Theory and Applications of Categories, Vol. 41, No. 48, 2024, pp. 1679–1697.

ENRICHED MORITA THEORY OF MONOIDS IN A CLOSED SYMMETRIC MONOIDAL CATEGORY

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ABSTRACT. We develop Morita theory of monoids in a closed symmetric monoidal category, in the context of enriched category theory.

1. Introduction

Let R, R' be rings. The Eilenberg-Watts theorem [4], [12] states that every cocontinuous functor $\mathcal{F} : Mod_R \to Mod_{R'}$ between the categories of right modules is naturally isomorphic to the functor $- \otimes_R {}_RM_{R'}$ of taking tensor product over R for some (R, R')-bimodule ${}_RM_{R'}$. We say R, R' are Morita equivalent if we have an equivalence of categories between Mod_R and $Mod_{R'}$. The main theorem of Morita theory [9] states that the following are equivalent:

- Rings R, R' are Morita equivalent;
- There exists a finitely generated projective generator $P_{R'}$ in $Mod_{R'}$ together with an isomorphism of rings $R \cong End_{R'}(P_{R'})$;
- There exists an (R, R')-bimodule ${}_RM_{R'}$ and an (R', R)-bimodule ${}_{R'}N_R$ together with isomorphisms of bimodules ${}_RM_{R'} \otimes_{R'} {}_{R'}N_R \cong {}_RR_R$ and ${}_{R'}N_R \otimes_R {}_RM_{R'} \cong {}_{R'}R'_{R'}$.

We generalize these results in the context of enriched category theory. We begin by establishing the Eilenberg-Watts theorem in an enriched context. We follow the approach introduced by Mark Hovey in [5, §1-2] using tensorial strengths of enriched functors between tensored enriched categories. After establishing the Eilenberg-Watts theorem, we provide a theorem which characterizes when an enriched category is equivalent to the enriched category of right modules over the given monoid of the base category. As a corollary, we obtain the main result of Morita in enriched context.

The base category that we consider in this paper is a closed symmetric monoidal category $\mathscr{C} = (\mathcal{C}, \otimes, c, [-,-])$ whose underlying category $\mathcal{C}$ is finitely complete and finitely

The authors are grateful to Emily Riehl and anonymous reviewers for their comments, suggestions and careful reading of our paper. This paper is supported by Basic Science Research Institute Fund, whose NRF grant number is 2021R1A6A1A10042944.

Received by the editors 2023-09-16 and, in final form, 2024-10-25.

Transmitted by Clemens Berger. Published on 2024-10-29.

2020 Mathematics Subject Classification: 18D20.

Key words and phrases: Morita theory, Eilenberg-Watts theorem, enriched category, tensorial strength.

cocomplete. Some examples are the closed symmetric monoidal categories $\mathcal{S}et/f\mathcal{S}et/s\mathcal{S}et$ of small sets/finite sets/simplicial sets, $\mathcal{C}at$ of small categories, $\mathcal{A}b/f\mathcal{A}b$ of abelian groups/finitely generated abelian groups, $\mathcal{V}ec_K/f\mathcal{V}ec_K$ of vector spaces/finite dimensional vector spaces over a field K, $\mathcal{M}od_R/\widehat{\mathcal{M}od}_R/dg\mathcal{M}od_R$ of modules/L-complete modules/dg-modules over a commutative ring R, $\mathcal{CGT}/\mathcal{CGT}_*$ of unbased/based compactly generated topological spaces, $\mathcal{S}p^{\Sigma}_{CGT_*}$ of topological symmetric spectra, $\mathcal{CGWH}/\mathcal{CGWH}_*$ of unbased/based compactly generated weakly Hausdorff spaces, $\mathcal{B}an$ of Banach spaces with linear contractions. Every elementary topos is also an example.

We explain our main ideas and results. A monoid in $\mathcal{C}$ is a triple $\mathfrak{b} = (b, u_b, m_b)$ where b is an object in $\mathcal{C}$ and u_b, m_b are the unit, product morphisms in $\mathcal{C}$. For each monoid $\mathfrak{b}$ in $\mathcal{C}$, we denote $\mathcal{M}od_{\mathfrak{b}}$ as the $\mathcal{C}$-enriched category of right $\mathfrak{b}$-modules. We can see b as a right $\mathfrak{b}$-module which we denote as $b_{\mathfrak{b}}$. Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. For each $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$, the object $\mathcal{F}(b_{\mathfrak{b}})$ in $\mathcal{D}$ is equipped with a left action of $\mathfrak{b}$, and we have the $\mathcal{C}$-enriched left adjoint functor

$$- \circledast_{\mathfrak{b}\,\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$$

of taking tensor product over $\mathfrak{b}$. We show that there is a canonical $\mathcal{C}$-enriched natural transformation

$$\lambda^{\mathcal{F}} :\ - \circledast_{\mathfrak{b}\,\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \Longrightarrow \mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D} \tag{1.1}$$

associated to $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ (Lemma 3.1). This was defined in [5, Proposition 1.1] as an ordinary natural transformation when $\mathcal{D} = \mathcal{M}od_{\mathfrak{b}'}$ for another monoid $\mathfrak{b}'$ in $\mathcal{C}$. Moreover, we show that the following are equivalent (Proposition 3.2):

- $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ is a $\mathcal{C}$-enriched left adjoint;
- $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ is $\mathcal{C}$-enriched cocontinuous;
- $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ preserves $\mathcal{C}$-tensors, i.e. its tensorial strength is invertible, and the underlying functor $\mathcal{F}_0$ preserves coequalizers;
- The $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}} : - \circledast_{\mathfrak{b}\,\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \Rightarrow \mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ in (1.1) is invertible.

Using this result, we prove the following generalization of the Eilenberg-Watts theorem. Left $\mathfrak{b}$-module objects in $\mathcal{D}$ are introduced in § 2.4.

1.1. THEOREM. *Let $\mathfrak{b}$ be a monoid in $\mathcal{C}$ and let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. We have a fully faithful left adjoint functor*

$${}_{\mathfrak{b}}\mathcal{D} \longrightarrow \mathcal{C}\text{-}Funct(\mathcal{M}od_{\mathfrak{b}}, \mathcal{D}) \tag{1.2}$$

from the category of left $\mathfrak{b}$-modules objects in $\mathcal{D}$ to the category of $\mathcal{C}$-enriched functors $\mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$. The essential image of the left adjoint functor (1.2) is the coreflective full *subcategory $\mathcal{C}$-$Funct_{cocon}(\mathcal{M}od_{\mathfrak{b}}, \mathcal{D})$ of cocontinuous $\mathcal{C}$-enriched functors $\mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$, and we have an adjoint equivalence of categories*

$$ {}_{\mathfrak{b}}\mathcal{D} \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathcal{C}\text{-}Funct_{cocon}(\mathcal{M}od_{\mathfrak{b}}, \mathcal{D}). $$

The coreflection of a $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ is the associated $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}}$ in (1.1).

Let us explain why Theorem 1.1 can be seen as a generalization of the Eilenberg-Watts theorem. Given another monoid $\mathfrak{b}'$ in $\mathcal{C}$, we define a $(\mathfrak{b}, \mathfrak{b}')$-bimodule as a left $\mathfrak{b}$-module object in $\mathcal{M}od_{\mathfrak{b}'}$ (Definition 2.6). After substituting $\mathcal{D} = \mathcal{M}od_{\mathfrak{b}'}$ in Theorem 1.1, we obtain the following corollary.

1.2. Corollary. *Let $\mathfrak{b}$, $\mathfrak{b}'$ be monoids in $\mathcal{C}$. We have an adjoint equivalence of categories*

$$ {}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'} \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathcal{C}\text{-}Funct_{cocon}(\mathcal{M}od_{\mathfrak{b}}, \mathcal{M}od_{\mathfrak{b}'}) $$

from the category of $(\mathfrak{b}, \mathfrak{b}')$-bimodules to the category of cocontinuous $\mathcal{C}$-enriched functors $\mathcal{M}od_{\mathfrak{b}} \to \mathcal{M}od_{\mathfrak{b}'}$.

The original Eilenberg-Watts theorem [4], [12] states that the functor from left to right in Corollary 1.2 is essentially surjective when $\mathcal{C} = \mathcal{A}b$. This has been generalized to the situation where the target category is a general tensored $\mathcal{A}b$-enriched category by Nyman and Smith [10]. The main result of their article is precisely our Theorem 1.1 in the special case $\mathcal{C} = \mathcal{A}b$. We mention that Corollary 1.2 has been discussed online when $\mathcal{C}$ is a Bénabou cosmos. [1]

In the original Eilenberg-Watts theorem, we only assume the cocontinuity of the underlying functor (i.e., preservation of sums and coequalizers). In a general $\mathcal{C}$-enriched setting this is not enough, and we use preservation of $\mathcal{C}$-tensors which is a more restrictive condition than preservation of sums. The reason why the weaker assumption is enough in the case of $\mathcal{C} = \mathcal{A}b$ is the following special property of abelian module categories: any natural transformation between cocontinuous functors out of an abelian module category is invertible as soon as it is invertible at a projective generator.

Next, we characterize when a $\mathcal{C}$-enriched category $\mathcal{D}$ is equivalent to $\mathcal{M}od_{\mathfrak{b}}$. We say an object X in a $\mathcal{C}$-enriched category $\mathcal{D}$ is a *$\mathcal{C}$-enriched compact generator* if the $\mathcal{C}$-enriched Hom functor $\mathcal{D}(X, -) : \mathcal{D} \to \mathcal{C}$ is conservative, preserves $\mathcal{C}$-tensors and the underlying functor $\mathcal{D}(X, -)_0$ preserves coequalizers (Definition 4.1).

1.3. Theorem. *Let $\mathfrak{b}$ be a monoid in $\mathcal{C}$, and let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. Then $\mathcal{D}$ is equivalent to $\mathcal{M}od_{\mathfrak{b}}$ as $\mathcal{C}$-enriched categories if and only if there exists a $\mathcal{C}$-enriched compact generator X in $\mathcal{D}$ inducing an isomorphism of monoids $f : \mathfrak{b} \cong End_{\mathcal{D}}(X)$ in $\mathcal{C}$.*

[1] See https://mathoverflow.net/questions/159735/in-what-generality-does-eilenberg-watts-hold and https://ncatlab.org/nlab/show/Eilenberg-Watts+theorem for the discussions of Corollary 1.2 over a Bénabou cosmos $\mathcal{C}$.

Using Theorem 1.1 and Theorem 1.3, we establish the main theorem of Morita theory in enriched context. We say monoids $\mathfrak{b}$ and $\mathfrak{b}'$ in $\mathcal{C}$ are *Morita equivalent* if $\mathcal{M}od_{\mathfrak{b}}$ and $\mathcal{M}od_{\mathfrak{b}'}$ are equivalent as $\mathcal{C}$-enriched categories.

1.4. Corollary. *Let $\mathfrak{b}$, $\mathfrak{b}'$ be monoids in $\mathcal{C}$. The following are equivalent:*

(i) Monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are Morita equivalent;

(ii) There exists a $\mathcal{C}$-enriched compact generator $x_{\mathfrak{b}'}$ in $\mathcal{M}od_{\mathfrak{b}'}$ together with an isomorphism of monoids $\mathfrak{b} \cong End_{\mathcal{M}od_{\mathfrak{b}'}}(x_{\mathfrak{b}'})$ in $\mathcal{C}$;

(iii) There exists a $(\mathfrak{b}, \mathfrak{b}')$-bimodule ${}_{\mathfrak{b}}x_{\mathfrak{b}'}$ and a $(\mathfrak{b}', \mathfrak{b})$-bimodule ${}_{\mathfrak{b}'}y_{\mathfrak{b}}$ together with isomorphisms of bimodules ${}_{\mathfrak{b}}x_{\mathfrak{b}'} \circledast_{\mathfrak{b}'} {}_{\mathfrak{b}'}y_{\mathfrak{b}} \cong {}_{\mathfrak{b}}b_{\mathfrak{b}}$ and ${}_{\mathfrak{b}'}y_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}x_{\mathfrak{b}'} \cong {}_{\mathfrak{b}'}b'_{\mathfrak{b}'}$.

If we consider $\mathcal{C} = \mathcal{A}b$ in Corollary 1.4, we recover the original result of Morita.

2. Enriched Categories

We fix a closed symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, c, [-,-])$ whose underlying category $\mathcal{C}$ is finitely complete and finitely cocomplete. We denote objects in $\mathcal{C}$ with small letters. Let z, x, y be objects in $\mathcal{C}$. We have the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and the unit object c in $\mathcal{C}$, together with coherence isomorphisms

$$\begin{aligned} a_{z,x,y} &: z\otimes(x\otimes y) \xrightarrow{\cong} (z\otimes x)\otimes y, & \imath_x &: c\otimes x \xrightarrow{\cong} x, \\ s_{x,y} &: x\otimes y \xrightarrow{\cong} y\otimes x, & \jmath_x &: x\otimes c \xrightarrow{\cong} x \end{aligned} \tag{2.1}$$

in $\mathcal{C}$ that are natural in variables z, x, y. For each object x in $\mathcal{C}$, the functor $-\otimes x : \mathcal{C} \to \mathcal{C}$ admits a right adjoint $[x,-] : \mathcal{C} \to \mathcal{C}$ and we have a functor $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.

We refer [3], [6] for the basics of enriched category theory. Let $\mathcal{D}$ be a $\mathcal{C}$-enriched category and let X, Y, Z be objects in $\mathcal{D}$. We denote $\mathcal{D}(X,Y)$ as the Hom object and $\mathbb{I}_X : c \to \mathcal{D}(X,X)$, $\mu_{X,Y,Z} : \mathcal{D}(Y,Z) \otimes \mathcal{D}(X,Y) \to \mathcal{D}(X,Z)$ as the identity, composition morphisms in $\mathcal{C}$. We denote $\mathcal{D}_0$ as the underlying category of $\mathcal{D}$. A morphism $X \to Y$ in $\mathcal{D}$ means a morphism from X to Y in the underlying category $\mathcal{D}_0$ of $\mathcal{D}$. We denote $\mathbb{I}_X : X \xrightarrow{\cong} X$ as the identity morphism $\mathbb{I}_X : c \to \mathcal{D}(X,X)$ of X in $\mathcal{D}$. For each morphism $l : X \to Y$ in $\mathcal{D}$, we have morphisms $l_\star : \mathcal{D}(Z,X) \to \mathcal{D}(Z,Y)$ and $l^\star : \mathcal{D}(Y,Z) \to \mathcal{D}(X,Z)$ in $\mathcal{C}$.

The category $\mathcal{C}$ has a canonical $\mathcal{C}$-enriched category structure whose Hom objects are given by $\mathcal{C}(x,y) = [x,y]$. We identify the underlying category of the $\mathcal{C}$-enriched category $\mathcal{C}$ with the original category $\mathcal{C}$.

Let $\mathcal{D}'$ be another $\mathcal{C}$-enriched category. For each $\mathcal{C}$-enriched functor $\alpha : \mathcal{D} \to \mathcal{D}'$, we have the underlying functor $\alpha_0 : \mathcal{D}_0 \to \mathcal{D}'_0$ and we denote $\alpha_{X,Y} : \mathcal{D}(X,Y) \to \mathcal{D}'(\alpha(X), \alpha(Y))$ as the morphism between Hom objects. We write $I_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ as the identity $\mathcal{C}$-enriched functor of $\mathcal{D}$. Let $\beta : \mathcal{D} \to \mathcal{D}'$ be another $\mathcal{C}$-enriched functor from

$\mathcal{D}$ to $\mathcal{D}'$. For each $\mathcal{C}$-enriched natural transformation $\xi : \alpha \Rightarrow \beta : \mathcal{D} \to \mathcal{D}'$, we have the underlying natural transformation $\xi_0 : \alpha_0 \Rightarrow \beta_0 : \mathcal{D}_0 \to \mathcal{D}'_0$ whose component at each object X in $\mathcal{D}$ is $(\xi_0)_X = \xi_X : \alpha(X) \to \beta(X)$. We denote $\mathcal{C}$-*Funct*$(\mathcal{D}, \mathcal{D}')$ as the category of $\mathcal{C}$-enriched functors from $\mathcal{D}$ to $\mathcal{D}'$.

2.1. Tensored enriched categories and tensorial strengths. We say a $\mathcal{C}$-enriched category $\mathcal{D}$ is *tensored* if for each object X in $\mathcal{D}$, the $\mathcal{C}$-enriched Hom functor $\mathcal{D}(X, -) : \mathcal{D} \to \mathcal{C}$ admits a left adjoint $\mathcal{C}$-enriched functor $- \circledast X : \mathcal{C} \to \mathcal{D}$. We denote the components of the unit, counit of the $\mathcal{C}$-enriched adjunction $- \circledast X \dashv \mathcal{D}(X, -)$ at $z \in \mathrm{Obj}(\mathcal{C})$, $Y \in \mathrm{Obj}(\mathcal{D})$ as

$$\mathcal{C} \underset{\mathcal{D}(X,-)}{\overset{-\circledast X}{\rightleftarrows}} \mathcal{D} \qquad Cv_{z,X} : z \longrightarrow \mathcal{D}(X, z \circledast X), \quad Ev_{X,Y} : \mathcal{D}(X, Y) \circledast Y \longrightarrow X.$$

For each morphism $l : z \circledast X \to Y$ in $\mathcal{D}$, we denote the corresponding morphism in $\mathcal{C}$ as $\bar{l} : z \to \mathcal{D}(X, Y)$ and call it as the *right adjunct* of l. We have a unique isomorphism $\imath_X : c \circledast X \xrightarrow{\cong} X$ in $\mathcal{D}$ whose right adjunct is the morphism $\bar{\imath}_X = \mathbb{I}_X : c \to \mathcal{D}(X, X)$ in $\mathcal{C}$.

Let $\mathcal{D}$, $\mathcal{D}'$ be tensored $\mathcal{C}$-enriched categories and let $z \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Obj}(\mathcal{D})$. For each $\mathcal{C}$-enriched functor $\beta : \mathcal{D} \to \mathcal{D}'$, the *tensorial strength associated to* β *at* z, X is a morphism $t^{\beta}_{z,X} : z \circledast \beta(X) \to \beta(z \circledast X)$ in $\mathcal{D}'$ defined as follows:

$$\begin{aligned} t^{\beta}_{z,X} &:= Ev_{\beta(X),\beta(z\circledast X)} \circ (\beta_{X,z\circledast X} \circledast \mathbb{I}_{\beta(X)}) \circ (Cv_{z,X} \circledast \mathbb{I}_{\beta(X)}) \\ &: z \circledast \beta(X) \longrightarrow \mathcal{D}(X, z \circledast X) \circledast \beta(X) \longrightarrow \mathcal{D}'\big(\beta(X), \beta(z \circledast X)\big) \circledast \beta(X) \longrightarrow \beta(z \circledast X). \end{aligned}$$

We say the $\mathcal{C}$-functor $\beta : \mathcal{D} \to \mathcal{D}'$ *preserves* $\mathcal{C}$*-tensors* if the tensorial strength $t^{\beta}_{z,X}$ associated to β is an isomorphism in $\mathcal{D}'$ for every pair z, X.

2.2. Example. *The $\mathcal{C}$-enriched category $\mathcal{C}$ is tensored. Let z, x, $y \in \mathrm{Obj}(\mathcal{C})$. The tensored object of x, y in $\mathcal{C}$ is $x \circledast y = x \otimes y$. Moreover,*

- *the coherence isomorphism $\imath_x : c \otimes x \xrightarrow{\cong} x$ in (2.1) corresponds to the unique isomorphism $\imath_x : c \circledast x \xrightarrow{\cong} x$ in $\mathcal{C}$;*
- *the coherence isomorphism $a_{z,x,y} : z \otimes (x \otimes y) \xrightarrow{\cong} (z \otimes x) \otimes y$ in (2.1) corresponds to the tensorial strength $t^{-\circledast y}_{z,x} : z \circledast (x \circledast y) \xrightarrow{\cong} (z \circledast x) \circledast y$ associated to the $\mathcal{C}$-enriched functor $- \circledast y : \mathcal{C} \to \mathcal{C}$ at z, x.*

Let x, y be objects in $\mathcal{C}$. Throughout this paper, we identify the object $x \otimes y$ in $\mathcal{C}$ with the tensored object $x \circledast y$ in $\mathcal{C}$. For instance, given a monoid $\mathfrak{b} = (b, u_b, m_b)$ in $\mathcal{C}$, we denote the product morphism as $m_b : b \circledast b \to b$.

Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category. For each object X in $\mathcal{D}$, the $\mathcal{C}$-enriched functor $- \circledast X : \mathcal{C} \to \mathcal{D}$ preserves $\mathcal{C}$-tensors. We denote the associated tensorial strength as

$$a_{w,z,X} := t^{-\circledast X}_{w,z} : \ w \circledast (z \circledast X) \xrightarrow{\ \cong\ } (w \circledast z) \circledast X, \qquad \forall w, z \in \mathrm{Obj}(\mathcal{C}).$$

We often omit this isomorphism and simply denote $w \circledast z \circledast X \in \mathrm{Obj}(\mathcal{D})$.

2.3. Example. *Let $\mathfrak{b} = (b, u_b, m_b)$ be a monoid in $\mathcal{C}$. We explain the tensored $\mathcal{C}$-enriched category $\mathcal{M}od_{\mathfrak{b}}$ of right $\mathfrak{b}$-modules. A* right $\mathfrak{b}$-module *is a pair $z_{\mathfrak{b}} = \big(z,\ z \circledast b \xrightarrow{\gamma_z} z\big)$ of an object z in $\mathcal{C}$, and a morphism $\gamma_z : z \circledast b \to z$ in $\mathcal{C}$ satisfying the right $\mathfrak{b}$-action relations. For instance, we have the right $\mathfrak{b}$-module $b_{\mathfrak{b}} := \big(b,\ b \circledast b \xrightarrow{\gamma_b = m_b} b\big)$. The Hom object between right $\mathfrak{b}$-modules $z_{\mathfrak{b}} = (z, \gamma_z)$ and $\tilde{z}_{\mathfrak{b}} = (\tilde{z}, \gamma_{\tilde{z}})$ is given by the equalizer*

$$\mathcal{M}od_{\mathfrak{b}}(z_{\mathfrak{b}}, \tilde{z}_{\mathfrak{b}}) \xhookrightarrow{\mathcal{U}_{z_{\mathfrak{b}},\tilde{z}_{\mathfrak{b}}}} \mathcal{C}(z, \tilde{z}) \underset{(\gamma_{\tilde{z}})_\star \circ (-\circledast b)_{z,\tilde{z}}}{\overset{(\gamma_z)^\star}{\rightrightarrows}} \mathcal{C}(z \circledast b, \tilde{z}). \tag{2.2}$$

The tensored object of $w \in \mathrm{Obj}(\mathcal{C})$ and $z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$ is the right $\mathfrak{b}$-module

$$w \circledast z_{\mathfrak{b}} = (w \circledast z,\ \gamma_{w \circledast z}), \qquad \gamma_{w \circledast z} = \mathbb{I}_w \circledast \gamma_z :\ w \circledast z \circledast b \longrightarrow w \circledast z.$$

For each right $\mathfrak{b}$-module $z_{\mathfrak{b}} = (z, \gamma_z)$, the morphism $\gamma_z : z \circledast b \to z$ in $\mathcal{C}$ becomes a morphism $\gamma_{z_{\mathfrak{b}}} : z \circledast b_{\mathfrak{b}} \to z_{\mathfrak{b}}$ in $\mathcal{M}od_{\mathfrak{b}}$. For instance, the morphism $\gamma_b = m_b : b \circledast b \to b$ in $\mathcal{C}$ becomes a morphism $\gamma_{b_{\mathfrak{b}}} : b \circledast b_{\mathfrak{b}} \to b_{\mathfrak{b}}$ in $\mathcal{M}od_{\mathfrak{b}}$. The underlying category $(\mathcal{M}od_{\mathfrak{b}})_0$ of $\mathcal{M}od_{\mathfrak{b}}$ has coequalizers. For each right $\mathfrak{b}$-module $z_{\mathfrak{b}} = (z, \gamma_z)$, we have the following coequalizer diagram in $(\mathcal{M}od_{\mathfrak{b}})_0$.

$$z \circledast b \circledast b_{\mathfrak{b}} \underset{\mathbb{I}_z \circledast \gamma_{b_{\mathfrak{b}}}}{\overset{\gamma_z \circledast \mathbb{I}_{b_{\mathfrak{b}}}}{\rightrightarrows}} z \circledast b_{\mathfrak{b}} \xrightarrow{\gamma_{z_{\mathfrak{b}}}} z_{\mathfrak{b}} \tag{2.3}$$

Let $\mathfrak{b}$ be a monoid in $\mathcal{C}$. We have the forgetful $\mathcal{C}$-enriched functor $\mathcal{U} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{C}$ whose morphism on Hom objects is given by the equalizer $\mathcal{U}_{z_{\mathfrak{b}},\tilde{z}_{\mathfrak{b}}} : \mathcal{M}od_{\mathfrak{b}}(z, \tilde{z}) \hookrightarrow \mathcal{C}(z, \tilde{z})$ defined in (2.2). The forgetful $\mathcal{C}$-enriched functor $\mathcal{U} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{C}$ preserves $\mathcal{C}$-tensors, as the associated tensorial strength at $w \in \mathrm{Obj}(\mathcal{C})$, $z_{\mathfrak{b}} = (z, \gamma_z) \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$ is the identity morphism $w \circledast z = w \circledast z$ in $\mathcal{C}$.

We introduce basic properties of tensorial strengths without proof. See [11, §3] for detailed explanations.

1. Let $\mathcal{D}$, $\mathcal{D}'$ be tensored $\mathcal{C}$-enriched categories and let $w, z \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Obj}(\mathcal{D})$. For each $\mathcal{C}$-enriched functor $\beta : \mathcal{D} \to \mathcal{D}'$, the tensorial strength associated to β satisfies the following relations.

$$\begin{array}{ccc} c \circledast \beta(X) & \xrightarrow{t^\beta_{c,X}} & \beta(c \circledast X) \\ & \underset{\imath_{\beta(X)}}{\searrow} \cong & \cong \downarrow \beta(\imath_X) \\ & & \beta(X) \end{array} \qquad \begin{array}{ccccc} w \circledast \big(z \circledast \beta(X)\big) & \xrightarrow{\mathbb{I}_w \circledast t^\beta_{z,X}} & w \circledast \beta(z \circledast X) & \xrightarrow{t^\beta_{w,z \circledast X}} & \beta\big(w \circledast (z \circledast X)\big) \\ a_{w,z,\beta(X)} \downarrow \cong & & & & \cong \downarrow \beta(a_{w,z,X}) \\ (w \circledast z) \circledast \beta(X) & & \xrightarrow{t^\beta_{w \circledast z, X}} & & \beta\big((w \circledast z) \circledast X\big) \end{array} \tag{2.4}$$

Conversely, suppose we have a functor $\mathcal{F}_0 : \mathcal{D}_0 \to \mathcal{D}'_0$ between the underlying categories of $\mathcal{D}$, $\mathcal{D}'$ together with a collection of morphisms in $\mathcal{D}'$

$$\big\{\ t_{z,X} : z \circledast \mathcal{F}_0(X) \to \mathcal{F}_0(z \circledast X)\ \big|\ z \in \mathrm{Obj}(\mathcal{C}),\ X \in \mathrm{Obj}(\mathcal{D})\ \big\}$$

which is natural in variables z, X and satisfies the relations (2.4). Then we have a unique $\mathcal{C}$-enriched functor $\beta : \mathcal{D} \to \mathcal{D}'$ whose underlying functor β_0 is equal to $\mathcal{F}_0$ and $t^{\beta}_{z,X} = t_{z,X}$ holds for every pair z, X.

2. Let α, $\beta : \mathcal{D} \to \mathcal{D}'$ be $\mathcal{C}$-enriched functors between tensored $\mathcal{C}$-enriched categories $\mathcal{D}$, $\mathcal{D}'$ and let $z \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Obj}(\mathcal{D})$. For each $\mathcal{C}$-enriched natural transformation $\xi : \alpha \Rightarrow \beta : \mathcal{D} \to \mathcal{D}'$, we have the following relation.

$$\begin{array}{ccc} z \circledast \alpha(X) & \xrightarrow{t^{\alpha}_{z,X}} & \alpha\big(z \circledast X\big) \\ {\scriptstyle \mathbb{I}_z \circledast \xi_X}\downarrow & & \downarrow{\scriptstyle \xi_{z \circledast X}} \\ z \circledast \beta(X) & \xrightarrow{t^{\beta}_{z,X}} & \beta\big(z \circledast X\big) \end{array} \tag{2.5}$$

Conversely, suppose we are given a natural transformation $\xi_0 : \alpha_0 \Rightarrow \beta_0 : \mathcal{D}_0 \to \mathcal{D}'_0$ between the underlying functors α_0, β_0. Then ξ_0 becomes a $\mathcal{C}$-enriched natural transformation $\xi : \alpha \Rightarrow \beta : \mathcal{D} \to \mathcal{D}'$ if and only if it satisfies the relation (2.5) for every pair z, X. This is precisely the correspondence between $\mathcal{C}$-enriched natural transformations and strong natural transformations, first introduced by Anders Kock in [7]. It is also explained in [2].

3. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$ be tensored $\mathcal{C}$-enriched categories and let $\mathcal{D} \xrightarrow{\beta} \mathcal{D}' \xrightarrow{\beta'} \mathcal{D}''$ be $\mathcal{C}$-enriched functors. The tensorial strength of the composition $\beta'\beta : \mathcal{D} \to \mathcal{D}''$ at $z \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Obj}(\mathcal{D})$ is given by

$$t^{\beta'\beta}_{z,X} = \beta'(t^{\beta}_{z,X}) \circ t^{\beta'}_{z,\beta(X)} : \; z \circledast \beta'\beta(X) \longrightarrow \beta'\big(z \circledast \beta(X)\big) \longrightarrow \beta'\beta(z \circledast X).$$

2.4. Left module objects. For the rest of this section, $\mathfrak{b} = (b, u_{\mathfrak{b}}, m_{\mathfrak{b}})$ is a monoid in $\mathcal{C}$.

2.5. Definition. *Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category. A* left $\mathfrak{b}$-module object in $\mathcal{D}$ *is a pair ${}_{\mathfrak{b}}X = \big(X,\, b \circledast X \xrightarrow{\rho_X} X\big)$ of an object X in $\mathcal{D}$, and a morphism $\rho_X : b \circledast X \to X$ in $\mathcal{D}$ satisfying the left $\mathfrak{b}$-action relations. A morphism ${}_{\mathfrak{b}}X \to {}_{\mathfrak{b}}\tilde{X}$ of left $\mathfrak{b}$-module objects in $\mathcal{D}$ is a morphism $X \to \tilde{X}$ in $\mathcal{D}$ which is compatible with the left $\mathfrak{b}$-action morphisms ρ_X, $\rho_{\tilde{X}}$. We denote*

$${}_{\mathfrak{b}}\mathcal{D}$$

as the category of left $\mathfrak{b}$-module objects in $\mathcal{D}$. We do not treat ${}_{\mathfrak{b}}\mathcal{D}$ as a $\mathcal{C}$-enriched category.

Let X be an object in a tensored $\mathcal{C}$-enriched category $\mathcal{D}$. Then the triple $End_{\mathcal{D}}(X) := (\mathcal{D}(X,X),\ \mathbb{I}_X,\ \mu_{X,X,X})$ is a monoid in $\mathcal{C}$. For each morphism $\rho_X : b \circledast X \to X$ in $\mathcal{D}$, the pair (X, ρ_X) is a left $\mathfrak{b}$-module object in $\mathcal{D}$ if and only if the right adjunct $\bar{\rho}_X : b \to \mathcal{D}(X,X)$ of ρ_X becomes a morphism $\bar{\rho}_X : \mathfrak{b} \to End_{\mathcal{D}}(X)$ of monoids in $\mathcal{C}$.

Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category and let ${}_{\mathfrak{b}}X = (X, \rho_X)$ be a left $\mathfrak{b}$-module object in $\mathcal{D}$. Then the $\mathcal{C}$-enriched Hom functor $\mathcal{D}(X, -) : \mathcal{D} \to \mathcal{C}$ factors through the forgetful $\mathcal{C}$-enriched functor $\mathcal{U} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{C}$. We have a $\mathcal{C}$-enriched functor

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathcal{D}({}_{\mathfrak{b}}X,-)} & \mathcal{M}od_{\mathfrak{b}} \\ & \searrow^{\mathcal{D}(X,-)} & \downarrow \mathcal{U} \\ & & \mathcal{C} \end{array} \qquad \mathcal{D}({}_{\mathfrak{b}}X,-) : \mathcal{D} \to \mathcal{M}od_{\mathfrak{b}} \tag{2.6}$$

which sends each object Y in $\mathcal{D}$ to the right $\mathfrak{b}$-module $\mathcal{D}({}_{\mathfrak{b}}X, Y) = (\mathcal{D}(X,Y), \gamma_{\mathcal{D}(X,Y)})$ whose right $\mathfrak{b}$-action is given by

$$\gamma_{\mathcal{D}(X,Y)} : \mathcal{D}(X,Y) \circledast b \xrightarrow{\mathbb{I}_{\mathcal{D}(X,Y)} \circledast \bar{\rho}_X} \mathcal{D}(X,Y) \circledast \mathcal{D}(X,X) \xrightarrow{\mu_{X,X,Y}} \mathcal{D}(X,Y).$$

2.6. Definition. *Let $\mathfrak{b}' = (b', u_{b'}, m_{b'})$ be another monoid in $\mathcal{C}$. We define a $(\mathfrak{b}, \mathfrak{b}')$-*bimodule ${}_{\mathfrak{b}}x_{\mathfrak{b}'}$ *as a left $\mathfrak{b}$-module object in the tensored $\mathcal{C}$-enriched category $\mathcal{M}od_{\mathfrak{b}'}$ of right $\mathfrak{b}'$-modules. Equivalently, it is a pair ${}_{\mathfrak{b}}x_{\mathfrak{b}'} = \big(x_{\mathfrak{b}'},\ b \circledast x_{\mathfrak{b}'} \xrightarrow{\rho_{x_{\mathfrak{b}'}}} x_{\mathfrak{b}'}\big)$ of a right $\mathfrak{b}'$-module $x_{\mathfrak{b}'} = \big(x,\ x \circledast b' \xrightarrow{\gamma'_x} x\,\big)$ and a morphism $\rho_{x_{\mathfrak{b}'}} : b \circledast x_{\mathfrak{b}'} \to x_{\mathfrak{b}'}$ in $\mathcal{M}od_{\mathfrak{b}'}$ satisfying the left $\mathfrak{b}$-action relations. We denote*

$${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$$

as the category of $(\mathfrak{b}, \mathfrak{b}')$-bimodules. We do not treat ${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ as a $\mathcal{C}$-enriched category. Note that we have the $(\mathfrak{b}, \mathfrak{b})$-bimodule ${}_{\mathfrak{b}}b_{\mathfrak{b}} := (b_{\mathfrak{b}},\ \gamma_{b_{\mathfrak{b}}} : b \circledast b_{\mathfrak{b}} \to b_{\mathfrak{b}})$.

2.7. Example. *We explain what $\mathcal{M}od_{\mathfrak{b}}$ and ${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ are in each example of the base category $\mathcal{C}$.*

1. *Let $\mathcal{C} = \mathcal{A}b$ be the closed symmetric monoidal category of abelian groups.*
    - *Monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are rings;*
    - *$\mathcal{M}od_{\mathfrak{b}}$ is the preadditive category of right modules over the ring $\mathfrak{b}$;*
    - *${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ is the category of $(\mathfrak{b}, \mathfrak{b}')$-bimodules.*
2. *Let $\mathcal{C} = f\mathcal{A}b$ be the closed symmetric monoidal category of finitely generated abelian groups.*
    - *Monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are rings finitely generated as abelian groups;*
    - *$\mathcal{M}od_{\mathfrak{b}}$ is the preadditive category of right modules over the ring $\mathfrak{b}$ which are finitely generated as abelian groups;*
    - *${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ is the category of $(\mathfrak{b}, \mathfrak{b}')$-bimodules which are finitely generated as abelian groups.*
3. *Let $\mathcal{C} = s\mathcal{S}et$ be the closed symmetric monoidal category of simplicial sets.*
    - *Monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are simplicial monoids;*

- *$\mathcal{M}od_{\mathfrak{b}}$ is the simplicially enriched category of simplicial sets equipped with a right action of the simplicial monoid $\mathfrak{b}$;*
- *${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ is the category of simplicial sets equipped with a bi-action of the simplicial monoids $\mathfrak{b}$, $\mathfrak{b}'$.*

4. *Let $\mathcal{C} = \mathcal{B}an$ be the closed symmetric monoidal category of Banach spaces and linear contractions between them, equipped with the projective tensor product.*
   - *Monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are associative unital Banach algebras;*
   - *$\mathcal{M}od_{\mathfrak{b}}$ is the $\mathcal{B}an$-enriched category of Banach spaces equipped with a right action of the Banach algebra $\mathfrak{b}$;*
   - *${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ is the category of Banach spaces equipped with a bi-action of the Banach algebras $\mathfrak{b}$, $\mathfrak{b}'$.*
5. *Let $\mathcal{C} = \mathcal{S}p^{\Sigma}_{CGT_*}$ be the closed symmetric monoidal category of topological symmetric spectra.*
   - *Monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are symmetric ring spectra;*
   - *$\mathcal{M}od_{\mathfrak{b}}$ is the $\mathcal{S}p^{\Sigma}_{CGT_*}$-enriched category of symmetric spectra equipped with a right action of the symmetric ring spectrum $\mathfrak{b}$;*
   - *${}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'}$ is the category of symmetric spectra equipped with a bi-action of the symmetric ring spectra $\mathfrak{b}$, $\mathfrak{b}'$.*

2.8. Definition. *Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. We define the functor*

$$- \circledast_{\mathfrak{b}} - : (\mathcal{M}od_{\mathfrak{b}})_0 \times {}_{\mathfrak{b}}\mathcal{D} \to \mathcal{D}_0$$

as follows. The functor sends each pair of a right $\mathfrak{b}$-module $z_{\mathfrak{b}} = (z, \gamma_z)$ and an object ${}_{\mathfrak{b}}X = (X, \rho_X)$ in ${}_{\mathfrak{b}}\mathcal{D}$ to the following coequalizer in $\mathcal{D}_0$.

$$z \circledast b \circledast X \underset{\mathbb{I}_z \circledast \rho_X}{\overset{\gamma_z \circledast \mathbb{I}_X}{\rightrightarrows}} z \circledast X \xrightarrow{cq_{z_{\mathfrak{b}}, {}_{\mathfrak{b}}X}} z_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X$$

The functor sends each pair of a morphism $l : z_{\mathfrak{b}} \to \tilde{z}_{\mathfrak{b}}$ in $\mathcal{M}od_{\mathfrak{b}}$ and a morphism $\tilde{l} : {}_{\mathfrak{b}}X \to {}_{\mathfrak{b}}\tilde{X}$ in ${}_{\mathfrak{b}}\mathcal{D}$ to the unique morphism $l \circledast_{\mathfrak{b}} \tilde{l} : z_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X \to \tilde{z}_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}\tilde{X}$ in $\mathcal{D}$ satisfying the relation

$$\begin{array}{ccc} z \circledast X & \xrightarrow{l \circledast \tilde{l}} & \tilde{z} \circledast \tilde{X} \\ {\scriptstyle cq_{z_{\mathfrak{b}}, {}_{\mathfrak{b}}X}} \downarrow & & \downarrow {\scriptstyle cq_{\tilde{z}_{\mathfrak{b}}, {}_{\mathfrak{b}}\tilde{X}}} \\ z_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X & \xrightarrow{\exists !\ l \circledast_{\mathfrak{b}} \tilde{l}} & \tilde{z}_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}\tilde{X} \end{array} \qquad cq_{\tilde{z}_{\mathfrak{b}}, {}_{\mathfrak{b}}\tilde{X}} \circ (l \circledast \tilde{l}) = (l \circledast_{\mathfrak{b}} \tilde{l}) \circ cq_{z_{\mathfrak{b}}, {}_{\mathfrak{b}}X}.$$

Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. For each object ${}_{\mathfrak{b}}X = (X, \rho_X)$ in ${}_{\mathfrak{b}}\mathcal{D}$, we have a unique isomorphism $\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X} : b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X \xrightarrow{\cong} X$ in $\mathcal{D}$ which satisfies the relation $\rho_X = \imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X} \circ cq_{b_{\mathfrak{b}}, {}_{\mathfrak{b}}X}$. The inverse of $\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X}$ is given by

$$\begin{array}{ccc} b\circledast X & \xrightarrow{\rho_X} & X \\ {\scriptstyle cq_{b_{\mathfrak{b}},{}_{\mathfrak{b}}X}}\downarrow & & \| \\ b_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X & \xrightarrow[\cong]{\exists!\ \imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X}} & X \end{array} \qquad \begin{aligned} (\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X})^{-1} &= cq_{b_{\mathfrak{b}},{}_{\mathfrak{b}}X} \circ (u_{\mathfrak{b}} \circledast \mathbb{I}_X) \circ \imath_X^{-1} \\ &: X \xrightarrow{\cong} c\circledast X \longrightarrow b\circledast X \twoheadrightarrow b_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X. \end{aligned} \tag{2.7}$$

2.9. Lemma. *Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. We have a well-defined functor*

$$\begin{aligned} {}_{\mathfrak{b}}\mathcal{D} &\longrightarrow \mathcal{C}\text{-}Funct(\mathcal{M}od_{\mathfrak{b}}, \mathcal{D}) \\ {}_{\mathfrak{b}}X &\longmapsto - \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D} \end{aligned} \tag{2.8}$$

from the category of left $\mathfrak{b}$-module objects in $\mathcal{D}$ to the category of $\mathcal{C}$-enriched functors $\mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$.

1. *For each object ${}_{\mathfrak{b}}X = (X, \rho_X)$ in ${}_{\mathfrak{b}}\mathcal{D}$, we have a $\mathcal{C}$-enriched functor $- \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ which is uniquely determined as follows. The underlying functor of $- \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X$ is defined in Definition 2.8, and the associated tensorial strength*

$$a_{w,z_{\mathfrak{b}},{}_{\mathfrak{b}}X} := t^{-\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X}_{w,z_{\mathfrak{b}}} :\ w\circledast(z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X) \xrightarrow{\cong} (w\circledast z_{\mathfrak{b}})\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X, \quad w \in \mathrm{Obj}(\mathcal{C}),\ z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$$

   is the unique isomorphism in $\mathcal{D}$ satisfying the relation

$$\begin{array}{ccc} w\circledast(z\circledast X) & \xrightarrow[\cong]{a_{w,z,X}} & (w\circledast z)\circledast X \\ {\scriptstyle \mathbb{I}_w\circledast cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}X}}\downarrow & & \downarrow{\scriptstyle cq_{w\circledast z_{\mathfrak{b}},{}_{\mathfrak{b}}X}} \\ w\circledast(z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X) & \xrightarrow[\cong]{\exists!\ a_{w,z_{\mathfrak{b}},{}_{\mathfrak{b}}X}} & (w\circledast z_{\mathfrak{b}})\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X \end{array} \qquad \begin{aligned} &cq_{w\circledast z_{\mathfrak{b}},{}_{\mathfrak{b}}X} \circ a_{w,z,X} \\ &= a_{w,z_{\mathfrak{b}},{}_{\mathfrak{b}}X} \circ (\mathbb{I}_w \circledast cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}X}). \end{aligned}$$

2. *For each morphism ${}_{\mathfrak{b}}X \to {}_{\mathfrak{b}}\tilde{X}$ in ${}_{\mathfrak{b}}\mathcal{D}$, the following collection of morphisms in $\mathcal{D}$*

$$\left\{\ z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X \to z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}\tilde{X}\ \middle|\ z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})\ \right\} \tag{2.9}$$

   defines a $\mathcal{C}$-enriched natural transformation $- \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X \Rightarrow - \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}\tilde{X} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$.

Proof. We leave for the readers to check that such isomorphisms $a_{w,z_{\mathfrak{b}},{}_{\mathfrak{b}}X}$ in $\mathcal{D}$ uniquely exist, and satisfy the relations (2.4). Thus we have a unique $\mathcal{C}$-enriched functor $-\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ as described in statement 1. Statement 2 is also true, as we can check that the collection (2.9) of morphisms in $\mathcal{D}$ satisfies the relation (2.5). We conclude that the functor (2.8) is well-defined. ∎

We will show in §3 that the functor (2.8) in Lemma 2.9 is the fully faithful left adjoint functor (1.2) described in Theorem 1.1.

2.10. Proposition. *Let $\mathcal{D}$ be a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. For each left $\mathfrak{b}$-module object ${}_{\mathfrak{b}}X$ in $\mathcal{D}$, we have a $\mathcal{C}$-enriched adjunction*

$$\mathcal{M}od_{\mathfrak{b}} \underset{\mathcal{D}({}_{\mathfrak{b}}X,-)}{\overset{-\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X}{\rightleftarrows}} \mathcal{D} \qquad -\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X \dashv \mathcal{D}({}_{\mathfrak{b}}X,-) : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$$

whose unit, counit $\mathcal{C}$-enriched natural transformations η, ε are described below.

- *The component of the unit η at each $z_{\mathfrak{b}} = (z, \gamma_z) \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$ is the unique morphism $\eta_{z_{\mathfrak{b}}} : z_{\mathfrak{b}} \to \mathcal{D}({}_{\mathfrak{b}}X, z_{\mathfrak{b}} \circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X)$ in $\mathcal{M}od_{\mathfrak{b}}$, whose corresponding morphism in $\mathcal{C}$ is*
$$\eta_{z_{\mathfrak{b}}} :\ z \xrightarrow{Cv_{z,X}} \mathcal{D}(X, z\circledast X) \xrightarrow{(cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}X})_{\star}} \mathcal{D}(X, z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X).$$
- *The component of the counit ε at each $Y \in \mathrm{Obj}(\mathcal{D})$ is the unique morphism $\varepsilon_Y : \mathcal{D}({}_{\mathfrak{b}}X, Y) \circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X \to Y$ in $\mathcal{D}$ which satisfies the relation*
$$\begin{array}{ccc} \mathcal{D}(X,Y)\circledast X & \xrightarrow{Ev_{X,Y}} & Y \\ {\scriptstyle cq_{\mathcal{D}({}_{\mathfrak{b}}X,Y),{}_{\mathfrak{b}}X}}\downarrow & & \| \\ \mathcal{D}({}_{\mathfrak{b}}X,Y)\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X & \xrightarrow{\exists!\ \varepsilon_Y} & Y \end{array} \qquad Ev_{X,Y} = \varepsilon_Y \circ cq_{\mathcal{D}({}_{\mathfrak{b}}X,Y),{}_{\mathfrak{b}}X}.$$

Proof. The components $\eta_{z_{\mathfrak{b}}}$, ε_Y are well-defined and are natural in variables $z_{\mathfrak{b}}$, Y, respectively. As their components $\eta_{z_{\mathfrak{b}}}$, ε_Y satisfy the relation (2.5), we obtain $\mathcal{C}$-enriched natural transformations η, ε. We leave for the readers to check that η, ε satisfy the triangular identities. ∎

3. The Eilenberg-Watts Theorem

In this section, we prove Theorem 1.1 which generalizes the Eilenberg-Watts theorem in enriched context. We also give a proof of Corollary 1.2. Throughout this section, $\mathfrak{b} = (b, u_{\mathfrak{b}}, m_{\mathfrak{b}})$ is a monoid in $\mathcal{C}$ and $\mathcal{D}$ is a tensored $\mathcal{C}$-enriched category whose underlying category $\mathcal{D}_0$ has coequalizers. We are going to show that the functor

$$\begin{aligned} {}_{\mathfrak{b}}\mathcal{D} &\xrightarrow{(2.8)} \mathcal{C}\text{-}\mathit{Funct}(\mathcal{M}od_{\mathfrak{b}}, \mathcal{D}) \\ {}_{\mathfrak{b}}X &\longmapsto -\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}X : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D} \end{aligned}$$

defined in Lemma 2.9 is left adjoint to the functor of evaluating at $b_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$. Let us explain the right adjoint functor in detail. Using the properties (2.4), (2.5) of tensorial strengths associated to $\mathcal{C}$-enriched functors $\mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$, one can check that the following are true.

- For each $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$, the object $\mathcal{F}(b_{\mathfrak{b}})$ in $\mathcal{D}$ becomes a left $\mathfrak{b}$-module object ${}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) = \big(\mathcal{F}(b_{\mathfrak{b}}), \rho_{\mathcal{F}(b_{\mathfrak{b}})}\big)$ in $\mathcal{D}$ whose left $\mathfrak{b}$-action morphism is

$$\rho_{\mathcal{F}(b_{\mathfrak{b}})} :\ b\circledast\mathcal{F}(b_{\mathfrak{b}}) \xrightarrow{t^{\mathcal{F}}_{b,b_{\mathfrak{b}}}} \mathcal{F}(b\circledast b_{\mathfrak{b}}) \xrightarrow{\mathcal{F}(\gamma_{b_{\mathfrak{b}}})} \mathcal{F}(b_{\mathfrak{b}}).$$

- For each $\mathcal{C}$-enriched natural transformation $\xi : \mathcal{F} \Rightarrow \widetilde{\mathcal{F}} : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$, the component $\xi_{b_{\mathfrak{b}}} : \mathcal{F}(b_{\mathfrak{b}}) \to \widetilde{\mathcal{F}}(b_{\mathfrak{b}})$ of ξ at $b_{\mathfrak{b}}$ becomes a morphism $\xi_{b_{\mathfrak{b}}} : {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \to {}_{\mathfrak{b}}\widetilde{\mathcal{F}}(b_{\mathfrak{b}})$ of left $\mathfrak{b}$-module objects in $\mathcal{D}$.

Thus we obtain a well-defined functor

$$\begin{array}{rcl} \mathcal{C}\text{-}Funct(\mathcal{Mod}_{\mathfrak{b}}, \mathcal{D}) & \longrightarrow & {}_{\mathfrak{b}}\mathcal{D} \\ \mathcal{F} : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D} & \longmapsto & {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \end{array} \tag{3.1}$$

of evaluating at $b_{\mathfrak{b}}$.

3.1. Lemma. *For each $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$, we have a $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}} : -\circledast_{\mathfrak{b}}\, {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \Rightarrow \mathcal{F} : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$ whose component $\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}$ at $z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{Mod}_{\mathfrak{b}})$ is the unique morphism in $\mathcal{D}$ satisfying the relation*

$$\begin{array}{ccc} z\circledast\mathcal{F}(b_{\mathfrak{b}}) & \xrightarrow{t^{\mathcal{F}}_{z,b_{\mathfrak{b}}}} & \mathcal{F}(z\circledast b_{\mathfrak{b}}) \\ {\scriptstyle cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}(\gamma_{z_{\mathfrak{b}}})} \\ z_{\mathfrak{b}}\circledast_{\mathfrak{b}}\,{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) & \xrightarrow{\exists!\ \lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}} & \mathcal{F}(z_{\mathfrak{b}}) \end{array} \qquad \mathcal{F}(\gamma_{z_{\mathfrak{b}}}) \circ t^{\mathcal{F}}_{z,b_{\mathfrak{b}}} = \lambda^{\mathcal{F}}_{z_{\mathfrak{b}}} \circ cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}.$$

Moreover, the component of $\lambda^{\mathcal{F}}$ at $b_{\mathfrak{b}}$ is given by $\lambda^{\mathcal{F}}_{b_{\mathfrak{b}}} = \imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})} :\ b_{\mathfrak{b}}\circledast_{\mathfrak{b}}\,{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \xrightarrow{\cong} \mathcal{F}(b_{\mathfrak{b}})$.

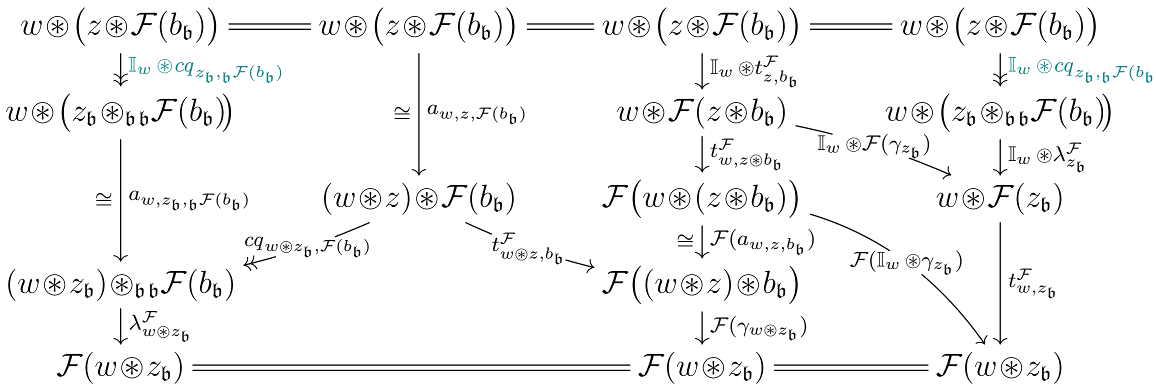
Proof. We leave for the readers to check that such morphism $\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}$ in $\mathcal{D}$ uniquely exists, and that the following diagram of morphisms in $\mathcal{D}$ commutes.

$$\begin{array}{ccc} z\circledast b\circledast\mathcal{F}(b_{\mathfrak{b}}) & \xrightarrow{t^{\mathcal{F}}_{z\circledast b,b_{\mathfrak{b}}}} & \mathcal{F}(z\circledast b\circledast b_{\mathfrak{b}}) \\ {\scriptstyle \gamma_z\circledast\mathbb{I}_{\mathcal{F}(b_{\mathfrak{b}})}}\big\downarrow\big\downarrow{\scriptstyle \mathbb{I}_z\circledast\rho_{\mathcal{F}(b_{\mathfrak{b}})}} & & {\scriptstyle \mathcal{F}(\gamma_z\circledast\mathbb{I}_{b_{\mathfrak{b}}})}\big\downarrow\big\downarrow{\scriptstyle \mathcal{F}(\mathbb{I}_z\circledast\gamma_{b_{\mathfrak{b}}})} \\ z\circledast\mathcal{F}(b_{\mathfrak{b}}) & \xrightarrow{t^{\mathcal{F}}_{z,b_{\mathfrak{b}}}} & \mathcal{F}(z\circledast b_{\mathfrak{b}}) \\ {\scriptstyle cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}(\gamma_{z_{\mathfrak{b}}})} \\ z_{\mathfrak{b}}\circledast_{\mathfrak{b}}\,{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) & \xrightarrow{\exists!\ \lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}} & \mathcal{F}(z_{\mathfrak{b}}) \end{array} \tag{3.2}$$

The collection $\{\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}\}$ of morphisms in $\mathcal{D}$ is natural in variable $z_{\mathfrak{b}}$. To show that the collection $\{\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}\}$ is $\mathcal{C}$-enriched natural in variable $z_{\mathfrak{b}}$, we need to verify the following relation for every pair $w \in \mathrm{Obj}(\mathcal{C})$, $z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{Mod}_{\mathfrak{b}})$.

$$\begin{array}{ccc} w\circledast\big(z_{\mathfrak{b}}\circledast_{\mathfrak{b}}\,{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})\big) & \xrightarrow[\cong]{a_{w,z_{\mathfrak{b}},{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}} & (w\circledast z_{\mathfrak{b}})\circledast_{\mathfrak{b}}\,{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \\ {\scriptstyle \mathbb{I}_w\circledast\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda^{\mathcal{F}}_{w\circledast z_{\mathfrak{b}}}} \\ w\circledast\mathcal{F}(z_{\mathfrak{b}}) & \xrightarrow{t^{\mathcal{F}}_{w,z_{\mathfrak{b}}}} & \mathcal{F}(w\circledast z_{\mathfrak{b}}) \end{array} \tag{3.3}$$

Consider the following commutative diagram.



After right-cancelling the epimorphism $\mathbb{I}_w \circledast cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}$ in the above diagram, we obtain the relation (3.3). Thus we have a well-defined $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}}$ as we claimed. From the definition of $\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}$ given in (2.7), we obtain that $\lambda^{\mathcal{F}}_{b_{\mathfrak{b}}} = \imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})} : b_{\mathfrak{b}} \circledast_{\mathfrak{b}}\, {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \xrightarrow{\cong} \mathcal{F}(b_{\mathfrak{b}})$. ∎

3.2. Proposition. *For each $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$, the following are equivalent:*

(i) $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ is a $\mathcal{C}$-enriched left adjoint;

(ii) $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ is $\mathcal{C}$-enriched cocontinuous;

(iii) $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ preserves $\mathcal{C}$-tensors, and the underlying functor $\mathcal{F}_0$ preserves coequalizers;

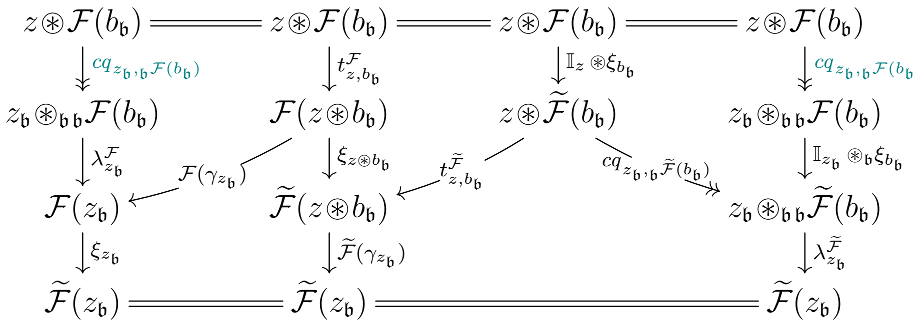
(iv) The $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}} : - \circledast_{\mathfrak{b}}\, {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \Rightarrow \mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ defined in Lemma 3.1 is invertible.

Proof. By Proposition 2.10, (iv) implies (i). It is straightforward that (i) implies (ii), and (ii) implies (iii). We claim that (iii) implies (iv). Assume that the $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ preserves $\mathcal{C}$-tensors, and the underlying functor $\mathcal{F}_0 : (\mathcal{M}od_{\mathfrak{b}})_0 \to \mathcal{D}_0$ preserves coequalizers. Recall the coequalizer diagram (2.3) in $(\mathcal{M}od_{\mathfrak{b}})_0$. If we look at the diagram in (3.2) we see that the top, middle horizontal morphisms in $\mathcal{D}$ are isomorphisms, and the right vertical morphisms also form a coequalizer diagram in the underlying category $\mathcal{D}_0$ of $\mathcal{D}$. This shows that $\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}$ is an isomorphism in $\mathcal{D}$ for every $z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$. We conclude that the $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}}$ is invertible. ∎

3.3. Lemma. *Let $\mathcal{F}, \widetilde{\mathcal{F}} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ be $\mathcal{C}$-enriched functors. For each $\mathcal{C}$-enriched natural transformation $\xi : \mathcal{F} \Rightarrow \widetilde{\mathcal{F}} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$, we have the following relation for every $z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$.*

$$\begin{array}{ccc} z_{\mathfrak{b}} \circledast_{\mathfrak{b}}\, {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) & \xrightarrow{\lambda^{\mathcal{F}}_{z_{\mathfrak{b}}}} & \mathcal{F}(z_{\mathfrak{b}}) \\ {\scriptstyle \mathbb{I}_{z_{\mathfrak{b}}} \circledast_{\mathfrak{b}} \xi_{b_{\mathfrak{b}}}} \downarrow & & \downarrow {\scriptstyle \xi_{z_{\mathfrak{b}}}} \\ z_{\mathfrak{b}} \circledast_{\mathfrak{b}}\, {}_{\mathfrak{b}}\widetilde{\mathcal{F}}(b_{\mathfrak{b}}) & \xrightarrow{\lambda^{\widetilde{\mathcal{F}}}_{z_{\mathfrak{b}}}} & \widetilde{\mathcal{F}}(z_{\mathfrak{b}}) \end{array} \tag{3.4}$$

PROOF. For each $z_{\mathfrak{b}} = (z, \gamma_z) \in \mathrm{Obj}(\mathcal{Mod}_{\mathfrak{b}})$, we have the following diagram.



After right-cancelling the epimorphism $cq_{z_{\mathfrak{b}}, {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}$ in the above diagram, we obtain the relation (3.4). ∎

Let ${}_{\mathfrak{b}}X$ be a left $\mathfrak{b}$-module object in $\mathcal{D}$. The functor $\mathcal{C}\text{-}Funct(\mathcal{Mod}_{\mathfrak{b}}, \mathcal{D}) \to {}_{\mathfrak{b}}\mathcal{D}$ of evaluating at $b_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{Mod}_{\mathfrak{b}})$ defined in (3.1) sends the $\mathcal{C}$-enriched functor $- \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$ to the left $\mathfrak{b}$-module object ${}_{\mathfrak{b}}b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X = (b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X, \rho_{b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X})$ in $\mathcal{D}$, where

$$\rho_{b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X} : b \circledast (b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X) \xrightarrow[\cong]{a_{b, b_{\mathfrak{b}}, {}_{\mathfrak{b}}X}} (b \circledast b_{\mathfrak{b}}) \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X \xrightarrow{\gamma_{b_{\mathfrak{b}}} \circledast_{\mathfrak{b}} \mathbb{I}_{{}_{\mathfrak{b}}X}} b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X.$$

One can check that the isomorphism $\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X} : b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X \xrightarrow{\cong} X$ in $\mathcal{D}$ defined in (2.7) becomes an isomorphism $\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X} : {}_{\mathfrak{b}}b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X \xrightarrow{\cong} {}_{\mathfrak{b}}X$ in ${}_{\mathfrak{b}}\mathcal{D}$. We are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. From the equivalence of statements *(i)-(iv)* in Proposition 3.2, we conclude that the map (2.8) induces an equivalence of categories

$${}_{\mathfrak{b}}\mathcal{D} \xrightarrow{\simeq} \mathcal{C}\text{-}Funct_{cocon}(\mathcal{Mod}_{\mathfrak{b}}, \mathcal{D})$$

between ${}_{\mathfrak{b}}\mathcal{D}$ and the category of cocontinuous $\mathcal{C}$-enriched functors $\mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$. The latter is a full, coreflective subcategory of the category $\mathcal{C}\text{-}Funct(\mathcal{Mod}_{\mathfrak{b}}, \mathcal{D})$ of all $\mathcal{C}$-enriched functors $\mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$ thanks to Lemma 3.3 by taking $\widetilde{\mathcal{F}}$ to be a general $\mathcal{C}$-enriched functor and $\mathcal{F}$ a $\mathcal{C}$-cocontinuous one. This completes the proof of Theorem 1.1.

One can also directly show that the functor ${}_{\mathfrak{b}}\mathcal{D} \to \mathcal{C}\text{-}Funct(\mathcal{Mod}_{\mathfrak{b}}, \mathcal{D})$ in (2.8) is left adjoint to the functor $\mathcal{C}\text{-}Funct(\mathcal{Mod}_{\mathfrak{b}}, \mathcal{D}) \to {}_{\mathfrak{b}}\mathcal{D}$ in (3.1). The component of the unit at each object ${}_{\mathfrak{b}}X$ in ${}_{\mathfrak{b}}\mathcal{D}$ is the isomorphism $(\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X})^{-1} : {}_{\mathfrak{b}}X \xrightarrow{\cong} {}_{\mathfrak{b}}b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X$ in ${}_{\mathfrak{b}}\mathcal{D}$. The component of the counit at each $\mathcal{C}$-enriched functor $\mathcal{F} : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{D}$ is the $\mathcal{C}$-enriched natural transformation $\lambda^{\mathcal{F}} : - \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}}) \Rightarrow \mathcal{F}$ defined in Lemma 3.1. One can check that the isomorphism $(\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X})^{-1}$ is natural in variable ${}_{\mathfrak{b}}X$, and by Lemma 3.3 $\lambda^{\mathcal{F}}$ is natural in variable $\mathcal{F}$. We can check the triangular identities using the relation $\lambda^{\mathcal{F}}_{b_{\mathfrak{b}}} = \imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}\mathcal{F}(b_{\mathfrak{b}})}$ and the explicit description of $(\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X})^{-1}$ given in (2.7). The rest of the statements in Theorem 1.1 are straightforward to check using Proposition 3.2. This is another proof of Theorem 1.1. ∎

Proof of Corollary 1.2. Let $\mathfrak{b}'$ be another monoid in $\mathcal{C}$. After substituting $\mathcal{D} = \mathcal{M}od_{\mathfrak{b}'}$ in Theorem 1.1, we obtain the adjoint equivalence of categories

$$ {}_{\mathfrak{b}}\mathcal{M}od_{\mathfrak{b}'} \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathcal{C}\text{-}Funct_{cocon}(\mathcal{M}od_{\mathfrak{b}}, \mathcal{M}od_{\mathfrak{b}'}) $$

whose right adjoint is the functor of evaluating at $b_{\mathfrak{b}}$. ∎

4. Morita Theory

In this section, we prove Theorem 1.3 which characterizes when a $\mathcal{C}$-enriched category $\mathcal{D}$ is equivalent to $\mathcal{M}od_{\mathfrak{b}}$ for a given monoid $\mathfrak{b}$ in $\mathcal{C}$. We also give a proof of Corollary 1.4 which generalizes the result of Morita in enriched context.

4.1. Definition. *Let $\mathcal{D}$ be a $\mathcal{C}$-enriched category and let $X \in \mathrm{Obj}(\mathcal{D})$. We say*

(i) X is a $\mathcal{C}$-enriched compact object *in $\mathcal{D}$ if the $\mathcal{C}$-enriched Hom functor $\mathcal{D}(X,-) : \mathcal{D} \to \mathcal{C}$ preserves $\mathcal{C}$-tensors, and the underlying functor $\mathcal{D}(X,-)_0$ preserves coequalizers;*

(ii) X is a $\mathcal{C}$-enriched generator *in $\mathcal{D}$ if the $\mathcal{C}$-enriched Hom functor $\mathcal{D}(X,-) : \mathcal{D} \to \mathcal{C}$ is conservative;*

(iii) X is a $\mathcal{C}$-enriched compact generator *in $\mathcal{D}$ if it is both a $\mathcal{C}$-enriched compact object and a $\mathcal{C}$-enriched generator in $\mathcal{D}$.*

4.2. Example. *Consider the case when $\mathcal{C} = \mathcal{A}b$ is the closed symmetric monoidal category of abelian groups. Let R be a ring and let $\mathcal{M}od_R$ be the preadditive category of right R-modules. For each right R-module N_R,*

(i) N_R is an $\mathcal{A}b$-enriched compact object in $\mathcal{M}od_R$ if and only if it is a finitely generated projective right R-module;

(ii) N_R is an $\mathcal{A}b$-enriched generator in $\mathcal{M}od_R$ if and only if it is a generator in the category of right R-modules;

(iii) N_R is an $\mathcal{A}b$-enriched compact generator in $\mathcal{M}od_R$ if and only if it is a finitely generated projective generator in the category of right R-modules.

Let us explain the 'only if' part of statement (i). Assume that N_R is an $\mathcal{A}b$-enriched compact object in $\mathcal{M}od_R$. By Proposition 3.2, the $\mathcal{A}b$-enriched Hom functor $\mathcal{M}od_R(N_R,-)$ is $\mathcal{A}b$-enriched cocontinuous. In particular, the underlying functor $\mathcal{M}od_R(N_R,-)_0$ is cocontinuous.

- *N_R is a projective right R-module if and only if the underlying functor $\mathcal{M}od_R(N_R,-)_0$ preserves coequalizers.*

- *A projective right R-module N_R is finitely generated if and only if the underlying functor $\mathcal{M}od_R(N_R,-)_0$ preserves arbitrary sums. This is explained in the proof of [1, Proposition1.2(c)].*

Therefore N_R is a finitely generated projective right R-module.

4.3. Lemma. *Let $\mathfrak{b} = (b, u_b, m_b)$ be a monoid in $\mathcal{C}$. The right $\mathfrak{b}$-module $b_\mathfrak{b}$ is a $\mathcal{C}$-enriched compact generator in $\mathcal{M}od_\mathfrak{b}$, and we have an isomorphism of monoids $\mathfrak{b} \cong End_{\mathcal{M}od_\mathfrak{b}}(b_\mathfrak{b})$ in $\mathcal{C}$.*

Proof. Recall that for each $z_\mathfrak{b} \in \mathrm{Obj}(\mathcal{M}od_\mathfrak{b})$, we have a morphism $\gamma_{z_\mathfrak{b}} : z \circledast b_\mathfrak{b} \to z_\mathfrak{b}$ in $\mathcal{M}od_\mathfrak{b}$. One can check that the corresponding right adjunct $\bar{\gamma}_{z_\mathfrak{b}} : \ z \xrightarrow{\cong} \mathcal{M}od_\mathfrak{b}(b_\mathfrak{b}, z_\mathfrak{b})$ is an isomorphism in $\mathcal{C}$, and is $\mathcal{C}$-enriched natural in variable $z_\mathfrak{b}$. Thus we have an isomorphism of $\mathcal{C}$-enriched functors $\mathcal{U} \cong \mathcal{M}od_\mathfrak{b}(b_\mathfrak{b}, -) : \mathcal{M}od_\mathfrak{b} \to \mathcal{C}$. The forgetful $\mathcal{C}$-enriched functor $\mathcal{U} : \mathcal{M}od_\mathfrak{b} \to \mathcal{C}$ is conservative, preserves $\mathcal{C}$-tensors, and its underlying functor $\mathcal{U}_0$ preserves coequalizers. We conclude that $b_\mathfrak{b}$ is a $\mathcal{C}$-enriched compact generator in $\mathcal{M}od_\mathfrak{b}$. We leave for the readers to check that the isomorphism $\bar{\gamma}_{b_\mathfrak{b}} : b \xrightarrow{\cong} \mathcal{M}od_\mathfrak{b}(b_\mathfrak{b}, b_\mathfrak{b})$ in $\mathcal{C}$ becomes an isomorphism of monoids $\bar{\gamma}_{b_\mathfrak{b}} : \mathfrak{b} \cong End_{\mathcal{M}od_\mathfrak{b}}(b_\mathfrak{b})$ in $\mathcal{C}$. ∎

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 4.3, the only if part is true. We prove the if part as follows. Let us denote $f : b \xrightarrow{\cong} \mathcal{D}(X,X)$ as the isomorphism in $\mathcal{C}$. Then we have a morphism $\rho_X : b \circledast X \xrightarrow[\cong]{f \circledast \mathbb{I}_X} \mathcal{D}(X,X) \circledast X \xrightarrow{Ev_{X,X}} X$ in $\mathcal{D}$ whose right adjunct is $\bar{\rho}_X = f : b \xrightarrow{\cong} \mathcal{D}(X,X)$, and the pair ${}_\mathfrak{b}X = (X, \rho_X)$ is a left $\mathfrak{b}$-module object in $\mathcal{D}$. By Proposition 2.10, we have the following adjoint pair of $\mathcal{C}$-enriched functors.

$$\mathcal{M}od_\mathfrak{b} \underset{\beta := \mathcal{D}({}_\mathfrak{b}X, -)}{\overset{\alpha := - \circledast_\mathfrak{b} {}_\mathfrak{b}X}{\rightleftarrows}} \mathcal{D} \tag{4.1}$$

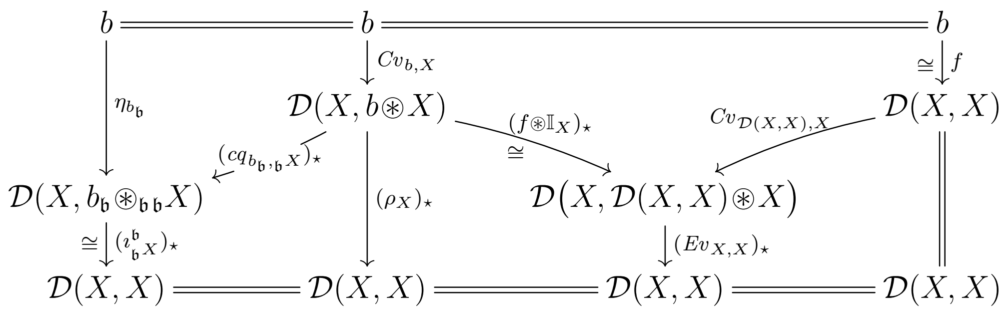
We are going to show that the $\mathcal{C}$-enriched adjunction (4.1) is an adjoint equivalence of $\mathcal{C}$-enriched categories. First, we show that $\beta\alpha : \mathcal{M}od_\mathfrak{b} \to \mathcal{M}od_\mathfrak{b}$ is $\mathcal{C}$-enriched cocontinuous as follows. Recall the diagram in (2.6).

- The $\mathcal{C}$-enriched functor $\mathcal{D}(X,-) : \mathcal{D} \to \mathcal{C}$ preserves $\mathcal{C}$-tensors, and the underlying functor $\mathcal{D}(X,-)_0$ preserves coequalizers.
- The $\mathcal{C}$-enriched category $\mathcal{M}od_\mathfrak{b}$ is tensored, and the underlying category $(\mathcal{M}od_\mathfrak{b})_0$ has coequalizers.
- The forgetful $\mathcal{C}$-enriched functor $\mathcal{U} : \mathcal{M}od_\mathfrak{b} \to \mathcal{C}$ is conservative, preserves $\mathcal{C}$-tensors, and the underlying functor $\mathcal{U}_0$ preserves coequalizers.

Thus we obtain that the $\mathcal{C}$-enriched functor $\beta = \mathcal{D}({}_\mathfrak{b}X, -) : \mathcal{D} \to \mathcal{M}od_\mathfrak{b}$ preserves $\mathcal{C}$-tensors, and the underlying functor β_0 preserves coequalizers. Then the $\mathcal{C}$-enriched functor

$\beta\alpha : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{M}od_{\mathfrak{b}}$ also has the same properties. By Proposition 3.2, we conclude that the $\mathcal{C}$-enriched functor $\beta\alpha : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{M}od_{\mathfrak{b}}$ is cocontinuous.

Next, we show that the adjunction (4.1) is an adjoint equivalence of $\mathcal{C}$-enriched categories. We begin by showing that the unit $\eta : I_{\mathcal{M}od_{\mathfrak{b}}} \Rightarrow \beta\alpha : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{M}od_{\mathfrak{b}}$ is a $\mathcal{C}$-enriched natural isomorphism. By Corollary 1.2, it suffices to show that the component $\eta_{b_{\mathfrak{b}}} : b_{\mathfrak{b}} \to \mathcal{D}({}_{\mathfrak{b}}X, b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X)$ at $b_{\mathfrak{b}}$ is an isomorphism in $\mathcal{M}od_{\mathfrak{b}}$. Consider the following diagram.



We obtain that the morphism $\eta_{b_{\mathfrak{b}}} : b \to \mathcal{D}(X, b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X)$ in $\mathcal{C}$ is equal to $(\imath^{\mathfrak{b}}_{{}_{\mathfrak{b}}X})_\star^{-1} \circ f : b \xrightarrow{\cong} \mathcal{D}(X,X) \xrightarrow{\cong} \mathcal{D}(X, b_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}X)$ which is an isomorphism. This shows that the unit $\eta : I_{\mathcal{M}od_{\mathfrak{b}}} \Rightarrow \beta\alpha$ is a $\mathcal{C}$-enriched natural isomorphism.

To conclude that the $\mathcal{C}$-enriched adjunction (4.1) is an equivalence of $\mathcal{C}$-enriched categories, it suffices to show that the right adjoint $\beta = \mathcal{D}({}_{\mathfrak{b}}X, -) : \mathcal{D} \to \mathcal{M}od_{\mathfrak{b}}$ is conservative. This is because any $\mathcal{C}$-enriched adjunction with fully faithful left adjoint and conservative right adjoint is an adjoint equivalence of $\mathcal{C}$-enriched categories due to the triangular identities. As we assumed that X is also a $\mathcal{C}$-enriched generator in $\mathcal{D}$, the $\mathcal{C}$-enriched functor $\mathcal{D}(X, -) : \mathcal{D} \to \mathcal{C}$ is conservative. From the relation (2.6), we obtain that $\beta = \mathcal{D}({}_{\mathfrak{b}}X, -) : \mathcal{D} \to \mathcal{M}od_{\mathfrak{b}}$ is also conservative. This completes the proof of Theorem 1.3. ■

4.4. REMARK. *Let us weaken the assumption of Theorem 1.3 and merely assume that X is a $\mathcal{C}$-enriched compact object in $\mathcal{D}$. Then the left adjoint $\mathcal{C}$-enriched functor $\alpha : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{D}$ in (4.1) induces an equivalence of $\mathcal{C}$-enriched categories from $\mathcal{M}od_{\mathfrak{b}}$ to a coreflective full $\mathcal{C}$-enriched subcategory of $\mathcal{D}$.*

4.5. REMARK. *Theorem 1.3 is related to the result in [2] which states that the Eilenberg-Moore category of a $\mathcal{C}$-enriched $\mathcal{C}$-tensor preserving monad $\mathcal{T}$ on $\mathcal{C}$ is equivalent to the category of right $\mathcal{T}(c)$-modules.*

Let $\mathfrak{b} = (b, u_b, m_b)$ be a monoid in $\mathcal{C}$. We have a $\mathcal{C}$-enriched natural isomorphism

$$\jmath^{\mathfrak{b}} : - \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}b_{\mathfrak{b}} \overset{\cong}{\Longrightarrow} I_{\mathcal{M}od_{\mathfrak{b}}} : \mathcal{M}od_{\mathfrak{b}} \to \mathcal{M}od_{\mathfrak{b}} \tag{4.2}$$

whose component at $z_{\mathfrak{b}} = (z, \gamma_z) \in \mathrm{Obj}(\mathcal{M}od_{\mathfrak{b}})$ is the unique isomorphism $\jmath^{\mathfrak{b}}_{z_{\mathfrak{b}}} : z_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}b_{\mathfrak{b}} \xrightarrow{\cong}$

$z_{\mathfrak{b}}$ in $\mathcal{D}$ satisfying the relation

$$\begin{array}{ccc} z\circledast b_{\mathfrak{b}} & \xrightarrow{\gamma_{z_{\mathfrak{b}}}} & z_{\mathfrak{b}} \\ {\scriptstyle cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}b_{\mathfrak{b}}}}\downarrow & & \| \\ z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}b_{\mathfrak{b}} & \underset{\cong}{\xrightarrow{\exists!\, \jmath^{\mathfrak{b}}_{z_{\mathfrak{b}}}}} & z_{\mathfrak{b}} \end{array} \qquad \gamma_{z_{\mathfrak{b}}} = \jmath^{\mathfrak{b}}_{z_{\mathfrak{b}}} \circ cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}b_{\mathfrak{b}}}.$$

Let $\mathfrak{b}'$, $\mathfrak{b}''$ be additional monoids in $\mathcal{C}$. For each pair of a $(\mathfrak{b},\mathfrak{b}')$-bimodule ${}_{\mathfrak{b}}x_{\mathfrak{b}'} = (x_{\mathfrak{b}'}, \rho_{x_{\mathfrak{b}'}})$ and a $(\mathfrak{b}',\mathfrak{b}'')$-bimodule ${}_{\mathfrak{b}'}y_{\mathfrak{b}''}$, we have the $(\mathfrak{b},\mathfrak{b}'')$-bimodule

$${}_{\mathfrak{b}}x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''} = \left(x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''},\quad \rho_{x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}} : b\circledast(x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}) \longrightarrow x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}\right)$$

whose left $\mathfrak{b}$-action is given by

$$\rho_{x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}} : b\circledast(x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}) \underset{\cong}{\xrightarrow{a_{b,x_{\mathfrak{b}'},{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}}} (b\circledast x_{\mathfrak{b}'})\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''} \xrightarrow{\rho_{x_{\mathfrak{b}'}}\circledast_{\mathfrak{b}'}\mathbb{I}_{{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}} x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}.$$

We have a $\mathcal{C}$-enriched natural isomorphism

$$a_{-,{}_{\mathfrak{b}}x_{\mathfrak{b}'},{}_{\mathfrak{b}'}y_{\mathfrak{b}''}} : -\circledast_{\mathfrak{b}}({}_{\mathfrak{b}}x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}) \overset{\cong}{\Longrightarrow} (-\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}x_{\mathfrak{b}'})\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''} \; : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{Mod}_{\mathfrak{b}''} \tag{4.3}$$

whose component $a_{z_{\mathfrak{b}},{}_{\mathfrak{b}}x_{\mathfrak{b}'},{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}$ at $z_{\mathfrak{b}} \in \mathrm{Obj}(\mathcal{Mod}_{\mathfrak{b}})$ is the unique morphism in $\mathcal{Mod}_{\mathfrak{b}''}$ which makes the following diagram commutative.

$$\begin{array}{ccc} z\circledast(x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}) & \underset{\cong}{\xrightarrow{a_{z,x_{\mathfrak{b}'},{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}}} & (z\circledast x_{\mathfrak{b}'})\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''} \\ {\scriptstyle cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}}\downarrow & & \downarrow{\scriptstyle cq_{z_{\mathfrak{b}},{}_{\mathfrak{b}}x_{\mathfrak{b}'}}\circledast_{\mathfrak{b}'}\mathbb{I}_{{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}} \\ z_{\mathfrak{b}}\circledast_{\mathfrak{b}}({}_{\mathfrak{b}}x_{\mathfrak{b}'}\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''}) & \underset{\cong}{\xrightarrow{\exists!\, a_{z_{\mathfrak{b}},{}_{\mathfrak{b}}x_{\mathfrak{b}'},{}_{\mathfrak{b}'}y_{\mathfrak{b}''}}}} & (z_{\mathfrak{b}}\circledast_{\mathfrak{b}}{}_{\mathfrak{b}}x_{\mathfrak{b}'})\circledast_{\mathfrak{b}'}{}_{\mathfrak{b}'}y_{\mathfrak{b}''} \end{array}$$

We are ready to prove Corollary 1.4.

Proof of Corollary 1.4. By substituting $\mathcal{D} = \mathcal{Mod}_{\mathfrak{b}'}$ in Theorem 1.3, we immediately obtain that statements (i), (ii) are equivalent. We are left to show that statements (i), (iii) are equivalent. The monoids $\mathfrak{b}$, $\mathfrak{b}'$ in $\mathcal{C}$ are Morita equivalent if and only if there exist a pair of cocontinuous $\mathcal{C}$-enriched functors $\alpha : \mathcal{Mod}_{\mathfrak{b}} \to \mathcal{Mod}_{\mathfrak{b}'}$, $\beta : \mathcal{Mod}_{\mathfrak{b}'} \to \mathcal{Mod}_{\mathfrak{b}}$ together with a pair of $\mathcal{C}$-enriched natural isomorphisms $\beta\alpha \cong I_{\mathcal{Mod}_{\mathfrak{b}}}$, $\alpha\beta \cong I_{\mathcal{Mod}_{\mathfrak{b}'}}$. By Corollary 1.2 and using the $\mathcal{C}$-enriched natural isomorphisms (4.2), (4.3), we obtain that the existence of such pair α, β is equivalent to the existence of bimodules ${}_{\mathfrak{b}}x_{\mathfrak{b}'}$, ${}_{\mathfrak{b}'}y_{\mathfrak{b}}$ together with isomorphisms of bimodules ${}_{\mathfrak{b}}x_{\mathfrak{b}'} \circledast_{\mathfrak{b}'} {}_{\mathfrak{b}'}y_{\mathfrak{b}} \cong {}_{\mathfrak{b}}b_{\mathfrak{b}}$ and ${}_{\mathfrak{b}'}y_{\mathfrak{b}} \circledast_{\mathfrak{b}} {}_{\mathfrak{b}}x_{\mathfrak{b}'} \cong {}_{\mathfrak{b}'}b'_{\mathfrak{b}'}$. ■

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