon A \to 0 \to B$ for the zero arrow. Assume also that the zero arrows absorbe nullhomotopies, in the sense that, for all arrows $A \xrightarrow{f} B \xrightarrow{g} C$ and for all 2-cells $\alpha \colon 0^A_B \Rightarrow f$ and $\beta \colon 0^B_C \Rightarrow g$ one has

$$0^A_B \circ \beta = 0^A_C \text{ and } \alpha \circ 0^B_C = 0^A_C.$$

Then reduced interchange precisely gives Condition 2.1.2 by taking $k = 0^B_C$ and $h = 0^A_B$. Finally, observe that, in a 2-category with zero object, the conditions $0^A_B \circ \beta = 0^A_C$ and $\alpha \circ 0^B_C = 0^A_C$ are themselves particular instances of Condition 2.1.2. Indeed, assuming Condition 2.1.2 and using that the horizontal composition of two identities 2-cells gives the identity 2-cell on the composite arrow, we have $0^A_B \circ \beta = 0^A_B \circ g = 0^A_B \cdot g = 0^A_C$ and $\alpha \circ 0^B_C = f \circ 0^B_C = f \cdot 0^B_C = 0^A_C$.

In any category with nullhomotopies $(\mathcal{B}, \Theta)$, we can express the notions of (strong) Θ-kernel and (strong) Θ-cokernel. We recall them following [37, 53].

2.1.4. Definition. Let $g \colon B \to C$ be an arrow in a category with nullhomotopies $(\mathcal{B}, \Theta)$.

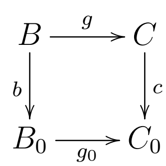
1. A homotopy cokernel of g with respect to Θ (or Θ-cokernel) is a universal triple
$$\mathcal{C}(g) \in \mathcal{B}, c_g \colon C \to \mathcal{C}(g), \gamma_g \in \Theta(g \cdot c_g)$$
This means that, for any other triple $(D \in \mathcal{B}, h \colon C \to D, \varphi \in \Theta(g \cdot h))$, there exists a unique arrow $h' \colon \mathcal{C}(g) \to D$ such that $c_g \cdot h' = h$ and $\gamma_g \circ h' = \varphi$
2. A Θ-cokernel $(\mathcal{C}(g), c_g, \gamma_g)$ is strong if, for any triple $(D, h \colon \mathcal{C}(g) \to D, \varphi \in \Theta(c_g \cdot h))$ such that $g \circ \varphi = \gamma_g \circ h$, there exists a unique nullhomotopy $\varphi' \in \Theta(h)$ such that $c_g \circ \varphi' = \varphi$.
3. The notion of (strong) Θ-kernel is dual of the notion of (strong) Θ-cokernel. The notation is
$$\mathcal{N}(g) \in \mathcal{B}, n_g \colon \mathcal{N}(g) \to B, \nu_g \in \Theta(n_g \cdot g)$$

2.1.5. . In order to make easier the comparison with other kinds of limits discussed in this section, we recall from [37, 53] the cancellation properties satisfied by a Θ-cokernel. Those for a Θ-kernel are dual.

1. Given arrows $f: A \to B$ and $g, h: \mathcal{C}(f) \to C$, if $c_f \cdot g = c_f \cdot h$ and $\gamma_f \circ g = \gamma_f \circ h$, then $g = h$.

2. Given arrows $f: A \to B$ and $g: \mathcal{C}(f) \to C$ and nullhomotopies $\varphi, \psi \in \Theta(g)$ such that $c_f \circ \varphi = c_f \circ \psi$, if the Θ-cokernel $\mathcal{C}(f)$ is strong and if Θ satisfies the reduced interchange, then $\varphi = \psi$.

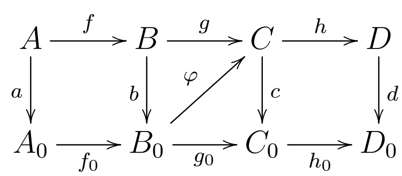
2.1.6. Example. For a given category $\mathcal{A}$, objects and arrows of the category of arrows $\mathbf{Arr}(\mathcal{A})$ are written as $(g, g_0): (B, b, B_0) \to (C, c, C_0)$, where



commutes. As set of nullhomotopies $\Theta_\Delta(g, g_0)$ we take the set of diagonals:

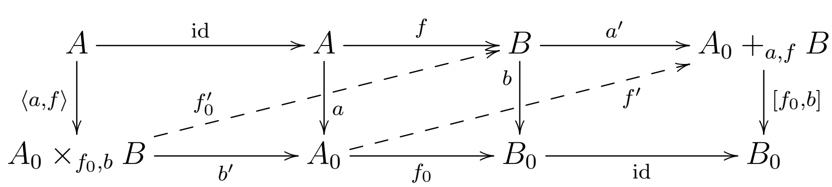
$$\Theta_\Delta(g, g_0) = \{\varphi: B_0 \to C \mid b \cdot \varphi = g,\ \varphi \cdot c = g_0\}$$

In the situation of the following diagram



the composition is given by the formula $(f, f_0) \circ \varphi \circ (h, h_0) = f_0 \cdot \varphi \cdot h$. It is easy to check that the reduced interchange 2.1.2 is satisfied.

Let us recall from [37] that, if $\mathcal{A}$ has pullbacks and pushouts, then $\mathbf{Arr}(\mathcal{A})$ has strong Θ_Δ-kernels and strong Θ_Δ-cokernels. For an arrow (f, f_0), they are depicted in the following diagram, where $A_0 \times_{f_0, b} B$ is the pullback of f_0 and b, $A_0 +_{a,f} B$ is the pushout of a and f, and the dashed arrows are the structural nullhomotopies



Finally, recall that in a category with nullhomotopies $(\mathcal{B}, \Theta)$, an object A is Θ-trivial if $\Theta(\mathrm{id}_A) \neq \emptyset$, and two objects A and B are Θ-orthogonal if $\Theta(f) = \{*\}$ for all arrows

$f\colon A \to B$. In $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, an object (A, a, A_0) is Θ_Δ-trivial iff it is orthogonal (on both sides) to any other object, and this is the case iff the arrow $a\colon A \to A_0$ is an isomorphism.

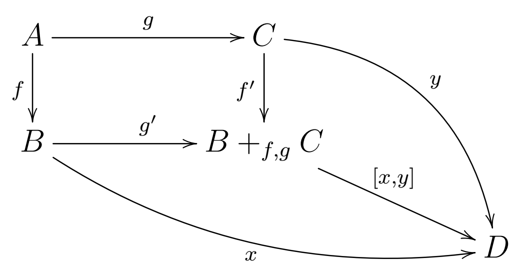
The general notion of Θ-strong (co)limit in a category with nullhomotopies $(\mathcal{B}, \Theta)$ has been introduced in [53]. We need three special cases. We treat the case of colimits, the situation for limits is dual.

2.1.7. Definition. Let $(\mathcal{B}, \Theta)$ be a category with nullhomotopies.

1. An initial object $\emptyset$ of $\mathcal{B}$ is Θ-strong if $\Theta(\emptyset \to X) = \{*\}$ for every object $X \in \mathcal{B}$.

2. A zero object 0 of $\mathcal{B}$ is Θ-strong if $\Theta(0 \to X) = \{*\} = \Theta(X \to 0)$ for all $X \in \mathcal{B}$.

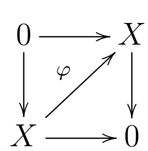
3. Consider the following commutative diagram, where the square is a pushout:



The pushout is Θ-strong if, given $\varphi \in \Theta(x)$ and $\psi \in \Theta(y)$ such that $f \circ \varphi = g \circ \psi$, there exists a unique $[\varphi, \psi] \in \Theta([x, y])$ such that $g' \circ [\varphi, \psi) = \varphi$ and $f' \circ [\varphi, \psi] = \psi$.

2.1.8. . Here is the obvious cancellation property of a Θ-strong pushout with respect to nullhomotopies: (with the notation of Definition 2.1.7) given an arrow $h\colon B +_{f,g} C \to D$ and nullhomotopies $\alpha, \beta \in \Theta(h)$, if $g' \circ \alpha = g' \circ \beta$ and $f' \circ \alpha = f' \circ \beta$, then $\alpha = \beta$.

2.1.9. . To avoid any confusion, observe that, for a category with nullhomotopies $(\mathcal{B}, \Theta)$, to have a Θ-strong zero object does not imply that there is a unique nullhomotopy on the zero arrow $0_Y^X\colon X \to Y$. An easy counterexample is provided by $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$ under the assumption that $\mathcal{A}$ has a zero object 0. In this case, $(0, \mathrm{id}, 0)$ is a Θ_Δ-strong zero object in $\mathbf{Arr}(\mathcal{A})$ and the commutative diagram



shows that, for any object $X \in \mathcal{A}$, the nullhomotopies on the zero arrow from the object $(0, 0 \to X, X)$ to the object $(X, X \to 0, 0)$ correspond to the endomorphisms on X.

2.1.10. Remark. It is possible to restate Definition 2.1.1 of nullhomotopy structure using a variant of the category $\mathbf{Arr}(\mathcal{A})$, called twisted arrow category in [19] and category of factorizations in [4]. Objects and arrows $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ are depicted by the following commutative diagram

$$\begin{array}{ccc} A & \xleftarrow{\;f\;} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

With the obvious composition and identities, they form a category that we denote by $\mathbf{Arr}^{\leftrightarrows}(\mathcal{A})$. Clearly, a structure of nullhomotopies Θ on a category $\mathcal{A}$ is nothing but a functor

$$\Theta\colon \mathbf{Arr}^{\leftrightarrows}(\mathcal{A}) \to \mathbf{Set}$$

the action of Θ on an arrow (f, f_0) being given by the map

$$\Theta(f, f_0) = f \circ - \circ f_0\colon \Theta(a) \to \Theta(f \cdot a \cdot f_0) = \Theta(b)$$

(size conditions are not relevant here). In other words, nullhomotopy structures are the non-linear version of the natural systems of abelian groups introduced in [4], as pointed out to me by the referee. In fact, the whole 2-category of categories with nullhomotopies introduced in [53] can be expressed using the $\mathbf{Arr}^{\leftrightarrows}$-construction. For this, observe that any functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ induces a functor $\mathcal{F}^{\leftrightarrows}\colon \mathbf{Arr}^{\leftrightarrows}(\mathcal{A}) \to \mathbf{Arr}^{\leftrightarrows}(\mathcal{B})$. Now, a morphism of categories with nullhomotopies $(\mathcal{F}, \{\mathcal{F}_a\})\colon (\mathcal{A}, \Theta_{\mathcal{A}}) \to (\mathcal{B}, \Theta_{\mathcal{B}})$ in the sense of Definition 2.4 in [53] amounts to a functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ together with a natural transformation

$$\begin{array}{ccc} \mathbf{Arr}^{\leftrightarrows}(\mathcal{A}) & \xrightarrow{\;\mathcal{F}^{\leftrightarrows}\;} & \mathbf{Arr}^{\leftrightarrows}(\mathcal{B}) \\ & \overset{\mathcal{F}_\bullet}{\Rightarrow} & \\ {\scriptstyle \Theta_{\mathcal{A}}}\searrow & & \swarrow{\scriptstyle \Theta_{\mathcal{B}}} \\ & \mathbf{Set} & \end{array}$$

Explicitly, the naturality of $\mathcal{F}_\bullet$ means that $\mathcal{F}_b(f \circ \varphi \circ f_0) = \mathcal{F}(f) \circ \mathcal{F}_a(\varphi) \circ \mathcal{F}(f_0)$ for all $\varphi \in \Theta_{\mathcal{A}}(a)$. As far as 2-morphisms are concerned, observe that any natural transformation $\alpha\colon \mathcal{F} \Rightarrow \mathcal{G}\colon \mathcal{A} \to \mathcal{B}$ induces a functor

$$\alpha^{\leftrightarrows}\colon \mathbf{Arr}^{\leftrightarrows}(\mathcal{A}) \to \mathbf{Arr}^{\leftrightarrows}(\mathcal{B})$$

which sends an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ on the arrow

$$(\mathcal{F}(f), \mathcal{G}(f_0))\colon (\mathcal{F}A, \alpha_A \cdot \mathcal{G}(a) = \mathcal{F}(a) \cdot \alpha_{A_0}, \mathcal{G}A_0) \to (\mathcal{F}B, \alpha_B \cdot \mathcal{G}(b) = \mathcal{F}(b) \cdot \alpha_{B_0}, \mathcal{G}B_0)$$

Moreover, the natural transformation $\alpha\colon \mathcal{F} \Rightarrow \mathcal{G}$ induces also two natural transformations

$$\mathcal{F} \star \alpha\colon \mathcal{F}^{\leftrightarrows} \cdot \Theta_{\mathcal{B}} \Rightarrow \alpha^{\leftrightarrows} \cdot \Theta_{\mathcal{B}}\,, \quad (\mathcal{F} \star \alpha)(a) = \Theta_{\mathcal{B}}(\mathrm{id}_{\mathcal{F}A}, \alpha_{A_0})$$

$$\alpha \star \mathcal{G} \colon \mathcal{G}^{\leftrightarrows} \cdot \Theta_{\mathcal{B}} \Rightarrow \alpha^{\leftrightarrows} \cdot \Theta_{\mathcal{B}} \,, \quad (\alpha \star \mathcal{G})(a) = \Theta_{\mathcal{B}}(\alpha_A, \mathrm{id}_{\mathcal{G}A_0})$$

Finally, a 2-morphism of categories with nullhomotopies

$$\alpha \colon (\mathcal{F}, \{\mathcal{F}_a\}) \Rightarrow (\mathcal{G}, \{\mathcal{G}_a\}) \colon (\mathcal{A}, \Theta_{\mathcal{A}}) \to (\mathcal{B}, \Theta_{\mathcal{B}})$$

in the sense of Definition 2.4 in [53] amounts to a natural transformation $\alpha \colon \mathcal{F} \Rightarrow \mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc} \Theta_{\mathcal{A}} & \xRightarrow{\ \mathcal{F}_\bullet\ } & \mathcal{F}^{\leftrightarrows} \cdot \Theta_{\mathcal{B}} \\ {\scriptstyle \mathcal{G}_\bullet} \Downarrow & & \Downarrow {\scriptstyle \mathcal{F} \star \alpha} \\ \mathcal{G}^{\leftrightarrows} \cdot \Theta_{\mathcal{B}} & \xRightarrow[\ \alpha \star \mathcal{G}\ ]{} & \alpha^{\leftrightarrows} \cdot \Theta_{\mathcal{B}} \end{array}$$

Explicitly, this commutativity condition means that $\mathcal{F}_a(\varphi) \circ \alpha_{A_0} = \alpha_A \circ \mathcal{G}_a(\varphi)$ for all $\varphi \in \Theta_{\mathcal{A}}(a)$.

2.2. 2-categories, 2-limits and H-limits. With the following three examples, we establish the link between categories with nullhomotopies and 2-categories. For the definition of 2-category, see Chapter 7 in [6].

2.2.1. Example. Let $\mathcal{B}$ be a 2-category.

1. Assume that the underlying category of $\mathcal{B}$ has a zero object. The category $\mathcal{B}$ can be seen as a category with nullhomotopies by putting

$$\Theta_0(g \colon B \to C) = \{\text{2-cells } \varphi \colon 0_C^B \Rightarrow g\}$$

   The map $\Theta_0(g) \to \Theta_0(f \cdot g \cdot h)$ is given by horizontal composition with identities 2-cells. This makes sense because $f \cdot 0_C^B \cdot h = 0_D^A$ for any $f \colon A \to B$ and $h \colon C \to D$.

2. More in general, let $\mathcal{Z}$ be an ideal of arrows in the underlying category of $\mathcal{B}$. This means that, if f and g are composable arrows and one of them is in $\mathcal{Z}$, then the composite $f \cdot g$ also is in $\mathcal{Z}$. We get a nullhomotopy structure on $\mathcal{B}$ by putting

$$\Theta_{\mathcal{Z}}(g \colon B \to C) = \{\text{2-celles } \varphi \colon s \Rightarrow g \mid s \in \mathcal{Z}\}$$

   The map $\Theta_{\mathcal{Z}}(g) \to \Theta_{\mathcal{Z}}(f \cdot g \cdot h)$ is as above.

3. We can take as ideal $\mathcal{Z}$ the class of all arrows in $\mathcal{B}$. The corresponding structure of nullhomotopies $\Theta_{\mathcal{B}}$ provides an example of a structure which does not satisfy the reduced interchange (Condition 2.1.2). Indeed, in this case the reduced interchange would imply that, for any arrow f, there exists a unique 2-cell from f to f.

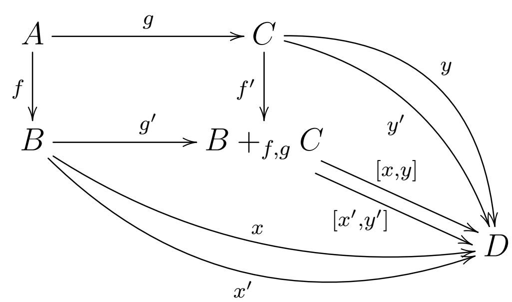
For a 2-functor between 2-categories, the general notion of 2-limit can be found in Chapter 7 of [6]. In this paper we need the following particular cases (and their duals).

2.2.2. Definition. Let $\mathcal{B}$ be a 2-category.

1. An initial object $\emptyset$ in $\mathcal{B}$ is 2-initial if it is $\Theta_{\mathcal{B}}$-strong: for every object $X \in \mathcal{B}$, there is a unique 2-cell on the unique arrow $\emptyset_X \colon \emptyset \to X$.

2. A zero object 0 in $\mathcal{B}$ is 2-zero if it is 2-initial and 2-terminal.

3. Consider a pushout diagram in $\mathcal{B}$ and two commutative squares $f \cdot x = g \cdot y$ and $f \cdot x' = g \cdot y'$ together with the corresponding factorizations

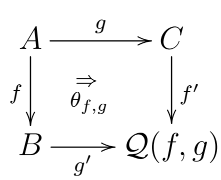


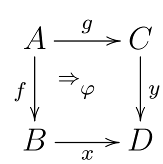
The pushout is a 2-pushout if it is $\Theta_{\mathcal{B}}$-strong: given 2-cells $\varphi \colon x \Rightarrow x'$ and $\psi \colon y \Rightarrow y'$ such that $f \circ \varphi = g \circ \psi$, there exists a unique 2-cell $[\varphi, \psi] \colon [x, y] \Rightarrow [x', y']$ such that $g' \circ [\varphi, \psi] = \varphi$ and $f' \circ [\varphi, \psi] = \psi$.

In a 2-category, it is possible to express also the notion of homotopy limit (H-limit, for short). We consider the case of H-pushouts in a 2-category with invertible 2-cells. Basically, the difference between 2-pushouts and H-pushouts is that a 2-pushout is also an ordinary pushout in the underlying category, whereas this is not the case for (strong) H-pushouts.

2.2.3. Definition. Let $\mathcal{B}$ be a 2-category with invertible 2-cells. Consider two arrows with same domain $f \colon A \to B$ and $g \colon A \to C$.

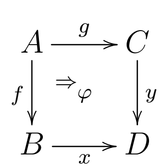
1. The diagram hereunder on the left is the H-pushout of f and g if, for every diagram like the one hereunder on the right

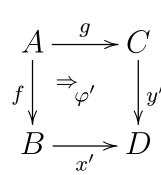




there exists a unique arrow $[x, \varphi, y] \colon \mathcal{Q}(f, g) \to D$ such that $g' \cdot [x, \varphi, y] = x$ and $f' \cdot [x, \varphi, y] = y$ and $\theta_{f,g} \circ [x, \varphi, y] = \varphi$.

2. The H-pushout of f and g is strong if, given

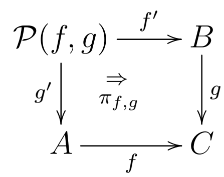




and 2-cells $\alpha\colon x \Rightarrow x'$ and $\beta\colon y \Rightarrow y'$ such that $\varphi \cdot (g \circ \beta) = (f \circ \alpha) \cdot \varphi'$, there exists a unique 2-cell $[\alpha, \beta]\colon [x, \varphi, y] \Rightarrow [x', \varphi', y']$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$.

2'. Equivalently, the H-pushout of f and g is strong if, given two parallel arrows $h, k\colon \mathcal{Q}(f, g) \rightrightarrows D$ and 2-cells $\alpha\colon g' \cdot h \Rightarrow g' \cdot k$ and $\beta\colon f' \cdot h \Rightarrow f' \cdot k$ such that $(\theta_{f,g} \circ h) \cdot (g \circ \beta) = (f \circ \alpha) \cdot (\theta_{f,g} \circ k)$, there exists a unique 2-cell $[\alpha, \beta]\colon h \Rightarrow k$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$.

3. The definition of (strong) H-pullback is dual. Here is the notation:



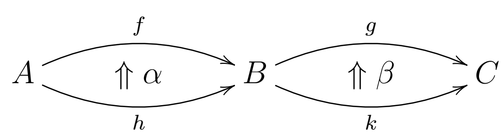
2.2.4. . A strong H-pushout satisfies the following cancellation property: given arrows $h, k\colon \mathcal{Q}(f, g) \rightrightarrows D$ and 2-cells $\lambda\colon h \Rightarrow k$ and $\mu\colon h \Rightarrow k$, if $g' \circ \lambda = g' \circ \mu$ and $f' \circ \lambda = f' \circ \mu$, then $\lambda = \mu$.

2.2.5. . What about strong H-(co)products in a 2-category $\mathcal{B}$ with invertible 2-cells?

1. If one writes the definition of strong H-initial object in the same spirit of the Definition 2.2.3 of strong H-pushout, since the base diagram is empty one recovers the definition of 2-initial object as given in point 1 of Definition 2.2.2. For the same reason, strong H-terminal is the same as 2-terminal and strong H-zero is the same as 2-zero.

2. As a consequence, if we define strong H-coproducts as strong H-pushouts of arrows out from the 2-initial objects, we get 2-coproducts. Dually, strong H-products are nothing but 2-products.

2.2.6. Remark. Consider once again a 2-category $\mathcal{B}$ with invertible 2-cells and with a 2-zero object 0. The reduced interchange (Condition 2.1.2) holds in $(\mathcal{B}, \Theta_0)$.
More precisely, in the situation



we have:

(a) $\alpha \circ 0^B_C = 0^A_C$ and $0^A_B \circ \beta = 0^A_C$,

(b) if $h = 0^A_B$ and $k = 0^B_C$, then (a) gives the absorption conditions of 2.1.3,

(c) if $h = 0^A_B$ and $k = 0^B_C$, then $\alpha \circ g = f \circ \beta$, that is Condition 2.1.2.

Proof. We prove the first of the two equations in (a), the proof of the second one is similar.

$$\alpha \circ 0^B_C = \alpha \circ (0^B \circ 0_C) = (\alpha \circ 0^B) \circ 0_C = 0^A \circ 0_C = 0^A_C$$

where the first and the last equality are because the horizontal composition of two identity 2-cells is the identity 2-cell on the composite arrow, the second equality is the associativity of the horizontal composition, and the third equality is because the object 0 is 2-terminal. Point (b) is just a special case of (a).
To prove (c), from (a) we have $\alpha \circ g = (0^A_B \circ \beta) \cdot (\alpha \circ g) = \alpha \circ \beta = (\alpha \circ 0^B_C) \cdot (f \circ \beta) = f \circ \beta$. ■

2.2.7. . It is worthwhile to write explicitly the definition of strong H-cokernel and strong H-kernel as special cases of Definition 2.2.3. The strong H-cokernel and the strong H-kernel of an arrow $f\colon A \to B$ are, respectively, the following strong H-pushout and strong H-pullback

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 0^A}\downarrow & \overset{\Rightarrow}{\gamma_f} & \downarrow{\scriptstyle c_f} \\ 0 & \xrightarrow[0_{\mathcal{C}(f)}]{} & \mathcal{C}(f) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{N}(f) & \xrightarrow{n_f} & A \\ {\scriptstyle 0^{\mathcal{N}(f)}}\downarrow & \overset{\Rightarrow}{\nu_f} & \downarrow{\scriptstyle f} \\ 0 & \xrightarrow[0_B]{} & B \end{array}$$

Let us do the job for the strong H-cokernel. Since a 2-zero object is, in particular, a zero object in the usual sense, when we make explicit the conditions of Definition 2.2.3 for the above strong H-pushout, we get the following conditions:

1. The diagram hereunder on the left is the H-cokernel of $f\colon A \to B$ if, for every diagram like the one hereunder on the right

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & \Uparrow \gamma_f & \searrow{\scriptstyle c_f} & \\ A & & \xrightarrow[0^A_{\mathcal{C}(f)}]{} & & \mathcal{C}(f) \end{array} \qquad\qquad \begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & \Uparrow \varphi & \searrow{\scriptstyle x} & \\ A & & \xrightarrow[0^A_D]{} & & D \end{array}$$

   there is a unique arrow $[x, \varphi]\colon \mathcal{C}(f) \to D$ such that $c_f \cdot [x, \varphi] = x$ and $\gamma_f \circ [x, \varphi] = \varphi$.

2. The H-cokernel of $f\colon A \to B$ is strong if, given

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & \Uparrow \varphi & \searrow{\scriptstyle x} & \\ A & & \xrightarrow[0^A_D]{} & & D \end{array} \qquad\qquad \begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & \Uparrow \varphi' & \searrow{\scriptstyle x'} & \\ A & & \xrightarrow[0^A_D]{} & & D \end{array}$$

   and a 2-cell $\alpha\colon x \Rightarrow x'$ such that $\varphi \cdot (f \circ \alpha) = \varphi'$, there exists a unique 2-cell $[\alpha]\colon [x, \varphi] \Rightarrow [x', \varphi']$ such that $c_f \circ [\alpha] = \alpha$.

2'. Equivalently, the H-cokernel of $f\colon A \to B$ is strong if, given arrows $h, k\colon \mathcal{C}(f) \rightrightarrows D$ and a 2-cell $\alpha\colon c_f \cdot h \Rightarrow c_f \cdot k$ such that $(\gamma_f \circ h) \cdot (f \circ \alpha) = \gamma_f \circ k$, there exists a unique 2-cell $[\alpha]\colon k \Rightarrow k$ such that $c_f \circ [\alpha] = \alpha$.

2.2.8. . Observe that, if in the above condition 2.2.7.2' we take $h = 0_D^{\mathcal{C}(f)}$, we recover the condition to be Θ_0-strong (see Example 2.2.1.1 for the structure Θ_0). To check this fact, we need the absorption rule $\gamma_f \circ 0_D^{\mathcal{C}(f)} = 0_D^A$ which is guaranteed by Remark 2.2.6.(a). In other words, the notion of strong H-cokernel coming from Definition 2.2.3 coincides with the one of strong Θ_0-cokernel coming from Definition 2.1.4 if the zero object is indeed a 2-zero object.

2.2.9. . The cancellation property of a strong H-cokernel is as follows: given parallel arrows $h, k \colon \mathcal{C}(f) \rightrightarrows D$ and 2-cells $\lambda \colon h \Rightarrow k$ and $\mu \colon h \Rightarrow k$, if $c_f \circ \lambda = c_f \circ \mu$, then $\lambda = \mu$.

2.3. Bicategories and bilimits. In this subsection, we adapt to bipushouts and bicokernels what we did in Definition 2.2.3 and in item 2.2.7 for H-pushouts and H-cokernels. We work in a bicategory with invertible 2-cells. Note that, thanks to coherence theorems for bicategories, see [36] or [30], here and in the rest of the paper we treat bicategories as 2-categories, that is, coherence isomorphisms will not be written explicitly.

2.3.1. Definition. Let $\mathcal{B}$ be a bicategory with invertible 2-cells. An object $\emptyset \in \mathcal{B}$ is biinitial if, for any other object $X \in \mathcal{B}$, there exists an arrow $x \colon \emptyset \to X$ and, moreover, for any pair of arrows $x, x' \colon \emptyset \rightrightarrows X$, there exists a unique 2-cell $\overline{x} \colon x \Rightarrow x'$.
The notion of biterminal object is dual. Bizero means biinitial and biterminal.

2.3.2. Definition. Let $\mathcal{B}$ be a bicategory with invertible 2-cells. Fix two arrows with same domain $f \colon A \to B$ and $g \colon A \to C$ in $\mathcal{B}$ and consider the following diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\;g\;} & C \\
{\scriptstyle f}\downarrow & \overset{\Rightarrow}{\theta_{f,g}} & \downarrow{\scriptstyle f'} \\
B & \xrightarrow[\;g'\;]{} & \mathcal{Q}(f,g)
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\;g\;} & C \\
{\scriptstyle f}\downarrow & \Rightarrow \varphi & \downarrow{\scriptstyle y} \\
B & \xrightarrow[\;x\;]{} & D
\end{array}
$$

The diagram on the left is a bipushout of f and g if it satisfies the two following conditions:

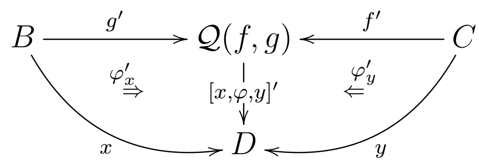
1. For any diagram like the one on the right, there exists a fill-in, that is, an arrow $[x, \varphi, y]$ and two 2-cells φ_x and φ_y as in

$$
\begin{array}{ccccc}
B & \xrightarrow{\;g'\;} & \mathcal{Q}(f,g) & \xleftarrow{\;f'\;} & C \\
 & \overset{\varphi_x}{\Rightarrow} & \downarrow{\scriptstyle [x,\varphi,y]} & \overset{\varphi_y}{\Leftarrow} & \\
 & \searrow^{x} & D & {}^{y}\swarrow &
\end{array}
$$

   such that the following diagram commutes

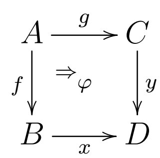
$$
\begin{array}{ccc}
f \cdot g' \cdot [x,\varphi,y] & \xRightarrow{\;\theta_{f,g} \circ [x,\varphi,y]\;} & g \cdot f' \cdot [x,\varphi,y] \\
{\scriptstyle f \circ \varphi_x}\Uparrow & & \Uparrow{\scriptstyle g \circ \varphi_y} \\
f \cdot x & \xRightarrow[\;\varphi\;]{} & g \cdot y
\end{array}
$$

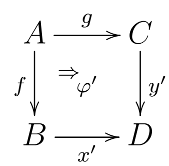
2. The fill-in is essentially unique: if



is another fill-in for the same diagram $\varphi: f \cdot x \Rightarrow g \cdot y$, then there exists a unique 2-cell $\overline{\varphi}: [x, \varphi, y] \Rightarrow [x, \varphi, y]'$ such that $\varphi_x \cdot (g' \circ \overline{\varphi}) = \varphi'_x$ and $\varphi_y \cdot (f' \circ \overline{\varphi}) = \varphi'_y$.

2'. Equivalently, given

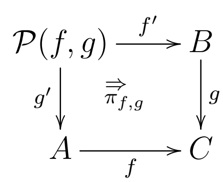




and 2-cells $\alpha: x \Rightarrow x'$ and $\beta: y \Rightarrow y'$ such that $\varphi \cdot (g \circ \beta) = (f \circ \alpha) \cdot \varphi'$, there exists a unique 2-cell $[\alpha, \beta]: [x, \varphi, y] \Rightarrow [x', \varphi', y']$ such that $\varphi_x \cdot (g' \circ [\alpha, \beta]) = \alpha \cdot \varphi_{x'}$ and $\varphi_y \cdot (f' \circ [\alpha, \beta]) = \beta \cdot \varphi_{y'}$.

2". Equivalently, given two arrows $h, k: \mathcal{Q}(f, g) \rightrightarrows D$ and 2-cells $\alpha: g' \cdot h \Rightarrow g' \cdot k$ and $\beta: f' \cdot h \Rightarrow f' \cdot k$ such that $(\theta_{f,g} \circ h) \cdot (g \circ \beta) = (f \circ \alpha) \cdot (\theta_{f,g} \circ k)$, there exists a unique 2-cell $[\alpha, \beta]: h \Rightarrow k$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$.

The notion of bipullback is dual of that of bipushout. Here is the notation for the bipullback of two arrows with same codomain $f: A \to C$ and $g: B \to C$ in $\mathcal{B}$:



2.3.3. . Comparing Definition 2.2.3 and Definition 2.3.2, we see that a strong H-pushout satisfies also the universal property of the bipushout. Clearly, the same holds also for strong H-pullbacks and bipullbacks. This fact will be exploited in the sequel in order to construct bi(co)limits in the bicategory of fractions starting from strong H-(co)limits in the base category. This also explains why we use the same notation for strong H-(co)limits and bi(co)limits.

2.3.4. . Some other comments on Definition 2.3.2:

1. The equivalence between conditions 2, 2' and 2" in Definition 2.3.2 is a particular case of a general argument: let $\mathbb{B}$ be a groupoid and $E: \mathbb{A} \to \mathbb{B}$ a functor essentially surjective on objects. Then E is fully faithful iff, for every object $B \in \mathbb{B}$ and for every choice of pre-images $(X, x: E(X) \to B), (Y, y: E(Y) \to B)$, there exists a unique arrow $f: X \to Y$ such that $x = E(f) \cdot y$.

2. The notions introduced in Definition 2.3.2 are relaxed if compared with the notion of bilimit given in Chapter 7 of [6]. To recover the definition of bilimit in [6], one should require that the 2-cell of the universal diagram is an identity. In fact, some authors would use the prefix pseudo instead of bi in Definition 2.3.2.

3. A bipushout has the same cancellation property with respect to 2-cells than a strong H-pushout (see item 2.2.4): given arrows $h, k\colon \mathcal{Q}(f,g) \rightrightarrows D$ and 2-cells $\lambda\colon h \Rightarrow k$ and $\mu\colon h \Rightarrow k$, if $g' \circ \lambda = g' \circ \mu$ and $f' \circ \lambda = f' \circ \mu$, then $\lambda = \mu$.

4. If $\mathcal{B}$ has a biinitial object or a bizero object, as special cases of bipushouts we get bicoproducts and bicokernels. Dually, we get biproducts (not to be confused with the bi-product $X \oplus Y$ in an additive category) and bikernels as special cases of bipullbacks. Note that different choices of biinitial, biterminal or bizero objects give rise to equivalent biuniversal constructions.

2.3.5. Remark. Despite what we have just said about the choice of a bizero object, the case of bikernels and bicokernels deserves a bit more of attention. Contrary to what happens with a 2-zero object, a bizero object does not satisfy the usual universal property of the zero object in the underlying category. Therefore, in order to express the universal property of the bicokernel in the same shape as the universal property of the strong H-cokernel (see item 2.2.7), we have to choose in a coherent way a bizero object 0 and, for every other object $X \in \mathcal{B}$, two arrows $0^X\colon X \to 0$ and $0_X\colon 0 \to X$. Here, coherent means just that $0^0 = \mathrm{id}_0 = 0_0$ and $0_X \cdot 0^X = \mathrm{id}_0$ for every object X. These choices determine:

- for every pair of objects X, Y in $\mathcal{B}$, a canonical arrow $0^X_Y = 0^X \cdot 0_Y\colon X \to 0 \to Y$,
- for every arrow $f\colon X \to Y$ in $\mathcal{B}$, unique 2-cells $0^f\colon 0^X \Rightarrow f \cdot 0^Y$ and $0_f\colon 0_Y \Rightarrow 0_X \cdot f$,
- for every arrow $f\colon X \to Y$ and object $Z \in \mathcal{B}$, canonical 2-cells $0^f \circ 0_Z\colon 0^X_Z \Rightarrow f \cdot 0^Y_Z$ and $0^Z \circ 0_f\colon 0^Z_Y \Rightarrow 0^Z_X \cdot f$.

Using the above convention, we can deduce some useful facts:

1. For every object X, we have $0^{\mathrm{id}_X} = 0^X = 0^X_0$ and $0_{\mathrm{id}_X} = 0_X = 0^0_X$.

2. For every pair of objects X and Y, we have $0^{0^X_Y} = 0^X$ and $0_{0^X_Y} = 0_Y$.

3. For every pair of composable arrows f and g, we have $0^{f \cdot g} = 0^f \cdot (f \circ 0^g)$ and $0_{f \cdot g} = 0_g \cdot (0_f \circ g)$.

4. For every pair of composable arrows f and g and for every object D, we have $0^{f \cdot g} \circ 0_D = (0^f \circ 0_D) \cdot (f \circ 0^g \circ 0_D)$ and $0^D \circ 0_{f \cdot g} = (0^D \circ 0_g) \cdot (0^D \circ 0_f \circ g)$.

5. For every 2-cell $A \underset{h}{\overset{f}{\rightrightarrows}} B$ ($\Uparrow \alpha$), we have $0^h \cdot (\alpha \circ 0^B) = 0^f$ and $0_h \cdot (0_A \circ \alpha) = 0_f$.

   In particular, when $h = 0^A_B$, we have $\alpha \circ 0^B = 0^f$ and $0_A \circ \alpha = 0_f$.

6. For every 2-cell $A \underset{h}{\overset{f}{\Uparrow \alpha}} B$ and for every object C, we have

   $0^f \circ 0_C = (0^h \circ 0_C) \cdot (\alpha \circ 0^B_C)$ and $0^C \circ 0_f = (0^C \circ 0_h) \cdot (0^C_A \circ \alpha)$. In particular, when $h = 0^A_B$, we have $0^f \circ 0_C = \alpha \circ 0^B_C$ and $0^C \circ 0_f = 0^C_A \circ \alpha$.

7. For every pair of 2-cells $A \underset{0^A_B}{\overset{f}{\Uparrow \alpha}} B \underset{0^B_C}{\overset{g}{\Uparrow \beta}} C$, we have

   $(0^A \circ 0_g) \cdot (\alpha \circ g) = (0^f \circ 0_C) \cdot (f \circ \beta)$.

Note that the equations in point 6 are the relaxed version of the absorption conditions of 2.1.3, and the equation in point 7 is the relaxed version of the reduced interchange of 2.1.2.

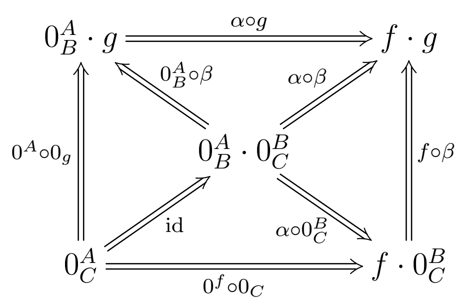
Proof. Points 1, 2 and 3 are obvious using that 0 is either biterminal or biinitial.
Point 4 follows from point 3 using distributivity:

$$0^{f \cdot g} \circ 0_D = (0^f \cdot (f \circ 0^g)) \circ 0_D = (0^f \circ 0_D) \cdot (f \circ 0^g \circ 0_D)$$

and similarly for the second equation.
Point 5 is obvious using that 0 is either biterminal or biinitial. The particular case follows from the general case using point 2.
Point 6 follows from point 5 using distributivity:

$$0^f \circ 0_C = (0^h \cdot (\alpha \circ 0^D)) \circ 0_C = (0^h \circ 0_C) \cdot (\alpha \circ 0^B \circ 0_C) = (0^h \circ 0_C) \cdot (\alpha \circ 0^B_C)$$

and similarly for the second equation. The particular case follows from the general case using point 2.
The proof of point 7 is depicted in the following commutative diagram, where we use both equations of point 6.



■

2.3.6. . Let $\mathcal{B}$ be a bicategory with invertible 2-cells. Assume that $\mathcal{B}$ has a bizero object. As we have pointed out in item 2.3.4.4, the bicokernel and the bikernel of an arrow $f\colon A \to B$ are defined, respectively, by the following bipushout and bipullback

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 0^A}\downarrow & \overset{\Rightarrow}{\gamma_f} & \downarrow{\scriptstyle c_f} \\ 0 & \xrightarrow[0_{\mathcal{C}(f)}]{} & \mathcal{C}(f) \end{array} \qquad \begin{array}{ccc} \mathcal{N}(f) & \xrightarrow{n_f} & A \\ {\scriptstyle 0^{\mathcal{N}(f)}}\downarrow & \overset{\Rightarrow}{\nu_f} & \downarrow{\scriptstyle f} \\ 0 & \xrightarrow[0_B]{} & B \end{array}$$

We can assume that a coherent choice of a bizero object 0 has been made and, using Remark 2.3.5, we can make explicit the conditions defining the bicokernel (those for the bikernel are dual). Consider the following diagrams:

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & \Uparrow \gamma_f & \searrow{\scriptstyle c_f} & \\ A & & \xrightarrow[0^A_{\mathcal{C}(f)}]{} & & \mathcal{C}(f) \end{array} \qquad \begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & \Uparrow \varphi & \searrow{\scriptstyle x} & \\ A & & \xrightarrow[0^A_D]{} & & D \end{array}$$

The diagram on the left is a bicokernel of f if it satisfies the two following conditions:

1. For any diagram like the one on the right, there exists a fill-in, that is, an arrow $[x, \varphi]$ and a 2-cell φ_x as in

$$\begin{array}{ccccc} & & \mathcal{C}(f) & & \\ & {\scriptstyle c_f}\nearrow & \Uparrow \varphi_x & \searrow{\scriptstyle [x,\varphi]} & \\ B & & \xrightarrow[x]{} & & D \end{array}$$

   such that the following diagram commutes

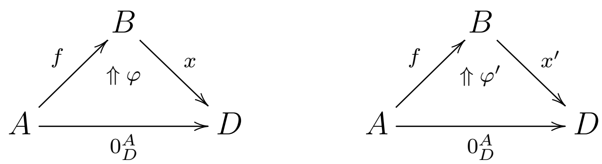
$$\begin{array}{ccc} f \cdot c_f \cdot [x,\varphi] & \xLeftarrow{f \circ \varphi_x} & f \cdot x \\ {\scriptstyle \gamma_f \circ [x,\varphi]}\Uparrow & & \Uparrow{\scriptstyle \varphi} \\ 0^A_{\mathcal{C}(f)} \cdot [x,\varphi] & \xLeftarrow[0^A \circ 0_{[x,\varphi]}]{} & 0^A_D \end{array}$$

2. The fill-in is essentially unique: if

$$\begin{array}{ccccc} & & \mathcal{C}(f) & & \\ & {\scriptstyle c_f}\nearrow & \Uparrow \varphi'_x & \searrow{\scriptstyle [x,\varphi]'} & \\ B & & \xrightarrow[x]{} & & D \end{array}$$

   is another fill-in for the same diagram $\varphi\colon 0^A_D \Rightarrow f \cdot x$, then there exists a unique 2-cell $\overline{\varphi}\colon [x, \varphi] \Rightarrow [x, \varphi]'$ such that $\varphi_x \cdot (c_f \circ \overline{\varphi}) = \varphi'_x$.

2'. Equivalently, given

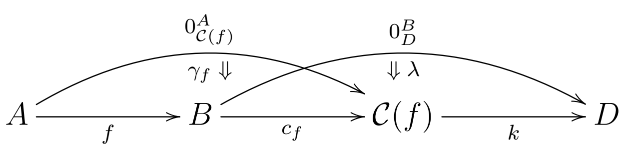


and a 2-cell $\alpha\colon x \Rightarrow x'$ such that $\varphi \cdot (f \circ \alpha) = \varphi'$, there exists a unique 2-cell $[\alpha]\colon [x,\varphi] \Rightarrow [x',\varphi']$ such that $\varphi_x \cdot (c_f \circ [\alpha]) = \alpha \cdot \varphi_{x'}$.

2". Equivalently, given two arrows $h, k\colon \mathcal{C}(f) \rightrightarrows D$ and a 2-cell $\alpha\colon c_f \cdot h \Rightarrow c_f \cdot k$ such that $(0^A \circ 0_h) \cdot (\gamma_f \circ h) \cdot (f \circ \alpha) = (0^A \circ 0_k) \cdot (\gamma_f \circ k)$, there exists a unique 2-cell $[\alpha]\colon h \Rightarrow k$ such that $c_f \circ [\alpha] = \alpha$.

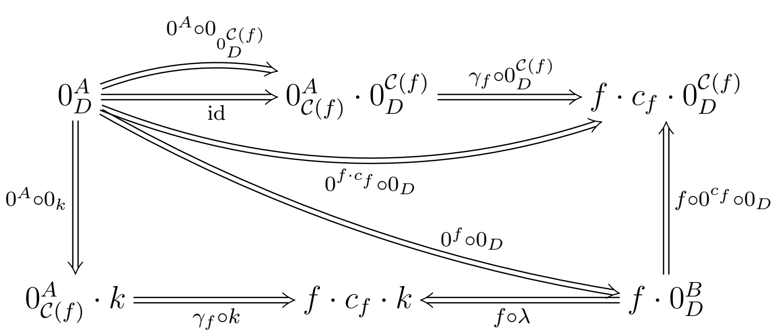
2.3.7. . A bicokernel has the same cancellation property with respect to 2-cells than a strong H-cokernel (see item 2.2.9): given arrows $h, k\colon \mathcal{C}(f) \rightrightarrows D$ and 2-cells $\lambda\colon h \Rightarrow k$ and $\mu\colon h \Rightarrow k$, if $c_f \circ \lambda = c_f \circ \mu$, then $\lambda = \mu$.

2.3.8. Remark. Later on, we will need that bicokernels satisfy a relaxed version of the condition 2 in Definition 2.1.4. Here it is: in the situation



if $(0^A \circ 0_k) \cdot (\gamma_f \circ k) = (0^f \circ 0_D) \cdot (f \circ \lambda)$, then there exists a unique 2-cell $[\lambda]\colon 0_D^{\mathcal{C}(f)} \Rightarrow k$ such that $(0^{c_f} \circ 0_D) \cdot (c_f \circ [\lambda]) = \lambda$.

Proof. Let us see how the above condition follows from condition 2" in item 2.3.6. In 2", put $h = 0_D^{\mathcal{C}(f)}$ and $\alpha = (0^{c_f} \circ 0_D)^{-1} \cdot \lambda$. We have to show that α satisfies the condition in 2", which amounts to the commutativity of the following diagram:



To check that the four regions commute use, from the top to the bottom, point 2, point 6 and point 4 of Remark 2.3.5 and the assumption on λ. ∎

2.3.9. Example. To end this section, we give an example which fits into the situation discussed in Remark 2.3.5. The example is provided by the 2-category $\mathbf{Grpd}_*$ of pointed groupoids, lax pointed functors and pointed natural transformations. Objects are pairs $(\mathbb{A}, I)$ with I a chosen object of the groupoid $\mathbb{A}$. Arrows are pairs $(F, F_0)\colon (\mathbb{A}, I) \to (\mathbb{B}, I)$ with $F\colon \mathbb{A} \to \mathbb{B}$ a functor and $F_0\colon I \to F(I)$ a chosen arrow. The composition of (F, F_0) and (G, G_0) is $(F \cdot G, G_0 \cdot G(F_0))$. The 2-cells $\alpha\colon (H, H_0) \Rightarrow (F, F_0)$ are pointed natural transformations, that is, natural transformations $\alpha\colon H \Rightarrow F$ satisfying the condition $H_0 \cdot \alpha_I = F_0$. Despite the fact that $\mathbf{Grpd}_*$ does not have a zero object (indeed, the one arrow groupoid is bizero and terminal, but not initial), in $\mathbf{Grpd}_*$ there is a canonical arrow $(0, \mathrm{id}_I)\colon (\mathbb{A}, I) \to (\mathbb{B}, I)$, where $0\colon \mathbb{A} \to \mathbb{B}$ is the constant functor with value I.
Categorical groups and categorical crossed modules are special kinds of pointed groupoids and they fit into the same pattern. This is the reason why, when working with categorical groups and categorical crossed modules, one can adopt the version of bikernels and bicokernels described in item 2.3.6, see for example [50, 18, 11].

3. Fractions

3.1. Generalities on the bicategory of fractions. In this section, we recall and complete some general facts on bicategories of fractions, a tool introduced by Dorette Pronk in [43] (see also [48]).

3.1.1. . If $\mathcal{B}$ is a bicategory with invertible 2-cells and if Σ is a class of arrows in $\mathcal{B}$, the bicatgeory of fractions of $\mathcal{B}$ with respect to Σ is the universal solution to the problem of turning each element of Σ into an equivalence. The current notation is

$$\mathcal{P}_\Sigma\colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

This means that $\mathcal{P}_\Sigma(w)$ is an equivalence for any $w \in \Sigma$ and that any other morphism of bicategories sharing such a property factorizes through $\mathcal{P}_\Sigma$ in an essentially unique way.
As it is already the case for categories of fractions (see [21] or Chapter 5 in [6]), the description of $\mathcal{B}[\Sigma^{-1}]$ can be quite complicated, but it simplifies drastically if the class Σ has some good properties. For the purpose of this paper, the main result from [43] is an explicit description of the bicategory of fractions under the assumption that Σ has a right calculus of fractions. We are not going to recall the definition of right calculus of fractions because the example we are interested in satisfies a stronger condition introduced in [51] under the name of bipullback congruence (see [5] for the original categorical version).

3.1.2. Remark. The condition on $\mathcal{B}$ that all 2-cells are invertible does not appear in [43]. This assumption on 2-cells makes a bit easier the construction of the bicategory of fractions partially recalled in 3.1.5 and is part of the definition of 2-abelian bicategory (Definition 7.1.5). The price to pay is Lemma 3.1.8, which ensures that 2-cells in the bicategory of fractions are invertible if they are invertible in the original bicategory. Lemma 3.2.1 holds without the extra assumption on 2-cells and the same is true for Lemma 3.2.2, even if the proof is to be adapted. What is less clear to me is what happens with some other notions of higher dimensional limits when we pass from a general bicategory to the bicategory of fractions.

3.1.3. DEFINITION. Let $\mathcal{B}$ be a bicategory with invertible 2-cells and Σ a class of arrows in $\mathcal{B}$. We say that Σ is a bipullback congruence if the following conditions are satisfied:

1) Σ contains the equivalences.

2) If there exists a 2-cell $f \Rightarrow g$, then $f \in \Sigma$ iff $g \in \Sigma$.

3) If two of f, g and $f \cdot g$ are in Σ, then the third one also is in Σ.

4) Σ is stable under bipullbacks.

We say that Σ is a bipushout congruence if it satisfies conditions 1), 2) and 3) above and

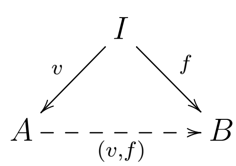
4') Σ is stable under bipushouts.

3.1.4. . In [51] is has been proved that, if $\mathcal{B}$ has the needed bipullbacks, then any bipullback congruence has a right calculus of fractions. Dually, if $\mathcal{B}$ has the needed bipushouts, then any bipushout congruence has a left calculus of fractions.

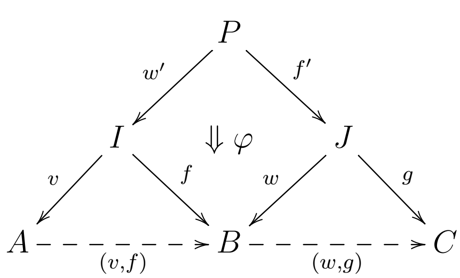
3.1.5. . Here is (part of) the explicit description of $\mathcal{P}^r_\Sigma \colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ from [43]. We assume that Σ has a right calculus of fractions. (If we assume that $\mathcal{B}$ has bipullbacks and that Σ is a bipullback congruence, the description of the bicategory of fractions does not change, but the computation of the vertical composition of 2-cells is a bit easier than in [43].) We add an r (for right) to the exponent because we will need also the left version.

1. The objects of $\mathcal{B}^r[\Sigma^{-1}]$ are those of $\mathcal{B}$.

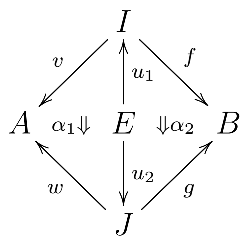
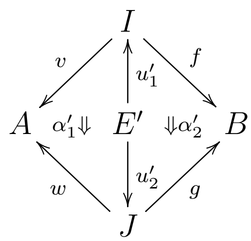
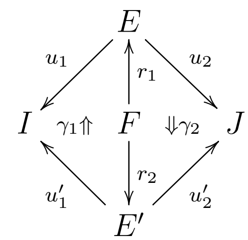
2. An arrow $(v, f) \colon A \to B$ in $\mathcal{B}^r[\Sigma^{-1}]$ is a span with $v \in \Sigma$ and f an arbitrary arrow in $\mathcal{B}$:



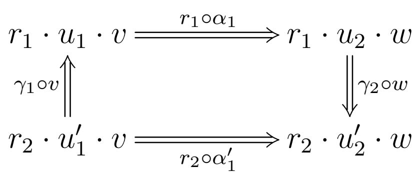
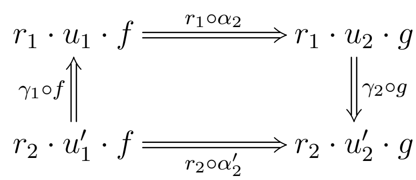
3. Composition of arrows in $\mathcal{B}^r[\Sigma^{-1}]$ is depicted in the following diagram, where the existence of the square with $w' \in \Sigma$ comes from the right calculus of fractions (one can chose as P any bipullback of f and w if Σ is a bipullback congruence)



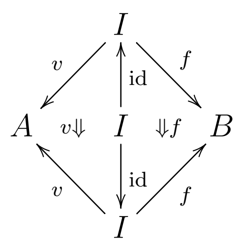
4. A 2-cell $(v,f) \Rightarrow (w,g)$ in $\mathcal{B}^r[\Sigma^{-1}]$ is an equivalence class of 4-tuples $(u_1,u_2,\alpha_1,\alpha_2)$ as in the diagram hereunder on the left, with $u_1 \cdot v \in \Sigma$:

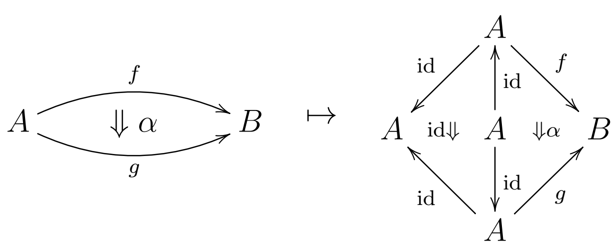
Two 4-tuples $(u_1,u_2,\alpha_1,\alpha_2)$ and $(u'_1,u'_2,\alpha'_1,\alpha'_2)$ are equivalent if there exists a 4-tuple $(r_1,r_2,\gamma_1,\gamma_2)$ as above on the right, with $r_1 \cdot u_1 \cdot v \in \Sigma$ and such that the following diagrams commute

5. The identity 2-cell is represented by



6. $\mathcal{P}^r_\Sigma \colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ can be defined as follows, since Σ contains the equivalences:



7. An arrow $(v,f) \colon A \to B$ in $\mathcal{B}^r[\Sigma^{-1}]$ with both v and f in Σ is an equivalence with quasi-inverse $(f,v) \colon B \to A$. In particular, for any $v \in \Sigma$, $\mathcal{P}^r_\Sigma(v)$ is an equivalence with quasi-inverse (v,id). It follows that, for any arrow (v,f) in $\mathcal{B}^r[\Sigma^{-1}]$, we have

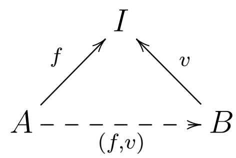
$$(v,f) \simeq \mathcal{P}^r_\Sigma(v)^{-1} \cdot \mathcal{P}^r_\Sigma(f)$$

8. Finally, the extension $\overline{\mathcal{F}} \colon \mathcal{B}^r[\Sigma^{-1}] \to \mathcal{D}$ of $\mathcal{F} \colon \mathcal{B} \to \mathcal{D}$ along $\mathcal{P}^r_\Sigma$ is defined, on objects and arrows, by

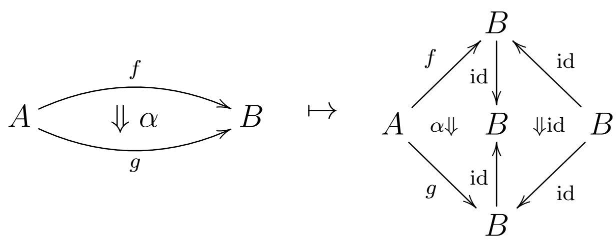
$$\overline{\mathcal{F}}\,(A \xrightarrow{(v,f)} B) \;=\; \mathcal{F}A \xrightarrow{\mathcal{F}(v)^{-1}} \mathcal{F}C \xrightarrow{\mathcal{F}(f)} \mathcal{F}B$$

3.1.6. . Without entering into details, here is what we are going to use if the class Σ has a left calculus of fractions or is a bipushout congruence:

1. Objects of $\mathcal{B}^l[\Sigma^{-1}]$ are those of $\mathcal{B}$.
2. An arrow $(f, v)\colon A \to B$ in $\mathcal{B}^l[\Sigma^{-1}]$ is a cospan with $v \in \Sigma$ and f an arbitrary arrow in $\mathcal{B}$:



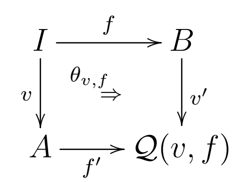
3. $\mathcal{P}^l_\Sigma\colon \mathcal{B} \to \mathcal{B}^l[\Sigma^{-1}]$ can be defined as follows:



3.1.7. . Observe that $\mathcal{P}^r_\Sigma\colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ and $\mathcal{P}^l_\Sigma\colon \mathcal{B} \to \mathcal{B}^l[\Sigma^{-1}]$ are two different descriptions of the universal solution of the same problem. As a consequence, if Σ has a right calculus of fractions and a left calculus of fractions, there is a biequivalence of bicategories $\mathcal{H}\colon \mathcal{B}^r[\Sigma^{-1}] \to \mathcal{B}^l[\Sigma^{-1}]$ commuting with $\mathcal{P}^r_\Sigma$ and $\mathcal{P}^l_\Sigma$. Passing through 3.1.1 and 3.1.6, the biequivalence $\mathcal{H}$ can be described, on objects and arrows, by

$$\mathcal{H}\,(A \xleftarrow{\;v\;} I \xrightarrow{\;f\;} B) \;=\; A \xrightarrow{\;f'\;} \mathcal{Q}(v,f) \xleftarrow{\;v'\;} B$$

where we can use the following pushout if Σ is a pushout congruence

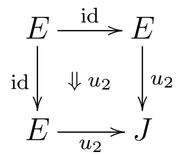


In Section 7, we will need the second part of the following simple lemma.

3.1.8. Lemma. *Let $\mathcal{B}$ be a bicategory with invertible 2-cells and Σ a class of arrows with a right calculus of fractions.*

1. *The 2-cells of $\mathcal{B}^r[\Sigma^{-1}]$ represented by $(u_1, u_2, \alpha_1, \alpha_2)$ and $(s \cdot u_1, s \cdot u_2, s \circ \alpha_1, s \circ \alpha_2)$, where s is any arrow in Σ whose codomain is the domain of u_1 and u_2, are equal.*
2. *The 2-cells of $\mathcal{B}^r[\Sigma^{-1}]$ are invertible. Indeed, the inverse of a 2-cell represented by $(u_1, u_2, \alpha_1, \alpha_2)$ is represented by $(u_2, u_1, \alpha_1^{-1}, \alpha_2^{-1})$.*

PROOF. 1. The equivalence between the two representatives is attested by the 4-tuple $(s, \mathrm{id}, s \cdot u_1, s \cdot u_2)$.
2. To compute the vertical composition of $(u_1, u_2, \alpha_1, \alpha_2)$ and $(u_2, u_1, \alpha_1^{-1}, \alpha_2^{-1})$ we can choose the diagram



and we get the 4-tuple $(u_1, u_1, \alpha_1 \cdot \alpha_1^{-1} = u_1 \cdot v, \alpha_2 \cdot \alpha_2^{-1} = u_1 \cdot f)$. By the first part of the Lemma, this 4-tuple represents the identity 2-cell. ∎

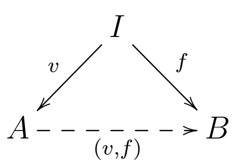
3.2. BIPULLBACKS AND BIPUSHOUTS IN THE BICATEGORY OF FRACTIONS. We recall here the main result from [27] together with the constructive part of the proof, which will be needed later.

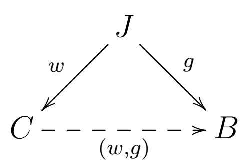
3.2.1. LEMMA. *Let $\mathcal{B}$ be a bicategory with invertible 2-cells and Σ a class of arrows in $\mathcal{B}$.*

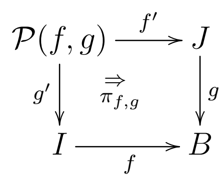
1. *If $\mathcal{B}$ has bipullbacks and if Σ has a right calculus of fractions, then $\mathcal{B}^r[\Sigma^{-1}]$ has bipullbacks and $\mathcal{P}^r_\Sigma \colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ preserves bipullbacks.*

2. *If $\mathcal{B}$ has bipushouts and if Σ has a left calculus of fractions, then $\mathcal{B}^l[\Sigma^{-1}]$ has bipushouts and $\mathcal{P}^l_\Sigma \colon \mathcal{B} \to \mathcal{B}^l[\Sigma^{-1}]$ preserves bipushouts.*

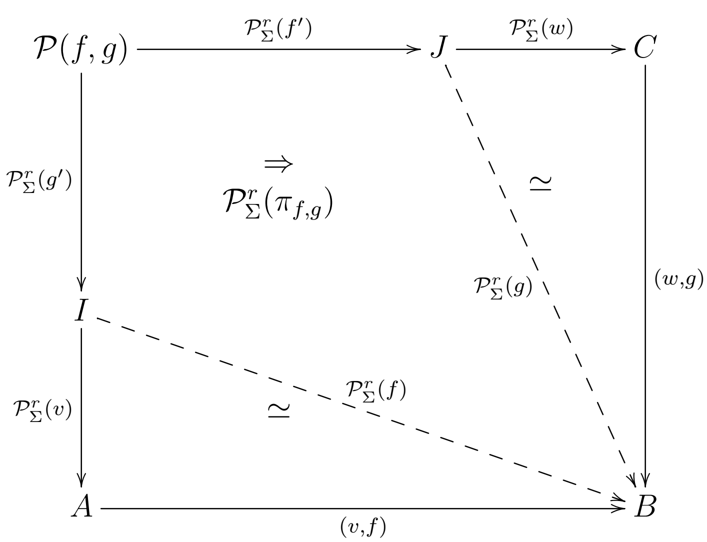
PROOF. 1. Start with two arrows in $\mathcal{B}^r[\Sigma^{-1}]$ and consider the bipullback in $\mathcal{B}$:







The bipullback in $\mathcal{B}^r[\Sigma^{-1}]$ is given by



■

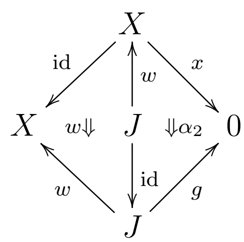
And here is the analogous statement for biterminal and biinitial objects, a simple result which has been "forgotten" in [27].

3.2.2. LEMMA. *Let $\mathcal{B}$ be a bicategory with invertible 2-cells and Σ a class of arrows in $\mathcal{B}$.*

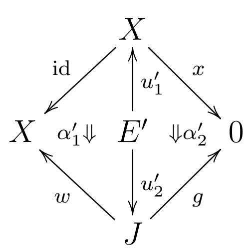
1. *Assume that Σ has a right calculus of fractions. If $\mathcal{B}$ has a biterminal object or a biinitial object, then $\mathcal{P}^r_\Sigma$ preserves them.*

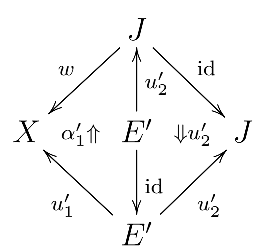
2. *Assume that Σ has a left calculus of fractions. If $\mathcal{B}$ has a biterminal object or a biinitial object, then $\mathcal{P}^l_\Sigma$ preserves them.*

PROOF. 1. Let 0 be a biterminal object in $\mathcal{B}$ and fix an object $X \in \mathcal{B}$. There exists an arrow $x\colon X \to 0$ in $\mathcal{B}$, so that we get an arrow $\mathcal{P}^r_\Sigma(x)\colon X \to 0$ in $\mathcal{B}^r[\Sigma^{-1}]$. Let now $(w, g)\colon X \to 0$ be another arrow in $\mathcal{B}^r[\Sigma^{-1}]$. We get a 2-cell $\mathcal{P}^r_\Sigma(x) \Rightarrow (w, g)$ represented by

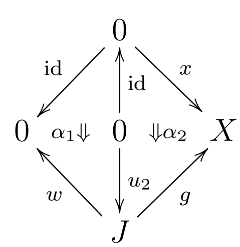


where $\alpha_2\colon w \cdot x \Rightarrow g$ is the unique 2-cell produced by the fact that 0 is biterminal in $\mathcal{B}$. Given another 2-cell $\mathcal{P}^r_\Sigma(x) \Rightarrow (w, g)$ represented by the diagram hereunder on the left, the equality of the two 2-celles is attested by the diagram hereunder on the right

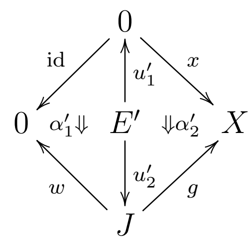


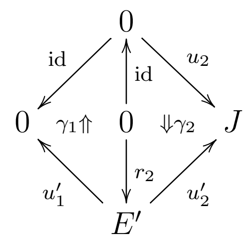


Indeed, $u'_2 \cdot w \in \Sigma$, one of the conditions on 2-cells at the end of 3.1.1.4 reduces to $\alpha'_1 = \alpha'_1$ and the other one is automatically satisfied because 0 is biterminal in $\mathcal{B}$.
Let 0 be a biinitial object in $\mathcal{B}$ and fix an object $X \in \mathcal{B}$. There exists an arrow $x\colon 0 \to X$ in $\mathcal{B}$, so that we get an arrow $\mathcal{P}^r_\Sigma(x)\colon 0 \to X$ in $\mathcal{B}^r[\Sigma^{-1}]$. Let now $(w, g)\colon 0 \to X$ be another arrow in $\mathcal{B}^r[\Sigma^{-1}]$. We get a 2-cell $\mathcal{P}^r_\Sigma(x) \Rightarrow (w, g)$ represented by



where the arrow u_2 and the (unique) 2-cells α_1 and α_2 are produced by the fact that 0 is biinitial in $\mathcal{B}$. Given another 2-cell $\mathcal{P}^r_\Sigma(x) \Rightarrow (w, g)$ represented by the diagram hereunder on the left, the equality of the two 2-celles is attested by the diagram hereunder on the right





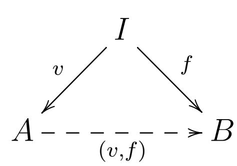
where the arrow r_2 and the (unique) 2-cells γ_1 and γ_2 are produced by the fact that 0 is biinitial in $\mathcal{B}$. Indeed, both the conditions on 2-cells at the end of 3.1.1.4 are automatically satisfied because 0 is biinitial in $\mathcal{B}$. ■

3.2.3. Corollary. *Let $\mathcal{B}$ be a bicategory with invertible 2-cells. Assume that $\mathcal{B}$ has bipullbacks, bipushouts and a bizero object. Let Σ be a class of arrows in $\mathcal{B}$ having a right calculus of fractions and a left calculus of fractions.*

1. *$\mathcal{B}^r[\Sigma^{-1}]$ has bipullbacks, bipushouts and a bizero object. Moreover, $\mathcal{P}^r_\Sigma \colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ preserves bipullbacks, bipushouts and the bizero object.*
2. *$\mathcal{B}^l[\Sigma^{-1}]$ has bipullbacks, bipushouts and a bizero object. Moreover, $\mathcal{P}^l_\Sigma \colon \mathcal{B} \to \mathcal{B}^l[\Sigma^{-1}]$ preserves bipullbacks, bipushouts and the bizero object.*

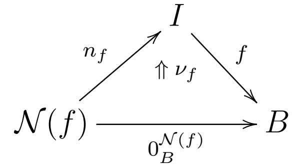
Proof. Everything follows from Lemma 3.2.1, Lemma 3.2.2 and the fact that $\mathcal{P}^r_\Sigma$ and $\mathcal{P}^l_\Sigma$ are connected by a biequivalence (see 3.1.7). ■

3.2.4. . For later use, let us describe explicitly the bikernel and the bicokernel of an arrow

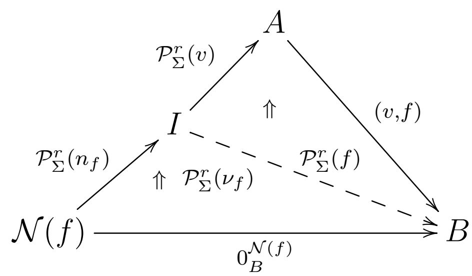


in $\mathcal{B}^r[\Sigma^{-1}]$. We assume that a choice of a bizero object 0 and arrows 0^X and 0_X has been done, as discussed in Remark 2.3.5.

1. Assume that $\mathcal{B}$ has bikernels and that Σ has a right calculus of fractions. Consider a bikernel of f in $\mathcal{B}$:

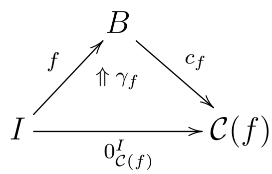


Since, by Lemma 3.2.1.1 and Lemma 3.2.2.1, $\mathcal{P}^r_\Sigma\colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ preserves bikernels, and bikernels are determined up to equivalence, a bikernel in $\mathcal{B}^r[\Sigma^{-1}]$ of (v, f) is given by

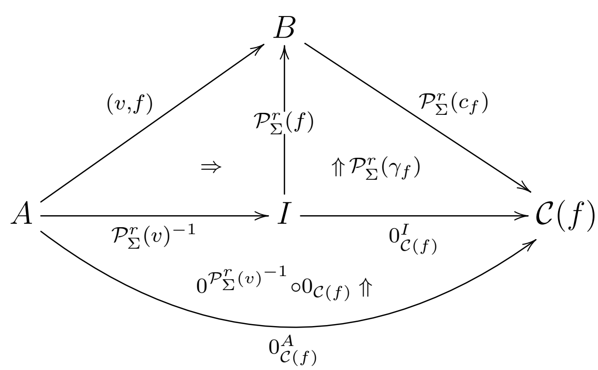


(which, in fact, is a special case of the construction in the proof of Lemma 3.2.1).

2. Assume that $\mathcal{B}$ has bicokernels and that Σ has a right calculus of fractions and a left calculus of fractions. Consider a bicokernel of f in $\mathcal{B}$:



Since, by Lemma 3.2.1.2 and Lemma 3.2.2.2, $\mathcal{P}^r_\Sigma\colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ preserves bicokernels, and bicokernels are determined up to equivalence, and keeping in mind the biequivalence $\mathcal{B}^r[\Sigma^{-1}] \simeq \mathcal{B}^l[\Sigma^{-1}]$ of 3.1.7, a bicokernel in $\mathcal{B}^r[\Sigma^{-1}]$ of (v, f) is given by

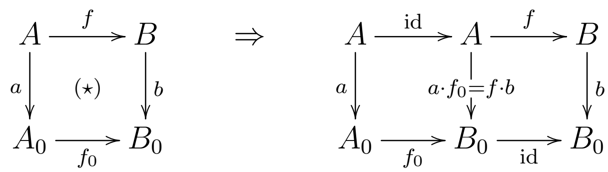


3. Observe that, if we call $\psi\colon P^r_\Sigma(f) \Rightarrow P^r_\Sigma(v) \cdot (v, f)$ and $\varphi\colon (v, f) \Rightarrow P^r_\Sigma(v)^{-1} \cdot P^r_\Sigma(f)$ the unlabelled 2-cells involved in the previous description of the bikernel and of the bicokernel, then ψ and φ are related by $\psi = (\alpha \circ P^r_\Sigma(f)) \cdot (P^r_\Sigma(v) \circ \varphi^{-1})$, where $\alpha\colon \mathrm{id}_I \Rightarrow P^r_\Sigma(v) \cdot P^r_\Sigma(v)^{-1}$ is a 2-cell attesting that $P^r_\Sigma(v)$ and $P^r_\Sigma(v)^{-1}$ are quasi-inverse equivalences.

4. Factorization systems in $\mathbf{Arr}(\mathcal{A})$

4.1. Two-step factorizations.

4.1.1. . From [35], we known that, with no assumption on $\mathcal{A}$, always exists a free orthogonal factorization system in $\mathbf{Arr}(\mathcal{A})$. Here is the factorization of an arrow:



The classes of arrows which determine this factorization can be described as:

$$\mathcal{E}_0 = \{(f, f_0) \mid f \text{ is an isomorhism } \} \quad , \quad \mathcal{M}_0 = \{(f, f_0) \mid f_0 \text{ is an isomorphism } \}$$

If $\mathcal{A}$ has pushouts or pullbacks, the classes $\mathcal{E}_0$ and $\mathcal{M}_0$ enter in two other orthogonal factorization systems. In this section, we discuss these factorization systems and their relation with strong Θ_Δ-kernels and strong Θ_Δ-cokernels.

4.1.2. . First factorization system. We take as classes of arrows the following classes:

- $\mathcal{E}_1 = \{(f, f_0) \mid (\star) \text{ is a pushout } \}$
- $\mathcal{M}_1 = \mathcal{E}_0 = \{(f, f_0) \mid f \text{ is an isomorphism } \}$

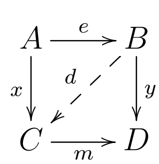
4.1.3. . Second factorization system. We take as classes of arrows the following classes:

- $\mathcal{E}_2 = \mathcal{M}_0 = \{(f, f_0) \mid f_0 \text{ is an isomorphism } \}$
- $\mathcal{M}_2 = \{(f, f_0) \mid (\star) \text{ is a pullback } \}$

We recall the definition of orthogonal factorization system as it appears in [24] (see also Chapter 5 in [6]).

4.1.4. Definition. Two classes $(\mathcal{E}, \mathcal{M})$ of arrows in a category $\mathcal{A}$ are an orthogonal factorization system if they satisfy the following conditions:

1) Both classes are stable under composition with isomorphisms.
2) Each arrow of $\mathcal{A}$ can be factorized as an arrow in $\mathcal{E}$ followed by an arrow in $\mathcal{M}$.
3) Orthogonality: for each solid commutative diagram with $e \in \mathcal{E}$ and $m \in \mathcal{M}$



there exists a unique arrow d such that $e \cdot d = x$ and $d \cdot m = y$.

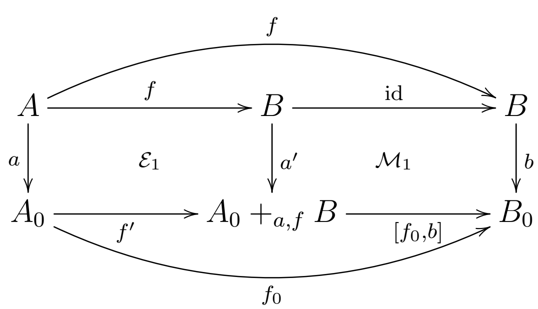
4.1.5. . For later use, let us recall that, if $(\mathcal{E}, \mathcal{M})$ is an orthogonal factorization system, then:

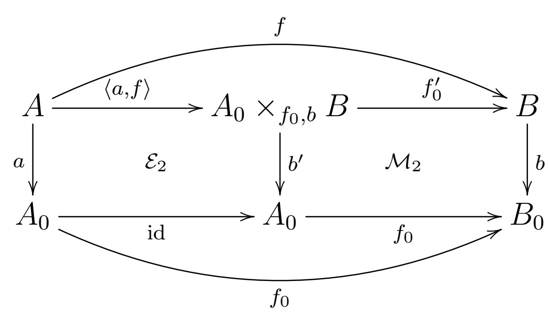
1. $\mathcal{E}$ and $\mathcal{M}$ contain identites and are closed under composition,
2. if $f \cdot g$ and f are in $\mathcal{E}$, then g is in $\mathcal{E}$,
3. if $f \cdot g$ and g are in $\mathcal{M}$, then f is in $\mathcal{M}$.

4.1.6. Proposition. *Let $\mathcal{A}$ be a category.*

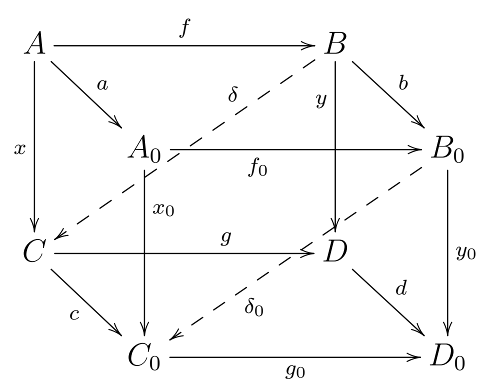
1. *If $\mathcal{A}$ has pushouts, then $(\mathcal{E}_1, \mathcal{M}_1)$ is an orthogonal factorization system in* $\mathbf{Arr}(\mathcal{A})$.
2. *If $\mathcal{A}$ has pullbacks, then $(\mathcal{E}_2, \mathcal{M}_2)$ is an orthogonal factorization system in* $\mathbf{Arr}(\mathcal{A})$.

Proof. The first condition in Definition 4.1.4 is easy to check for the four classes. As far as the $(\mathcal{E}_1, \mathcal{M}_1)$ and the $(\mathcal{E}_2, \mathcal{M}_2)$ factorizations of an arrow $(f, f_0): (A, a, A_0) \to (B, b, B_0)$ are concerned, they are obtained from the factorizations of the commutative square $f \cdot b = a \cdot f_0$ through, respectively, the pushout and the pullback as in the following diagrams:





We check the orthogonality condition for $(\mathcal{E}_1, \mathcal{M}_1)$, that for $(\mathcal{E}_2, \mathcal{M}_2)$ is dual and we omit it. Consider the following solid commutative square in $\mathbf{Arr}(\mathcal{A})$, where the top face is a pushout and g is an isomorphism:



We look for the dashed arrows δ and δ_0 which have to satisfy the following conditions:

$$\delta \cdot g = y \ , \quad f \cdot \delta = x \ , \quad \delta_0 \cdot g_0 = y_0 \ , \quad f_0 \cdot \delta_0 = x_0 \ , \quad \delta \cdot c = b \cdot \delta_0$$

Since g is an isomorphism, from the fist condition we get necessarily $\delta = y \cdot g^{-1}$. From the fourth and the fifth conditions, we get $f \cdot \delta \cdot c = a \cdot x_0$. Since the top face is a pushout, this implies that there exists a unique δ_0 such that $b \cdot \delta_0 = \delta \cdot c$ and $f_0 \cdot \delta_0 = x_0$. It remains to verify the second and the third conditions: for the second one, compose with g; for the third one, precompose with b and f_0. ∎

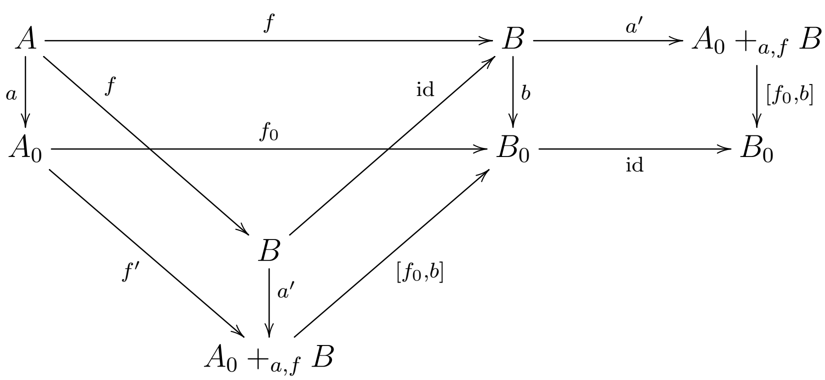
4.1.7. . Recall that an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ is proper if the arrows of $\mathcal{E}$ are epis and the arrows of $\mathcal{M}$ are monos. The factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ in $\mathbf{Arr}(\mathcal{A})$ are not proper, but they are "nullhomotopically proper" in the following sense (here is the case of $(\mathcal{E}_1, \mathcal{M}_1)$, the case of $(\mathcal{E}_2, \mathcal{M}_2)$ is dual):

1. Given two composable arrows $(A, a, A_0) \xrightarrow{(f,f_\ddagger)} (B, b, B_0) \xrightarrow{(g,g_0)} (C, c, C_0)$ and a nullhomotopy $\varphi \in \Theta_\Delta((f, f_0) \cdot (g, g_0))$, if $(f, f_0) \in \mathcal{E}_1$, then there exists a unique nullhomotopy $\varphi' \in \Theta_\Delta(g, g_0)$ such that $(f, f_0) \circ \varphi' = \varphi$. In other words, the arrows in $\mathcal{E}_1$ are an example of what we could call strong Θ_Δ-epimorphisms.

2. Given two arrows $(A, a, A_0) \xrightarrow{(f,f_\ddagger)} (B, b, B_0) \xrightarrow{(g,g_0)} (C, c, C_0)$ and two nullhomotopies $\varphi, \psi \in \Theta_\Delta((f, f_0))$, if $\varphi \circ (g, g_0) = \psi \circ (g, g_0)$ and if $(g, g_0) \in \mathcal{M}_1$, then $\varphi = \psi$. In other words, the arrows in $\mathcal{M}_1$ are (not necessarily strong) Θ_Δ-monomorphisms.

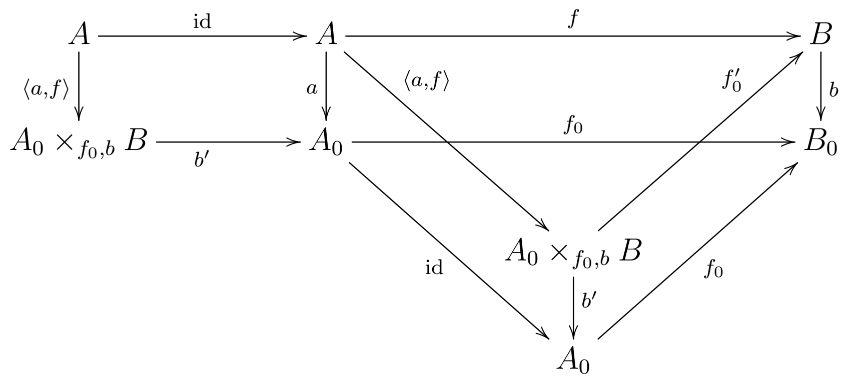
4.1.8. . The fundamental link between, on one side, the factorization systems $(\mathcal{E}_1, \mathcal{M}_1)$ and $(\mathcal{E}_2, \mathcal{M}_2)$ and, on the other side, Θ_Δ-cokernels and Θ_Δ-kernels in $\mathbf{Arr}(\mathcal{A})$ is easy to see: just compare how an arrow is factorized (proof of Proposition 4.1.6) and how Θ_Δ-cokernels and Θ_Δ-kernels are constructed in Example 2.1.6.

1. The $(\mathcal{E}_1, \mathcal{M}_1)$ factorization of an arrow $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ is the factorization of (f, f_0) through the Θ_Δ-kernel of the Θ_Δ-cokernel of (f, f_0):



2. The $(\mathcal{E}_2, \mathcal{M}_2)$ factorization of an arrow $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ is the factor-

ization of (f, f_0) through the Θ_Δ-cokernel of the Θ_Δ-kernel of (f, f_0):



Using 4.1.8, we can say something more on the four classes of arrows involved in the factorisation systems of $\mathbf{Arr}(\mathcal{A})$.

4.1.9. Proposition. *Let $\mathcal{A}$ be a category with pushouts and pullbacks and consider an arrow $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ in $\mathbf{Arr}(\mathcal{A})$.*

1. *The following conditions are equivalent:*
   - *(a) $(f, f_0) \in \mathcal{E}_1$, that is, $(*)$ is a pushout,*
   - *(b) the object part of its Θ_Δ-cokernel is Θ_Δ-trivial.*
2. *The following conditions are equivalent:*
   - *(a) $(f, f_0) \in \mathcal{M}_1$, that is, f is an isomorphism,*
   - *(b) (f, f_0) is a Θ_Δ-kernel,*
   - *(c) (f, f_0) is the Θ_Δ-kernel of its Θ_Δ-cokernel.*
3. *The following conditions are equivalent:*
   - *(a) $(f, f_0) \in \mathcal{E}_2$, that is, f_0 is an isomorphism,*
   - *(b) (f, f_0) is a Θ_Δ-cokernel,*
   - *(c) (f, f_0) is the Θ_Δ-cokernel of its Θ_Δ-kernel.*
4. *The following conditions are equivalent:*
   - *(a) $(f, f_0) \in \mathcal{M}_2$, that is, $(*)$ is a pullback,*
   - *(b) the object part of its Θ_Δ-kernel is Θ_Δ-trivial.*

Proof. 1. This follows directly from the description of Θ_Δ-cokernels in $\mathbf{Arr}(\mathcal{A})$ and the characterization of Θ_Δ-trivial objects, see Example 2.1.6.

2. The implication (c) $\Rightarrow$ (b) is obvious and the implication (b) $\Rightarrow$ (a) immediately follows from the description of Θ_Δ-cokernel in Example 2.1.6. As far as the implication (a) $\Rightarrow$ (c) is concerned, if $(f, f_0) \in \mathcal{E}_2$, then the $\mathcal{M}_2$ part of its $(\mathcal{E}_2, \mathcal{M}_2)$-factorization is an isomorphism. Therefore, from 4.1.8.2 it follows that (f, f_0) is the Θ_Δ-cokernel of its Θ_Δ-kernel. ■

4.1.10. Corollary. *Let $\mathcal{A}$ be a category with pushouts and pullbacks.*

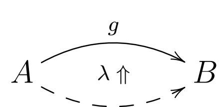
1. *If $(g, g_0) \in \mathcal{E}_1$, then the Θ_Δ-cokernel of a composite arrow $(g, g_0) \cdot (f, f_0)$ is the Θ_Δ-cokernel of (f, f_0). In particular, the Θ_Δ-cokernel of an arrow is the Θ_Δ-cokernel of the $\mathcal{M}_1$-component of the arrow.*

2. *If $(f, f_0) \in \mathcal{M}_2$, then the Θ_Δ-kernel of a composite arrow $(g, g_0) \cdot (f, f_0)$ is the Θ_Δ-kernel of (g, g_0). In particular, the Θ_Δ-kernel of an arrow is the Θ_Δ-kernel of the $\mathcal{E}_2$-component of the arrow.*

Proof. 1. This follows from 4.1.8.1 and Proposition 4.1.9.1. Alternatively, this facts can also be deduced from 4.1.7. ■

4.2. Three-step factorizations.

4.2.1. . To prepare the abelian case, which will be studied in the next sections, we have to analyze more the $\mathcal{E}_1$ and the $\mathcal{M}_2$ components of an arrow. We consider first the $\mathcal{E}_1$-component: our aim is to further decompose it in two components, the first one lying in $\mathcal{E}_1 \cap \mathcal{E}_2$ and the second one lying (in the abelian case) in $\mathcal{E}_1 \cap \mathcal{M}_2$. For this, we work as far as possible in a category with nullhomotopies, before specializing to the case of $\mathbf{Arr}(\mathcal{A})$. In this section, we will sometimes depict a nullhomotopy $\lambda \in \Theta(g)$ as

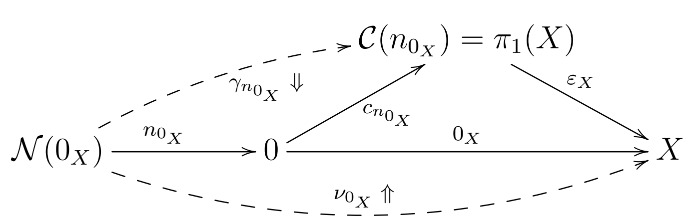


Indeed, this is the situation of Example 2.2.1.1 if we see the dashed arrow as a zero arrow.

4.2.2. . Let $(\mathcal{B}, \Theta)$ be a category with nullhomotopies satisfying the reduced interchange. Assume the existence in $\mathcal{B}$ of Θ-kernels, strong Θ-cokernels and a Θ-strong initial object 0. As a preliminary step, fix an object $X \in \mathcal{B}$. We denote by

$$\varepsilon_X \colon \pi_1(X) \to X$$

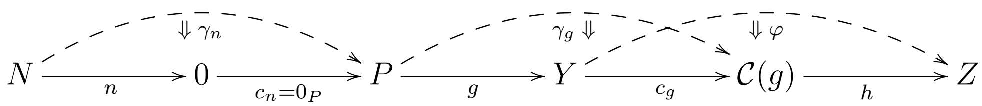
the factorization of the unique arrow $0_X \colon 0 \to X$ through the Θ-cokernel of its Θ-kernel:



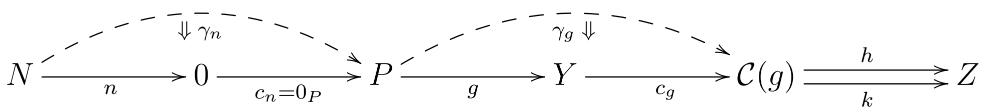
The object $\pi_1(X)$ is therefore the Θ-cokernel of an arrow whose codomain is the initial object 0. The interest of this fact lies in the following result.

4.2.3. Lemma. *Let $(\mathcal{B}, \Theta)$ be as in 4.2.2 and let P be the Θ-cokernel of an arrow with codomain 0. For any arrow $g: P \to Y$, the arrow $c_g: Y \to \mathcal{C}(g)$ is a strong Θ-epimorphism (see 4.1.7.1) and an epimorphism.*

Proof. We start showing that c_g is a strong Θ-epimorphism. Consider the situation depicted by the following diagram

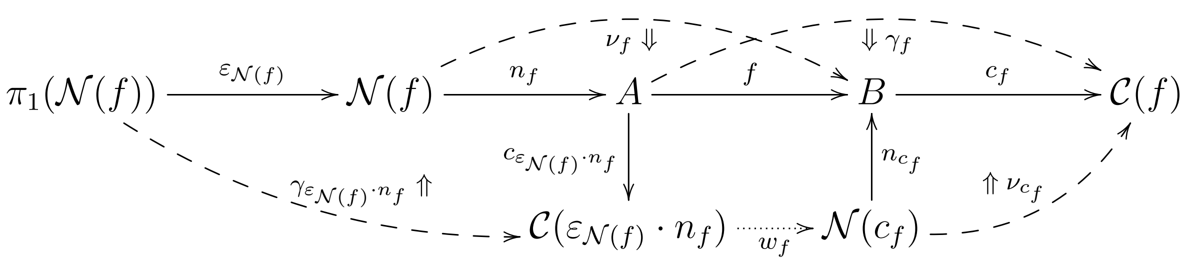


We have to prove that there exists a unique nullhomotopy $\varphi' \in \Theta(h)$ such that $c_g \circ \varphi' = \varphi$. This will follow from the universal property of the strong Θ-cokernel $\mathcal{C}(g)$ if we can verify the equation $\gamma_g \circ h = g \circ \varphi$. By 2.1.5.2, it is enough to check this equation precomposed with $c_n: 0 \to P$. Now the equality $c_n \circ \gamma_g \circ h = c_n \cdot g \circ \varphi$ holds because 0 is a Θ-strong initial object.
Now we show that c_g is an epimorphism. Consider the situation depicted by the following diagram



and assume that $c_g \cdot h = c_g \cdot k$. By 2.1.5.1, to prove that $h = k$ we have to prove that $\gamma_g \circ h = \gamma_g \circ k$. By 2.1.5.2, it is enough to check this equation precomposed with $c_n: 0 \to P$ and we conclude as in the first part of the proof. ∎

4.2.4. Lemma. *Let $(\mathcal{B}, \Theta)$ be as in 4.2.2. Fix an arrow $f: A \to B$ in $\mathcal{B}$ and consider the diagram*



There exists a unique arrow $w_f: \mathcal{C}(\varepsilon_{\mathcal{N}(f)} \cdot n_f) \to \mathcal{N}(c_f)$ such that $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \cdot n_{c_f} = f$ and $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \circ \nu_{c_f} = \gamma_f$.

Proof. By the universal property of the Θ-kernel $\mathcal{N}(c_f)$, there exists a unique arrow $f_1 \colon A \to \mathcal{N}(c_f)$ such that $f_1 \cdot n_{c_f} = f$ and $f_1 \circ \nu_{c_f} = \gamma_f$. By the universal property of the Θ-cokernel $\mathcal{C}(\varepsilon_{\mathcal{N}(f)} \cdot n_f)$, there exists a unique arrow $t_f \colon \mathcal{C}(\varepsilon_{\mathcal{N}(f)} \cdot n_f) \to B$ such that $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot t_f = f$ and $\gamma_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ t_f = \varepsilon_{\mathcal{N}(f)} \circ \nu_f$. Now observe that $\gamma_f \in \Theta(f \cdot c_f) = \Theta(c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot t_f \cdot c_f)$. We can apply the first part of Lemma 4.2.3 by taking $\pi_1(\mathcal{N}(f))$ as P and $\varepsilon_{\mathcal{N}(f)} \cdot n_f$ as g. We obtain a unique nullhomotopy $\gamma'_f \in \Theta(t_f \cdot c_f)$ such that $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \gamma'_f = \gamma_f$. By the universal property of $\mathcal{N}(c_f)$, we get a unique arrow w_f such that $w_f \cdot n_{c_f} = t_f$ and $w_f \circ \nu_{c_f} = \gamma'_f$. Therefore, the square in the above diagram commutes:

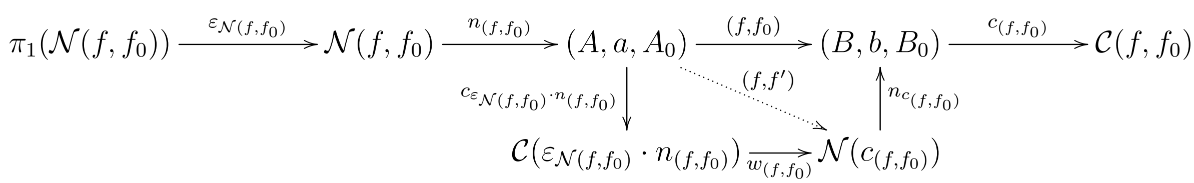
$$c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \cdot n_{cf} = c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot t_f = f.$$

Moreover, the four triangles determined, inside the square, by f_1 and t_f, commute. The only one which remains to check is $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f = f_1$. For this, it suffices to go back to the definition of f_1:

$$c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \cdot n_{cf} = f \quad \text{and} \quad c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \circ \nu_{c_f} = c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \gamma'_f = \gamma_f.$$

Since, by the second part of Lemma 4.2.3, $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f}$ is an epimorphism, the arrow w_f is in fact characterized by the condition $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f = f_1$. Putting together this condition with the conditions defining f_1, we can conclude that w_f is the unique arrow such that $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \cdot n_{c_f} = f$ and $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f \circ \nu_{c_f} = \gamma_f$. ■

4.2.5. . Now we can specialize the previous construction in order to get a three-step factorization of an arrow $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ of $\mathbf{Arr}(\mathcal{A})$. We assume that the category $\mathcal{A}$ has zero object 0, pushouts and pullbacks. With compact notation, we get:



In this case, the down arrow is in $\mathcal{E}_1 \cap \mathcal{E}_2$ by Lemma 4.2.6, the arrow $w_{(f,f_0)}$ is in $\mathcal{E}_1$ because its composition with the down arrow is the $\mathcal{E}_1$-component (f, f') of (f, f_0) (use 4.1.5.2), and the up arrow is in $\mathcal{M}_1$.
Let us point out that the notation used in 4.2.2 can be justified by the fact that, in the special case given by $\mathcal{B} = \mathbf{Arr}(\mathcal{A})$, ε is indeed the counit of the adjunction

$$\mathcal{A} \underset{\Lambda}{\overset{\text{Ker}}{\leftrightarrows}} \mathbf{Arr}(\mathcal{A}) \quad \Lambda \dashv \text{Ker} \qquad \Lambda(A) = (A, 0^A, 0)$$

Explicitly, the counit

$$\varepsilon_{(A,a,A_0)} \colon \pi_1(A, a, A_0) = \Lambda(\text{Ker}(A, a, A_0)) \to (A, a, A_0)$$

is the $\mathcal{M}_2$-component in of the unique arrow $\Lambda(0) \to (A, a, A_0)$, that is,

$$\begin{array}{ccc} \mathrm{Ker}(a) & \xrightarrow{k_a} & A \\ \downarrow & & \downarrow{\scriptstyle a} \\ 0 & \longrightarrow & A_0 \end{array}$$

Finally, here is the formulation of Lemma 4.2.3 in the special case of $(\mathcal{B}, \Theta) = (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$. We state it explicitly since its proof is particularly simple and illustrative.

4.2.6. LEMMA. *Let $\mathcal{A}$ be a category with zero object and cokernels. Any arrow in $\mathbf{Arr}(\mathcal{A})$ with domain an object of the form $\Lambda(X)$ has, as Θ_Δ-cokernel, an arrow in $\mathcal{E}_1 \cap \mathcal{E}_2$.*

PROOF. Following Example 2.1.6, in the diagram hereunder the square on the right is the Θ_Δ-cokernel of the square on the left

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{q_f} & \mathrm{Coker}(f) \\ \downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle [0,y]} \\ 0 & \longrightarrow & Y_0 & \xrightarrow[\mathrm{id}]{} & Y_0 \end{array}$$

The square on the right is a pushout, that is, an arrow in $\mathcal{E}_1$. Moreover, all Θ_Δ-cokernels are in $\mathcal{E}_2$. ∎

4.2.7. . We sketch now the dual analysis for the $\mathcal{M}_2$ component of an arrow in $\mathbf{Arr}(\mathcal{A})$, in order to get another three-step factorization. We treat directly the case of $\mathbf{Arr}(\mathcal{A})$ and we leave to the reader the general discussion dual to that in 4.2.2, 4.2.3 and 4.2.4. Assume that the category $\mathcal{A}$ has zero object, pushouts and pullbacks. We use the unit of the adjonction

$$\mathcal{A} \underset{\Gamma}{\overset{\mathrm{Coker}}{\rightleftarrows}} \mathbf{Arr}(\mathcal{A}) \quad \mathrm{Coker} \dashv \Gamma \qquad \Gamma(A) = (0, 0_A, A)$$

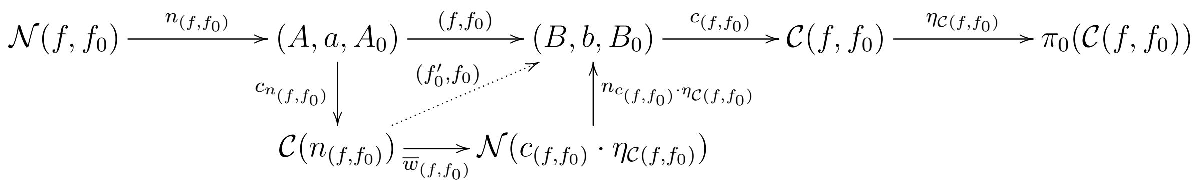
which, for a given object $(A, a, A_0) \in \mathbf{Arr}(\mathcal{A})$, is given by the factorization through the Θ_Δ-kernel of the Θ_Δ-cokernel of the unique arrow $(A, a, A_0) \to \Gamma(0)$, that is, the $\mathcal{E}_1$-component of such an arrow. It will be denoted by

$$\eta_{(A,a,A_0)} \colon (A, a, A_0) \to \Gamma(\mathrm{Coker}(A, a, A_0)) = \pi_0(A, a, A_0)$$

Explicitly, this is just

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ {\scriptstyle a}\downarrow & & \downarrow \\ A_0 & \xrightarrow[q_a]{} & \mathrm{Coker}(a) \end{array}$$

The new three-step factorization of an arrow (f, f_0) in $\mathbf{Arr}(\mathcal{A})$ is provided by the commutative square in the following diagram:



where the down arrow is in $\mathcal{E}_2$, the arrow $\overline{w}_{(f,f_0)}$ is in $\mathcal{M}_2$ and the up arrow is in $\mathcal{M}_1 \cap \mathcal{M}_2$ being the Θ_Δ-kernel of an arrow with codomain of the form $\Gamma(X)$ (use the dual of Lemma 4.2.6).

5. $\mathbf{Arr}(\mathcal{A})$: the additive case

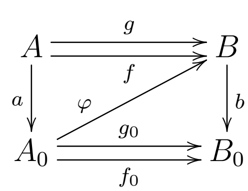
5.1. The 2-dimensional structure of $\mathbf{Arr}(\mathcal{A})$.

5.1.1. . From [10, 32], we know that, if $\mathcal{A}$ is an additive category with finite limits, there are equivalences

$$\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{RG}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$$

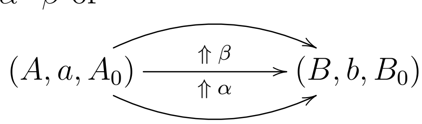
where $\mathbf{RG}(\mathcal{A})$ is the category of reflexive graphs in $\mathcal{A}$, and $\mathbf{Grpd}(\mathcal{A})$ is the 2-category of internal groupoids in $\mathcal{A}$. Transporting the 2-categorical structure from $\mathbf{Grpd}(\mathcal{A})$ to $\mathbf{Arr}(\mathcal{A})$, we get the following structure:

1. A 2-cell $(A,a,A_0) \underset{(f,f_0)}{\overset{(g,g_0)}{\Uparrow \varphi}} (B,b,B_0)$ is an arrow $\varphi\colon A_0 \to B$, as in the following diagram, such that $f + a\cdot\varphi = g$ and $f_0 + \varphi\cdot b = g_0$



   In particular, if the domain (f, f_0) of the 2-cell φ is the pair of zero arrows $(0_B^A, 0_{B_0}^{A_0})$, then φ is nothing but a nullhomotopy on (g, g_0) in the sense of Example 2.1.6.

2. The vertical composition $\alpha \cdot \beta$ of



   is $\alpha + \beta$. The identity 2-cell on an arrow $(f, f_0)\colon (A,a,A_0) \to (B,b,B_0)$ is the zero arrow $0_B^{A_0}\colon A_0 \to B$. Note that 2-cells are invertible with respect to vertical composition: the inverse of α is $-\alpha$.

3. The horizontal composition $\alpha \circ \beta$ of

$$(A,a,A_0) \underset{(f,f_0)}{\overset{(g,g_0)}{\underset{\Uparrow \alpha}{\rightrightarrows}}} (B,b,B_0) \underset{(h,h_0)}{\overset{(k,k_0)}{\underset{\Uparrow \beta}{\rightrightarrows}}} (C,c,C_0)$$

is $\alpha \cdot h + g_0 \cdot \beta$ or, equivalently, $f_0 \cdot \beta + \alpha \cdot k$. In particular, the horizontal composition of a 2-cell with identities 2-cells corresponds to the formula $(f, f_0) \circ \varphi \circ (h, h_0) = f_0 \cdot \varphi \cdot h$ for nullhomotopies in $\mathbf{Arr}(\mathcal{A})$, and this even if the domain of φ is not a zero arrow.

5.1.2. . Let us point out two aspects of the 2-categorical structure of $\mathbf{Arr}(\mathcal{A})$ which will be useful later.

1. The absorption conditions of 2.1.3 hold in $\mathbf{Arr}(\mathcal{A})$ not only with respect to nullhomotopies, but with respect to arbitrary 2-cells. Explicitly, for all

$$(A,a,A_0) \underset{(f,f_0)}{\overset{(g,g_0)}{\underset{\Uparrow \alpha}{\rightrightarrows}}} (B,b,B_0) \underset{(h,h_0)}{\overset{(k,k_0)}{\underset{\Uparrow \beta}{\rightrightarrows}}} (C,c,C_0)$$

we have

$$\alpha \circ 0^{(B,b,B_0)}_{(C,c,C_0)} = 0^{(A,a,A_0)}_{(C,c,C_0)} = 0^{(A,a,A_0)}_{(B,b,B_0)} \circ \beta$$

This remark fits into the general situation discussed in Remark 2.2.6 and, in an even more general context, in Remark 2.3.5.

2. For any 2-cell $(A,a,A_0) \underset{(\mathrm{id},\mathrm{id})}{\overset{(f,f_0)}{\underset{\Uparrow \varphi}{\rightrightarrows}}} (A,a,A_0)$ with domain an identity arrow, one has $\varphi \circ (f, f_0) = (f, f_0) \circ \varphi$. Indeed,

$$\varphi \circ (f, f_0) = \varphi \cdot f = \varphi \cdot (\mathrm{id} + a \cdot \varphi) = \varphi + \varphi \cdot a \cdot \varphi = (\mathrm{id} + \varphi \cdot a) \cdot \varphi = f_0 \cdot \varphi = (f, f_0) \circ \varphi$$

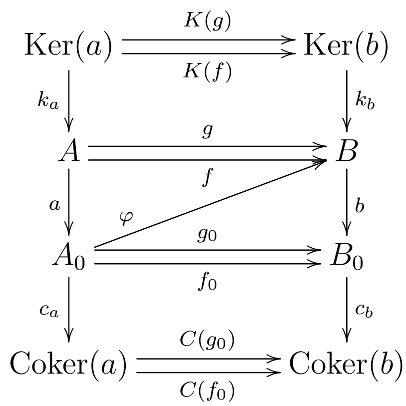
Now that we dispose of a 2-categorical structure on $\mathbf{Arr}(\mathcal{A})$, we can look at equivalences in $\mathbf{Arr}(\mathcal{A})$. We recall an easy lemma and a corollary, which express a well-known homotopy invariance. (The notation used in the statement of the lemma is made explicit in the proof.)

5.1.3. Lemma. *Let $\mathcal{A}$ be an additive category with kernels and cokernels. If there exists a 2-cell*

$$(A,a,A_0) \underset{(f,f_0)}{\overset{(g,g_0)}{\underset{\Uparrow \varphi}{\rightrightarrows}}} (B,b,B_0)$$

in $\mathbf{Arr}(\mathcal{A})$, then $K(f) = K(g)$ and $C(f_0) = C(g_0)$.

Proof. The whole situation is depicted by the following diagram, where the arrow φ satisfies the conditions $f + a \cdot \varphi = g$ and $f_0 + \varphi \cdot b = g_0$,



It follows that

$$K(g) \cdot k_b = k_a \cdot g = k_a \cdot f + k_a \cdot a \cdot \varphi = k_a \cdot f = K(f) \cdot k_b$$

$$c_a \cdot C(g_0) = g_0 \cdot c_b = f_0 \cdot c_b + \varphi \cdot b \cdot c_b = f_0 \cdot c_b = c_a \cdot C(f_{\ddagger})$$

From the first equation, we get $K(g) = K(f)$ because k_b is a monomorphism. From the second equation, we get $C(g_0) = C(f_0)$ because c_a is an epimorphism. ∎

5.1.4. Corollary. *Let $\mathcal{A}$ be an additive category with kernels and cokernels. If we have an equivalence*

$$(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$$

in the 2-category $\mathbf{Arr}(\mathcal{A})$, *then* $K(f) \colon \mathrm{Ker}(a) \to \mathrm{Ker}(b)$ *and* $C(f_0) \colon \mathrm{Coker}(a) \to \mathrm{Coker}(b)$ *are isomorphisms.*

Proof. Consider an equivalence $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ in $\mathbf{Arr}(\mathcal{A})$ with quasi-inverse $(f^*, f_0^*) \colon (B, b, B_0) \to (A, a, A_0)$. Using Lemma 5.1.3 to justify the second equality, we have

$$K(f) \cdot K(f^*) = K(f \cdot f^*) = K(\mathrm{id}_A) = \mathrm{id}_{\mathrm{Ker}(a)}$$

and similarly $K(f^*) \cdot K(f) = \mathrm{id}_{\mathrm{Ker}(b)}$, so that $K(f)$ is an isomorphism. In the same way, $C(f_0)$ is an isomorphism. ∎

5.1.5. . The converse of Corollary 5.1.4 is not true, even if we assume $\mathcal{A}$ to be abelian. This fact is at the heart of Section 7. Using the terminology which will be introduced in Proposition 6.1.3, Corollary 5.1.4 can be restated saying that every equivalence in $\mathbf{Arr}(\mathcal{A})$ is a weak equivalence.

Before going on with the study of H-limits in $\mathbf{Arr}(\mathcal{A})$, which is the object of Subsection 5.2, we state the next result, which reinforces Proposition 3.10 in [53]. The proof is very close to that of Proposition 3.10 in [53] and can be omitted.

5.1.6. PROPOSITION. *Let $\mathcal{A}$ be an additive category.*

1. *If $\mathcal{A}$ has finite colimits, then finite colimits in $\mathbf{Arr}(\mathcal{A})$ are 2-colimits.*

2. *If $\mathcal{A}$ has finite limits, then finite limits in $\mathbf{Arr}(\mathcal{A})$ are 2-limits.*

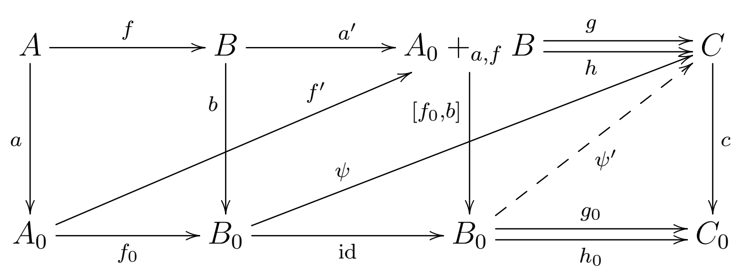
5.2. STRONG H-LIMITS IN $\mathbf{Arr}(\mathcal{A})$. In this section, we start observing that, when $\mathcal{A}$ is additive with finite (co)limits, then the Θ_Δ–(co)kernels in $\mathbf{Arr}(\mathcal{A})$ are strong H-(co)kernels. Then, we go further by constructing, always under the assumption that $\mathcal{A}$ is additive with finite limits and finite colimits, strong H-pushouts and strong H-pullbacks in $\mathbf{Arr}(\mathcal{A})$.

5.2.1. PROPOSITION. *Let $\mathcal{A}$ be an additive category.*

1. *If $\mathcal{A}$ has pushouts, then Θ_Δ-cokernels in $\mathbf{Arr}(\mathcal{A})$ are strong H-cokernels.*

2. *If $\mathcal{A}$ has pullbacks, then Θ_Δ-kernels in $\mathbf{Arr}(\mathcal{A})$ are strong H-kernels.*

PROOF. 1. In $\mathbf{Arr}(\mathcal{A})$, the condition 2.2.7.2 expressing the fact that a H-cokernel is strong gives the following diagram, where the dashed arrow is the 2-cell we are looking for:



The hypothesis on ψ are $h_0+\psi\cdot c=g_0$. and $a'\cdot h+b\cdot\psi=a'\cdot g$. The compatibility between $\gamma_{(f,f_0)}$ and ψ gives $f'\cdot h+f_0\cdot\psi=f'\cdot g$. The condition $c_{(f,f_0)}\circ\psi'=\psi$ becomes $\mathrm{id}\cdot\psi'=\psi$, so that it remains only to prove that we can see ψ as a 2-cell $(h,h_0)\Rightarrow(g,g_0)$. The first condition is $h_0+\psi\cdot c=g_0$, which is the first hypothesis on ψ. The second condition is $h+[f_0,b]\cdot\psi=g$ and we check it precomposing with a' and f':

- $a'\cdot(h+[f_0,b]\cdot\psi)=a'\cdot h+a'\cdot[f_0,b]\cdot\psi=a'\cdot h+b\cdot\psi=a'\cdot g$, where the last equality is the second hypothesis on ψ.

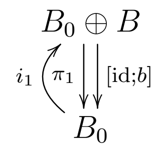
- $f'\cdot(h+[f_0,b]\cdot\psi)=f'\cdot h+f'\cdot[f_0,b]\cdot\psi=f'\cdot h+f_0\cdot\psi=f'\cdot g$, where the last equality precisely is the compatibility condition.

The proof of part 2 is dual. ∎

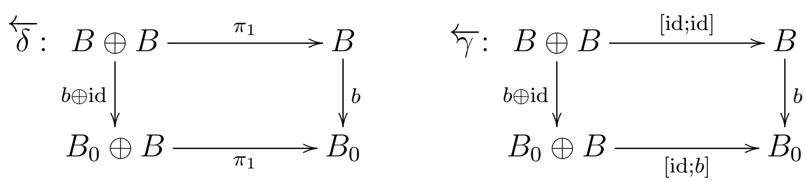
5.2.2. . We are going to construct strong H-pullbacks and strong H-pushouts in $\mathbf{Arr}(\mathcal{A})$. The strategy is as follows: from [51] and [26], we know how to construct strong H-pullbacks in $\mathbf{Grpd}(\mathcal{A})$ if $\mathcal{A}$ is a category with finite limits. If we assume that $\mathcal{A}$ is also additive, then we can translate the construction from $\mathbf{Grpd}(\mathcal{A})$ to $\mathbf{Arr}(\mathcal{A})$ via the equivalences $\mathbf{Grpd}(\mathcal{A}) \simeq \mathbf{RG}(\mathcal{A}) \simeq \mathbf{Arr}(\mathcal{A})$. Finally, we dualize what we get in $\mathbf{Arr}(\mathcal{A})$ so to have a description also for strong H-pushouts. Now some more details.

5.2.3. . Let $b\colon B \to B_0$ be an object in $\mathbf{Arr}(\mathcal{A})$, with $\mathcal{A}$ additive. Consider its image by the denormalization functor $\mathcal{D}\colon \mathbf{Arr}(\mathcal{A}) \to \mathbf{RG}(\mathcal{A})$:

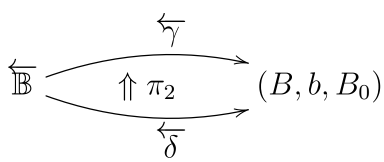
$\mathcal{D}(B,b,B_0) =$



By pulling back b along the domain π_1 and along the codomain $[\mathrm{id};b]$ of the reflexive graph $\mathcal{D}(B,b,B_0)$, we get two arrows in $\mathbf{Arr}(\mathcal{A})$:

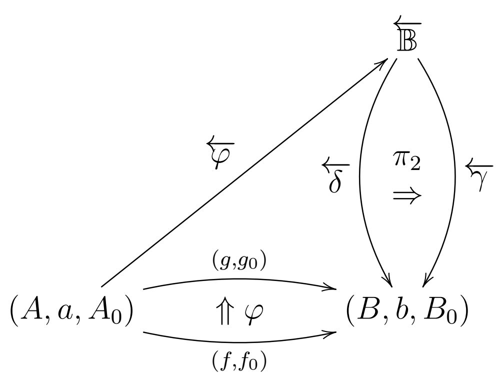


Let us write $\overleftarrow{\mathbb{B}}$ for the domain $(B\oplus B, b\oplus\mathrm{id}, B_0 \oplus B)$ of $\overleftarrow{\delta}$ and $\overleftarrow{\gamma}$. The second projection $\pi_2\colon B_0 \oplus B \to B$ can be interpreted as a 2-cell in $\mathbf{Arr}(\mathcal{A})$



5.2.4. Lemma. *Let $\mathcal{A}$ be an additive category.*

1. *In the situation hereunder, for every 2-cell φ there exists a unique arrow $\overleftarrow{\varphi}$ such that $\overleftarrow{\varphi} \cdot \overleftarrow{\delta} = (f,f_0)$, $\overleftarrow{\varphi} \cdot \overleftarrow{\gamma} = (g,g_0)$ and $\overleftarrow{\varphi} \circ \pi_2 = \varphi$*



2. The following diagram is a strong H-pullback in $\mathbf{Arr}(\mathcal{A})$:

$$\begin{array}{ccc} \overleftarrow{\mathbb{B}} & \xrightarrow{\overleftarrow{\gamma}} & (B,b,B_0) \\ {\scriptstyle \overleftarrow{\delta}}\downarrow & \overset{\Rightarrow}{\pi_2} & \downarrow{\scriptstyle \mathrm{id}} \\ (B,b,B_0) & \xrightarrow[\mathrm{id}]{} & (B,b,B_0) \end{array}$$

PROOF. 1. Here is the passage from the 2-cell φ to the arrow $\overleftarrow{\varphi}$:

$$\begin{array}{ccc} A & \overset{g}{\underset{f}{\rightrightarrows}} & B \\ {\scriptstyle a}\downarrow & \nearrow^{\varphi} & \downarrow{\scriptstyle b} \\ A_0 & \overset{g_0}{\underset{f_0}{\rightrightarrows}} & B_0 \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \xrightarrow{\langle f; a\cdot\varphi\rangle} & B\oplus B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b\oplus\mathrm{id}} \\ A_0 & \xrightarrow[\langle f_0;\varphi\rangle]{} & B_0\oplus B \end{array}$$

This is the unique possibility. Indeed, if an arrow

$$\begin{array}{ccc} A & \xrightarrow{h=\langle h_1;h_2\rangle} & B\oplus B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b\oplus\mathrm{id}} \\ A_0 & \xrightarrow[h_0=\langle h_{01};h_{02}\rangle]{} & B_0\oplus B \end{array}$$

satisfies the requested conditions, one has:

- $h\cdot b\oplus\mathrm{id} = a\cdot h_0$ implies $h_2 = a\cdot h_{02}$,
- $(h,h_0)\cdot\overleftarrow{\delta} = (f,f_0)$ implies $h_1 = f$ and $h_{01} = f_0$,
- and finally $(h,h_0)\circ\pi_2 = \varphi$ implies $h_{02} = \varphi$.

2. The first point of the present lemma precisely means that the diagram in the second point is a H-pullback. It remains to show that it is strong. For this, consider

$$\begin{array}{ccc} (A,a,A_0) & \xrightarrow{(g,g_0)} & (B,b,B_0) \\ {\scriptstyle (f,f_0)}\downarrow & \overset{\Rightarrow}{\varphi} & \downarrow{\scriptstyle \mathrm{id}} \\ (B,b,B_0) & \xrightarrow[\mathrm{id}]{} & (B,b,B_0) \end{array} \qquad \begin{array}{ccc} (A,a,A_0) & \xrightarrow{(g',g'_0)} & (B,b,B_0) \\ {\scriptstyle (f',f'_0)}\downarrow & \overset{\Rightarrow}{\varphi'} & \downarrow{\scriptstyle \mathrm{id}} \\ (B,b,B_0) & \xrightarrow[\mathrm{id}]{} & (B,b,B_0) \end{array}$$

together with 2-cells $\alpha\colon (f,f_0)\Rightarrow(f',f'_0)$ and $\beta\colon (g,g_0)\Rightarrow(g',g'_0)$ such that $\alpha\cdot\varphi' = \varphi\cdot\beta$. The unique 2-cell, say

$$(A,a,A_0) \underset{\overleftarrow{\varphi}}{\overset{\overleftarrow{\varphi'}}{\rightrightarrows}} \overleftarrow{\mathbb{B}} \qquad \Uparrow\mu$$

such that $\mu\circ\overleftarrow{\delta} = \alpha$ and $\mu\circ\overleftarrow{\gamma} = \beta$ is $\langle\alpha;\beta-\alpha\rangle\colon A_0\to B\oplus B$. Indeed, the first condition on μ forces the first component of μ to be α, and the second condition on μ forces the second component of μ to be $\beta-\alpha$. ∎

We need the following fact from Section 2 in [26].

5.2.5. Lemma. *In a 2-category $\mathcal{B}$ with invertible 2-cells, consider the following diagram*

$$\begin{array}{ccccc} D \times_{h,g'} \mathcal{P}(f,g) & \xrightarrow{h'} & \mathcal{P}(f,g) & \xrightarrow{f'} & B \\ {\scriptstyle g''}\downarrow & \overset{\Rightarrow}{\mathrm{id}} & {\scriptstyle g'}\downarrow & \overset{\Rightarrow}{\pi}_{f,g} & \downarrow{\scriptstyle g} \\ D & \xrightarrow[h]{} & A & \xrightarrow[f]{} & C \end{array}$$

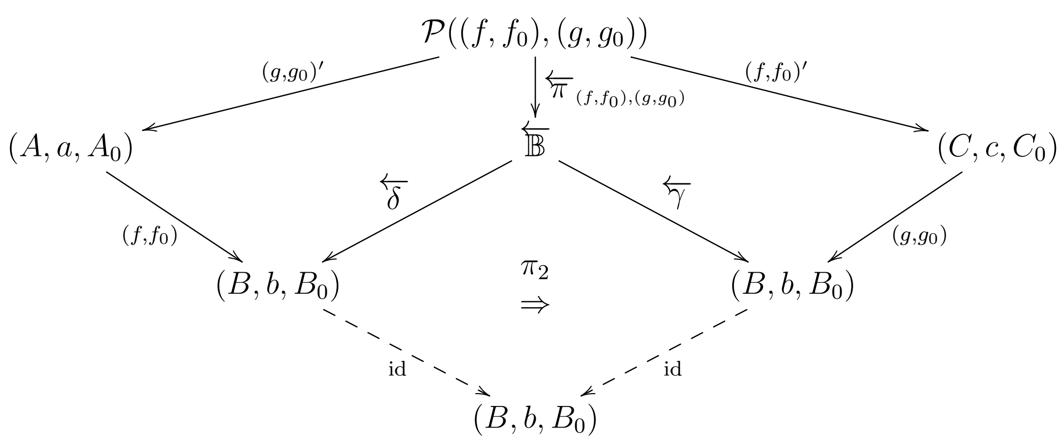
If the right-hand square is a strong H-pullback and the left-hand square is a 2-pullback, then the total diagram is a strong H-pullback.

5.2.6. Corollary. *If $\mathcal{A}$ is an additive category with finite limits, then* $\mathbf{Arr}(\mathcal{A})$ *has strong H-pullbacks.*

Proof. Thanks to Proposition 5.1.6, Lemma 5.2.4 and Lemma 5.2.5, the strong H-pullback

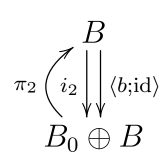
$$\begin{array}{ccc} \mathcal{P}((f,f_0),(g,g_0)) & \xrightarrow{(f,f_0)'} & (C,c,C_0) \\ {\scriptstyle (g,g_0)'}\downarrow & \overset{\Rightarrow}{\pi_{(f,f_0),(g,g_0)}} & \downarrow{\scriptstyle (g,g_0)} \\ (A,a,A_0) & \xrightarrow[(f,f_0)]{} & (B,b,B_0) \end{array}$$

is obtained by the following diagram, whose solid part is, by construction, a limit (and therefore a 2-limit) in $\mathbf{Arr}(\mathcal{A})$,



and $\pi_{(f,f_0),(g,g_0)} = \overleftarrow{\pi}_{(f,f_0),(g,g_0)} \circ \pi_2$. ■

5.2.7. . We pass now to strong H-pushouts in $\mathbf{Arr}(\mathcal{A})$. If $\mathcal{A}$ is an additive category, we can associate to an object $b \colon B \to B_0$ of $\mathbf{Arr}(\mathcal{A})$ a coreflexive graph



and we can construct two arrows in $\mathbf{Arr}(\mathcal{A})$ via the following pushouts:

$$\overrightarrow{\delta}\colon\ \begin{array}{ccc} B & \xrightarrow{\ i_2\ } & B_0 \oplus B \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\oplus b} \\ B_0 & \xrightarrow[\ i_2\ ]{} & B_0 \oplus B_0 \end{array} \qquad\qquad \overrightarrow{\gamma}\colon\ \begin{array}{ccc} B & \xrightarrow{\ \langle b;\mathrm{id}\rangle\ } & B_0 \oplus B \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\oplus b} \\ B_0 & \xrightarrow[\ \langle \mathrm{id};\mathrm{id}\rangle\ ]{} & B_0 \oplus B_0 \end{array}$$

Let us write $\overrightarrow{\mathbb{B}}$ for the codomain $(B_0 \oplus B, \mathrm{id} \oplus b, B_0 \oplus B_0)$ of $\overrightarrow{\delta}$ and $\overrightarrow{\gamma}$. The first injection $i_1\colon B_0 \to B_0 \oplus B$ can be interpreted as a 2-cell in $\mathbf{Arr}(\mathcal{A})$

$$(B, b, B_0) \ \underset{\overrightarrow{\delta}}{\overset{\overrightarrow{\gamma}}{\rightrightarrows}}\ \overrightarrow{\mathbb{B}} \qquad (\Uparrow i_1)$$

5.2.8. Lemma. *Let $\mathcal{A}$ be an additive category.*

1. *In the situation hereunder, for every 2-cell φ there exists a unique arrow $\overrightarrow{\varphi}$ such that $\overrightarrow{\delta} \cdot \overrightarrow{\varphi} = (f, f_0)$, $\overrightarrow{\gamma} \cdot \overrightarrow{\varphi} = (g, g_0)$ and $i_1 \circ \overrightarrow{\varphi} = \varphi$*

$$\begin{array}{ccc} & & \overrightarrow{\mathbb{B}} \\ & \swarrow{\scriptstyle \overrightarrow{\varphi}} & {\scriptstyle \overrightarrow{\delta}}\Big\uparrow \overset{i_1}{\Rightarrow} \Big\uparrow{\scriptstyle \overrightarrow{\gamma}} \\ (A, a, A_0) & \underset{(f,f_0)}{\overset{(g,g_0)}{\leftleftarrows}}\ (\Uparrow \varphi) & (B, b, B_0) \end{array}$$

2. *The following diagram is a strong H-pushout in $\mathbf{Arr}(\mathcal{A})$:*

$$\begin{array}{ccc} (B, b, B_0) & \xrightarrow{\ \mathrm{id}\ } & (B, b, B_0) \\ {\scriptstyle \mathrm{id}}\big\downarrow & \overset{\Rightarrow}{i_1} & \big\downarrow{\scriptstyle \overrightarrow{\gamma}} \\ (B, b, B_0) & \xrightarrow[\ \overrightarrow{\delta}\ ]{} & \overrightarrow{\mathbb{B}} \end{array}$$

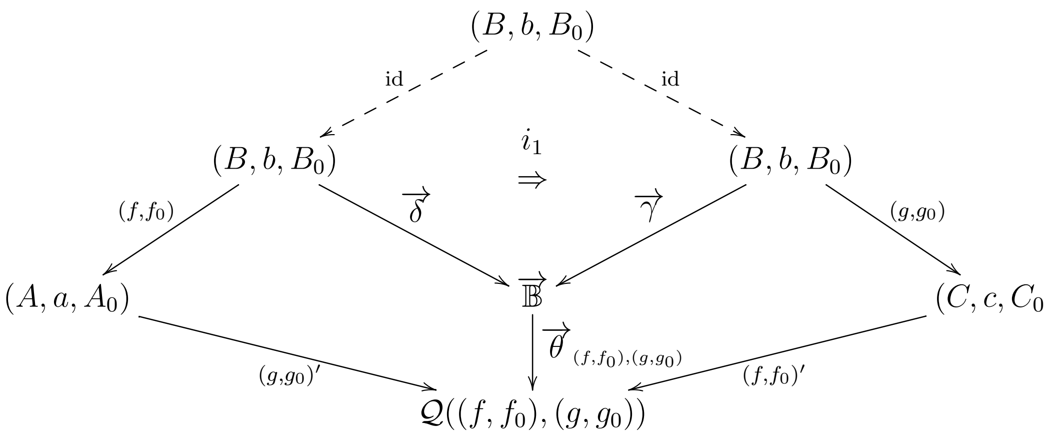
Thanks to Proposition 5.1.6, Lemma 5.2.8, whose proof is left to the reader, and to the dual of Lemma 5.2.5, whose formulation is also left to the reader, we can conclude to the existence of strong H-pushouts in $\mathbf{Arr}(\mathcal{A})$.

5.2.9. Corollary. *If $\mathcal{A}$ is an additive category with finite colimits, then $\mathbf{Arr}(\mathcal{A})$ has strong H-pushouts.*

PROOF. The strong H-pushout

$$\begin{array}{ccc} (B,b,B_0) & \xrightarrow{(g,g_0)} & (C,c,C_0) \\ {\scriptstyle (f,f_0)}\downarrow & \overset{\Rightarrow}{\theta_{(f,f_0),(g,g_0)}} & \downarrow{\scriptstyle (f,f_0)'} \\ (A,a,A_0) & \xrightarrow[(g,g_0)']{} & \mathcal{Q}((f,f_0),(g,g_0)) \end{array}$$

is obtained by the following diagram, whose solid part is, by construction, a colimit (and therefore a 2-colimit) in $\mathbf{Arr}(\mathcal{A})$,

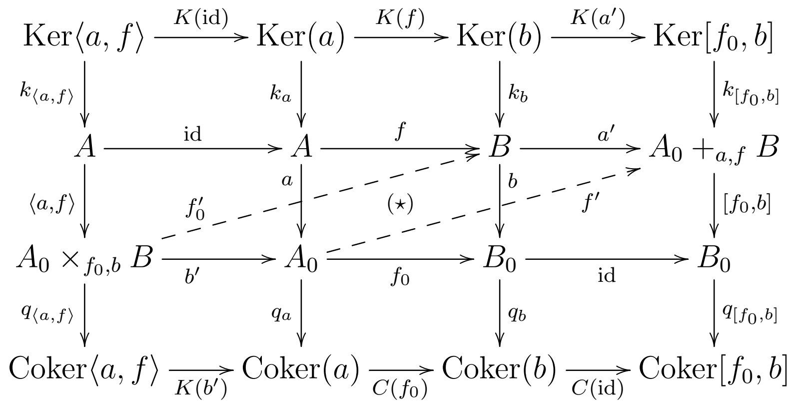


and $\theta_{(f,f_0),(g,g_0)} = i_1 \circ \overrightarrow{\theta}_{(f,f_0),(g,g_0)}$. ∎

6. $\mathbf{Arr}(\mathcal{A})$: the abelian case

6.1. WEAK EQUIVALENCES IN $\mathbf{Arr}(\mathcal{A})$. The aim of this section is to show that, when the base category $\mathcal{A}$ is abelian, in the three-step factorizations of an arrow of $\mathbf{Arr}(\mathcal{A})$ obtained in 4.2.5 and 4.2.7, the middle term is a weak equivalence, that is, an element of $\mathcal{E}_1 \cap \mathcal{M}_2$. For this, in 6.1.2 and 6.1.3, we collect and complete some known material on abelian categories and organize it in a way convenient for the proof of Proposition 6.1.5. We do not take care to state each single result in its greatest generality, since the overall result needs $\mathcal{A}$ to be abelian and to work with this assumption makes the exposition simpler.

6.1.1. . Starting from an arrow $(f,f_0)\colon (A,a,A_0) \to (B,b,B_0)$ in $\mathbf{Arr}(\mathcal{A})$ and assuming that $\mathcal{A}$ has zero object, finite colimits and finite limits, we can construct the following diagram

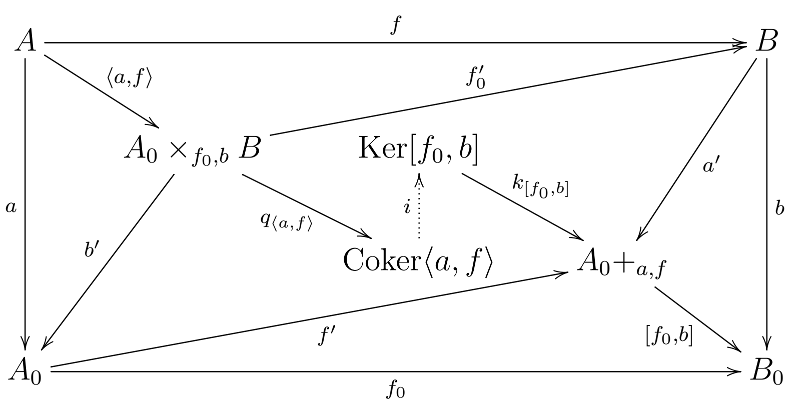


where the dashed arrows are the structural nullhomotopies of $\mathcal{N}(f, f_0)$ and $\mathcal{C}(f, f_0)$.

We recall two results from [52] (see also Lemma 4.5.1 in [7] for the first one and [28] for the second one). The second one is the snail lemma, a variant of the classical snake lemma.

6.1.2. Lemma. *(With the notation of 6.1.1.) Let $\mathcal{A}$ be an abelian category.*

1. *In the following diagram*



there exists a unique arrow $i\colon \mathrm{Coker}\langle a, f\rangle \to \mathrm{Ker}[f_0, b]$ *such that*

$$q_{\langle a,f\rangle} \cdot i \cdot k_{[f_0,b]} = b' \cdot f' - f'_0 \cdot a'$$

Moreover, the arrow i is an isomorphism.

2. *Let us write $H(f, f_0)$ for any of the isomorphic objects* $\mathrm{Coker}\langle a, f\rangle \simeq \mathrm{Ker}[f_0, b]$*. The following sequence is exact:*

$$0 \longrightarrow \mathrm{Ker}\langle a, f\rangle \xrightarrow{K(\mathrm{id})} \mathrm{Ker}(a) \xrightarrow{K(f)} \mathrm{Ker}(b) \xrightarrow{K(a')} H(f, f_0)$$

$$H(f, f_0) \xrightarrow[C(b')]{} \mathrm{Coker}(a) \xrightarrow[C(f_0)]{} \mathrm{Coker}(b) \xrightarrow[C(\mathrm{id})]{} \mathrm{Coker}[f_0, b] \longrightarrow 0$$

In the next proposition, we conclude the study of the classes $\mathcal{E}_1$ and $\mathcal{M}_2$ introduced in Section 4. The terminology we use comes from the equivalence $\mathbf{Arr}(\mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{A})$. In fact, Conditions 1.(c) and 2.(c) are intended to simplify the comparison with [26], where a more detailed discussion can be found.

6.1.3. Proposition. *(With the notation of 6.1.1 and 6.1.2.) Let $\mathcal{A}$ be an abelian category. Consider an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ in $\mathbf{Arr}(\mathcal{A})$*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & (\star) & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

1. *The following conditions are equivalent. When they are satisfied, we say that (f, f_0) is faithful.*
   - *(a)* $K(f)\colon \mathrm{Ker}(a) \to \mathrm{Ker}(b)$ *is a monomorphism,*
   - *(b)* $\langle a, f\rangle\colon A \to A_0 \times_{f_0,b} B$ *is a monomorphism,*
   - *(c)* $\langle a; f\rangle\colon A \to A_0 \oplus B$ *is a monomorphism.*
2. *The following conditions are equivalent. When they are satisfied, we say that (f, f_0) is essentially surjective.*
   - *(a)* $C(f_0)\colon \mathrm{Coker}(a) \to \mathrm{Coker}(b)$ *is an epimorphism,*
   - *(b)* $[f_0, b]\colon A_0 +_{a,f} B \to B_0$ *is an epimorphism,*
   - *(c)* $[f_0; b]\colon A_0 \oplus B \to B_0$ *is an epimorphism.*
3. *The following conditions are equivalent. When they are satisfied, we say that (f, f_0) is full.*
   - *(a)* $K(f)$ *is an epimorphism and* $C(f_0)$ *is a monomorphism,*
   - *(b)* $\langle a, f\rangle\colon A \to A_0 \times_{f_0,b} B$ *is an epimorphism,*
   - *(c)* $[f_0, b]\colon A_0 +_{a,f} B \to B_0$ *is a monomorphism,*
   - *(d)* $H(f, f_0) = 0$,
   - *(e) the structural nullhomotopies of $\mathcal{N}(f, f_0)$ and $\mathcal{C}(f, f_0)$ are compatible, that is,* $\nu_{(f,f_0)} \circ c_{(f,f_0)} = n_{(f,f_0)} \circ \gamma_{(f,f_0)}$.
4. *(f, f_0) is full and faithful iff it is in $\mathcal{M}_2$, that is, iff $(\star)$ is a pullback.*
5. *(f, f_0) is full and essentially surjective iff it is in $\mathcal{E}_1$, that is, iff $(\star)$ is a pushout.*
6. *The following conditions are equivalent. When they are satisfied, we say that (f, f_0) is a weak equivalence.*

(a) $K(f)$ *and* $C(f_0)$ *are isomorphisms,*

(b) (f, f_0) *is full, faithful and essentially surjective,*

(c) $(f, f_0) \in \mathcal{E}_1 \cap \mathcal{M}_2$, *that is,* $(\star)$ *is a pullback and a pushout.*

PROOF. 1. The equivalence between (a) and (b) comes from the snail exact sequence of Lemma 6.1.2.2, which gives that $\mathrm{Ker}\langle a, f\rangle$ is the kernel of $K(f)$. The equivalence between (b) and (c) comes from the commutativity of the following triangle, where the arrow e is the canonical equalizer defining the pullback as a subobject of the product

$$\begin{array}{ccc} A & \xrightarrow{\langle a,f\rangle} & A_0 \times_{f_0,b} B \\ & \searrow^{\langle a;f\rangle} & \downarrow e \\ & & A_0 \times B \end{array}$$

2. Dual of 1.
3. The equivalence between (a) and (d) comes from the exactness in $\mathrm{Ker}(b), H(f, f_0)$ and $\mathrm{Coker}(a)$ of the snail sequence in Lemma 6.1.2.2. The equivalences between (b) and (d) and between (c) and (d) come from Lemma 6.1.2.1 because $H(f, f_0) \simeq \mathrm{Coker}\langle a, f\rangle$ and $H(f, f_0) \simeq \mathrm{Ker}[f_0, b]$. The equivalence between (d) and (e) also comes from Lemma 6.1.2.1 because $\nu_{(f,f_0)} \circ c_{(f,f_0)} = n_{(f,f_0)} \circ \gamma_{(f,f_0)}$ iff $f_0' \cdot a' = b' \cdot f'$ iff $q_{\langle a,f\rangle} \cdot i \cdot k_{[f_0,b]} = 0$ iff $i = 0 \colon \mathrm{Coker}\langle a, f\rangle \to \mathrm{Ker}[f_0, b]$ and i is an isomorphism.
4. Obvious: the commutative square $(\star)$ is a pullback iff its factorization $\langle a, f\rangle$ through the pullback $A_0 \times_{f_0,b} B$ is an isomorphism, that is, a monomorphism and an epimorphism.
5. Dual of 4.
6. The implication (a) $\Rightarrow$ (b) follows from points 1., 2. and 3. The implication (b) $\Rightarrow$ (c) follows from points 4. and 5. The implication (c) $\Rightarrow$ (a) is obvious: in any category, if $(\star)$ is a pullback, then $K(f)$ is an isomorphism, if $(\star)$ is a pushout, then $C(f_0)$ is an isomorphism. ∎

6.1.4. COROLLARY. *Consider an arrow* $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ *in* $\mathbf{Arr}(\mathcal{A})$, *with* $\mathcal{A}$ *abelian.*

1. *If* $f \colon A \to B$ *is a monomorphism, then* (f, f_0) *is faithful.*

2. *If* $f_0 \colon A_0 \to B_0$ *is an epimorphism, then* (f, f_0) *is essentially surjective.*

PROOF. 1. Since $K(f) \cdot k_b = k_a \cdot f$, if f is a monomorphism then $K(f)$ also is a monomorphism. We can conclude by point 1. of Proposition 6.1.3. ∎

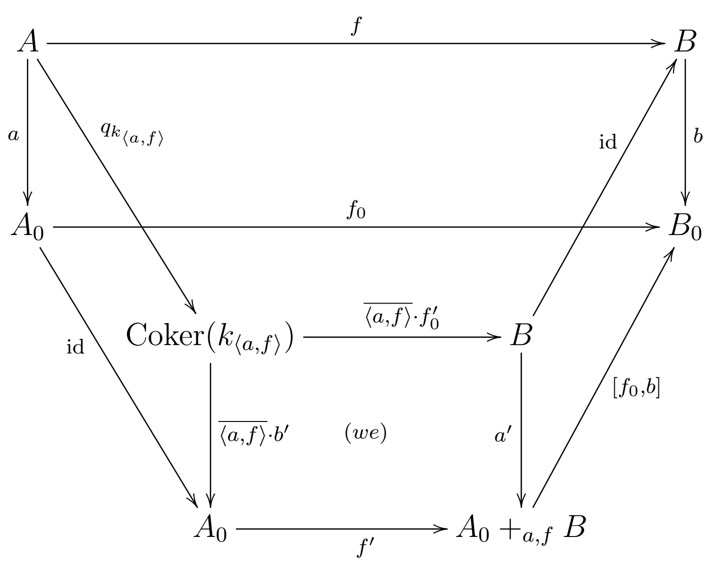
We can now prove a result, announced in 4.2.1, on the three-step factorizations of an arrow in $\mathbf{Arr}(\mathcal{A})$.

6.1.5. Proposition. *Let $\mathcal{A}$ be an abelian category and let $(f;f_0)\colon(A,a,A_0)\to(B,b,B_0)$ be an arrow in $\mathbf{Arr}(\mathcal{A})$. Consider the factorizations obtained in 4.2.5 and 4.2.7:*

$$
\begin{array}{ccc}
(A,a,A_0) & \xrightarrow{(f,f_0)} & (B,b,B_0) \\
{\scriptstyle c_{\varepsilon_{\mathcal{N}(f,f_0)}\cdot n_{(f,f_0)}}}\downarrow & & \uparrow{\scriptstyle n_{c_{(f,f_0)}}} \\
\mathcal{C}(\varepsilon_{\mathcal{N}(f,f_0)}\cdot n_{(f,f_0)}) & \xrightarrow[w_{(f,f_0)}]{} & \mathcal{N}(c_{(f,f_0)})
\end{array}
\qquad
\begin{array}{ccc}
(A,a,A_0) & \xrightarrow{(f,f_0)} & (B,b,B_0) \\
{\scriptstyle c_{n_{(f,f_0)}}}\downarrow & & \uparrow{\scriptstyle n_{c_{(f,f_0)}\cdot\eta_{\mathcal{C}(f,f_0)}}} \\
\mathcal{C}(n_{(f,f_0)}) & \xrightarrow[\overline{w}_{(f,f_0)}]{} & \mathcal{N}(c_{(f,f_0)}\cdot\eta_{\mathcal{C}(f,f_0)})
\end{array}
$$

The arrows $w_{(f,f_0)}$ and $\overline{w}_{(f,f_0)}$ are weak equivalences.

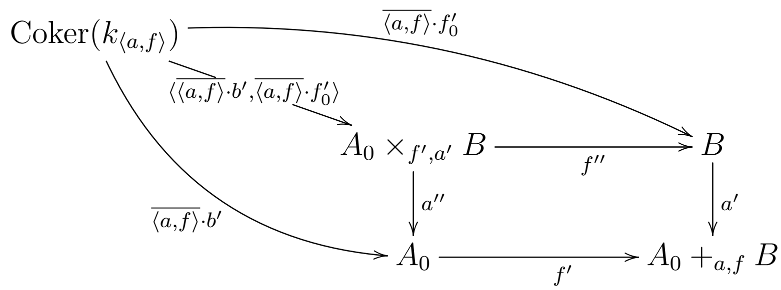
Proof. We prove that $w_{(f,f_0)}\colon\mathcal{C}(\varepsilon_{\mathcal{N}(f,f_0)}\cdot n_{(f,f_0)})\to\mathcal{N}(c_{(f,f_0)})$ is a weak equivalence, the proof for $\overline{w}_{(f,f_0)}\colon\mathcal{C}(n_{(f,f_0)})\to\mathcal{N}(c_{(f,f_0)}\cdot\eta_{\mathcal{C}(f,f_0)})$ is dual. Let us write explicitly the first diagram appearing in the statement:



where $\overline{\langle a,f\rangle}$ is the unique arrow making commutative the following diagram

$$
\begin{array}{ccccc}
\mathrm{Ker}\langle a,f\rangle & \xrightarrow{k_{\langle a,f\rangle}} & A & \xrightarrow{\langle a,f\rangle} & A_0\times_{f_0,b}B \\
 & & {\scriptstyle q_{k_{\langle a,f\rangle}}}\downarrow & \nearrow{\scriptstyle \overline{\langle a,f\rangle}} & \\
 & & \mathrm{Coker}(k_{\langle a,f\rangle}) & &
\end{array}
$$

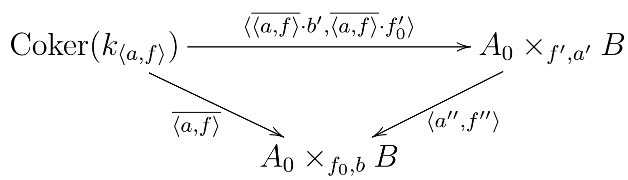
We have to prove that the commutative square (we) is a weak equivalence and we already know that it is a pushout (see 4.2.5). Following Proposition 6.1.3, to prove that (we) is a pullback it suffices to prove that its factorization through the pullback, as in the following diagram, is a monomorphism:



For this, observe that

$$a'' \cdot f_0 = a'' \cdot f' \cdot [f_0, b] = f'' \cdot a' \cdot [f_0, b] = f'' \cdot b$$

so that there is a unique arrow $\langle a'', f''\rangle\colon A_0 \times_{f',a'} B \to A_0 \times_{f_0,b} B$ such that $[a'', f''] \cdot b' = a''$ and $[a'', f''] \cdot f'_0 = f''$. Now we claim that the following triangle commutes



If this is the case, then we are done because $\overline{\langle a, f\rangle}$ is a monomorphism (being the mono part of the epi-mono factorization of $\langle a, f\rangle$ in $\mathcal{A}$), so that $\langle\overline{\langle a, f\rangle} \cdot b', \overline{\langle a, f\rangle} \cdot f'_0\rangle$ also is a monomorphism, as required. To check that the above triangle commutes, it suffices to compose with the projections of the pullback $A_0 \times_{f_0,b} B$:

$$\langle\overline{\langle a, f\rangle} \cdot b', \overline{\langle a, f\rangle} \cdot f'_0\rangle \cdot \langle a'', f''\rangle \cdot b' = \langle\overline{\langle a, f\rangle} \cdot b', \overline{\langle a, f\rangle} \cdot f'_0\rangle \cdot a'' = \overline{\langle a, f\rangle} \cdot b'$$
$$\langle\overline{\langle a, f\rangle} \cdot b', \overline{\langle a, f\rangle} \cdot f'_0\rangle \cdot \langle a'', f''\rangle \cdot f'_0 = \langle\overline{\langle a, f\rangle} \cdot b', \overline{\langle a, f\rangle} \cdot f'_0\rangle \cdot f'' = \overline{\langle a, f\rangle} \cdot f'_0$$

■

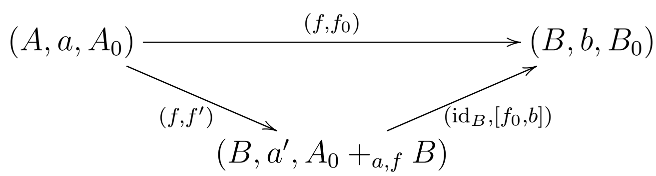
6.1.6. . Even if we will not use it in the rest of the paper, to complete the picture we recall here another characterization of the classes of arrows introduced in Proposition 6.1.3. All but the last point can be deduced from Proposition 6.1.3 using the fact that, in an abelian category, a complex $f \cdot g = 0$ is exact iff $k_g \cdot q_f = 0$.
Consider an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ in $\mathbf{Arr}(\mathcal{A})$ together with the complex

$$\mathcal{S}: \quad 0 \longrightarrow A \xrightarrow{\langle a;-f\rangle} A_0 \oplus B \xrightarrow{[f_0;b]} B_0 \longrightarrow 0$$

1. (f, f_0) is faithful iff $\mathcal{S}$ is exact in A.
2. (f, f_0) is essentially surjective iff $\mathcal{S}$ is exact in B_0.
3. (f, f_0) is full iff $\mathcal{S}$ is exact in $A_0 \oplus B$.
4. (f, f_0) is a weak equivalence iff $\mathcal{S}$ is exact.
5. (f, f_0) is an equivalence iff $\mathcal{S}$ is split exact.

6.1.7. Remark. Putting together Lemma 5.1.3 and Proposition 6.1.3, we can deduce that the classes of arrows studied in Proposition 6.1.3 are stable under 2-cells and under composition with weak equivalences. Moreover, if a composite arrow is faithful, then the first term is faithful, and if the composite is essentially surjective, then the last term is essentially surjective. The above stability conditions are not satisfied by the classes $\mathcal{M}_1$ and $\mathcal{E}_2$ studied in Section 4. In fact, the stabilization under composition with weak equivalences of the class $\mathcal{M}_1$ is the class of faithful arrows, and the stabilization under composition with weak equivalences of the class $\mathcal{E}_2$ is the class of essentially surjective arrows. This explains why, to get factorization systems in a bicategorical sense (see [31, 34, 17]) in the bicategory of fractions $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$, one can use the classes $\mathcal{E}_1$ and $\mathcal{M}_2$, but the classes $\mathcal{M}_1$ and $\mathcal{E}_2$ have to be replaced by the classes of faithful arrows and of essentially surjective arrows.

Proof. Let us explain the case of $\mathcal{M}_1$, the case of $\mathcal{E}_2$ is dual. Consider the $(\mathcal{E}_1, \mathcal{M}_1)$ factorization of a faithful arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ as in Proposition 4.1.6



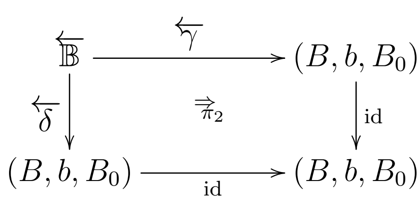
The $\mathcal{E}_1$-component is full and essentially surjective by Proposition 6.1.3, and it is faithful because (f, f_0) is faithful. Therefore, it is a weak equivalence. This means that every faithful arrow can be obtained by composing a weak equivalence with an arrow in $\mathcal{M}_1$. ■

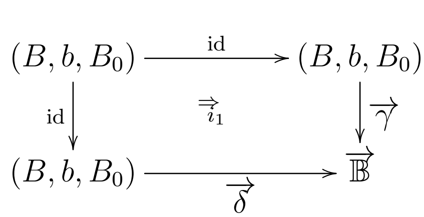
6.2. Calculus of fractions in $\mathbf{Arr}(\mathcal{A})$.

6.2.1. Lemma. *(With the notation of 5.2.3 and 5.2.7.) Let $\mathcal{A}$ be an additive category. Fix an object (B, b, B_0) in $\mathbf{Arr}(\mathcal{A})$. The following arrows are equiivalences:*

$$\overleftarrow{\gamma}\colon \overleftarrow{\mathbb{B}} \to (B, b, B_0)\,,\ \overleftarrow{\delta}\colon \overleftarrow{\mathbb{B}} \to (B, b, B_0)\,,\ \overrightarrow{\gamma}\colon (B, b, B_0) \to \overrightarrow{\mathbb{B}}\,,\ \overrightarrow{\delta}\colon (B, b, B_0) \to \overrightarrow{\mathbb{B}}$$

Proof. Recall, from Lemma 5.2.4 and Lemma 5.2.8, that the following diagrams are, respectively, a strong H-pullback and a strong H-pushout.





Since equivalences are stable under bipullbacks and bipushouts, we are done. ■

In the rest of this section, we denote by Σ the class of weak equivalences in $\mathbf{Arr}(\mathcal{A})$, for $\mathcal{A}$ an abelian category. This means that an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ is in Σ when the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

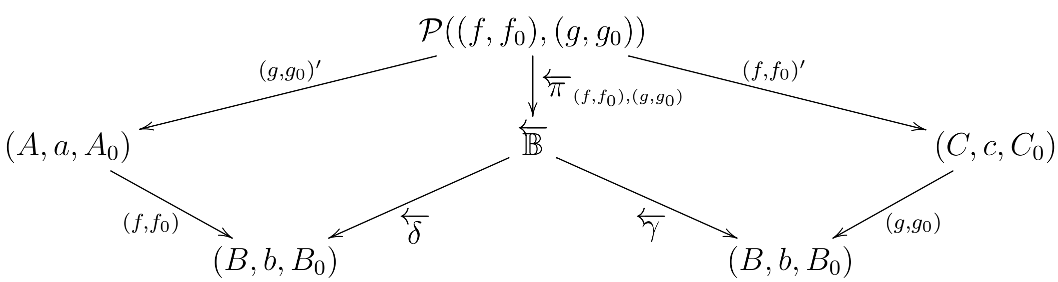
is a pullback and a pushout or, equivalently, when the induced arrows $K(f)$ and $C(f_0)$ are isomorphisms (see Proposition 6.1.3).

6.2.2. Proposition. *Let $\mathcal{A}$ be an abelian category. The class Σ of weak equivalences in $\mathbf{Arr}(\mathcal{A})$ is a bipullback congruence and a bipushout congruence.*

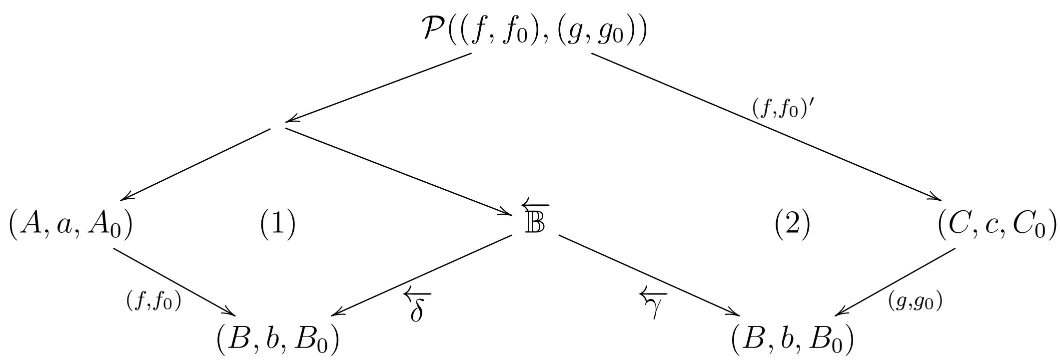
Proof. First of all, note that the statement makes sense because, by Corollary 5.2.6 and Corollary 5.2.9, $\mathbf{Arr}(\mathcal{A})$ has strong H-pullbacks and strong H-pushouts, and then it has bipullbacks and bipushouts. We are going to check conditions 1), 2), 3) and 4) of Definition 3.1.3. The proof of condition 4') is dual to that of condition 4) and we omit it.
1) Σ contains the equivalences. This follows from Corollary 5.1.4 and Proposition 6.1.3.6.
2) If there exists a 2-cell $\varphi\colon (f, f_0) \Rightarrow (g, g_0)$, then $(f, f_0) \in \Sigma$ iff $(g, g_0) \in \Sigma$. By Lemma 5.1.3, $K(f) = K(g)$ and $C(f_0) = C(g_0)$. By point 6 of Proposition 6.1.3, we are done.
3) Σ satisfies the "2 out of 3" rule. This follows once again from Proposition 6.1.3.6, because the class of isomorphisms obviously satisfies such a rule.
4) Σ is stable under bipullbacks. Recall from Corollary 5.2.6 that a bipullback of two arrows

$$(A, a, A_0) \xrightarrow{(f,f_0)} (B, b, B_0) \xleftarrow{(g,g_0)} (C, c, C_0)$$

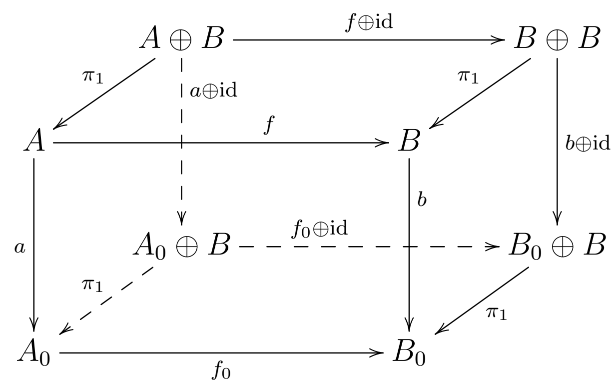
in $\mathbf{Arr}(\mathcal{A})$ can be obtained as the following limit



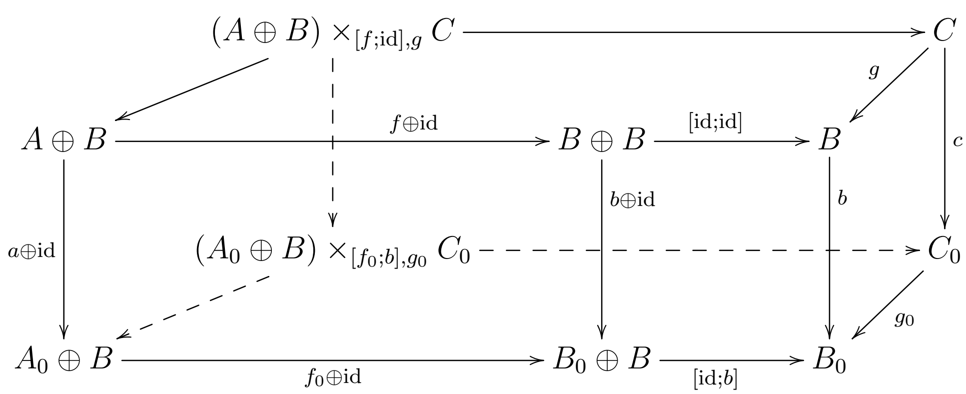
We assume that (f, f_0) is a weak equivalence and we have to show that $(f, f_0)'$ is a weak equivalence too. Observe that the previous limit can be computed by performing two pullbacks



Let us write explicitly the first pullback:



The front face is a pullback by assumption. The top and the bottom faces are pullbacks by construction. The right and the left faces commute by construction. As a consequence, the back face is a pullback. Now we can write explicitly the second pullback:



The front face is a pullback because it is the pasting of two pullbacks: the one on the left is the back face of the previous cube and the one on the right is a pullback by Lemma 6.2.1. The top and the bottom faces are pullbacks by construction. The right face commutes by assumption and the left face commutes by construction. As a consequence, the back face is a pullback, that is, as an arrow in $\mathbf{Arr}(\mathcal{A})$ it is full and faithful by point 4 of Proposition 6.1.3. Now recall that (f, f_0), being a weak equivalence, is in particular essentially surjective. This means that $[f_0; b]: A_0 \oplus B \to B_0$ is an epimorphism (see point 2. of Proposition 6.1.3). Therefore, in the previous diagram the unlabelled arrow

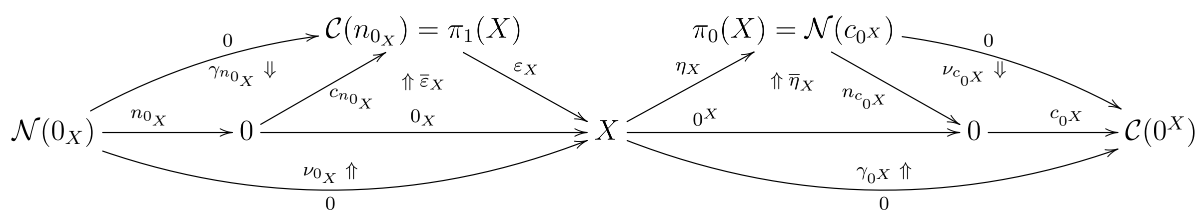
$$(A_0 \oplus B) \times_{[f_0;b],g_0} C_0 \longrightarrow C_0$$

is an epimorphism (because it is obtained by pulling back an epimorphism in an abelian category). By Corollary 6.1.4, this implies that the back face is essentially surjective. Using once again Proposition 6.1.3, we can conclude that the back face is a weak equivalence and the proof is complete. ■

7. $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$ is 2-abelian

7.1. 2-ABELIAN BICATEGORIES. In order to recall the definition of 2-abelian bicategory introduced by M. Dupont in his Ph.D. Thesis [15], we need a construction for bicategories parallel to the one done in 4.2.2-4.2.4 for categories with nullhomotopies.

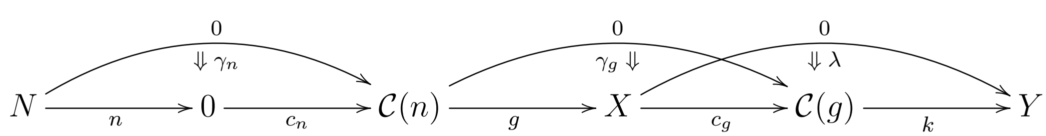
7.1.1. . Let $\mathcal{B}$ be a bicategory with invertible 2-cells. Assume that $\mathcal{B}$ has a bizero object 0, bikernels and bicokernels. For an object $X \in \mathcal{B}$, we define $\pi_0(X)$ and $\pi_1(X)$ by the following diagrams:



where $(\varepsilon_X, \overline{\varepsilon}_X)$ is the fill-in provided by the universal property of the bicokernel $\mathcal{C}(n_{0_X})$ and $(\eta_X, \overline{\eta}_X)$ is the fill-in provided by the universal property of the bikernel $\mathcal{N}(c_{0^X})$.

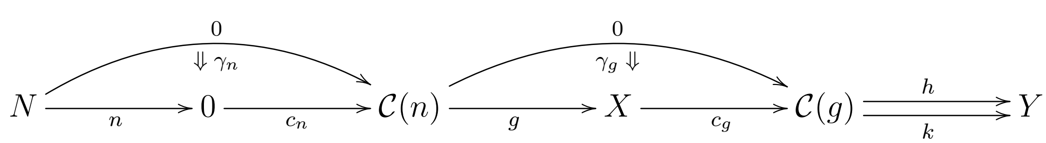
7.1.2. LEMMA. *Let $\mathcal{B}$ be as in 7.1.1.*

1. *(a) Consider arrows n, g, k and a 2-cell λ as in the following diagram:*



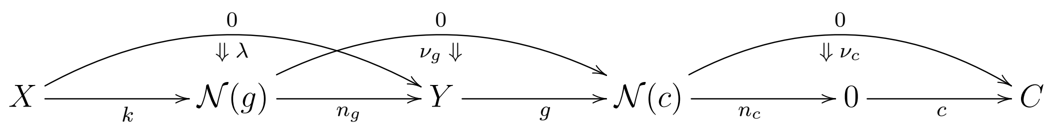
Then, there exists a unique 2-cell $\overline{\lambda}: 0 \Rightarrow k$ such that $(0^{c_g} \circ 0_Y) \cdot (c_g \circ \overline{\lambda}) = \lambda$.

(b) Consider arrows n, g, h and k as in the following diagram:



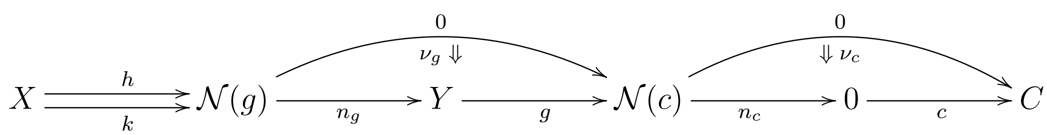
Let $\lambda\colon c_g \cdot h \Rightarrow c_g \cdot k$ *be a 2-cell. Then, there exists a unique 2-cell* $\overline{\lambda}\colon h \Rightarrow k$ *such that* $c_g \circ \overline{\lambda} = \lambda$.

2. (a) *Consider arrows* k, g, c *and a 2-cell* λ *as in the following diagram:*



Then, there exists a unique 2-cell $\overline{\lambda}\colon 0 \Rightarrow k$ *such that* $(0^X \circ 0_{n_g}) \cdot (\overline{\lambda} \circ n_g) = \lambda$.

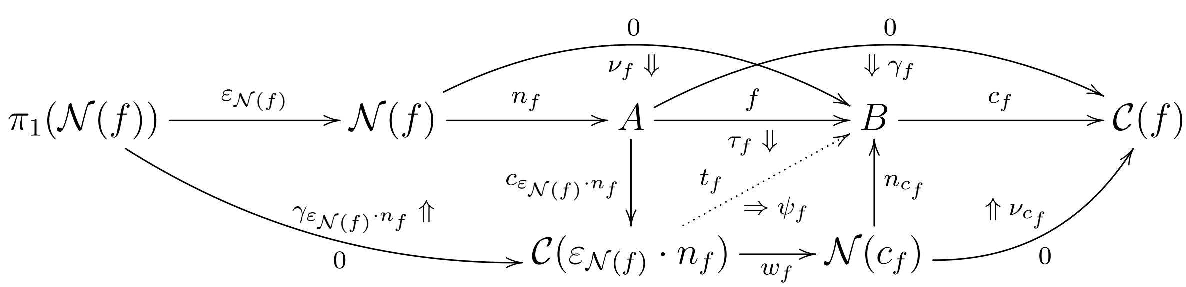
(b) *Consider arrows* h, k, g *and* c *as in the following diagram:*



Let $\lambda\colon h \cdot n_g \Rightarrow k \cdot n_g$ *be a 2-cell. Then, there exists a unique 2-cell* $\overline{\lambda}\colon h \Rightarrow k$ *such that* $\overline{\lambda} \circ n_g = \lambda$.

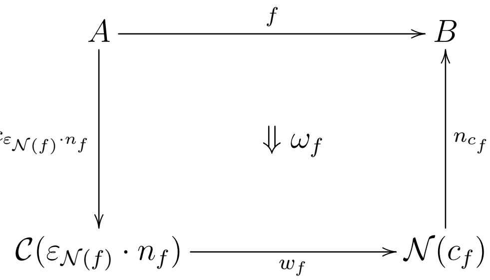
Proof. 1.(a) In order to apply Remark 2.3.8 to the 2-cell λ, we have to check the condition $(0^g \circ 0_Y) \cdot (g \circ \lambda) = (0^{\mathcal{C}(n)} \circ 0_k) \cdot (\gamma_g \circ k)$. Thanks to 2.3.7, it is enough to precompose with c_n and to verify the condition $c_n \cdot (0^g \circ 0_Y) \cdot (g \circ \lambda) = c_n \cdot (0^{\mathcal{C}(n)} \circ 0_k) \cdot (\gamma_g \circ k)$. This last equality is verified because it involves 2-cells between arrows whose domain is a bizero object.
1.(b) In order to apply condition 2.3.6.2" to the 2-cell λ, we have to check the condition $(0^{\mathcal{C}(n)} \circ 0_h) \cdot (\gamma_g \circ h) \cdot (g \circ \lambda) = (0^{\mathcal{C}(n)} \circ 0_k) \cdot (\gamma_g \circ k)$. Thanks to 2.3.7, it is enough to check this equation precomposed with $c_n\colon 0 \to P$ and we conclude as in the first part of the proof. ∎

7.1.3. . Let $\mathcal{B}$ be as in 7.1.1. Fix an arrow $f\colon A \to B$ in $\mathcal{B}$ and consider the following diagram



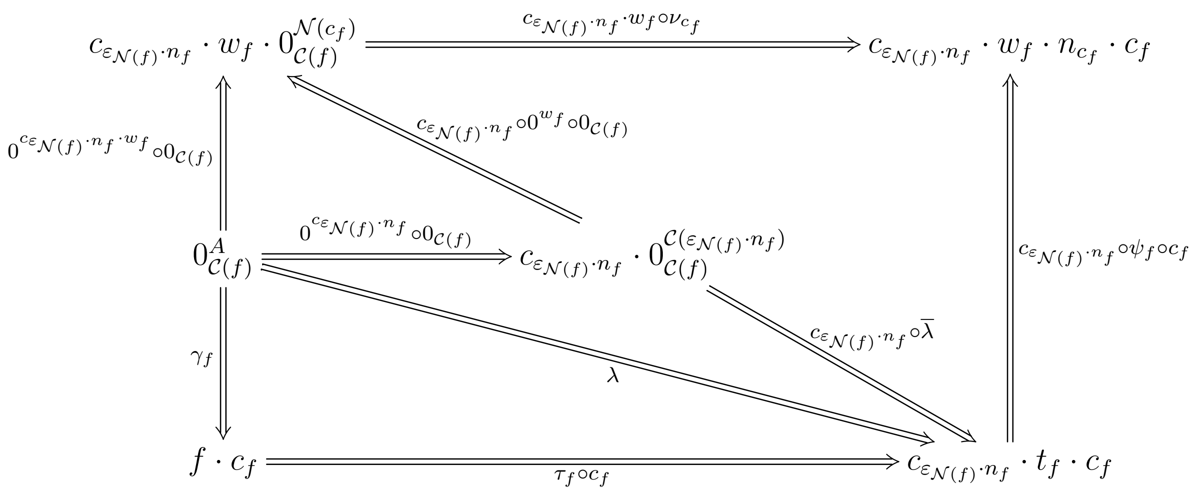
Let us start by explaining how to get the arrow w_f and a 2-celle ω_f to fill-in the square. The pair (t_f, τ_f) is a fill-in of $(f, (0^{\varepsilon_{\mathcal{N}(f)}} \circ 0_B) \cdot (\varepsilon_{\mathcal{N}(f)} \circ \nu_f))$ through the bicokernel of $\varepsilon_{\mathcal{N}(f)} \cdot n_f$. Now we put $X = \mathcal{N}(f)$ in 7.1.1 and we apply Lemma 7.1.2.1(a) by taking $n = n_{0_{\mathcal{N}(f)}}, g = \varepsilon_{\mathcal{N}(f)} \cdot n_f, k = t_f \cdot c_f$ and $\lambda = \gamma_f \cdot (\tau_f \circ c_f)$. We obtain a 2-cell $\overline{\lambda}\colon 0 \Rightarrow t_f \cdot c_f$.

Finally, the pair (w_f, ψ_f) is a fill-in of $(t_f, \overline{\lambda})$ through the bikernel of c_f. If we call $\omega_f = \tau_f \cdot (c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \psi_f)$, we get



We are going to prove that the pair $(c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f, \omega_f)$ is a fill-in of (f, γ_f) through the bikernel of c_f. Moreover, the pair (w_f, ω_f) is characterized by such a condition, in the sense that, if an arrow $x \colon \mathcal{C}(\varepsilon_{\mathcal{N}(f) \cdot n_f}) \to \mathcal{N}(c_f)$ and a 2-cell $\xi \colon f \Rightarrow c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot x \cdot n_{c_f}$ are such that the pair $(c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot x, \xi)$ is another fill-in of (f, γ_f) through the bikernel of c_f, then there exists a unique 2-cell $\overline{\xi} \colon x \Rightarrow w_f$ such that $\xi \cdot (c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \overline{\xi} \circ n_{c_f}) = \omega_f$.

Proof. The first thing to prove is that $(c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f, \omega_f)$ is a fill-in of (f, γ_f) through the bikernel of c_f. This amounts to the commutativity of the following diagram:

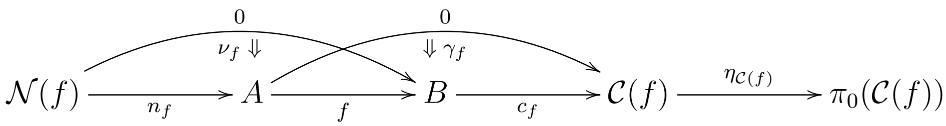


The four regions commute: by the condition on (w_f, ψ_f) to be a fill-in (the top-right triangle), by point 4 of Remark 2.3.5 (the top-left triangle), by the fact that λ factors through $\overline{\lambda}$ (the triangle in the middle), by the definition of λ (the bottom-left triangle). Now assume that the pair $(c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot x, \xi)$ is another fill-in of (f, γ_f) through the bikernel of c_f. By the essential uniqueness of the fill-in in the universal property of the bikernel, there exists a unique 2-cell $\hat{\xi} \colon c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot x \Rightarrow c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \cdot w_f$ such that $\xi \cdot (\hat{\xi} \circ n_{c_f}) = \omega_f$. We can apply 7.1.2.1(b) by taking $n = n_{0_{\mathcal{N}(f)}}, g = \varepsilon_{\mathcal{N}(f)} \cdot n_f, h = x, k = w_f$ and $\lambda = \hat{\xi}$. We get a unique 2-cell $\overline{\xi} \colon x \Rightarrow w_f$ such that $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \overline{\xi} = \hat{\xi}$. It follows that

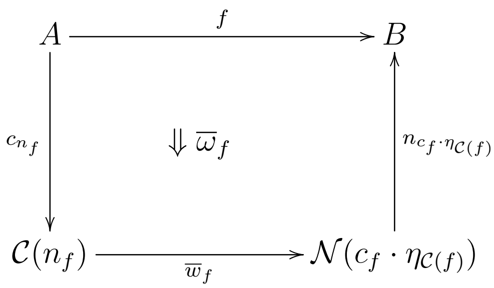
$$\xi \cdot (c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \overline{\xi} \circ n_{c_f}) = \xi \cdot (\hat{\xi} \circ n_{c_f}) = \omega_f$$

Uniqueness of $\overline{\xi}$: if $\psi\colon x \Rightarrow w_f$ is another 2-cell such that $\xi \cdot (c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \psi \circ n_{c_f}) = \omega_f$, then $c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f} \circ \psi = \hat{\xi}$ by definition of $\hat{\xi}$, and then $\psi = \overline{\xi}$ by definition of $\overline{\xi}$. ■

7.1.4. . Let $\mathcal{B}$ be as in 7.1.1 and consider once again an arrow $f\colon A \to B$ in $\mathcal{B}$. Starting from



and using Lemma 7.1.2.2, we get



which is a fill-in of (f, ν_f) through the bicokernel of n_f. Moreover, the pair $(\overline{w}_f, \overline{\omega}_f)$ is characterized by such a condition. The various steps to construct $\overline{w}_f$ and $\overline{\omega}_f$ are dual of those in 7.1.3.

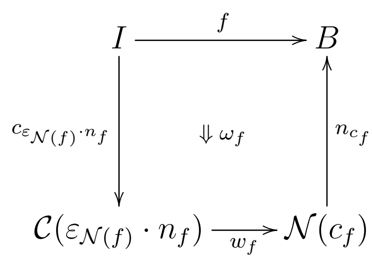
7.1.5. Definition. (M. Dupont [15]) A bicategory $\mathcal{B}$ is 2-abelian if:

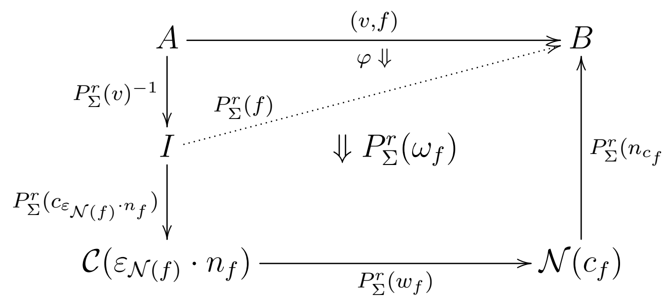
1) 2-cells in $\mathcal{B}$ are invertible;
2) $\mathcal{B}$ has a bizero object, bicokernels, bikernels, bicoproducts and biproducts;
3) $\mathcal{B}$ is 2-Puppe exact: for every arrow $f\colon A \to B$ in $\mathcal{B}$, the arrows w_f and $\overline{w}_f$ constructed in 7.1.3 and 7.1.4 are equivalences.

7.2. The main result. In this section, we prove that, when the base category $\mathcal{A}$ is abelian, the bicategory of fractions of $\mathbf{Arr}(\mathcal{A})$ with respect to the class Σ of weak equivalences is 2-abelian.

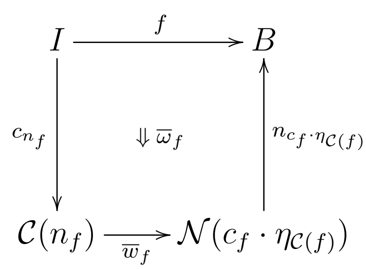
7.2.1. Lemma. *Let $\mathcal{B}$ be a bicategory with invertible 2-cells, zero object, bicokernels and bikernels. Consider a class Σ of arrows in $\mathcal{B}$ and assume that it is a bipullback congruence and a bipushout congruence. Fix an arrow $(v, f)\colon A \to B$ in $\mathcal{B}^r[\Sigma^{-1}]$ together with a chosen 2-cell $\varphi\colon (v, f) \Rightarrow P^r_\Sigma(u)^{-1} \cdot P^r_\Sigma(f)$ (see 3.1.1.8).*

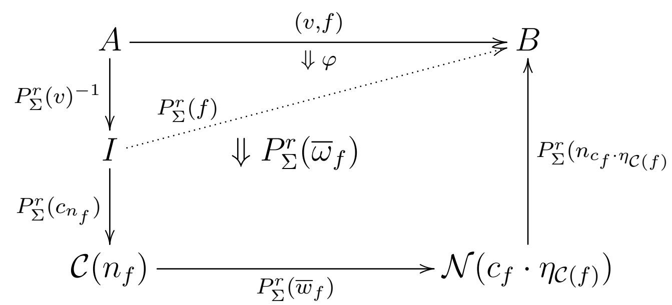
1. If the diagram hereunder on the left is the three-step factorization of f in $\mathcal{B}$ as in 7.1.3, then the diagram hereunder on the right is the corresponding three-step *factorization of* (v, f) *in* $\mathcal{B}^r[\Sigma^{-1}]$.



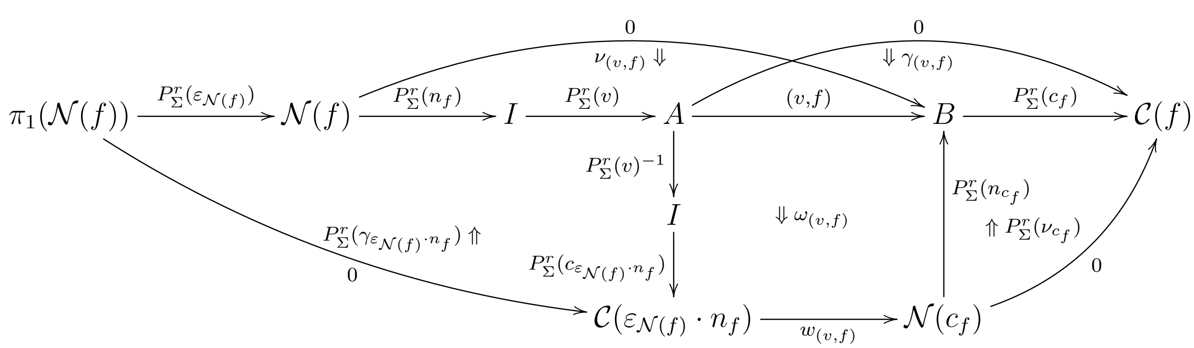


2. *If the diagram hereunder on the left is the three-step factorization of* f *in* $\mathcal{B}$ *as in 7.1.4, then the diagram hereunder on the right is the corresponding three-step factorization of* (v, f) *in* $\mathcal{B}^r[\Sigma^{-1}]$.





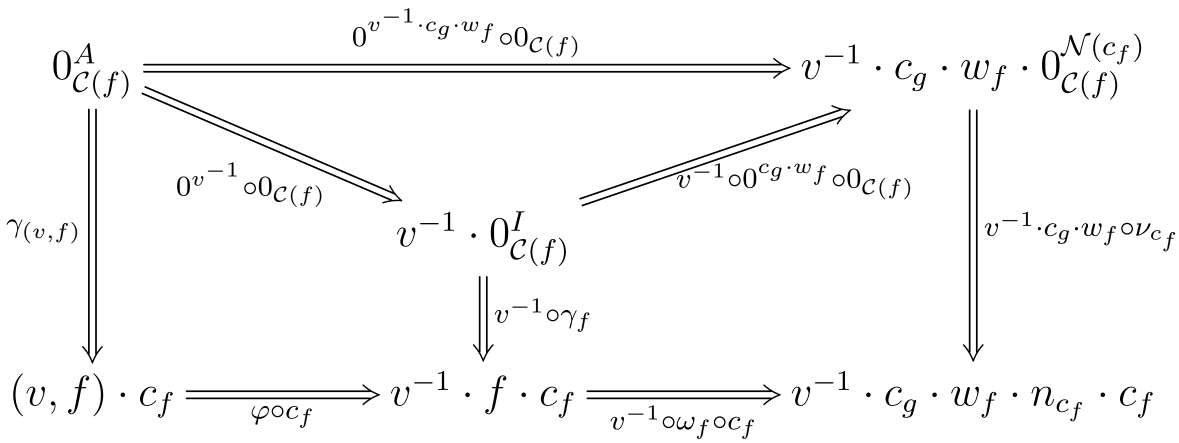
PROOF. Observe that, since $P^r_\Sigma \colon \mathcal{B} \to \mathcal{B}^r[\Sigma^{-1}]$ preserves zero object, bikernels and bicokernels (see Corollary 3.2.3), it preserves also the construction of π_0 and π_1 (see 7.1.1).
1. Since P^r_Σ preserves zero object, bikernels and bicokernels, and keeping in mind the description of bikernels and bicokernels in $\mathcal{B}^r[\Sigma^{-1}]$ established in 3.2.4, the factorisation of (v, f) fits into the following diagram (compare with 7.1.3)



Therefore, following 7.1.3, all what we have to do is to prove that the pair

$$(P^r_\Sigma(v)^{-1} \cdot P^r_\Sigma(c_{\varepsilon_{\mathcal{N}(f)} \cdot n_f}) \cdot P^r_\Sigma(w_f), \varphi \cdot (P^r_\Sigma(v)^{-1} \circ P^r_\Sigma(\omega_f)))$$

is a fill-in of $((v,f),\gamma_{(v,f)})$ through the bikernel of $c_{(v,f)} = P^r_\Sigma(c_f)$. This amounts to the commutativity of the following diagram (where we omit to write P^r_Σ and we write c_g for $c_{\varepsilon_{\mathcal{N}(f)}\cdot n_f}$ to make it readable):

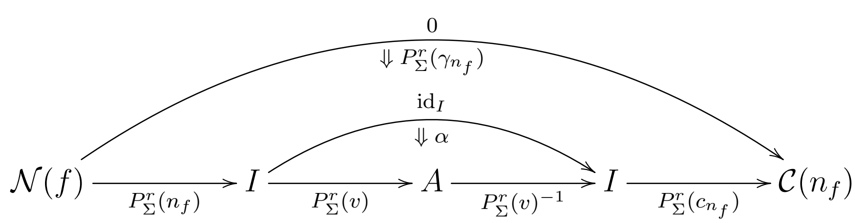


The three regions commute: the triangle on the top commutes by 2.3.5.4, the trapezoid on the left is the description of $\gamma_{(v,f)}$ given in 3.2.4.2, and the trapezoid on the right commutes because the pair $(c_{\varepsilon_{\mathcal{N}(f)}\cdot n_f}\cdot w_f, \omega_f)$ is a fill-in of (f,γ_f) through the bikernel of c_f.

2. Using once again 3.2.3 and 3.2.4, the proof reduces to check that the pair

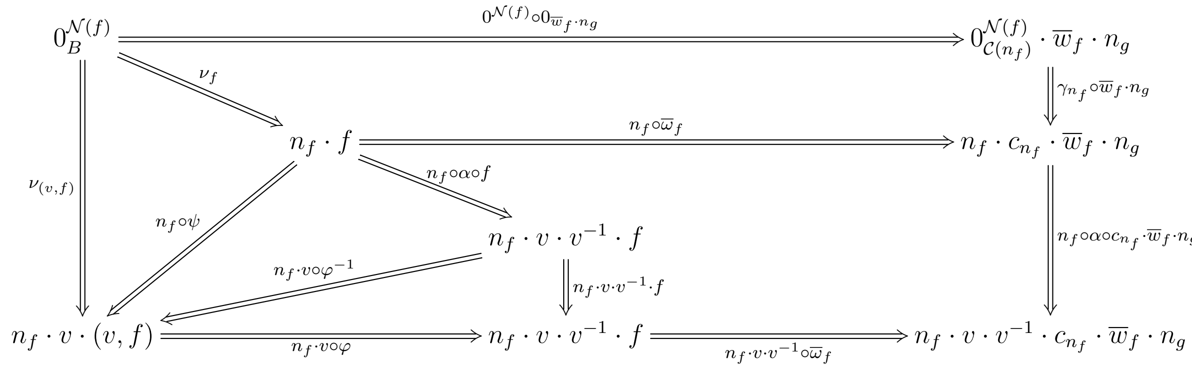
$$(P^r_\Sigma(\overline{w}_f)\cdot P^r_\Sigma(n_{c_f\cdot\eta_{\mathcal{C}(f)}}), \varphi\cdot(P^r_\Sigma(v)^{-1}\circ P^r_\Sigma(\overline{\omega}_f)))$$

is a fill-in of $((v,f),\nu_{(v,f)})$ through the bicokerne of $n_{(v,f)} = P^r_\Sigma(n_f)\cdot P^r_\Sigma(v)$. Since such a bicokernel can be described using the bicokernel of n_f as



where α attests that $P^r_\Sigma(v)$ and $P^r_\Sigma(v)^{-1}$ are quasi-inverse equivalences, the condition of fill-in amounts to the commutativity of the following diagram (where we omit to write P^r_Σ

and we write n_g for $n_{c_f \cdot \eta_{\mathcal{C}(f)}}$ to make it readable):



The five regions commute: the concave region commutes by interchange, the trapezoid on the top commutes because the pair $(\overline{w}_f \cdot n_{c_f \cdot \eta_{\mathcal{C}(f)}}, \overline{\omega}_f)$ is a fill-in of (f, ν_f) through the bicokernel of n_f, the first triangle is the description of $\nu_{(v,f)}$ given in 3.2.4.1, the second triangle commutes by 3.2.4.3, the commutativity of the third triangle is obvious. ■

7.2.2. Proposition. *Let $\mathcal{A}$ be an abelian category and let Σ be the class of weak equivalences in* $\mathbf{Arr}(\mathcal{A})$. *The bicategory of fractions* $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$ *is 2-abelian.*

Proof. Since $\mathbf{Arr}(\mathcal{A})$ has a bizero object, bipushouts and bipullbacks (see Corollary 5.2.6 and Corollary 5.2.9) and since the class Σ is a bipushout congruence and a bipullback congruence (see Proposition 6.2.2), we can use 3.1.4 and Corollary 3.2.3 to conclude that $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$ has a bizero object, bipushouts and bipullbacks. In particular, $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$ has bicokernels, bicoproducts, bikernels and biproducts. Moreover, since 2-cells are invertible in $\mathbf{Arr}(\mathcal{A})$, by Lemma 3.1.8 the same holds in $\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$. Finally, the conditions to be 2-Puppe exact follow from Proposition 6.1.5 and Lemma 7.2.1. ■

7.2.3. . Under the assumptions of item 7.1.1, a quick way to define discrete and connected objects in a bicategory $\mathcal{B}$ is to say that X is discrete if $\eta_X \colon X \to \pi_0(X)$ is an equivalence, and that X is connected if $\varepsilon_X \colon \pi_1(X) \to X$ is an equivalence. Let us recall from [15] that, if $\mathcal{B}$ is 2-abelian, then the sub-bicategories of discrete objects and of connected objects are abelian categories and they are equivalent. When $\mathcal{B} = \mathbf{Arr}(\mathcal{A})$, with this construction one precisely recovers the abelian category $\mathcal{A}$.

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