

YONEDA LEMMA AND REPRESENTATION THEOREM FOR DOUBLE CATEGORIES

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ABSTRACT. We study (vertically) normal lax double functors valued in the weak double category $\mathbb{C}at$ of small categories, functors, profunctors and natural transformations, which we refer to as *lax double presheaves*. We show that for the theory of double categories they play a similar role as 2-functors valued in $\mathbb{C}at$ for 2-categories. We first introduce representable lax double presheaves and establish a Yoneda lemma. Then we build a Grothendieck construction which gives a 2-equivalence between lax double presheaves and discrete double fibrations over a fixed double category. Finally, we prove a representation theorem showing that a lax double presheaf is represented by an object if and only if its Grothendieck construction has a double terminal object.

Contents

1	Introduction	1698
2	Background on double categories	1704
3	Lax double presheaves	1710
4	Yoneda lemma	1729
5	Discrete double fibrations	1741
6	Grothendieck construction for lax double presheaves	1750
7	Representation theorem for lax double presheaves	1775

1. Introduction

THE CLASSICAL STORY. Universal constructions are the core of the study of category theory. Such constructions are determined by universal properties which are obtained by requiring that a certain presheaf is representable. A *presheaf* on a category \mathcal{C} is a functor $X: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ into the category of sets and maps, and it is said to be *represented* by an object \hat{x} in \mathcal{C} if it is isomorphic to the representable functor $\mathcal{C}(-, \hat{x}): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ taking as values the hom sets of \mathcal{C} with target \hat{x} . Presheaves therefore play an important role in category theory and, for this reason, they have been extensively studied.

One of the most fundamental results in ordinary category theory is the Yoneda lemma. It says that, given a presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$, natural transformations $\mathcal{C}(-, \hat{x}) \Rightarrow X$ are

Received by the editors 2024-03-21 and, in final form, 2024-10-27.

Transmitted by Joachim Kock. Published on 2024-11-01.

2020 Mathematics Subject Classification: 18N10, 18D30, 18D60, 18D40.

Key words and phrases: Double categories, presheaves, Yoneda lemma, Grothendieck construction, representation theorem.

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completely determined by a single element of the set $X\hat{x}$, for every object \hat{x} in \mathcal{C} . More explicitly, this says that we have an isomorphism of sets

$$[\mathcal{C}^{\text{op}}, \text{Set}](\mathcal{C}(-, \hat{x}), X) \cong X\hat{x};$$

see e.g. [Rie17, Theorem 2.2.4]. As a consequence, we get a Yoneda embedding $\mathcal{C} \hookrightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$, allowing us to infer universal constructions in \mathcal{C} from an analogous setting in presheaves, where a very explicit computation of such constructions is possible.

Another approach to presheaves is given by studying them from a fibrational perspective through the Grothendieck construction. This construction encodes the structure of a presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ in a different way by compressing all of its data into a single category $\int_{\mathcal{C}} X$ coming with a canonical projection onto \mathcal{C} . This projection $\int_{\mathcal{C}} X \rightarrow \mathcal{C}$ is the prototypical example of a discrete fibration. A *discrete fibration* is a functor $P: \mathcal{E} \rightarrow \mathcal{C}$ such that every morphism in \mathcal{C} of the form $f: x \rightarrow Pe$ admits a unique lift $e' \rightarrow e$ in \mathcal{E} . In fact, the Grothendieck construction induces an equivalence

$$\int_{\mathcal{C}}: \text{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{\cong} \text{Fib}(\mathcal{C}) \subseteq \text{Cat}/\mathcal{C},$$

where $\text{Fib}(\mathcal{C})$ denotes the full subcategory of the slice Cat/\mathcal{C} of categories over \mathcal{C} spanned by the discrete fibrations; see e.g. [LR20, Theorem 2.1.2].

A powerful consequence of this equivalence is that one can derive another criterion to detect representability of a presheaf by testing whether its Grothendieck construction has a terminal object; see e.g. [Rie17, Proposition 2.4.8]. We refer to this result as the *representation theorem*.

A DRAMA IN 2-CATEGORIES. The categories that one considers in practice often have more structure. Typically, the category Cat of categories and functors further comes with a notion of natural transformations between its morphisms. This is an example of a *2-category*, which, in addition to objects and morphisms, also have 2-morphisms between their morphisms. In this context, morphisms between two objects in a 2-category \mathcal{C} now form a category rather than a set, and so the representable presheaves of a 2-category take values in the 2-category Cat . We refer to 2-functors $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ as *2-presheaves*. This allows for a refined version of universal properties by requiring that certain 2-presheaves $X: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ are *represented* by an object \hat{x} in \mathcal{C} in this higher sense, i.e., there is an isomorphism between X and the representable 2-presheaf $\mathcal{C}(-, \hat{x}): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$. This gives a better-behaved definition of universal constructions in the 2-categorical context, as these are now compatible with the higher structure of the 2-categories involved.

One can then formulate a 2-categorical version of the Yoneda lemma by upgrading isomorphisms of sets into isomorphisms of categories. It says that, given a 2-presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ and an object \hat{x} in \mathcal{C} , there is an isomorphism of categories

$$[\mathcal{C}^{\text{op}}, \text{Cat}](\mathcal{C}(-, \hat{x}), X) \cong X\hat{x};$$

see e.g. [JY21, Lemma 8.3.16]. Again, this yields a Yoneda embedding $\mathcal{C} \hookrightarrow [\mathcal{C}^{\text{op}}, \text{Cat}]$, which shows how the role of Set has been replaced by Cat in the 2-categorical world.

On the other hand, a first 2-categorical version of the Grothendieck construction was introduced by Buckley. As shown in [Buc14, Theorem 2.2.11], this Grothendieck construction induces a 2-equivalence of 2-categories

$$\int_{\mathcal{C}}: [\mathcal{C}^{\text{op}}, \text{Cat}] \xrightarrow{\cong} 2\mathcal{Fib}(\mathcal{C}) \subseteq 2\text{Cat}/\mathcal{C},$$

where $2\mathcal{Fib}(\mathcal{C})$ denotes the 2-subcategory of the slice $2\text{Cat}/\mathcal{C}$ of 2-categories onto \mathcal{C} spanned by certain 2-fibrations; see [Buc14, Definition 2.1.10]. While this gives a fibrational perspective on 2-presheaves, there is a priori no obvious generalization of the representation theorem in this setting. Indeed, as clingman and the second author show in [cM22a], it is not true that a 2-presheaf X is represented by an object if and only if its Grothendieck construction $\int_{\mathcal{C}} X$ admits a 2-terminal object. Instead, one would need to generalize results by Gagna–Harpaz–Lanari [GHL22] to obtain a more appropriate characterization.

THE SAVIOR: DOUBLE CATEGORIES. However, passing to a double categorical context allows for such a characterization. Double categories are another type of 2-dimensional categorical structure, namely, the internal categories to Cat , which have objects, two different kinds of morphisms between objects—the *horizontal* and *vertical* morphisms—, and 2-dimensional morphisms called *squares*. In particular, every 2-category can be seen as a double category with only trivial vertical morphisms. Then, Grandis–Paré introduce in [GP19, §1.2] a double categorical Grothendieck construction which takes a 2-presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ to a double category $\iint_{\mathcal{C}} X$ coming with a canonical projection onto \mathcal{C} (seen as a double category). Analogously to ordinary categories, this projection $\iint_{\mathcal{C}} X \rightarrow \mathcal{C}$ is a *discrete double fibration*, i.e., the induced functors on the categories of objects and horizontal morphisms, and on the categories of vertical morphisms and squares are discrete fibrations of categories. Then, in [MSV23, Theorem 5.1], the second author with Sarazola–Verdugo show that this double Grothendieck construction also induces a 2-equivalence of 2-categories

$$\iint_{\mathcal{C}}: [\mathcal{C}^{\text{op}}, \text{Cat}] \xrightarrow{\cong} \text{D}\mathcal{Fib}(\mathcal{C}) \subseteq \text{DblCat}_v/\mathcal{C},$$

where $\text{D}\mathcal{Fib}(\mathcal{C})$ denotes the 2-full 2-subcategory of $\text{DblCat}_v/\mathcal{C}$ spanned by the discrete double fibrations. Here, DblCat_v denotes the 2-category of double categories, double functors, and vertical transformations; see Notation 2.17.

Using this double Grothendieck construction, a representation theorem can now be formulated by saying that a 2-presheaf is represented by an object if and only if its double Grothendieck construction admits a double terminal object; see [cM22b, Theorem 6.8] and [MSV23, Theorem 6.12]. Since these results already require the language of double categories, one might wonder whether the whole situation extends to double categories and a suitable notion of double presheaves.

THE STORY CONTINUES FOR DOUBLE CATEGORIES. In the double categorical world, Paré [Par11] and Fiore–Gambino–Kock [FGK12] prove a Yoneda lemma for “set-valued” double presheaves. In this context, the target of the double presheaves is given by the (weak) double category Set of sets, maps, spans, and maps of spans. Given a double

category \mathbb{C} , the representable presheaf at an object \hat{x} in \mathbb{C} is then given by associating to each object x in \mathbb{C} the set of horizontal morphisms $x \rightarrow \hat{x}$ in \mathbb{C} . Moreover, to each vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , one can associate a span of sets whose middle object is the set of all squares in \mathbb{C} of the form

$$\begin{array}{ccc}
 x & \xrightarrow{f} & \hat{x} \\
 u \downarrow & \alpha & \Downarrow \\
 x' & \xrightarrow{f'} & \hat{x}
 \end{array} \tag{1.1}$$

However, such an assignment fails to be functorial in vertical morphisms, and therefore this forces the correct notion of double presheaves to be lax in the vertical direction. Hence a *lax double presheaf* is a lax double functor $\mathbb{C}^{\text{op}} \rightarrow \text{Set}$. Using this notion, the Yoneda lemma proven as [Par11, Theorem 2.3] and [FGK12, Proposition 3.10], respectively, shows that, for every object \hat{x} in \mathbb{C} , there is an isomorphism of sets

$$[\mathbb{C}^{\text{op}}, \text{Set}]^{\text{lax}}(\mathbb{C}(-, \hat{x}), X) \cong X\hat{x}.$$

However, since those presheaves only detect sets of morphisms, when taking $\mathbb{C} = \mathcal{C}$ to be a 2-category, there is no obvious way on how to retrieve the 2-categorical Yoneda lemma from this statement.

In this paper, we want to enhance the structure of those double presheaves by considering “category-valued” double presheaves instead. For this, we consider the (weak) double category Cat of categories, functors, profunctors, and natural transformations, in place of Set . Then, the representable presheaf at an object \hat{x} in \mathbb{C} will now associate to each object x in \mathbb{C} the category $\mathbb{C}(x, \hat{x})$ of horizontal morphisms $x \rightarrow \hat{x}$ and globular squares in \mathbb{C} of the form

$$\begin{array}{ccc}
 x & \xrightarrow{f} & \hat{x} \\
 \Downarrow & \alpha & \Downarrow \\
 x & \xrightarrow{f'} & \hat{x}
 \end{array}$$

Then, similarly to before, to each vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , one can associate a profunctor which assigns to any two horizontal morphisms $f: x \rightarrow \hat{x}$ and $f': x' \rightarrow \hat{x}$, the set of squares of the form (1.1). While this assignment is still not compatible with vertical composition (see Remark 3.44), by adding the extra structure of a category to the values of the representable presheaf, it now becomes *normal*, i.e., it preserves vertical identities strictly. Therefore, we define *lax double presheaves* as normal lax double functors $\mathbb{C}^{\text{op}} \rightarrow \text{Cat}$.

We then obtain the following Yoneda lemma, which appears as Theorem 4.2. Here we denote by $[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}}$ the 2-category of (normal) lax double presheaves, horizontal transformations, and globular modifications; see Notation 3.30.

1.1. THEOREM. *Given a double category \mathbb{C} , an object \hat{x} in \mathbb{C} , and a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, there is an isomorphism of categories*

$$[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}}(\mathbb{C}(-, \hat{x}), X) \cong X\hat{x},$$

which is 2-natural in \hat{x} and X .

This gives a 2-categorical enhancement of the Yoneda lemma for double categories shown by Paré and Fiore–Gambino–Kock. Moreover, by taking $\mathbb{C} = \mathcal{C}$ to be a 2-category, we retrieve as a straightforward consequence the Yoneda lemma for 2-categories mentioned above, as shown in Corollary 4.10.

Continuing the story, Paré introduces in [Par11, §3.7] a Grothendieck construction for “set-valued” lax double presheaves $X: \mathbb{C}^{\text{op}} \rightarrow \text{Set}$ in the form of double categories $\int_{\mathbb{C}} X$ with a canonical projection onto \mathbb{C} . Lambert then shows in [Lam21, Lemma 2.10] that this projection $\int_{\mathbb{C}} X \rightarrow \mathbb{C}$ is a discrete double fibration, and in [Lam21, Theorem 2.27] that the Grothendieck construction induces an equivalence of categories

$$\int_{\mathbb{C}}: [\mathbb{C}^{\text{op}}, \text{Set}]^{\text{lax}} \xrightarrow{\cong} \text{DFib}(\mathbb{C}) \subseteq \text{DblCat}/\mathbb{C}, \quad (1.2)$$

where $\text{DFib}(\mathbb{C})$ denotes the full (1-)subcategory of DblCat/\mathbb{C} spanned by the discrete double fibrations.

In this paper, we extend Paré’s Grothendieck construction to our “category-valued” lax double presheaves. Given a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, we also build in Construction 6.1 a double category $\int_{\mathbb{C}} X$ coming with a canonical projection onto \mathbb{C} . Surprisingly, despite the fact that X is a (normal) lax double functor rather than a strict one, its Grothendieck construction $\int_{\mathbb{C}} X$ is an actual strict double category; note that this already happens for Paré’s Grothendieck construction (see [Par11, Theorem 3.8]). Even though we now consider presheaves valued in categories rather than in sets, the projection $\int_{\mathbb{C}} X \rightarrow \mathbb{C}$ is still a discrete double fibration.

Furthermore, by extending the Grothendieck construction to a 2-functor, we can then show the following equivalence with discrete double fibrations, which appears as Theorem 6.13.

1.2. THEOREM. *Given a double category \mathbb{C} , the Grothendieck construction induces a 2-equivalence of 2-categories*

$$\int_{\mathbb{C}}: [\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}} \xrightarrow{\cong} \text{DFib}(\mathbb{C}) \subseteq \text{DblCat}_v/\mathbb{C},$$

which is pseudo-natural in \mathbb{C} .

Again, by taking $\mathbb{C} = \mathcal{C}$ to be a 2-category, we retrieve as a straightforward consequence the 2-equivalence between 2-presheaves and discrete double fibrations from [MSV23], as shown in Corollary 6.30.

There is now an intriguing relation between Lambert’s equivalence from (1.2) and our 2-equivalence from Theorem 1.2. At its heart is the fact that the fibers of a discrete double fibration are categories—seen as double categories with trivial horizontal morphisms and squares—; see Proposition 5.11. Therefore, to obtain an equivalence with a certain type of double presheaves, one needs to encode both the objects and the morphisms of those fibers into the presheaf. While the inverse of the equivalence (1.2) only detects the underlying sets of objects of the fibers as the values of the presheaf, the morphisms are encoded in the

lax unitality condition. On the other hand, the inverse of the equivalence from Theorem 1.2 directly chooses the values of the presheaf to be the categories given by the fibers, and the lax unitality condition now becomes strict. In fact, the relation between the 2-categories $[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}}$ and $[\mathbb{C}^{\text{op}}, \text{Set}]^{\text{lax}}$ can be understood from the universal property of the adjunction developed by Cruttwell–Shulman in [CS10, Proposition 5.14], using that $\text{Mod}(\text{Set}) = \text{Cat}$.

The idea of enhancing “set-valued” lax double presheaves to “category-valued” ones is already studied in a paper by Cruttwell–Lambert–Pronk–Szyld, where they consider lax double *pseudo* functors $\mathbb{C}^{\text{op}} \rightarrow \text{Span}(\text{Cat})$ into the (weak) double category $\text{Span}(\text{Cat})$ of categories, functors, spans of categories, and morphisms of spans. For such presheaves, they show in [CLPS22, Theorem 3.45] that there is an equivalence of categories

$$[\mathbb{C}^{\text{op}}, \text{Span}(\text{Cat})]^{\text{lax}} \xrightarrow{\cong} \text{DbfFib}(\mathbb{C}) \subseteq \text{DbfCat}/\mathbb{C},$$

where $\text{DbfFib}(\mathbb{C})$ denotes the subcategory of DbfCat/\mathbb{C} spanned by the *double fibrations*; see [CLPS22, Definition 3.38]. These double fibrations generalize discrete double fibrations in the same vein that Grothendieck fibrations generalize discrete fibrations for ordinary categories, meaning that their fibers are double categories instead of categories. By restricting the above equivalence to discrete double fibrations, they retrieve Lambert’s equivalence (1.2). Hence, this suggests that the approach taken in this paper is different from theirs.

Finally, we use the Grothendieck equivalence from Theorem 1.2 to prove a representation theorem for double categories. Namely, we want to find a criterion for a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ to be *represented* by an object \hat{x} of \mathbb{C} , i.e., isomorphic to the representable lax double presheaf $\mathbb{C}(-, \hat{x}): \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$. As for the case of 2-categories, these will be determined by double terminal objects in the Grothendieck construction. The following result appears as Theorem 7.9.

1.3. THEOREM. *A lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ is represented by an object if and only if its Grothendieck construction $\int_{\mathbb{C}} X$ has a double terminal object.*

Again, by choosing $\mathbb{C} = \mathcal{C}$ to be a 2-category, we retrieve the representation theorem for 2-categories from [cM22b, MSV23], as shown in Corollary 7.11.

NOTATIONS. Throughout the paper, we will assume basic knowledge about 2-category theory. We refer the reader to [JY21] for a complete account of the theory of 2-categories. We use the following notations:

- we write Set for the category of sets and maps,
- we write Cat for the (2-)category of categories, functors, (and natural transformations),
- we write 2Cat for the 2-category of 2-categories, 2-functors, and 2-natural transformations.

ACKNOWLEDGMENT. We would like to thank Denis-Charles Cisinski, Johannes Glossner, and Nima Rasekh for helpful discussions related to the subject of this paper. We are also grateful to Nathanael Arkor and David Kern for insightful comments on a first draft of this paper, and in particular for pointing to the result of Cruttwell–Shulman [CS10] shedding light on the relation between normal lax double functors into \mathbf{Cat} and lax double functors into \mathbf{Set} . We also thank the anonymous referee for their careful reading, their insightful comments, and for catching many typos.

During the realization of this work, the second author was a member of the Collaborative Research Centre “SFB 1085: Higher Invariants” funded by the Deutsche Forschungsgemeinschaft (DFG).

2. Background on double categories

In this section, we give a concise introduction of the main notions of double category theory needed in this paper. We refer the reader to [Gra19] for more details.

As our lax double presheaves will be lax double functors taking value in the weak double category \mathbf{Cat} , we start by introducing in Sections 2.1 to 2.6 the notions of weak double categories and (vertically) lax double functors between them. We further introduce horizontal and vertical transformations between such functors, as well as modifications.

By imposing strictness conditions, we retrieve in Sections 2.7 to 2.19 the notions of double categories, double functors, horizontal and vertical transformations between double functors, and modifications.

WEAK DOUBLE CATEGORIES. We start by introducing (vertically) weak double categories.

2.1. DEFINITION. A *weak double category* \mathbb{C} consists of

- objects x, y, x', y', \dots ,
- horizontal morphisms $f: x \rightarrow y$, with a horizontal identity 1_x at each object x , and an associative and unital composition gf for all composable horizontal morphisms f, g ,
- vertical morphisms $u: x \bullet \rightarrow x'$, with a vertical identity e_x at each object x , and a composition $u' \bullet u$ for all composable vertical morphisms u, u' ,
- squares α as depicted below, written inline as $\alpha: \left[u \begin{array}{c} f \\ f' \end{array} v \right] : \left[\begin{array}{cc} x & y \\ x' & y' \end{array} \right]$ or simply as $\alpha: \left[u \begin{array}{c} f \\ f' \end{array} v \right]$,

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow u & \alpha & \downarrow v \\ x' & \xrightarrow{f'} & y' \end{array}$$

with a horizontal identity square 1_u at each vertical morphism u , and an associative and unital horizontal composition $\beta \circ \alpha$ along vertical morphisms for all horizontally composable squares α, β , as well as a vertical identity square e_f at each horizontal morphism f , and a vertical composition $\alpha' \bullet \alpha$ along horizontal morphisms for all vertically composable squares α, α' .

If the vertical boundaries of a square are identities, i.e., $x = x', y = y', u = e_x$ and $v = e'_y$, we call such a square *globular*.

- for all composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x'' \xrightarrow{u''} x'''$, a horizontally invertible *associator square*

$$\alpha_{u,u',u''}: \left[(u'' \bullet u') \bullet u \begin{array}{c} 1_x \\ 1_{x'''} \end{array} u'' \bullet (u' \bullet u) \right],$$

- for every vertical morphism $u: x \rightarrow x'$, horizontally invertible *left* and *right unitor squares*

$$\lambda_u: \left[u \bullet e_x \begin{array}{c} 1_x \\ 1_{x'} \end{array} u \right] \quad \text{and} \quad \rho_u: \left[e_{x'} \bullet u \begin{array}{c} 1_x \\ 1_{x'} \end{array} u \right],$$

satisfying the following conditions:

- (1) horizontal and vertical compositions of squares satisfy the interchange law,
- (2) for all composable horizontal morphisms f, g , we have $e_g \circ e_f = e_{gf}$, and for every object x , we have $e_{1_x} = 1_{e_x}$,
- (3) for all composable vertical morphisms u, u' , we have $1_{u'} \bullet 1_u = 1_{u' \bullet u}$,
- (4) the associator square $\alpha_{u,u',u''}$ is natural in (u, u', u'') , and satisfies the pentagon axiom,
- (5) the unitor squares λ_u and ρ_u are natural in u , compatible with associator squares, and for an object x , one has $\lambda_{e_x} = \rho_{e_x}$.

If the unitor squares are horizontal identities, we call \mathbb{C} a *unitary weak double category*.

For a more detailed description of the coherence conditions, we refer the reader to [Gra19, Definition 3.3.1].

Next, we introduce a suitable notion of functors between weak double categories, namely the notion of a lax double functor. As horizontal composition in a weak double category is strictly associative and unital while the vertical one is not, it is not surprising that lax double functors only preserve strictly horizontal compositions and identities while, in the vertical direction, we get comparison squares.

2.2. DEFINITION. Given weak double categories \mathbb{C}, \mathbb{D} , a *lax double functor* $X: \mathbb{C} \rightarrow \mathbb{D}$ consists of

- assignments on objects, horizontal morphisms, vertical morphisms, and squares, which are compatible with sources and targets,
- for all composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x''$ in \mathbb{C} , a *composition comparison square* in \mathbb{D}

$$\mu_{u,u'}: \left[Xu' \bullet Xu \begin{array}{c} 1_{Xx} \\ 1_{Xx''} \end{array} X(u' \bullet u) \right] : \left[\begin{array}{cc} Xx & Xx \\ Xx'' & Xx'' \end{array} \right],$$

- for every object x in \mathbb{C} , an *identity comparison square* in \mathbb{D}

$$\varepsilon_x: \left[e_{Xx} \begin{array}{c} 1_{Xx} \\ 1_{Xx} \end{array} X e_x \right] : \left[\begin{array}{cc} Xx & Xx \\ Xx & Xx \end{array} \right],$$

satisfying the following conditions:

- (1) it preserves horizontal identities and horizontal compositions strictly,
- (2) composition comparison squares $\mu_{u,u'}$ are natural with respect to (u, u') , and compatible with associator squares,
- (3) identity comparison squares ε_x are natural with respect to x , and compatible with composition comparison squares and unitor squares.

If all identity comparison squares are horizontal identities, i.e., if X preserves strictly vertical identities, we call X a *normal lax double functor*. Consequently, the comparison squares $\mu_{u,e_{x'}}$ and $\mu_{e_x,u}$ agree with the identity.

For a more detailed description of the coherence conditions, we refer the reader to [Gra19, Definition 3.5.1].

Lax double functors between two weak double categories assemble into a weak double category, and we introduce here its horizontal morphisms, vertical morphisms, and squares.

2.3. DEFINITION. Given lax double functors $X, Y: \mathbb{C} \rightarrow \mathbb{D}$, a *horizontal transformation* $F: X \Rightarrow Y$ consists of

- for every object x in \mathbb{C} , a horizontal morphism $Fx: Xx \rightarrow Yx$ in \mathbb{D} ,
- for every vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , a square $F_u: \left[Xu \begin{array}{c} F_x \\ F_{x'} \end{array} Yu \right]$ in \mathbb{D} ,

such that the components F_x are natural in x , and the components F_u are natural in u and compatible with composition and identity comparison squares.

For a more detailed description of the coherence conditions, we refer the reader to [Gra19, §3.5.4].

2.4. DEFINITION. Given lax double functors $X, X': \mathbb{C} \rightarrow \mathbb{D}$, a *colax vertical transformation* $U: X \rightrightarrows X'$ consists of

- for every object x in \mathbb{C} , a vertical morphism $U_x: Xx \dashrightarrow X'x$ in \mathbb{D} ,
- for every horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , a square $U_f: \begin{bmatrix} U_x & Xf \\ X'f & U_y \end{bmatrix}$ in \mathbb{D} ,
- for every vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} , a *naturality comparison square* in \mathbb{D}

$$U_u: \begin{bmatrix} U_{x'} \bullet Xu & 1_{Xx} \\ 1_{X'x'} & X'u \bullet U_x \end{bmatrix},$$

satisfying the following conditions:

- (1) the components U_f are compatible with composition of horizontal morphisms and horizontal identities,
- (2) the components U_u are natural in u and compatible with composition and identity comparison squares.

If the naturality comparison squares are invertible, we call U a *pseudo vertical transformation*.

Moreover, if a pseudo vertical transformation is strictly natural in the vertical direction, i.e., if the natural comparison squares U_u are horizontal identities, we call U a *vertical transformation*.

For a more detailed description of the coherence conditions, we refer the reader to the transposed version of [Fem23, Definition 2.2]. See also [Gra19, Definition 3.8.2].

2.5. DEFINITION. Given lax double functors $X, X', Y, Y': \mathbb{C} \rightarrow \mathbb{D}$, horizontal transformations $F: X \rightrightarrows Y$, $F': X' \rightrightarrows Y'$ and colax vertical transformations $U: X \rightrightarrows X'$, $V: Y \rightrightarrows Y'$, then a *modification* $A: \begin{bmatrix} U & F \\ F' & V \end{bmatrix}$ consists of, for every object x in \mathbb{C} , a square in \mathbb{D}

$$A_x: \begin{bmatrix} U_x & F_x \\ F'_x & V_x \end{bmatrix} : \begin{bmatrix} Xx & Yx \\ X'x & Y'x \end{bmatrix}$$

satisfying the following conditions:

- (1) (horizontal compatibility) for every horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , one has

$$A_y \circ U_f = V_f \circ A_x,$$

- (2) (vertical compatibility) for every vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} , one has

$$(F'_u \bullet A_x) \circ U_u = V_u \circ (A_{x'} \bullet F_u).$$

If the vertical boundaries of a modification are identities, i.e., $X = X'$, $Y = Y'$, $U = e_X$ and $V = e_Y$, we call such a modification *globular*.

As claimed above, lax double functors assemble into a weak double category. As we will not make use of the vertical structure of this double category throughout the paper, we state the result as it appears in [Gra19, Theorem 3.8.4], where the vertical morphisms are given by the pseudo vertical transformation. A version with colax vertical transformations also exists, as proven in [Fem23, §2].

2.6. PROPOSITION. *Given two weak double categories \mathbb{C} and \mathbb{D} , there is a weak double category $\llbracket \mathbb{C}, \mathbb{D} \rrbracket^{\text{lax}}$ of lax double functors $\mathbb{C} \rightarrow \mathbb{D}$, horizontal transformations, pseudo vertical transformations and modifications.*

We denote by $\mathbf{H}\llbracket \mathbb{C}, \mathbb{D} \rrbracket^{\text{lax}}$ its underlying 2-category of lax double functors, horizontal transformations, and globular modifications.

(STRICT) DOUBLE CATEGORIES. To simplify computations, except for the weak double category Cat , all of our double categories will be strict. Therefore, we further present here the strict version of double categories. We refer the reader to [Gra19, §3.2] for a more detailed introduction to strict double categories.

2.7. DEFINITION. A *double category* is a weak double category, where the composition of vertical morphisms and the vertical composition of squares are strictly associative and unital. In other words, the associator and unitor squares in Definition 2.1 are horizontal identities.

2.8. DEFINITION. A *double functor* is a lax double functor between double categories which preserves vertical compositions and identities strictly. In other words, the composition and identity comparison squares in Definition 2.2 are horizontal identities.

2.9. NOTATION. We denote by DbCat the category of double categories and double functors.

Due to the strictness of vertical compositions, we can now define two different underlying categories of a double category.

2.10. DEFINITION. Given a double category \mathbb{C} , we define

- its *underlying vertical category* $\text{Ver}_0 \mathbb{C}$ to be the category of objects and vertical morphisms in \mathbb{C} ,
- its *underlying horizontal category* $\text{Hor}_0 \mathbb{C}$ to be the category of objects and horizontal morphisms in \mathbb{C} .

These constructions extend to functors

$$\text{Ver}_0, \text{Hor}_0: \text{DbCat} \rightarrow \text{Cat}.$$

2.11. REMARK. The functors $\text{Ver}_0, \text{Hor}_0: \text{DbCat} \rightarrow \text{Cat}$ admit left adjoints

$$\mathbb{V}, \mathbb{H}: \text{Cat} \rightarrow \text{DbCat},$$

respectively. We call a category in the image of \mathbb{V} (resp. \mathbb{H}) a *vertical* (resp. *horizontal*) double category.

2.12. **REMARK.** Given a double category \mathbb{C} , we can further define a category $\text{Ver}_1 \mathbb{C}$ of horizontal morphisms and squares, with composition given by the vertical composition of squares in \mathbb{C} . We can then see \mathbb{C} as an internal category to Cat

$$\text{Ver}_0 \mathbb{C} \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} \text{Ver}_1 \mathbb{C} \xleftarrow{c} \text{Ver}_1 \mathbb{C} \times_{\text{Ver}_0 \mathbb{C}} \text{Ver}_1 \mathbb{C}.$$

We can, in fact, upgrade the underlying horizontal category to a 2-category as follows.

2.13. **DEFINITION.** Given a double category \mathbb{C} , we define its *underlying horizontal 2-category* \mathbf{HC} to be the 2-category of objects, horizontal morphisms, and globular squares in \mathbb{C} . This construction extends to a functor

$$\mathbf{H}: \text{DblCat} \rightarrow 2\text{Cat}.$$

2.14. **REMARK.** The functor $\mathbf{H}: \text{DblCat} \rightarrow 2\text{Cat}$ admits a left adjoint

$$\mathbb{H}: 2\text{Cat} \rightarrow \text{DblCat},$$

which sees a 2-category as a double category with only trivial vertical morphisms.

The following result can be deduced from [Joh02, Lemma B2.3.15(ii)], using that double categories are internal categories in Cat .

2.15. **PROPOSITION.** *The category DblCat is cartesian closed.*

2.16. **NOTATION.** Given double categories \mathbb{C} and \mathbb{D} , we denote by $\llbracket \mathbb{C}, \mathbb{D} \rrbracket$ the internal hom in DblCat . It is the double category whose

- objects are double functors $\mathbb{C} \rightarrow \mathbb{D}$,
- horizontal morphisms are horizontal transformations between (strict) double functors as defined in Definition 2.3,
- vertical morphisms are vertical transformations between double functors as defined in Definition 2.4,
- squares are modifications as defined in Definition 2.5.

We refer the reader to [Gra19, §3.2.7] for more details.

Using these structures, we can define two different 2-categories of double categories, by picking the 2-morphisms to be either the horizontal or vertical transformations. Each of them will play an important role in this paper: the one with horizontal transformations will be giving the naturality of the constructions, while the one with the vertical transformations will be used to define the 2-category of double categorical discrete fibrations.

2.17. NOTATION. The category DblCat can be upgraded into a 2-category in two different ways:

- we write DblCat_h for the 2-category of double categories, double functors, and horizontal transformations,
- we write DblCat_v for the 2-category of double categories, double functors, and vertical transformations.

2.18. REMARK. By [FPP08, Proposition 2.5], the adjunctions $\mathbb{V} \dashv \text{Ver}_0$ from Remark 2.11 and $\mathbb{H} \dashv \mathbf{H}$ from Remark 2.14 extend to 2-adjunctions

$$\mathbb{V}: \text{Cat} \rightleftarrows \text{DblCat}_v: \text{Ver}_0 \quad \text{and} \quad \mathbb{H}: 2\text{Cat} \rightleftarrows \text{DblCat}_h: \mathbf{H}.$$

We will also make use of the horizontal opposite of a double category, which we now recall.

2.19. DEFINITION. We define a 2-functor $(-)^{\text{op}}: \text{DblCat}_h^{\text{co}} \rightarrow \text{DblCat}_h$, where $\text{DblCat}_h^{\text{co}}$ is the 2-category obtained from DblCat_h by reversing the 2-morphisms, sending

- a double category \mathbb{C} to its *horizontal opposite double category* \mathbb{C}^{op} which consists of the same objects and vertical morphisms as \mathbb{C} , but where the direction of horizontal morphisms and squares is horizontally reversed,
- a double functor $G: \mathbb{C} \rightarrow \mathbb{D}$ to the double functor $G^{\text{op}}: \mathbb{C}^{\text{op}} \rightarrow \mathbb{D}^{\text{op}}$, which acts as G on objects, horizontal morphisms, vertical morphisms, and squares,
- a horizontal transformation $B: G \Rightarrow G': \mathbb{C} \rightarrow \mathbb{D}$ to the horizontal transformation

$$B^{\text{op}}: G'^{\text{op}} \Rightarrow G^{\text{op}}: \mathbb{C}^{\text{op}} \rightarrow \mathbb{D}^{\text{op}},$$

whose components are the same as those of B .

It is straightforward to check that this construction is 2-functorial.

3. Lax double presheaves

In this section, we introduce our lax double presheaves as lax double functors $\mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ with \mathbb{C} a (strict) double category and Cat the weak double category of categories. To introduce the weak double category Cat , we need two notions of morphisms between categories. The horizontal ones will simply be the functors, and the vertical ones will be the profunctors. For this, we first recall in Sections 3.1 to 3.23 the notion of profunctors and their relations to two-sided discrete fibrations. Then, in Sections 3.24 to 3.28, we review the construction of the weak double category Cat .

In Sections 3.29 to 3.36, we introduce the notion of lax double presheaves, and finally, in Sections 3.37 to 3.45, we study a first class of examples of lax double presheaves, the *representable lax double presheaves*.

PROFUNCTORS AND TWO-SIDED DISCRETE FIBRATIONS. We start by recalling profunctors, as well as their weakly associative and unital composition.

3.1. DEFINITION. A *profunctor* $U: \mathcal{C} \dashrightarrow \mathcal{C}'$ between categories \mathcal{C} and \mathcal{C}' is a functor

$$U: \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}.$$

3.2. NOTATION. We denote by $\text{Prof}(\mathcal{C}, \mathcal{C}')$ the category of profunctors $\mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}$ and natural transformations between them.

Composition of profunctors is defined via coends; see e.g. [Rie14, §1.2] for a definition. As the category Set is cocomplete, coends in Set always exist and are given very explicitly by the following formula, as mentioned in [Rie14, (1.2.4)].

3.3. PROPOSITION. *Given a functor $U: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$, a coend of U exists and can be computed as the coequalizer*

$$\int^x U(x, x) \cong \text{coeq} \left(\begin{array}{ccc} \bigsqcup_{x \xrightarrow{f} x' \in \mathcal{C}} U(x', x) & \rightrightarrows & \bigsqcup_{x \in \mathcal{C}} U(x, x) \end{array} \right),$$

where the two parallel maps are induced by $U(f, x)$ and $U(x', f)$, respectively.

3.4. REMARK. As universal constructions, coends are unique up to a unique isomorphism. In what follows, we choose a specific coend for each functor $U: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$, and speak of *the coend* of U .

We can now use the notion of coends to define composition of profunctors.

3.5. CONSTRUCTION. Given profunctors $U: \mathcal{C} \dashrightarrow \mathcal{C}'$ and $U': \mathcal{C}' \dashrightarrow \mathcal{C}''$, their composition is the profunctor $U' \bullet U: \mathcal{C}^{\text{op}} \times \mathcal{C}'' \rightarrow \text{Set}$ sending

- an object (x, x'') in $\mathcal{C} \times \mathcal{C}''$ to the set $\int^{x' \in \mathcal{C}'} U'(x', x'') \times U(x, x')$ given by the coend of the functor $U'(-, x'') \times U(x, -): \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}$,
- a morphism $(f, f''): (x, x'') \rightarrow (y, y'')$ in $\mathcal{C} \times \mathcal{C}''$ to the unique induced map between coends

$$\int^{x' \in \mathcal{C}'} U'(x', f'') \times U(f, x'): \int^{x' \in \mathcal{C}'} U'(x', x'') \times U(y, x') \rightarrow \int^{x' \in \mathcal{C}'} U'(x', y'') \times U(x, x').$$

This construction extends to a functor

$$\bullet: \text{Prof}(\mathcal{C}, \mathcal{C}') \times \text{Prof}(\mathcal{C}', \mathcal{C}'') \rightarrow \text{Prof}(\mathcal{C}, \mathcal{C}'').$$

3.6. REMARK. Composition of profunctors is not strictly associative as the order in which we take coequalizers matters. However, as coends are unique up to a unique isomorphism, composition of profunctors is associative up to a unique invertible comparison cell.

Moreover, this composition of profunctors admits as identities the following.

3.7. DEFINITION. The *identity profunctor* $e_{\mathcal{C}}: \mathcal{C} \dashrightarrow \mathcal{C}$ at a category \mathcal{C} is given by the hom functor $\mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$.

It will be convenient if composition of profunctors is strictly unital. The following result implies that we can pick $U \bullet e_{\mathcal{C}} := U$ and $e_{\mathcal{C}'} \bullet U := U$, for every profunctor $U: \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}$.

3.8. LEMMA. *Given a profunctor $U: \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}$ and objects x in \mathcal{C} , x'' in \mathcal{C}' , there are canonical natural bijections*

$$U(x, x'') \cong \int^{x' \in \mathcal{C}} U(x', x'') \times \mathcal{C}(x, x') \quad \text{and} \quad U(x, x'') \cong \int^{x' \in \mathcal{C}'} \mathcal{C}'(x', x'') \times U(x, x').$$

In particular, this implies that there are canonical natural isomorphisms of profunctors

$$U \cong U \bullet e_{\mathcal{C}} \quad \text{and} \quad U \cong e_{\mathcal{C}'} \bullet U.$$

PROOF. The first bijection can be shown by proving that the set $U(x, x'')$ together with the maps

$$\iota_{x'}: U(x', x'') \times \mathcal{C}(x, x') \rightarrow U(x, x''), \quad (u, f) \mapsto U(f, x'')(u),$$

for objects x' in \mathcal{C} , satisfies the universal property of the coend of

$$U(-, x'') \times \mathcal{C}(x, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}.$$

The second bijection can be shown analogously. ■

In the remainder of this section, we study the connection between profunctors and two-sided discrete fibrations, which will be useful later. For this, let us first recall the definition of a two-sided discrete fibration.

3.9. DEFINITION. Given a functor $P: \mathcal{E} \rightarrow \mathcal{C}$, we say that a morphism $g: e \rightarrow e'$ in \mathcal{E}

- is a *P-lift* of a morphism $f: c \rightarrow c'$ in \mathcal{C} if $Pg = f$,
- *lies in the fiber of P at an object c in C* if it is a *P-lift* of the identity 1_c .

3.10. DEFINITION. A functor $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$ is a *two-sided discrete fibration* over $\mathcal{C} \times \mathcal{C}'$, if the following conditions hold:

- (1) for every object e in \mathcal{E} and every morphism $f: x \rightarrow Pe$ in \mathcal{C} , there is a unique *P-lift* of f with target e lying in the fiber of Q at Qe , i.e., there is a unique morphism $P^*f: f^*e \rightarrow e$ in \mathcal{E} such that $P(P^*f) = f$ and $Q(P^*f) = 1_{Qe}$,
- (2) for every object e in \mathcal{E} and every morphism $f': Qe \rightarrow x'$ in \mathcal{C}' , there is a unique *Q-lift* of f' with source e lying in the fiber of P at Pe , i.e., there is a unique morphism $Q_!f': e \rightarrow f'_!e$ in \mathcal{E} such that $Q(Q_!f') = f'$ and $P(Q_!f') = 1_{Pe}$,
- (3) for every morphism $g: e \rightarrow e'$ in \mathcal{E} , the source of the unique *P-lift* of Pg agrees with the target of the unique *Q-lift* of Qg and their composite is g , i.e., one has $(Pg)^*e' = (Qg)_!e$ and $P^*(Pg) \circ Q_!(Qg) = g$.

3.11. NOTATION. We denote by $\text{TSFib}(\mathcal{C}, \mathcal{C}')$ the full subcategory of the slice category $\text{Cat}/_{\mathcal{C} \times \mathcal{C}'}$ spanned by the two-sided discrete fibrations over $\mathcal{C} \times \mathcal{C}'$.

There is an equivalence of categories between profunctors and two-sided discrete fibrations, that we now recall.

3.12. CONSTRUCTION. We construct a functor

$$\text{fib}: \text{TSFib}(\mathcal{C}, \mathcal{C}') \rightarrow \text{Prof}(\mathcal{C}, \mathcal{C}').$$

It sends a two-sided discrete fibration $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$ to the profunctor

$$\text{fib}(P, Q): \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}$$

sending

- an object (x, x') in $\mathcal{C} \times \mathcal{C}'$ to the fiber $(P, Q)^{-1}(x, x')$ of (P, Q) at (x, x') ,
- a morphism $(f, f'): (x, x') \rightarrow (y, y')$ in $\mathcal{C}^{\text{op}} \times \mathcal{C}'$ to the map

$$(P, Q)^{-1}(x, x') \rightarrow (P, Q)^{-1}(y, y')$$

given by sending e to $f'_! f^* e = f^* f'_! e$,

and a morphism of two-sided discrete fibrations

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ & \searrow^{(P, Q)} & \swarrow_{(P', Q')} \\ & \mathcal{C} \times \mathcal{C}' & \end{array}$$

to the natural transformation $\text{fib}(F): \text{fib}(P, Q) \Rightarrow \text{fib}(P', Q')$ whose component at an object (x, x') in $\mathcal{C} \times \mathcal{C}'$ is the unique induced map between fibers

$$\text{fib}(F)_{x, x'} := F_{x, x'}: (P, Q)^{-1}(x, x') \rightarrow (P', Q')^{-1}(x, x').$$

A proof of the following result can be found in [LR20, Theorem 2.3.2].

3.13. THEOREM. *The functor fib induces an equivalence of categories*

$$\text{TSFib}(\mathcal{C}, \mathcal{C}') \xrightarrow[\cong]{\text{fib}} \text{Prof}(\mathcal{C}, \mathcal{C}').$$

In particular, this equivalence is compatible with precomposition and pullback, as follows.

3.14. PROPOSITION. *Given functors $G: \mathcal{C} \rightarrow \mathcal{D}$ and $G': \mathcal{C}' \rightarrow \mathcal{D}'$, the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \text{TSFib}(\mathcal{D}, \mathcal{D}') & \xrightarrow[\cong]{\text{fib}} & \text{Prof}(\mathcal{D}, \mathcal{D}') \\ (G \times G')^* \downarrow & & \downarrow (G^{\text{op}} \times G')^* \\ \text{TSFib}(\mathcal{C}, \mathcal{C}') & \xrightarrow[\cong]{\text{fib}} & \text{Prof}(\mathcal{C}, \mathcal{C}') \end{array}$$

where the left-hand functor is induced by taking pullbacks along $G \times G': \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D} \times \mathcal{D}'$ and the right-hand functor is induced by precomposing along $G^{\text{op}} \times G': \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D}'$.

As a consequence, we get the following result.

3.15. COROLLARY. *Let $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$ and $(R, S): \mathcal{F} \rightarrow \mathcal{D} \times \mathcal{D}'$ be two-sided discrete fibrations. Then a commutative square of functors of the form*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{F} \\ (P, Q) \downarrow & & \downarrow (R, S) \\ \mathcal{C} \times \mathcal{C}' & \xrightarrow{G \times G'} & \mathcal{D} \times \mathcal{D}' \end{array}$$

corresponds to a natural transformation

$$F: \text{fib}(P, Q) \Rightarrow \text{fib}(R, S)(G^{\text{op}} \times G'): \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set},$$

whose component at an object (x, x') in $\mathcal{C}^{\text{op}} \times \mathcal{C}'$ is given by the unique induced map between fibers

$$F_{x, x'}: (P, Q)^{-1}(x, x') \rightarrow (R, S)^{-1}(Gx, Gx').$$

PROOF. By the universal property of pullback, such a square of functors corresponds to a functor $\hat{F}: \mathcal{E} \rightarrow (G \times G')^* \mathcal{F}$ over $\mathcal{C} \times \mathcal{C}'$ as depicted in the following diagram:

$$\begin{array}{ccccc} & & & & F \\ & & & & \curvearrowright \\ \mathcal{E} & & & & \mathcal{F} \\ & \searrow \hat{F} & & & \downarrow (R, S) \\ & & (G \times G')^* \mathcal{F} & \longrightarrow & \mathcal{F} \\ & & \downarrow \lrcorner & & \downarrow (R, S) \\ (P, Q) \downarrow & & \mathcal{C} \times \mathcal{C}' & \xrightarrow{G \times G'} & \mathcal{D} \times \mathcal{D}' \end{array}$$

Using Proposition 3.14, the functor \hat{F} over $\mathcal{C} \times \mathcal{C}'$ corresponds to a natural transformation

$$\text{fib}(P, Q) \Rightarrow \text{fib}((G \times G')^*(R, S)) \cong (G^{\text{op}} \times G')^*(\text{fib}(R, S)) = \text{fib}(R, S)(G^{\text{op}} \times G').$$

Note that the description of the components of this natural transformation follows directly from the definition of the functor fib. ■

Finally, similarly to profunctors one can also define a composition for two-sided discrete fibrations as follows.

3.16. CONSTRUCTION. Let $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$ and $(P', Q'): \mathcal{E}' \rightarrow \mathcal{C}' \times \mathcal{C}''$ be two-sided discrete fibrations, and let \mathcal{A} be the wide subcategory of the pullback $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$ with morphisms given by

$$\{(e, g^*e') \xrightarrow{(Q!g, P'^*g)} (g!e, e') \mid Qe \xrightarrow{g} P'e' \in \mathcal{C}'\}.$$

We define the composition $(P', Q') \bullet (P, Q): \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{\sim} \rightarrow \mathcal{C} \times \mathcal{C}''$ to be the unique functor obtained by the universal property of the pushout $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{\sim}$ defined by the following commutative diagram in Cat .

$$\begin{array}{ccc}
 \mathcal{A} & \longrightarrow & \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}' \\
 \downarrow & \lrcorner & \downarrow \\
 \pi_0 \mathcal{A} & \longrightarrow & \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{\sim} \\
 & & \dashrightarrow \\
 & & \mathcal{C} \times \mathcal{C}''
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \xrightarrow{(P\pi_1, Q'\pi_2)} \\
 \\
 \xrightarrow{(P', Q') \bullet (P, Q)}
 \end{array}$$

Here π_i , for $i = 1, 2$ denote the canonical projections from the pullback $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$, and $\pi_0 \mathcal{A}$ is the set of path components of the category \mathcal{A} and $\mathcal{A} \rightarrow \pi_0 \mathcal{A}$ is the canonical projection. Note that the composite

$$\mathcal{A} \rightarrow \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}' \xrightarrow{(P\pi_1, Q'\pi_2)} \mathcal{C} \times \mathcal{C}''$$

factors through $\mathcal{A} \rightarrow \pi_0 \mathcal{A}$ since, for every morphism $g: Qe \rightarrow P'e'$ in \mathcal{C}' , we have

$$P\pi_1(Q!g, P'^*g) = P(Q!g) = \text{id}_{Pe} \quad \text{and} \quad Q'\pi_2(Q!g, P'^*g) = Q'(P'^*g) = \text{id}_{Q'e'},$$

and so we get an outer commutative diagram, as desired.

Using Lemma 3.18 below, this construction extends to a functor

$$\bullet: \text{TSFib}(\mathcal{C}, \mathcal{C}') \times \text{TSFib}(\mathcal{C}', \mathcal{C}'') \rightarrow \text{TSFib}(\mathcal{C}, \mathcal{C}'').$$

3.17. REMARK. Unpacking the universal property of the pushout, we observe that the category $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{\sim}$ is obtained from the pullback $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$ by making all morphisms of \mathcal{A} into identities. In particular, for every morphism $g: Qe \rightarrow P'e'$ in \mathcal{C}' , the objects (e, g^*e') and $(g!e, e')$ in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$ are identified in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{\sim}$.

3.18. LEMMA. Let $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$ and $(P', Q'): \mathcal{E}' \rightarrow \mathcal{C}' \times \mathcal{C}''$ be two-sided discrete fibrations. Then their composition

$$(P', Q') \bullet (P, Q): \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{\sim} \rightarrow \mathcal{C} \times \mathcal{C}''$$

is a two-sided discrete fibration.

PROOF. By definition, the functor $(P', Q') \bullet (P, Q): \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim} \rightarrow \mathcal{C} \times \mathcal{C}''$ acts as $P: \mathcal{E} \rightarrow \mathcal{C}$ and $Q': \mathcal{E}' \rightarrow \mathcal{C}''$ on each component, and so we write $(P', Q') \bullet (P, Q) = (P\pi_1, Q'\pi_2)$, where π_i , for $i = 1, 2$, denote the canonical projections from the pullback $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$.

We show that $(P\pi_1, Q'\pi_2): \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim} \rightarrow \mathcal{C} \times \mathcal{C}'$ satisfies the conditions of a two-sided discrete fibration from Definition 3.10. To prove (1), let $[\hat{e}, \hat{e}']$ be an object in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim}$ and $f: x \rightarrow P\hat{e}$ be a morphism in \mathcal{C} . Then a $P\pi_1$ -lift of f with target $[\hat{e}, \hat{e}']$ that lies in the fiber of $Q'\pi_2$ at $Q'\hat{e}'$ is given by the morphism in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim}$

$$[P^*f, 1_{\hat{e}'}]: [f^*\hat{e}, \hat{e}'] \rightarrow [\hat{e}, \hat{e}'].$$

Note that this is well-defined as $Q(P^*f) = 1_{Q\hat{e}} = 1_{P'\hat{e}'} = P'(1_{\hat{e}'})$ in \mathcal{C}' . Now let

$$[g, g']: [e, e'] \rightarrow [\hat{e}, \hat{e}'],$$

be another $P\pi_1$ -lift of f that lies in the fiber of $Q'\pi_2$ at $Q'\hat{e}'$, i.e., a morphism $[g, g']$ in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim}$ such that $Pg = f$ and $Q'g' = 1_{Q'\hat{e}'}$. By definition of the pullback $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$, we further have $Qg = P'g'$ in \mathcal{C}' . Using condition (3) of Definition 3.10 for the two-sided discrete fibrations (P, Q) and (P', Q') , we have factorizations

$$g = P^*(Pg) \circ Q_!(Qg) = P^*f \circ Q_!(Qg)$$

and

$$g' = P'^*(P'g') \circ Q'_!(Q'g') = P'^*(Qg) \circ Q'_!(1_{Q'\hat{e}'}) = P'^*(Qg) \circ 1_{\hat{e}'} = P'^*(Qg),$$

where we use that $Q'_!(1_{Q'\hat{e}'}) = 1_{\hat{e}'}$ by unicity of lifts. Since the morphism $[Q_!(Qg), P'^*(Qg)]$ is, by definition, an identity in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim}$, we have that

$$[g, g'] = [P^*f \circ Q_!(Qg), P'^*(Qg)] = [P^*f, 1_{\hat{e}'}] \circ [Q_!(Qg), P'^*(Qg)] = [P^*f, 1_{\hat{e}'}].$$

This shows that $[P^*f, 1_{\hat{e}'}]$ is the unique such lift. Condition (2) can be shown analogously.

To show (3), let $[g, g']: [e, e'] \rightarrow [\hat{e}, \hat{e}']$ be a morphism in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim}$. Then the unique lifts of Pg and $Q'g'$ provided by conditions (1) and (2) are the morphisms in $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim}$

$$[P^*(Pg), 1_{\hat{e}'}]: [(Pg)^*\hat{e}, \hat{e}'] \rightarrow [\hat{e}, \hat{e}'] \quad \text{and} \quad [1_e, Q'_!(Q'g')]: [e, e'] \rightarrow [e, (Q'g')_!e'].$$

Recall that, by definition of the pullback $\mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'$, we have $Qg = P'g'$ in \mathcal{C}' . Using property (3) for the two-sided discrete fibrations (P, Q) and (P', Q') , we observe that

$$(Pg)^*\hat{e} = (Qg)_!e \quad \text{and} \quad (Q'g')_!e' = (P'g')^*\hat{e}' = (Qg)^*\hat{e}',$$

and so the source of the first lift agrees with the target of the second lift by Remark 3.17. Therefore we can form their composite and we get

$$\begin{aligned} & [P^*(Pg), 1_{\hat{e}'}] \circ [1_e, Q'_!(Q'g')] \\ &= [P^*(Pg), 1_{\hat{e}'}] \circ [Q_!(Qg), P'^*(Qg)] \circ [1_e, Q'_!(Q'g')] \quad [Q_!(Qg), P'^*(Qg)] \sim 1_{[(Qg)_!e, \hat{e}']} \\ &= [P^*(Pg), \circ Q_!(Qg), P'^*(Qg) \circ Q'_!(Q'g')] \quad \text{Composition in } \mathcal{E} \times_{\mathcal{C}'} \mathcal{E}'_{/\sim} \\ &= [g, g']. \quad (3) \text{ for } (P, Q) \text{ and } (P', Q') \end{aligned}$$

This shows the desired result. ■

3.19. **REMARK.** As for profunctors, composition of two-sided discrete fibrations is not strictly associative. However, as pullbacks and pushouts are unique up to unique isomorphism, composition of two-sided discrete fibrations is associative up to a unique invertible comparison cell.

Composition of two-sided discrete fibrations is defined in such a way that it corresponds to composition of profunctors through the equivalence fib.

3.20. **PROPOSITION.** *Given categories $\mathcal{C}, \mathcal{C}', \mathcal{C}''$, the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathrm{TSFib}(\mathcal{C}, \mathcal{C}') \times \mathrm{TSFib}(\mathcal{C}', \mathcal{C}'') & \xrightarrow[\cong]{\mathrm{fib} \times \mathrm{fib}} & \mathrm{Prof}(\mathcal{C}, \mathcal{C}') \times \mathrm{Prof}(\mathcal{C}', \mathcal{C}'') \\ \bullet \downarrow & & \downarrow \bullet \\ \mathrm{TSFib}(\mathcal{C}, \mathcal{C}'') & \xrightarrow[\cong]{\mathrm{fib}} & \mathrm{Prof}(\mathcal{C}, \mathcal{C}'') \end{array}$$

PROOF. Given two-sided discrete fibrations $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$ and $(P', Q'): \mathcal{E}' \rightarrow \mathcal{C}' \times \mathcal{C}''$, we want to show that there is an isomorphism in $\mathrm{Prof}(\mathcal{C}, \mathcal{C}'')$

$$\mathrm{fib}((P', Q') \bullet (P, Q)) \cong \mathrm{fib}(P', Q') \bullet \mathrm{fib}(P, Q) \quad (3.1)$$

which is natural in (P, Q) and (P', Q') . When evaluated at an object (x, x'') in $\mathcal{C} \times \mathcal{C}''$, this amounts to showing that there is an isomorphism in Set

$$\mathrm{fib}((P', Q') \bullet (P, Q))(x, x'') \cong \mathrm{fib}(P', Q') \bullet \mathrm{fib}(P, Q)(x, x'') \quad (3.2)$$

which is natural in (x, x'') . By definition, we have that the set $\mathrm{fib}(P', Q') \bullet \mathrm{fib}(P, Q)(x, x'')$ is the coequalizer of the diagram

$$\bigsqcup_{x' \xrightarrow{g} \hat{x}' \in \mathcal{C}'} (P', Q')^{-1}(\hat{x}', x'') \times (P, Q)^{-1}(x, x') \rightrightarrows \bigsqcup_{x' \in \mathcal{C}'} (P', Q')^{-1}(x', x'') \times (P, Q)^{-1}(x, x')$$

where the two parallel maps are induced by $g^* \times 1_{(P, Q)^{-1}(x, x')}$ and $1_{(P', Q')^{-1}(\hat{x}', x'')} \times g_!$, respectively. Hence, to obtain the isomorphism (3.2), it suffices to show that the set $\mathrm{fib}((P', Q') \bullet (P, Q))(x, x'')$ is also a coequalizer of the above diagram. However, by Remark 3.17, we see that the fiber $\mathrm{fib}((P', Q') \bullet (P, Q))(x, x'')$ consists of the quotient of the set

$$\bigsqcup_{x' \in \mathcal{C}'} (P', Q')^{-1}(x', x'') \times (P, Q)^{-1}(x, x')$$

by the relation $(e, g^*e') \sim (g_!e, e')$, for all morphisms $g: Qe \rightarrow P'e'$ in \mathcal{C}' . But these are precisely the relations enforced by the coequalizer. Hence we have a canonical isomorphism (3.2). Moreover, it is natural in (x, x'') since, for both profunctors, the actions of morphisms in \mathcal{C} and \mathcal{C}'' are determined by taking unique lifts along P and Q' , therefore yielding the natural isomorphism (3.1).

Finally, the naturality of the isomorphism (3.1) in (P, Q) and (P', Q') follows directly from the universal property of the coequalizers. \blacksquare

Moreover, this composition of two-sided discrete fibration admits as identities the following.

3.21. DEFINITION. The *identity two-sided discrete fibration* at a category \mathcal{C} is the two-sided discrete fibration $(s, t): \mathcal{C}^{[1]} \rightarrow \mathcal{C} \times \mathcal{C}$, where $[1]$ denotes the category associated with the poset $\{0 < 1\}$, often called the *walking morphism*—so $\mathcal{C}^{[1]}$ is the category of morphisms and commutative squares of morphisms in \mathcal{C} —and the functor (s, t) is induced by taking source and target.

3.22. REMARK. Note that, given a category \mathcal{C} , the image of the identity two-sided discrete fibration $(s, t): \mathcal{C}^{[1]} \rightarrow \mathcal{C} \times \mathcal{C}$ under the equivalence $\text{fib}: \text{TSFib}(\mathcal{C}, \mathcal{C}) \rightarrow \text{Prof}(\mathcal{C}, \mathcal{C})$ is the identity profunctor $e_{\mathcal{C}} = \mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$.

As a consequence, we get the following.

3.23. LEMMA. *Given a two-sided discrete fibration $(P, Q): \mathcal{E} \rightarrow \mathcal{C} \times \mathcal{C}'$, there are canonical isomorphisms in $\text{TSFib}(\mathcal{C}, \mathcal{C}')$*

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{\cong} & \mathcal{C}^{[1]} \times_{\mathcal{C}} \mathcal{E}_{/\sim} \\
 \searrow^{(P,Q)} & & \swarrow_{(P,Q) \bullet (s,t)} \\
 & \mathcal{C} \times \mathcal{C}' &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \mathcal{E} & \xrightarrow{\cong} & \mathcal{E} \times_{\mathcal{C}'} \mathcal{C}'^{[1]}_{/\sim} \\
 \searrow^{(P,Q)} & & \swarrow_{(s,t) \bullet (P,Q)} \\
 & \mathcal{C} \times \mathcal{C}' &
 \end{array}$$

PROOF. We show the first isomorphism, and the second can be shown analogously.

Let $(s, t): \mathcal{C}^{[1]} \rightarrow \mathcal{C} \times \mathcal{C}$ denote the identity two-sided discrete fibration. We have the following natural isomorphisms

$$\begin{array}{ll}
 \text{fib}((P, Q) \bullet (s, t)) \cong \text{fib}(P, Q) \bullet \text{fib}(s, t) & \text{Proposition 3.20} \\
 \cong \text{fib}(P, Q) \bullet e_{\mathcal{C}} & \text{Remark 3.22} \\
 \cong \text{fib}(P, Q) & \text{Lemma 3.8}
 \end{array}$$

Since fib is an equivalence, it reflects isomorphisms and so we get an isomorphism

$$\mathcal{C}^{[1]} \times_{\mathcal{C}} \mathcal{E}_{/\sim} \cong \mathcal{E}$$

over $\mathcal{C} \times \mathcal{C}'$, as desired. ■

THE WEAK DOUBLE CATEGORY $\mathbb{C}\text{at}$. With the notion of profunctors at hand, we are now ready to introduce the weak double category $\mathbb{C}\text{at}$.

3.24. DEFINITION. We define $\mathbb{C}\text{at}$ to be the unitary weak double category whose

- objects are (small) categories $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}', \dots$,
- horizontal morphisms $\mathcal{C} \rightarrow \mathcal{D}$ are functors $F: \mathcal{C} \rightarrow \mathcal{D}$, with their ordinary identity and composition,

- vertical morphisms $\mathcal{C} \rightarrow \mathcal{C}'$ are profunctors $U: \mathcal{C}^{\text{op}} \times \mathcal{C}' \rightarrow \text{Set}$, with identity at \mathcal{C} given by the hom functor $e_{\mathcal{C}} := \mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$ and composition defined as in Construction 3.5; using Lemma 3.8, we impose $e_{\mathcal{C}} \bullet U := U$ and $U \bullet e_{\mathcal{C}'} := U$, for every profunctor $U: \mathcal{C} \rightarrow \mathcal{C}'$,
- squares

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ U \downarrow & \alpha & \downarrow V \\ \mathcal{C}' & \xrightarrow{F'} & \mathcal{D}' \end{array}$$

are natural transformations

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{C}' & \xrightarrow{F^{\text{op}} \times F'} & \mathcal{D}^{\text{op}} \times \mathcal{D}' \\ & \searrow U & \nearrow V \\ & \text{Set} & \end{array}$$

with horizontal composition of two horizontally composable squares $\alpha: [U \xrightarrow{F} V]$ and $\beta: [V \xrightarrow{G'} W]$ given by the natural transformation

$$(F^{\text{op}} \times F')\beta \circ \alpha: U \Rightarrow W \circ ((GF)^{\text{op}} \times (G'F')),$$

and vertical composition of two vertically composable squares $\alpha: [U \xrightarrow{F} V]$ and $\alpha': [U' \xrightarrow{F'} V']$ given by the natural transformation

$$\alpha' \bullet \alpha: U' \bullet U \Rightarrow (V' \bullet V) \circ (F^{\text{op}} \times F'')$$

whose component at an object (x, x'') of $\mathcal{C} \times \mathcal{C}''$ is the unique map between coends

$$\int^{x' \in \mathcal{C}'} \alpha'_{x', x''} \times \alpha_{x, x'}: \int^{x' \in \mathcal{C}'} U'(x', x'') \times U(x, x') \rightarrow \int^{y' \in \mathcal{D}'} V'(y', F''x'') \times V(Fx, y').$$

induced by $\alpha'_{x', x''}: U'(x', x'') \rightarrow V'(F'x', F''x'')$ and $\alpha_{x, x'}: U(x, x') \rightarrow V(Fx, F'x')$,

- for composable profunctors U, U', U'' , the associator square

$$\alpha_{U, U', U''}: \left[(U'' \bullet U') \bullet U \xrightarrow{1_{\mathcal{C}}} U'' \bullet (U' \bullet U) \right]$$

is the natural isomorphism

$$\alpha_{U, U', U''}: ((U'' \bullet U') \bullet U) \xrightarrow{\cong} (U'' \bullet (U' \bullet U))$$

whose component at an object (x, x''') in $\mathcal{C} \times \mathcal{C}'''$ is the unique isomorphism between coends

$$(\alpha_{U, U', U''})_{x, x'''}: ((U'' \bullet U') \bullet U)(x, x''') \xrightarrow{\cong} (U'' \bullet (U' \bullet U))(x, x''').$$

The following result is mentioned in [Gra19, §3.4.3], and can be deduced from [Bén73, §2].

3.25. LEMMA. *The construction Cat is a unitary weak double category.*

Finally, we give a more convenient description of the globular squares in Cat .

3.26. LEMMA. *A globular square $\alpha: [e_{\mathcal{C}} \begin{smallmatrix} F \\ F' \end{smallmatrix} e_{\mathcal{D}}]$ in Cat is equivalently a natural transformation $\alpha: F \Rightarrow F': \mathcal{C} \rightarrow \mathcal{D}$.*

PROOF. By definition, a globular square $\alpha: [e_{\mathcal{C}} \begin{smallmatrix} F \\ F' \end{smallmatrix} e_{\mathcal{D}}]$ in Cat is a natural transformation

$$\alpha: \mathcal{C}(-, -) \Rightarrow \mathcal{D}(F(-), F'(-)): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set},$$

corresponding uniquely to a family of natural transformations

$$\{\alpha_{-,x}: \mathcal{C}(-, x) \Rightarrow \mathcal{D}(F(-), F'x)\}_{x \in \mathcal{C}}$$

natural in x . By the usual Yoneda lemma, such a family corresponds then uniquely to a family $\{\alpha_x \in \mathcal{D}(Fx, F'x)\}_{x \in \mathcal{C}}$ natural in x . But such a family defines precisely the components of a natural transformation $\alpha: F \Rightarrow F'$. ■

3.27. REMARK. Under the bijection of Lemma 3.26, the identity natural transformation id_F at a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ corresponds to the globular square $\text{id}_F: [e_{\mathcal{C}} \begin{smallmatrix} F \\ F \end{smallmatrix} e_{\mathcal{D}}]$ in Cat given by the natural transformation $\text{id}_F: \mathcal{C}(-, -) \Rightarrow \mathcal{D}(F(-), F(-)): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$ whose component at objects x, x' in \mathcal{C} is the map of sets

$$\mathcal{C}(x, x') \rightarrow \mathcal{D}(Fx, Fx')$$

sending a morphism $g: x \rightarrow x'$ in \mathcal{C} to the morphism $Fg: Fx \rightarrow Fx'$ in \mathcal{D} .

3.28. REMARK. As a consequence of Lemma 3.26, we see that the underlying horizontal 2-category \mathbf{HCat} is simply the 2-category Cat itself.

THE 2-CATEGORY OF LAX DOUBLE PRESHEAVES. Having constructed the weak double category Cat , we can now introduce our notion of lax double presheaves.

3.29. DEFINITION. Given a (strict) double category \mathbb{C} , a *lax double presheaf* over \mathbb{C} is a normal lax double functor $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$.

3.30. NOTATION. We denote by $\mathcal{P}_{\mathbb{C}} := \mathbf{H}[[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}}$ the 2-category of lax double presheaves over \mathbb{C} , horizontal transformations, and globular modifications. Here we have that $[[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}}$ is the full double subcategory of the double category $[[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{lax}}$ from Proposition 2.6 spanned by the normal lax double functors and, consequently, $\mathbf{H}[[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{nlax}}$ is the corresponding 2-subcategory of the 2-category $\mathbf{H}[[\mathbb{C}^{\text{op}}, \text{Cat}]^{\text{lax}}$ from Proposition 2.6.

In the case where $\mathbb{C} = \mathbf{HC}$ with \mathcal{C} a 2-category, we can provide another description of $\mathcal{P}_{\mathbf{HC}}$.

3.31. NOTATION. Given a 2-category \mathcal{C} , we denote by $[\mathcal{C}^{\text{op}}, \text{Cat}]$ the 2-category of 2-presheaves, i.e., 2-functors $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}$, 2-natural transformations, and modifications; see e.g. [RV22, §B.2].

3.32. LEMMA. *Given a 2-category \mathcal{C} , there is an isomorphism of 2-categories*

$$\mathcal{P}_{\mathbb{H}\mathcal{C}} \cong [\mathcal{C}^{\text{op}}, \text{Cat}],$$

which is natural in \mathcal{C} .

PROOF. By Remark 2.18, we have a 2-adjunction

$$\mathbb{H}: 2\text{Cat} \rightleftarrows \text{DblCat}_h: \mathbf{H}.$$

Hence this gives an isomorphism of categories

$$\text{Hor}_0[[\mathbb{H}\mathcal{C}^{\text{op}}, \text{Cat}]] \cong 2\text{Cat}(\mathcal{C}^{\text{op}}, \mathbf{H}\text{Cat})$$

which can be promoted to an isomorphism of 2-categories

$$\mathbf{H}[[\mathbb{H}\mathcal{C}^{\text{op}}, \text{Cat}]] \cong [\mathcal{C}^{\text{op}}, \mathbf{H}\text{Cat}].$$

Then, noticing that normal lax double functors out of $\mathbb{H}\mathcal{C}^{\text{op}}$ are simply (strict) double functors and using Remark 3.28, we get an isomorphism of 2-categories

$$\mathcal{P}_{\mathbb{H}\mathcal{C}} = \mathbf{H}[[\mathbb{H}\mathcal{C}^{\text{op}}, \text{Cat}]]^{\text{nlax}} = \mathbf{H}[[\mathbb{H}\mathcal{C}^{\text{op}}, \text{Cat}]] \cong [\mathcal{C}^{\text{op}}, \mathbf{H}\text{Cat}] = [\mathcal{C}^{\text{op}}, \text{Cat}],$$

as desired. ■

To simplify computations later, we unpack here the data of the 2-category $\mathcal{P}_{\mathbb{C}}$.

3.33. REMARK. A lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ consists of

- for every object x in \mathbb{C} , a category Xx ,
- for every horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , a functor $Xf: Xy \rightarrow Xx$,
- for every vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , a profunctor $Xu: Xx^{\text{op}} \times Xx' \rightarrow \text{Set}$,
- for every square $\alpha: \begin{bmatrix} u & f \\ f' & v \end{bmatrix}$ in \mathbb{C} , a natural transformation

$$X\alpha: Xv \Rightarrow Xu(Xf^{\text{op}} \times Xf'),$$

- for all composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x''$ in \mathbb{C} , a composition comparison natural transformation

$$\mu_{u,u'}: Xu' \bullet Xu \Rightarrow X(u' \bullet u): Xx^{\text{op}} \times Xx'' \rightarrow \text{Set}.$$

such that the following conditions are satisfied:

- (1) it preserves horizontal identities and horizontal compositions strictly,
- (2) composition comparison natural transformations $\mu_{u,u'}$ are natural with respect to (u, u') and compatible with the associator natural transformations in Cat ,
- (3) it preserves vertical identities strictly; and, for every vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , the natural transformations $\mu_{u, e_{x'}}$ and $\mu_{e_x, u}$ agree with the identity natural transformation at Xu .

3.34. REMARK. Given lax double presheaves $X, Y: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, then a horizontal transformation $F: X \Rightarrow Y$ consists of

- for every object x in \mathbb{C} , a functor $F_x: Xx \rightarrow Yx$,
- for every vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} , a natural transformation

$$F_u: Xu \Rightarrow Yu \circ (F_x^{\text{op}} \times F_{x'}): Xx^{\text{op}} \times Xx' \rightarrow \text{Set},$$

such that the following conditions are satisfied:

- (1) the functors F_x are natural in x : for every horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , the following diagram of functors commutes

$$\begin{array}{ccc} Xy & \xrightarrow{Xf} & Xx \\ F_y \downarrow & & \downarrow F_x \\ Yy & \xrightarrow{Yf} & Yx \end{array}$$

- (2) the natural transformations F_u are natural in u : for every square $\alpha: \begin{bmatrix} u & f \\ & v \end{bmatrix}: \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ in \mathbb{C} , the following diagram in $\text{Prof}(Xy, Xy')$ commutes,

$$\begin{array}{ccc} Xv & \xrightarrow{X\alpha} & Xu(Xf^{\text{op}} \times Xf') \\ F_v \Downarrow & & \Downarrow F_u(Xf^{\text{op}} \times Xf') \\ Yv(F_y^{\text{op}} \times F_{y'}) & \xrightarrow{Y\alpha(F_y^{\text{op}} \times F_{y'})} & Yu((Yf \circ F_y)^{\text{op}} \times (Yf' \circ F_{y'})) \end{array}$$

where we use that $Yf \circ F_y = F_x \circ Xf$ and $Yf' \circ F_{y'} = F_{x'} \circ Xf'$ by (1),

- (3) the natural transformations F_u are compatible with composition comparisons: for all composable vertical morphisms $x \dashrightarrow x' \dashrightarrow x''$ in \mathbb{C} , the following diagram in $\text{Prof}(Xx, Xx'')$ commutes,

$$\begin{array}{ccc} Xu' \bullet Xu & \xrightarrow{\mu_{u,u'}} & X(u' \bullet u) \\ F_{u'} \bullet F_u \Downarrow & & \Downarrow F_{u' \bullet u} \\ (Yu' \bullet Yu)(F_x^{\text{op}} \times F_{x''}) & \xrightarrow{\mu_{u,u'}(F_x^{\text{op}} \times F_{x''})} & Y(u' \bullet u)(F_x^{\text{op}} \times F_{x''}) \end{array}$$

- (4) the natural transformations F_u are compatible with vertical identities: for every object x in \mathbb{C} , the natural transformation F_{e_x} is the identity at F_x .

3.35. **REMARK.** Given horizontal transformations $F, F': X \rightrightarrows Y: \mathbb{C}^{\text{op}} \rightarrow \mathbb{C}\text{at}$, then a globular modification $A: [e_X \xrightarrow{F} F' e_Y]$ consists of, for every object x in \mathbb{C} , a natural transformation

$$A_x: F_x \rightrightarrows F'_x: Xx \rightarrow Yx,$$

such that the following conditions are satisfied:

- (1) the natural transformations A_x are compatible with horizontal morphisms: for every horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , the following pasting diagram of functors and natural transformations commutes,

$$\begin{array}{ccc} Xy & \xrightarrow{Xf} & Xx \\ \left. \begin{array}{c} \downarrow \\ F_y \xrightarrow{A_y} F'_y \\ \downarrow \end{array} \right\} & & \left. \begin{array}{c} \downarrow \\ F_x \xrightarrow{A_x} F'_x \\ \downarrow \end{array} \right\} \\ Yy & \xrightarrow{Yf} & Yx \end{array}$$

- (2) the natural transformations A_x are compatible with vertical morphisms: for every vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , the following diagram in $\text{Prof}(Xx, Xx')$ commutes.

$$\begin{array}{ccc} Xu & \xrightarrow{F_u} & Yu(F_x^{\text{op}} \times F_{x'}) \\ \left. \begin{array}{c} \downarrow \\ F'_u \\ \downarrow \end{array} \right\} & & \left. \begin{array}{c} \downarrow \\ Yu(F_x^{\text{op}} \times A_{x'}) \\ \downarrow \end{array} \right\} \\ Yu(F_x^{\text{op}} \times F_{x'}) & \xrightarrow{Yu(A_x^{\text{op}} \times F_{x'})} & Yu(F_x^{\text{op}} \times F_{x'}) \end{array}$$

Finally, we extend the construction $\mathcal{P}_{\mathbb{C}}$ into a 2-functor.

3.36. **CONSTRUCTION.** We define a 2-functor

$$\mathcal{P}: \text{DblCat}_h^{\text{coop}} \rightarrow 2\text{Cat},$$

where $\text{DblCat}_h^{\text{coop}}$ is the 2-category obtained from DblCat_h by reversing the morphisms and the 2-morphisms, which sends

- a double category \mathbb{C} to the weak double category $\mathcal{P}_{\mathbb{C}}$ of lax double presheaves over \mathbb{C} , horizontal transformations, and globular modifications,
- a double functor $G: \mathbb{C} \rightarrow \mathbb{D}$ to the 2-functor

$$\mathcal{P}_G := (G^{\text{op}})^*: \mathcal{P}_{\mathbb{D}} \rightarrow \mathcal{P}_{\mathbb{C}},$$

induced by precomposition with G^{op} ,

- a horizontal transformation $B: G \Rightarrow G': \mathbb{C} \rightarrow \mathbb{D}$ to the 2-natural transformation

$$\mathcal{P}_B := (B^{\text{op}})^*: \mathcal{P}_{G'} \Rightarrow \mathcal{P}_G$$

whose component at a lax double presheaf $X: \mathbb{D}^{\text{op}} \rightarrow \mathbb{C}\text{at}$ is the horizontal transformation

$$(\mathcal{P}_B)_X := X \circ B^{\text{op}}: \mathcal{P}_{G'}(X) = X \circ G'^{\text{op}} \rightarrow \mathcal{P}_G(X) = X \circ G^{\text{op}}$$

of lax double presheaves $\mathbb{C}^{\text{op}} \rightarrow \mathbb{C}\text{at}$.

It is straightforward to check that this construction is 2-functorial.

REPRESENTABLE LAX DOUBLE PRESHEAVES. We now want to introduce our first class of examples. They will be induced by the following construction.

3.37. CONSTRUCTION. Given a double category \mathbb{C} , we define a normal lax double functor

$$\mathbb{C}(-, -): \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{C}\text{at}$$

by the following data:

- it sends a pair (x, \hat{x}) of objects in \mathbb{C} to the category $\mathbb{C}(x, \hat{x})$ whose
 - objects are horizontal morphisms $g: x \rightarrow \hat{x}$ in \mathbb{C} ,
 - morphisms $g \rightarrow g'$ are globular squares $\eta: [e_x \xrightarrow{g} e_{\hat{x}}]$ in \mathbb{C} ,
 - composition is given by the vertical composition of squares in \mathbb{C}
- it sends a pair $(x \xrightarrow{f} y, \hat{x} \xrightarrow{\hat{f}} \hat{y})$ of horizontal morphisms in \mathbb{C} to the functor

$$\mathbb{C}(f, \hat{f}): \mathbb{C}(y, \hat{x}) \rightarrow \mathbb{C}(x, \hat{y})$$

sending

- a horizontal morphism $g: y \rightarrow \hat{x}$ to the composite $\hat{f} \circ g \circ f: x \rightarrow \hat{y}$,
- a square $\eta: [e_y \xrightarrow{g} e_{\hat{x}}]$ to the square $e_{\hat{f}} \circ \eta \circ e_f: [e_x \xrightarrow{\hat{f} \circ g \circ f} e_{\hat{y}}]$,

- it sends a pair $(x \xrightarrow{u} x', \hat{x} \xrightarrow{\hat{u}} \hat{x}')$ of vertical morphisms in \mathbb{C} to the profunctor

$$\mathbb{C}(u, \hat{u}): \mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x', \hat{x}') \rightarrow \text{Set}$$

sending

- an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x}')$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x}')$ to the set of squares

$$\{[u \xrightarrow{g} \hat{u}] : [x \xrightarrow{\hat{x}} x']\},$$

– a pair of squares $(\alpha: [e_x \xrightarrow{g} e_{\hat{x}}], \alpha': [e_{x'} \xrightarrow{g'} e_{\hat{x}'}])$ to the map

$$\mathbb{C}(u, \hat{u})(h, g') \rightarrow \mathbb{C}(u, \hat{u})(g, h')$$

sending a square $\eta: [u \xrightarrow{h} \hat{u}]$ to the vertical composite of squares

$$\begin{array}{ccc} x & \xrightarrow{g} & \hat{x} \\ \Downarrow & \alpha & \Downarrow \\ x & \xrightarrow{h} & \hat{x} \\ u \downarrow & \eta & \downarrow \hat{u} \\ x' & \xrightarrow{g'} & \hat{x}' \\ \Downarrow & \alpha' & \Downarrow \\ x' & \xrightarrow{h'} & \hat{x}' \end{array}$$

- it sends a pair $(\alpha: [v \xrightarrow{f} u], \hat{\alpha}: [\hat{u} \xrightarrow{\hat{f}} \hat{v}])$ of squares in \mathbb{C} to the natural transformation

$$\mathbb{C}(\alpha, \hat{\alpha}): \mathbb{C}(u, \hat{u}) \Rightarrow \mathbb{C}(v, \hat{v})(\mathbb{C}(f, \hat{f})^{\text{op}} \times \mathbb{C}(f', \hat{f}')): \mathbb{C}(y, \hat{x})^{\text{op}} \times \mathbb{C}(y', \hat{x}') \rightarrow \text{Set}$$

whose component at an object $(y \xrightarrow{g} \hat{x}, y' \xrightarrow{g'} \hat{x}')$ of $\mathbb{C}(y, \hat{x}) \times \mathbb{C}(y', \hat{x}')$ is the map

$$\mathbb{C}(\alpha, \hat{\alpha})_{g, g'}: \mathbb{C}(u, \hat{u})(g, g') \rightarrow \mathbb{C}(v, \hat{v})(\hat{f} \circ g \circ f, \hat{f}' \circ g' \circ f'),$$

sending a square $\eta: [u \xrightarrow{g} \hat{u}]$ to the horizontal composite of squares

$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & \hat{x} & \xrightarrow{\hat{f}} & \hat{y} \\ v \downarrow & \alpha & u \downarrow & \eta & \downarrow \hat{u} & \hat{\alpha} & \downarrow \hat{v} \\ x' & \xrightarrow{f'} & y' & \xrightarrow{g'} & \hat{x}' & \xrightarrow{\hat{f}'} & \hat{y}' \end{array}$$

- its composition comparison square at a pair $(x, \hat{x}) \xrightarrow{(u, \hat{u})} (x', \hat{x}') \xrightarrow{(u', \hat{u}')} (x'', \hat{x}'')$ of composable vertical morphisms in \mathbb{C} is the natural transformation

$$\mu_{(u, \hat{u}), (u', \hat{u}')}: \mathbb{C}(u', \hat{u}') \bullet \mathbb{C}(u, \hat{u}) \Rightarrow \mathbb{C}(u' \bullet u, \hat{u}' \bullet \hat{u}): \mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x'', \hat{x}'') \rightarrow \text{Set},$$

whose component at an object $(x \xrightarrow{g} \hat{x}, x'' \xrightarrow{g''} \hat{x}'')$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x'', \hat{x}'')$ is the map

$$\mathbb{C}(u', \hat{u}') \bullet \mathbb{C}(u, \hat{u})(g, g'') \rightarrow \mathbb{C}(u' \bullet u, \hat{u}' \bullet \hat{u})(g, g''),$$

induced by the universal property of the coend from the family of maps, given at an object $x' \xrightarrow{g'} \hat{x}'$ of $\mathbb{C}(x', \hat{x}')$, by the map

$$\mathbb{C}(u', \hat{u}')(g', g'') \times \mathbb{C}(u, \hat{u})(g, g') \longrightarrow \mathbb{C}(u' \bullet u, \hat{u}' \bullet \hat{u})(g, g''),$$

sending an element $(\alpha': [u' \xrightarrow{g'} \hat{u}'], \alpha: [u \xrightarrow{g} \hat{u}])$ of $\mathbb{C}(u', \hat{u}')(g', g'') \times \mathbb{C}(u, \hat{u})(g, g')$ to the vertical composite $\alpha' \bullet \alpha$ in $\mathbb{C}(u' \bullet u, \hat{u}' \bullet \hat{u})(g, g'')$.

3.38. LEMMA. *The construction*

$$\mathbb{C}(-, -): \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \text{Cat}$$

is a normal lax double functor.

PROOF. It is straightforward to see that $\mathbb{C}(-, -): \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \text{Cat}$ preserves horizontal compositions and identities. The fact that, given vertical morphisms u, \hat{u} in \mathbb{C} , the composition comparison natural transformations $\mu_{u,u'}$ are natural in (u, u') follows from the interchange law that the compositions of squares in \mathbb{C} satisfy, and the fact that the natural transformations $\mu_{u,u'}$ are compatible with associator natural transformations follows from the associativity of vertical composition of squares in \mathbb{C} . Finally, to see that $\mathbb{C}(-, -)$ is normal, observe that, given objects x, \hat{x} in \mathbb{C} , then $\mathbb{C}(e_x, e_{\hat{x}})$ is the identity profunctor $e_{\mathbb{C}(x, \hat{x})}$, and similarly, given horizontal morphisms f, \hat{f} in \mathbb{C} , then $\mathbb{C}(e_f, e_{\hat{f}})$ is the identity natural transformation at $\mathbb{C}(f, \hat{f})$. ■

3.39. REMARK. We note that $\mathbb{C}(-, -)$ is in general not a pseudo double functor. If this was the case, the composition comparison natural transformations

$$\mu_{(u, \hat{u}), (u', \hat{u}')}: \mathbb{C}(u', \hat{u}') \bullet \mathbb{C}(u, \hat{u}) \Rightarrow \mathbb{C}(u' \bullet u, \hat{u}' \bullet \hat{u}): \mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x'', \hat{x}'') \rightarrow \text{Set},$$

would be invertible. This would imply that every square $[u' \bullet u \xrightarrow{g} \hat{u}' \bullet \hat{u}]$ in \mathbb{C} could be factorized into a vertical composite of two squares of the form $[u \xrightarrow{g} \hat{u}]$ and $[u' \xrightarrow{g'} \hat{u}']$. However, such a factorization does not exist in general, as we show in Example 3.40 below.

3.40. EXAMPLE. We consider the double category \mathbb{C} generated by the following data

$$\begin{array}{ccc} x & \xrightarrow{g} & \hat{x} \\ u \bullet \downarrow & & \bullet \downarrow \hat{u} \\ x' & \alpha & \hat{x}' \\ u' \bullet \downarrow & & \bullet \downarrow \hat{u}' \\ x'' & \xrightarrow{g''} & \hat{x}'' \end{array}$$

Then, note that we have

$$\mathbb{C}(x', \hat{x}') = \emptyset, \quad \mathbb{C}(x, \hat{x}) = \{g\} \quad \text{and} \quad \mathbb{C}(x'', \hat{x}'') = \{g''\}.$$

Hence we get that

$$\mathbb{C}(u', \hat{u}') \bullet \mathbb{C}(u, \hat{u})(g, g'') = \int^{g' \in \mathbb{C}(x', \hat{x}') = \emptyset} \mathbb{C}(u', \hat{u}')(g', g'') \times \mathbb{C}(u, \hat{u})(g, g') = \emptyset,$$

while

$$\mathbb{C}(u' \bullet u, \hat{u}' \bullet \hat{u})(g, g'') = \{\alpha\}.$$

Hence $(\mu_{(u, \hat{u}), (u', \hat{u}')})_{g, g''}$ is not a bijection, and therefore $\mu_{(u, \hat{u}), (u', \hat{u}'})$ is not invertible.

By fixing components, we get the following.

3.41. CONSTRUCTION. Given a double category \mathbb{C} and an object \hat{x} in \mathbb{C} , the *representable lax double presheaf at \hat{x}* is the composite of normal lax double functors

$$\mathbb{C}(-, \hat{x}): \mathbb{C}^{\text{op}} \cong \mathbb{C}^{\text{op}} \times [0] \xrightarrow{\mathbb{C}^{\text{op}} \times \hat{x}} \mathbb{C}^{\text{op}} \times \mathbb{C} \xrightarrow{\mathbb{C}(-, -)} \text{Cat}.$$

This construction can be extended on horizontal morphisms and globular squares as follows.

3.42. CONSTRUCTION. Given a double category \mathbb{C} and a horizontal morphism $\hat{f}: \hat{x} \rightarrow \hat{y}$ in \mathbb{C} , the *representable horizontal transformation at \hat{f}*

$$\mathbb{C}(-, \hat{f}): \mathbb{C}(-, \hat{x}) \Rightarrow \mathbb{C}(-, \hat{y}): \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$$

is the horizontal transformation whose

- component at an object x in \mathbb{C} is the functor $\mathbb{C}(x, \hat{f}) := \mathbb{C}(1_x, \hat{f}): \mathbb{C}(x, \hat{x}) \rightarrow \mathbb{C}(x, \hat{y})$,
- component at a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} is the natural transformation of functors $\mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x', \hat{x}) \rightarrow \text{Set}$

$$\mathbb{C}(u, \hat{f}) := \mathbb{C}(1_u, e_{\hat{f}}): \mathbb{C}(u, e_{\hat{x}}) \Rightarrow \mathbb{C}(u, e_{\hat{y}})(\mathbb{C}(x, \hat{f})^{\text{op}} \times \mathbb{C}(x', \hat{f})).$$

The fact that $\mathbb{C}(-, \hat{f})$ is a horizontal transformation follows from the coherences of $\mathbb{C}(-, -)$.

3.43. CONSTRUCTION. Given a double category \mathbb{C} and a globular square $\hat{\alpha}: \left[\begin{array}{c} e_{\hat{x}} \quad \hat{f} \\ \hat{f}' \quad e_{\hat{x}'} \end{array} \right]$ in \mathbb{C} , the *representable globular modification at $\hat{\alpha}$*

$$\mathbb{C}(-, \hat{\alpha}): \left[\begin{array}{c} e_{\mathbb{C}(-, \hat{x})} \quad \mathbb{C}(-, \hat{f}) \\ \mathbb{C}(-, \hat{f}') \quad e_{\mathbb{C}(-, \hat{x}')} \end{array} \right]$$

is the globular modification whose component at an object x in \mathbb{C} is the natural transformation

$$\mathbb{C}(x, \hat{\alpha}) := \mathbb{C}(e_{1_x}, \hat{\alpha}): \mathbb{C}(x, \hat{f}) \Rightarrow \mathbb{C}(x, \hat{f}').$$

The fact that $\mathbb{C}(-, \hat{\alpha})$ is a globular modification follows from the coherences of $\mathbb{C}(-, -)$.

However, this construction fails to extend to vertical morphisms.

3.44. REMARK. Given a double category \mathbb{C} and a vertical morphism $\hat{u}: \hat{x} \rightarrow \hat{x}'$ in \mathbb{C} , we could consider the “vertical transformation” whose

- component at an object x in \mathbb{C} is the profunctor

$$\mathbb{C}(x, \hat{u}) := \mathbb{C}(e_x, \hat{u}): \mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x, \hat{x}') \rightarrow \text{Set},$$

- component at a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} is the natural transformation

$$\mathbb{C}(f, \hat{u}) := \mathbb{C}(e_f, 1_{\hat{u}}): \mathbb{C}(y, \hat{u}) \Rightarrow \mathbb{C}(x, \hat{u})(\mathbb{C}(f, \hat{x})^{\text{op}} \times \mathbb{C}(f, \hat{x}')).$$

One might expect to get a colax vertical transformation $\mathbb{C}(-, \hat{u}): \mathbb{C}(-, \hat{x}) \rightrightarrows \mathbb{C}(-, \hat{x}')$. However, there are in general no naturality comparison squares. Indeed, given a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , such a naturality comparison square would be a natural transformation of the form

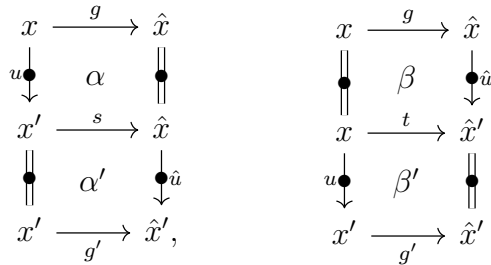
$$\mathbb{C}(x', \hat{u}) \bullet \mathbb{C}(u, \hat{x}) \rightrightarrows \mathbb{C}(u, \hat{x}') \bullet \mathbb{C}(x, \hat{u}): \mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x', \hat{x}') \rightarrow \text{Set}.$$

Thus, when evaluated at an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x}')$ in $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x}')$, we would get a map

$$\int^{s \in \mathbb{C}(x', \hat{x})} \mathbb{C}(x', \hat{u})(s, g') \times \mathbb{C}(u, \hat{x})(g, s) \longrightarrow \int^{t \in \mathbb{C}(x, \hat{x}')} \mathbb{C}(u, \hat{x}')(t, g') \times \mathbb{C}(x, \hat{u})(g, t). \quad (3.3)$$

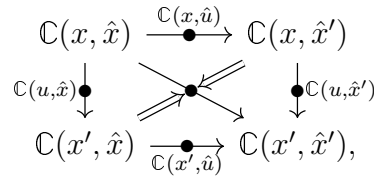
But in general, such a map does not exist, as we show in Example 3.45 below.

The core issue here is that elements in the left-hand set of (3.3) are represented by pairs (α, α') of squares as depicted below left for some horizontal morphism $s: x' \rightarrow \hat{x}$ in \mathbb{C} , while elements of the right-hand set of (3.3) are represented by pairs (β, β') of squares as depicted below right for some horizontal morphism $t: x \rightarrow \hat{x}'$.



But there is no natural assignment between such pairs.

Can we evade this problem by introducing a new notion of vertical transformation? Note that the vertical composites $\alpha' \bullet \alpha$ and $\beta' \bullet \beta$ of the above squares share the same boundaries, so the composition comparison transformations of $\mathbb{C}(-, -)$ fit into the following picture



where the diagonal vertical morphism is given by $\mathbb{C}(u, \hat{u})$, which might be a good candidate for replacing the naturality comparison squares. This fixes our issue regarding naturality comparison squares, but introduces yet another problem: such vertical transformations do not compose! Indeed, the fact that the 2-morphisms in the above diagram face opposite directions prevents the composition of two such transformations.

These issues are the core of our decision to work in the 2-category $\mathcal{P}_{\mathbb{C}}$ of lax double presheaves, in which vertical transformations are not considered, rather than in the double category $[[\mathbb{C}^{\text{op}}, \text{Cat}]]^{\text{nlax}}$.

3.45. EXAMPLE. We consider the double category \mathbb{C} generated by the following data

$$\begin{array}{ccc}
 x & \xrightarrow{g} & \hat{x} \\
 u \downarrow & \alpha & \Downarrow \\
 x' & \xrightarrow{h} & \hat{x} \\
 \Downarrow & \alpha' & \downarrow \hat{u} \\
 x' & \xrightarrow{g'} & \hat{x}',
 \end{array}$$

Then, note the we have

$$\mathbb{C}(x, \hat{x}') = \emptyset, \quad \mathbb{C}(x, \hat{x}) = \{g\}, \quad \mathbb{C}(x', \hat{x}) = \{h\} \quad \text{and} \quad \mathbb{C}(x', \hat{x}') = \{g'\}.$$

Hence we get that

$$\int^{s \in \mathbb{C}(x', \hat{x}) = \{h\}} \mathbb{C}(x', \hat{u})(s, g') \times \mathbb{C}(u, \hat{x})(g, s) = \{(\alpha', \alpha)\}$$

while

$$\int^{t \in \mathbb{C}(x, \hat{x}') = \emptyset} \mathbb{C}(u, \hat{x}')(t, g') \times \mathbb{C}(x, \hat{u})(g, t) = \emptyset.$$

This implies that a map as in (3.3) cannot exist.

4. Yoneda lemma

In this section, we show a 2-categorical version of the Yoneda lemma for double categories using the weak double category Cat . More precisely, given a double category \mathbb{C} and a double lax presheaf X over \mathbb{C} , we prove that there is a 2-natural isomorphism between the category of morphisms from a representable lax double presheaf $\mathbb{C}(-, \hat{x})$ to X and the category $X\hat{x}$.

In Sections 4.1 to 4.4, we construct the Yoneda comparison map and show that it is 2-natural. In Sections 4.5 to 4.10, we construct its inverse henceforth proving the desired Yoneda lemma. Finally, in Sections 4.11 and 4.12, we deduce that there is a Yoneda embedding from the underlying horizontal 2-category of \mathbb{C} into its 2-category of lax double presheaves $\mathcal{P}_{\mathbb{C}}$.

STATEMENT OF YONEDA LEMMA. We start by constructing the components of the Yoneda isomorphism.

4.1. CONSTRUCTION. Given a double category \mathbb{C} , a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, and an object $\hat{x} \in \mathbb{C}$, we construct a functor

$$\Psi_{\hat{x}, X}: \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) \rightarrow X\hat{x}.$$

It sends a horizontal transformation $\varphi: \mathbb{C}(-, \hat{x}) \Rightarrow X$ to the object $\varphi_{\hat{x}}(1_{\hat{x}})$ of $X\hat{x}$, i.e., the image of the identity $1_{\hat{x}}$ under the functor $\varphi_{\hat{x}}: \mathbb{C}(\hat{x}, \hat{x}) \rightarrow X\hat{x}$. It sends a modification

$$\begin{array}{ccc} \mathbb{C}(-, \hat{x}) & \xrightarrow{\varphi} & X \\ \Downarrow & \nu & \Downarrow \\ \mathbb{C}(-, \hat{x}) & \xrightarrow{\varphi'} & X \end{array}$$

to the morphism $(\nu_{\hat{x}})_{1_{\hat{x}}}: \varphi_{\hat{x}}(1_{\hat{x}}) \rightarrow \varphi'_{\hat{x}}(1_{\hat{x}})$ in $X\hat{x}$, i.e., the component of the natural transformation $\nu_{\hat{x}}: \varphi_{\hat{x}} \Rightarrow \varphi'_{\hat{x}}: \mathbb{C}(\hat{x}, \hat{x}) \rightarrow X\hat{x}$ at $1_{\hat{x}}$.

We now state the Yoneda lemma.

4.2. THEOREM. *Given a double category \mathbb{C} , a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, and an object \hat{x} in \mathbb{C} , the functor*

$$\Psi_{\hat{x}, X}: \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) \rightarrow X\hat{x}$$

is an isomorphism, which is 2-natural in \hat{x} in \mathbf{HC} and in X in $\mathcal{P}_{\mathbb{C}}$.

We show that it is 2-natural in \hat{x} and X , separately.

4.3. PROPOSITION. *The functors*

$$\Psi_{\hat{x}, X}: \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) \rightarrow X\hat{x}$$

are 2-natural in \hat{x} in \mathbf{HC} .

PROOF. To prove 2-naturality in \hat{x} , we first have to show that, given a horizontal morphism $\hat{f}: \hat{x} \rightarrow \hat{y}$ in \mathbb{C} , the following diagram of functors commutes.

$$\begin{array}{ccc} \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{y}), X) & \xrightarrow{\Psi_{\hat{y}, X}} & X\hat{y} \\ \mathbb{C}(-, \hat{f})^* \downarrow & & \downarrow X\hat{f} \\ \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) & \xrightarrow{\Psi_{\hat{x}, X}} & X\hat{x} \end{array}$$

To do so, we evaluate both composites at a horizontal transformation $\varphi: \mathbb{C}(-, \hat{y}) \Rightarrow X$:

$$\begin{aligned} \Psi_{\hat{x}, X}(\mathbb{C}(-, \hat{f})^*(\varphi)) &= \Psi_{\hat{x}, X}(\varphi \circ \mathbb{C}(-, \hat{f})) && \text{Definition of } \mathbb{C}(-, \hat{f})^* \\ &= (\varphi \circ \mathbb{C}(-, \hat{f}))_{\hat{x}}(1_{\hat{x}}) && \text{Definition of } \Psi_{\hat{x}, X} \\ &= \varphi_{\hat{x}}(\mathbb{C}(\hat{x}, \hat{f})(1_{\hat{x}})) && \text{Composition of transformations} \\ &= \varphi_{\hat{x}}(\hat{f}) && \text{Definition of } \mathbb{C}(\hat{x}, \hat{f}) \\ &= \varphi_{\hat{x}}(\mathbb{C}(\hat{f}, \hat{y})(1_{\hat{y}})) && \text{Definition of } \mathbb{C}(\hat{f}, \hat{y}) \\ &= X\hat{f}(\varphi_{\hat{y}}(1_{\hat{y}})) && \text{Naturality of } \varphi \\ &= X\hat{f}(\Psi_{\hat{y}, X}(\varphi)), && \text{Definition of } \Psi_{\hat{y}, X} \end{aligned}$$

as well as at a globular modification $\nu: [e_{\mathbb{C}(-,\hat{y})} \xrightarrow{\varphi} e_X]$:

$$\begin{aligned}
 \Psi_{\hat{x},X}(\mathbb{C}(-,\hat{f})^*(\nu)) &= \Psi_{\hat{x},X}(\nu \circ \mathbb{C}(-,\hat{f})) && \text{Definition of } \mathbb{C}(-,\hat{f})^* \\
 &= ((\nu \circ \mathbb{C}(-,\hat{f}))_{\hat{x}})_{1_{\hat{x}}} && \text{Definition of } \Psi_{\hat{x},X} \\
 &= (\nu_{\hat{x}})_{\mathbb{C}(\hat{x},\hat{f})(1_{\hat{x}})} && \text{Whiskering} \\
 &= (\nu_{\hat{x}})_{\hat{f} \circ 1_{\hat{x}}} && \text{Definition of } \mathbb{C}(\hat{x},\hat{f}) \\
 &= (\nu_{\hat{x}})_{1_{\hat{y}} \circ \hat{f}} && \hat{f} \circ 1_{\hat{x}} = \hat{f} = 1_{\hat{y}} \circ \hat{f} \\
 &= (\nu_{\hat{x}})_{\mathbb{C}(\hat{f},\hat{y})(1_{\hat{y}})} && \text{Definition of } \mathbb{C}(\hat{f},\hat{y}) \\
 &= X\hat{f}((\nu_{\hat{y}})_{1_{\hat{y}}}) && \text{Horizontal compatibility of } \nu \\
 &= X\hat{f}(\Psi_{\hat{y},X}(\nu)). && \text{Definition of } \Psi_{\hat{y},X}
 \end{aligned}$$

This shows that the two composites agree on objects and morphisms, as desired.

Next, we have to show that, given a globular square $\hat{\alpha}: [e_{\hat{x}} \xrightarrow{\hat{f}} e_{\hat{y}}]$ in \mathbb{C} , the following pasting diagram of functors and natural transformations commutes.

$$\begin{array}{ccc}
 \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-,\hat{y}), X) & \xrightarrow{\Psi_{\hat{y},X}} & X\hat{y} \\
 \mathbb{C}(-,\hat{f})^* \left(\begin{array}{c} \mathbb{C}(-,\hat{\alpha})^* \\ \xrightarrow{\quad} \\ \mathbb{C}(-,\hat{f}')^* \end{array} \right) & & X\hat{f} \left(\begin{array}{c} X\hat{\alpha} \\ \xrightarrow{\quad} \\ X\hat{f}' \end{array} \right) \\
 \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-,\hat{x}), X) & \xrightarrow{\Psi_{\hat{x},X}} & X\hat{x}
 \end{array}$$

To do so, we evaluate both whiskerings at a horizontal transformation $\varphi: \mathbb{C}(-,\hat{y}) \Rightarrow X$:

$$\begin{aligned}
 (\Psi_{\hat{x},X} \circ \mathbb{C}(-,\hat{\alpha})^*)_{\varphi} &= \Psi_{\hat{x},X}(\varphi \circ \mathbb{C}(-,\hat{\alpha})) && \text{Definition of } \mathbb{C}(-,\hat{\alpha})^* \\
 &= ((\varphi \circ \mathbb{C}(-,\hat{\alpha}))_{\hat{x}})_{1_{\hat{x}}} && \text{Definition of } \Psi_{\hat{x},X} \\
 &= \varphi_{\hat{x}}(\mathbb{C}(\hat{x},\hat{\alpha})_{1_{\hat{x}}}) && \text{Whiskering} \\
 &= \varphi_{\hat{x}}(\hat{\alpha} \circ e_{1_{\hat{x}}}) && \text{Definition of } \mathbb{C}(\hat{x},\hat{\alpha}) \\
 &= \varphi_{\hat{x}}(e_{1_{\hat{y}}} \circ \hat{\alpha}) && \hat{\alpha} \circ e_{1_{\hat{x}}} = \hat{\alpha} = e_{1_{\hat{y}}} \circ \hat{\alpha} \\
 &= \varphi_{\hat{x}}(\mathbb{C}(\hat{\alpha},\hat{y})_{1_{\hat{y}}}) && \text{Definition of } \mathbb{C}(\hat{\alpha},\hat{y}) \\
 &= (X\hat{\alpha})_{\varphi_{\hat{y}}(1_{\hat{y}})} && \text{2-Naturality of } \varphi \\
 &= (X\hat{\alpha})_{\Psi_{\hat{y},X}(\varphi)} && \text{Definition of } \Psi_{\hat{y},X} \\
 &= (X\hat{\alpha} \circ \Psi_{\hat{y},X})_{\varphi}. && \text{Whiskering}
 \end{aligned}$$

This shows that the components of the two whiskerings agree, as desired. ■

4.4. PROPOSITION. *The functors*

$$\Psi_{\hat{x},X}: \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-,\hat{x}), X) \rightarrow X\hat{x}$$

are 2-natural in X in $\mathcal{P}_{\mathbb{C}}$.

PROOF. To prove 2-naturality in X , we first have to show that, given a horizontal transformation $F: X \Rightarrow Y$ of lax double presheaves over \mathbb{C} , the following diagram of functors commutes.

$$\begin{array}{ccc} \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) & \xrightarrow{\Psi_{\hat{x}, X}} & X \hat{x} \\ F_* \downarrow & & \downarrow F_{\hat{x}} \\ \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), Y) & \xrightarrow{\Psi_{\hat{x}, Y}} & Y \hat{x} \end{array}$$

To do so, we evaluate both composites at a horizontal transformation $\varphi: \mathbb{C}(-, \hat{x}) \Rightarrow X$:

$$\begin{aligned} \Psi_{\hat{x}, Y}(F_*(\varphi)) &= \Psi_{\hat{x}, Y}(F \circ \varphi) && \text{Definition of } F_* \\ &= (F \circ \varphi)_{\hat{x}}(1_{\hat{x}}) && \text{Definition of } \Psi_{\hat{x}, Y} \\ &= F_{\hat{x}}(\varphi_{\hat{x}}(1_{\hat{x}})) && \text{Composition of horizontal transformations} \\ &= F_{\hat{x}}(\Psi_{\hat{x}, X}(\varphi)) && \text{Definition of } \Psi_{\hat{x}, X} \end{aligned}$$

as well as at a globular modification $\nu: [e_{\mathbb{C}(-, \hat{x})} \xrightarrow{\varphi} e_X]$:

$$\begin{aligned} \Psi_{\hat{x}, Y}(F_*(\nu)) &= \Psi_{\hat{x}, Y}(F \circ \nu) && \text{Definition of } F_* \\ &= ((F \circ \nu)_{\hat{x}})_{1_{\hat{x}}} && \text{Definition of } \Psi_{\hat{x}, X} \\ &= F_{\hat{x}}((\nu_{\hat{x}})_{1_{\hat{x}}}) && \text{Whiskering} \\ &= F_{\hat{x}}(\Psi_{\hat{x}, X}(\nu)). && \text{Definition of } \Psi_{\hat{x}, X} \end{aligned}$$

This shows that the two composites agree on objects and morphisms, as desired.

Next, we have to show that, given a globular modification $A: [e_X \xrightarrow{F'} e_Y]$ of lax double presheaves over \mathbb{C} , the following pasting diagram of functors and natural transformations commutes.

$$\begin{array}{ccc} \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) & \xrightarrow{\Psi_{\hat{x}, X}} & X \hat{x} \\ \begin{array}{c} \left(\begin{array}{ccc} & & \\ & \xrightarrow{A_*} & \\ & & \end{array} \right) \\ F_* \left(\begin{array}{ccc} & & \\ & \xrightarrow{A_*} & \\ & & \end{array} \right) F'_* \end{array} & & \begin{array}{c} \left(\begin{array}{ccc} & & \\ & \xrightarrow{A_{\hat{x}}} & \\ & & \end{array} \right) \\ F_{\hat{x}} \left(\begin{array}{ccc} & & \\ & \xrightarrow{A_{\hat{x}}} & \\ & & \end{array} \right) F'_{\hat{x}} \end{array} \\ \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), Y) & \xrightarrow{\Psi_{\hat{x}, Y}} & Y \hat{x} \end{array}$$

To do so, we evaluate both whiskerings at a horizontal transformation $\varphi: \mathbb{C}(-, \hat{x}) \Rightarrow X$:

$$\begin{aligned} (\Psi_{\hat{x}, Y} \circ A_*)_{\varphi} &= \Psi_{\hat{x}, Y}(A \circ \varphi) && \text{Definition of } A_* \\ &= ((A \circ \varphi)_{\hat{x}})_{1_{\hat{x}}} && \text{Definition of } \Psi_{\hat{x}, Y} \\ &= (A_{\hat{x}})_{\varphi_{\hat{x}}(1_{\hat{x}})} && \text{Whiskering} \\ &= (A_{\hat{x}})_{\Psi_{\hat{x}, X}(\varphi)} && \text{Definition of } \Psi_{\hat{x}, X} \\ &= (A_{\hat{x}} \circ \Psi_{\hat{x}, X})_{\varphi}. && \text{Whiskering} \end{aligned}$$

This shows that the components of the two whiskerings agree, as desired. ■

PROOF OF YONEDA LEMMA. To prove the Yoneda lemma, namely Theorem 4.2, let us fix a double category \mathbb{C} , a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ and an object $\hat{x} \in \mathbb{C}$. We construct an inverse $\Phi_{\hat{x}, X}: X\hat{x} \rightarrow \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X)$ of the functor $\Psi_{\hat{x}, X}$ from Construction 4.1. We start with its assignment on objects.

4.5. CONSTRUCTION. Given an object x_- in $X\hat{x}$, we construct a horizontal transformation

$$\Phi_{\hat{x}, X}(x_-): \mathbb{C}(-, \hat{x}) \Rightarrow X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$$

such that

- its component at an object x in \mathbb{C} is given by the functor

$$\Phi_{\hat{x}, X}(x_-)_x: \mathbb{C}(x, \hat{x}) \rightarrow Xx$$

sending an object $x \xrightarrow{g} \hat{x}$ in $\mathbb{C}(x, \hat{x})$ to the object $Xg(x_-)$ in Xx , i.e., the image of the object x_- under the functor $Xg: X\hat{x} \rightarrow Xx$, and a square $\eta: [e_x \xrightarrow{g} e_{\hat{x}}]$ to the morphism $(X\eta)_{x_-}: Xg(x_-) \rightarrow Xg'(x_-)$ in Xx , i.e., the component at x_- of the natural transformation $X\eta: Xg \Rightarrow Xg': X\hat{x} \rightarrow Xx$,

- its component at a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} is given by the natural transformation

$$\Phi_{\hat{x}, X}(x_-)_u: \mathbb{C}(u, \hat{x}) \Rightarrow Xu(\Phi_{\hat{x}, X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x}, X}(x_-)_{x'}): \mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x', \hat{x}) \rightarrow \text{Set}$$

whose component at an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x})$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x})$ is the map

$$\mathbb{C}(u, \hat{x})(g, g') \longrightarrow Xu(Xg(x_-), Xg'(x_-))$$

sending a square $\eta: [u \xrightarrow{g} e_{\hat{x}}]$ to the element $(X\eta)_{x_-, x_-}(1_{x_-})$, i.e., the image of the identity 1_{x_-} under the component $(X\eta)_{x_-, x_-}: X\hat{x}(x_-, x_-) \rightarrow Xu(Xg(x_-), Xg'(x_-))$ of the natural transformation $X\eta: Xe_{\hat{x}} = e_{X\hat{x}} \Rightarrow Xu(Xg^{\text{op}} \times Xg')$ at x_- .

4.6. LEMMA. *The construction*

$$\Phi_{\hat{x}, X}(x_-): \mathbb{C}(-, \hat{x}) \Rightarrow X$$

is a horizontal transformation.

PROOF. We first show naturality of the components $\Phi_{\hat{x}, X}(x_-)_x$ in x . For this, let $f: y \rightarrow x$ be a horizontal morphism in \mathbb{C} . We want to show that the following diagram of functors commutes.

$$\begin{array}{ccc} \mathbb{C}(x, \hat{x}) & \xrightarrow{\mathbb{C}(f, \hat{x})} & \mathbb{C}(y, \hat{x}) \\ \Phi_{\hat{x}, X}(x_-)_x \downarrow & & \downarrow \Phi_{\hat{x}, X}(x_-)_y \\ Xx & \xrightarrow{Xf} & Xy \end{array}$$

To do so, we evaluate both composites at an object $x \xrightarrow{g} \hat{x}$ of $\mathbb{C}(x, \hat{x})$:

$$\begin{aligned} \Phi_{\hat{x}, X}(x_-)_y \circ \mathbb{C}(f, \hat{x})(g) &= \Phi_{\hat{x}, X}(x_-)_y(g \circ f) && \text{Definition of } \mathbb{C}(f, \hat{x}) \\ &= X(g \circ f)(x_-) && \text{Definition of } \Phi_{\hat{x}, X}(x_-) \\ &= Xf \circ Xg(x_-) && \text{Functoriality of } X \\ &= Xf \circ \Phi_{\hat{x}, X}(x_-)_x(g) && \text{Definition of } \Phi_{\hat{x}, X}(x_-) \end{aligned}$$

as well as at a morphism $\eta: [e_x \xrightarrow{g} e_{\hat{x}}]$ of $\mathbb{C}(x, \hat{x})$:

$$\begin{aligned} \Phi_{\hat{x}, X}(x_-)_y \circ \mathbb{C}(f, \hat{x})(\eta) &= \Phi_{\hat{x}, X}(x_-)_y(\eta \circ e_f) && \text{Definition of } \mathbb{C}(f, \hat{x}) \\ &= X(\eta \circ e_f)_{x_-, x_-}(1_{x_-}) && \text{Definition of } \Phi_{\hat{x}, X}(x_-) \\ &= Xe_f \circ X\eta_{x_-, x_-}(1_{x_-}) && \text{Functoriality of } X \\ &= Xf \circ X\eta_{x_-, x_-}(1_{x_-}) && \text{Normality of } X \\ &= Xf \circ \Phi_{\hat{x}, X}(x_-)_x(\eta) && \text{Definition of } \Phi_{\hat{x}, X}(x_-) \end{aligned}$$

This shows that the two composites agree on objects and morphisms, as desired.

Next, we show naturality of the components $\Phi_{\hat{x}, X}(x_-)_u$ in u . For this, consider a square $\alpha: [v \xrightarrow{f} u] : [y' \xrightarrow{x} x']$ in \mathbb{C} . We want to show that the following diagram in $\text{Prof}(\mathbb{C}(x, \hat{x}), \mathbb{C}(x', \hat{x}))$ commutes.

$$\begin{array}{ccc} \mathbb{C}(u, \hat{x}) & \xrightarrow{\mathbb{C}(\alpha, \hat{x})} & \mathbb{C}(v, \hat{x})(\mathbb{C}(f, \hat{x})^{\text{op}} \times \mathbb{C}(f', \hat{x})) \\ \Phi_{\hat{x}, X}(x_-)_u \downarrow & & \downarrow \Phi_{\hat{x}, X}(x_-)_v(\mathbb{C}(f, \hat{x})^{\text{op}} \times \mathbb{C}(f', \hat{x})) \\ Xu(\Phi_{\hat{x}, X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x}, X}(x_-)_{x'}) & \xrightarrow{X\alpha(\Phi_{\hat{x}, X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x}, X}(x_-)_{x'})} & Xv((Xf \circ \Phi_{\hat{x}, X}(x_-)_x)^{\text{op}} \times (Xf' \circ \Phi_{\hat{x}, X}(x_-)_{x'})) \end{array}$$

where we recall that, by the first part, we have the relations $Xf \circ \Phi_{\hat{x}, X}(x_-)_x = \Phi_{\hat{x}, X}(x_-)_y \circ \mathbb{C}(f, \hat{x})$ and $Xf' \circ \Phi_{\hat{x}, X}(x_-)_{x'} = \Phi_{\hat{x}, X}(x_-)_{y'} \circ \mathbb{C}(f', \hat{x})$. When evaluated at an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x})$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x})$, this amounts to showing that the following diagram in Set commutes.

$$\begin{array}{ccc} \mathbb{C}(u, \hat{x})(g, g') & \xrightarrow{(\mathbb{C}(\alpha, \hat{x}))_{g, g'}} & \mathbb{C}(u, \hat{x})(gf, g'f') \\ (\Phi_{\hat{x}, X}(x_-)_u)_{g, g'} \downarrow & & \downarrow (\Phi_{\hat{x}, X}(x_-)_v)_{gf, g'f'} \\ Xv(Xg(x_-), Xg'(x_-)) & \xrightarrow{(X\alpha)_{Xg(x_-), Xg'(x_-)}} & Xu(X(gf)(x_-), X(g'f')(x_-)) \end{array}$$

To do so, we evaluate both composites at an element $\eta: [u \xrightarrow{g} e_{\hat{x}}]$ in $\mathbb{C}(u, \hat{x})(g, g')$:

$$\begin{aligned} &(\Phi_{\hat{x}, X}(x_-)_v)_{gf, g'f'}(\mathbb{C}(\alpha, \hat{x})(\eta)) \\ &= X(\mathbb{C}(\alpha, \hat{x})(\eta))_{x_-, x_-}(1_{x_-}) && \text{Definition of } \Phi_{\hat{x}, X}(x_-)_v \\ &= X(\eta \circ \alpha)_{x_-, x_-}(1_{x_-}) && \text{Definition of } \mathbb{C}(\alpha, \hat{x}) \\ &= (X\alpha)_{Xg(x_-), Xg'(x_-)}((X\eta)_{x_-, x_-}(1_{x_-})) && \text{Functoriality of } X \\ &= (X\alpha)_{Xg(x_-), Xg'(x_-)}(\Phi_{\hat{x}, X}(x_-)_u(\eta)). && \text{Definition of } \Phi_{\hat{x}, X}(x_-)_u \end{aligned}$$

This shows that the two composites agree, as desired.

Next, we show compatibility of $\Phi_{\hat{x},X}(x_-)_u$ with composition comparison squares. For this, let $u: x \rightarrow x'$ and $u': x' \rightarrow x''$ be composable vertical morphisms in \mathbb{C} . We want to show that the following diagram in $\text{Prof}(\mathbb{C}(x, \hat{x}), \mathbb{C}(x'', \hat{x}))$ commutes.

$$\begin{array}{ccc} \mathbb{C}(u', \hat{x}) \bullet \mathbb{C}(u, \hat{x}) & \xrightarrow{\mu_{u,u'}} & \mathbb{C}(u' \bullet u, \hat{x}) \\ \Phi_{\hat{x},X}(x_-)_{u'} \bullet \Phi_{\hat{x},X}(x_-)_u \downarrow & & \downarrow \Phi_{\hat{x},X}(x_-)_{u' \bullet u} \\ (Xu' \bullet Xu)(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x_-)_{x''}) & \xrightarrow{\mu_{u,u'}(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x_-)_{x''})} & X(u' \bullet u)(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x_-)_{x''}) \end{array}$$

When evaluated at an object $(x \xrightarrow{g} \hat{x}, x'' \xrightarrow{g''} \hat{x})$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x'', \hat{x})$, this amounts to showing that the following diagram in Set commutes.

$$\begin{array}{ccc} (\mathbb{C}(u', \hat{x}) \bullet \mathbb{C}(u, \hat{x}))(g, g'') & \xrightarrow{(\mu_{u,u'})_{g,g''}} & \mathbb{C}(u' \bullet u, \hat{x})(g, g'') \\ (\Phi_{\hat{x},X}(x_-)_{u'} \bullet \Phi_{\hat{x},X}(x_-)_u)_{g,g''} \downarrow & & \downarrow (\Phi_{\hat{x},X}(x_-)_{u' \bullet u})_{g,g''} \\ (Xu' \bullet Xu)(Xg(x_-), Xg''(x_-)) & \xrightarrow{(\mu_{u,u'})_{Xg(x_-), Xg''(x_-)}} & X(u' \bullet u)(Xg(x_-), Xg''(x_-)) \end{array}$$

By the universal properties of the coequalizers, it is enough to show that, given an object $x' \xrightarrow{g'} \hat{x}$ of $\mathbb{C}(x', \hat{x})$, the following diagram in Set commutes.

$$\begin{array}{ccc} \mathbb{C}(u', \hat{x})(g', g'') \times \mathbb{C}(u, \hat{x})(g, g') & \xrightarrow{\bullet} & \mathbb{C}(u' \bullet u, \hat{x})(g, g'') \\ \downarrow (\Phi_{\hat{x},X}(x_-)_{u'})_{g',g''} \times (\Phi_{\hat{x},X}(x_-)_u)_{g,g'} & & \downarrow (\Phi_{\hat{x},X}(x_-)_{u' \bullet u})_{g,g''} \\ Xu'(Xg'(x_-), Xg''(x_-)) \times Xu(Xg(x_-), Xg'(x_-)) & \xrightarrow{(\mu_{u,u'})_{Xg(x_-), Xg''(x_-)}} & X(u' \bullet u)(Xg(x_-), Xg''(x_-)) \end{array}$$

To do so, we evaluate both composites at an element $(\eta': [u' \xrightarrow{g'} e_{\hat{x}}], \eta: [u \xrightarrow{g} e_{\hat{x}}])$ of the set $\mathbb{C}(u', \hat{x})(g', g'') \times \mathbb{C}(u, \hat{x})(g, g')$:

$$\begin{aligned} & (\mu_{u,u'})_{Xg(x_-), Xg''(x_-)}((\Phi_{\hat{x},X}(x_-)_{u'} \times \Phi_{\hat{x},X}(x_-)_u)((\eta', \eta))) \\ &= (\mu_{u,u'})_{Xg(x_-), Xg''(x_-)}((X\eta')_{x_-, x_-}(1_{x_-}), (X\eta)_{x_-, x_-}(1_{x_-})) && \text{Definition of } \Phi_{\hat{x},X}(x_-) \\ &= X(\eta' \bullet \eta)_{x_-, x_-}(1_{x_-}) && \text{Naturality of } \mu_{u,u'} \\ &= \Phi_{\hat{x},X}(x_-)(\eta' \bullet \eta) && \text{Definition of } \Phi_{\hat{x},X}(x_-)_{u' \bullet u} \end{aligned}$$

This shows that both composites agree, as desired.

Finally, we show compatibility of $\Phi_{\hat{x},X}(x_-)_u$ with vertical identities. Given an object x of \mathbb{C} , we want to show that the natural transformation of functors $\mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x, \hat{x}) \rightarrow \text{Set}$

$$\Phi_{\hat{x},X}(x_-)_{e_x}: \mathbb{C}(e_x, \hat{x}) \Rightarrow Xe_x(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x_-)_x)$$

is the identity at $\Phi_{\hat{x},X}(x_-)_x$. When evaluated at an object $(x \xrightarrow{g} \hat{x}, x \xrightarrow{g'} \hat{x})$ of the category $\mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x, \hat{x})$, this is given by the map of sets

$$(\Phi_{\hat{x},X}(x_-)_{e_x})_{g,g'}: \mathbb{C}(x, \hat{x})(g, g') \rightarrow Xx(Xg(x_-), Xg'(x_-))$$

sending an element $\eta: [e_x \xrightarrow{g} e_{\hat{x}}]$ in $\mathbb{C}(x, \hat{x})(g, g')$ to $(X\eta)_{x_-,x_-}(1_{x_-}) = \Phi_{\hat{x},X}(x_-)_x(\eta)$. So, by Remark 3.27, we see that $\Phi_{\hat{x},X}(x_-)_{e_x}$ is the identity at $\Phi_{\hat{x},X}(x_-)_x$. ■

We now turn to the assignment of $\Psi_{\hat{x},X}$ on morphisms.

4.7. CONSTRUCTION. Given a morphism $u_-: x_- \rightarrow x'_-$ in $X\hat{x}$, we construct a globular modification in $\mathcal{P}_{\mathbb{C}}$

$$\begin{array}{ccc} \mathbb{C}(-, \hat{x}) & \xrightarrow{\Phi_{\hat{x},X}(x_-)} & X \\ \Downarrow & \Phi_{\hat{x},X}(u_-) & \Downarrow \\ \mathbb{C}(-, \hat{x}) & \xrightarrow{\Phi_{\hat{x},X}(x'_-)} & X \end{array}$$

such that its component at an object $x \in \mathbb{C}$ is given by the natural transformation

$$\Phi_{\hat{x},X}(u_-)_x: \Phi_{\hat{x},X}(x_-)_x \Rightarrow \Phi_{\hat{x},X}(x'_-)_x: \mathbb{C}(x, \hat{x}) \rightarrow Xx$$

whose component at an object $x \xrightarrow{g} \hat{x}$ of $\mathbb{C}(x, \hat{x})$ is given by the morphism

$$Xg(u_-): Xg(x_-) = \Phi_{\hat{x},X}(x_-)_x(g) \longrightarrow Xg(x'_-) = \Phi_{\hat{x},X}(x'_-)_x(g),$$

i.e., the image of the morphism u_- under the functor $Xg: X\hat{x} \rightarrow Xx$.

4.8. LEMMA. *The construction $\Phi_{\hat{x},X}(u_-)$ is a modification.*

PROOF. We start by showing horizontal compatibility of $\Phi_{\hat{x},X}(u_-)$. Given a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , we have to show that the following pasting diagram of functors and natural transformations commutes.

$$\begin{array}{ccc} \mathbb{C}(y, \hat{x}) & \xrightarrow{\mathbb{C}(f, \hat{x})} & \mathbb{C}(x, \hat{x}) \\ \Phi_{\hat{x},X}(x_-)_y \left(\begin{array}{c} \downarrow \\ \xrightarrow{\Phi_{\hat{x},X}(u_-)_y} \\ \downarrow \end{array} \right) & \Phi_{\hat{x},X}(x'_-)_y & \Phi_{\hat{x},X}(x_-)_x \left(\begin{array}{c} \downarrow \\ \xrightarrow{\Phi_{\hat{x},X}(u_-)_x} \\ \downarrow \end{array} \right) & \Phi_{\hat{x},X}(x'_-)_x \\ Xy & \xrightarrow{Xf} & Xx \end{array}$$

To do so, we evaluate both whiskerings at an object $y \xrightarrow{g} \hat{x}$ of $\mathbb{C}(y, \hat{x})$:

$$\begin{aligned} (Xf \circ \Phi_{\hat{x},X}(u_-)_y)_g &= Xf((\Phi_{\hat{x},X}(u_-)_y)_g) && \text{Whiskering} \\ &= Xf(Xg(u_-)) && \text{Definition of } \Phi_{\hat{x},X}(u_-)_y \\ &= X(gf)(u_-) && \text{Functoriality of } X \\ &= (\Phi_{\hat{x},X}(u_-)_x)_{gf} && \text{Definition of } \Phi_{\hat{x},X}(u_-)_x \\ &= (\Phi_{\hat{x},X}(u_-)_x)_{\mathbb{C}(f, \hat{x})(g)} && \text{Definition of } \mathbb{C}(f, \hat{x}) \\ &= (\Phi_{\hat{x},X}(u_-)_x \circ \mathbb{C}(f, \hat{x}))_g. && \text{Whiskering} \end{aligned}$$

This shows that the components of the two whiskerings agree, as desired.

Now, we show vertical compatibility of $\Phi_{\hat{x},X}(u_-)$. Given a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , we have to show that the following diagram commutes in $[\mathbb{C}(x, \hat{x})^{\text{op}} \times \mathbb{C}(x', \hat{x}), \text{Set}]$:

$$\begin{array}{ccc} \mathbb{C}(u, \hat{x}) & \xrightarrow{\Phi_{\hat{x},X}(x_-)_u} & Xu(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x_-)_{x'}) \\ \Phi_{\hat{x},X}(x'_-)_u \downarrow & & \downarrow Xu(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(u_-)_{x'}) \\ Xu(\Phi_{\hat{x},X}(x'_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x'_-)_{x'}) & \xrightarrow{Xu(\Phi_{\hat{x},X}(u_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x'_-)_{x'})} & Xu(\Phi_{\hat{x},X}(x_-)_x^{\text{op}} \times \Phi_{\hat{x},X}(x'_-)_{x'}) \end{array}$$

When evaluated at an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x})$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x})$, this amounts to showing that the following diagram in Set commutes.

$$\begin{array}{ccc} \mathbb{C}(u, \hat{x})(g, g') & \xrightarrow{(\Phi_{\hat{x},X}(x_-)_u)_{g,g'}} & Xu(Xg(x_-), Xg'(x_-)) \\ (\Phi_{\hat{x},X}(x'_-)_u)_{g,g'} \downarrow & & \downarrow Xg'(u_-)_* \\ Xu(Xg(x'_-), Xg'(x'_-)) & \xrightarrow{Xg(u_-)_*} & Xu(Xg(x_-), Xg'(x'_-)) \end{array}$$

To do so, we evaluate both composites at an element $\eta: [u \xrightarrow{g'} e_{\hat{x}}]$ in $\mathbb{C}(u, \hat{x})(g, g')$:

$$\begin{aligned} Xg'(u_-)_*((\Phi_{\hat{x},X}(x_-)_u)_{g,g'}(\eta)) &= Xg'(u_-)_*((X\eta)_{x_-,x_-}(1_{x_-})) && \text{Definition of } \Phi_{\hat{x},X}(x_-)_u \\ &= (X\eta)_{x_-,x'_-}((u_-)_*(1_{x_-})) && \text{Naturality of } X\eta \\ &= (X\eta)_{x_-,x'_-}(u_-) && \text{Definition of } (u_-)_* \\ &= (X\eta)_{x_-,x'_-}((u_-)^*(1_{x'_-})) && \text{Definition of } (u_-)^* \\ &= Xg(u_-)^*((X\eta)_{x'_-,x'_-}(1_{x'_-})) && \text{Naturality of } X\eta \\ &= Xg(u_-)^*((\Phi_{\hat{x},X}(x'_-)_u)_{g,g'}(\eta)) && \text{Definition of } \Phi_{\hat{x},X}(x'_-)_u \end{aligned}$$

This shows that the two composites agree, as desired. \blacksquare

Putting everything together, we get the following.

4.9. CONSTRUCTION. Given a double category \mathbb{C} , a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, and an object $\hat{x} \in \mathbb{C}$, we construct a functor

$$\Phi_{\hat{x},X}: X\hat{x} \rightarrow \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X)$$

which sends

- an object x_- in $X\hat{x}$ to the horizontal transformation $\Phi_{\hat{x},X}(x_-): \mathbb{C}(-, \hat{x}) \Rightarrow X$ as in Construction 4.5,
- a morphism $u_-: x_- \rightarrow x'_-$ in $X\hat{x}$ to the modification $\Phi_{\hat{x},X}(u_-): \left[e_{\mathbb{C}(-, \hat{x})} \begin{array}{c} \Phi_{\hat{x},X}(x_-) \\ \Phi_{\hat{x},X}(x'_-) \end{array} e_X \right]$ as in Construction 4.7.

We are now ready to prove the Yoneda lemma.

PROOF OF THEOREM 4.2. We show that the functors

$$\Psi_{\hat{x},X}: \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) \rightarrow X\hat{x} \quad \text{and} \quad \Phi_{\hat{x},X}: X\hat{x} \rightarrow \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X)$$

from Constructions 4.1 and 4.9 are inverses to each other.

We first consider the composite

$$X\hat{x} \xrightarrow{\Phi_{\hat{x},X}} \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) \xrightarrow{\Psi_{\hat{x},X}} X\hat{x}.$$

It sends an object x_- in $X\hat{x}$ to the object

$$\begin{aligned} \Psi_{\hat{x},X}(\Phi_{\hat{x},X}(x_-)) &= \Phi_{\hat{x},X}(x_-)_{\hat{x}}(1_{\hat{x}}) && \text{Definition of } \Psi_{\hat{x},X} \\ &= X1_{\hat{x}}(x_-) && \text{Definition of } \Phi_{\hat{x},X} \\ &= 1_{X\hat{x}}(x_-) = x_- && \text{Functoriality of } X \end{aligned}$$

and a morphism $u_-: x_- \rightarrow x'_-$ in $X\hat{x}$ to the morphism

$$\begin{aligned} \Psi_{\hat{x},X}(\Phi_{\hat{x},X}(u_-)) &= (\Phi_{\hat{x},X}(u_-))_{\hat{x}}1_{\hat{x}} && \text{Definition of } \Psi_{\hat{x},X} \\ &= X1_{\hat{x}}(u_-) && \text{Definition of } \Phi_{\hat{x},X}(u_-) \\ &= 1_{X\hat{x}}(u_-) = u_-. && \text{Functoriality of } X \end{aligned}$$

This shows that the composite $\Psi_{\hat{x},X} \circ \Phi_{\hat{x},X}$ is the identity.

We then consider the composite

$$\mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X) \xrightarrow{\Psi_{\hat{x},X}} X\hat{x} \xrightarrow{\Phi_{\hat{x},X}} \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), X)$$

It sends a horizontal transformation $\varphi: \mathbb{C}(-, \hat{x}) \Rightarrow X$ to the horizontal transformation

$$\Phi_{\hat{x},X}(\Psi_{\hat{x},X}(\varphi)) = \Phi_{\hat{x},X}(\varphi_{\hat{x}}(1_{\hat{x}})): \mathbb{C}(-, \hat{x}) \Rightarrow X$$

whose

- component at an object x in \mathbb{C} is given by the functor

$$\Phi_{\hat{x},X}(\varphi_{\hat{x}}(1_{\hat{x}}))_x: \mathbb{C}(x, \hat{x}) \rightarrow Xx$$

defined as follows. It sends an object $x \xrightarrow{g} \hat{x}$ in $\mathbb{C}(x, \hat{x})$ to the object

$$\begin{aligned} \Phi_{\hat{x},X}(\varphi_{\hat{x}}(1_{\hat{x}}))_x(g) &= Xg(\varphi_{\hat{x}}(1_{\hat{x}})) && \text{Definition of } \Phi_{\hat{x},X} \\ &= \varphi_x(\mathbb{C}(g, \hat{x})(1_{\hat{x}})) && \text{Naturality of } \varphi \\ &= \varphi_x(1_{\hat{x}} \circ g) = \varphi_x(g) && \text{Definition of } \mathbb{C}(g, \hat{x}) \end{aligned}$$

and a morphism $\eta: [e_x \xrightarrow{g} e_{\hat{x}}]$ in $\mathbb{C}(x, \hat{x})$ to the morphism

$$\begin{aligned} \Phi_{\hat{x},X}(\varphi_{\hat{x}}(1_{\hat{x}}))_x(\eta) &= (X\eta)_{\varphi_{\hat{x}}(1_{\hat{x}})} && \text{Definition of } \Phi_{\hat{x},X} \\ &= \varphi_x(\mathbb{C}(\eta, \hat{x})_{1_{\hat{x}}}) && \text{Naturality of } \varphi \\ &= \varphi_x(\eta). && \text{Definition of } \mathbb{C}(\eta, \hat{x}) \end{aligned}$$

This shows that $\Phi_{\hat{x},X}(\Psi_{\hat{x},X}(\varphi))_x = \Phi_{\hat{x},X}(\varphi_{\hat{x}}(1_{\hat{x}}))_x = \varphi_x$.

- component at a vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} is given by the natural transformation

$$\Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))_u: \mathbb{C}(u, \hat{x}) \Rightarrow Xu(\Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))_x^{\text{op}} \times \Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))_{x'}) = Xu(\varphi_x^{\text{op}} \times \varphi'_{x'})$$

defined as follows. Its component at an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x})$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x})$ is the map

$$(\Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))_u)_{g, g'}: \mathbb{C}(u, \hat{x})(g, g') \rightarrow Xu(\varphi_x(g), \varphi'_{x'}(g'))$$

which sends an element $\eta: [u \xrightarrow{g'} e_{\hat{x}}]$ of $\mathbb{C}(u, \hat{x})(g, g')$ to the element

$$\begin{aligned} (\Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))_u)_{g, g'}(\eta) &= (X\eta)_{\varphi_{\hat{x}}(1_{\hat{x}}), \varphi_{\hat{x}}(1_{\hat{x}})}(1_{\varphi_{\hat{x}}(1_{\hat{x}})}) && \text{Definition of } \Phi_{\hat{x}, X} \\ &= (X\eta)_{\varphi_{\hat{x}}(1_{\hat{x}}), \varphi_{\hat{x}}(1_{\hat{x}})}(e_{1_{\hat{x}}}) && \text{Functoriality of } \varphi_{\hat{x}} \\ &= \varphi_u(\mathbb{C}(\eta, \hat{x})_{1_{\hat{x}}, 1_{\hat{x}}}(e_{1_{\hat{x}}})) && \text{Naturality of } \varphi \\ &= \varphi_u(e_{1_{\hat{x}}} \circ \eta) = \varphi_u(\eta). && \text{Definition of } \mathbb{C}(\eta, \hat{x}) \end{aligned}$$

This shows that $\Phi_{\hat{x}, X}(\Psi_{\hat{x}, X}(\varphi))_u = \Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))_u = \varphi_u$.

Putting these two together, we deduce that $\Phi_{\hat{x}, X}(\Psi_{\hat{x}, X}(\varphi)) = \varphi$.

The composite $\Phi_{\hat{x}, X} \circ \Psi_{\hat{x}, X}$ sends a globular modification $\nu: [e_{\mathbb{C}(-, \hat{x})} \xrightarrow{\varphi} e_X]$ to the globular modification

$$\begin{array}{ccc} \mathbb{C}(-, \hat{x}) & \xrightarrow{\Phi_{\hat{x}, X}(\varphi_{\hat{x}}(1_{\hat{x}}))=\varphi} & X \\ \parallel & \Phi_{\hat{x}, X} \circ \Psi_{\hat{x}, X}(\nu) = \Phi_{\hat{x}, X}((\nu_{\hat{x}})_{1_{\hat{x}}}) & \parallel \\ \mathbb{C}(-, \hat{x}) & \xrightarrow{\Phi_{\hat{x}, X}(\varphi'_{\hat{x}}(1_{\hat{x}}))=\varphi'} & X \end{array}$$

defined as follows. Its component at an object x in \mathbb{C} is given by the natural transformation

$$\Phi_{\hat{x}, X}((\nu_{\hat{x}})_{1_{\hat{x}}})_x: \varphi_x \Rightarrow \varphi'_x: \mathbb{C}(x, \hat{x}) \rightarrow Xx.$$

whose component at an object $x \xrightarrow{g} \hat{x}$ of $\mathbb{C}(x, \hat{x})$ is

$$\begin{aligned} (\Phi_{\hat{x}, X}((\nu_{\hat{x}})_{1_{\hat{x}}})_x)_g &= Xg((\nu_{\hat{x}})_{1_{\hat{x}}}) && \text{Definition of } \Phi_{\hat{x}, X} \\ &= (\nu_x)_{\mathbb{C}(g, \hat{x})(1_{\hat{x}})} && \text{Horizontal compatibility of } \nu \\ &= (\nu_x)_{1_{\hat{x}} \circ g} = (\nu_x)_g. && \text{Definition of } \mathbb{C}(g, \hat{x}) \end{aligned}$$

This shows that $\Phi_{\hat{x}, X}(\Phi_{\hat{x}, X}(\nu)) = \nu$. Hence this proves that the composite $\Phi_{\hat{x}, X} \circ \Psi_{\hat{x}, X}$ is the identity, and concludes the proof. \blacksquare

As a consequence of Lemma 3.32 and Theorem 4.2, when taking $\mathbb{C} = \mathbb{H}\mathbb{C}$ with \mathcal{C} a 2-category, we retrieve the usual Yoneda lemma for 2-categories; see [JY21, Lemma 8.3.16].

4.10. COROLLARY. *Given a 2-category \mathcal{C} , a 2-presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$, and an object \hat{x} in \mathcal{C} , the functor*

$$\Psi_{\hat{x}, X}: [\mathcal{C}^{\text{op}}, \text{Cat}](\mathcal{C}(-, \hat{x}), X) \rightarrow X\hat{x}$$

is an isomorphism, which is 2-natural in \hat{x} in \mathcal{C} and in X in $[\mathcal{C}^{\text{op}}, \text{Cat}]$.

YONEDA EMBEDDING. Using the Yoneda lemma, we can construct a Yoneda embedding from the underlying horizontal 2-category of a double category \mathbb{C} to its 2-category of lax double presheaves.

4.11. CONSTRUCTION. Given a double category \mathbb{C} , there is a Yoneda 2-functor

$$\mathcal{Y}_{\mathbb{C}}: \mathbb{H}\mathbb{C} \rightarrow \mathcal{P}_{\mathbb{C}}$$

sending

- an object \hat{x} of \mathbb{C} to the representable lax double presheaf $\mathbb{C}(-, \hat{x}): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$,
- a horizontal morphism $\hat{f}: \hat{x} \rightarrow \hat{y}$ of \mathbb{C} to the representable horizontal transformation

$$\mathbb{C}(-, \hat{f}): \mathbb{C}(-, \hat{x}) \Rightarrow \mathbb{C}(-, \hat{y}),$$

- a globular square $\hat{\alpha}: \left[e_{\hat{x}} \begin{array}{c} \hat{f} \\ \hat{f}' \end{array} e_{\hat{y}} \right]$ to the representable globular modification

$$\mathbb{C}(-, \hat{\alpha}): \left[e_{\mathbb{C}(-, \hat{x})} \begin{array}{c} \mathbb{C}(-, \hat{f}) \\ \mathbb{C}(-, \hat{f}') \end{array} e_{\mathbb{C}(-, \hat{y})} \right].$$

It is straightforward to check that this construction is 2-functorial.

4.12. THEOREM. *Given a double category \mathbb{C} , the Yoneda 2-functor*

$$\mathcal{Y}_{\mathbb{C}}: \mathbb{H}\mathbb{C} \rightarrow \mathcal{P}_{\mathbb{C}}$$

is an embedding of 2-categories.

PROOF. Let \hat{x} and \hat{y} be objects in \mathbb{C} . Then by Theorem 4.2, we have a canonical isomorphism of categories

$$\mathbb{C}(\hat{x}, \hat{y}) \cong \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), \mathbb{C}(-, \hat{y})).$$

This shows that $\mathcal{Y}_{\mathbb{C}}: \mathbb{H}\mathbb{C} \rightarrow \mathcal{P}_{\mathbb{C}}$ is 2-fully-faithful, as desired. ■

5. Discrete double fibrations

We now want lax double presheaves from a fibrational point of view. For this, we recall in this section the notion of discrete double fibrations. In Sections 5.1 to 5.9, we recollect the definition and main properties of discrete double fibrations. In particular, we construct a 2-category of discrete double fibrations over a double category \mathbb{C} and show that it is 2-functorial in \mathbb{C} . In Sections 5.10 to 5.20, we study the fibers of a discrete double fibration at both an object and a vertical morphism, which will be useful in the next section to construct a lax double presheaf from a discrete double fibration. Finally, in Sections 5.21 to 5.23, we present a first class of examples of discrete double fibrations, which are given by canonical projections from slice double categories. As these will correspond to representable lax double presheaves, we refer to them as *representable discrete double fibrations*.

THE 2-CATEGORY OF DISCRETE DOUBLE FIBRATIONS. We start by introducing the 2-category of discrete double fibrations.

5.1. **DEFINITION.** A double functor $P: \mathbb{E} \rightarrow \mathbb{C}$ is a *discrete double fibration* if the following commutative square in Cat is a pullback

$$\begin{array}{ccc} \text{Ver}_1 \mathbb{E} & \xrightarrow{t} & \text{Ver}_0 \mathbb{E} \\ \text{ver}_1 P \downarrow & \lrcorner & \downarrow \text{ver}_0 P \\ \text{Ver}_1 \mathbb{C} & \xrightarrow{t} & \text{Ver}_0 \mathbb{C} \end{array}$$

5.2. **NOTATION.** We denote by $\text{DFib}(\mathbb{C})$ the 2-full 2-subcategory of the slice 2-category $\text{DblCat}_v/\mathbb{C}$ spanned by the discrete double fibrations over \mathbb{C} . In other words,

- an object in $\text{DFib}(\mathbb{C})$ is a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ over \mathbb{C} ,
- given discrete double fibrations $P: \mathbb{E} \rightarrow \mathbb{C}$ and $Q: \mathbb{F} \rightarrow \mathbb{C}$, a morphism from P to Q in $\text{DFib}(\mathbb{C})$ is a double functor $F: \mathbb{E} \rightarrow \mathbb{F}$ such that $Q \circ F = P$,
- given discrete double fibrations $P: \mathbb{E} \rightarrow \mathbb{C}$, $Q: \mathbb{F} \rightarrow \mathbb{C}$ as well as double functors $F: \mathbb{E} \rightarrow \mathbb{F}$, $F': \mathbb{E} \rightarrow \mathbb{F}$ such that $Q \circ F = P = Q \circ F'$, a 2-morphism from F to F' in $\text{DFib}(\mathbb{C})$ is a vertical transformation $A: F \Rightarrow F'$ such that $Q \circ A = P$, i.e., $Q(A_x) = e_{P_x}$ for all objects x in \mathbb{E} and $Q(A_f) = e_{P_f}$ for all horizontal morphisms f in \mathbb{E} .

5.3. **REMARK.** Unpacking the pullback condition, a double functor $P: \mathbb{E} \rightarrow \mathbb{C}$ is a discrete double fibration if and only if the following conditions hold:

- (1) for every object y_- in \mathbb{E} and every horizontal morphism $f: x \rightarrow Py_-$ in \mathbb{C} , there is a unique horizontal morphism $f_-: x_- \rightarrow y_-$ in \mathbb{E} such that $Pf_- = f$. We write $f^*y_- := x_-$ and $P^*f := f_-$.

(2) for every vertical morphism $v_- : x_- \dashrightarrow x'_-$ in \mathbb{E} and every square $\alpha : \left[u \begin{smallmatrix} f \\ f' \end{smallmatrix} Pv_- \right]$ in \mathbb{C} , there is a unique square $\alpha_- : \left[u_- \begin{smallmatrix} f_- \\ f'_- \end{smallmatrix} v_- \right]$ in \mathbb{E} such that $P\alpha_- = \alpha$. We write $\alpha^*v_- := u_-$ and $P^*\alpha := \alpha_-$.

5.4. REMARK. Note that, by unicity of the lifts, if $f : x \rightarrow y$ and $g : y \rightarrow z$ are two composable horizontal morphisms in \mathbb{C} and z_- is an object of \mathbb{E} such that $Pz_- = z$, then

$$(g \circ f)^*z_- = f^*g^*z_-.$$

Similarly, if $\alpha : \left[u \begin{smallmatrix} f \\ f' \end{smallmatrix} v \right]$ and $\beta : \left[v \begin{smallmatrix} g \\ g' \end{smallmatrix} w \right]$ are two horizontally composable squares in \mathbb{C} and w_- is a vertical morphism in \mathbb{E} such that $Pw_- = w$, then

$$(\beta \circ \alpha)^*w_- = \alpha^*\beta^*w_-.$$

The construction $D\mathcal{Fib}(\mathbb{C})$ extends to a pseudo-functor $\text{DbCat}_h^{\text{coop}} \rightarrow 2\text{Cat}$. To define its action on morphisms, we use the following pullback stability of discrete double fibrations. This result follows from [MSV23, Proposition 4.11] by taking $\mathcal{V} = \text{Cat}$ and the fact that pullbacks in 2Cat are 2-categorical pullbacks.

5.5. LEMMA. *Discrete double fibrations are stable under pullback. In particular, by univ-ersality of pullbacks, a double functor $G : \mathbb{C} \rightarrow \mathbb{D}$ induces a 2-functor*

$$G^* : D\mathcal{Fib}(\mathbb{D}) \rightarrow D\mathcal{Fib}(\mathbb{C})$$

by taking pullbacks along G .

We now describe its action on 2-morphisms.

5.6. CONSTRUCTION. Given a horizontal transformation $B : G \rightrightarrows G' : \mathbb{C} \rightarrow \mathbb{D}$, we define a 2-natural transformation

$$B^* : G'^* \rightrightarrows G^* : D\mathcal{Fib}(\mathbb{D}) \rightarrow D\mathcal{Fib}(\mathbb{C}),$$

whose component at a discrete double fibration $P : \mathbb{E} \rightarrow \mathbb{D}$ is given by the morphism of discrete double fibrations over \mathbb{C}

$$\begin{array}{ccc} G'^*\mathbb{E} & \xrightarrow{B_P^*} & G^*\mathbb{E} \\ & \searrow^{G'^*P} & \swarrow_{G^*P} \\ & & \mathbb{C} \end{array}$$

which sends

- an object (x, x_-) in $G'^*\mathbb{E}$, i.e., a pair of objects x in \mathbb{C} and x_- in \mathbb{E} such that $Px_- = G'x$, to the object $(x, (B_x)^*x_-)$ in $G^*\mathbb{E}$, where $(B_x)^*x_-$ denotes the source of the unique lift of the horizontal morphism $B_x : Gx \rightarrow G'x$ in \mathbb{D} with target x_- ,

- a horizontal morphism $(f, f_-): (x, x_-) \rightarrow (y, y_-)$ in $G'^*\mathbb{E}$, i.e., a pair of horizontal morphisms $f: x \rightarrow y$ in \mathbb{C} and $f_-: x_- \rightarrow y_-$ in \mathbb{E} such that $Pf_- = G'f$, to the horizontal morphism in $G^*\mathbb{E}$

$$(f, P^*Gf): (x, (B_x)^*x_-) \rightarrow (y, (B_y)^*y_-),$$

where $P^*Gf: (B_x)^*x_- \rightarrow (B_y)^*y_-$ denotes the unique lift of Gf ,

- a vertical morphism $(u, u_-): (x, x_-) \dashrightarrow (x', x'_-)$ in $G'^*\mathbb{E}$, i.e., a pair of vertical morphisms $u: x \dashrightarrow x'$ in \mathbb{C} and $u_-: x_- \dashrightarrow x'_-$ in \mathbb{E} such that $Pu_- = G'u$, to the vertical morphism in $G^*\mathbb{E}$

$$(u, (B_u)^*u_-): (x, (B_x)^*x_-) \dashrightarrow (x', (B_{x'})^*x'_-),$$

where $(B_u)^*u_-$ denotes the source of the unique lift of the square $B_u: \left[\begin{array}{ccc} Gu & B_x & G'u \\ & B_{x'} & \end{array} \right]$ in \mathbb{D} with target u_- ,

- a square $(\alpha, \alpha_-): \left[\begin{array}{ccc} (u, u_-) & \begin{array}{c} (f, f_-) \\ (f', f'_-) \end{array} & (v, v_-) \end{array} \right]: \left[\begin{array}{ccc} (x, x_-) & \begin{array}{c} (y, y_-) \\ (y', y'_-) \end{array} \end{array} \right]$ in $G'^*\mathbb{E}$, i.e., a pair of squares $\alpha: \left[\begin{array}{ccc} u & f & v \\ & f' & \end{array} \right]$ in \mathbb{C} and $\alpha_-: \left[\begin{array}{ccc} u_- & f_- & v_- \end{array} \right]$ in \mathbb{E} such that $P\alpha_- = G'\alpha$, to the square in $G^*\mathbb{E}$

$$\begin{array}{ccccc} (x, (B_x)^*x_-) & \xrightarrow{(f, P^*f)} & (y, (B_y)^*y_-) & & \\ (u, (B_u)^*u_-) \downarrow & & (\alpha, P^*G\alpha) & & \downarrow (v, (B_v)^*v_-) \\ (x', (B_{x'})^*x'_-) & \xrightarrow{(f', P^*f')} & (y', (B_{y'})^*y'_-) & & \end{array}$$

where $P^*G\alpha: \left[\begin{array}{ccc} (B_u)^*u_- & P^*f & (B_v)^*v_- \end{array} \right]$ denotes the unique lift of $G\alpha$.

5.7. LEMMA. *The construction $B^*: G'^* \Rightarrow G^*: \mathcal{DFib}(\mathbb{D}) \rightarrow \mathcal{DFib}(\mathbb{C})$ is a 2-natural transformation.*

PROOF. First note that the components $B_P^*: G'^*\mathbb{E} \rightarrow G^*\mathbb{E}$ are well-defined double functors over \mathbb{C} . The fact that it preserves compositions and identities is a consequence of the unicity of lifts.

Similarly, the 2-naturality of B_P^* in P follows from the unicity of lifts. ■

Putting everything together, we get the following.

5.8. CONSTRUCTION. We define a pseudo-functor

$$\mathcal{DFib}: \text{DblCat}_h^{\text{coop}} \rightarrow 2\text{Cat},$$

which sends

- a double category \mathbb{C} to the 2-category $\mathcal{DFib}(\mathbb{C})$ of discrete double fibrations, double functors over \mathbb{C} , and vertical transformations over \mathbb{C} ,

- a double functor $G: \mathbb{C} \rightarrow \mathbb{D}$ to the 2-functor

$$D\mathcal{Fib}(G) := G^*: D\mathcal{Fib}(\mathbb{D}) \rightarrow D\mathcal{Fib}(\mathbb{C})$$

induced by taking pullback along G ,

- a horizontal transformation $B: G \Rightarrow G': \mathbb{C} \rightarrow \mathbb{D}$ to the 2-natural transformation

$$D\mathcal{Fib}(B): D\mathcal{Fib}(G') = G'^* \Rightarrow D\mathcal{Fib}(G) = G^*: D\mathcal{Fib}(\mathbb{D}) \rightarrow D\mathcal{Fib}(\mathbb{C})$$

from Construction 5.6.

It is straightforward to check that this construction is pseudo-functorial.

Finally, we state the following result, which will be useful later on. A proof can be found in [MSV23, Proposition 4.12] by taking $\mathcal{V} = \text{Cat}$.

5.9. LEMMA. *A morphism $F: \mathbb{E} \rightarrow \mathbb{F}$ between discrete double fibrations $P: \mathbb{E} \rightarrow \mathbb{C}$ and $Q: \mathbb{F} \rightarrow \mathbb{C}$ is an isomorphism of double categories if and only if the induced functor $\text{Ver}_0 F: \text{Ver}_0 \mathbb{E} \rightarrow \text{Ver}_0 \mathbb{F}$ is an isomorphism of categories.*

FIBERS OF DISCRETE DOUBLE FIBRATIONS. We now want to study the fibers of a discrete double fibration at both an object and a vertical morphism, which will be useful to construct a lax double presheaf out of a discrete double fibration. Let us first define these.

5.10. CONSTRUCTION. Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$, an object x in \mathbb{C} , and a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , we define the following pullbacks in DbCat ,

$$\begin{array}{ccc} P^{-1}x & \longrightarrow & \mathbb{E} \\ \downarrow & \lrcorner & \downarrow P \\ [0] & \xrightarrow{x} & \mathbb{C} \end{array} \qquad \begin{array}{ccc} P^{-1}u & \longrightarrow & \mathbb{V}[1, \mathbb{E}] \\ \downarrow & \lrcorner & \downarrow \mathbb{V}[1, P] \\ [0] & \xrightarrow{u} & \mathbb{V}[1, \mathbb{C}] \end{array}$$

where $\mathbb{V}[1]$ denotes the double category free on a vertical morphism. Note that there is a canonical induced map $P_u: P^{-1}u \rightarrow P^{-1}x \times P^{-1}x'$ obtained using the universal property of pullback as follows:

$$\begin{array}{ccccc} P^{-1}u & \longrightarrow & \mathbb{V}[1, \mathbb{E}] & & \\ \downarrow & \searrow \exists! P_u & \downarrow & \searrow (s,t) & \\ & & P^{-1}x \times P^{-1}x' & \longrightarrow & \mathbb{E} \times \mathbb{E} \\ & & \downarrow & \lrcorner & \downarrow \mathbb{V}[1, P] \\ [0] & \xrightarrow{u} & \mathbb{V}[1, \mathbb{C}] & & \\ \downarrow & & \downarrow & \searrow (s,t) & \downarrow P \times P \\ [0] & \xrightarrow{(x,x')} & \mathbb{C} \times \mathbb{C} & & \end{array}$$

We start by showing that the fiber $P^{-1}x$ at an object is a category, seen as an object in DblCat through the vertical embedding $\mathbb{V}: \text{Cat} \rightarrow \text{DblCat}$. Explicitly, this means that its horizontal morphisms and squares are all trivial.

5.11. PROPOSITION. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and an object x in \mathbb{C} , the double category $P^{-1}x$ is a vertical double category. We also denote by $P^{-1}x$ the corresponding category.*

PROOF. Consider the following diagram in Cat

$$\begin{array}{ccccc} \text{Ver}_1 P^{-1}x & \rightarrow & \text{Ver}_1 \mathbb{E} & \xrightarrow{t} & \text{Ver}_0 \mathbb{E} & & \text{Ver}_1 P^{-1}x & \xrightarrow{t} & \text{Ver}_0 P^{-1}x & \rightarrow & \text{Ver}_0 \mathbb{E} \\ \downarrow & \lrcorner & \text{Ver}_1 P \downarrow & \lrcorner & \downarrow_{\text{Ver}_0 P} & = & \downarrow & & \downarrow & \lrcorner & \downarrow_{\text{Ver}_0 P} \\ [0] & \xrightarrow{x} & \text{Ver}_1 \mathbb{C} & \xrightarrow{t} & \text{Ver}_0 \mathbb{C} & & [0] & \xlongequal{\quad} & [0] & \xrightarrow{x} & \text{Ver}_0 \mathbb{C} \end{array}$$

Here, the leftmost and rightmost squares are pullbacks by definition of $P^{-1}x$ and the fact that the functor Ver_i preserves pullbacks for $i = 0, 1$, and the middle left square is a pullback by definition of P being a discrete double fibration. Hence, by cancellation of pullbacks, the middle right square is also a pullback and so $\text{Ver}_1 P^{-1}x \cong \text{Ver}_0 P^{-1}x$, showing that $P^{-1}x$ is a vertical double category. ■

Unpacking definition, we have the following explicit description of the fiber $P^{-1}x$.

5.12. REMARK. Given an object x in \mathbb{C} , the category $P^{-1}x$ is the category whose

- objects are objects x_- in \mathbb{E} such that $Px_- = x$,
- morphisms are vertical morphisms $s_-: x_- \rightarrow \hat{x}_-$ in \mathbb{E} such that $Ps_- = e_x$.

Moreover, every horizontal morphism in \mathbb{C} acts on fibers by taking unique lifts as follows.

5.13. CONSTRUCTION. Given a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , there is an induced functor

$$f^*: P^{-1}y \rightarrow P^{-1}x$$

given by sending

- an object y_- in $P^{-1}y$ to the object f^*y_- in $P^{-1}x$, i.e., the source of the unique lift of the horizontal morphism f with target y_- ,
- a morphism $s_-: y_- \rightarrow \hat{y}_-$ in $P^{-1}y$ to the morphism $e_f^*s_-: f^*y_- \rightarrow f^*\hat{y}_-$ in $P^{-1}x$, i.e., the source of the unique lift of the identity square e_f with target s_- .

Next, we show that the fiber $P^{-1}u$ at a vertical morphism is also a category, embedded vertically in DblCat .

5.14. LEMMA. *Given a double category \mathbb{C} and $i = 0, 1$, there is an isomorphism of categories*

$$\text{Ver}_i \llbracket \mathbb{V}[1], \mathbb{E} \rrbracket \cong (\text{Ver}_i \mathbb{E})^{[1]}$$

natural in \mathbb{C} .

PROOF. By Remark 2.18, we have a 2-adjunction

$$\mathbb{V} : \text{Cat} \rightleftarrows \text{DblCat}_v : \text{Ver}_0.$$

In particular, for the Cat -enriched homs $\text{Ver}_0 \llbracket -, - \rrbracket$ of DblCat_v and $(-)^{(-)}$ of Cat , this means that we have an isomorphism of categories

$$\text{Ver}_0 \llbracket \mathbb{V}[1], \mathbb{E} \rrbracket \cong (\text{Ver}_0 \mathbb{E})^{[1]}.$$

This shows the case $i = 0$. For the case $i = 1$, observe that, for every double category \mathbb{C} , we have $\text{Ver}_1 \mathbb{C} \cong \text{Ver}_0 \llbracket \mathbb{H}[1], \mathbb{C} \rrbracket$, where $\mathbb{H}[1]$ denotes the double category free on a horizontal morphism. Then we have natural isomorphisms of categories

$$\begin{aligned} \text{Ver}_1 \llbracket \mathbb{V}[1], \mathbb{E} \rrbracket &\cong \text{Ver}_0 \llbracket \mathbb{H}[1], \llbracket \mathbb{V}[1], \mathbb{E} \rrbracket \rrbracket && \text{Observation} \\ &\cong \text{Ver}_0 \llbracket \mathbb{H}[1] \times \mathbb{V}[1], \mathbb{E} \rrbracket && (-) \times \mathbb{V}[1] \dashv \llbracket \mathbb{H}[1], - \rrbracket \\ &\cong \text{Ver}_0 \llbracket \mathbb{V}[1], \llbracket \mathbb{H}[1], \mathbb{E} \rrbracket \rrbracket && \mathbb{H}[1] \times (-) \dashv \llbracket \mathbb{H}[1], - \rrbracket \\ &\cong (\text{Ver}_0 \llbracket \mathbb{H}[1], \mathbb{E} \rrbracket)^{[1]} && \text{Case } i = 0 \\ &\cong (\text{Ver}_1 \mathbb{E})^{[1]}, && \text{Observation} \end{aligned}$$

where the product-internal homs adjunctions hold by Proposition 2.15. ■

5.15. LEMMA. *Given a discrete double fibration $P : \mathbb{E} \rightarrow \mathbb{C}$, the induced double functor*

$$\llbracket \mathbb{V}[1], P \rrbracket : \llbracket \mathbb{V}[1], \mathbb{E} \rrbracket \rightarrow \llbracket \mathbb{V}[1], \mathbb{C} \rrbracket$$

is also a discrete double fibration.

PROOF. We have to show that the following square is a pullback in Cat

$$\begin{array}{ccc} \text{Ver}_1(\llbracket \mathbb{V}[1], \mathbb{E} \rrbracket) & \xrightarrow{t} & \text{Ver}_0(\llbracket \mathbb{V}[1], \mathbb{E} \rrbracket) \\ \text{Ver}_1(\llbracket \mathbb{V}[1], P \rrbracket) \downarrow & & \downarrow \text{Ver}_0(\llbracket \mathbb{V}[1], P \rrbracket) \\ \text{Ver}_1(\llbracket \mathbb{V}[1], \mathbb{C} \rrbracket) & \xrightarrow{t} & \text{Ver}_0(\llbracket \mathbb{V}[1], \mathbb{C} \rrbracket) \end{array}$$

By the natural isomorphism in Lemma 5.14, this is equivalent to showing that the following square is a pullback in Cat

$$\begin{array}{ccc} (\text{Ver}_1 \mathbb{E})^{[1]} & \xrightarrow{t^{[1]}} & (\text{Ver}_0 \mathbb{E})^{[1]} \\ (\text{Ver}_1 P)^{[1]} \downarrow & & \downarrow (\text{Ver}_0 P)^{[1]} \\ (\text{Ver}_1 \mathbb{C})^{[1]} & \xrightarrow{t^{[1]}} & (\text{Ver}_0 \mathbb{C})^{[1]}, \end{array}$$

But this follows directly from the facts that P is a discrete double fibration and that $(-)^{[1]}$ preserves pullbacks, as a right adjoint. ■

5.16. **PROPOSITION.** *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , the double category $P^{-1}u$ is a vertical double category. We also denote by $P^{-1}u$ the corresponding category.*

PROOF. This follows from applying Proposition 5.11 to the discrete double fibration $[[\mathbb{V}[1], P]]$ from Lemma 5.15. \blacksquare

Unpacking definition, we have the following explicit description of the fiber $P^{-1}u$.

5.17. **REMARK.** Given a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , the category $P^{-1}u$ is the category whose

- objects are vertical morphisms $u_-: x_- \rightarrow x'_-$ in \mathbb{E} such that $Pu_- = u$,
- morphisms $u_- \rightarrow \hat{u}_-$ are pairs (s_-, s'_-) of vertical morphisms s_- and s'_- in \mathbb{E} such that $Ps_- = e_x$ and $Ps'_- = e_{x'}$, and making the following diagram in \mathbb{E} commutes.

$$\begin{array}{ccc}
 x_- & \xrightarrow{s_-} & \hat{x}_- \\
 u \downarrow & & \downarrow \hat{u}_- \\
 x'_- & \xrightarrow{s'_-} & \hat{x}'_-
 \end{array} \tag{5.1}$$

Moreover, every square in \mathbb{C} acts on fibers by taking unique lifts as follows.

5.18. **CONSTRUCTION.** Given a square $\alpha: \begin{bmatrix} u & f \\ & v \end{bmatrix}$ in \mathbb{C} , there is an induced functor

$$\alpha^*: P^{-1}v \rightarrow P^{-1}u$$

given by sending

- an object $v_-: y_- \rightarrow y'_-$ in $P^{-1}v$ to the object $\alpha^*v_-: f^*y_- \rightarrow f'^*y'_-$ in $P^{-1}u$, i.e., the source of the unique lift of α with target v_- ,
- a morphism $(s_-, s'_-): v_- \rightarrow v'_-$ in $P^{-1}v$ to the morphism $(e_f^*s_-, e_{f'}^*s'_-): \alpha^*v_- \rightarrow \alpha^*v'_-$ in $P^{-1}u$, where $e_f^*s_-$ and $e_{f'}^*s'_-$ are the sources of the unique lifts of the identity squares e_f and $e_{f'}$ with targets s_- and s'_- , respectively.

Note that the assignment on morphisms is well-defined as

$$\alpha^*v'_- \bullet e_f^*s_- = \alpha^*(v'_- \bullet s_-) = \alpha^*(s'_- \bullet v_-) = e_{f'}^*s'_- \bullet \alpha^*v_-$$

by unicity of lifts and by definition of the morphism (s_-, s'_-) in $P^{-1}v$.

Next, we study the double functor P_u , which now amounts simply to an ordinary functor. We show that this is a two-sided discrete fibration. Hence, using the equivalence from Theorem 3.13, this will yield a profunctor associated with each vertical morphism u .

5.19. PROPOSITION. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and a vertical morphism $u: x \bullet \rightarrow x'$ in \mathbb{C} , the functor*

$$P_u: P^{-1}u \rightarrow P^{-1}x \times P^{-1}x'$$

is a two-sided discrete fibration.

PROOF. First note that the functor $P_u: P^{-1}u \rightarrow P^{-1}x \times P^{-1}x'$ is given by sending

- an object $u_-: x_- \bullet \rightarrow x'_-$ in $P^{-1}u$ to the object (x_-, x'_-) of $P^{-1}x \times P^{-1}x'$,
- a morphism $(s_-, s'_-): u_- \rightarrow \hat{u}_-$ in $P^{-1}u$ to the morphism (s_-, s'_-) of $P^{-1}x \times P^{-1}x'$.

We write $P_u = (P_u^1, P_u^2)$ for its components.

We show that (P_u^1, P_u^2) satisfies the conditions of a two-sided discrete fibration from Definition 3.10. To prove (1), let $\hat{u}_-: \hat{x}_- \rightarrow \hat{x}'_-$ be an object in $P^{-1}u$ and $s_-: x_- \bullet \rightarrow \hat{x}_-$ be a morphism in $P^{-1}x$. Then the unique P_u^1 -lift of s_- with target \hat{u}_- that lies in the fiber of P_u^2 at \hat{x}'_- is given by the following commutative square.

$$\begin{array}{ccc} x_- & \xrightarrow{s_-} & \hat{x}_- \\ \hat{u}_- \bullet s_- \downarrow & & \downarrow \hat{u}_- \\ \hat{x}'_- & \xrightarrow{e_{\hat{x}'_-}} & \hat{x}'_- \end{array}$$

Condition (2) can be shown analogously.

To see (3), let $(s_-, s'_-): u_- \rightarrow \hat{u}_-$ be a morphism in $P^{-1}u$ as in (5.1). Then the unique lifts of s_- and s'_- provided by conditions (1) and (2) are the commutative squares

$$\begin{array}{ccc} x_- & \xrightarrow{s_-} & \hat{x}_- \\ \hat{u}_- \bullet s_- \downarrow & & \downarrow \hat{u}_- \\ \hat{x}'_- & \xrightarrow{e_{\hat{x}'_-}} & \hat{x}'_- \end{array} \quad \text{and} \quad \begin{array}{ccc} x_- & \xrightarrow{e_{x_-}} & x_- \\ u_- \downarrow & & \downarrow s'_- \bullet u_- \\ x'_- & \xrightarrow{s'_-} & \hat{x}'_- \end{array}$$

respectively. Then, by definition of the morphism (s_-, s'_-) in $P^{-1}u$, we have that $\hat{u}_- \bullet s_- = s'_- \bullet u_-$ and, moreover, the two squares compose to the commutative square (5.1) defining (s_-, s'_-) . This proves that P_u is a two-sided discrete fibration. ■

In particular, as lax double presheaves are normal, we want to be able to associate the identity profunctor, i.e., the hom functor, to a vertical identity. The following result allows us to do so.

5.20. LEMMA. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and an object x in \mathbb{C} , the two-sided discrete fibration*

$$P_{e_x}: P^{-1}e_x \rightarrow P^{-1}x \times P^{-1}x$$

is canonically isomorphic to the identity two-sided discrete fibration

$$(s, t): (P^{-1}x)^{[1]} \rightarrow P^{-1}x \times P^{-1}x.$$

PROOF. By definition of P_{e_x} (see Construction 5.10), to get the desired isomorphism it suffices to show that the following commutative square in DblCat is a pullback.

$$\begin{array}{ccc} \llbracket \mathbb{V}[1], P^{-1}x \rrbracket = (P^{-1}x)^{[1]} & \longrightarrow & \llbracket \mathbb{V}[1], \mathbb{E} \rrbracket \\ \downarrow & \lrcorner & \downarrow \llbracket \mathbb{V}[1], P \rrbracket \\ [0] & \xrightarrow{e_x} & \llbracket \mathbb{V}[1], \mathbb{C} \rrbracket \end{array}$$

However, this follows directly from the definition of the fiber $P^{-1}x$ and the fact that $\llbracket \mathbb{V}[1], - \rrbracket$ preserves pullbacks, as a right adjoint. \blacksquare

REPRESENTABLE DISCRETE DOUBLE FIBRATIONS. We now introduce our first examples of discrete double fibrations.

5.21. DEFINITION. Given a double category \mathbb{C} and an object \hat{x} in \mathbb{C} , we define the *slice double category* over \hat{x} to be the following pullback in DblCat .

$$\begin{array}{ccc} \mathbb{C}/\hat{x} & \longrightarrow & \llbracket \mathbb{H}[1], \mathbb{C} \rrbracket \\ \downarrow & \lrcorner & \downarrow (s,t) \\ \mathbb{C} & \xrightarrow{\text{id}_{\mathbb{C}} \times \{\hat{x}\}} & \mathbb{C} \times \mathbb{C} \end{array}$$

5.22. REMARK. Explicitly, given a double category \mathbb{C} and an object \hat{x} , we have that \mathbb{C}/\hat{x} is the double category whose

- objects are pairs (x, g) of an object x in \mathbb{C} and a horizontal morphism $g: x \rightarrow \hat{x}$ in \mathbb{C} ,
- horizontal morphisms $(x, g) \rightarrow (y, h)$ are horizontal morphisms $f: x \rightarrow y$ in \mathbb{C} such that $g = h \circ f$,
- vertical morphisms $(x, g) \dashrightarrow (x', g')$ are pairs (u, η) of a vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} and a square $\eta: \left[u \begin{array}{c} g \\ g' \end{array} e_{\hat{x}} \right]$ in \mathbb{C} ,
- squares $\alpha: \left[(u, \eta) \begin{array}{c} f \\ f' \end{array} (v, \theta) \right]$ are squares $\alpha: \left[u \begin{array}{c} f \\ f' \end{array} v \right]$ in \mathbb{C} such that $\eta = \theta \circ \alpha$.

Moreover, the canonical projection double functor $\mathbb{C}/\hat{x} \rightarrow \mathbb{C}$ is given by projecting onto the first component.

As claimed before, the canonical projection is a discrete double fibration. A proof can be found in [MSV23, Proposition 4.17] by taking $\mathcal{V} = \text{Cat}$.

5.23. PROPOSITION. *Given a double category \mathbb{C} and an object \hat{x} in \mathbb{C} , the canonical projection from the slice double category over \hat{x}*

$$\mathbb{C}/\hat{x} \rightarrow \mathbb{C}$$

is a discrete double fibration.

6. Grothendieck construction for lax double presheaves

In this section, we study the link between lax double presheaves and discrete double fibrations. More precisely, given a double category \mathbb{C} , we prove that there is a pseudo-natural 2-equivalence—induced by the Grothendieck construction—between the 2-category of lax double presheaves over \mathbb{C} and the 2-category of discrete double fibrations over \mathbb{C} .

In Sections 6.1 to 6.12, we define the Grothendieck construction of a lax double presheaf over a double category \mathbb{C} . We show, using the coherence conditions for normal lax double functors, that this actually defines a discrete double fibration over \mathbb{C} , where the base is an actual strict double category. This may sound surprising at first since only the base double category of a lax double presheaf is strict, but not the presheaf itself. We further show that this construction is pseudo-natural in \mathbb{C} . In Sections 6.13 to 6.30, we construct the weak inverse of the Grothendieck construction induced by taking fibers of discrete double fibrations. We show that there are 2-natural isomorphisms relating the Grothendieck construction and its weak inverse, henceforth giving the desired 2-equivalence.

GROTHENDIECK CONSTRUCTION. We start by defining the Grothendieck construction of a lax double presheaf.

6.1. **CONSTRUCTION.** Given a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, its *Grothendieck construction* is the double category $\int_{\mathbb{C}} X$ whose

- objects are pairs (x, x_-) of objects x in \mathbb{C} and x_- in Xx ,
- horizontal morphisms $(x, x_-) \rightarrow (y, y_-)$ are horizontal morphisms $f: x \rightarrow y$ in \mathbb{C} satisfying $x_- = Xf(y_-)$, where we recall that Xf is a functor $Xf: Xy \rightarrow Xx$,
- vertical morphisms $(x, x_-) \rightarrow (x', x'_-)$ are given by pairs (u, u_-) of a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} and an element u_- in $Xu(x_-, x'_-)$, where we recall that Xu is a profunctor $Xu: Xx^{\text{op}} \times Xx' \rightarrow \text{Set}$,
- squares

$$\begin{array}{ccc}
 (x, x_-) & \xrightarrow{f} & (y, y_-) \\
 \downarrow (u, u_-) & \alpha & \downarrow (v, v_-) \\
 (x', x'_-) & \xrightarrow{f'} & (y', y'_-)
 \end{array}$$

are squares $\alpha: [u \xrightarrow{f'} v]$ in \mathbb{C} satisfying $u_- = (X\alpha)_{y_-, y'_-}(v_-)$, where we recall that $X\alpha$ is a natural transformation $X\alpha: Xv \Rightarrow Xu \circ (Xf^{\text{op}} \times Xf'): Xy^{\text{op}} \times Xy' \rightarrow \text{Set}$.

Compositions and identities for horizontal morphisms and squares are defined as in \mathbb{C} . The vertical composite of two composable vertical morphisms $(x, x_-) \xrightarrow{(u, u_-)} (x', x'_-) \xrightarrow{(u', u'_-)} (x'', x''_-)$ is given by the pair $(u' \bullet u, u'_- \bullet u_-)$ of the composite $u' \bullet u$ in \mathbb{C} and the element

$$u'_- \bullet u_- := (\mu_{u, u'})_{x_-, x''_-}([u'_-, u_-]),$$

where $[u'_-, u_-]$ is the element in $(Xu' \bullet Xu)(x_-, x''_-) = \int^{\hat{x}'_- \in X'x'} Xu'(\hat{x}'_-, x''_-) \times Xu(x_-, \hat{x}'_-)$ represented by the element (u'_-, u_-) of $Xu'(x', x'') \times Xu(x, x')$, and $\mu_{u, u'}$ is the composition comparison natural transformation

$$\mu_{u, u'}: Xu' \bullet Xu \Rightarrow X(u' \bullet u): Xx^{\text{op}} \times Xx'' \rightarrow \text{Set}.$$

The vertical identity at an object (x, x_-) is given by the pair $(e_x, 1_{x_-})$, where we recall that $Xe_x = e_{Xx}$ is the hom functor $Xx(-, -): Xx^{\text{op}} \times Xx \rightarrow \text{Set}$, as X is normal.

It comes with a canonical projection double functor $\pi_X: \iint_{\mathbb{C}} X \rightarrow \mathbb{C}$ given by projecting onto the first component.

6.2. LEMMA. *The Grothendieck construction $\iint_{\mathbb{C}} X$ is a strict double category.*

PROOF. We first show that composition of horizontal morphisms and squares is well-defined. To see this, given composable horizontal morphisms $f: (x, x_-) \rightarrow (y, y_-)$ and $g: (y, y_-) \rightarrow (z, z_-)$ in $\iint_{\mathbb{C}} X$, then the composite $g \circ f: (x, x_-) \rightarrow (z, z_-)$ is also a horizontal morphism in $\iint_{\mathbb{C}} X$ since

$$X(g \circ f)(z_-) = Xf(Xg(z_-)) = Xf(y_-) = x_-.$$

Then, given horizontally composable squares in $\iint_{\mathbb{C}} X$

$$\begin{array}{ccccc} (x, x_-) & \xrightarrow{f} & (y, y_-) & \xrightarrow{g} & (z, z_-) \\ (u, u_-) \downarrow & & \alpha & & \downarrow (v, v_-) \quad \beta & & \downarrow (w, w_-) \\ (x', x'_-) & \xrightarrow{f'} & (y', y'_-) & \xrightarrow{g'} & (z', z'_-) \end{array}$$

their horizontal composite $\beta \circ \alpha$ is also a square in $\iint_{\mathbb{C}} X$ since

$$\begin{aligned} X(\beta \circ \alpha)_{z_-, z'_-}(w_-) &= (X\alpha)_{Xg(z_-), Xg'(z'_-)}((X\beta)_{z_-, z'_-}(w_-)) && \text{Functorality of } X \\ &= (X\alpha)_{y_-, y'_-}((X\beta)_{z_-, z'_-}(w_-)) && \text{Horizontal morphisms in } \iint_{\mathbb{C}} X \\ &= (X\alpha)_{y_-, y'_-}(v_-) = u_- && \text{Squares in } \iint_{\mathbb{C}} X \end{aligned}$$

Finally, given vertically composable squares in $\iint_{\mathbb{C}} X$

$$\begin{array}{ccc} (x, x_-) & \xrightarrow{f} & (y, y_-) \\ (u, u_-) \downarrow & & \alpha & & \downarrow (v, v_-) \\ (x', x'_-) & \xrightarrow{f'} & (y', y'_-) \\ (u', u'_-) \downarrow & & \alpha' & & \downarrow (v', v'_-) \\ (x'', x''_-) & \xrightarrow{f''} & (y'', y''_-) \end{array}$$

their vertical composite $\alpha' \bullet \alpha$ is also a square in $\mathbb{J}\mathbb{C} X$ since

$$\begin{aligned}
 X(\alpha' \bullet \alpha)_{y_-, y''_-} (v'_- \bullet v_-) & \\
 &= X(\alpha' \bullet \alpha)_{y_-, y''_-} ((\mu_{v, v'})_{y_-, y''_-} ([v'_-, v_-])) && \text{Definition of } \bullet \\
 &= (\mu_{u, u'})_{x_-, x''_-} ((X\alpha' \bullet X\alpha)_{y_-, y''_-} ([v'_-, v_-])) && \text{Naturality of } \mu \\
 &= (\mu_{u, u'})_{x_-, x''_-} ([(X\alpha')_{y'_-, y''_-} (v'_-), (X\alpha)_{y_-, y'_-} (v_-)]) && \text{Comp. of profunctors} \\
 &= (\mu_{u, u'})_{x_-, x''_-} ([u'_-, u_-]) && \text{Squares in } \mathbb{J}\mathbb{C} X \\
 &= u'_- \bullet u_- && \text{Definition of } \bullet
 \end{aligned}$$

Now, it remains to show that composition of vertical morphisms in $\mathbb{J}\mathbb{C} X$ is strictly associative and unital. We start by showing associativity. Given three composable vertical morphisms $(x, x_-) \xrightarrow{(u, u_-)} (x', x'_-) \xrightarrow{(u', u'_-)} (x'', x''_-) \xrightarrow{(u'', u''_-)} (x''', x'''_-)$ in $\mathbb{J}\mathbb{C} X$, we have that

$$\begin{aligned}
 (u''_- \bullet u'_-) \bullet u_- &= (\mu_{u, u'' \bullet u'})_{x'_-, x'''_-} ([u''_- \bullet u'_-, u_-]) && \text{Definition of } \bullet \\
 &= (\mu_{u, u'' \bullet u'})_{x'_-, x'''_-} ([(\mu_{u', u''_-})_{x'_-, x''_-} ([u''_-, u'_-]), u_-]) && \text{Definition of } \bullet \\
 &= (\mu_{u' \bullet u, u''_-})_{x_-, x'''_-} ([u''_-, (\mu_{u, u'})_{x_-, x''_-} ([u'_-, u_-])]) && \text{Associativity of } \mu \\
 &= (\mu_{u' \bullet u, u''_-})_{x_-, x'''_-} ([u''_-, u'_- \bullet u_-]) && \text{Definition of } \bullet \\
 &= u''_- \bullet (u'_- \bullet u_-) && \text{Definition of } \bullet
 \end{aligned}$$

Finally, we show unitality. Given a vertical morphism $(u, u_-): (x, x_-) \rightarrow (x', x'_-)$ in $\mathbb{J}\mathbb{C} X$, we have that

$$u_- \bullet 1_{x_-} = (\mu_{u, e_x})_{x_-, x'_-} ([u_-, 1_{x_-}]) = (1_{Xu})_{x_-, x'_-} (u_-) = 1_{Xu(x_-, x'_-)}(u_-) = u_-$$

using that μ_{u, e_x} is the identity at Xu . Similarly, we have $1_{x'_-} \bullet u_- = u_-$. This concludes the proof that $\mathbb{J}\mathbb{C} X$ is a strict double category. ■

As a first example, we compute the Grothendieck construction of a representable lax double presheaf.

6.3. PROPOSITION. *Given a double category \mathbb{C} and an object \hat{x} in \mathbb{C} , we have*

$$\mathbb{J}\mathbb{C} \mathbb{C}(-, \hat{x}) = \mathbb{C}/\hat{x}.$$

PROOF. Explicitly, the double category $\mathbb{J}\mathbb{C} \mathbb{C}(-, \hat{x})$ is such that

- an object (x, x_-) in $\mathbb{J}\mathbb{C} \mathbb{C}(-, \hat{x})$ is the data of an object x in \mathbb{C} and an object x_- in $\mathbb{C}(x, \hat{x})$, i.e., a horizontal morphism $x_-: x \rightarrow \hat{x}$ in \mathbb{C} ,
- a horizontal morphism $f: (x, x_-) \rightarrow (y, y_-)$ in $\mathbb{J}\mathbb{C} \mathbb{C}(-, \hat{x})$ is the data of a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} satisfying $x_- = \mathbb{C}(f, \hat{x})(y_-) = y_- \circ f$,

- a vertical morphism $(u, u_-): (x, x_-) \rightarrow (x', x'_-)$ in $\mathbb{J}_{\mathbb{C}} \mathbb{C}(-, \hat{x})$ is the data of a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} and an element u_- in $\mathbb{C}(u, \hat{x})(x_-, x'_-)$, i.e., a square $u_-: \begin{bmatrix} u & x_- \\ x'_- & e_{\hat{x}} \end{bmatrix}$ in \mathbb{C} ,
- a square $\alpha: \left[(u, u_-) \xrightarrow{f} (v, v_-) \right]$ in $\mathbb{J}_{\mathbb{C}} \mathbb{C}(-, \hat{x})$ is the data of a square $\alpha: \begin{bmatrix} u & f \\ x'_- & v \end{bmatrix}$ in \mathbb{C} satisfying $u_- = \mathbb{C}(\alpha, \hat{x})_{y_-, y'_-}(v_-) = v_- \circ \alpha$.

By comparing with the data of the double category \mathbb{C}/\hat{x} as described in Definition 5.22, we get the desired result. \blacksquare

By Proposition 5.23, we know that the canonical projection $\mathbb{J}_{\mathbb{C}} \mathbb{C}(-, \hat{x}) = \mathbb{C}/\hat{x} \rightarrow \mathbb{C}$ is a discrete double fibration. The following shows that the Grothendieck construction of any lax double presheaf is also a discrete double fibration.

6.4. PROPOSITION. *The canonical projection $\pi_X: \mathbb{J}_{\mathbb{C}} X \rightarrow \mathbb{C}$ is a discrete double fibration.*

PROOF. We check the conditions of a discrete double fibration from Remark 5.3.

- (1) Given an object (y, y_-) in $\mathbb{J}_{\mathbb{C}} X$ as well as a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , the unique lift of f with target (y, y_-) is given by the horizontal morphism $f: (x, Xf(y_-)) \rightarrow (y, y_-)$ in $\mathbb{J}_{\mathbb{C}} X$.
- (2) Given a vertical morphism $(v, v_-): (y, y_-) \rightarrow (y', y'_-)$ in $\mathbb{J}_{\mathbb{C}} X$ as well as a square $\alpha: \begin{bmatrix} u & f \\ x'_- & v \end{bmatrix}$ in \mathbb{C} , the unique lift of α with target (v, v_-) is the square

$$\begin{array}{ccc} (x, Xf(y_-)) & \xrightarrow{f} & (y, y_-) \\ \downarrow (u, (X\alpha)_{y_-, y'_-}(v_-)) & \alpha & \downarrow (v, v_-) \\ (x', Xf'(y'_-)) & \xrightarrow{f'} & (y', y'_-) \end{array}$$

This shows that $\pi_X: \mathbb{J}_{\mathbb{C}} X \rightarrow \mathbb{C}$ is a discrete double fibration. \blacksquare

We can extend the Grothendieck construction to a 2-functor $\mathbb{J}_{\mathbb{C}}: \mathcal{P}_{\mathbb{C}} \rightarrow \mathcal{DFib}(\mathbb{C})$. We first describe its assignment on morphisms.

6.5. CONSTRUCTION. Given a horizontal transformation $F: X \Rightarrow Y: \mathbb{C}^{\text{op}} \rightarrow \mathbb{C}\text{at}$, we define a double functor over \mathbb{C}

$$\mathbb{J}_{\mathbb{C}} F: \mathbb{J}_{\mathbb{C}} X \rightarrow \mathbb{J}_{\mathbb{C}} Y$$

sending

- an object (x, x_-) in $\mathbb{J}_{\mathbb{C}} X$ to the object $(x, F_x x_-)$ in $\mathbb{J}_{\mathbb{C}} Y$, where $F_x x_-$ is the image of x_- under the functor $F_x: Xx \rightarrow Yx$,

- a horizontal morphism $f: (x, x_-) \rightarrow (y, y_-)$ in $\mathbb{J}\mathbb{C} X$ to the horizontal morphism in $\mathbb{J}\mathbb{C} Y$

$$f: (x, F_x x_-) \rightarrow (y, F_y y_-),$$

where $Yf(F_y y_-) = F_x(Xf(y_-)) = F_x x_-$ by naturality of F_x ,

- a vertical morphism $(u, u_-): (x, x_-) \dashrightarrow (x', x'_-)$ in $\mathbb{J}\mathbb{C} X$ to the vertical morphism in $\mathbb{J}\mathbb{C} Y$

$$(u, (F_u)_{x_-, x'_-} u_-): (x, F_x x_-) \dashrightarrow (x', F_{x'} x'_-),$$

where $(F_u)_{x_-, x'_-} u_-$ is the image of u_- under the component

$$(F_u)_{x_-, x'_-}: Xu(x_-, x'_-) \rightarrow Yu(F_x x_-, F_{x'} x'_-)$$

of the natural transformation $F_u: Xu \Rightarrow Yu(F_x^{\text{op}} \times F_{x'}): Xx^{\text{op}} \times Xx' \rightarrow \text{Set}$.

- a square $\alpha: \left[(u, u_-) \xrightarrow{f} (v, v_-) \right]: \left[\begin{smallmatrix} (x, x_-) & (y, y_-) \\ (x', x'_-) & (y', y'_-) \end{smallmatrix} \right]$ in $\mathbb{J}\mathbb{C} X$ to the square in $\mathbb{J}\mathbb{C} Y$

$$\begin{array}{ccc} (x, F_x x_-) & \xrightarrow{f} & (y, F_y y_-) \\ \downarrow (u, (F_u)_{x_-, x'_-} u_-) & \alpha & \downarrow (v, (F_v)_{y_-, y'_-} v_-) \\ (x', F_{x'} x'_-) & \xrightarrow{f'} & (y', F_{y'} y'_-) \end{array}$$

where $(Y\alpha)_{F_y y_-, F_{y'} y'_-} ((F_v)_{y_-, y'_-} v_-) = (F_u)_{x_-, x'_-} ((X\alpha)_{y_-, y'_-} v_-) = (F_u)_{x_-, x'_-} u_-$ by naturality of F_u .

Note that, by construction, the double functor $\mathbb{J}\mathbb{C} F$ clearly lies over \mathbb{C} .

6.6. LEMMA. *The construction $\mathbb{J}\mathbb{C} F: \mathbb{J}\mathbb{C} X \rightarrow \mathbb{J}\mathbb{C} Y$ is a strict double functor.*

PROOF. By construction, the double functor $\mathbb{J}\mathbb{C} F$ clearly preserves compositions and identities of horizontal morphisms and squares. It remains to show that it preserves compositions and identities of vertical morphisms.

Given composable vertical morphisms $(x, x_-) \dashrightarrow^{(u, u_-)} (x', x'_-) \dashrightarrow^{(u', u'_-)} (x'', x''_-)$ in $\mathbb{J}\mathbb{C} X$, we have:

$$\begin{aligned} \mathbb{J}\mathbb{C} F((u', u'_-) \bullet (u, u_-)) & & & \\ = (u' \bullet u, (F_{u' \bullet u})_{x_-, x''_-} (u'_- \bullet u_-)) & & & \text{Definition of } \mathbb{J}\mathbb{C} F \\ = (u' \bullet u, (F_{u' \bullet u})_{x_-, x''_-} (\mu_{u, u'})([u'_-, u_-])) & & & \text{Composition in } \mathbb{J}\mathbb{C} X \\ = (u' \bullet u, (\mu_{u, u'})_{x_-, x''_-} (F_{u'} \bullet F_u)_{x_-, x''_-} ([u'_-, u_-])) & & & \text{Compatibility of } F \text{ and } \mu \\ = (u' \bullet u, (\mu_{u, u'})_{x_-, x''_-} ([(F_{u'})_{x'_-, x''_-} u'_-, (F_u)_{x_-, x'_-} u_-])) & & & \text{Comp. of profunctors} \\ = (u', (F_{u'})_{x'_-, x''_-} u'_-) \bullet (u, (F_u)_{x_-, x'_-} u_-) & & & \text{Composition in } \mathbb{J}\mathbb{C} Y \\ = \mathbb{J}\mathbb{C} F(u', u'_-) \bullet \mathbb{J}\mathbb{C} F(u, u_-). & & & \text{Definition of } \mathbb{J}\mathbb{C} F \end{aligned}$$

This shows that $\int_{\mathbb{C}} F$ preserves vertical composition. Now, given an object (x, x_-) of $\int_{\mathbb{C}} X$, we have:

$$\begin{aligned}
 \int_{\mathbb{C}} F(e_{(x, x_-)}) &= \int_{\mathbb{C}} F(e_x, 1_{x_-}) && \text{Vertical identity in } \int_{\mathbb{C}} X \\
 &= (e_x, (F_{e_x})_{x_-, x_-} 1_{x_-}) && \text{Definition of } \int_{\mathbb{C}} F \\
 &= (e_x, 1_{F_x(x_-)}) && F_{e_x} = 1_{F_x} \\
 &= e_{(x, F_x x_-)} && \text{Vertical identity in } \int_{\mathbb{C}} Y \\
 &= e_{\int_{\mathbb{C}} F(x, x_-)}. && \text{Definition of } \int_{\mathbb{C}} F
 \end{aligned}$$

This shows that $\int_{\mathbb{C}} F$ preserves vertical identities and concludes the proof that $\int_{\mathbb{C}} F$ is a strict double functor. ■

Now, we describe the assignment of $\int_{\mathbb{C}}$ on 2-morphisms.

6.7. CONSTRUCTION. Given a globular modification $A: [e_X \begin{smallmatrix} F \\ F' \end{smallmatrix} e_Y] : \begin{bmatrix} X & Y \\ X & Y \end{bmatrix}$ of lax double presheaves $\mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, we define a vertical transformation over \mathbb{C}

$$\int_{\mathbb{C}} A: \int_{\mathbb{C}} F \Rightarrow \int_{\mathbb{C}} F': \int_{\mathbb{C}} X \rightarrow \int_{\mathbb{C}} Y$$

whose

- component at an object (x, x_-) in $\int_{\mathbb{C}} X$ is given by the vertical morphism in $\int_{\mathbb{C}} Y$

$$(e_x, (A_x)_{x_-}): (x, F_x x_-) \rightarrow (x, F'_x x_-),$$

where $(A_x)_{x_-}$ denotes the component at the object x_- of Xx of the natural transformation $A_x: F_x \Rightarrow F'_x: Xx \rightarrow Yx$,

- component at a horizontal morphism $f: (x, x_-) \rightarrow (y, y_-)$ in $\int_{\mathbb{C}} X$ is given by the square in $\int_{\mathbb{C}} Y$

$$\begin{array}{ccc}
 (x, F_x x_-) & \xrightarrow{f} & (y, F_y y_-) \\
 \downarrow (e_x, (A_x)_{x_-}) & e_f & \downarrow (e_y, (A_y)_{y_-}) \\
 (x, F'_x x_-) & \xrightarrow{f} & (y, F'_y y_-)
 \end{array}$$

where $(Y e_f)_{F_y y_-, F'_y y_-}((A_y)_{y_-}) = Y f((A_y)_{y_-}) = (A_x)_{X f(y_-)} = (A_x)_{x_-}$ by normality of Y and horizontal compatibility of A_x .

Note that, by construction, the vertical transformation $\int_{\mathbb{C}} A$ clearly lies over \mathbb{C} .

6.8. LEMMA. *The construction $\int_{\mathbb{C}} A: \int_{\mathbb{C}} F \Rightarrow \int_{\mathbb{C}} F': \int_{\mathbb{C}} X \rightarrow \int_{\mathbb{C}} Y$ is a vertical transformation.*

PROOF. We first show that the components $(\int_{\mathbb{C}} A)_{(x,x_-)}$ are natural in objects (x, x_-) . Given a vertical morphism $(u, u_-): (x, x_-) \rightarrow (x', x'_-)$ in $\int_{\mathbb{C}} X$, we have to show that the following diagram in $\int_{\mathbb{C}} Y$ commutes.

$$\begin{array}{ccc} \int_{\mathbb{C}} F(x, x_-) & \xrightarrow{\bullet (\int_{\mathbb{C}} A)_{(x,x_-)}} & \int_{\mathbb{C}} F'(x, x_-) \\ \int_{\mathbb{C}} F(u, u_-) \downarrow \bullet & & \bullet \downarrow \int_{\mathbb{C}} F'(u, u_-) \\ \int_{\mathbb{C}} F(x', x'_-) & \xrightarrow{\bullet (\int_{\mathbb{C}} A)_{(x',x'_-)}} & \int_{\mathbb{C}} F'(x', x'_-) \end{array}$$

This we can do by a short computation:

$$\begin{aligned} \int_{\mathbb{C}} F'(u, u_-) \circ (\int_{\mathbb{C}} A)_{(x,x_-)} & && \\ = (u, (F'_u)_{x_-,x'_-}(u_-)) \bullet (e_x, (A_x)_{x_-}) & && \text{Definition of } \int_{\mathbb{C}} A, \int_{\mathbb{C}} F' \\ = (u \bullet e_x, (\mu_{e_x, u})_{x_-,x'_-}([(F'_u)_{x_-,x'_-}(u_-), (A_x)_{x_-}])) & && \text{Composition in } \int_{\mathbb{C}} Y \\ = (e_{x'} \bullet u, (\mu_{u, e_{x'}})_{x_-,x'_-}([(A_{x'})_{x'_-}, (F_u)_{x_-,x'_-}(u_-)])) & && \text{Vertical compatibility of } A \\ = (e_{x'}, (A_{x'})_{x'_-}) \bullet (u, (F_u)_{x_-,x'_-}(u_-)) & && \text{Composition in } \int_{\mathbb{C}} Y \\ = (\int_{\mathbb{C}} A)_{(x',x'_-)} \bullet \int_{\mathbb{C}} F(u, u_-) & && \text{Definition of } \int_{\mathbb{C}} A, \int_{\mathbb{C}} F \end{aligned}$$

The naturality of the components $(\int_{\mathbb{C}} A)_f$ in horizontal morphisms f is then straightforward from the fact that these components are given by the vertical identity squares e_f . ■

Putting everything together, we get the following.

6.9. CONSTRUCTION. Given a double category \mathbb{C} , we define a 2-functor

$$\int_{\mathbb{C}} : \mathcal{P}_{\mathbb{C}} \rightarrow \text{DFib}(\mathbb{C}),$$

sending

- a double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ to the discrete double fibration $\pi_X: \int_{\mathbb{C}} X \rightarrow \mathbb{C}$ from Construction 6.1,
- a horizontal transformation $F: X \Rightarrow Y$ to the double functor $\int_{\mathbb{C}} F: \int_{\mathbb{C}} X \rightarrow \int_{\mathbb{C}} Y$ over \mathbb{C} from Construction 6.5,
- a globular modification $A: [e_X \xrightarrow{F} e_{X'}]$ to the vertical transformation over \mathbb{C}

$$\int_{\mathbb{C}} A: \int_{\mathbb{C}} F \Rightarrow \int_{\mathbb{C}} F'$$

from Construction 6.7.

It is straightforward to check that this construction is 2-functorial.

Next, we want to show that the construction $\int_{\mathbb{C}}$ is pseudo-natural in \mathbb{C} . For this, we start by constructing the pseudo-naturality comparison cells.

6.10. CONSTRUCTION. Given a double functor $G: \mathbb{C} \rightarrow \mathbb{D}$, we construct a 2-natural isomorphism

$$\begin{array}{ccc} \mathcal{P}_{\mathbb{D}} & \xrightarrow{(G^{\text{op}})^*} & \mathcal{P}_{\mathbb{C}} \\ \mathcal{J}_{\mathbb{D}} \downarrow & \swarrow \cong & \downarrow \mathcal{J}_{\mathbb{C}} \\ \text{DFib}(\mathbb{D}) & \xrightarrow{G^*} & \text{DFib}(\mathbb{C}) \end{array}$$

whose component at a lax double presheaf $X: \mathbb{D}^{\text{op}} \rightarrow \text{Cat}$ is the invertible double functor over \mathbb{C}

$$(\mathcal{J}_G)_X: \mathcal{J}_{\mathbb{C}}(G^{\text{op}})^* X = \mathcal{J}_{\mathbb{C}} X G^{\text{op}} \longrightarrow G^* \mathcal{J}_{\mathbb{D}} X = \mathbb{C} \times_{\mathbb{D}} \mathcal{J}_{\mathbb{D}} X$$

given by sending

- an object (x, x_-) in $\mathcal{J}_{\mathbb{C}} X G^{\text{op}}$, i.e., a pair of objects x in \mathbb{C} and x_- in $XG(x)$, to the object $(x, (Gx, x_-))$ in $\mathbb{C} \times_{\mathbb{D}} \mathcal{J}_{\mathbb{D}} X$,
- a horizontal morphism $f: (x, x_-) \rightarrow (y, y_-)$ in $\mathcal{J}_{\mathbb{C}} X G^{\text{op}}$, i.e., a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} such that $XGf(y_-) = x_-$, to the horizontal morphism in $\mathbb{C} \times_{\mathbb{D}} \mathcal{J}_{\mathbb{D}} X$

$$(f, Gf): (x, (Gx, x_-)) \rightarrow (y, (Gy, y_-)),$$

- a vertical morphism $(u, u_-): (x, x_-) \rightarrow (x', x'_-)$ in $\mathcal{J}_{\mathbb{C}} X G^{\text{op}}$, i.e., a pair of a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} and an element u_- in $XGu(x_-, x'_-)$, to the vertical morphism in $\mathbb{C} \times_{\mathbb{D}} \mathcal{J}_{\mathbb{D}} X$

$$(u, (Gu, u_-)): (x, (Gx, x_-)) \rightarrow (x', (Gx', x'_-)),$$

- a square $\alpha: \left[\begin{array}{c} (u, u_-) \\ \xrightarrow{f} \\ (v, v_-) \end{array} \right] : \left[\begin{array}{c} (x, x_-) \\ \xrightarrow{f'} \\ (x', x'_-) \end{array} \right] \rightarrow \left[\begin{array}{c} (y, y_-) \\ \xrightarrow{f'} \\ (y', y'_-) \end{array} \right]$ in $\mathcal{J}_{\mathbb{C}} X G^{\text{op}}$, i.e., a square $\alpha: \left[\begin{array}{c} u \\ \xrightarrow{f'} \\ v \end{array} \right]$ in \mathbb{C} such that $(XG\alpha)_{x_-, x'_-}(v_-) = u_-$, to the square in $\mathbb{C} \times_{\mathbb{D}} \mathcal{J}_{\mathbb{D}} X$

$$\begin{array}{ccc} (x, (Gx, x_-)) & \xrightarrow{(f, Gf)} & (y, (Gy, y_-)) \\ \downarrow (u, (Gu, u_-)) & & \downarrow (v, (Gv, v_-)) \\ (x', (Gx', x'_-)) & \xrightarrow{(f', Gf')} & (y', (Gy', y'_-)). \end{array}$$

Note that it admits an obvious inverse given by projecting onto the first and last components, so that the components $(\mathcal{J}_G)_X$ are invertible.

6.11. LEMMA. *The construction $\mathcal{J}_G: \mathcal{J}_{\mathbb{C}} \circ (G^{\text{op}})^* \Rightarrow G^* \circ \mathcal{J}_{\mathbb{D}}$ is a 2-natural isomorphism.*

PROOF. It remains to prove that the components $(\int_G)_X$ are 2-natural in X . Given a horizontal transformation $F: X \Rightarrow Y: \mathbb{D}^{\text{op}} \rightarrow \text{Cat}$, we have to show that the following diagram of double functors over \mathbb{C} commutes.

$$\begin{array}{ccc} \int_{\mathbb{C}} XG^{\text{op}} & \xrightarrow{(\int_G)_X} & G^* \int_{\mathbb{D}} X = \mathbb{C} \times_{\mathbb{D}} \int_{\mathbb{D}} X \\ \int_{\mathbb{C}} (G^{\text{op}})^* F = \int_{\mathbb{C}} F \circ G^{\text{op}} \downarrow & & \downarrow G^*(\int_{\mathbb{D}} F) = \mathbb{C} \times_{\mathbb{D}} \int_{\mathbb{D}} F \\ \int_{\mathbb{C}} YG^{\text{op}} & \xrightarrow{(\int_G)_Y} & G^* \int_{\mathbb{D}} Y = \mathbb{C} \times_{\mathbb{D}} \int_{\mathbb{D}} Y \end{array}$$

We check that the above diagram commutes on objects, and the proof is similar for horizontal morphisms, vertical morphisms, and squares. To do so, we evaluate both composites at an object (x, x_-) in $\int_{\mathbb{C}} XG^{\text{op}}$:

$$\begin{aligned} (\int_G)_Y \circ \int_{\mathbb{C}} F \circ G^{\text{op}}(x, x_-) &= (\int_G)_Y(x, (F \circ G^{\text{op}})_x x_-) && \text{Definition of } \int_{\mathbb{C}} F \circ G^{\text{op}} \\ &= (\int_G)_Y(x, F_{Gx} x_-) && \text{Whiskering} \\ &= (x, (Gx, F_{Gx} x_-)) && \text{Definition of } (\int_G)_Y \\ &= (\mathbb{C} \times_{\mathbb{D}} \int_{\mathbb{D}} F)(x, (Gx, x_-)) && \text{Definition of } \mathbb{C} \times_{\mathbb{D}} \int_{\mathbb{D}} F \\ &= (\mathbb{C} \times_{\mathbb{D}} \int_{\mathbb{D}} F) \circ (\int_G)_X(x, x_-). && \text{Definition of } (\int_G)_X \end{aligned}$$

This shows that the desired diagram commutes.

Next, given a globular modification $A: [e_X \xrightarrow{F} e_Y] : \begin{bmatrix} X & Y \\ X & Y \end{bmatrix}$ in $\mathcal{P}_{\mathbb{D}}$, we have to check that the following diagram of double functors and vertical transformations commutes

$$\begin{array}{ccc} \int_{\mathbb{C}} XG^{\text{op}} & \xrightarrow{(\int_G)_X} & G^* \int_{\mathbb{D}} X \\ \int_{\mathbb{C}} F \circ G^{\text{op}} \left(\begin{array}{c} \int_{\mathbb{C}} A \circ G^{\text{op}} \\ \bullet \\ \int_{\mathbb{C}} F' \circ G^{\text{op}} \end{array} \right) & & G^* \int_{\mathbb{D}} F \left(\begin{array}{c} G^* \int_{\mathbb{D}} A \\ \bullet \\ G^* \int_{\mathbb{D}} F' \end{array} \right) \\ \int_{\mathbb{C}} YG^{\text{op}} & \xrightarrow{(\int_G)_Y} & G^* \int_{\mathbb{D}} Y \end{array}$$

To do so, we first evaluate both whiskerings at an object (x, x_-) of $\int_{\mathbb{C}} XG^{\text{op}}$:

$$\begin{aligned} (\int_G)_Y(\int_{\mathbb{C}} A \circ G^{\text{op}})_{(x, x_-)} &= (\int_G)_Y(e_x, ((A \circ G^{\text{op}})_x)_{x_-}) && \text{Definition of } \int_{\mathbb{C}} A \circ G^{\text{op}} \\ &= (\int_G)_Y(e_x, (A_{Gx})_{x_-}) && \text{Whiskering} \\ &= (e_x, (Ge_x, (A_{Gx})_{x_-})) && \text{Definition of } (\int_G)_Y \\ &= (e_x, (e_{Gx}, (A_{Gx})_{x_-})) && \text{Functorality of } G \\ &= (G^* \int_{\mathbb{D}} A)_{(x, (Gx, x_-))} && \text{Definition of } G^* \int_{\mathbb{D}} A \\ &= (G^* \int_{\mathbb{D}} A)_{(\int_G)_X(x, x_-)}. && \text{Definition of } (\int_G)_X \end{aligned}$$

This shows that the components at objects of $\int_{\mathbb{C}} XG^{\text{op}}$ of the two whiskerings agree. The fact that their components at horizontal morphisms agree is then straightforward from the fact that they are both given by the vertical identity square (e_f, e_{Gf}) in $G^* \int_{\mathbb{D}} Y$. ■

Putting the components $\mathbb{J}_{\mathbb{C}}$ and \mathbb{J}_G together, we get the following.

6.12. PROPOSITION. *The Grothendieck construction defines a pseudo-natural transformation of pseudo-functors*

$$\mathbb{J}: \mathcal{P} \Rightarrow \text{DFib}: \text{DblCat}_h^{\text{coop}} \rightarrow 2\text{Cat},$$

where \mathcal{P} is the 2-functor from Construction 3.36 and DFib is the pseudo-functor from Construction 5.8.

PROOF. We first show that the components \mathbb{J}_G are functorial in G . Given composable double functors $G: \mathbb{C} \rightarrow \mathbb{D}$ and $H: \mathbb{D} \rightarrow \mathbb{E}$, we have to show that the following composite coincides with the 2-natural transformation \mathbb{J}_{HG}

$$\mathbb{J}_{\mathbb{C}}((HG)^{\text{op}})^* = \mathbb{J}_{\mathbb{C}}(G^{\text{op}})^*(H^{\text{op}})^* \xrightarrow{\mathbb{J}_G(H^{\text{op}})^*} G^* \mathbb{J}_{\mathbb{D}}(H^{\text{op}})^* \xrightarrow{G^* \mathbb{J}_H} G^* H^* \mathbb{J}_{\mathbb{E}} \cong (HG)^* \mathbb{J}_{\mathbb{E}}.$$

When evaluated at a lax double presheaf $X: \mathbb{E}^{\text{op}} \rightarrow \text{Cat}$, this amounts to showing that the following composite coincides with the double functor $(\mathbb{J}_{HG})_X$

$$\mathbb{J}_{\mathbb{C}} X H^{\text{op}} G^{\text{op}} \xrightarrow{(\mathbb{J}_G)_{X H^{\text{op}}}} G^* \mathbb{J}_{\mathbb{D}} X H^{\text{op}} \xrightarrow{G^* (\mathbb{J}_H)_X} G^* H^* \mathbb{J}_{\mathbb{E}} X \cong (HG)^* \mathbb{J}_{\mathbb{E}} X.$$

But this is straightforward from the definitions of \mathbb{J}_G , \mathbb{J}_H , and \mathbb{J}_{HG} . Moreover, given a double category \mathbb{C} , a similar computation shows that $\mathbb{J}_{\text{id}_{\mathbb{C}}} = \text{id}_{\mathbb{J}_{\mathbb{C}}}$.

Finally, we show that the components \mathbb{J}_G are natural in G . Given a horizontal transformation $B: G \Rightarrow G': \mathbb{C} \rightarrow \mathbb{D}$, we have to show that the following diagram of 2-natural transformations commutes,

$$\begin{array}{ccc} \mathbb{J}_{\mathbb{C}}(G'^{\text{op}})^* & \xrightarrow{\mathbb{J}_{G'}} & G'^* \mathbb{J}_{\mathbb{D}} \\ \mathbb{J}_{\mathbb{C}}(B^{\text{op}})^* \downarrow & & \downarrow B^* \mathbb{J}_{\mathbb{D}} \\ \mathbb{J}_{\mathbb{C}}(G^{\text{op}})^* & \xrightarrow{\mathbb{J}_G} & G^* \mathbb{J}_{\mathbb{D}}. \end{array}$$

where we recall that the left-hand $(B^{\text{op}})^*$ is as in Construction 3.36 and the right-hand B^* is as in Construction 5.6. When evaluated at a lax double presheaf $X: \mathbb{D}^{\text{op}} \rightarrow \text{Cat}$, this amounts to showing that the following diagram of double functors commutes.

$$\begin{array}{ccc} \mathbb{J}_{\mathbb{C}} X G'^{\text{op}} & \xrightarrow{(\mathbb{J}_{G'})_X} & G'^* \mathbb{J}_{\mathbb{D}} X \\ \mathbb{J}_{\mathbb{C}} X \circ B^{\text{op}} \downarrow & & \downarrow B^* \mathbb{J}_{\mathbb{D}} X \\ \mathbb{J}_{\mathbb{C}} X G^{\text{op}} & \xrightarrow{(\mathbb{J}_G)_X} & G^* \mathbb{J}_{\mathbb{D}} X. \end{array}$$

We check that the above diagram commutes on objects, and the proof is similar for horizontal morphisms, vertical morphisms, and squares. To do so, we evaluate both composites at an object (x, x_-) in $\mathbb{J}_{\mathbb{C}} XG'^{\text{op}}$:

$$\begin{aligned}
 B_{\mathbb{J}_{\mathbb{D}} X}^*(\mathbb{J}_{G'})(x, x_-) &= B_{\mathbb{J}_{\mathbb{D}} X}^*(x, (G'x, x_-)) && \text{Definition of } \mathbb{J}_{G'} \\
 &= (x, (B_x)^*(G'x, x_-)) && \text{Definition of } B^* \\
 &= (x, (Gx, X(B_x)x_-)) && \text{Unique lifts in } \mathbb{J}_{\mathbb{D}} X \\
 &= (\mathbb{J}_G)_X(x, X(B_x)x_-) && \text{Definition of } (\mathbb{J}_G)_X \\
 &= (\mathbb{J}_G)_X(x, (X \circ B^{\text{op}})_x x_-) && \text{Whiskering} \\
 &= (\mathbb{J}_G)_X(\mathbb{J}_{\mathbb{C}}(X \circ B^{\text{op}}))(x, x_-) && \text{Definition of } \mathbb{J}_{\mathbb{C}} X \circ B^{\text{op}}
 \end{aligned}$$

This shows that the desired diagram commutes. ■

GROTHENDIECK 2-EQUIVALENCE. We now show that the Grothendieck construction establishes a 2-equivalence between the 2-categories $\mathcal{P}_{\mathbb{C}}$ of lax double presheaves and $\text{DFib}(\mathbb{C})$ of discrete double fibrations over a fixed double category \mathbb{C} .

6.13. THEOREM. *Given a double category \mathbb{C} , the Grothendieck construction*

$$\mathbb{J}_{\mathbb{C}}: \mathcal{P}_{\mathbb{C}} \longrightarrow \text{DFib}(\mathbb{C})$$

is a 2-equivalence of 2-categories, which is pseudo-natural in \mathbb{C} .

In order to show this, we construct a weak inverse 2-functor $\mathfrak{D}_{\mathbb{C}}: \text{DFib}(\mathbb{C}) \rightarrow \mathcal{P}_{\mathbb{C}}$ of the 2-functor $\mathbb{J}_{\mathbb{C}}$ from Construction 6.9. We start with its assignment on objects.

6.14. CONSTRUCTION. Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$, we construct a lax double presheaf

$$\mathfrak{D}_{\mathbb{C}}(P): \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$$

which sends

- an object x in \mathbb{C} to the category $P^{-1}x$ given by the fiber of P at x ,
- a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} to the functor

$$\mathfrak{D}_{\mathbb{C}}(P)f := f^*: \mathfrak{D}_{\mathbb{C}}(P)y = P^{-1}y \rightarrow \mathfrak{D}_{\mathbb{C}}(P)x = P^{-1}x,$$

from Construction 5.13,

- a vertical morphism $u: x \bullet \rightarrow x'$ in \mathbb{C} to the profunctor

$$\mathfrak{D}_{\mathbb{C}}(P)u: \mathfrak{D}_{\mathbb{C}}(P)x^{\text{op}} \times \mathfrak{D}_{\mathbb{C}}(P)x' = P^{-1}x^{\text{op}} \times P^{-1}x' \rightarrow \text{Set}$$

given by the image under the equivalence from Theorem 3.13

$$\text{fib}: \text{TSFib}(P^{-1}x, P^{-1}x') \xrightarrow{\simeq} \text{Prof}(P^{-1}x, P^{-1}x')$$

of the two-sided discrete fibration from Proposition 5.19

$$P^{-1}u \rightarrow P^{-1}x \times P^{-1}x';$$

using Lemma 5.20, we impose $\mathfrak{D}_{\mathbb{C}}(P)e_x := e_{P^{-1}x}$, for every object x in \mathbb{C} ,

- a square $\alpha: \begin{bmatrix} u & f \\ & f' \end{bmatrix} v : \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ in \mathbb{C} to the natural transformation

$$\mathfrak{D}_{\mathbb{C}}(P)\alpha: \mathfrak{D}_{\mathbb{C}}(P)v \Rightarrow \mathfrak{D}_{\mathbb{C}}(P)u \circ (\mathfrak{D}_{\mathbb{C}}(P)f^{\text{op}} \times \mathfrak{D}_{\mathbb{C}}(P)f')$$

which is induced as in Corollary 3.15 by the following commutative square

$$\begin{array}{ccc} P^{-1}v & \xrightarrow{\alpha^*} & P^{-1}u \\ \downarrow & & \downarrow \\ P^{-1}y \times P^{-1}y' & \xrightarrow{(f^*, f'^*)} & P^{-1}x \times P^{-1}x' \end{array}$$

where α^* is the functor from Construction 5.18; note that the above diagram indeed commutes by construction of f^*, f'^*, α^* ,

- for composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x''$ in \mathbb{C} , the composition comparison square is given by the natural transformation

$$\mu_{u, u'}: \mathfrak{D}_{\mathbb{C}}(P)u' \bullet \mathfrak{D}_{\mathbb{C}}(P)u \Rightarrow \mathfrak{D}_{\mathbb{C}}(P)(u' \bullet u)$$

given by the image under the equivalence from Theorem 3.13

$$\text{fib}: \text{TSEFib}(P^{-1}x, P^{-1}x'') \xrightarrow{\simeq} \text{Prof}(P^{-1}x, P^{-1}x'')$$

of the morphism of two-sided discrete fibrations

$$\begin{array}{ccc} P^{-1}u \times_{P^{-1}x'} P^{-1}u' /_{\sim} & \xrightarrow{-\bullet-} & P^{-1}(u' \bullet u) \\ & \searrow & \swarrow \\ & P^{-1}x \times P^{-1}x'' & \end{array}$$

induced by vertical composition in \mathbb{E} .

6.15. LEMMA. *The construction $\mathfrak{D}_{\mathbb{C}}(P): \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ is a lax double presheaf.*

PROOF. First note that $\mathfrak{D}_{\mathbb{C}}(P)$ preserves horizontal compositions and identities for horizontal morphisms and squares by unicity of lifts.

We first show that the composition comparison natural transformations $\mu_{u,u'}$ are natural in (u, u') . Given squares $\alpha: \begin{bmatrix} u & f \\ & f' \end{bmatrix} v$ and $\alpha': \begin{bmatrix} u' & f' \\ & f'' \end{bmatrix} v'$ in \mathbb{C} , we have to show that the following diagram in $\text{Prof}(P^{-1}x, P^{-1}x'')$ commutes.

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{C}}(P)v' \bullet \mathcal{D}_{\mathbb{C}}(P)v & \xrightarrow{\mathcal{D}_{\mathbb{C}}(P)\alpha' \bullet \mathcal{D}_{\mathbb{C}}(P)\alpha} & (\mathcal{D}_{\mathbb{C}}(P)u \bullet \mathcal{D}_{\mathbb{C}}(P)u)(f^{*\text{op}} \times f''^*) \\ \mu_{v,v'} \downarrow & & \downarrow \mu_{u,u'}(f^{*\text{op}} \times f''^*) \\ \mathcal{D}_{\mathbb{C}}(P)(v' \bullet v) & \xrightarrow{\mathcal{D}_{\mathbb{C}}(P)(\alpha' \bullet \alpha)} & \mathcal{D}_{\mathbb{C}}(P)(u' \bullet u)(f^{*\text{op}} \times f''^*) \end{array}$$

Under the natural equivalence fib from Theorem 3.13 and Proposition 3.14 and using Proposition 3.20, this amounts to showing that the following diagram of functors commutes.

$$\begin{array}{ccc} P^{-1}v \times_{P^{-1}y'} P^{-1}v'_{/\sim} & \xrightarrow{\alpha^* \times \alpha'^*} & P^{-1}u \times_{P^{-1}x'} P^{-1}u'_{/\sim} \\ \text{---} \downarrow & & \downarrow \text{---} \\ P^{-1}(v' \bullet v) & \xrightarrow{(\alpha' \bullet \alpha)^*} & P^{-1}(u' \bullet u) \end{array}$$

But this follows directly from the unicity of lifts.

Next, we show that the composition comparison natural transformations $\mu_{u,u'}$ are compatible with vertical composition. To see this, given three composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x'' \xrightarrow{u''} x'''$ in \mathbb{C} , we have to show that the following diagram in the category $\text{TSFib}(P^{-1}x, P^{-1}x''')$ commutes.

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{C}}(P)u'' \bullet \mathcal{D}_{\mathbb{C}}(P)u' \bullet \mathcal{D}_{\mathbb{C}}(P)u & \xrightarrow{\mu_{u',u''} \bullet \mathcal{D}_{\mathbb{C}}(P)u} & \mathcal{D}_{\mathbb{C}}(P)(u'' \bullet u') \bullet \mathcal{D}_{\mathbb{C}}(P)u \\ \mathcal{D}_{\mathbb{C}}(P)u'' \bullet \mu_{u,u'} \downarrow & & \downarrow \mu_{u,u''} \bullet u' \\ \mathcal{D}_{\mathbb{C}}(P)u'' \bullet \mathcal{D}_{\mathbb{C}}(P)(u' \bullet u) & \xrightarrow{\mu_{u',u''}} & \mathcal{D}_{\mathbb{C}}(P)(u'' \bullet u' \bullet u)(f^{*\text{op}} \times f''^*) \end{array}$$

Under the equivalence fib from Theorem 3.13, this amounts to showing that the following diagram in $\text{TSFib}(P^{-1}x, P^{-1}x''')$ commutes.

$$\begin{array}{ccc} P^{-1}u \times_{P^{-1}x'} P^{-1}u' \times_{P^{-1}x''} P^{-1}u''_{/\sim} & \xrightarrow{1_{P^{-1}u} \times (-\bullet-)} & P^{-1}u \times_{P^{-1}x'} P^{-1}(u'' \bullet u')_{/\sim} \\ (-\bullet-) \times 1_{P^{-1}u''} \downarrow & & \downarrow \text{---} \\ P^{-1}(u' \bullet u) \times_{P^{-1}x''} P^{-1}u''_{/\sim} & \xrightarrow{\text{---}} & P^{-1}(u'' \bullet u' \bullet u) \end{array}$$

But this follows directly from the fact that vertical composition in \mathbb{E} is strictly associative.

Next, we show that $\mathcal{D}_{\mathbb{C}}(P)$ preserves vertical identities. By construction, it preserves the vertical identity at an object. Given a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , we have to show that the natural transformation $\mathcal{D}_{\mathbb{C}}(P)e_f$ is the identity at $\mathcal{D}_{\mathbb{C}}(P)f = f^*$. By

definition, we have that $\mathfrak{D}_{\mathbb{C}}(P)e_f$ is the natural transformation induced as in Corollary 3.15 by the below left commutative square.

$$\begin{array}{ccc}
 P^{-1}e_y & \xrightarrow{e_f^*} & P^{-1}e_x \\
 P e_y \downarrow & & \downarrow P e_x \\
 P^{-1}y \times P^{-1}y & \xrightarrow{(f^*, f^*)} & P^{-1}x \times P^{-1}x
 \end{array}
 \qquad
 \begin{array}{ccc}
 (P^{-1}y)^{[1]} & \xrightarrow{(f^*)^{[1]}} & (P^{-1}x)^{[1]} \\
 (s,t) \downarrow & & \downarrow (s,t) \\
 P^{-1}y \times P^{-1}y & \xrightarrow{(f^*, f^*)} & P^{-1}x \times P^{-1}x
 \end{array}$$

Since the above left commutative square is canonically isomorphic to the above right commutative square by Lemma 5.20, we see that the desired natural transformation is the identity at f^* .

Finally, we show that the composition comparison natural transformations $\mu_{u, e_{x'}}$ and μ_{u, e_x} agree with the identity natural transformation at $\mathfrak{D}_{\mathbb{C}}(P)u$. To see this, given a vertical morphism $u: x \bullet \rightarrow x'$ in \mathbb{C} , by Lemmas 3.23 and 5.20, we have canonical isomorphisms over $P^{-1}x \times P^{-1}x'$

$$P^{-1}e_x \times_{P^{-1}x} P^{-1}u /_{\sim} \cong (P^{-1}x)^{[1]} \times_{P^{-1}x} P^{-1}u /_{\sim} \cong P^{-1}u$$

and

$$P^{-1}u \times_{P^{-1}x'} P^{-1}e_{x'} /_{\sim} \cong P^{-1}u \times_{P^{-1}x'} (P^{-1}x')^{[1]} /_{\sim} \cong P^{-1}u.$$

Note that these isomorphisms are induced by vertical composition, therefore we have $\mu_{e_x, u} = 1_{\mathfrak{D}_{\mathbb{C}}(P)u} = \mu_{u, e_{x'}}$, concluding the proof. \blacksquare

We now turn to the assignment of $\mathfrak{D}_{\mathbb{C}}$ on morphisms.

6.16. CONSTRUCTION. Given a morphism of discrete double fibrations

$$\begin{array}{ccc}
 \mathbb{E} & \xrightarrow{F} & \mathbb{F} \\
 & \searrow P & \swarrow Q \\
 & & \mathbb{C}
 \end{array}$$

we construct a horizontal transformation

$$\mathfrak{D}_{\mathbb{C}}(F): \mathfrak{D}_{\mathbb{C}}(P) \Rightarrow \mathfrak{D}_{\mathbb{C}}(Q)$$

whose

- component at an object x is given by the unique induced functor between fibers

$$\mathfrak{D}_{\mathbb{C}}(F)_x := F_x: \mathfrak{D}_{\mathbb{C}}(P)x = P^{-1}x \rightarrow \mathfrak{D}_{\mathbb{C}}(Q)x = Q^{-1}x$$

- component at a vertical morphism $u: x \bullet \rightarrow x'$ is given by the natural transformation

$$\mathfrak{D}_{\mathbb{C}}(F)_u: \mathfrak{D}_{\mathbb{C}}(P)u \Rightarrow \mathfrak{D}_{\mathbb{C}}(Q)u(\mathfrak{D}_{\mathbb{C}}(F)_x^{\text{op}} \times \mathfrak{D}_{\mathbb{C}}(F)_{x'})$$

which is induced as in Corollary 3.15 by the following commutative square

$$\begin{array}{ccc}
 P^{-1}u & \xrightarrow{F_u := \llbracket \mathbb{V}[1], F \rrbracket_u} & Q^{-1}u \\
 \downarrow & & \downarrow \\
 P^{-1}x \times P^{-1}x' & \xrightarrow{F_x \times F_{x'}} & Q^{-1}x \times Q^{-1}x'
 \end{array}$$

where $\llbracket \mathbb{V}[1], F \rrbracket_u$ denotes the unique induced functor between fibers.

6.17. LEMMA. *The construction $\mathfrak{D}_{\mathbb{C}}(F): \mathfrak{D}_{\mathbb{C}}(P) \Rightarrow \mathfrak{D}_{\mathbb{C}}(Q)$ is a horizontal transformation.*

PROOF. We first show that the components $\mathfrak{D}_{\mathbb{C}}(F)_x$ are natural in x . Given a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , we have to show that the following diagram of functors commutes.

$$\begin{array}{ccc}
 P^{-1}y & \xrightarrow{F_y} & Q^{-1}y \\
 f^* \downarrow & & \downarrow f^* \\
 P^{-1}x & \xrightarrow{F_x} & Q^{-1}x
 \end{array}$$

But this follows directly from the fact that every morphism of discrete double fibrations over \mathbb{C} preserves unique lifts.

Next, we show that the components $\mathfrak{D}_{\mathbb{C}}(F)_u$ are natural in u . Consider a square $\alpha: \begin{bmatrix} u & f \\ x' & y' \end{bmatrix} : \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ in \mathbb{C} . We have to show that the following diagram in $\text{Prof}(P^{-1}y, P^{-1}y')$ commutes.

$$\begin{array}{ccc}
 \mathfrak{D}_{\mathbb{C}}(P)v & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F)v} & \mathfrak{D}_{\mathbb{C}}(Q)v(F_y^{\text{op}} \times F_{y'}) \\
 \mathfrak{D}_{\mathbb{C}}(P)\alpha \downarrow & & \downarrow \mathfrak{D}_{\mathbb{C}}(Q)\alpha(F_y^{\text{op}} \times F_{y'}) \\
 \mathfrak{D}_{\mathbb{C}}(P)u(f^{*\text{op}} \times f'^*) & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F)_u(f^{*\text{op}} \times f'^*)} & \mathfrak{D}_{\mathbb{C}}(Q)u((F_x f^*)^{\text{op}} \times F_{x'} f')
 \end{array}$$

Under the natural equivalence fib from Theorem 3.13 and Proposition 3.14, this amounts to showing that the following diagram of functors commutes.

$$\begin{array}{ccc}
 P^{-1}v & \xrightarrow{F_v} & Q^{-1}v \\
 \alpha^* \downarrow & & \downarrow \alpha^* \\
 P^{-1}u & \xrightarrow{F_u} & Q^{-1}u
 \end{array}$$

But this follows directly from the fact that every morphism of discrete double fibrations over $\llbracket \mathbb{V}[1], \mathbb{C} \rrbracket$ preserves unique lifts.

We now show that the components $\mathfrak{D}_{\mathbb{C}}(F)_u$ are compatible with composition comparison natural transformations. Given composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x''$ in \mathbb{C} ,

we have to show that the following diagram in $\text{Prof}(P^{-1}x, P^{-1}x'')$ commutes.

$$\begin{array}{ccc} \mathfrak{D}_{\mathbb{C}}(P)u' \bullet \mathfrak{D}_{\mathbb{C}}(P)u & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F)_{u'} \bullet \mathfrak{D}_{\mathbb{C}}(F)_u} & (\mathfrak{D}_{\mathbb{C}}(Q)u' \bullet \mathfrak{D}_{\mathbb{C}}(Q)u)(F_x^{\text{op}} \times F_{x''}) \\ \mu_{u,u'} \Downarrow & & \Downarrow \mu_{u,u'}(F_x^{\text{op}} \times F_{x''}) \\ \mathfrak{D}_{\mathbb{C}}(P)(u' \bullet u) & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F)_{u' \bullet u}} & \mathfrak{D}_{\mathbb{C}}(Q)(u' \bullet u)(F_x^{\text{op}} \times F_{x''}) \end{array}$$

Under the equivalence fib from Theorem 3.13, this amounts to showing that the following diagram of functors commutes.

$$\begin{array}{ccc} P^{-1}u \times_{P^{-1}x'} P^{-1}u' /_{\sim} & \xrightarrow{F_u \times F_{u'}} & Q^{-1}u \times_{Q^{-1}x'} Q^{-1}u' /_{\sim} \\ \bullet \downarrow & & \downarrow \bullet \\ P^{-1}(u' \bullet u) & \xrightarrow{F_{u' \bullet u}} & Q^{-1}(u' \bullet u) \end{array}$$

But this follows directly from the fact that F preserves vertical composition.

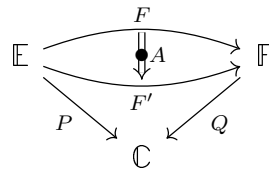
Finally, we show that the components $\mathfrak{D}_{\mathbb{C}}(F)_u$ are compatible with vertical identities. Given an object x in \mathbb{C} , we have to show that the natural transformation $\mathfrak{D}_{\mathbb{C}}(F)_{e_x}$ is the identity at $\mathfrak{D}_{\mathbb{C}}(F)_x = F_x$. By definition, we have that $\mathfrak{D}_{\mathbb{C}}(F)_{e_x}$ is the natural transformation induced as in Corollary 3.15 by the below left commutative square.

$$\begin{array}{ccc} P^{-1}e_x & \xrightarrow{\llbracket \mathbb{V}[1], F \rrbracket_{e_x}} & Q^{-1}e_x & & (P^{-1}x)^{[1]} & \xrightarrow{F_x^{[1]}} & (Q^{-1}x)^{[1]} \\ P_{e_x} \downarrow & & \downarrow Q_{e_x} & & (s,t) \downarrow & & \downarrow (s,t) \\ P^{-1}x \times P^{-1}x & \xrightarrow{F_x \times F_x} & Q^{-1}x \times Q^{-1}x & & P^{-1}x \times P^{-1}x & \xrightarrow{F_x \times F_x} & Q^{-1}x \times Q^{-1}x \end{array}$$

Since the above left commutative square is canonically isomorphic to the above right commutative square by Lemma 5.20, we see that the desired natural transformation is the identity at F_x , concluding the proof. ■

Finally, we turn to the assignment of $\mathfrak{D}_{\mathbb{C}}$ on 2-morphisms.

6.18. CONSTRUCTION. Given a vertical transformation of discrete double fibrations



we construct a globular modification

$$\begin{array}{ccc} \mathfrak{D}_{\mathbb{C}}(P) & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F)} & \mathfrak{D}_{\mathbb{C}}(Q) \\ \Downarrow & \mathfrak{D}_{\mathbb{C}}(A) & \Downarrow \\ \mathfrak{D}_{\mathbb{C}}(P) & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F')} & \mathfrak{D}_{\mathbb{C}}(Q) \end{array}$$

defined as follows. Its component at an object x in \mathbb{C} is given by the natural transformation

$$\mathfrak{D}_{\mathbb{C}}(A)_x: \mathfrak{D}_{\mathbb{C}}(F)_x = F_x \Rightarrow \mathfrak{D}_{\mathbb{C}}(F')_x = F'_x: P^{-1}x \rightarrow Q^{-1}x$$

whose component at an object x_- of $P^{-1}x$ is given by the morphism in $Q^{-1}x$

$$A_{x_-}: Fx_- \dashrightarrow F'x_-.$$

6.19. LEMMA. *The construction $\mathfrak{D}_{\mathbb{C}}(A): \left[e_{\mathfrak{D}_{\mathbb{C}}(P)} \frac{\mathfrak{D}_{\mathbb{C}}(F)}{\mathfrak{D}_{\mathbb{C}}(F')} e_{\mathfrak{D}_{\mathbb{C}}(Q)} \right]$ is a globular modification.*

PROOF. We first show horizontal compatibility of $\mathfrak{D}_{\mathbb{C}}(A)_x$. Given a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , we have to show that the following pasting diagram of functors and natural transformations commutes.

$$\begin{array}{ccc} P^{-1}y & \xrightarrow{f^*} & P^{-1}x \\ \left(\begin{array}{c} \downarrow F_y \\ \xrightarrow{\mathfrak{D}_{\mathbb{C}}(A)_y} \\ \downarrow F'_y \end{array} \right) & & \left(\begin{array}{c} \downarrow F_x \\ \xrightarrow{\mathfrak{D}_{\mathbb{C}}(A)_x} \\ \downarrow F'_x \end{array} \right) \\ Q^{-1}y & \xrightarrow{f^*} & Q^{-1}x \end{array}$$

To see this, we first note the following. Since $A: F \dashrightarrow F'$ is a vertical transformation over \mathbb{C} , given an object y_- in $P^{-1}y$, the component of A at the horizontal morphism $P^*f: f^*y_- \rightarrow y_-$ is a square $A_{P^*f}: \left[A_{f^*y_-} \begin{array}{c} F_{P^*f} \\ \xrightarrow{F'P^*f} \\ A_{y_-} \end{array} \right]$ such that $QA_{P^*f} = e_f$. Hence it has to be the unique lift of e_f along Q with target A_{y_-} . When evaluating both whiskerings at an object y_- in $P^{-1}y$, we therefore have:

$$\begin{aligned} e_f^*(\mathfrak{D}_{\mathbb{C}}(A)_y)_{y_-} &= e_f^*A_{y_-} && \text{Definition of } \mathfrak{D}_{\mathbb{C}}(A)_y \\ &= A_{f^*y_-} && \text{Unicity of lifts} \\ &= (\mathfrak{D}_{\mathbb{C}}(A)_x)_{f^*y_-}. && \text{Definition of } \mathfrak{D}_{\mathbb{C}}(A)_x \end{aligned}$$

This shows that the components of the two whiskerings agree, as desired.

Next, we show vertical compatibility of $\mathfrak{D}_{\mathbb{C}}(A)_x$. Given a vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} , we have to show that the following diagram in $\text{Prof}(P^{-1}, P^{-1}x')$ commutes.

$$\begin{array}{ccc} \mathfrak{D}_{\mathbb{C}}(P)u & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(F)u} & \mathfrak{D}_{\mathbb{C}}(Q)u(F_x^{\text{op}} \times F_{x'}) \\ \mathfrak{D}_{\mathbb{C}}(F')u \downarrow & & \downarrow \mathfrak{D}_{\mathbb{C}}(Q)u(F_x^{\text{op}} \times A_{x'}) \\ \mathfrak{D}_{\mathbb{C}}(Q)u(F_x^{\text{op}} \times F_{x'}) & \xrightarrow{\mathfrak{D}_{\mathbb{C}}(Q)u(A_x^{\text{op}} \times F_{x'})} & \mathfrak{D}_{\mathbb{C}}(Q)u(F_x^{\text{op}} \times F_{x'}) \end{array}$$

Under the natural equivalence fib from Theorem 3.13 and Proposition 3.14, this amounts to showing that the following diagram of functors and natural transformations commutes.

$$\begin{array}{ccc}
 P^{-1}u & \xrightarrow{\quad} & P^{-1}x \times P^{-1}x' \\
 \left(\begin{array}{c} \curvearrowright \\ \text{ } \\ \curvearrowleft \end{array} \right) & & \left(\begin{array}{c} \curvearrowright \\ \text{ } \\ \curvearrowleft \end{array} \right) \\
 F_u \left(\begin{array}{c} \xrightarrow{\quad} \\ \text{ } \\ \xrightarrow{\quad} \end{array} \right) F'_u & F_x \times F_{x'} \left(\begin{array}{c} \xrightarrow{\quad} \\ \text{ } \\ \xrightarrow{\quad} \end{array} \right) F'_x \times F'_{x'} & (6.1) \\
 Q^{-1}u & \xrightarrow{\quad} & Q^{-1}x \times Q^{-1}x'
 \end{array}$$

Here, the left-hand transformation is defined as follows: using the description of the morphisms in $Q^{-1}u$ from Remark 5.17, there is a natural transformation

$$(\mathcal{D}_{\mathbb{C}}(A)_x, \mathcal{D}_{\mathbb{C}}(A)_{x'}) : F_u \Rightarrow F'_u$$

whose component at an object $u_- : x_- \rightarrow x'_-$ in $P^{-1}u$ is given by the commutative diagram in \mathbb{F}

$$\begin{array}{ccc}
 Fx_- & \xrightarrow{A_{x_-}} & F'x_- \\
 Fu_- \downarrow & & \downarrow F'u_- \\
 Fx'_- & \xrightarrow{A_{x'_-}} & F'x'_-
 \end{array}$$

Note that this gives a well-defined morphism of $Q^{-1}u$ since the above diagram of vertical morphisms commutes by naturality of A_{x_-} in x_- , and $QA_{x_-} = e_x$ and $QA_{x'_-} = e_{x'}$ by definition of the vertical transformation A living over \mathbb{C} . The fact that the diagram (6.1) commutes is then straightforward from the fact that the horizontal arrows are picking source and target as described in Proposition 5.19. \blacksquare

Putting everything together, we get the following.

6.20. CONSTRUCTION. Given a double category \mathbb{C} , we define a 2-functor

$$\mathcal{D}_{\mathbb{C}} : \text{DFib}(\mathbb{C}) \rightarrow \mathcal{P}_{\mathbb{C}}$$

which sends

- a discrete double fibration $P : \mathbb{E} \rightarrow \mathbb{C}$ to the lax double presheaf $\mathcal{D}_{\mathbb{C}}(P) : \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ from Construction 6.14,
- a double functor $F : \mathbb{E} \rightarrow \mathbb{F}$ over \mathbb{C} to the horizontal transformation

$$\mathcal{D}_{\mathbb{C}}(F) : \mathcal{D}_{\mathbb{C}}(P) \Rightarrow \mathcal{D}_{\mathbb{C}}(Q)$$

from Construction 6.16,

- a vertical transformation $A : F \Rightarrow F'$ over \mathbb{C} to the globular modification

$$\mathcal{D}_{\mathbb{C}}(A) : \left[e_{\mathcal{D}_{\mathbb{C}}(P)} \begin{array}{c} \mathcal{D}_{\mathbb{C}}(F) \\ \mathcal{D}_{\mathbb{C}}(F') \end{array} e_{\mathcal{D}_{\mathbb{C}}(Q)} \right]$$

from Construction 6.18.

It is straightforward to check that this construction is 2-functorial.

We are now ready to show that $\mathcal{D}_{\mathbb{C}}$ is a weak inverse of $\mathbb{J}_{\mathbb{C}}$. We start by constructing a 2-natural isomorphism $\varepsilon: \mathcal{D}_{\mathbb{C}} \mathbb{J}_{\mathbb{C}} \cong \text{id}_{\mathcal{P}_{\mathbb{C}}}$. For this, we first describe the composite $\mathcal{D}_{\mathbb{C}} \mathbb{J}_{\mathbb{C}}$ on objects, morphisms, and 2-morphisms explicitly.

6.21. LEMMA. *Given a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, the lax double presheaf*

$$\mathcal{D}_{\mathbb{C}}(\pi_X): \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$$

associated with the Grothendieck construction $\pi_X: \mathbb{J}_{\mathbb{C}} X \rightarrow \mathbb{C}$ admits the following description.

- *Given an object x in \mathbb{C} , the category $\mathcal{D}_{\mathbb{C}}(\pi_X)x = \pi_X^{-1}x$ is given by $\{x\} \times Xx$.*
- *Given a horizontal morphism $f: x \rightarrow y$ in \mathbb{C} , the functor $\mathcal{D}_{\mathbb{C}}(\pi_X)f: \pi_X^{-1}y \rightarrow \pi_X^{-1}x$ is given by*

$$[y \mapsto x] \times Xf: \{y\} \times Xy \rightarrow \{x\} \times Xx.$$

- *Given a vertical morphism $u: x \bullet \rightarrow x'$ in \mathbb{C} , the profunctor*

$$\mathcal{D}_{\mathbb{C}}(\pi_X)u: \pi_X^{-1}x^{\text{op}} \times \pi_X^{-1}x' \rightarrow \text{Set}$$

is given by

$$\{u\} \times Xu: \{x\} \times Xx^{\text{op}} \times \{x'\} \times Xx' \cong Xx^{\text{op}} \times Xx' \rightarrow \text{Set}.$$

- *Given a square $\alpha: \begin{bmatrix} u & f \\ u' & f' \end{bmatrix} v: \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ in \mathbb{C} , the natural transformation*

$$\mathcal{D}_{\mathbb{C}}(\pi_X)\alpha: \mathcal{D}_{\mathbb{C}}(\pi_X)v \Rightarrow \mathcal{D}_{\mathbb{C}}(\pi_X)u(\mathcal{D}_{\mathbb{C}}(\pi_X)f^{\text{op}} \times \mathcal{D}_{\mathbb{C}}(\pi_X)f')$$

is given by

$$[v \mapsto u] \times X\alpha: \{v\} \times Xv \Rightarrow \{u\} \times Xu(Xf^{\text{op}} \times Xf').$$

- *Given composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x''$ in \mathbb{C} , the compositor comparison natural transformation $\mathcal{D}_{\mathbb{C}}(\pi_X)u' \bullet \mathcal{D}_{\mathbb{C}}(\pi_X)u \Rightarrow \mathcal{D}_{\mathbb{C}}(\pi_X)(u' \bullet u)$ is given by*

$$[(u, u') \mapsto u' \bullet u] \times \mu_{u, u'}: (\{u\} \times Xu) \bullet (\{u'\} \times Xu') \rightarrow \{u' \bullet u\} \times X(u' \bullet u)$$

where $\mu_{u, u'}$ denotes the composition comparison natural transformation of X and we use the isomorphism $(\{u\} \times Xu) \bullet (\{u'\} \times Xu') \cong \{(u, u')\} \times (Xu \bullet Xu')$.

PROOF. By unpacking Constructions 6.1 and 6.14, we observe the following.

Given an object x in \mathbb{C} , then $\pi_X^{-1}x$ is the category whose

- objects are pairs (x, x_-) with x_- an object in Xx ,
- morphisms $(x, x_-) \rightarrow (x, x'_-)$ are pairs (e_x, s_-) with s_- an element of $Xe_x(x_-, x'_-) = Xx(x_-, x'_-)$, i.e., a morphism $s_- : x_- \rightarrow x'_-$ in Xx .

Hence, we have that $\pi_X^{-1}x = \{x\} \times Xx$, as desired.

Next, given a horizontal morphism $f : x \rightarrow y$ in \mathbb{C} , the functor $\mathcal{D}_{\mathbb{C}}(\pi_X)f$ is given by assigning the source of the unique lift of f or e_f along the discrete double fibration $\pi_X : \int_{\mathbb{C}} X \rightarrow \mathbb{C}$. Hence, by the proof of Proposition 6.4, we get the desired description, namely $\mathcal{D}_{\mathbb{C}}(\pi_X)f = [y \mapsto x] \times Xf$.

Now, given a vertical morphism $u : x \bullet \rightarrow x'$ in \mathbb{C} , the profunctor $\mathcal{D}_{\mathbb{C}}(\pi_X)u$ is the image under the equivalence fib from Theorem 3.13 of the two-sided discrete fibrations

$$\pi_X^{-1}u \rightarrow \pi_X^{-1}x \times \pi_X^{-1}x'.$$

Here $\pi_X^{-1}u$ is the category whose

- objects are pairs (u, u_-) with u_- an element of $Xu(x_-, x'_-)$, for some object (x_-, x'_-) in $Xx \times Xx'$,
- morphisms $(u, u_-) \rightarrow (u, \hat{u}_-)$ with $u_- \in Xu(x_-, x'_-)$ and $\hat{u}_- \in Xu(\hat{x}_-, \hat{x}'_-)$ are tuples $(u, s_-, s'__-)$ with $(x_- \xrightarrow{s_-} \hat{x}_-, x'_- \xrightarrow{s'_-} \hat{x}'_-)$ a morphism in $Xx \times Xx'$ making the following square in $\int_{\mathbb{C}} X$ commutes.

$$\begin{array}{ccc} (x, x_-) & \xrightarrow{(e_x, s_-)} & (x, \hat{x}_-) \\ \downarrow (u, s_-) \bullet & & \downarrow \bullet (u, \hat{u}_-) \\ (x', x'_-) & \xrightarrow{(e_{x'}, s'_-)} & (x', \hat{x}'_-) \end{array}$$

Hence, by applying the functor fib from Construction 3.12, we see that the profunctor $\mathcal{D}_{\mathbb{C}}(\pi_X)u$ assigns to an object (x_-, x'_-) in $\pi_X^{-1}x \times \pi_X^{-1}x'$ the set $\{u\} \times Xu(x_-, x'_-)$. It remains to show that the action on morphisms of $\mathcal{D}_{\mathbb{C}}(\pi_X)u$ and $\{u\} \times Xu$ agree.

By the proof of Proposition 5.19, the action of a morphism $(x_- \xrightarrow{s_-} \hat{x}_-, x'_- \xrightarrow{s'_-} \hat{x}'_-)$ in $\pi_X^{-1}x \times \pi_X^{-1}x'$ on fibers is given by taking an element (u, u_-) in $\{u\} \times Xu(\hat{x}_-, \hat{x}'_-)$ to the composite $(e_x, s') \bullet (u, u_-) \bullet (e_{x'}, s_-) = (u, s' \bullet u_- \bullet s_-)$ of $\int_{\mathbb{C}} X$. However, we have:

$$\begin{aligned} s'_- \bullet u_- \bullet s_- &= (\mu_{u, e_{x'}})_{\hat{x}_-, \hat{x}'_-}([s'_-, u_- \bullet s_-]) && \text{Definition of } \bullet \\ &= (\mu_{u, e_{x'}})_{x_-, \hat{x}'_-}([s'_-, (\mu_{e_x, u})_{x_-, x'_-}([u_-, s_-])]) && \text{Definition of } \bullet \\ &= (\mu_{u, e_{x'}})_{x_-, \hat{x}'_-}([s'_-, Xu(s_-, x'_-)(u_-)]) && \mu_{e_x, u} = 1_{Xu} \\ &= Xu(x_-, s'_-)(Xu(s_-, x'_-)(u_-)) && \mu_{u, e_{x'}} = 1_{Xu} \\ &= Xu(s_-, s'_-)(u_-). && \text{Functoriality of } Xu \end{aligned}$$

Hence, this shows that $\mathcal{D}_{\mathbb{C}}(\pi_X)u = \{u\} \times Xu$, as desired.

Next, given a square $\alpha: \begin{bmatrix} u & f \\ & f' \end{bmatrix} v: \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ in \mathbb{C} , the natural transformation $\mathcal{D}_{\mathbb{C}}(\pi_X)\alpha$ is induced as in Corollary 3.15 by the following commutative square

$$\begin{array}{ccc} \pi_X^{-1}v & \xrightarrow{\alpha^*} & \pi_X^{-1}u \\ \downarrow & & \downarrow \\ \pi_X^{-1}y \times \pi_X^{-1}y' & \xrightarrow{f^* \times f'^*} & \pi_X^{-1}x \times \pi^{-1}x' \end{array}$$

where α^* is given by assigning the source of the unique lift of α along the discrete double fibration $\pi_X: \int_{\mathbb{C}} X \rightarrow \mathbb{C}$. Hence, by the proof of Proposition 6.4, we get the desired description, namely $\mathcal{D}_{\mathbb{C}}(\pi_X)\alpha = [u \mapsto v] \times X\alpha$.

Finally, given composable vertical morphisms $x \xrightarrow{u} x' \xrightarrow{u'} x''$ in \mathbb{C} , the composition comparison natural transformation of $\mathcal{D}_{\mathbb{C}}(\pi_X)$ is the image under the equivalence fib from Theorem 3.13 of the morphism of two-sided discrete fibrations

$$\begin{array}{ccc} \pi_X^{-1}u \times_{\pi_X^{-1}x'} \pi_X^{-1}u'_{/\sim} & \xrightarrow{\quad \bullet \quad} & \pi_X^{-1}(u' \bullet u) \\ & \searrow \quad \swarrow & \\ & \pi_X^{-1}x \times \pi_X^{-1}x'' & \end{array}$$

induced by vertical composition in $\int_{\mathbb{C}} X$. However, by definition, the composite of two vertical morphisms (u, u_-) and (u', u'_-) with $u_- \in Xu(x_-, x'_-)$ and $u'_- \in Xu(x'_-, x''_-)$ in $\int_{\mathbb{C}} X$ is given by the pair $(u' \bullet u, u'_- \bullet u_-)$ with $u'_- \bullet u_- = (\mu_{u,u'})_{x_-, x''_-}([u'_-, u_-])$. Hence, we get the desired description, namely the composition comparison is given by the assignment $[(u, u') \mapsto u' \bullet u] \times \mu_{u,u'}$. ■

6.22. LEMMA. *Given a horizontal transformation $F: X \Rightarrow Y$ of lax double presheaves $X, Y: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, the horizontal transformation*

$$\mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} F: \mathcal{D}_{\mathbb{C}}(\pi_X) \Rightarrow \mathcal{D}_{\mathbb{C}}(\pi_Y)$$

admits the following description.

- *Given an object x in \mathbb{C} , the functor $(\mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} F)_x: \mathcal{D}_{\mathbb{C}}(\pi_X)x \rightarrow \mathcal{D}_{\mathbb{C}}(\pi_Y)x$ is given by*

$$\{x\} \times F_x: \{x\} \times Xx \rightarrow \{x\} \times Yx.$$

- *Given a vertical morphisms $u: x \rightarrow x'$ in \mathbb{C} , the natural transformation*

$$(\mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} F)_u: \mathcal{D}_{\mathbb{C}}(\pi_X)u \Rightarrow \mathcal{D}_{\mathbb{C}}(\pi_Y)u((\mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} F)_x^{\text{op}} \times (\mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} F)_{x'})$$

is given by

$$\{u\} \times F_u: \{u\} \times Xu \Rightarrow \{u\} \times Yu(F_x^{\text{op}} \times F_{x'}).$$

PROOF. Using Lemma 6.21 and unpacking Constructions 6.5 and 6.16, we observe the following.

Given an object x in \mathbb{C} , the functor $(\partial_{\mathbb{C}} \int_{\mathbb{C}} F)_x$ is the unique induced functor between fibers

$$(\int_{\mathbb{C}} F)_x = \{x\} \times F_x : \pi_X^{-1}(x) = \{x\} \times Xx \rightarrow \pi_Y^{-1}x = \{x\} \times Yx.$$

Hence, we get the desired description, namely $(\partial_{\mathbb{C}} \int_{\mathbb{C}} F)_x = \{x\} \times F_x$.

Given a vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} , the natural transformation $(\partial_{\mathbb{C}} \int_{\mathbb{C}} F)_u$ is induced as in Corollary 3.15 by the following commutative square

$$\begin{array}{ccc} \pi_X^{-1}u & \xrightarrow{(\int_{\mathbb{C}} F)_u} & \pi_Y^{-1}u \\ (s_X, t_X) \downarrow & & \downarrow (s_Y, t_Y) \\ \pi_X^{-1}x \times \pi_X^{-1}x' & \xrightarrow{F_x \times F_{x'}} & \pi_Y^{-1}x \times \pi_Y^{-1}x', \end{array}$$

where $(\int_{\mathbb{C}} F)_u$ is the unique induced functor between fibers

$$[\mathbb{V}[1], \int_{\mathbb{C}} F]_u = \{u\} \times F_u =: \pi_X^{-1}u = \{u\} \times Xu \rightarrow \pi_Y^{-1}u = \{u\} \times Yu.$$

Hence, we get the desired description, namely $(\partial_{\mathbb{C}} \int_{\mathbb{C}} F)_u = \{u\} \times F_u$. ■

6.23. LEMMA. *Given a globular modification $A: [e_X \xrightarrow{F} e_Y]$ between lax double presheaves $\mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, the globular modification*

$$\partial_{\mathbb{C}} \int_{\mathbb{C}} A: \left[e_{\partial_{\mathbb{C}}(\pi_X)} \xrightarrow{\partial_{\mathbb{C}} \int_{\mathbb{C}} F} e_{\partial_{\mathbb{C}}(\pi_Y)} \right]$$

admits the following description. Given an object x in \mathbb{C} , the natural transformation

$$(\partial_{\mathbb{C}} \int_{\mathbb{C}} A)_x: (\partial_{\mathbb{C}} \int_{\mathbb{C}} F)_x \Rightarrow (\partial_{\mathbb{C}} \int_{\mathbb{C}} F')_x$$

is given by

$$\{x\} \times A_x: \{x\} \times Fx \Rightarrow \{x\} \times F'x.$$

PROOF. Using Lemma 6.22 and unpacking Constructions 6.7 and 6.18, we observe the following.

Given an object x in \mathbb{C} , the natural transformation

$$(\partial_{\mathbb{C}} \int_{\mathbb{C}} A)_x: (\partial_{\mathbb{C}} \int_{\mathbb{C}} F)_x = \{x\} \times F_x \Rightarrow (\partial_{\mathbb{C}} \int_{\mathbb{C}} F')_x = \{x\} \times F'_x$$

is given at an object (x, x_-) in $\pi_X^{-1}x = \{x\} \times Xx$ by the vertical morphism in $\int_{\mathbb{C}} Y$

$$(\int_{\mathbb{C}} A)_{(x, x_-)} = (e_x, (A_x)_{x_-}): (x, F_x x_-) \dashrightarrow (x, F'_x x_-).$$

Hence, we get the desired description, namely $(\partial_{\mathbb{C}} \int_{\mathbb{C}} A)_x = \{x\} \times A_x$. ■

Using the explicit description of the composite $\mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}}$, we can now construct the desired 2-natural isomorphism.

6.24. PROPOSITION. *There is a 2-natural isomorphism $\varepsilon: \mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} \Rightarrow \text{id}_{\mathcal{P}_{\mathbb{C}}}$.*

PROOF. Given a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, we define the component of the 2-natural isomorphism $\varepsilon: \mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} \Rightarrow \text{id}_{\mathcal{P}_{\mathbb{C}}}$ at X to be the invertible horizontal transformation

$$\varepsilon_X: \mathcal{D}_{\mathbb{C}} \int_{\mathbb{C}} X \Rightarrow X$$

whose components at an object x in \mathbb{C} and at a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} are given by the canonical isomorphisms

$$\{x\} \times Xx \cong Xx \quad \text{and} \quad \{u\} \times Xu \cong Xu.$$

Using the description from Lemma 6.21, it is straightforward to check that this gives a well-defined horizontal transformation.

Moreover, it follows easily from the descriptions in Lemmas 6.22 and 6.23 that the components ε_X are 2-natural in X . ■

Next, we want to construct a 2-natural isomorphism $\eta: \text{id}_{\text{DFib}(\mathbb{C})} \cong \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}$. As before, we first describe the composite $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}$ on objects, morphisms, and 2-morphisms.

6.25. LEMMA. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$, the discrete double fibration*

$$\pi_{\mathcal{D}_{\mathbb{C}}(P)}: \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P) \rightarrow \mathbb{C}$$

admits the following description. We have that $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ is the double category whose

- *objects are pairs (x, x_-) of objects x in \mathbb{C} and x_- in \mathbb{E} such that $Px_- = x$,*
- *horizontal morphisms $(x, x_-) \rightarrow (y, y_-)$ are horizontal morphisms $f: x \rightarrow y$ in \mathbb{C} such that $f^*y_- = x_-$, i.e., x_- is the source of the unique lift $P^*f: x_- \rightarrow y_-$ in \mathbb{E} of f ,*
- *vertical morphisms $(x, x_-) \rightarrow (x', x'_-)$ are given by pairs (u, u_-) of vertical morphisms $u: x \rightarrow x'$ in \mathbb{C} and $u_-: x_- \rightarrow x'_-$ in \mathbb{E} such that $Pu_- = u$,*
- *squares $\left[(u, u_-) \xrightarrow{f} (v, v_-) \right]$ are squares $\alpha: \left[u \xrightarrow{f} v \right]$ in \mathbb{C} such that $\alpha^*v_- = u_-$, i.e., u_- is the source of the unique lift $P^*\alpha: \left[u_- \xrightarrow{f_-} v_- \right]$ in \mathbb{E} of α ,*

and that $\pi_{\mathcal{D}_{\mathbb{C}}(P)}: \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P) \rightarrow \mathbb{C}$ is given by projecting onto the first variable.

PROOF. By unpacking Constructions 6.1 and 6.14, we observe the following.

We have that $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ is the double category whose

- objects are pairs (x, x_-) of objects x in \mathbb{C} and x_- in $\mathcal{D}_{\mathbb{C}}(P)x = P^{-1}x$, i.e., x_- is an object in \mathbb{E} such that $Px_- = x$,
- horizontal morphisms $(x, x_-) \rightarrow (y, y_-)$ are horizontal morphisms $f: x \rightarrow y$ in \mathbb{C} such that $x_- = \mathcal{D}_{\mathbb{C}}(P)f(y_-) = f^*y_-$,
- vertical morphisms $(x, x_-) \dashrightarrow (x', x'_-)$ are given by pairs (u, u_-) of a vertical morphism $u: x \dashrightarrow x'$ in \mathbb{C} and an element u_- in $\mathcal{D}_{\mathbb{C}}(P)u(x_-, x'_-)$, i.e., u_- is an element in the fiber of the two-sided discrete fibration $P^{-1}u \rightarrow P^{-1}x \times P^{-1}x'$ at (x_-, x'_-) , namely a vertical morphism $u_-: x_- \dashrightarrow x'_-$ in \mathbb{E} such that $Pu_- = u$,
- squares $\left[(u, u_-) \xrightarrow{f} (v, v_-) \right] : \left[\begin{smallmatrix} (x, x_-) & (y, y_-) \\ (x', x'_-) & (y', y'_-) \end{smallmatrix} \right]$ are squares $\alpha: \left[u \xrightarrow{f} v \right] : \left[\begin{smallmatrix} x & y \\ x' & y' \end{smallmatrix} \right]$ in \mathbb{C} such that $u_- = \mathcal{D}_{\mathbb{C}}(P)\alpha_{y_-, y'_-}(v_-) = \alpha^*v_-$.

Moreover, the double functor $\pi_{\mathcal{D}_{\mathbb{C}}(P)}: \iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P) \rightarrow \mathbb{C}$ is given by projecting onto the first variable. This gives the desired description. \blacksquare

6.26. LEMMA. *Given a double functor $F: \mathbb{E} \rightarrow \mathbb{F}$ of discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and $Q: \mathbb{F} \rightarrow \mathbb{C}$, the double functor*

$$\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(F): \iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P) \rightarrow \iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$$

admits the following description. It sends

- an object (x, x_-) in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ to the object (x, Fx_-) in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$,
- a horizontal morphism $f: (x, x_-) \rightarrow (y, y_-)$ in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ to the horizontal morphism in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$

$$f: (x, Fx_-) \rightarrow (y, Fy_-),$$

- a vertical morphism $(u, u_-): (x, x_-) \dashrightarrow (x', x'_-)$ in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ to the vertical morphism in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$

$$(u, Fu_-): (x, Fx_-) \dashrightarrow (x', Fx'_-),$$

- a square $\alpha: \left[(u, u_-) \xrightarrow{f} (v, v_-) \right]$ in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ to the square in $\iint_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$

$$\alpha: \left[(u, Fu_-) \xrightarrow{f} (v, Fv_-) \right].$$

PROOF. This is straightforward from unpacking Constructions 6.5 and 6.16. \blacksquare

6.27. LEMMA. *Given a vertical transformation $A: F \rightrightarrows F': \mathbb{E} \rightarrow \mathbb{F}$ between morphisms of discrete double fibrations $P: \mathbb{E} \rightarrow \mathbb{C}$ and $Q: \mathbb{F} \rightarrow \mathbb{C}$, the vertical transformation*

$$\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(A): \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(F) \rightrightarrows \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(F')$$

admits the following description.

- *Its component at an object (x, x_-) in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ is the vertical morphism in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$*

$$(e_x, A_{x_-}): (x, Fx_-) \rightarrow (x, F'x_-),$$

- *Its component at a horizontal morphism $f: (x, x_-) \rightarrow (y, y_-)$ in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ is the square $e_f: \left[(e_x, A_{x_-}) \begin{smallmatrix} f \\ f \end{smallmatrix} (e_y, A_{y_-}) \right]$ in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(Q)$.*

PROOF. This is straightforward from unpacking Constructions 6.7 and 6.18. ■

We are now ready to construct the desired 2-natural isomorphism.

6.28. CONSTRUCTION. We construct a 2-natural transformation $\eta: \text{id}_{\text{DFib}(\mathbb{C})} \Rightarrow \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}$ whose component at a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ is given by the functor over \mathbb{C}

$$\eta_P: \mathbb{E} \rightarrow \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$$

sending, using the description of $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ from Lemma 6.25,

- an object x_- in \mathbb{E} to the object (Px_-, x_-) in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$,
- a horizontal morphism $f_-: x_- \rightarrow y_-$ in \mathbb{E} to the horizontal morphism in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$

$$Pf_-: (Px_-, x_-) \rightarrow (Py_-, y_-),$$

where $(Pf_-)^*(y_-) = x_-$ by unicity of the lift f_- of Pf_- with target y_- ,

- a vertical morphism $u_-: x_- \rightarrow x'_-$ in \mathbb{E} to the vertical morphism in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$

$$(Pu_-, u_-): (Px_-, x_-) \rightarrow (Px'_-, x'_-),$$

- a square $\alpha_-: \left[u_- \begin{smallmatrix} f_- \\ f'_- \end{smallmatrix} v_- \right]$ in \mathbb{E} to the square in $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$

$$P\alpha_-: \left[(Pu_-, u_-) \begin{smallmatrix} Pf_- \\ Pf'_- \end{smallmatrix} (Pv_-, v_-) \right],$$

where $(P\alpha_-)^*(v_-) = u_-$ by unicity of the lift α_- of $P\alpha_-$ with target v_- .

6.29. PROPOSITION. *The construction $\eta: \text{id}_{\text{DFib}(\mathbb{C})} \Rightarrow \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}$ is a 2-natural isomorphism.*

PROOF. First note that the double functor $\eta_P: \mathbb{E} \rightarrow \int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P)$ admits as an inverse the double functor $\int_{\mathbb{C}} \mathcal{D}_{\mathbb{C}}(P) \rightarrow \mathbb{E}$ given by projecting onto the second component on objects and vertical morphisms, and by picking the unique lift of horizontal morphisms and squares.

Moreover, it follows easily from the descriptions in Lemmas 6.26 and 6.27 that the components η_P are 2-natural in P . ■

PROOF OF THEOREM 6.13. The data $(\mathbb{J}_{\mathbb{C}}, \mathcal{D}_{\mathbb{C}}, \eta, \varepsilon)$ from Constructions 6.9 and 6.20 and Propositions 6.24 and 6.29 provides the desired 2-equivalence. ■

As a consequence of Lemma 3.32 and Theorem 6.13, when taking $\mathbb{C} = \mathbb{H}\mathcal{C}$ with \mathcal{C} a 2-category, we retrieve the Grothendieck 2-equivalence from [MSV23, Theorem 5.1] in the case of $\mathcal{V} = \text{Cat}$.

6.30. COROLLARY. *Given a 2-category \mathcal{C} , the Grothendieck construction*

$$\mathbb{J}_{\mathbb{H}\mathcal{C}}: [\mathcal{C}^{\text{op}}, \text{Cat}] \longrightarrow \text{DFib}(\mathbb{H}\mathcal{C})$$

is a 2-equivalence of 2-categories, which is pseudo-natural in \mathcal{C} .

7. Representation theorem for lax double presheaves

In this last section, we turn to a representation theorem for double categories. While in the presheaf world a lax double functor is said to be represented by an object if it is isomorphic to a representable lax double functor at the same object, in the fibrational world we can reformulate this representation condition in terms of *double terminal objects*. In Sections 7.1 to 7.7, we first introduce double terminal objects and show that a discrete double fibration is represented, i.e., it is isomorphic to a double slice, if and only if it has a double terminal object. Then, in Sections 7.8 to 7.11, we use this result to show that a lax double presheaf is represented by an object if and only if its Grothendieck construction has a double terminal object. This gives a nice criterion to test representability of a given lax double presheaf, and formulate universal properties for double categories.

DOUBLE TERMINAL OBJECTS. Let us fix a double category \mathbb{C} . We start by recalling the definition of a double terminal object.

7.1. DEFINITION. An object \hat{x} in \mathbb{C} is *double terminal* if the canonical projection $\mathbb{C}/\hat{x} \rightarrow \mathbb{C}$ is an isomorphism of double categories.

7.2. REMARK. Unpacking the double isomorphism, an object \hat{x} in \mathbb{C} is double terminal if and only if the following conditions hold:

- (1) for every object x in \mathbb{C} , there is a unique horizontal morphism $t_x: x \rightarrow \hat{x}$ in \mathbb{C} ,
- (2) for every vertical morphism $u: x \rightarrow x'$ in \mathbb{C} , there is a unique square $\tau_u: \begin{bmatrix} u & t_x \\ & t_{x'} \end{bmatrix}$ in \mathbb{C} .

7.3. NOTATION. In what follows, for an object \hat{x} in \mathbb{C} and a double functor $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$, we abuse notation and denote the discrete double fibrations $\mathbb{C}/\hat{x} \rightarrow \mathbb{C}$ and $\pi_X: \mathbb{J}_{\mathbb{C}} X \rightarrow \mathbb{C}$ simply by their sources \mathbb{C}/\hat{x} and $\mathbb{J}_{\mathbb{C}} X$.

We can translate the Yoneda lemma in the fibrational setting as follows.

7.4. THEOREM. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and an object \hat{x} in \mathbb{C} , there is an isomorphism of categories*

$$\Psi: \mathcal{DFib}(\mathbb{C})(\mathbb{C}/\hat{x}, P) \xrightarrow{\cong} P^{-1}\hat{x}$$

which is 2-natural in \hat{x} in \mathbf{HC} and in P in $\mathcal{DFib}(\mathbb{C})$.

PROOF. We have the following 2-natural isomorphisms

$$\begin{aligned} \mathcal{DFib}(\mathbb{C})(\mathbb{C}/\hat{x}, P) &= \mathcal{DFib}(\mathbb{C})(\int_{\mathbb{C}} \mathbb{C}(-, \hat{x}), P) && \text{Proposition 6.3} \\ &\cong \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), \mathcal{D}_{\mathbb{C}}(P)) && \int_{\mathbb{C}} \dashv \mathcal{D}_{\mathbb{C}} \\ &\cong \mathcal{D}_{\mathbb{C}}(P)\hat{x} = P^{-1}\hat{x}, && \text{Theorem 4.2} \end{aligned}$$

as desired. ■

In particular, we are interested in the inverse of Ψ

$$\Phi: P^{-1}\hat{x} \xrightarrow{\cong} \mathcal{DFib}(\mathbb{C})(\mathbb{C}/\hat{x}, P).$$

We describe its action on objects explicitly.

7.5. LEMMA. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$, an object \hat{x} in \mathbb{C} and an object \hat{x}_- in $P^{-1}\hat{x}$, then the double functor over \mathbb{C}*

$$\Phi(\hat{x}_-): \mathbb{C}/\hat{x} \rightarrow \mathbb{E}$$

can be described as follows. It sends

- an object (x, g) in \mathbb{C}/\hat{x} to the object $g^*\hat{x}_-$ in \mathbb{E} , i.e., the source of the unique lift of the horizontal morphism $g: x \rightarrow \hat{x}$ with target \hat{x}_- ,
- a horizontal morphism $f: (x, g) \rightarrow (y, h)$ in \mathbb{C}/\hat{x} to the horizontal morphism in \mathbb{E}

$$P^*f: g^*\hat{x}_- = f^*h^*\hat{x}_- \rightarrow h^*\hat{x}_-,$$

i.e., the unique lift of the horizontal morphism $f: x \rightarrow y$ with target $h^*\hat{x}_-$,

- a vertical morphism $(u, \eta): (x, g) \dashrightarrow (x', g')$ in \mathbb{C}/\hat{x} to the vertical morphism in \mathbb{E}

$$\eta^*e_{\hat{x}_-}: g^*\hat{x}_- \dashrightarrow g'^*\hat{x}_-,$$

i.e., the source of the unique lift of the square $\eta: [u \begin{smallmatrix} g \\ g' \end{smallmatrix} e_{\hat{x}}]$ with target $e_{\hat{x}_-}$,

- a square $\alpha: [(u, \eta) \begin{smallmatrix} f \\ f' \end{smallmatrix} (v, \theta)]$ in \mathbb{C}/\hat{x} to the square in \mathbb{E}

$$P^*\alpha: \left[\eta^*e_{\hat{x}_-} = \alpha^*\theta^*e_{\hat{x}_-} \begin{smallmatrix} P^*f \\ P^*f' \end{smallmatrix} \theta^*e_{\hat{x}_-} \right],$$

i.e., the unique lift of the square $\alpha: [u \begin{smallmatrix} f \\ f' \end{smallmatrix} v]$ with target $\theta^*e_{\hat{x}_-}$.

PROOF. By construction, the functor Φ is given by the composite

$$P^{-1}\hat{x} \xrightarrow{\Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}} \mathcal{P}_{\mathbb{C}}(\mathbb{C}(-, \hat{x}), \mathfrak{D}_{\mathbb{C}}(P)) \xrightarrow{\mathbb{J}_{\mathbb{C}}} \text{DFib}(\mathbb{C})(\mathbb{C}/\hat{x}, \mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P)) \xrightarrow{(\eta_P^{-1})^*} \text{DFib}(\mathbb{C})(\mathbb{C}/\hat{x}, P).$$

and therefore $\Phi(\hat{x}_-) = \eta_P^{-1}(\mathbb{J}_{\mathbb{C}}(\Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-)))$. By Construction 4.5, the horizontal transformation

$$\Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-): \mathbb{C}(-, \hat{x}) \Rightarrow \mathfrak{D}_{\mathbb{C}}(P)$$

is defined as follows.

- Its component at an object x in \mathbb{C} is the functor $\mathbb{C}(x, \hat{x}) \rightarrow \mathfrak{D}_{\mathbb{C}}(P)x = P^{-1}x$ which sends
 - a horizontal morphism $g: x \rightarrow \hat{x}$ in \mathbb{C} to the object $\mathfrak{D}_{\mathbb{C}}(P)g(\hat{x}_-) = g^*\hat{x}_-$ in $P^{-1}x$, i.e., the source of the unique lift of g with target \hat{x}_- ,
 - a globular square $\eta: [e_x \xrightarrow{g} e_{\hat{x}}]$ to the vertical morphism $(\mathfrak{D}_{\mathbb{C}}(P)\eta)_{\hat{x}_-} = \eta^*e_{\hat{x}_-}$ in $P^{-1}x$, i.e., the source of the unique lift of η with target $e_{\hat{x}_-}$.
- Its component at a vertical morphism $u: x \rightarrow x'$ in \mathbb{C} is the natural transformation

$$\Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-)_u: \mathbb{C}(u, \hat{x}) \Rightarrow \mathfrak{D}_{\mathbb{C}}(P)u \circ (\Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-)_x^{\text{op}} \times \Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-)_x'),$$

whose component at an object $(x \xrightarrow{g} \hat{x}, x' \xrightarrow{g'} \hat{x})$ of $\mathbb{C}(x, \hat{x}) \times \mathbb{C}(x', \hat{x})$ is given by the map

$$\mathbb{C}(u, \hat{x})(g, g') \rightarrow \mathfrak{D}_{\mathbb{C}}(P)u(g^*\hat{x}_-, g'^*\hat{x}_-) = \{g^*\hat{x}_- \xrightarrow{u_-} g'^*\hat{x}_- \mid Pu_- = u\},$$

sending a square $\eta: [u \xrightarrow{g} e_{\hat{x}}]$ in \mathbb{C} to the vertical morphism $(\mathfrak{D}_{\mathbb{C}}(P)\eta)_{\hat{x}_-, \hat{x}_-}(1_{\hat{x}_-}) = \eta^*e_{\hat{x}_-}$, i.e., the source of the unique lift of η with target $e_{\hat{x}_-}$.

Next, using the description from Construction 6.5, the double functor

$$\mathbb{J}_{\mathbb{C}} \Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-): \mathbb{J}_{\mathbb{C}} \mathbb{C}(-, \hat{x}) = \mathbb{C}/\hat{x} \rightarrow \mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P)$$

is given by sending

- an object (x, g) of \mathbb{C}/\hat{x} to the object $(x, g^*\hat{x}_-)$ in $\mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P)$,
- a horizontal morphism f in \mathbb{C}/\hat{x} to the corresponding horizontal morphism f in $\mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P)$,
- a vertical morphism $(u, \eta): (x, g) \rightarrow (x', g')$ in \mathbb{C}/\hat{x} to the vertical morphism in $\mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P)$

$$(u, \eta^*e_{\hat{x}_-}): (x, g^*\hat{x}_-) \rightarrow (x', g'^*\hat{x}_-),$$
- a square α in \mathbb{C}/\hat{x} to the corresponding square α in $\mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P)$.

Finally, applying the inverse $\eta_P^{-1}: \mathbb{J}_{\mathbb{C}} \mathfrak{D}_{\mathbb{C}}(P) \rightarrow \mathbb{E}$ of the unit as described in Proposition 6.29, we can observe that the double functor $\Phi(\hat{x}_-) = \eta_P^{-1}(\mathbb{J}_{\mathbb{C}} \Phi_{\hat{x}, \mathfrak{D}_{\mathbb{C}}(P)}(\hat{x}_-)): \mathbb{C}/\hat{x} \rightarrow \mathbb{E}$ is as described in the statement. ■

Finally, we can prove that double terminal objects detect representability. For this, we first prove the following result.

7.6. LEMMA. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and an object \hat{x}_- in \mathbb{E} with $\hat{x} := P(\hat{x}_-)$, then the induced double functor between slices*

$$P_{/\hat{x}_-}: \mathbb{E}/\hat{x}_- \rightarrow \mathbb{C}/\hat{x}$$

is an isomorphism of double categories.

PROOF. By definition, the double functor $P_{/\hat{x}_-}$ fits into a commutative square in DblCat

$$\begin{array}{ccc} \mathbb{E}/\hat{x}_- & \xrightarrow{P_{/\hat{x}_-}} & \mathbb{C}/\hat{x} \\ \downarrow & & \downarrow \\ \mathbb{E} & \xrightarrow{P} & \mathbb{C} \end{array}$$

where the vertical canonical projections are discrete double fibrations by Proposition 6.4. Hence $P_{/\hat{x}_-}$ is a morphism of the discrete double fibrations over \mathbb{C} . Therefore, to show that it is an isomorphism, by Lemma 5.9, it suffices to show that the induced functor

$$\text{Ver}_0(P_{/\hat{x}_-}): \text{Ver}_0(\mathbb{E}/\hat{x}_-) \rightarrow \text{Ver}_0(\mathbb{C}/\hat{x})$$

is an isomorphism of categories.

We construct an inverse of $\text{Ver}_0(P_{/\hat{x}_-})$

$$L: \text{Ver}_0(\mathbb{C}/\hat{x}) \rightarrow \text{Ver}_0(\mathbb{E}/\hat{x}_-)$$

as follows. It is the functor sending

- an object (x, g) of \mathbb{C}/\hat{x} to the object $(g^*\hat{x}_-, P^*g)$ of \mathbb{E}/\hat{x}_- , where $P^*g: g^*\hat{x}_- \rightarrow \hat{x}_-$ denotes the unique lift of the horizontal morphism $g: x \rightarrow \hat{x}$ in \mathbb{C} ,
- a vertical morphism $(u, \eta): (x, g) \dashrightarrow (x', g')$ in \mathbb{C}/\hat{x} to the vertical morphism in \mathbb{E}/\hat{x}_-

$$(\eta^*e_{\hat{x}_-}, P^*\eta): (g^*\hat{x}_-, P^*g) \dashrightarrow (g'^*\hat{x}_-, P^*g'),$$

where $P^*\eta: \left[\eta^*e_{\hat{x}_-} \begin{array}{c} P^*g \\ P^*g' \end{array} e_{\hat{x}_-} \right]$ denotes the unique lift of the square $\eta: \left[u \begin{array}{c} g \\ g' \end{array} e_{\hat{x}} \right]$.

Clearly, the functor L defines an inverse of $\text{Ver}_0(P_{/\hat{x}_-})$ by unicity of lifts. ■

7.7. THEOREM. *Given a discrete double fibration $P: \mathbb{E} \rightarrow \mathbb{C}$ and an object \hat{x}_- in \mathbb{E} with $\hat{x} := P(\hat{x}_-)$, then the object \hat{x}_- is double terminal in \mathbb{E} if and only if the double functor*

$$\Phi(\hat{x}_-): \mathbb{C}/\hat{x} \rightarrow \mathbb{E}$$

is an isomorphism of double categories.

PROOF. We first show that the following triangle of double functors commutes.

$$\begin{array}{ccc} \mathbb{E}/\hat{x}_- & \longrightarrow & \mathbb{E} \\ P_{/\hat{x}_-} \downarrow & \nearrow \Phi(\hat{x}_-) & \\ \mathbb{C}/\hat{x} & & \end{array}$$

Unpacking the description of $\Phi(\hat{x}_-)$ from Lemma 7.5 and using unicity of lifts, we note that the composition $\Phi(\hat{x}_-) \circ P_{/\hat{x}_-}$ sends

- an object $(x_-, x_- \xrightarrow{g_-} \hat{x}_-)$ in \mathbb{E}/\hat{x}_- to the object in \mathbb{E}

$$\Phi(\hat{x}_-)(P_{/\hat{x}_-}(x_-, g_-)) = \Phi(\hat{x}_-)(Px_-, Pg_-) = (Pg_-)^* \hat{x}_- = x_-,$$

- a horizontal morphism $f_-: (x_-, g_-) \rightarrow (y_-, h_-)$ in \mathbb{E}/\hat{x}_- to the horizontal morphism in \mathbb{E}

$$\Phi(\hat{x}_-)(P_{/\hat{x}_-}(f_-)) = \Phi(\hat{x}_-)(Pf_-) = P^*(Pf_-) = f_-,$$

- a vertical morphism $(u_-, \eta_-): (x_-, g_-) \rightarrow (x'_-, g'_-)$ in \mathbb{E}/\hat{x}_- to the vertical morphism in \mathbb{E}

$$\Phi(\hat{x}_-)(P_{/\hat{x}_-}(u_-, \eta_-)) = \Phi(\hat{x}_-)(Pu_-, P\eta_-) = (P\eta_-)^* e_{\hat{x}_-} = u_-,$$

- a square $\alpha_-: \left[(u_-, \eta_-) \begin{array}{c} f_- \\ f'_- \end{array} (v_-, \theta_-) \right]$ to the square in \mathbb{E}

$$\Phi(\hat{x}_-)(P_{/\hat{x}_-}(\alpha_-)) = \Phi(\hat{x}_-)(P\alpha_-) = P^*(P\alpha_-) = \alpha_-.$$

Hence the composite $\Phi(\hat{x}_-) \circ P_{/\hat{x}_-}$ coincides with the canonical projection $\mathbb{E}/\hat{x} \rightarrow \mathbb{E}$, as desired.

Now, by Lemma 7.6, the double functor $P_{/\hat{x}_-}$ is an isomorphism of double categories. Therefore, by 2-out-of-3 for isomorphisms, the double functor $\Phi(\hat{x}_-): \mathbb{C}/\hat{x} \rightarrow \mathbb{E}$ is an isomorphism if and only if the canonical projection $\mathbb{E}/\hat{x}_- \rightarrow \mathbb{E}$ is an isomorphism. However, by definition, this means that the object \hat{x}_- is double terminal in \mathbb{E} . \blacksquare

REPRESENTATION THEOREM. Finally, we state our representation theorem. For this, we first introduce the notion of a represented lax double presheaf.

7.8. DEFINITION. A lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ is *represented by an object \hat{x} in \mathbb{C}* if there is an object \hat{x}_- in $X\hat{x}$ such that the induced horizontal transformation

$$\Phi_{\hat{x}, X}(\hat{x}_-): \mathbb{C}(-, \hat{x}) \Rightarrow X$$

from Construction 4.5 is invertible. In this case, we also say that X is *represented by (\hat{x}, \hat{x}_-)* .

7.9. THEOREM. *Given a lax double presheaf $X: \mathbb{C}^{\text{op}} \rightarrow \text{Cat}$ and objects \hat{x} in \mathbb{C} and \hat{x}_- in $X\hat{x}$, then X is represented by (\hat{x}, \hat{x}_-) if and only if the object (\hat{x}, \hat{x}_-) is double terminal in the Grothendieck construction $\int_{\mathbb{C}} X$.*

PROOF. By definition, the lax double presheaf X is represented by (\hat{x}, \hat{x}_-) if and only if the induced horizontal transformation

$$\Phi_{\hat{x}, X}(\hat{x}_-): \mathbb{C}(-, \hat{x}) \Rightarrow X$$

is invertible. By applying the 2-equivalence $\int_{\mathbb{C}}: \mathcal{P}_{\mathbb{C}} \rightarrow \text{DFib}(\mathbb{C})$ from Theorem 6.13, this is the case if and only if the induced morphism of discrete double fibrations over \mathbb{C}

$$\int_{\mathbb{C}} \Phi_{\hat{x}, X}(\hat{x}_-): \int_{\mathbb{C}} \mathbb{C}(-, \hat{x}) = \mathbb{C}/\hat{x} \rightarrow \int_{\mathbb{C}} X$$

is an isomorphism. Noticing that $\Phi(\hat{x}, \hat{x}_-) = \int_{\mathbb{C}} \Phi_{\hat{x}, X}(\hat{x}_-)$ using Lemma 7.5, it follows from Theorem 7.7 that this is equivalent to the object (\hat{x}, \hat{x}_-) being double terminal in $\int_{\mathbb{C}} X$, as desired. ■

As a consequence, we get that the double slice always has a double terminal object given by the identity.

7.10. COROLLARY. *The object $(\hat{x}, 1_{\hat{x}})$ is double terminal in \mathbb{C}/\hat{x} .*

PROOF. As $\Phi_{\hat{x}, \mathbb{C}(-, \hat{x})}(1_{\hat{x}})$ is the identity horizontal transformation $\mathbb{C}(-, \hat{x}) \Rightarrow \mathbb{C}(-, \hat{x})$ by functoriality of $\Phi_{\hat{x}, \mathbb{C}(-, \hat{x})}$, the lax double presheaf $\mathbb{C}(-, \hat{x})$ is represented by $(\hat{x}, 1_{\hat{x}})$. Then, by Theorem 7.9, the object $(\hat{x}, 1_{\hat{x}})$ is double terminal in the double category $\int_{\mathbb{C}} \mathbb{C}(-, \hat{x}) = \mathbb{C}/\hat{x}$, where the last equality holds by Proposition 6.3. ■

Finally, as a consequence of Lemma 3.32 and Theorem 7.9, when taking $\mathbb{C} = \mathbb{HC}$ with \mathcal{C} a 2-category, we retrieve the representation theorem for 2-presheaves from [MSV23, Theorem 6.12] in the case of $\mathcal{V} = \text{Cat}$. This is a stricter version of [cM22b, Theorem 6.8].

7.11. COROLLARY. *Given a 2-presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ and objects \hat{x} in \mathcal{C} and \hat{x}_- in $X\hat{x}$, then X is represented by (\hat{x}, \hat{x}_-) if and only if the object (\hat{x}, \hat{x}_-) is double terminal in the Grothendieck construction $\int_{\mathbb{HC}} X$.*

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