GRADED BRAIDED COMMUTATIVITY IN HOCHSCHILD COHOMOLOGY

JAVIER CÓPPOLA AND ANDREA SOLOTAR

ABSTRACT. We prove the graded braided commutativity of the Hochschild cohomology of A with trivial coefficients, where A is a braided Hopf algebra in the category of Yetter-Drinfeld modules over the group algebra of an abelian group, under some finiteness conditions on a projective resolution of A as A-bimodule. This is a generalization of a result by Mastnak, Pevtsova, Schauenburg and Witherspoon to a context which includes Nichols algebras such as the Jordan and the super Jordan plane. We prove this result by constructing a coduoid-up-to-homotopy structure on the aforementioned projective resolution in the duoidal category of chain complexes of A-bimodules. We also prove that the Hochschild complex of a braided bialgebra A in an arbitrary braided monoidal category is a cocommutative comonoid up to homotopy with the deconcatenation product which induces the cup product in Hochschild cohomology.

1. Introduction

Given a field $\Bbbk$ and an associative algebra A, the Hochschild cohomology $\mathrm{H}^\bullet(A, A)$ has a very rich structure, which has been widely studied. In particular it is a nonnegatively graded algebra with the cup product which is also graded commutative; this fact allows using methods of commutative algebra. The graded commutativity has been first proved by Gerstenhaber [G64], this proof gave rise to the Gerstenhaber bracket. Another proof of the graded commutativity was obtained via the Eckmann-Hilton argument [SÁ04]. In fact the cup product is also defined for $\mathrm{H}^\bullet(A, R)$, where R is an A-bimodule with an associative product $R \otimes_A R \longrightarrow R$, but graded commutativity is in general no longer true in this context. If A is an augmented algebra, one can set $R = \Bbbk$, where $\Bbbk$ is an A-bimodule via the augmentation. It has been proven that for a Hopf algebra A, $\mathrm{H}^\bullet(A, \Bbbk)$ is also graded commutative [FS04, SÁ04, T04]. The proof in [FS04] uses the fact that $\mathrm{H}^\bullet(A, \Bbbk)$ is a subalgebra of $\mathrm{H}^\bullet(A, A)$. The question of what happens when A is a Hopf algebra in a different category has been raised in [MPSW10]. In that article, the authors

work with Hopf algebras in braided categories. They prove that in this situation, if certain internal hom objects exist, then $\mathrm{H}^\bullet(A, \Bbbk)$ is a graded braided commutative algebra. An example of this situation is considering Nichols algebras; these are Hopf algebras in the braided category of Yetter-Drinfeld modules over a Hopf algebra H. They have been thoroughly studied in the case $H = \Bbbk G$ for an abelian or finite group G (see for example [AAPW22, AnSch02, Her20, ŞV16]). The additional hypothesis about internal homs translates here to the condition of either A or H being finite dimensional.

Looking at the computations of the graded algebra structure of $\mathrm{H}^\bullet(A, \Bbbk)$ for A the Jordan plane [LS21] or the super Jordan plane [RS18], we realized that even if the Hochschild cohomology algebra is not graded commutative, it is still graded braided commutative, even though the Jordan and the super Jordan plane are both infinite dimensional and the Hopf algebra is $\Bbbk\mathbb{Z}$, which is also infinite dimensional. The computations are detailed in the Appendix of this paper.

Our main aim is to prove that graded braided commutativity still holds in a more general context. For this, we show how to make use of arbitrary resolutions. The key step is moving from braided monoidal categories to duoidal categories.

The contents of this work are the following:

In Section 2 we recall some preliminary definitions and results, together with some examples that are relevant in the sequel: Hochschild cohomology in Subsection 2.1, monoidal and braided monoidal categories in Subsections 2.2 and 2.3 respectively, Nichols algebras in Subsection 2.4.

The subject of Section 3 is graded braided commutativity. We review some results and definitions from [MPSW10] and finally prove a theorem (see Theorem 3.32) which slightly generalizes Theorem 3.12 and Corollary 3.13 of [MPSW10]. An intermediate step for this result is a theorem of braided graded cocommutativity up to homotopy for the complex analog to the bar resolution (see Theorem 3.30) which makes no use of internal homs.

Section 4 contains our main results about graded braided commutativity, Theorem 4.27 and Corollary 4.28. The cases of the Jordan and of the super Jordan plane are particular instances of these last results.

In this paper we will denote by $\Bbbk$ an arbitrary field. It will be clear from the context whether the symbol $\otimes$ represents the usual tensor product of $\Bbbk$-vector spaces, or stands for the product in an abstract monoidal category.

We thank Marcelo Aguiar and Estanislao Herscovich for fruitful discussions on this topic, and Florencia Cubría and Marcelo Fiori for their help on giving this article its final shape. Special thanks to the anonymous referee, for their detailed report and useful suggestions.

2. Preliminaries

2.1. HOCHSCHILD COHOMOLOGY. In this section we will recall some well-known facts about Hochschild cohomology that will be useful in this article, see for example [Wit19].

2.1.1. DEFINITION. *Let A be a $\Bbbk$-algebra and M be an A-bimodule. The* Hochschild cohomology *of A with coefficients in M is defined as*

$$\mathrm{H}^\bullet(A, M) = \mathrm{Ext}^\bullet_{AA}(A, M),$$

where $\mathrm{Ext}^\bullet_{AA}(A, -)$ denotes the left derived functors of $\mathrm{Hom}_{A\otimes A^\mathrm{op}}(A, -)$, that will be denoted $\mathrm{Hom}_{AA}(A, -)$.

2.1.2. DEFINITION. *Consider A as a bimodule over itself with actions given by its multiplication μ. The* bar resolution *of A is the following free resolution of A as A-bimodule:*

$$\cdots \xrightarrow{b'_3} B_3(A) \xrightarrow{b'_2} B_2(A) \xrightarrow{b'_1} B_1(A) \xrightarrow{b'_0} B_0(A) \longrightarrow 0,$$

where $B_n(A) = A \otimes A^{\otimes n} \otimes A$ for all $n \in \mathbb{N}_0$ and the differential $b'_n : B_{n+1}(A) \longrightarrow B_n(A)$ is given by the formula

$$b'_n(a_0 \otimes \cdots \otimes a_{n+2}) = \sum_{i=0}^{n+1} (-1)^i \, a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}.$$

The quasi-isomorphism from this resolution to A is induced by $\mu : B_0(A) = A \otimes A \longrightarrow A$.

Thus, when computing Hochschild cohomology via this resolution one gets $\mathrm{H}^\bullet(A, M) = H(\mathrm{Hom}_{AA}(B_\bullet(A), M), (b'_\bullet)^*)$. The naturality of the adjunction $\mathrm{Hom}_{AA}(A \otimes V \otimes A, M) \simeq \mathrm{Hom}_\Bbbk(V, M)$ implies that the complexes $(\mathrm{Hom}_{AA}(B_\bullet(A), M), (b'_\bullet)^*)$ and $(C^\bullet(A), d^\bullet)$ are isomorphic, where $C^n(A) = \mathrm{Hom}_\Bbbk(A^{\otimes n}, M)$ and $d^n : C^n(A) \longrightarrow C^{n+1}(A)$ is:

$$
\begin{aligned}
d^n f(a_1 \otimes \cdots \otimes a_{n+1}) \;=\; & a_1 \cdot f(a_2 \otimes \cdots \otimes a_{n+1}) \\
& + \sum_{i=1}^n (-1)^i \, f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
& + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}.
\end{aligned}
$$

2.1.3. REMARK. Let R be a $\Bbbk$-algebra, and $\rho : A \longrightarrow R$ an algebra homomorphism. There is an A-bimodule structure on R defining $a \cdot r \cdot b = \rho(a) r \rho(b), \forall a, b \in A, \ r \in R$. Moreover, with this bimodule structure the multiplication $\mu : R \otimes R \longrightarrow R$ factors through $R \otimes_A R$.

. Let $P_\bullet \longrightarrow A$ be a projective resolution of A as A-bimodule. The canonical identification $A \xrightarrow{\;\cong\;} A \otimes_A A$ lifts to a chain morphism $\omega : P_\bullet \longrightarrow (P \otimes_A P)_\bullet$, unique up to a unique chain homotopy. This map allows the definition of a convolution product in $\mathrm{Hom}_{AA}(P_\bullet, R)$, denoted by $\cup$, which passes to cohomology. This is the *cup product* of the Hochschild cohomology of A with coefficients in R, denoted by $\cup : \mathrm{H}^\bullet(A, R) \otimes \mathrm{H}^\bullet(A, R) \longrightarrow \mathrm{H}^\bullet(A, R)$.

For the bar resolution, one can explicitly define a coproduct as follows:

$$\omega(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = \sum_{p+q=n} (1 \otimes a_1 \otimes \cdots \otimes a_p \otimes 1) \otimes_A (1 \otimes a_{p+1} \otimes \cdots \otimes a_{p+q} \otimes 1),$$

which gives the following well-known expression for the cup product:

$$(\varphi \cup \psi)(1 \otimes a_1 \otimes \cdots \otimes a_{p+q} \otimes 1) = \varphi(1 \otimes a_1 \otimes \cdots \otimes a_p \otimes 1)\psi(1 \otimes a_{p+1} \otimes \cdots \otimes a_{p+q} \otimes 1),$$

for all $\varphi \in \mathrm{Hom}_{AA}(B_p(A), R), \psi \in \mathrm{Hom}_{AA}(B_q(A), R), a_1, \ldots, a_{p+q} \in A$.

Translating this to the complex $(C^\bullet(A), d^\bullet)$ gives the following formula:

$$(\alpha \cup \beta)(a_1 \otimes \cdots \otimes a_{p+q}) = \alpha(a_1 \otimes \cdots \otimes a_p)\beta(a_{p+1} \otimes \cdots \otimes a_{p+q}), \ \forall a_1, \ldots, a_{p+q} \in A,$$

$\alpha \in C^p(A), \beta \in C^q(A), a_1, \ldots a_{p+q} \in A$.

2.1.4. DEFINITION. *An* augmented $\Bbbk$-algebra *is a $\Bbbk$-algebra A equipped with an algebra homomorphism $\varepsilon : A \longrightarrow \Bbbk$. We will omit ε when it is clear from the context.*
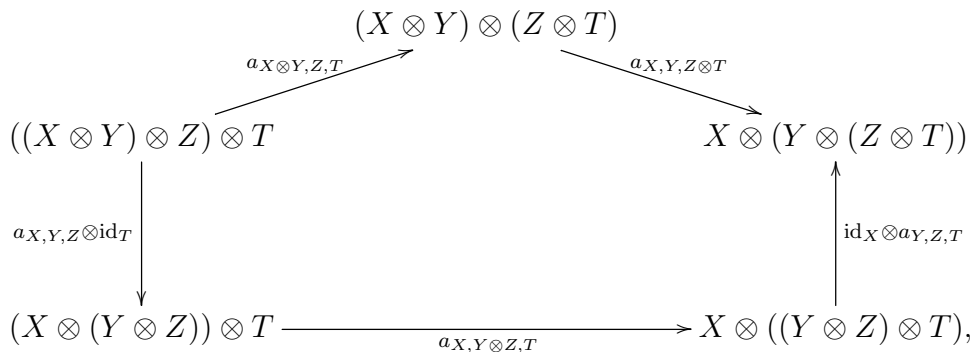
Notice that if A is an augmented $\Bbbk$-algebra then $\Bbbk$ becomes an A-bimodule. We will focus our attention in the algebra $\mathrm{H}^\bullet(A, \Bbbk)$.

Every Hopf algebra A is an augmented algebra via its counit. Its cohomology with trivial coefficients is isomorphic to $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$, which is the so called *Hopf algebra cohomology* [Wit19, Definition 9.3.5]. Note that the category of A-bimodules is monoidal with the usual tensor product whenever A is a Hopf algebra. After reviewing some concepts on braided monoidal categories, we will see a generalization of this fact in the last section.

2.2. MONOIDAL CATEGORIES. In this section we will recall some facts about monoidal categories. For a more complete study in this topic we refer to [Mac98] and [AM10].

2.2.1. DEFINITION. *A* monoidal category *is a 6-tuple $(\mathcal{C}, \otimes, I, a, \ell, r)$, where*

- $\mathcal{C}$ *is a category,*

- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *is a bifunctor, called its* product,

- I *is an object in $\mathcal{C}$, called its* unit,

- a *is a natural isomorphism, called* associator, *given by a family of morphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$ for each $X, Y, Z \in \mathcal{C}$, such that the following pentagonal diagram commutes for all $X, Y, Z, T \in \mathcal{C}$:*

$$
\begin{array}{ccc}
 & (X \otimes Y) \otimes (Z \otimes T) & \\
\nearrow a_{X \otimes Y, Z, T} & & \searrow a_{X, Y, Z \otimes T} \\
((X \otimes Y) \otimes Z) \otimes T & & X \otimes (Y \otimes (Z \otimes T)) \\
\downarrow a_{X,Y,Z} \otimes \mathrm{id}_T & & \uparrow \mathrm{id}_X \otimes a_{Y,Z,T} \\
(X \otimes (Y \otimes Z)) \otimes T \xrightarrow[a_{X, Y \otimes Z, T}]{} & & X \otimes ((Y \otimes Z) \otimes T),
\end{array}
$$

- ℓ *and r are natural isomorphisms, called* unitors, *given by families of morphisms $\ell_X : I \otimes X \longrightarrow X$ and $r_X : X \otimes I \longrightarrow X$ such that the following triangular diagram*

commutes for all $X, Y \in \mathcal{C}$:

$$(X \otimes I) \otimes Y \xrightarrow{\quad a_{X,I,Y} \quad} X \otimes (I \otimes Y)$$

$$r_X \otimes \mathrm{id}_Y \searrow \qquad \swarrow \mathrm{id}_X \otimes \ell_Y$$

$$X \otimes Y.$$

A monoidal category is called strict *if a, ℓ, r are identities.*

When the rest of the structure is clear from its context, we may refer to a monoidal category just by the name of the underlying category $\mathcal{C}$. We will also say that it is a monoidal structure *on* $\mathcal{C}$.

MacLane's Coherence Theorem [Mac98, Chapter VII, Section 2] states that these two relations imply that every composition of these morphisms from one arbitrary product of objects in $\mathcal{C}$ to another one yields the same morphism.

2.2.2. REMARK. If $(\mathcal{C}, \otimes, I, a, \ell, r)$ is a monoidal category, then its *opposite category* $\mathcal{C}^{\mathrm{op}}$ admits a monoidal structure given by $(\mathcal{C}^{\mathrm{op}}, \otimes, I, a^{-1}, \ell^{-1}, r^{-1})$.

2.2.3. DEFINITION. *Let $(\mathcal{C}, \otimes, I, a, \ell, r)$ be a monoidal category. Its* transpose *is the monoidal category $(\mathcal{C}, \widetilde{\otimes}, I, a^{-1}, r, \ell)$, where $X \widetilde{\otimes} Y = Y \otimes X$ for all $X, Y \in \mathcal{C}$.*

Without loss of generality, from now on we will consider strict monoidal categories, since every monoidal category is equivalent to a strict one via functors that preserve the monoidal structure [Mac98, Chapter XI, Section 3], in a sense that will be precisely stated later. This is also the way monoidal categories are usually dealt with.

In this work, all monoidal categories will be $\Bbbk$-linear, being the product bilinear with respect to morphisms and distributive over direct sums. In order to have (co)chain complexes and their (co)homology inside these categories, we will work with an abelian structure as well. A more recent reference book for this setting is [EGNO15]. One possible extension of this approach could be to consider exact categories, we will not develop it in this article.

2.2.4. DEFINITION. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A* monoid *in $\mathcal{C}$ is a triple (A, μ, η), where*

- *A is an object in $\mathcal{C}$,*

- *$\mu : A \otimes A \longrightarrow A$ is a morphism in $\mathcal{C}$, called* multiplication *or* product*, such that the following diagram commutes:*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}_A} & A \otimes A \\ {\scriptstyle \mathrm{id}_A \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad \mu \quad} & A, \end{array}$$

This property is called associativity *of the product.*

- $\eta : I \longrightarrow A$ is a morphism in $\mathcal{C}$, called unit, such that the following diagram commutes:

$$A \otimes I \xrightarrow{\mathrm{id}_A \otimes \eta} A \otimes A \xleftarrow{\eta \otimes \mathrm{id}_A} I \otimes A$$

with id_A, μ, id_A to A.

This property is called unitality of the product.

In several texts in the literature, for instance [EGNO15] and [MPSW10], monoids are called algebras.

2.2.5. DEFINITION. A comonoid in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$.

2.2.6. REMARK. A monoid in a monoidal category is a monoid in its transpose, and viceversa. The same happens for comonoids.

2.2.7. DEFINITION. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let (A, μ, η) and (A', μ', η') be monoids in $\mathcal{C}$. A morphism of monoids from A to A' is a morphism $f : A \longrightarrow A'$, such that the following diagrams commute:

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\
\downarrow{\mu} & & \downarrow{\mu'} \\
A & \xrightarrow{f} & A',
\end{array}
\qquad
\begin{array}{ccc}
 & I & \\
\eta \swarrow & & \searrow \eta' \\
A & \xrightarrow{f} & A'.
\end{array}
$$

2.2.8. REMARK. It can be proved directly that identities and composition of morphisms of monoids are morphisms of monoids. Thus, they form a category, often named $\mathrm{Mon}(\mathcal{C})$.

Next we give examples of monoidal categories which give rise to (co)monoids found in the literature.

2.2.9. EXAMPLE. For the monoidal category of $\Bbbk$-vector spaces $(\mathrm{Vect}, \otimes_{\Bbbk}, \Bbbk)$, (co)monoids are ordinary $\Bbbk$-(co)algebras.

2.2.10. EXAMPLE. Let R be a $\Bbbk$-algebra. There is a monoidal structure in the category of R-bimodules $({}_R\mathrm{Mod}_R, \otimes_R, R)$, where R acts on itself by multiplication. Its monoids are called R-algebras.

2.2.11. EXAMPLE. The category of $\mathbb{Z}$-graded $\Bbbk$-vector spaces has a monoidal structure $(\mathrm{Vect}^{\mathbb{Z}}, \otimes, \Bbbk_0)$, where the product of two $\mathbb{Z}$-graded vector spaces $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and $W = \bigoplus_{n \in \mathbb{Z}} W_n$ is given by

$$(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes_{\Bbbk} W_q,$$

and the unit $\Bbbk_0$ is the graded vector space $\Bbbk$ concentrated in degree 0. (Co)monoids in this monoidal category are known as graded $\Bbbk$-(co)algebras.

2.2.12. EXAMPLE. Let $(C^\bullet, d_C^\bullet)$ and $(D^\bullet, d_D^\bullet)$ be two nonnegatively graded cochain complexes of vector spaces. The product $C \otimes D$ of their underlying $\mathbb{N}_0$-graded spaces can be defined as in the previous example, and the map $\delta^\bullet$ defined by

$$\delta^{p+q}|_{C^p \otimes D^q} = d_C^p \otimes \mathrm{id}_{D^q} + (-1)^p \, \mathrm{id}_{C^p} \otimes d_D^q$$

is a differential. This happens to be the total complex of the product of both complexes. A unit element for this product is again the graded vector space $\mathbb{k}_0$ with zero differential, giving rise to the monoidal category $(\mathrm{Coch}(\mathrm{Vect}), \otimes, \mathbb{k}_0)$. (Co)monoids in this category are called *differential graded $\mathbb{k}$-(co)algebras.*

One can define a monoidal structure in the category of chain complexes in an analogous way.

2.2.13. REMARK. The previous constructions can be made for $\mathcal{C}^\mathbb{Z}, \mathrm{Ch}(\mathcal{C}), \mathrm{Coch}(\mathcal{C})$ whenever $\mathcal{C}$ is a monoidal category with a $\mathbb{k}$-linear structure and arbitrary direct sums.

2.2.14. EXAMPLE. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a $\mathbb{k}$-bialgebra. It is well-known that the category $_H\mathrm{Mod}$ of left H-modules is a monoidal category with the usual tensor product of vector spaces. If V and W are two H-modules, the action on $V \otimes W$ is given by

$$h \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w, \ \forall h \in H, v \in V, w \in W,$$

where $\Delta(h) = h_{(1)} \otimes h_{(2)} \in H \otimes H$ is the usual Sweedler notation for the coproduct, with the sum ommitted. The unit of the product $\mathbb{k}$ can be given an H-module structure via

$$h \cdot 1 = \varepsilon(h), \ \forall h \in H.$$

(Co)-monoids in this monoidal category are called *H-module (co)algebras.*

2.2.15. EXAMPLE. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a $\mathbb{k}$-bialgebra. The category $^H\mathrm{Mod}$ of left H-comodules is a monoidal category with the usual tensor product of vector spaces. If V and W are two H-modules, the coaction on $V \otimes W$ is given by

$$(v \otimes w)_{[-1]} \otimes (v \otimes w)_{[0]} = v_{[-1]}w_{[-1]} \otimes v_{[0]} \otimes w_{[0]}, \ \forall v \in V, w \in W,$$

where $v_{[-1]} \otimes v_{[0]} \in H \otimes V$ is the usual Sweedler notation for the coaction, with the sum omitted. The unit of the product $\mathbb{k}$ can be given an H-comodule structure defining

$$1_{[-1]} \otimes 1_{[0]} = 1 \otimes 1.$$

(Co)-monoids in this monoidal category are called *H-comodule (co)algebras.*

If $H = \mathbb{k}G$ is a group algebra, the equality

$$V_g = \{v \in V : \ v_{[-1]} \otimes v_{[0]} = g \otimes v\}$$

gives a $1 - 1$ correspondence between left H-comodules and *G-graded vector spaces*, that is, vector spaces with a decomposition $V = \bigoplus_{g \in G} V_g$. The formula above for the tensor

product of two H-comodules V and W yields a grading in the tensor product $V \otimes W = \bigoplus_{g \in G} (V \otimes W)_g$, where $(V \otimes W)_g = \bigoplus_{st=g} V_s \otimes W_t$. The grading on the unit $\Bbbk$ induced by its $\Bbbk G$-coaction is concentrated in degree 1_G. In this case, an H-comodule algebra is known as a G-graded algebra, that is, a $\Bbbk$-algebra $A = \bigoplus_{g \in G} A_g$ such that $A_s A_t \subseteq A_{st}$, $\forall s, t \in G$, and $1_A \in A_{1_G}$. If $G = \mathbb{Z}$, the monoidal category obtained this way is exactly the category $\mathrm{Vect}^{\mathbb{Z}}$ of Example 2.2.11. Since this case includes the examples we are focused on, in order to avoid confusion with homological degree, we will say that an element v of a G-graded vector space V has *internal degree* g if $v \in V_g$.

2.2.16. **Example.** Let $(H, \mu, \eta, \Delta, \varepsilon, s)$ be a $\Bbbk$-Hopf algebra. A *left-left Yetter-Drinfeld module over H* is a vector space V endowed with a left H-action and a left H-coaction such that

$$(h \cdot v)_{[-1]} \otimes (h \cdot v)_{[0]} = h_{(1)} v_{[-1]} s(h_{(3)}) \otimes h_{(2)} \cdot v_{[0]}, \ \forall h \in H, v \in V.$$

In the particular case where $H = \Bbbk G$ is a group algebra, the compatibiliy condition translates as $g \cdot V_h \subseteq V_{ghg^{-1}}$. If, in addition, G is abelian, this condition results in $g \cdot V_h \subseteq V_h$, so a Yetter-Drinfeld module over $\Bbbk G$ is a G-module $V = \bigoplus_{h \in G} V_h$, where V_h is a G-submodule of V for every $h \in G$.

A *morphism of left-left Yetter-Drinfeld modules over H* is a linear map that is simultaneously a morphism of left H-modules and a morphism of left H-comodules. The category consisting of the class of left-left Yetter Drinfeld modules over H and their morphisms is denoted by $_H^H \mathcal{YD}$. It can be given a monoidal structure with the usual tensor of vector spaces, defining an action and a coaction on the tensor product of two Yetter-Drinfeld modules in the same way as in the previous examples. It is straightforward to check that the new action and coaction are compatible.

A fact that will be used later is the following: For a Yetter-Drinfeld module $V \in {}_H^H \mathcal{YD}$, let $T(V) = \bigoplus_{n \in \mathbb{N}_0} V^{\otimes n}$ denote the tensor algebra of V with the concatenation product and the inclusion of $\Bbbk = V^{\otimes 0}$ as unit. Since $T(V)$ is a direct sum of tensor products of a Yetter-Drinfeld module over H, it can also be regarded as a Yetter-Drinfeld module over H, and it is immediate to check that the aforementioned product and unit preserve the Yetter-Drinfeld module structure. Therefore, one can consider $T(V)$ as a monoid in $_H^H \mathcal{YD}$.

2.2.17. **Example.** The *Jordan plane* is the algebra $A = \frac{\Bbbk\langle x,y \rangle}{\langle yx - xy + \frac{1}{2} x^2 \rangle}$. Let us define an action of the Hopf algebra $\Bbbk \mathbb{Z} = \Bbbk[t, t^{-1}]$ on A by $t \cdot x = x$, $t \cdot y = x + y$, and a $\mathbb{Z}$-grading on A setting $x, y \in A_1$. These can be extended multiplicatively to $\Bbbk\langle x, y \rangle$, and are well defined after taking the quotient because the relation is homogeneous and preserved by the action. Since the action preserves the grading, A is a Yetter-Drinfeld module over $\Bbbk \mathbb{Z}$. By definition, the algebra structure of A is compatible with the action and the coaction of $\Bbbk \mathbb{Z}$, so A is a monoid in the monoidal category $_{\Bbbk \mathbb{Z}}^{\Bbbk \mathbb{Z}} \mathcal{YD}$.

2.2.18. **Example.** This is an example of the same type, with a different action of $\mathbb{Z}$. The *super Jordan plane* is the algebra $A = \frac{\Bbbk\langle x,y \rangle}{\langle x^2, y^2 x - xy^2 - xyx \rangle}$ with $\Bbbk \mathbb{Z}$-action on A given by $t \cdot x = -x$, $t \cdot y = x - y$, and $\mathbb{Z}$-grading on A setting $x, y \in A_1$. As in the previous example,

they induce on A the structure of Yetter-Drinfeld module over $\Bbbk\mathbb{Z}$ which is compatible with its product and unit, so A is a monoid in $^{\Bbbk\mathbb{Z}}_{\Bbbk\mathbb{Z}}\mathcal{YD}$.

2.2.19. EXAMPLE. Let $\rho : A \longrightarrow R$ be a $\Bbbk$-algebra homomorphism. From Remark 2.1.3, we know that ρ endows R with an A-bimodule structure. Now let $P_\bullet \xrightarrow{f} A$ be a projective resolution of A as A-bimodule and consider the chain map $\omega : P_\bullet \longrightarrow (P \otimes_A P)_\bullet$ used in the definition of the cup product in Hochschild cohomology. The triple $(P_\bullet, \omega, f)$ is a comonoid in the monoidal category $\left(\overline{\mathrm{Ch}}\left(_A\mathrm{Mod}_A\right), \otimes_A, A_0\right)$, where the bar indicates that morphisms in this category are those of $\mathrm{Ch}\left(_A\mathrm{Mod}_A\right)$ modulo homotopy (this is necessary for the associativity and unitality conditions).

The convolution product makes $\left(\mathrm{Hom}_{AA}(P_\bullet, R), \cup, (1 \mapsto \rho \circ f)\right)$ a monoid in the monoidal category $(\overline{\mathrm{Coch}}(\mathrm{Vect}), \otimes, \Bbbk_0)$, and taking cohomology makes $\mathrm{H}^\bullet(A, R)$ a monoid in the monoidal category $(\mathrm{Vect}^{\mathbb{Z}}, \otimes, \Bbbk_0)$.

2.2.20. DEFINITION. *Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \times, J)$ be monoidal categories. A* lax monoidal functor *from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, φ, φ_0), where*

- $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is a functor,*

- φ *is a natural transformation given by a family of morphisms $\{\varphi_{X,Y}\}_{X,Y \in \mathcal{C}}$, where $\varphi_{X,Y} : F(X) \times F(Y) \longrightarrow F(X \otimes Y)$, such that for every $X, Y, Z \in \mathcal{C}$ the following diagram commutes:*

$$
\begin{array}{ccc}
F(X) \times F(Y) \times F(Z) & \xrightarrow{\varphi_{X,Y} \times \mathrm{id}_Z} & F(X \otimes Y) \times Z \\
{\scriptstyle \mathrm{id}_X \times \varphi_{Y,Z}} \downarrow & & \downarrow {\scriptstyle \varphi_{X \otimes Y, Z}} \\
F(X) \times F(Y \otimes Z) & \xrightarrow[\varphi_{X, Y \otimes Z}]{} & F(X \otimes Y \otimes Z),
\end{array}
$$

- $\varphi_0 : J \longrightarrow F(I)$ *is a morphism in $\mathcal{D}$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
F(X) \xrightarrow{=} J \times F(X) & \qquad & F(X) \xrightarrow{=} F(X) \times J \\
{\scriptstyle =} \downarrow \qquad \downarrow {\scriptstyle \varphi_0 \times \mathrm{id}_{F(X)}} & & {\scriptstyle =} \downarrow \qquad \downarrow {\scriptstyle \mathrm{id}_{F(X)} \times \varphi_0} \\
F(I \otimes X) \xleftarrow[\varphi_{I,X}]{} F(I) \times F(X), & & F(X \otimes I) \xleftarrow[\varphi_{X,I}]{} F(X) \times F(I).
\end{array}
$$

We will denote a lax monoidal functor by $(F, \varphi, \varphi_0) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \times, J)$, and eventually call it a lax monoidal structure on F.

2.2.21. DEFINITION. *A* strong monoidal functor *is a lax monoidal functor (F, φ, φ_0) such that φ is a natural isomorphism and φ_0 is an isomorphism.*

2.2.22. EXAMPLE. We may consider the functor $V \mapsto V^* = \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$ as a monoidal functor $((-)^*, \varphi, \varphi_0) : (\operatorname{Vect}^{\operatorname{op}}, \otimes, \Bbbk) \longrightarrow (\operatorname{Vect}, \otimes, \Bbbk)$. The natural transformation φ given by $\varphi_{V,W} : V^* \otimes W^* \longrightarrow (V \otimes W)^*$ is defined by $\varphi_{V,W}(f \otimes g) = f \otimes g$, considering the first $f \otimes g$ as the tensor product of two elements and the last one as the tensor product of two morphisms. The morphism $\varphi_0 : \Bbbk \longrightarrow \Bbbk^*$ is defined by $\varphi_0(1) = \operatorname{id}_{\Bbbk}$. This monoidal functor is not strong, but its restriction to the opposite category of finite dimensional vector spaces is so.

2.2.23. REMARK. Defining monoidal structures like in Examples 2.2.11 to 2.2.16, adding extra structure to the usual tensor product of $\Bbbk$-vector spaces, automatically gives their respective forgetful functors strong monoidal structures via identities.

2.2.24. EXAMPLE. In the case considered in Example 2.2.16 where $H = \Bbbk\mathbb{Z}$, the correspondence between left-left Yetter Drinfeld modules over $\Bbbk\mathbb{Z}$ and $\mathbb{Z}$-graded $\mathbb{Z}$-modules can be stated by saying that the identity induces a strong monoidal functor from $^{\Bbbk\mathbb{Z}}_{\Bbbk\mathbb{Z}}\mathcal{YD}$ to $(_{\Bbbk\mathbb{Z}}\operatorname{Mod})^{\mathbb{Z}}$, where the monoidal structure of the latter is as given in Remark 2.2.13.

2.2.25. EXAMPLE. Let R be a $\Bbbk$-algebra. The forgetful functor $U : {}_R\operatorname{Mod}_R \longrightarrow \operatorname{Vect}$ admits a lax monoidal structure $(U, \pi, \eta) : ({}_R\operatorname{Mod}_R, \otimes_R, R) \longrightarrow (\operatorname{Vect}, \otimes, \Bbbk)$, where $\pi_{M,N} : M \otimes N \longrightarrow M \otimes_R N$ is the canonical projection and $\eta : \Bbbk \longrightarrow R$ is the unit morphism of R. This lax monoidal functor is not strong in general.

2.2.26. EXAMPLE. Let us recall the Künneth Theorem for chain complexes from [Wei94, Thm 3.6.3]: Let R be a ring and let $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ be right and left R-module chain complexes, respectively. If P_n and $d_{n+1}(P_n)$ are flat for each n, then there is an exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \longrightarrow H_n(P \otimes_R Q) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(P), H_q(Q)) \longrightarrow 0.$$

When $R = \Bbbk$ is a field, the term involving Tor functors vanishes, hence the other two terms are isomorphic. As a consequence, the homology functor $H_\bullet : \operatorname{Ch}(\operatorname{Vect}) \longrightarrow \operatorname{Vect}^{\mathbb{Z}}$ is strong monoidal.

2.3. BRAIDED MONOIDAL CATEGORIES. In this section we will pay attention to an additional structure in some monoidal categories: braidings.

2.3.1. DEFINITION. *A* braided monoidal category *is a 7-tuple* $(\mathcal{C}, \otimes, I, a, \ell, r, c)$, *where*

- $(\mathcal{C}, \otimes, I, a, \ell, r)$ *is a monoidal category,*

- *c is a natural isomorphism, called* braiding, *given by a family of morphisms* $c_{X,Y} :$ $X \otimes Y \longrightarrow Y \otimes X$ *for each* $X, Y \in \mathcal{C}$, *such that the following diagrams commute for*

every $X, Y, Z \in \mathcal{C}$:

$$
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\
\downarrow{a_{X,Y,Z}^{-1}} & & \uparrow{a_{Y,Z,X}^{-1}} \\
(X \otimes Y) \otimes Z & & Y \otimes (Z \otimes X) \\
\downarrow{c_{X,Y} \otimes \mathrm{id}_Z} & & \uparrow{\mathrm{id}_Y \otimes c_{X,Z}} \\
(Y \otimes X) \otimes Z & \xrightarrow{a_{Y,X,Z}} & Y \otimes (X \otimes Z),
\end{array}
$$

$$
\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{c_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) \\
\downarrow{a_{X,Y,Z}} & & \uparrow{a_{Z,X,Y}} \\
X \otimes (Y \otimes Z) & & (Z \otimes X) \otimes Y \\
\downarrow{\mathrm{id}_X \otimes c_{Y,Z}} & & \uparrow{c_{X,Z} \otimes \mathrm{id}_Y} \\
X \otimes (Z \otimes Y) & \xrightarrow{a_{X,Z,Y}^{-1}} & (X \otimes Z) \otimes Y.
\end{array}
$$

2.3.2. REMARK. It is proven in Proposition 1.1 of [JS93] that commutativity of the previous diagrams implies commutativity of the following diagram for every $X \in \mathcal{C}$:

$$
\begin{array}{ccc}
X \otimes I & \xrightarrow{c_{X,I}} & I \otimes X \ . \\
& \searrow{r_X} \qquad \swarrow{\ell_X} & \\
& X &
\end{array}
$$

2.3.3. REMARK. Let $(\mathcal{C}, \otimes, I, a, \ell, r, c)$ be a braided monoidal category, and consider the monoidal structures on its opposite and transpose categories as in Remark 2.2.2 and Definition 2.2.3, respectively. They both admit the braiding $\tilde{c}$, where $\tilde{c}_{X,Y} = c_{Y,X}$ for all $X, Y \in \mathcal{C}$.

2.3.4. REMARK. For a braided monoidal category $(\mathcal{C}, \otimes, I, a, \ell, r, c)$, inverting the braiding gives another braided monoidal category $(\mathcal{C}, \otimes, I, a, \ell, r, c^{inv})$, where $c_{X,Y}^{inv} = (c_{Y,X})^{-1}$ for all $X, Y \in \mathcal{C}$.

2.3.5. REMARK. If a braided monoidal category is strict as a monoidal category, the hexagonal diagrams above turn into triangles, yielding the following equalities for every $X, Y, Z \in \mathcal{C}$:

$$
c_{X, Y \otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{id}_Z),
$$
$$
c_{X \otimes Y, Z} = (c_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z}).
$$

From these equalities one can deduce the *braid equation*, also referred to as the commutativity of the Yang-Baxter hexagon [JS91, Chapter 3]:

$$
(c_{Y,Z} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{X,Y}) \circ (c_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z}).
$$

All braided monoidal categories considered in this work will be strict as monoidal categories.

2.3.6. DEFINITION. *Let $(\mathcal{C}, \otimes, I, c)$ be a braided monoidal category.*

- *A monoid (A, μ, η) in $\mathcal{C}$ is* commutative *if the following diagram commutes:*

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\ c_{A,A}\ } & A \otimes A \ . \\
& \searrow^{\mu} \quad \swarrow^{\mu} & \\
& A &
\end{array}
$$

- *A comonoid (A, Δ, ε) in $\mathcal{C}$ is* cocommutative *if the following diagram commutes:*

$$
\begin{array}{ccc}
& A & \\
& \swarrow^{\Delta} \quad \searrow^{\Delta} & \\
A \otimes A & \xrightarrow[\ c_{A,A}\ ]{} & A \otimes A \ .
\end{array}
$$

Next we will discuss some relevant examples.

2.3.7. EXAMPLE. The formula $\tau_{V,W}(v \otimes w) = w \otimes v$, $\forall v \in V, w \in W$, defines a braiding τ in the categories Vect, $^H\mathrm{Mod}$ for a commutative $\Bbbk$-bialgebra H, and $_H\mathrm{Mod}$ for a cocommutative $\Bbbk$-bialgebra H. In these cases, (co)commutative (co)monoids are (co)monoids in their respective monoidal categories whose underlying $\Bbbk$-(co)algebra is (co)commutative in the usual sense.

2.3.8. REMARK. When working in braided monoidal categories whose objects are $\Bbbk$-vector spaces with additional structure, commutativity may be called *braided commutativity*, and analogously with cocommutativity. This is useful to emphasize that the braiding involved is not τ, and so the notion of commutativity involved differs from the usual one.

2.3.9. EXAMPLE. In the categories $\mathrm{Ch}(\mathrm{Vect}), \mathrm{Coch}(\mathrm{Vect})$ and $\mathrm{Vect}^{\mathbb{Z}}$, one can define a braiding τ^{gr} by the formula $\tau^{\mathrm{gr}}(v \otimes w) = (-1)^{pq} w \otimes v$, $\forall v \in V_p, w \in W_q$. The concept of (co)commutativity for this braiding is called *graded (co)commutativity*. In the categories $\overline{\mathrm{Ch}}(\mathrm{Vect})$ and $\overline{\mathrm{Coch}}(\mathrm{Vect})$, it is called *graded (co)commutativity up to homotopy*.

Note that the category $\mathrm{Vect}^{\mathbb{Z}}$ also admits the braiding τ, since it can also be regarded as the category of comodules over the cocommutative $\Bbbk$-Hopf algebra $\Bbbk\mathbb{Z}$. In this work, we will not use the braiding τ for this category, and the name $\mathrm{Vect}^{\mathbb{Z}}$ will be used only for the braided monoidal category of $\mathbb{Z}$-graded $\Bbbk$-vector spaces with braiding τ^{gr}.

2.3.10. EXAMPLE. The previous construction can be generalized to an arbitrary braided monoidal category $(\mathcal{C}, \otimes, I, c)$ with a $\Bbbk$-linear structure: in this case, one can define a braiding c^{gr} in the categories $\mathrm{Ch}(\mathcal{C}), \mathrm{Coch}(\mathcal{C})$ and $\mathcal{C}^{\mathbb{Z}}$ by the formula $c^{\mathrm{gr}}_{V,W}|_{V_p \otimes W_q} = (-1)^{pq} c_{V_p, W_q}$. As in Remark 2.3.8, (co)commutativity in these categories may be called *graded braided (co)commutativity*, or *braided graded (co)commutativity*, and also its versions up to homotopy in $\overline{\mathrm{Ch}}(\mathcal{C})$ and $\overline{\mathrm{Coch}}(\mathcal{C})$.

2.3.11. EXAMPLE. Let H be a $\Bbbk$-Hopf algebra. The category $(^H_H\mathcal{YD}, \otimes, \Bbbk)$ admits a braiding c such that

$$c_{V,W}(v \otimes w) = v_{[0]} \cdot w \otimes v_{[1]}, \ \forall v \in V, w \in W.$$

In the case $H = \Bbbk G$ for a group G, the braiding has the following expression:

$$c_{V,W}(v \otimes w) = g \cdot w \otimes v, \ \forall g \in G, v \in V_g, w \in W.$$

In the following sections we will define certain (co)monoids in the monoidal categories $\overline{\mathrm{Ch}}\left(^H_H\mathcal{YD}\right), \overline{\mathrm{Coch}}\left(^H_H\mathcal{YD}\right)$ and $\left(^H_H\mathcal{YD}\right)^{\mathbb{Z}}$, and prove that they are (co)commutative. To introduce them, we will need the following concept, which generalizes the definition of bialgebra to arbitrary braided monoidal categories.

2.3.12. DEFINITION. *Let $(\mathcal{C}, \otimes, I, c)$ be a braided monoidal category. A* bimonoid *in $\mathcal{C}$ is a 5-tuple $(A, \mu, \eta, \Delta, \varepsilon)$, where*

- *(A, μ, η) is a monoid,*

- *(A, Δ, ε) is a comonoid,*

- *the following diagrams commute:*

$$
\begin{array}{ccccc}
A \otimes A & \xrightarrow{\ \mu\ } & A & \xrightarrow{\ \Delta\ } & A \otimes A \\
{\scriptstyle \Delta \otimes \Delta}\downarrow & & & & \uparrow{\scriptstyle \mu \otimes \mu} \\
A \otimes A \otimes A \otimes A & \xrightarrow[\mathrm{id}_A \otimes c_{A,A} \otimes \mathrm{id}_A]{} & & & A \otimes A \otimes A \otimes A,
\end{array}
$$

$$
\begin{array}{ccc}
I \xrightarrow{\ \simeq\ } I \otimes I & \quad A \otimes A \xrightarrow{\ \mu\ } A & \quad I \xrightarrow{\ \mathrm{id}_I\ } I \\
{\scriptstyle \eta}\downarrow \quad \downarrow{\scriptstyle \eta \otimes \eta} & {\scriptstyle \varepsilon}\downarrow \quad \downarrow{\scriptstyle \varepsilon \otimes \varepsilon} & {\scriptstyle \eta}\searrow \quad \nearrow{\scriptstyle \varepsilon} \\
A \xrightarrow{\ \Delta\ } A \otimes A, & I \otimes I \xrightarrow{\ \simeq\ } I, & A.
\end{array}
$$

2.3.13. REMARK. If (A, μ, η) and (A', μ', η') are two monoids in a braided monoidal category $(\mathcal{C}, \otimes, I, c)$, then the triple $(A \otimes A', (\mu \otimes \mu') \circ (\mathrm{id}_A \otimes c_{A',A} \otimes \mathrm{id}_{A'}), \eta \otimes \eta')$ is also a monoid.

Analogously, if (A, Δ, ε) and $(A', \Delta', \varepsilon')$ are two comonoids, their product admits the comonoid structure $(A \otimes A', (\mathrm{id} \otimes c_{A,A'} \otimes \mathrm{id}_{A'}) \circ (\Delta \otimes \Delta'), \varepsilon \otimes \varepsilon')$.

As for ordinary bialgebras, the axioms of a bimonoid $(A, \mu, \eta, \Delta, \varepsilon)$ are equivalent to Δ and ε being morphisms of monoids, and they are also equivalent to μ and η being morphisms of comonoids.

2.3.14. DEFINITION. *A Hopf monoid is a 6-tuple* $(A, \mu, \eta, \Delta, \varepsilon, s)$*, where* $(A, \mu, \eta, \Delta, \varepsilon)$ *is a bimonoid, and* $s : A \longrightarrow A$ *is a morphism in* $\mathcal{C}$*, called* antipode*, such that the following diagram commutes*

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\;s \otimes \mathrm{id}_A\;} & A \otimes A \\
{\scriptstyle \Delta}\big\uparrow & & \big\downarrow{\scriptstyle \mu} \\
A \xrightarrow{\;\varepsilon\;} I & \xrightarrow{\;\eta\;} & A \\
{\scriptstyle \Delta}\big\downarrow & & \big\uparrow{\scriptstyle \mu} \\
A \otimes A & \xrightarrow[\;\mathrm{id}_A \otimes s\;]{} & A \otimes A.
\end{array}
$$

2.3.15. REMARK. Bimonoids and Hopf monoids in a braided monoidal category $\mathcal{C}$ are often named *braided bialgebras* and *braided Hopf algebras*, respectively, see for example [EGNO15].

2.3.16. REMARK. Unlike previous examples, the so called graded Hopf algebras in the literature do not correspond to Hopf monoids in the category $\mathrm{Vect}^{\mathbb{Z}}$, since the braiding used in their definition is just the braiding τ of vector spaces.

We finish this section with the definition of braided lax monoidal functor, and recall a proposition that states that these functors preserve commutative monoids.

2.3.17. DEFINITION. *Let* $(\mathcal{C}, \otimes, I, c^{\mathcal{C}})$ *and* $(\mathcal{D}, \times, J, c^{\mathcal{D}})$ *be two braided monoidal categories. A lax monoidal functor* $(F, \varphi, \varphi_0) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \times, J)$ *is a* braided lax monoidal functor *if for every* $X, Y \in \mathcal{C}$ *the following diagram commutes:*

$$
\begin{array}{ccc}
F(X) \times F(Y) & \xrightarrow{\;c^{\mathcal{D}}_{F(X), F(Y)}\;} & F(Y) \times F(X) \\
{\scriptstyle \varphi_{X,Y}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{Y,X}} \\
F(X \otimes Y) & \xrightarrow[\;F(c^{\mathcal{C}}_{X,Y})\;]{} & F(Y \otimes X).
\end{array}
$$

2.3.18. EXAMPLE. The braiding τ defined in Example 2.3.7 on the categories $^H\mathrm{Mod}$ for a commutative $\Bbbk$-bialgebra H and $_H\mathrm{Mod}$ for a cocommutative $\Bbbk$-bialgebra H coincides with the one of their underlying vector spaces, which automatically implies that their respective forgetful functors to the category of $\Bbbk$-vector spaces are braided.

2.3.19. EXAMPLE. Unlike the previous examples, the forgetful functors to Vect from the categories $\mathrm{Vect}^{\mathbb{Z}}, \mathrm{Ch}(\mathrm{Vect}), \mathrm{Coch}(\mathrm{Vect})$, and their respective versions up to homotopy, are not braided, since the braiding considered in these categories is τ^{gr} and not τ.

Another forgetful functor which is not braided is the one corresponding to $^H_H\mathcal{YD}$ for a Hopf algebra H, since the braiding in this category involves the H-module and H-comodule structure of its objects.

2.3.20. Proposition. *[AM10, Proposition 3.37] Given two braided monoidal categories $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \times, J)$, let (F, φ, φ_0) be a braided lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$, and let (A, μ, η) be a commutative monoid in $\mathcal{C}$. The triple $(F(A), F(\mu) \circ \varphi_{A,A}, F(\eta) \circ \varphi_0)$ is a commutative monoid in $\mathcal{D}$.*

2.4. Nichols Algebras. In this section we will recall the construction of the Nichols algebra of a Yetter-Drinfeld module V over a Hopf algebra H. Nichols algebras are examples of braided Hopf algebras in the category $^H_H\mathcal{YD}$ for some Hopf algebra H.

Let $T(V) = \bigoplus_{n \in \mathbb{N}_0} V^{\otimes n}$ be the tensor algebra of V as defined in Example 2.2.16, where we have also mentioned that it can be regarded as a monoid in $^H_H\mathcal{YD}$. Now, the braiding of this category induces a monoid structure on $T(V) \otimes T(V)$, which by Remark 2.2.23 can be seen as an ordinary algebra structure.

Since $T(V)$ is the free algebra generated by V, defining $\Delta(v) = 1 \otimes v + v \otimes 1$ extends to a unique algebra morphism $\Delta : T(V) \longrightarrow T(V) \otimes T(V)$. As a consequence of the definition of the H-action and H-coaction on tensor products, the map Δ turns out to be a morphism of Yetter-Drinfeld modules. The map $\varepsilon : T(V) \longrightarrow \Bbbk$ defined as the projection onto $V^{\otimes 0}$ is a counit for Δ, and trivially preserves the algebra structure and the Yetter-Drinfeld module structure. Therefore, $T(V)$ with this structure is a bimonoid in $^H_H\mathcal{YD}$, which happens to be a Hopf monoid.

There exists a maximal Hopf ideal and Yetter-Drinfeld submodule J of $T(V)$ among those contained in $\bigoplus_{n \geq 2} V^{\otimes n}$, see [AnSch02, Proposition 2.2]. The *Nichols algebra of V* is the quotient $\mathcal{B}(V) = \frac{T(V)}{J}$.

The Jordan plane of Example 2.2.17 and the super Jordan plane of Example 2.2.18 are the Nichols algebras corresponding to the $\Bbbk\mathbb{Z}$-Yetter Drinfeld modules, called $\mathcal{V}(1, 2)$ and $\mathcal{V}(-1, 2)$ respectively.

The Nichols algebra $\mathcal{B}(V)$ also inherits a natural $\mathbb{Z}$-grading from $T(V)$, which is preserved by its product and coproduct. In the Jordan and super Jordan plane it coincides with the $\mathbb{Z}$-grading given by their $\Bbbk\mathbb{Z}$-comodule structure.

3. Graded braided (co)commutativity

The aim of this section is to provide a setting where the Hochschild cohomology of A with coefficients in $\Bbbk$ could be given a graded Yetter-Drinfeld module structure over H, where A is a Nichols algebra in the category $^H_H\mathcal{YD}$. In this context, the $\mathbb{Z}$-graded vector space $\mathrm{H}^\bullet(A, \Bbbk)$ can be seen as a monoid in the category $\left(^H_H\mathcal{YD}\right)^{\mathbb{Z}}$, which happens to be graded braided commutative. We will follow the notation in Section 3 of [MPSW10], whose main result is recovered in Corollary 3.22.

3.1. Definition. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and let V, Y be objects in $\mathcal{C}$. An internal hom, or inner hom, is a pair $(\hom(V, Y), \Theta_{V,Y})$, where*

- $\hom(V, Y)$ *is an object in $\mathcal{C}$,*

- $\Theta_{V,Y} : \mathrm{Hom}_{\mathcal{C}}(- \otimes V, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, \hom(V, Y))$ *is a natural isomorphism.*

For $X \in \mathcal{C}$, we will write $\Theta_{X,V,Y} = (\Theta_{V,Y})_X$.

3.2. REMARK. Wherever $\hom(V, Y)$ exists, it is unique up to isomorphism and it is functorial in both variables, due to Yoneda's Lemma. See, for example, [Mac98, Chapter III, Section 2]. As a consequence, if $V \in \mathcal{C}$ is such that $\hom(V, Y)$ exists for all $Y \in \mathcal{C}$, then $\hom(V, -)$ is a functor which is right adjoint to $- \otimes V$.

3.3. DEFINITION. *A monoidal category $(\mathcal{C}, \otimes, I)$ is said to be* right closed *when the inner homs $(\hom(V, Y), \Theta_{V,Y})$ exist for all objects V, Y in $\mathcal{C}$.*

3.4. EXAMPLE. The monoidal category $(\mathrm{Vect}^{\mathbb{Z}}, \otimes, \Bbbk_0)$ is right closed monoidal, its internal hom functor is called *graded hom*. For a graded $\Bbbk$-vector space Y and $n \in \mathbb{Z}$, let $Y[n]$ denote the graded vector space such that $Y[n]_i = Y_{i+n}$. The graded hom object $\hom(V, Y) = \bigoplus_{n \in \mathbb{Z}} \hom(V, Y)_n$, is defined as follows:

$$\hom(V, Y)_n = \mathrm{Hom}_{\mathrm{Vect}^{\mathbb{Z}}}(V, Y[n]) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\Bbbk}(V_i, Y_{i+n}), \quad \forall\, V, Y \in \mathrm{Vect}^{\mathbb{Z}}.$$

3.5. REMARK. The canonical inclusion $\hom(V, Y) \subseteq \mathrm{Hom}_{\Bbbk}(V, Y)$ is an equality if and only if V is finite dimensional. This is equivalent to the condition that V_n is finite dimensional for all n, and nonzero only for finitely many values of n.

. The internal hom objects in the following examples can be defined more generally for a Hopf algebra H with bijective antipode. We will focus in the case $H = \Bbbk G$.

3.6. EXAMPLE. The category $(_{\Bbbk G}\mathrm{Mod}, \otimes, \Bbbk)$ is right closed monoidal, with its internal hom defined by $\hom(V, Y) = \mathrm{Hom}_{\Bbbk}(V, Y)$ and G-action given by

$$(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x), \ \forall g \in G, \ \forall x \in V, \ \forall f \in \mathrm{Hom}_{\Bbbk}(V, Y)$$

3.7. EXAMPLE. Let G be a group. The construction for $\mathbb{Z}$-graded vector spaces in Example 3.4 can be generalized to the monoidal category $(^{\Bbbk G}\mathrm{Mod}, \otimes, \Bbbk)$, which is isomorphic to the category Vect^G of G-graded vector spaces (see Example 2.2.15). It has an internal hom functor defined by $\hom(V, Y) = \bigoplus_{h \in G} \hom(V, Y)_h$, where

$$\hom(V, Y)_h = \prod_{s \in G} \mathrm{Hom}_{\Bbbk}(V_s, Y_{hs}).$$

As in Remark 3.5, there is a canonical inclusion $\hom(V, Y) \subseteq \mathrm{Hom}_{\Bbbk}(V, Y)$, which is an equality either when V is finite dimensional or when G is finite.

3.8. EXAMPLE. The category $(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}, \otimes, \Bbbk)$ of Yetter-Drinfeld modules over a group algebra has an internal hom which combines the $\Bbbk G$-module and $\Bbbk G$-comodule structures of its objects. For $V, Y \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, the G-graded space $\hom(V, Y)$ is defined as in the previous example, and the action of $g \in G$ on $f = (f_s)_{s \in G} \in \hom(V, Y)_h$ is given by

$$(g \cdot f)_s(v) = g \cdot (f_{g^{-1}sg}(g^{-1} \cdot v)), \ \forall s \in G, \ \forall v \in V_s.$$

It is straightforward to check that $g \cdot f \in \hom(V, Y)_{ghg^{-1}} \; \forall g, h \in G, f \in \hom(V, Y)_h$, so $\hom(V, Y)$ is indeed an object in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. It also holds that $\hom(V, Y) \simeq \mathrm{Hom}_\Bbbk(V, Y)$ when V is finite dimensional or G is finite.

The following remark is generalized in [MPSW10] to arbitrary monoidal categories. Their aim is, when possible, to replace the functor $\mathrm{Hom}_\Bbbk(-, \Bbbk)$ by an internal hom functor, obtaining the desired construction in the cases in which they coincide.

3.9. REMARK. Let (A, ε) be an augmented $\Bbbk$-algebra, and consider the cochain complex $(C^\bullet(A), d^\bullet)$. Taking into account the action of A on $\Bbbk$, the map $d^n : C^n(A) \longrightarrow C^{n+1}(A)$ is as follows:

$$
\begin{aligned}
d^n f(a_1 \otimes \cdots \otimes a_{n+1}) \; &= \; \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) \\
&+ \; \textstyle\sum_{i=1}^n (-1)^i \; f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} \; f(a_1 \otimes \cdots \otimes a_n) \varepsilon(a_{n+1}) \\
&= \; f\big(\; \varepsilon(a_1) a_2 \otimes \cdots \otimes a_{n+1} + \textstyle\sum_{i=1}^n (-1)^i \; a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\
&+ \; (-1)^{n+1} \; a_1 \otimes \cdots \otimes a_n \varepsilon(a_{n+1}) \; \big).
\end{aligned}
$$

This can also be read as $d^n f = f \circ \delta_n$, where $\delta_n : A^{\otimes n+1} \longrightarrow A^{\otimes n}$ is

$$
\begin{aligned}
\delta_n(a_1 \otimes \cdots \otimes a_{n+1}) \; &= \; \varepsilon(a_1) a_2 \otimes \cdots \otimes a_{n+1} \\
&+ \; \textstyle\sum_{i=1}^n (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \; + \; (-1)^{n+1} a_1 \otimes \cdots \otimes a_n \varepsilon(a_{n+1}).
\end{aligned}
$$

Setting $S_n(A) = A^{\otimes n}$ makes $(S_\bullet(A), \delta_\bullet)$ a chain complex of $\Bbbk$-vector spaces satisfying the equality $(C^\bullet(A), d^\bullet) = (\mathrm{Hom}_\Bbbk(S_\bullet(A), \Bbbk), (\delta_\bullet)^*)$.

Next we follow the general construction given in [MPSW10, Section 3].

3.10. DEFINITION. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, where I is a monoid with the canonical isomorphism $I \otimes I \simeq I$. An augmented monoid in $\mathcal{C}$ is a 4-tuple $(A, \mu, \eta, \varepsilon)$, where (A, μ, η) is a monoid in $\mathcal{C}$ and $\varepsilon : A \longrightarrow I$ is a morphism of monoids. As usual, we omit part of the structure when it is clear from the context.*

3.11. REMARK. If $(\mathcal{C}, \otimes, I, c)$ is a braided monoidal category and $(A, \mu, \eta, \Delta, \varepsilon)$ is a bimonoid in $\mathcal{C}$, then $(A, \mu, \eta, \varepsilon)$ is an augmented monoid in $\mathcal{C}$.

3.12. DEFINITION. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and let $(A, \mu, \eta, \varepsilon)$ be an augmented monoid in $\mathcal{C}$. The triple $(S(A), \{\partial_i\}_i, \{\sigma_j\}_j)$ defined as follows is a simplicial object in $\mathcal{C}$:*

- *Objects: $S_n(A) = A^{\otimes n}$.*

- *Faces: $\partial_i : S_{n+1}(A) \longrightarrow S_n(A)$*

$$
\partial_i = \begin{cases}
\varepsilon \otimes \mathrm{id}_A^{\otimes n} & \text{if } i = 0 \\
\mathrm{id}_A^{\otimes i-1} \otimes \mu \otimes \mathrm{id}_A^{n-i} & \text{if } i = 1, \ldots, n \\
\mathrm{id}_A^{\otimes n} \otimes \varepsilon & \text{if } i = n+1.
\end{cases}
$$

- *Degeneracies:* $\sigma_j \colon S_n(A) \longrightarrow S_{n+1}(A)$

$$\sigma_j = \mathrm{id}_A^{\otimes j} \otimes \eta \otimes \mathrm{id}_A^{\otimes n-j}, \quad \forall j = 0, \ldots, n.$$

We will denote its associated chain complex in $\mathcal{C}$ *by* $(S_\bullet(A), \delta_\bullet)$, *where the differential* $\delta_n \colon S_{n+1}(A) \longrightarrow S_n(A)$ *is defined as* $\delta_n = \sum_{i=o}^{n+1}(-1)^i \partial_i$.

3.13. REMARK. This construction is functorial, therefore it gives functors S and $S_\bullet$ from the category of augmented monoids in $\mathcal{C}$ to the categories of simplicial objects in $\mathcal{C}$ and of chain complexes in $\mathcal{C}$, respectively.

3.14. DEFINITION. *[MPSW10, Definition 3.4] Let* $(\mathcal{C}, \otimes, I)$ *be a monoidal category and let* $(A, \mu, \eta, \varepsilon)$ *be an augmented monoid in* $\mathcal{C}$. *The* Hochschild cohomology of A with trivial coefficients *is the cohomology of the complex* $(\hom(S_\bullet(A), I), \hom(\delta_\bullet, I))$. *It is denoted by* $H^\bullet(A)$.

The following are the final steps to obtain a monoid structure in $\mathcal{C}^{\mathbb{Z}}$ analogous to the Hochschild cohomology with trivial coefficients on a bialgebra, with the cup product.

3.15. DEFINITION. *Let* $(\mathcal{C}, \otimes, I)$ *be a monoidal category, and let* $V, Y \in \mathcal{C}$ *be such that* $(\hom(V, Y), \Theta_{V,Y})$ *exists. The inverse of the isomorphism*

$$\Theta_{\hom(V,Y),V,Y} \colon \mathrm{Hom}_{\mathcal{C}}(\hom(V, Y) \otimes V, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\hom(V, Y), \hom(V, Y)).$$

gives a morphism

$$ev_{V,Y} = \Theta^{-1}_{\hom(V,Y),V,Y}\left(\mathrm{id}_{\hom(V,Y)}\right) \colon \hom(V, Y) \otimes V \longrightarrow Y,$$

called evaluation morphism.

3.16. DEFINITION. *[MPSW10, Remark 3.6] Let* $(\mathcal{C}, \otimes, I)$ *be a monoidal category, and let* $V, W \in \mathcal{C}$ *be such that* $\hom(V, I), \hom(W, I)$ *and* $\hom(W \otimes V, I)$ *exist. The map*

$$\xi_{V,W} \colon \hom(V, I) \otimes \hom(W, I) \longrightarrow \hom(W \otimes V, I)$$

is defined as follows:

$$\xi_{V,W} = \Theta_{\hom(V,I)\otimes\hom(W,I),W\otimes V,I}\left(ev_{V,I} \circ (\mathrm{id}_{\hom(V,I)} \otimes ev_{W,I} \otimes \mathrm{id}_V)\right).$$

3.17. REMARK. If we restrict to those pairs V, W for which $\hom(V, I) \otimes \hom(W, I)$ and $\hom(W \otimes V, I)$ exist, we can say that the collection of maps $\xi_{V,W}$ define a natural transformation ξ.

3.18. REMARK. As remarked before, in the monoidal categories of Examples 3.4 to 3.8 the object $\hom(V, Y)$ is naturally included in $\mathrm{Hom}_{\Bbbk}(V, Y)$ for all objects V and Y. In these cases, the evaluation morphism turns out to coincide with the usual evaluation of functions. Now, let us consider two elements $f \in \hom(V, \Bbbk) \subseteq \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and $g \in \hom(W, \Bbbk) \subseteq \mathrm{Hom}_{\Bbbk}(W, \Bbbk)$. The equality

$$\xi_{V,W}(f \otimes g)(w \otimes v) = g(w)f(v), \quad \forall v \in V, w \in W,$$

shows that $\xi_{V,W}$ is in fact the canonical inclusion $V^* \otimes W^* \hookrightarrow (W \otimes V)^*$.

3.19. DEFINITION. *[MPSW10, Definition 3.7] Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, let A be an augmented monoid in $\mathcal{C}$, and let $(S_\bullet(A), \delta_\bullet)$ be the chain complex of Definition 3.12. The* cup product $\smile: \mathrm{H}^\bullet(A) \otimes \mathrm{H}^\bullet(A) \longrightarrow \mathrm{H}^\bullet(A)$ *is the collection of morphisms*

$$\smile_{p,q}: \mathrm{H}^p(A) \otimes \mathrm{H}^q(A) \longrightarrow \mathrm{H}^{q+p}(A)$$

induced by

$$\xi_{S_p(A), S_q(A)} : \hom(S_p(A), I) \otimes \hom(S_q(A), I) \longrightarrow \hom(S_q(A) \otimes S_p(A), I).$$

3.20. REMARK. Let $(\mathcal{C}, \otimes, \Bbbk)$ be a monoidal category like in Remark 3.18, and let A be an augmented monoid in $\mathcal{C}$, which implies that A is also an augmented $\Bbbk$-algebra. If for every $n \in \mathbb{N}_0$ the objects $\hom(A^{\otimes n}, \Bbbk)$ exist and their canonical inclusions in $\mathrm{Hom}_\Bbbk(A^{\otimes n}, \Bbbk)$ are isomorphisms of $\Bbbk$-vector spaces, then the cochain complexes $(\hom(S_\bullet(A), \Bbbk), \hom(\delta_\bullet, \Bbbk))$ and $(\mathrm{Hom}_\Bbbk(S_\bullet(A), \Bbbk), (\delta_\bullet)^*)$ are isomorphic. Taking homology, one gets that $\mathrm{H}^\bullet(A)$ and the usual Hochschild cohomology with trivial coefficients $\mathrm{H}^\bullet(A, \Bbbk)$ are isomorphic as $\mathbb{Z}$-graded $\Bbbk$-vector spaces. In addition, according to the expression of ξ, the cup product defined in $\mathrm{H}^\bullet(A)$ is the opposite of the usual cup product in $\mathrm{H}^\bullet(A, \Bbbk)$. To avoid confusion, we will use the symbol $\smile$ to refer to the cup product of definition 3.19, instead of the symbol $\cup$ used to define the usual cup product in Hochschild cohomology in Section 2.1.

Next we will recall one of the main results of [MPSW10], together with a corollary. We will prove a generalization of the latter in the end of this section.

3.21. THEOREM. *[MPSW10, Theorem 3.12] Let A be a bimonoid in a braided monoidal category $(\mathcal{C}, \otimes, I, c)$, such that $\hom(A^{\otimes n}, I)$ exists for all $n \in \mathbb{N}_0$. Then $(\mathrm{H}^\bullet(A), \smile)$ is a graded braided commutative algebra in $\mathcal{C}^{\mathbb{Z}}$.*

3.22. COROLLARY. *[MPSW10, Corollary 3.13] Let H be a Hopf algebra with bijective antipode, and let A be a bimonoid in ${}^H_H\mathcal{YD}$. If either H or A is finite dimensional, then $H^\bullet(A, \Bbbk)$ is a $\mathbb{Z}$-graded object in ${}^H_H\mathcal{YD}$, and the opposite of its cup product is graded braided commutative.*

The authors use this theorem to prove finite generation of the cohomology algebra of some pointed Hopf algebras.

On the other hand, the Jordan and super Jordan plane are infinite dimensional and they are bimonoids in the category of Yetter-Drinfeld modules over $\Bbbk\mathbb{Z}$, which is an infinite dimensional Hopf algebra. The internal hom objects needed in the construction in [MPSW10] do not exist, so a priori $\mathrm{H}^\bullet(A, \Bbbk)$ cannot be regarded as a $\mathbb{Z}$-graded Yetter-Drinfeld module over $\Bbbk\mathbb{Z}$, and therefore a priori it cannot be asked whether it is graded braided commutative. However, there is a property that can be checked, which coincides with graded braided commutativity of the cup product when it can be defined. The next propositions will lead to that property.

3.23. LEMMA. *Let $(\mathcal{C}, \otimes, I, c)$ be a braided monoidal category, and let $V, W \in \mathcal{C}$. The following diagram commutes:*

$$
\begin{array}{ccc}
\hom(V,I) \otimes \hom(W,I) \otimes W \otimes V & \xrightarrow{\left(c_{\hom(W,I),\hom(V,I)}\right)^{-1} \otimes c_{W,V}} & \hom(W,I) \otimes \hom(V,I) \otimes V \otimes W \\
\downarrow{\scriptstyle \mathrm{id}_{\hom(V,I)} \otimes ev_{W,I} \otimes \mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_{\hom(W,I)} \otimes ev_{V,I} \otimes \mathrm{id}_W} \\
\hom(V,I) \otimes V \xrightarrow{ev_{V,I}} I & \xleftarrow{ev_{W,I}} & \hom(W,I) \otimes W.
\end{array}
$$

PROOF. Given an object U in $\mathcal{C}$, we will denote $U^* = \hom(U, I)$. In fact, we will prove that the following diagram is commutative, where the arrows that appear in the opposite direction are isomorphisms.

Both triangles on top of the diagram commute by using the axioms in the definition of the braiding c, the quadrilaterals on both sides commute by naturality of c, and the remaining subdiagrams commute by the axioms of a monoidal category. ∎

3.24. PROPOSITION. *Let $(\mathcal{C}, \otimes, I, c)$ be a braided monoidal category, and let V and W be objects in $\mathcal{C}$ such that $\hom(U, I)$ exists for $U = V, W, V \otimes W, W \otimes V$. The following diagram commutes:*

$$
\begin{array}{ccc}
\hom(W,I) \otimes \hom(V,I) & \xrightarrow{\xi_{W,V}} & \hom(V \otimes W, I) \\
\downarrow{\scriptstyle c_{\hom(W,I),\hom(V,I)}} & & \downarrow{\scriptstyle \hom\left(c_{W,V}, \mathrm{id}_I\right)} \\
\hom(V,I) \otimes \hom(W,I) & \xrightarrow[\xi_{V,W}]{} & \hom(W \otimes V, I).
\end{array}
$$

PROOF. Let us consider the following diagram in Vect, which commutes by naturality of the adjunction isomorphism Θ of Definition 3.1.

$\operatorname{Hom}_{\mathcal{C}}\left(\hom(W,I) \otimes \hom(V,I) \otimes V \otimes W, I\right)$

$\Theta_{\hom(W,I)\otimes\hom(V,I),V\otimes W,I}$

$\operatorname{Hom}_{\mathcal{C}}\left(\hom(W,I) \otimes \hom(V,I), \hom(V \otimes W, I)\right)$

$\operatorname{Hom}_{\mathcal{C}}\left(\left(c_{\hom(W,I),\hom(V,I)}\right)^{-1}\otimes c_{W,V},\operatorname{id}_I\right)$

$\operatorname{Hom}_{\mathcal{C}}\left(\left(c_{\hom(W,I),\hom(V,I)}\right)^{-1},\hom\left(c_{W,V},\operatorname{id}_I\right)\right)$

$\operatorname{Hom}_{\mathcal{C}}\left(\hom(V,I) \otimes \hom(W,I) \otimes W \otimes V, I\right)$

$\Theta_{\hom(V,I)\otimes\hom(W,I),W\otimes V,I}$

$\operatorname{Hom}_{\mathcal{C}}\left(\hom(V,I) \otimes \hom(W,I), \hom(W \otimes V, I)\right).$

Using the previous lemma and chasing the element $ev_{W,I} \circ \left(\operatorname{id}_{\hom(W,I)} \otimes ev_{V,I} \otimes \operatorname{id}_W\right)$ in this diagram gives

$ev_{W,I} \circ \left(\operatorname{id}_{\hom(W,I)} \otimes ev_{V,I} \otimes \operatorname{id}_W\right)$

$\xi_{W,V}$

$ev_{V,I} \circ \left(\operatorname{id}_{\hom(V,I)} \otimes ev_{W,I} \otimes \operatorname{id}_V\right)$

$\xi_{V,W},$

from which precomposing with $c_{\hom(W,I),\hom(V,I)}$ gives

$$\xi_{V,W} \circ c_{\hom(W,I),\hom(V,I)} \quad = \quad \hom\left(c_{W,V},\operatorname{id}_I\right) \circ \xi_{W,V},$$

which is what we wanted to prove.   ∎

.

3.25. REMARK. Let us consider a bialgebra A in the braided monoidal category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of Yetter-Drinfeld modules over a group algebra $\Bbbk G$. Suppose that either A is finite dimensional or G is finite, so that $\hom(A^{\otimes n}, \Bbbk) = \operatorname{Hom}_{\Bbbk}(A^{\otimes n}, \Bbbk)$ (see Example 3.8). It

follows from Theorem 3.21 that the diagram induced in cohomology by the following one commutes:

$$\mathrm{Hom}_{\Bbbk}(A^{\otimes p},\Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(A^{\otimes q},\Bbbk)$$

$$c_{\mathrm{Hom}_{\Bbbk}(A^{\otimes p},\Bbbk),\mathrm{Hom}_{\Bbbk}(A^{\otimes q},\Bbbk)} \downarrow \qquad \xrightarrow{(-1)^{pq} \smile_{p,q}}$$

$$\mathrm{Hom}_{\Bbbk}(A^{\otimes q},\Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(A^{\otimes p},\Bbbk) \xrightarrow[\smile_{q,p}]{} \mathrm{Hom}_{\Bbbk}(A^{\otimes p+q},\Bbbk),$$

which, due to the previous proposition, is equivalent to the commutativity of the following diagram after taking cohomology:

$$\mathrm{Hom}_{\Bbbk}(A^{\otimes p},\Bbbk) \otimes \mathrm{Hom}_{\Bbbk}(A^{\otimes q},\Bbbk) \xrightarrow{\smile_{p,q}} \mathrm{Hom}_{\Bbbk}(A^{\otimes p+q},\Bbbk)$$

$$\xrightarrow{(-1)^{pq} \smile_{p,q}} \qquad \downarrow \mathrm{Hom}_{\Bbbk}\big(c_{A^{\otimes p},A^{\otimes q}},\Bbbk\big)$$

$$\mathrm{Hom}_{\Bbbk}(A^{\otimes p+q},\Bbbk),$$

In order to check the commutativity of the first diagram, it is necessary to evaluate the braiding c in the objects $\mathrm{Hom}_{\Bbbk}(A^{\otimes p},\Bbbk)$ and $\mathrm{Hom}_{\Bbbk}(A^{\otimes q},\Bbbk)$, which can be done because they coincide with their respective inner hom objects, so they can be regarded as objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.

On the other hand, checking the commutativity of the second diagram only requires evaluating the braiding in the objects $A^{\otimes p}$ and $A^{\otimes q}$. No inner hom objects are needed, so this condition can be checked even when the objects $\hom(A^{\otimes p},\Bbbk)$ and $\hom(A^{\otimes q},\Bbbk)$ do not exist, or when they exist but differ from $\mathrm{Hom}_{\Bbbk}(A^{\otimes p},\Bbbk)$ and $\mathrm{Hom}_{\Bbbk}(A^{\otimes q},\Bbbk)$, respectively. This is the case of the Jordan and the super Jordan plane. For these algebras, we have checked that the second diagram commutes at the level of cohomology, following the steps we give in detail in the Appendix.

. In what follows, we will prove that the condition checked above holds up to homotopy for any bialgebra A in a braided monoidal category $(\mathcal{C}, \otimes, I, c)$, and give a categorical interpretation of the result.

For this purpose, we will perform some calculations using graphical calculus. Tensor products between objects are represented as their horizontal juxtaposition, and morphisms as sets of strings going from top to bottom. In the following table we show notations that will be used later. Here, c is a braiding in a braided monoidal category, (A, μ, η) is a monoid, (A, Δ, ε) is a comonoid and (M, χ_ℓ, χ_r) is an A-bimodule.

.

Morphism	Symbol	Morphism	Symbol
$c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$		$\Delta : A \longrightarrow A \otimes A$	
$(c_{Y,X})^{-1} : X \otimes Y \longrightarrow Y \otimes X$		$\varepsilon : A \longrightarrow I$	
$\mu : A \otimes A \longrightarrow A$		$\chi_\ell : A \otimes M \longrightarrow M$	
$\eta : I \longrightarrow A$		$\chi_r : M \otimes A \longrightarrow M$	

. These are examples of properties that we will use in the calculations:

- Naturality of the braiding, as well as the axiom involving it and the unit constraints, can be expressed graphically when tensoring any morphism with a morphism from the unit object to another object of $\mathcal{C}$ or vice versa:

- One can also visualize naturality of the braiding in the following examples:

- The following equalities represent the axioms of associativity and unit of a monoid:

and the corresponding ones for a comonoid are the ones above turned upside down.

3.26. EXAMPLE. The simplicial object $(S(A), \{\partial_i\}_i, \{\sigma_j\}_j)$ introduced in Definition 3.12 can be written using graphical notation as follows:

- Its objects are $S_n(A) = A \cdot \overset{n}{\cdots} \cdot A$,

- its faces $\partial_i \colon S_{n+1}(A) \longrightarrow S_n(A)$ are given by

$$
\partial_0 = \quad \begin{matrix} \overset{n}{A\ A \cdots A} \\ \downarrow\ |\cdots| \\ \bullet \\ A\cdots A \end{matrix}\ , \qquad
\partial_i = \quad \begin{matrix} \overset{i-1}{A\cdots A}\ \overset{n-i}{A\cdots A} \\ |\cdots\ \cup\ \cdots| \\ A\cdots A\cdots A \end{matrix}\ , \qquad
\partial_{n+1} \quad \begin{matrix} \overset{n}{A\cdots A\ A} \\ |\cdots|\ \downarrow \\ \bullet \\ A\cdots A \end{matrix}\ ,
$$

- its degeneracies $\sigma_j \colon S_n(A) \longrightarrow S_{n+1}(A)$ are given by

$$
\sigma_j = \quad \begin{matrix} \overset{j-1}{A\cdots A}\quad \overset{n+1-j}{A\cdots A} \\ |\cdots|\ \ |\ \uparrow\ |\cdots| \\ \bullet \\ A\cdots A\ A\ A\cdots A \end{matrix}\ .
$$

.

3.27. DEFINITION. *Let* $(S(A) \otimes S(A))_\bullet$ *denote the product of* $S_\bullet(A)$ *with itself in* $\mathrm{Ch}(\mathcal{C})$, *defined analogously to Example 2.2.12. The* deconcatenation coproduct *is the map* $\mathrm{dec}_\bullet \colon S_\bullet(A) \longrightarrow (S(A) \otimes S(A))_\bullet$ *given componentwise by* $\mathrm{dec}_n = \sum_{p+q=n} \mathrm{dec}_{p,q}$, *where*

$$
\mathrm{dec}_{p,q} = \mathrm{id}_{A^{\otimes p+q}} \colon S_{p+q}(A) \longrightarrow S_p(A) \otimes S_q(A).
$$

. Next we will give a theorem which makes use of the deconcatenation coproduct to state a generalization of the previous computations. Together with additional structure on the functor $\hom(-, I)$, this theorem will allow us to recover the main result in Section 3 of [MPSW10]. In order to prove it, we will introduce in our context the Alexander-Whitney map (defined more generally in [Wei94, Section 8.5]) and recall Lemma 3.9 of [MPSW10].

3.28. DEFINITION. *For an augmented monoid* A *in a monoidal category* $\mathcal{C}$, *the product of the simplicial object* $S(A)$ *in* $\mathcal{C}$ *with itself is the simplicial object denoted by* $S(A) \times S(A)$, *such that* $(S(A) \times S(A))_n = S(A)_n \otimes S(A)_n$ *and whose faces and degeneracies are respectively* $\{\partial_i \otimes \partial_i\}_i$ *and* $\{\sigma_j \otimes \sigma_j\}_j$.

Let $(S(A) \times S(A))_\bullet$ *denote its associated chain complex.*

- *The* Alexander-Whitney map $\mathrm{AW}_\bullet \colon (S(A) \times S(A))_\bullet \longrightarrow (S(A) \otimes S(A))_\bullet$ *is defined componentwise by the maps* $\mathrm{AW}_{p,q} \colon S(A)_{p+q} \otimes S(A)_{p+q} \longrightarrow S(A)_p \otimes S(A)_q$,

$$
\mathrm{AW}_{p,q} = \quad \begin{matrix} \overset{p}{A\cdots A}\ \overset{q}{A\cdots A}\ \overset{p}{A\cdots A}\ \overset{q}{A\cdots A} \\ |\cdots|\ \downarrow\cdots\downarrow\ \downarrow\cdots\downarrow\ |\cdots| \\ \bullet\quad\quad \bullet \\ A\cdots A \quad\quad A\cdots A \end{matrix}\ .
$$

- *The* twisted Alexander-Whitney map $\overline{\mathrm{AW}}_\bullet : (S(A) \times S(A))_\bullet \longrightarrow (S(A) \otimes S(A))_\bullet$ *is defined componentwise by the maps* $\overline{\mathrm{AW}}_{p,q} : S(A)_{p+q} \otimes S(A)_{p+q} \longrightarrow S(A)_p \otimes S(A)_q,$

$$
\overline{\mathrm{AW}}_{p,q} = (-1)^{pq} \;
\begin{array}{cccc}
\overset{q}{A\cdots A} & \overset{p}{A\cdots A} & \overset{q}{A\cdots A} & \overset{p}{A\cdots A} \\
\Big\downarrow\cdots\Big\downarrow & \Big|\cdots\Big| & \Big|\cdots\Big| & \Big\downarrow\cdots\Big\downarrow \\
A\cdots A & & A\cdots A &
\end{array}.
$$

As stated in [Wei94], the maps $\mathrm{AW}_\bullet$ and $\overline{\mathrm{AW}}_\bullet$ are homotopic, since there is a unique natural chain map from $(S(A) \times S(A))_\bullet$ to $(S(A) \otimes S(A))_\bullet$ up to natural homotopy. Therefore, precomposing either of them with any chain map $f_\bullet : S_\bullet(A) \longrightarrow (S(A) \times S(A))_\bullet$ will give the same map up to homotopy. In the case A is a bialgebra, we will use the map $f_\bullet$ as the composition of the functor $S_\bullet$ applied to the coproduct of A, and the map $g_\bullet^{A,A}$ obtained from the simplicial map defined in the following lemma.

3.29. LEMMA. *[MPSW10, Lemma 3.9] Let A and B be two augmented algebras in a braided monoidal category $(\mathcal{C}, \otimes, I, c)$. For each $n \in \mathbb{N}_0$, let us define recursively a map* $g_n^{A,B} : (A \otimes B)^{\otimes n} \longrightarrow A^{\otimes n} \otimes B^{\otimes n}$ *as follows:*

- $g_0^{A,B} = \mathrm{id}_I,$

- $g_{n+1}^{A,B}$ *is the following composition:*

$$
(A \otimes B)^{\otimes n+1} \xrightarrow{\mathrm{id}_A \otimes c_{B,A}^{\otimes n} \otimes \mathrm{id}_B} A \otimes (A \otimes B)^{\otimes n} \otimes B \xrightarrow{\mathrm{id}_A \otimes g_n^{A,B} \otimes \mathrm{id}_B} A^{\otimes n+1} \otimes B^{\otimes n+1}
$$

The collection of maps $g^{A,B} = (g_n^{A,B})_{n \in \mathbb{N}_0}$ is an isomorphism of simplicial objects in $\mathcal{C}$ from $S(A \otimes B)$ to $S(A) \times S(B)$.

Now we are able to prove the following theorem.

3.30. THEOREM. *The triple $(S(A)_\bullet, \mathrm{dec}_\bullet, (\mathrm{id}_\mathbb{k})_0)$ is a comonoid in $\mathrm{Ch}(\mathcal{C})$, which is co-commutative up to homotopy.*

PROOF. Coassociativity and counitality of $(S(A)_\bullet, \mathrm{dec}_\bullet, (\mathrm{id}_\mathbb{k})_0)$ follow immediately from the definition, and checking that the maps involved are chain morphisms is straightforward.
Proving cocommutativity up to homotopy is checking that the following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
& S_\bullet(A) & \\
{\scriptstyle \mathrm{dec}_\bullet}\swarrow & & \searrow{\scriptstyle \mathrm{dec}_\bullet} \\
(S(A) \otimes S(A))_\bullet \xrightarrow{\;\;c_{S_\bullet(A),S_\bullet(A)}^{\mathrm{gr}}\;\;} & & (S(A) \otimes S(A))_\bullet,
\end{array}
$$

which in the component $S_p(A) \otimes S_q(A)$ reads as follows:

$$
\begin{array}{ccc}
& S_{p+q}(A) & \\
{\scriptstyle \mathrm{dec}_{q,p}} \swarrow & & \searrow {\scriptstyle \mathrm{dec}_{p,q}} \\
S_q(A) \otimes S_p(A) \xrightarrow[{(-1)^{pq}\ c_{S_q(A),S_p(A)}}]{} & & S_p(A) \otimes S_q(A).
\end{array}
$$

Notice that

$$
\mathrm{id}_{A^{\otimes p+q}} = \mathrm{AW}_{p,q} \circ g_{p+q}^{A,A} \circ S_{p+q}(\Delta) \sim \overline{\mathrm{AW}}_{p,q} \circ g_{p+q}^{A,A} \circ S_{p+q}(\Delta) = (-1)^{pq} c_{A^{\otimes q}, A^{\otimes p}},
$$

where the equalities are due to counitality. ∎

To illustrate the last steps of the previous proof, we show an example for $p = 2, q = 1$. In the second and third diagrams, the level at the top corresponds to the morphism $S_3(\Delta)$, the middle ones all together correspond to the morphism $g_3^{A,A}$, and the one at the bottom corresponds to the morphisms $\mathrm{AW}_{2,1}$ and $\overline{\mathrm{AW}}_{2,1}$, respectively.

To conclude this section, in the following example we will interpret Proposition 3.24 using the definition of a braided monoidal functor given in Definition 2.3.17. This will allow us to recover the result of Corollary 3.22.

3.31. EXAMPLE. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let $\mathrm{Adj}(\mathcal{C})$ denote the full subcategory of $\mathcal{C}$ whose objects are the $V \in \mathcal{C}$ such that the functor $- \otimes V$ has right adjoint $\hom(V, -)$.

If V and W are objects in $\mathrm{Adj}(\mathcal{C})$, the composition of natural isomorphisms

$$
\mathrm{Hom}_{\mathcal{C}}(- \otimes V \otimes W, -) \simeq \mathrm{Hom}_{\mathcal{C}}(- \otimes V, \hom(W, -)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, \hom(V, \hom(W, -)))
$$

implies that $V \otimes W$ is an object of $\mathrm{Adj}(\mathcal{C})$, and $\hom(V \otimes W, Y) = \hom(V, \hom(W, Y))$ for every object Y of $\mathcal{C}$. In addition, the natural isomorphism

$$
\mathrm{Hom}_{\mathcal{C}}(- \otimes I, -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, -)
$$

implies that $I \in \mathrm{Adj}(\mathcal{C})$ and $\hom(I, Y) = Y$ for every object $Y \in \mathcal{C}$. Hence, one can consider the monoidal category $(\mathrm{Adj}(\mathcal{C}), \otimes, I)$ and the restriction of the functor $\hom(-, I) : \mathrm{Adj}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathcal{C}$, which has a monoidal structure

$$
(\hom(-, I), \xi, \xi_0) : (\mathrm{Adj}(\mathcal{C})^{\mathrm{op}}, \widetilde{\otimes}, I) \longrightarrow (\mathcal{C}, \otimes, I).
$$

Here $\tilde{\otimes}$ denotes the transposed monoidal structure of Definition 2.2.3, ξ is the natural transformation of Definition 3.16, and $\xi_0 = \mathrm{id}_I$. Proposition 3.24 is the condition for this lax monoidal functor to be braided.

The last step is applying Proposition 2.3.20 to the lax monoidal functor defined above.

We finish this section with a result of graded braided commutativity up to homotopy in $\overline{\mathrm{Coch}}\left({}^H_H\mathcal{YD}\right)$. It slightly generalizes Corollary 3.13 of [MPSW10], which can be obtained from this theorem by taking cohomology.

3.32. Theorem. *Let H be a Hopf algebra, and let A be a bimonoid in the braided monoidal category $\left({}^H_H\mathcal{YD}, \otimes, \Bbbk, c\right)$. Suppose that either H or A is finite dimensional. Then the differential graded algebra $\mathrm{Hom}_\Bbbk(S(A)_\bullet, \Bbbk)$ with the opposite of the cup product is graded braided commutative up to homotopy.*

Proof. Let us consider the chain complex $S_\bullet(A)$ with the deconcatenation coproduct dec and its counit. By Theorem 3.30, it is a graded braided cocommutative comonoid in $\overline{\mathrm{Ch}}\left({}^H_H\mathcal{YD}\right)$.

On the other hand, the finite dimension of either H or A guarantees that the objects $S_n(A)$ lie in the subcategory $\mathrm{Adj}\left({}^H_H\mathcal{YD}\right)$ defined in Example 3.31. Let us denote this subcategory by $\mathcal{B}$. One can think of $S_\bullet(A)$ as a chain complex in $\mathcal{B}$. The functor $\hom(-, I)$ applied to the objects and the differentials of a chain complex, allows us to define a functor $\hom(-, I)^{\mathrm{Coch}} : (\mathrm{Ch}(\mathcal{B}))^{\mathrm{op}} \simeq \mathrm{Coch}(\mathcal{B}^{\mathrm{op}}) \longrightarrow \mathrm{Coch}(\mathcal{C})$.

One can now define componentwise a lax monoidal functor

$$\left(\hom(-, I)^{\mathrm{Coch}}, \xi^{\mathrm{Coch}}, \xi_0^{\mathrm{Coch}}\right) : \left(\mathrm{Coch}(\mathcal{B}^{\mathrm{op}}), \tilde{\otimes}, I_0\right) \longrightarrow \left(\mathrm{Coch}(\mathcal{C}), \otimes, I_0\right),$$

where the monoidal structures in the categories of cochain complexes are those of Remark 2.2.13, making $\xi^{\mathrm{Coch}}_{V_\bullet, W_\bullet}$ a morphism of cochain complexes. Associativity and unitality are checked componentwise, as well as the fact that this lax monoidal functor is braided with the graded braiding on both categories.

The same construction can be made modulo homotopy. Since $(S_\bullet(A), \mathrm{dec}, \mathrm{id}_\Bbbk)$ is a commutative monoid in the braided monoidal category $(\overline{\mathrm{Coch}}(\mathcal{B}^{\mathrm{op}}), \tilde{\otimes}, \Bbbk_0, \tilde{c}^{\mathrm{gr}})$, Proposition 2.3.20 gives a commutative monoid in $(\overline{\mathrm{Coch}}(\mathcal{C}), \otimes, \Bbbk_0, c^{\mathrm{gr}})$, which coincides as a cochain complex with $\hom(S_\bullet(A), \Bbbk)$. It is straightforward to check that the product of Proposition 2.3.20 coincides with the cup product $\smile$.

Finiteness of the dimension of either H or A also guarantees that the vector spaces $\hom(S_n(A), \Bbbk)$ and $\mathrm{Hom}_\Bbbk(S_n(A), \Bbbk))$ are isomorphic, and that the product $\smile$ is the opposite of the usual cup product $\cup$.

Therefore, $\mathrm{Hom}_\Bbbk(S_\bullet(A), \Bbbk)$ with the opposite of the usual cup product is a graded braided commutative algebra up to homotopy, and so it is after taking cohomology. ∎

4. An approach using duoidal categories

In the previous section we proved a cocommutativity theorem which holds in several braided monoidal categories, and which is also independent of the existence of cochain

complexes and cohomology inside the category. However, it is still desirable to have commutativity at the level of cohomology. For this reason, our aim in this section is to show a result of this kind for a more general class of algebras than in [MPSW10].

One problem of the previous approach is that it depends on the bar resolution, which is in general difficult to handle. Although some computations can be made, it would be better to be able to work with smaller projective resolutions. There is a technical issue when defining the comultiplication of a projective resolution: it is well-defined only on the tensor product over the algebra A, which in general admits no braidings. Instead, it is possible to deal with it using a generalization of braided monoidal categories: duoidal categories.

4.1. DEFINITION. *A duoidal category is a 15-tuple* $(\mathcal{C}, \diamond, I, a, \ell, r, \star, J, \alpha, \lambda, \rho, \zeta, \Delta_I, \mu_J, \zeta_0)$, *where*

- $(\mathcal{C}, \diamond, I, a, \ell, r)$ *and* $(\mathcal{C}, \star, J, \alpha, \lambda, \rho)$ *are monoidal categories.*

- (I, Δ_I, ζ_0) *is a comonoid in* $(\mathcal{C}, \star, J, \alpha, \lambda, \rho)$.

- (J, μ_J, ζ_0) *is a monoid in* $(\mathcal{C}, \diamond, I, a, \ell, r)$

- ζ *is a natural transformation, called* interchange law, *given by a family of morphisms* $\zeta_{X,Y,Z,T} : (X \star Y) \diamond (Z \star T) \longrightarrow (X \diamond Z) \star (Y \diamond T)$ *for each* $X, Y, Z, T \in \mathcal{C}$, *such that the following diagrams commute for every* $X, Y, Z, T, U, V \in \mathcal{C}$:

$$
\begin{array}{ccc}
((X \star T) \diamond (Y \star U)) \diamond (Z \star V) & \xrightarrow{a_{X \star T, Y \star U, Z \star V}} & (X \star T) \diamond ((Y \star U) \diamond (Z \star V)) \\
\downarrow{\zeta_{X,T,Y,U} \diamond \mathrm{id}_{Z \star V}} & & \downarrow{\mathrm{id}_{X \star T} \diamond \zeta_{Y,U,Z,V}} \\
((X \diamond Y) \star (T \diamond U)) \diamond (Z \star V) & & (X \star T) \diamond ((Y \diamond Z) \star (U \diamond V)) \\
\downarrow{\zeta_{X \diamond Y, T \diamond U, Z, V}} & & \downarrow{\zeta_{X,T,Y \diamond Z, U \diamond V}} \\
((X \diamond Y) \diamond Z) \star ((T \diamond U) \diamond V) & \xrightarrow{a_{X,Y,Z} \diamond a_{T,U,V}} & (X \diamond (Y \diamond Z)) \star (T \diamond (U \diamond V)),
\end{array}
$$

$$
\begin{array}{ccc}
((X \star Y) \star Z) \diamond ((T \star U) \star V) & \xrightarrow{\alpha_{X,Y,Z} \diamond \alpha_{T,U,V}} & (X \star (Y \star Z)) \diamond (T \star (U \star V)) \\
\downarrow{\zeta_{X \star Y, Z, T \star U, V}} & & \downarrow{\zeta_{X, Y \star Z, T, U \star V}} \\
((X \star Y) \diamond (T \star U)) \star (Z \diamond V) & & (X \diamond T) \star ((Y \star Z) \diamond (U \star V)) \\
\downarrow{\zeta_{X,T,Y,U} \star \mathrm{id}_{Z \diamond V}} & & \downarrow{\mathrm{id}_{X \diamond T} \star \zeta_{Y,Z,U,V}} \\
((X \diamond T) \star (Y \diamond U)) \star (Z \diamond V) & \xrightarrow{\alpha_{X \diamond T, Y \diamond U, Z \diamond W}} & (X \diamond T) \star ((Y \diamond U) \star (Z \diamond V)),
\end{array}
$$

$$
\begin{array}{ccc}
I \diamond (X \star Y) & \xrightarrow{\ell_{X \star Y}} & X \star Y \\
\downarrow{\Delta_I \diamond \mathrm{id}_{X \star Y}} & & \uparrow{\ell_X \star \ell_Y} \\
(I \star I) \diamond (X \star Y) & \xrightarrow{\zeta_{I,I,X,Y}} & (I \diamond X) \star (I \diamond Y),
\end{array}
\qquad
\begin{array}{ccc}
(X \star Y) \diamond I & \xrightarrow{r_{X \star Y}} & X \star Y \\
\downarrow{\mathrm{id}_{X \star Y} \diamond \Delta_I} & & \uparrow{r_X \star r_Y} \\
(X \star Y) \diamond (I \star I) & \xrightarrow{\zeta_{X,Y,I,I}} & (X \diamond I) \star (Y \diamond I),
\end{array}
$$

$$(J \star X) \diamond (J \star Y) \xrightarrow{\zeta_{J,X,J,Y}} (J \diamond J) \star (X \diamond Y) \qquad (X \star J) \diamond (Y \star J) \xrightarrow{\zeta_{X,J,X,J}} (X \diamond Y) \star (J \diamond J)$$

with vertical maps $\lambda_X \diamond \lambda_Y$, $\mu_J \star \mathrm{id}_{X \diamond Y}$, $\rho_X \diamond \rho_Y$, $\mathrm{id}_{X \diamond Y} \star \mu_J$ and

$$X \diamond Y \xleftarrow{\lambda_{X \diamond Y}} J \star (X \diamond Y), \qquad X \diamond Y \xleftarrow{\rho_{X \diamond Y}} (X \diamond Y) \star J,$$

We will omit in general the structure on the units, denoting the duoidal category by $(\mathcal{C}, \diamond, I, \star, J, \zeta, \zeta_0)$.

While this nomenclature was introduced in [BM12], the same structure is referred to in [AM10] as *2-monoidal category.*

4.2. EXAMPLE. Any braided monoidal category $(\mathcal{C}, \otimes, I, a, \ell, r, c)$ gives rise to the duoidal category $(\mathcal{C}, \otimes, I, a, \ell, r, \otimes, I, a, \ell, r, \zeta^c, \ell_I^{-1} = r_I^{-1}, \ell_I = r_I, \mathrm{id}_I)$, where the interchange law is $\zeta^c_{X,Y,Z,T} = \mathrm{id}_X \otimes c_{Y,Z} \otimes \mathrm{id}_T$.

4.3. REMARK. Note that all morphisms involved in the previous example are isomorphisms. There is a categorical Eckmann-Hilton argument which states that essentially every duoidal category whose structure morphisms are isomorphisms comes from a braided monoidal category. See, for example, [AM10, Proposition 6.11].

4.4. REMARK. If $(\mathcal{C}, \diamond, I, a, \ell, r, \star, J, \alpha, \lambda, \rho, \zeta, \Delta_I, \mu_J, \zeta_0)$ is a duoidal category, then so is its opposite category $(\mathcal{C}^{\mathrm{op}}, \star, J, \alpha^{-1}, \lambda^{-1}, \rho^{-1}, \diamond, I, a^{-1}, \ell^{-1}, r^{-1}, \zeta^{\mathrm{op}}, \mu_J, \Delta_I, \zeta_0)$, where the interchange law is $\zeta^{\mathrm{op}}_{X,Y,Z,T} = \zeta_{X,Z,Y,T}$.

4.5. REMARK. Transposing one or both of the monoidal structures of a duoidal category $(\mathcal{C}, \diamond, I, a, \ell, r, \star, J, \alpha, \lambda, \rho, \zeta, \Delta_I, \mu_J, \zeta_0)$ gives three different duoidal categories:

- $\left(\mathcal{C}, \widetilde{\diamond}, I, a^{-1}, r, \ell, \star, J, \alpha, \lambda, \rho, \zeta^{\widetilde{\diamond}}, \Delta_I, \mu_J, \zeta_0 \right)$, where $\zeta^{\widetilde{\diamond}}_{X,Y,Z,T} = \zeta_{Z,T,X,Y}$,

- $\left(\mathcal{C}, \diamond, I, a, \ell, r, \widetilde{\star}, J, \alpha^{-1}, \rho, \lambda, \zeta^{\widetilde{\star}}, \Delta_I, \mu_J, \zeta_0 \right)$, where $\zeta^{\widetilde{\star}}_{X,Y,Z,T} = \zeta_{Y,X,T,Z}$,

- $\left(\mathcal{C}, \widetilde{\diamond}, I, a^{-1}, r, \ell, \widetilde{\star}, J, \alpha^{-1}, \rho, \lambda, \zeta^{\widetilde{\diamond},\widetilde{\star}}, \Delta_I, \mu_J, \zeta_0 \right)$, where $\zeta^{\widetilde{\diamond},\widetilde{\star}}_{X,Y,Z,T} = \zeta_{T,Z,Y,X}$.

4.6. DEFINITION. *A duoidal category is* strict *if both underlying monoidal structures are strict.*

All duoidal categories considered in this work will be strict.

4.7. REMARK. For a duoidal category $\mathcal{C}$, both monoidal structures can be lifted to $\mathrm{Ch}(\mathcal{C})$, $\mathrm{Coch}(\mathcal{C})$ and $\mathcal{C}^{\mathbb{Z}}$, with the structures on the units concentrated in degree 0 and the interchange law ζ^{gr} defined by $\zeta^{\mathrm{gr}}_{X,Y,Z,T}|_{(X_m \star Y_n) \diamond (Z_k \star T_\ell)} = (-1)^{nk} \zeta_{X_m, Y_n, Z_k, T_\ell}$.

. Next we will exhibit the duoidal category we are interested in.

4.8. DEFINITION.

1. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let (A, μ, η) be a monoid in $\mathcal{C}$. An A-bimodule is a triple (M, χ_ℓ, χ_r), where*

   - *M is an object in $\mathcal{C}$,*
   - *$\chi_\ell : A \otimes M \longrightarrow M$ and $\chi_r : M \otimes A \longrightarrow M$ are morphisms in $\mathcal{C}$, respectively called* left *and* right *action, such that the following diagrams commute:*

$$
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{\mu \otimes \mathrm{id}_M} & A \otimes M \\
{\scriptstyle \mathrm{id}_A \otimes \chi_\ell}\downarrow & & \downarrow{\scriptstyle \chi_\ell} \\
A \otimes M & \xrightarrow{\chi_\ell} & M,
\end{array}
\qquad
\begin{array}{ccc}
M \otimes A \otimes A & \xrightarrow{\chi_r \otimes \mathrm{id}_A} & M \otimes A \\
{\scriptstyle \mathrm{id}_M \otimes \mu}\downarrow & & \downarrow{\scriptstyle \chi_r} \\
M \otimes A & \xrightarrow{\chi_r} & M,
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes M \otimes A & \xrightarrow{\chi_\ell \otimes \mathrm{id}_A} & M \otimes A \\
{\scriptstyle \mathrm{id}_A \otimes \chi_r}\downarrow & & \downarrow{\scriptstyle \chi_r} \\
A \otimes M & \xrightarrow{\chi_\ell} & M,
\end{array}
\qquad
\begin{array}{ccc}
I \otimes M & \xrightarrow{\eta \otimes \mathrm{id}_M} & A \otimes M \\
{\scriptstyle \mathrm{id}_M}\searrow & & \downarrow{\scriptstyle \chi_\ell} \\
 & & M.
\end{array}
\qquad
\begin{array}{ccc}
M \otimes A & \xleftarrow{\mathrm{id}_M \otimes \eta} & M \otimes I \\
{\scriptstyle \chi_r}\downarrow & & \swarrow{\scriptstyle \mathrm{id}_M} \\
M. & &
\end{array}
$$

2. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let (A, μ, η) be a monoid in $\mathcal{C}$. The* category $_A\mathcal{C}_A$ *of A-bimodules in $\mathcal{C}$ is defined as follows:*

   - *Objects are A-bimodules (M, χ_ℓ, χ_r).*
   - *For each pair of A-bimodules (M, χ_ℓ, χ_r) and $(M', \chi'_\ell, \chi'_r)$, the set of morphisms $\mathrm{Hom}_{_A\mathcal{C}_A}(M, M')$ consists of the maps $f : M \longrightarrow M'$ in $\mathcal{C}$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
A \otimes M & \xrightarrow{\chi_\ell} & M \\
{\scriptstyle \mathrm{id}_A \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\
A \otimes M' & \xrightarrow{\chi'_\ell} & M',
\end{array}
\qquad
\begin{array}{ccc}
M \otimes A & \xrightarrow{\chi_r} & M \\
{\scriptstyle f \otimes \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle f} \\
M' \otimes A & \xrightarrow{\chi'_r} & M'.
\end{array}
$$

Although mentioning $\mathcal{C}$ is redundant in this definition, it will be necessary when dealing at the same time with different categories with the same underlying monoidal structures (e.g. vector spaces, graded vector spaces, Yetter Drinfeld modules).

4.9. DEFINITION. *Under the conditions of Definition 4.8, let (M, χ_ℓ, χ_r) and $(M', \chi'_\ell, \chi'_r)$ be two A-bimodules in $\mathcal{C}$. The* product over A *of M and M' is the coequalizer $\pi_{M,M'} : M \otimes M' \longrightarrow M \otimes_A M'$ of the maps $\chi_r \otimes \mathrm{id}_{M'}$ and $\mathrm{id}_M \otimes \chi'_\ell : M \otimes A \otimes M' \longrightarrow M \otimes M'$. The left and right actions of A on $M \otimes_A M'$ are respectively induced by χ_ℓ and χ'_r.*

4.10. REMARK. It is easy to see that the product $\otimes_A$ is associative with unit A and functorial, therefore $(_A\mathcal{C}_A, \otimes_A, A)$ is a monoidal category.

4.11. DEFINITION. *Let $(\mathcal{C}, \otimes, I, c)$ be a braided monoidal category, let $(A, \mu, \eta, \Delta, \varepsilon)$ be a bimonoid in $\mathcal{C}$, and let (M, χ_ℓ, χ_r) and $(M', \chi'_\ell, \chi'_r)$ be A-bimodules in $\mathcal{C}$. The A-bimodule $M \odot M'$ is $\left(M \otimes M', \chi_\ell^\odot, \chi_r^\odot\right)$, with actions given by*

$$
\begin{aligned}
\chi_\ell^\odot &= (\chi_\ell \otimes \chi'_\ell) \circ (\mathrm{id}_A \otimes c_{A,M} \otimes \mathrm{id}_{M'}) \circ (\Delta \otimes \mathrm{id}_M \otimes \mathrm{id}_{M'}), \\
\chi_r^\odot &= (\chi_r \otimes \chi'_r) \circ (\mathrm{id}_M \otimes c_{M',A} \otimes \mathrm{id}_A) \circ (\mathrm{id}_M \otimes \mathrm{id}_{M'} \otimes \Delta).
\end{aligned}
$$

4.12. REMARK. It is clear that the product of two A-bimodule morphisms is also an A-bimodule morphism, so $\odot$ is a bifunctor. Coassociativity and counitality of A and the axioms of the braiding imply that $\odot$ is associative with unit $(I, \varepsilon, \varepsilon)$. Therefore, $({}_A\mathcal{C}_A, \odot, I)$ is a monoidal category.

We now obtain a duoidal category that will be used in what follows.

4.13. PROPOSITION. *Let $(\mathcal{C}, \otimes, I, c)$ be a braided monoidal category, and let $(A, \mu, \eta, \Delta, \varepsilon)$ be a bimonoid in $\mathcal{C}$. For any four A-bimodules M, N, K, L, let*

$$
\zeta^A_{M,N,K,L} : (M \odot N) \otimes_A (K \odot L) \longrightarrow (M \otimes_A K) \odot (N \otimes_A L)
$$

be the unique map such that the following diagram in $\mathcal{C}$ commutes:

$$
\begin{array}{ccc}
M \otimes N \otimes K \otimes L & \xrightarrow{\zeta^c_{M,N,K,L}} & M \otimes K \otimes N \otimes L \\
\downarrow{\scriptstyle \pi_{M \odot N, K \odot L}} & & \downarrow{\scriptstyle \pi_{M,K} \otimes \pi_{N,L}} \\
(M \odot N) \otimes_A (K \odot L) & \xrightarrow{\zeta^A_{M,N,K,L}} & (M \otimes_A N) \odot (K \otimes_A L),
\end{array}
$$

where we abuse notation by using the name of some A-bimodules and A-bimodule morphisms for their images under the canonical forgetful functor from ${}_A\mathcal{C}_A$ to $\mathcal{C}$.

Then, the 9-tuple $\left({}_A\mathcal{C}_A, \otimes_A, A, \odot, I, \zeta^A, \Delta, \mathrm{id}_I, \varepsilon\right)$ is a duoidal category.

PROOF. We need to prove that ζ^A is well-defined, that is, the map $(\pi_{M,K} \otimes \pi_{N,L}) \circ \zeta^c_{M,N,K,L}$ factors through $(M \odot N) \otimes_A (K \odot L)$. This is shown in the following graphical calculation. On one side, precomposing $\zeta^c_{M,N,K,L}$ with the right action of A on $M \odot N$ yields:

On the other side, precomposing $\zeta^c_{M,N,K,L}$ with the left action of A on $K \odot L$ yields:

which equals the previous calculation when postcomposed with $\pi_{M,K} \otimes \pi_{N,L}$.

Although ζ^A is initially defined as a map in $\mathcal{C}$, one can see immediately that it preserves bimodule structures, since they are induced by the respective left and right actions on A on the leftmost and rightmost factors, and ζ^A acts on them by the identity.

Compatibility of ζ^A with associativity and units of both products is a consequence of the axioms of the braiding c.   ∎

. This result holds in general for a duoidal category $\mathcal{C}$ not necessarily arising from a braided monoidal category (see [GLF16, Section 7.1]).

. The following definitions correspond to objects in a duoidal category that have one (co)product for each monoidal structure, in a compatible way.

4.14. DEFINITION. *Let* $(\mathcal{C}, \diamond, I, \star, J, \zeta, \Delta_I, \mu_J, \zeta_0)$ *be a duoidal category. A* duoid *in* $\mathcal{C}$ *is a 5-tuple* $(A, \mu, \eta, \nu, \iota)$, *where*

- (A, μ, η) *is a monoid in* $(\mathcal{C}, \diamond, I)$

- (A, ν, ι) *is a monoid in* $(\mathcal{C}, \star, I)$

- *the following diagrams commute:*

$$
\begin{array}{ccc}
(A \star A) \diamond (A \star A) & \xrightarrow{\ \zeta_{A,A,A,A}\ } & (A \diamond A) \star (A \diamond A) \\
{\scriptstyle \nu \diamond \nu}\downarrow & & \downarrow{\scriptstyle \mu \star \mu} \\
A \diamond A \ \xrightarrow{\ \mu\ } \ A \ \xleftarrow{\ \nu\ } \ A \star A,
\end{array}
$$

$$
\begin{array}{ccc}
I \xrightarrow{\ \Delta_I\ } I \star I & \quad J \diamond J \xrightarrow{\ \mu_J\ } J & \quad I \xrightarrow{\ \zeta_0\ } J \\
{\scriptstyle \eta}\downarrow \quad \downarrow{\scriptstyle \eta \star \eta} & {\scriptstyle \iota \diamond \iota}\downarrow \quad \downarrow{\scriptstyle \iota} & {\scriptstyle \eta}\searrow \quad \swarrow{\scriptstyle \iota} \\
A \xleftarrow{\ \nu\ } A \star A, & A \diamond A \xrightarrow{\ \mu\ } A, & A.
\end{array}
$$

Also, for a duoidal category $(\mathcal{C}, \diamond, I, \star, J, \zeta)$, a *coduoid* in $\mathcal{C}$ is a duoid in its opposite category $\mathcal{C}^{\mathrm{op}}$.

4.15. REMARK. The notion of a (co)duoid generalizes the one of a (co)commutative (co)monoid to arbitrary duoidal categories: If (A, μ, η) is a commutative monoid in a braided monoidal category $(\mathcal{C}, \otimes, I, c)$, then $(A, \mu, \eta, \mu, \eta)$ is a duoid in the duoidal category $(\mathcal{C}, \otimes, I, \otimes, I, \zeta^c)$, and analogously a cocommutative comonoid induces a coduoid. The converse is also true, due to the Eckmann-Hilton argument (see [AM10, Proposition 6.29]).

. In the sequel, we will focus on giving a coduoid structure to certain projective resolutions of $\Bbbk$ as A-bimodule, when A is a bimonoid in the category $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of Yetter-Drinfeld modules over an abelian group G. One of the coproducts will be the one of Example 2.2.19, which is related to the definition of the usual cup product (see Remark 2.1.3). First we will define the duoidal category whose coduoids we are interested in.

4.16. Definition. *Let A be a bimonoid in the category $\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}$. We define the duoidal category $\left(\underline{\mathrm{Ch}}\left(\,_A\left(\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}\right)_A\right), \otimes_A, A_0, \odot, \Bbbk_0, \left(\zeta^A\right)^{\mathrm{gr}}\right)$ as follows:*

- *Objects are chain complexes of A-bimodules in $\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}$.*

- *Morphisms are $\Bbbk$-linear homotopy classes of chain morphisms in $_A\mathrm{Mod}_A$.*

- *The products $\otimes_A$ and $\odot$ are defined componentwise as in Remark 2.2.13 with their units concentrated in degree 0.*

- *The graded interchange law $\left(\zeta^A\right)^{\mathrm{gr}}$ is defined as in Remark 4.7.*

4.17. Theorem. *Let A be a bimonoid in the braided monoidal category $\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}$ for a group G, and let $P_\bullet \overset{f}{\to} A$ be a chain complex in $_A\left(\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}\right)_A$ which is also a projective resolution of A in $_A\mathrm{Mod}_A$. The map $\omega : P_\bullet \longrightarrow (P \otimes_A P)_\bullet$ lifting the isomorphism $A \simeq A \otimes_A A$ as in Example 2.2.19 and the map $\delta : P_\bullet \longrightarrow (P \odot P)_\bullet$ defined as a lifting of the map $\Delta : A \longrightarrow A \odot A$ provide a coduoid $(P_\bullet, \omega, f_0, \delta, \varepsilon \circ f_0)$ in $\left(\underline{\mathrm{Ch}}\left(\,_A\left(\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}\right)_A\right), \otimes_A, A_0, \odot, \Bbbk_0, \left(\zeta^A\right)^{\mathrm{gr}}\right)$.*

Proof. Consider the following commutative diagram in $_A\left(\overset{\Bbbk G}{\Bbbk G}\mathcal{YD}\right)_A$:

$$
\begin{array}{ccccc}
A \otimes_A A & \xleftarrow{\;\simeq\;} & A & \xrightarrow{\;\Delta\;} & A \odot A \\
{\scriptstyle \Delta \otimes_A \Delta}\downarrow & & & & \downarrow{\scriptstyle \simeq} \\
(A \odot A) \otimes_A (A \odot A) & \xrightarrow[\zeta^A_{A,A,A,A}]{} & & & (A \otimes_A A) \odot (A \otimes_A A).
\end{array}
$$

Since the forgetful functor $\overset{\Bbbk G}{\Bbbk G}\mathcal{YD} \longrightarrow \mathrm{Vect}$ is strong monoidal, the diagram is also commutative in $_A\mathrm{Mod}_A$. Lifting these morphisms along the exact complexes in $_A\mathrm{Mod}_A$ involving products of $P_\bullet$ gives the following diagram in $_A\mathrm{Mod}_A$:

$$
\begin{array}{ccccc}
(P \otimes_A P)_\bullet & \xleftarrow{\;\omega\;} & P_\bullet & \xrightarrow{\;\delta\;} & (P \odot P)_\bullet \\
{\scriptstyle \delta \otimes_A \delta}\downarrow & & & & \downarrow{\scriptstyle \omega \odot \omega} \\
((P \odot P) \otimes_A (P \odot P))_\bullet & \xrightarrow[\left(\zeta^A\right)^{\mathrm{gr}}_{P_\bullet, P_\bullet, P_\bullet, P_\bullet}]{} & & & ((P \otimes_A P) \odot (P \otimes_A P))_\bullet,
\end{array}
$$

which consequently commutes up to a $\Bbbk$-linear homotopy. ∎

. In the next examples we show how the Jordan and super Jordan plane fit into the hypotheses of the previous theorem.

4.18. Example. For the Jordan plane, we consider a resolution of A as A-bimodule that has been computed in [LS21]. In [KMOOW21], the Jordan plane appears as an example of twisted tensor product of algebras, which allows the authors to provide a resolution of A as A-bimodule by alternative methods. This algebra is also studied in [ArSch87] as one of the regular algebras of global dimension 2. Tensoring over A the A-bimodule

resolution of A by the free resolution of the base field k as left A-module given in this article is obtained. The resolution in [LS21] is the following:

$$0 \longrightarrow A \otimes R \otimes A \xrightarrow{d_1} A \otimes V \otimes A \xrightarrow{d_0} A \otimes A \longrightarrow 0,$$

where $V = \Bbbk\{x,y\}$ and $R = \Bbbk r$. The differentials are
$$d_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v, \ \forall v \in V,$$
$$d_1(1 \otimes r \otimes 1) = y \otimes x \otimes 1 + 1 \otimes y \otimes x - x \otimes y \otimes 1 - 1 \otimes x \otimes y$$
$$+ \tfrac{1}{2}\, x \otimes x \otimes 1 + \tfrac{1}{2}\, 1 \otimes x \otimes x,$$

and the quasi-isomorphism is the one induced by $\mu : A \otimes A \longrightarrow A$.

We only need to define an action of $\Bbbk\mathbb{Z} = \Bbbk[t, t^{-1}]$ and a compatible internal $\mathbb{Z}$-grading on the A-bimodules $A \otimes R \otimes A$ and $A \otimes V \otimes A$. We will define the action of t on the elements of the form $1 \otimes v \otimes 1$ and extend it by $t \cdot (a \cdot m \cdot b) = (t \cdot a) \cdot (t \cdot m) \cdot (t \cdot b)$. Analogously, we will define the grading on these elements and extend it additively over the tensor product.

- $t \cdot (1 \otimes x \otimes 1) = 1 \otimes x \otimes 1$, $t \cdot (1 \otimes y \otimes 1) = 1 \otimes x \otimes 1 + 1 \otimes y \otimes 1$ and $1 \otimes x \otimes 1, 1 \otimes y \otimes 1$ have internal degree 1,

- $t \cdot (1 \otimes r \otimes 1) = 1 \otimes r \otimes 1$ and $1 \otimes r \otimes 1$ has internal degree 2.

It is straightworward to check that the differentials preserve the action and the internal grading.

4.19. EXAMPLE. For the super Jordan plane, we will use the following resolution of A as A-bimodule, computed in [RS18]:

$$\cdots \xrightarrow{d_3} A \otimes V_3 \otimes A \xrightarrow{d_2} A \otimes V_2 \otimes A \xrightarrow{d_1} A \otimes V_1 \otimes A \xrightarrow{d_0} A \otimes A \longrightarrow 0,$$

where $V_1 = \Bbbk\{x,y\}$ and $V_n = \Bbbk\{x^n, y^2x^{n-1}\}$, $\forall n \geq 2$. It has differentials
$$d_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v, \ \forall v \in V,$$
$$d_1(1 \otimes x^2 \otimes 1) = 1 \otimes x \otimes x + x \otimes x \otimes 1,$$
$$d_1(1 \otimes y^2 x \otimes 1) = y^2 \otimes x \otimes 1 + y \otimes y \otimes x + 1 \otimes y \otimes yx - xy \otimes y \otimes 1 - x \otimes y \otimes y$$
$$- 1 \otimes x \otimes y^2 - xy \otimes x \otimes 1 - x \otimes y \otimes x - 1 \otimes x \otimes yx,$$
$$d_n(1 \otimes x^{n+1} \otimes 1) = x \otimes x^n \otimes 1 + (-1)^{n+1}\, 1 \otimes x^n \otimes x,$$
$$d_n(1 \otimes y^2 x^n \otimes 1) = y^2 \otimes x^n \otimes 1 + (-1)^{n+1}\, 1 \otimes y^2 x^{n-1} \otimes x - x \otimes y^2 x^{n-1} \otimes 1$$
$$- xy \otimes x^n \otimes 1 - 1 \otimes x^n \otimes y^2 - 1 \otimes x^n \otimes yx, \ \forall n \geq 2,$$

and quasi-isomorphism induced by $\mu : A \otimes A \longrightarrow A$ as in the previous example.

Now, we define a compatible $\Bbbk\mathbb{Z}$-action and internal $\mathbb{Z}$-grading on the elements $1 \otimes v \otimes 1$ of the A-bimodules $A \otimes V_n \otimes A$:

- $t \cdot (1 \otimes x \otimes 1) = -1 \otimes x \otimes 1$, $t \cdot (1 \otimes y \otimes 1) = 1 \otimes x \otimes 1 - 1 \otimes y \otimes 1$ and $1 \otimes x \otimes 1, 1 \otimes y \otimes 1$ have internal degree 1,

- $t \cdot (1 \otimes x^n \otimes 1) = (-1)^n 1 \otimes x^n \otimes 1$, and $1 \otimes x^n \otimes 1$ has internal degree n, $\forall n \geq 2$.

- $t \cdot (1 \otimes y^2 x^{n-1} \otimes 1) = -1 \otimes x^n \otimes y + (-1)^{n-1}(x-y) \otimes x^n \otimes 1 + 1 \otimes y^2 x^{n-1} \otimes 1$, and $1 \otimes y^2 x^{n-1} \otimes 1$ has internal degree $n + 1$, $\forall n \geq 2$.

As in the previous example, it is straightforward to check the needed compatibilities.

In [RS18], there are two general hypotheses: the field is algebraically closed and its characteristic is zero. While the first hypothesis is not important since computing Hochschild cohomology commutes with taking the algebraic closure of the base field, the second one is crucial for the actual computation of Hochschild cohomology of the super Jordan plane in [RS18]. However, this restriction does not appear for constructing the minimal resolution, since the coefficients appearing in the differentials are just 1 and -1, and therefore it has no consequences for us.

. So far in this section, we have worked on a duoidal structure in the category $_A\left(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A$ analogous to the braided monoidal structure in $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. We have also defined a coduoid up to homotopy in $\underline{\mathrm{Ch}}\left(_A\left(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A\right)$ using a free resolution $P_\bullet \longrightarrow A$, in a similar way as we had previously defined a braided comonoid up to homotopy in $\overline{\mathrm{Ch}}\left(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)$ using the complex $S_\bullet$.

Our next step will be to define a functor $\hom_{AA}(-, \Bbbk) : {_A\left(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A} \longrightarrow {^{\Bbbk G}_{\Bbbk G}\mathcal{YD}}$ in a similar way to the definition of $\hom(-, \Bbbk) : {^{\Bbbk G}_{\Bbbk G}\mathcal{YD}} \longrightarrow {^{\Bbbk G}_{\Bbbk G}\mathcal{YD}}$ in Example 3.8. Under weaker finiteness conditions than in [MPSW10], it will coincide at the $\Bbbk$-linear level with the functor $\mathrm{Hom}_{AA}(-, \Bbbk)$, sending a free resolution $P_\bullet \longrightarrow A$ to the cochain complex used to compute Hochschild cohomology. We will extend this to the monoidal structures on them. The following definition is an analogue in the context of duoidal categories of the concept of braided lax monoidal functors in braided monoidal categories.

4.20. DEFINITION. *Let $(\mathcal{C}, \diamond, I, \star, J, \zeta)$ and $(\mathcal{C}', \diamond', I', \star', J', \zeta')$ be duoidal categories. A double lax monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a 5-tuple $(F, \varphi, \varphi_0, \gamma, \gamma_0)$, where*

- *(F, φ, φ_0) is a lax monoidal functor from $(\mathcal{C}, \diamond, I)$ to $(\mathcal{C}', \diamond', I')$.*

- *(F, γ, γ_0) is a lax monoidal functor from $(\mathcal{C}, \star, J)$ to $(\mathcal{C}', \star', J')$.*

- *The following diagrams commute:*

$$
\begin{array}{ccc}
(F(X) \star' F(Y)) \diamond' (F(Z) \star' F(T)) & \xrightarrow{\zeta'_{F(X),F(Y),F(Z),F(T)}} & (F(X) \diamond' F(Z)) \star' (F(Y) \diamond' F(T)) \\
{\scriptstyle \gamma_{X,Y} \diamond' \gamma_{Z,T}} \downarrow & & \downarrow {\scriptstyle \varphi_{X,Z} \star' \varphi_{Y,T}} \\
F(X \star Y) \diamond' F(Z \star T) & & F(X \diamond Z) \star' F(Y \diamond T) \\
{\scriptstyle \varphi_{X \star Y, Z \star T}} \downarrow & & \downarrow {\scriptstyle \gamma_{X \diamond Z, Y \diamond T}} \\
F((X \star Y) \diamond (Z \star T)) & \xrightarrow[\quad F(\zeta_{X,Y,Z,T}) \quad]{} & F((X \diamond Z) \star (Y \diamond T)),
\end{array}
$$

$$
\begin{array}{ccc}
I' \xrightarrow{\Delta_{I'}} I' \star' I' & J' \diamond' J' \xrightarrow{\mu_{J'}} J & I' \xrightarrow{\zeta_0'} J' \\
\end{array}
$$

$$
\begin{array}{lcl}
I' \xrightarrow{\Delta_{I'}} I' \star' I' & \qquad J' \diamond' J' \xrightarrow{\mu_{J'}} J & \qquad I' \xrightarrow{\zeta_0'} J' \\
\varphi_0 \downarrow \qquad \downarrow \varphi_0 \star' \varphi_0 & \gamma_0 \diamond' \gamma_0 \downarrow \qquad \downarrow \gamma_0 & \varphi_0 \downarrow \qquad \downarrow \gamma_0 \\
F(I) & & F(J) \qquad F(I) \xrightarrow{F(\zeta_0)} F(J). \\
F(\Delta_I) \downarrow & & \uparrow F(\mu_J) \\
F(I \star I) \xleftarrow{\gamma_{I,I}} F(I) \star' F(I), & F(J) \diamond' F(J) \xrightarrow{\varphi_{J,J}} F(J \diamond J),
\end{array}
$$

Lax monoidal functors preserve monoids, braided lax monoidal functors preserve commutative monoids, and we now recall a proposition stating that double lax monoidal functors preserve duoids.

4.21. PROPOSITION. *[AM10, Corollary 6.58] Given two duoidal categories* $(\mathcal{C}, \diamond, I, \star, J, \zeta)$ *and* $(\mathcal{C}', \diamond', I', \star', J', \zeta')$, *let* $(F, \varphi, \varphi_0, \gamma, \gamma_0)$ *be a double lax monoidal functor from* $\mathcal{C}$ *to* $\mathcal{C}'$, *and let* $(A, \mu, \eta, \nu, \iota)$ *be a duoid in* $\mathcal{C}$. *The 5-tuple* $(F(A), F(\mu) \circ \varphi_{A,A}, F(\eta) \circ \varphi_0, F(\nu) \circ \gamma_{A,A}, F(\iota) \circ \gamma_0)$ *is a duoid in* $\mathcal{C}'$.

In order to define the functor mentioned before Definition 4.20, let us recall the internal hom in the category $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ from Examples 3.7 and 3.8. Componentwise, it is as follows:

$$
\hom(V, Y)_h = \prod_{s \in G} \mathrm{Hom}_{\Bbbk}(V_s, Y_{hs}) = \mathrm{Hom}_{\mathrm{Vect}^G}(V, Y[h]),
$$

Where the G-graded vector space $Y[h]$ is defined componentwise by $Y[h]_s = Y_{hs}$.

The next definition resembles this internal hom.

4.22. DEFINITION. *Let* A *be an augmented algebra and* M, N *be two* A-bimodules in *the monoidal category* $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, *where* G *is an abelian group. The Yetter-Drinfeld module* $\hom_{AA}(M, N)$ *is defined componentwise by*

$$
\hom_{AA}(M, N)_h = \mathrm{Hom}_{A(\mathrm{Vect}^G)_A}(M, N[h]) = \mathrm{Hom}_{AA}(M, N) \cap \hom(M, N)_h,
$$

with the following action of $\Bbbk G$:

$$
(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x), \ \forall x \in M, \ \forall g \in G, \ \forall f \in \hom_{AA}(M, N)_h.
$$

There are several details implicit in this definition:

- We are using the isomorphism $^{\Bbbk G}_{\Bbbk G}\mathcal{YD} \simeq (_{\Bbbk G}\mathrm{Mod})^G$ of Example 2.2.16 to define a Yetter-Drinfeld module by a G-grading and a $\Bbbk G$-action on each component.

- The aforementioned isomorphism is also used to define the object $N[h]$. It can be checked directly that $N[h]$ is an A-bimodule in $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ if N is so.

- In Remark 2.2.23, we noticed that the forgetful functors $^{\Bbbk G}_{\Bbbk G}\mathcal{YD} \longrightarrow \mathrm{Vect}^G \longrightarrow \mathrm{Vect}$ are strong monoidal, so A can be regarded as a monoid and $M, N, N[h]$ as A-bimodules in any of these categories.

- It is straightforward to check that if f is in $\hom_{AA}(M, N)_h$, then $g \cdot f$ belongs to $\hom_{AA}(M, N)_h$ as well. Thus, $\hom_{AA}(M, N)_h$ is a $\Bbbk G$-submodule of $\hom(M, N)_h$ for every $h \in G$, and consequently $\hom_{AA}(M, N)$ is a G-graded $\Bbbk G$-submodule of $\hom(M, N)$.

It can also be easily checked that precomposing with A-bimodule morphisms in the first coordinate and postcomposing with A-bimodule morphisms in the second one preserves the G-grading and the $\Bbbk G$-action. This allows us to regard $\hom_{AA}(-, -)$ as a bifunctor from $_A\big(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\big)_A^{\mathrm{op}} \times {}_A\big(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\big)_A$ to $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.

4.23. REMARK. Let M be a Yetter-Drinfeld module over $\Bbbk G$ isomorphic to the free A-bimodule $A \otimes V \otimes A$ as a bimodule in Vect^G. The adjunction between the functor $A \otimes - \otimes A$ and the forgetful functor from $_A(\mathrm{Vect}^G)_A$ to Vect^G gives a canonical isomorphism

$$\mathrm{Hom}_{_A(\mathrm{Vect}^G)_A}(A \otimes V \otimes A, N[h]) \simeq \mathrm{Hom}_{\mathrm{Vect}^G}(V, N[h]).$$

This implies that

$$\hom_{AA}(A \otimes V \otimes A, N) \simeq \hom(V, N)$$

as G-graded vector spaces. If, in addition, V is a finite dimensional vector space, then

$$\hom_{AA}(A \otimes V \otimes A, N) \simeq \hom(V, N) \simeq \mathrm{Hom}_\Bbbk(V, N) \simeq \mathrm{Hom}_{AA}(A \otimes V \otimes A, N).$$

Since all these isomorphisms are natural, so is their composition.

4.24. PROPOSITION. *Let A be an augmented algebra in the monoidal category $^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ for an abelian group G. Suppose that there exists a chain complex $P_\bullet \longrightarrow A$ in $_A\big(^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\big)_A$ which is a resolution of A in $_A\mathrm{Mod}_A$, and such that for every $n \in \mathbb{N}_0$ the A-bimodule P_n is isomorphic as a G-graded vector space to $A \otimes V_n \otimes A$, with V_n finite dimensional. There is an isomorphism of $\mathbb{Z}$-graded vector spaces*

$$\mathrm{H}^\bullet(A, \Bbbk) \simeq H(\hom_{AA}(P_\bullet, \Bbbk)).$$

PROOF. Since V_n is finite dimensional, by the previous remark $\hom_{AA}(A \otimes V_n \otimes A, \Bbbk)$ is isomorphic to $\mathrm{Hom}_{AA}(A \otimes V_n \otimes A, \Bbbk)$.

Using that $P_\bullet \longrightarrow A$ is a projective resolution in Vect gives

$$\mathrm{H}^\bullet(A, \Bbbk) = \mathrm{Ext}^\bullet_{AA}(A, \Bbbk) = H(\mathrm{Hom}_{AA}(P_\bullet, \Bbbk)) \simeq H(\hom_{AA}(P_\bullet, \Bbbk)).$$

∎

Now we know that, under some finiteness conditions, the functor $\hom_{AA}(-, \Bbbk)$ can be used to compute Hochschild cohomology. The following step is giving this functor a double lax monoidal structure relating the coproducts ω and δ of Theorem 4.17 to the cup product of $\mathrm{H}^\bullet(A, \Bbbk)$.

4.25. Definition. *Let* $(A, \mu, \eta, \Delta, \varepsilon)$ *be a bimonoid in* ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, *and let* $M, N \in {}_A\big({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\big)_A$.

- *The morphism* $\varphi_{M.N} : \hom_{AA}(M, \Bbbk) \otimes \hom_{AA}(N, \Bbbk) \longrightarrow \hom_{AA}(N \odot M, \Bbbk)$ *is defined by* $\varphi_{M,N}(f \otimes g) = g \odot f$.

- *The morphism* $\varphi_0 : \Bbbk \longrightarrow \hom_{AA}(\Bbbk, \Bbbk)$ *is defined by* $\varphi_0(1) = \mathrm{id}_\Bbbk$.

- *The morphism* $\gamma_{M.N} : \hom_{AA}(M, \Bbbk) \otimes \hom_{AA}(N, \Bbbk) \longrightarrow \hom_{AA}(N \otimes_A M, \Bbbk)$ *is defined by* $\gamma_{M,N}(f \otimes g) = g \otimes_A f$.

- *The morphism* $\gamma_0 : \Bbbk \longrightarrow \hom_{AA}(A, \Bbbk)$ *is defined by* $\gamma_0(1) = \varepsilon$.

4.26. Theorem. *Let A be a bimonoid in* ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. *The 5-tuple* $\big(\hom_{AA}(-, \Bbbk), \varphi, \varphi_0, \gamma, \gamma_0\big)$ *is a double lax monoidal functor from* $\Big({}_A\big({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\big)_A^{\mathrm{op}}, \widetilde{\odot}, \Bbbk, \widetilde{\otimes}_A, A, \big(\zeta^A\big)^{\widetilde{\odot}, \widetilde{\otimes}_A} \Big)$ *to the duoidal category* $\Big({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}, \otimes, \Bbbk, , \otimes, \Bbbk, \zeta^c \Big)$.

Proof. First of all, notice that φ and γ are natural transformations due to the functoriality of the products $\odot, \otimes_A$ and $\otimes$. Associativity an unitality of both lax monoidal structures can be checked directly, as well as the axioms involving the respective comonoid and monoid structures on the first and second units. Let us pay attention to the relation between the functor $\hom_{AA}(-, \Bbbk)$ and the interchange laws $\big(\zeta^A\big)^{\widetilde{\odot}, \widetilde{\otimes}_A}$ and ζ^c. We need to prove the commutativity of the following diagram for every $M, N, K, L \in {}_A\big({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\big)_A$, where for $X \in \{M, N, K, L\}$ the object $\hom_{AA}(X, \Bbbk)$ is denoted by X°.

$$
\begin{array}{ccc}
M^\circ \otimes N^\circ \otimes K^\circ \otimes L^\circ & \xrightarrow{\mathrm{id}_{M^\circ} \otimes c_{N^\circ, K^\circ} \otimes \mathrm{id}_{L^\circ}} & M^\circ \otimes K^\circ \otimes N^\circ \otimes L^\circ \\
\downarrow{\scriptstyle \gamma_{M,N} \otimes \gamma_{K,L}} & & \downarrow{\scriptstyle \varphi_{M,K} \otimes \varphi_{N,L}} \\
(M \widetilde{\otimes}_A N)^\circ \otimes (K \widetilde{\otimes}_A L)^\circ & & (M \widetilde{\odot} K)^\circ \otimes (N \widetilde{\odot} L)^\circ \\
\downarrow{\scriptstyle \varphi_{M \widetilde{\otimes}_A N, K \widetilde{\otimes}_A L}} & & \downarrow{\scriptstyle \gamma_{M \widetilde{\odot} K, N \widetilde{\odot} L}} \\
\big[(M \widetilde{\otimes}_A N) \widetilde{\odot} (K \widetilde{\otimes}_A L)\big]^\circ & \xrightarrow{\big((\zeta^A)^{\widetilde{\odot}, \widetilde{\otimes}_A}_{M,K,N,L} \big)^\circ} & \big[(M \widetilde{\odot} K) \widetilde{\otimes}_A (N \widetilde{\odot} L)\big]^\circ,
\end{array}
$$

which translates into

$$
\begin{array}{ccc}
M^\circ \otimes N^\circ \otimes K^\circ \otimes L^\circ & \xrightarrow{\mathrm{id}_{M^\circ} \otimes c_{N^\circ, K^\circ} \otimes \mathrm{id}_{L^\circ}} & M^\circ \otimes K^\circ \otimes N^\circ \otimes L^\circ \\
\downarrow{\scriptstyle \gamma_{M,N} \otimes \gamma_{K,L}} & & \downarrow{\scriptstyle \varphi_{M,K} \otimes \varphi_{N,L}} \\
(N \otimes_A M)^\circ \otimes (L \otimes_A K)^\circ & & (K \odot M)^\circ \otimes (L \odot N)^\circ \\
\downarrow{\scriptstyle \varphi_{N \otimes_A M, L \otimes_A K}} & & \downarrow{\scriptstyle \gamma_{K \odot M, L \odot N}} \\
\big[(L \otimes_A K) \odot (N \otimes_A M)\big]^\circ & \xrightarrow{(\zeta^A_{L,N,K,M})^\circ} & \big[(L \odot N) \otimes_A (K \odot M)\big]^\circ,
\end{array}
$$

Now, let a, b, c, d be elements of G. We will chase an element $f \otimes f' \otimes g \otimes g'$ along the diagram above, where $f \in \hom_{AA}(M_a, \Bbbk)$, $f' \in \hom_{AA}(N_b, \Bbbk)$, $g \in \hom_{AA}(K_c, \Bbbk)$ and $g' \in \hom_{AA}(L_d, \Bbbk)$ respectively. Notice that, since $\Bbbk$ is concentrated in degree 1_G, the respective internal degrees of f, f', g, g' are $a^{-1}, b^{-1}, c^{-1}, d^{-1}$. Following the top and right sides of the diagram gives the element $(g' \odot f') \otimes_A (b^{-1} \cdot g \odot f)$, while following its left and bottom sides gives $((g' \otimes_A g) \odot (f' \otimes_A f)) \circ \zeta^A_{L,N,K,M}$. To show that both terms are equal, it suffices to evaluate them in a generic element $(w' \odot v') \otimes_A (w \odot v)$, where v, v', w, w' lie in M_a, N_b, K_c, L_d respectively. Evaluating the first term yields

$$
\begin{aligned}
(g' \odot f') \otimes_A (b^{-1} \cdot g \odot f)\left((w' \odot v') \otimes_A (w \odot v)\right) &= g'(w')f'(v')(b^{-1} \cdot g)(w)f(v) \\
&= g'(w')f'(v')g(b \cdot w)f(v),
\end{aligned}
$$

since the action of $\Bbbk G$ on $\Bbbk$ is trivial, while evaluating the second term yields

$$
\begin{aligned}
((g' \otimes_A g) &\odot (f' \otimes_A f))\left(\zeta^A_{L,N,K,M}((w' \otimes_A v') \odot (w \otimes v))\right) \\
&= ((g' \otimes_A g) \odot (f' \otimes_A f))\left((w' \otimes_A b \cdot w) \odot (v' \otimes_A v)\right) = g'(w')g(b \cdot w)f'(v')f(v).
\end{aligned}
$$

This shows that both compositions of the diagram coincide when evaluated in a generic element $f \otimes f' \otimes g \otimes g'$, proving the compatibility of the functor $\hom_{AA}(-, \Bbbk)$ with the interchange laws. ∎

. In the following theorem, we prove the result on graded braided commutativity up to $\Bbbk$-linear homotopy which leads to the desired property of graded braided commutativity of the Hochschild cohomology with trivial coefficients, under hypotheses that include Nichols algebras such as the Jordan and super Jordan plane.

4.27. THEOREM. *Let A be a bimonoid in the braided monoidal category $\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}, \otimes, \Bbbk, c\right)$ for an abelian group G. Suppose that there exists a complex $P_\bullet \longrightarrow A$ in $_A\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}\right)_A$ which is a projective resolution of A in $_A\mathrm{Mod}_A$, and such that for every $n \in \mathbb{N}_0$ the A-bimodule P_n is isomorphic as a G-graded vector space to $A \otimes V_n \otimes A$, where V_n is finite dimensional. Then the differential graded algebra $\mathrm{Hom}_{AA}(P_\bullet, \Bbbk)$ with the opposite of the cup product is graded braided commutative up to $\Bbbk$-linear homotopy.*

PROOF. In the same way as in Theorem 3.32, one can consider the opposite of the category $\underline{\mathrm{Ch}}\left(_A\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}\right)_A\right)$, which is isomorphic to the category of cochain complexes in $_A\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}\right)_A$ with cochain morphisms in $_A\mathrm{Mod}_A$ in the opposite direction.

On the other hand, let $\underline{\mathrm{Ch}}\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}\right)$ denote the category whose objects are cochain complexes of Yetter-Drinfeld modules over $\Bbbk G$, and whose morphisms are $\Bbbk$-linear homotopy classes of $\Bbbk$-linear cochain morphisms. It is a braided monoidal category with the tensor product defined as in Remark 2.2.13 and the graded braiding.

It is posible to define componentwise a cochain version of the functor $\hom_{AA}(-, \Bbbk)$ from the duoidal category $\left(\underline{\mathrm{Ch}}\left(_A\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}\right)_A\right)^{\mathrm{op}}, \widetilde{\odot}, \Bbbk_0, \widetilde{\otimes}_A, A_0, \left(\zeta^A\right)^{\widetilde{\odot}, \widetilde{\otimes}_A, gr}\right)$ to the duoidal category $\left(\underline{\mathrm{Coch}}\left(\begin{smallmatrix}\Bbbk G \\ \Bbbk G\end{smallmatrix}\mathcal{YD}\right), \otimes, \Bbbk_0, \otimes, \Bbbk_0, (\zeta^c)^{\mathrm{gr}}\right)$. All the axioms of a double lax monoidal

functor hold at the cochain level because they do componentwise, and the graded inter-change laws are compatible with the signs of the differentials given as in Remark 2.2.13, observing that they are being taken on the transpose product.

Translating the result of Theorem 4.17 to this context, one has that $(P_\bullet, \delta, f_0 \circ \varepsilon_0, \omega, f_0)$ is a duoid in $\underline{\mathrm{Coch}}\left({}_A\left({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right)_A{}^{\mathrm{op}}, \widetilde{\odot}, \Bbbk_0, \widetilde{\otimes}_A, A_0, \left(\zeta^A\right)^{\widetilde{\odot}, \widetilde{\otimes}_A, gr}\right)$. Proposition 4.21 implies that the double lax functor $\hom_{AA}(-, \Bbbk)$ sends the duoid $P_\bullet$ to a duoid $\hom_{AA}(P_\bullet, \Bbbk)$ in $\left(\underline{\mathrm{Coch}}\left({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right), \otimes, \Bbbk_0, \otimes, \Bbbk_0, (\zeta^c)^{\mathrm{gr}}\right)$. This duoid turns out to be a commutative monoid in the braided monoidal category $\left(\underline{\mathrm{Coch}}\left({}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}\right), \otimes, \Bbbk_0, c^{\mathrm{gr}}\right)$ by the Eckmann-Hilton ar-gument in Example 4.15. As proved in Proposition 4.24, the finiteness conditions on the free resolution $P_\bullet$ imply that $\hom_{AA}(P_\bullet, \Bbbk)$ is isomorphic to $\mathrm{Hom}_{AA}(P_\bullet, \Bbbk)$ as a cochain complex of $\Bbbk$-vector spaces. Regarding the monoid structure on $\mathrm{Hom}_{AA}(P_\bullet, \Bbbk)$ as the one induced by the comonoid $(P_\bullet, \omega, (\mathrm{id}_A)_0)$ via the lax monoidal functor $(\hom_{AA}(-, \Bbbk), \gamma, \gamma_0)$, one deduces that it coincides with the opposite of the usual cup product as defined in Example 2.2.19. ∎

4.28. COROLLARY. *Under the conditions above,* $\mathrm{H}^\bullet(A, \Bbbk)$ *is a graded braided commutative algebra in* ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$.

The projective resolutions of the Jordan and super Jordan plane shown respectively in Examples 4.18 and 4.19 verify the hypotheses of the previous theorem, therefore their re-spective Hochschild cohomology algebras with the opposite of the cup product are graded braided commutative algebras in ${}^{\Bbbk \mathbb{Z}}_{\Bbbk \mathbb{Z}}\mathcal{YD}$.

A. Computations for the Jordan and super Jordan planes

In this Appendix we make the computations needed to check the commutativity of the second diagram in Remark 3.25, that is of:

$$\mathrm{Hom}_{AA}(A^{\otimes p+2}, \Bbbk) \otimes \mathrm{Hom}_{AA}(A^{\otimes q+2}, \Bbbk) \xrightarrow{\smile_{p,q}} \mathrm{Hom}_{AA}(A^{\otimes p+q+2}, \Bbbk)$$

$$\xrightarrow{(-1)^{pq}\smile_{p,q}} \Bigg\downarrow {}^{\mathrm{Hom}_{AA}\left(\mathrm{id}_A \otimes c_{A^{\otimes p}, A^{\otimes q}} \otimes \mathrm{id}_A, \Bbbk\right)}$$

$$\mathrm{Hom}_{AA}(A^{\otimes p+q+2}, \Bbbk)..$$

In order to check the commutativity of this diagram after taking cohomology, we will choose in each case a suitable projective resolution of A as A-bimodule, denoted $P_\bullet \twoheadrightarrow A$, along with comparison morphisms $f_\bullet : P_\bullet \longrightarrow B_\bullet(A)$ and $g_\bullet : B_\bullet(A) \longrightarrow P_\bullet$.

For any two cocycles α and β, let us write $\alpha \sim \beta$ if their cohomology classes coincide. Commutativity means that

$$\mathrm{Hom}_{AA}\left(\mathrm{id}_A \otimes c_{A^{\otimes p}, A^{\otimes q}} \otimes \mathrm{id}_A, \Bbbk\right)(\smile_{p,q}(\psi' \otimes \varphi')) \sim (-1)^{pq} \smile_{p,q}(\psi' \otimes \varphi'),$$

for every $\psi' \in \mathrm{Hom}_{AA}(A^{\otimes p+2}, \Bbbk)$ and $\varphi' \in \mathrm{Hom}_{AA}(A^{\otimes q+2}, \Bbbk)$, that is:

$$(\psi' \smile \varphi') \circ \left(\mathrm{id}_A \otimes c_{A^{\otimes p}, A^{\otimes q}} \otimes \mathrm{id}_A\right) \sim (-1)^{pq}(\psi' \smile \varphi').$$

Replacing the bar resolution by $P_\bullet$ this reads as

$$(\psi \circ g_p \smile \varphi \circ g_q) \circ \left(\mathrm{id}_A \otimes c_{A^{\otimes p}, A^{\otimes q}} \otimes \mathrm{id}_A\right) \circ f_{p+q} \sim (-1)^{pq}(\psi \circ g_p \smile \varphi \circ g_q) \circ f_{p+q},$$

for every $\psi \in \mathrm{Hom}_{AA}(P_p, \Bbbk)$ and $\varphi \in \mathrm{Hom}_{AA}(P_q, \Bbbk)$. This equivalence of A-bimodule morphisms from P_{p+q} to $\Bbbk$ is what we will check.

. Let us start with the Jordan plane. We will use the resolution of Example 4.18:

$$0 \longrightarrow A \otimes R \otimes A \xrightarrow{d_1} A \otimes V \otimes A \xrightarrow{d_0} A \otimes A \longrightarrow 0,$$

where $V = \Bbbk\{x, y\}$ and $R = \Bbbk r$.

It is possible to define comparison morphisms $f_\bullet : P_\bullet \longrightarrow B_\bullet(A)$ and $g_\bullet : B_\bullet(A) \longrightarrow P_\bullet$ such that:

$$f_0 = g_0 = \mathrm{id}_{A\otimes A},$$
$$f_1(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1, \ \forall v \in V,$$
$$g_1(1 \otimes x^k y^l \otimes 1) = \sum_{i=0}^{k-1} x^i \otimes x \otimes x^{k-1-i} y^l + \sum_{j=0}^{l-1} x^k \otimes y \otimes y^{l-1-j}$$
$$f_2(1 \otimes r \otimes 1) = 1 \otimes y \otimes x \otimes 1 - 1 \otimes x \otimes y \otimes 1 + \tfrac{1}{2} 1 \otimes x \otimes x \otimes 1.$$

. Notice that the commutativity needs to be checked only for the cup products of $\mathrm{H}^1(A, \Bbbk)$ with itself, since $\mathrm{H}^0(A, \Bbbk)$ consists only of scalar multiples of the unit object and the product of an element in $\mathrm{H}^2(A, \Bbbk)$ with one in $\mathrm{H}^1(A, \Bbbk)$ or $\mathrm{H}^2(A, \Bbbk)$ is zero since $\mathrm{H}^3(A, \Bbbk) = 0$. Now, let $\psi, \varphi \in \mathrm{Hom}_{AA}(P_1, \Bbbk)$ be two 1-cocycles:

$$
\begin{aligned}
(\psi &\circ g_1 \smile \varphi \circ g_1)(f_2(1 \otimes r \otimes 1)) \\
&= (\psi \circ g_1 \smile \varphi \circ g_1)\left(1 \otimes y \otimes x \otimes 1 - 1 \otimes x \otimes y \otimes 1 + \tfrac{1}{2} 1 \otimes x \otimes x \otimes 1\right) \\
&= \varphi(g_1(1 \otimes y \otimes 1))\psi(g_1(1 \otimes x \otimes 1)) - \varphi(g_1(1 \otimes x \otimes 1))\psi(g_1(1 \otimes y \otimes 1)) \\
&\quad + \tfrac{1}{2}\varphi(g_1(1 \otimes x \otimes 1))\psi(g_1(1 \otimes x \otimes 1)) \\
&= \varphi(1 \otimes y \otimes 1)\psi(1 \otimes x \otimes 1) - \varphi(1 \otimes x \otimes 1)\psi(1 \otimes y \otimes 1) \\
&\quad + \tfrac{1}{2}\varphi(1 \otimes x \otimes 1)\psi(1 \otimes x \otimes 1).
\end{aligned}
$$

To compute the element we want to compare with the one above, recall that we write $\Bbbk\mathbb{Z} = \Bbbk[t, t^{-1}]$, and the braiding is given by $c_{V,W}(v \otimes w) = t^n \cdot w \otimes v$, where n is the degree of v. From now on, let $c_{p,q}$ denote the map $\mathrm{id}_A \otimes c_{A^{\otimes p}, A^{\otimes q}} \otimes \mathrm{id}_A$ for every $p, q \in \mathbb{N}_0$.

$$
\begin{aligned}
(\psi &\circ g_1 \smile \varphi \circ g_1)(c_{1,1}(f_2(1 \otimes r \otimes 1)) \\
&= (\psi \circ g_1 \smile \varphi \circ g_1)\left(c_{1,1}\left(1 \otimes y \otimes x \otimes 1 - 1 \otimes x \otimes y \otimes 1 + \tfrac{1}{2} 1 \otimes x \otimes x \otimes 1\right)\right) \\
&= (\psi \circ g_1 \smile \varphi \circ g_1)\left(1 \otimes t \cdot x \otimes y \otimes 1 - 1 \otimes t \cdot y \otimes x \otimes 1 + \tfrac{1}{2} 1 \otimes t \cdot x \otimes x \otimes 1\right) \\
&= (\varphi \circ g_1 \smile \psi \circ g_1)\left(1 \otimes x \otimes y \otimes 1 - 1 \otimes (x + y) \otimes x \otimes 1 + \tfrac{1}{2} 1 \otimes x \otimes x \otimes 1\right) \\
&= -\varphi(1 \otimes y \otimes 1)\psi(1 \otimes x \otimes 1) + \varphi(1 \otimes x \otimes 1)\psi(1 \otimes y \otimes 1) \\
&\quad - \tfrac{1}{2}\varphi(1 \otimes x \otimes 1)\psi(1 \otimes x \otimes 1) \\
&= -(\psi \circ g_1 \smile \varphi \circ g_1)(f_2(1 \otimes r \otimes 1)),
\end{aligned}
$$

as we wanted to prove.

. The computation for the super Jordan plane is more involved, since the minimal resolution, which is free, is as follows (see Example 4.19):

$$\cdots \xrightarrow{d_3} A \otimes V_3 \otimes A \xrightarrow{d_2} A \otimes V_2 \otimes A \xrightarrow{d_1} A \otimes V_1 \otimes A \xrightarrow{d_0} A \otimes A \longrightarrow 0,$$

where $V_1 = \Bbbk\{x, y\}$ and $V_n = \Bbbk\{x^n, y^2 x^{n-1}\}$, $\forall n \geq 2$.

For checking the commutativity condition, we will use the comparison morphisms $f_\bullet : P_\bullet \longrightarrow B_\bullet(A)$ and $g_\bullet : B_\bullet(A) \longrightarrow P_\bullet$ defined in [RS18, Section 5]. Since the chain morphism $g_\bullet$ is only partially defined in [RS18], we need to extend the definition of g_n for $n \geq 2$ as follows, :

$$g_n(1 \otimes x^{\otimes i} \otimes xy \otimes x^{\otimes n-1-i} \otimes 1) = 0 \quad \forall i: \ 0 \leq i \leq n - 2,$$
$$g_n(1 \otimes x^{\otimes n-1} \otimes xy \otimes 1) = 1 \otimes x^n \otimes y.$$

Again, $\mathrm{H}^0(A, \Bbbk) = \Bbbk$, so we will check the condition for cocycles in degrees greater or equal to 1. The cup products to be computed lie in $\mathrm{H}^n(A, \Bbbk)$ for $n \geq 2$, so we are going to evaluate them on the basis of P_n. Let $\psi \in \mathrm{Hom}_{AA}(P_p, \Bbbk)$ and $\varphi \in \mathrm{Hom}_{AA}(P_q, \Bbbk)$ be two cocycles, for $p, q \geq 1$. We need to consider four cases. In all of them, when applying the morphisms g_n, we keep in mind that $\psi(g_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1})) = 0$ if a_0 or a_{n+1} are homogeneous of positive degree, because they act by 0 on $\Bbbk$, and the same holds for φ. For $p = q = 1$:

Evaluating in $1 \otimes x^2 \otimes 1$ gives

$$
\begin{aligned}
(\psi \circ g_1 \smile \varphi \circ g_1)(f_2(1 \otimes x^2 \otimes 1)) &= (\psi \circ g_1 \smile \varphi \circ g_1)(1 \otimes x^{\otimes 2} \otimes 1) \\
&= \varphi(g_1(1 \otimes x \otimes 1))\psi(g_1(1 \otimes x \otimes 1)) \\
&= \varphi(1 \otimes x \otimes 1)\psi(1 \otimes x \otimes 1),
\end{aligned}
$$

$$
\begin{aligned}
(\psi \circ g_1 \smile \varphi \circ g_1)(c_{1,1}(f_2(1 \otimes x^2 \otimes 1))) &= (\psi \circ g_1 \smile \varphi \circ g_1)(c_{1,1}(1 \otimes x \otimes x \otimes 1)) \\
&= (\psi \circ g_1 \smile \varphi \circ g_1)(1 \otimes t \cdot x \otimes x \otimes 1) \\
&= (\psi \circ g_1 \smile \varphi \circ g_1)(1 \otimes (-x) \otimes x \otimes 1) \\
&= -\varphi(g_1(1 \otimes x \otimes 1))\psi(g_1(1 \otimes x \otimes 1)) \\
&= -\varphi(1 \otimes x \otimes 1)\psi(1 \otimes x \otimes 1) \\
&= -(\psi \circ g_1 \smile \varphi \circ g_1)(f_2(1 \otimes x^2 \otimes 1)),
\end{aligned}
$$

while evaluating in $1 \otimes y^2 x \otimes 1$ gives

$$
\begin{aligned}
(\psi \circ g_1 &\smile \varphi \circ g_1)(f_2(1 \otimes y^2 x \otimes 1)) \\
&= (\psi \circ g_1 \smile \varphi \circ g_1)(1 \otimes y \otimes yx \otimes 1 - 1 \otimes x \otimes y^2 \otimes 1 - 1 \otimes x \otimes yx \otimes 1) \\
&= \varphi(g_1(1 \otimes y \otimes 1))\psi(g_1(1 \otimes yx \otimes 1)) - \varphi(g_1(1 \otimes x \otimes 1))\psi(g_1(1 \otimes y^2 \otimes 1)) \\
&\quad - \varphi(g_1(1 \otimes x \otimes 1))\psi(g_1(1 \otimes yx \otimes 1)) \\
&= 0,
\end{aligned}
$$

since $g_1(1 \otimes a \otimes 1) = 0$ if a is homogeneous of degree greater than 1. On the other hand,

$$(\psi \circ g_1 \smile \varphi \circ g_1)(c_{1,1}(f_2(1 \otimes y^2 x \otimes 1)))$$
$$= (\psi \circ g_1 \smile \varphi \circ g_1)(c_{1,1}(1 \otimes y \otimes yx \otimes 1 - 1 \otimes x \otimes y^2 \otimes 1 - 1 \otimes x \otimes yx \otimes 1))$$
$$= (\psi \circ g_1 \smile \varphi \circ g_1)(1 \otimes t \cdot yx \otimes y \otimes 1 - 1 \otimes t \cdot y^2 \otimes x \otimes 1 - 1 \otimes t \cdot yx \otimes x \otimes 1)$$
$$= (\psi \circ g_1 \smile \varphi \circ g_1)(1 \otimes yx \otimes y \otimes 1 - 1 \otimes (y^2 - xy - yx) \otimes x \otimes 1 - 1 \otimes yx \otimes x \otimes 1)$$
$$= \varphi(g_1(1 \otimes yx \otimes 1))\psi(g_1(1 \otimes y \otimes 1) - \varphi(g_1(1 \otimes (y^2 - xy - yx) \otimes 1))\psi(g_1(1 \otimes x \otimes 1)$$
$$\quad - \varphi(g_1(1 \otimes yx \otimes 1))\psi(g_1(1 \otimes x \otimes 1))$$
$$= 0 = -(\psi \circ g_1 \smile \varphi \circ g_1)(f_2(1 \otimes y^2 x \otimes 1)).$$

For $p = 1, q \geq 2$, evaluating in $1 \otimes x^{1+q} \otimes 1$ gives on one hand

$$(\psi \circ g_1 \smile \varphi \circ g_q)(f_{1+q}(1 \otimes x^{1+q} \otimes 1)) = (\psi \circ g_1 \smile \varphi \circ g_q)(1 \otimes x^{\otimes 1+q} \otimes 1)$$
$$= \varphi(g_q(1 \otimes x^{\otimes q} \otimes 1)(\psi(g_1(1 \otimes x \otimes 1))$$
$$= \varphi(1 \otimes x^{\otimes q} \otimes 1)\psi(1 \otimes 1),$$

and on the other hand

$$(\psi \circ g_1 \smile \varphi \circ g_q)(c_{1,q}(f_{1+q}(1 \otimes x^{1+q} \otimes 1)))$$
$$= (\varphi \circ g_1 \smile \psi \circ g_q)(c_{1,q}(1 \otimes x^{\otimes 1+q} \otimes 1))$$
$$= (\varphi \circ g_q \smile \psi \circ g_1)(c_{1,q}(1 \otimes x \otimes x^{\otimes q} \otimes 1))$$
$$= (\psi \circ g_1 \smile \varphi \circ g_q)(1 \otimes t \cdot (x^{\otimes q}) \otimes x \otimes 1)$$
$$= (\psi \circ g_1 \smile \varphi \circ g_q)(1 \otimes (t \cdot x)^{\otimes q} \otimes x \otimes 1)$$
$$= (\psi \circ g_1 \smile \varphi \circ g_q)(1 \otimes (-x)^{\otimes q} \otimes x \otimes 1)$$
$$= (-1)^q (\psi \circ g_1 \smile \varphi \circ g_q)(1 \otimes x^{\otimes 1+q} \otimes 1)$$
$$= (-1)^q (\psi \circ g_1 \smile \varphi \circ g_q)(f_{1+q}(1 \otimes x^{1+q} \otimes 1)),$$

while evaluating in $1 \otimes y^2 x^q \otimes 1$ gives

$$(\psi g_1 \smile \varphi g_q)(f_{1+q}(1 \otimes y^2 x^q \otimes 1))$$
$$= (\psi g_1 \smile \varphi g_q)(1 \otimes y \otimes yx \otimes x^{\otimes q-1} \otimes 1 - 1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes q-1} \otimes 1$$
$$\quad + \sum_{i=0}^{q-2}(-1)^i \, 1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes q-2-i} \otimes 1)$$
$$= \varphi(g_q(1 \otimes y \otimes yx \otimes x^{\otimes q-2} \otimes 1))\psi(g_1(1 \otimes x \otimes 1))$$
$$\quad - \varphi(g_q(1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes q-2} \otimes 1))\psi(g_1(1 \otimes x \otimes 1))$$
$$\quad + \sum_{i=0}^{q-3}(-1)^i \, \varphi(g_q(1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes q-3-i} \otimes 1))\psi(g_1(1 \otimes x \otimes 1))$$
$$\quad + (-1)^{q-2} \, \varphi(g_q(1 \otimes x^{\otimes q} \otimes 1))\psi(g_1(1 \otimes (y^2 + yx) \otimes 1))$$
$$= \varphi(1 \otimes y^2 x^{q-1} \otimes 1)\psi(1 \otimes x \otimes 1),$$

$$(\psi g_1 \smile \varphi g_q)(c_{1,q}(f_{1+q}(1 \otimes y^2 x^q \otimes 1)))$$
$$= (\psi g_1 \smile \varphi g_q)(c_{1,q}(1 \otimes y \otimes yx \otimes x^{\otimes q-1} \otimes 1 - 1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes q-1} \otimes 1$$
$$+ \sum_{i=0}^{q-2}(-1)^i \, 1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes q-2-i} \otimes 1))$$
$$= (\psi g_1 \smile \varphi g_q)(1 \otimes t \cdot yx \otimes (t \cdot x)^{\otimes q-1} \otimes y \otimes 1$$
$$- 1 \otimes t \cdot (y^2 + yx) \otimes (t \cdot x)^{\otimes q-1} \otimes x \otimes 1$$
$$+ \sum_{i=0}^{q-2}(-1)^i \, 1 \otimes (t \cdot x)^{\otimes 1+i} \otimes t \cdot (y^2 + yx) \otimes (t \cdot x)^{\otimes q-2-i} \otimes x \otimes 1)$$
$$= (\psi g_1 \smile \varphi g_q)(1 \otimes yx \otimes (-x)^{\otimes q-1} \otimes y \otimes 1$$
$$- 1 \otimes (y^2 - xy) \otimes (-x)^{\otimes q-1} \otimes x \otimes 1$$
$$+ \sum_{i=0}^{q-2}(-1)^i \, 1 \otimes (-x)^{\otimes 1+i} \otimes (y^2 - xy) \otimes (-x)^{\otimes q-2-i} \otimes x \otimes 1)$$
$$= (-1)^{q-1} \, \varphi(g_q(1 \otimes yx \otimes x^{\otimes q-1}))\psi(g_1(1 \otimes y \otimes 1))$$
$$- (-1)^{q-1} \, \varphi(g_q(1 \otimes y^2 \otimes x^{\otimes q-1} \otimes 1))\psi(g_1(1 \otimes 1))$$
$$+ \varphi(g_q(1 \otimes x^{\otimes q-1} \otimes xy \otimes 1))\psi(g_1(1 \otimes x \otimes 1))$$
$$= (-1)^q \, \varphi(1 \otimes y^2 x^{q-1} \otimes 1)\psi(1 \otimes x \otimes 1)$$
$$= (-1)^q \, (\psi g_1 \smile \varphi g_q)(f_{1+q}(1 \otimes y^2 x^q \otimes 1))$$

For $p \geq 2, q = 1$, evaluating in $1 \otimes x^{p+1} \otimes 1$ gives

$$(\psi \circ g_p \smile \varphi \circ g_1)(f_{p+1}(1 \otimes x^{p+1} \otimes 1)) \; = \; (\psi \circ g_p \smile \varphi \circ g_1)(1 \otimes x^{\otimes p+1} \otimes 1)$$
$$= \; \varphi(g_1(1 \otimes x^{\otimes 1} \otimes 1)(\psi(g_p(1 \otimes x^{\otimes p} \otimes 1))$$
$$= \; \varphi(1 \otimes x \otimes 1)\psi(1 \otimes x^p \otimes 1)$$

$$(\psi \circ g_p \smile \varphi \circ g_1)(c_{p,1}(f_{p+1}(1 \otimes x^{p+1} \otimes 1))) \; = \; (\psi \circ g_p \smile \varphi \circ g_1)(c_{p,1}(1 \otimes x^{\otimes p+1} \otimes 1))$$
$$= \; (\psi \circ g_p \smile \varphi \circ g_1)(c_{p,1}(1 \otimes x^{\otimes p} \otimes x \otimes 1))$$
$$= \; (\psi \circ g_p \smile \varphi \circ g_1)(1 \otimes t^p \cdot x \otimes x^{\otimes p} \otimes 1)$$
$$= \; (\psi \circ g_p \smile \varphi \circ g_1)(1 \otimes (-1)^p x \otimes x^{\otimes p} \otimes 1)$$
$$= \; (-1)^p(\psi \circ g_p \smile \varphi \circ g_1)(1 \otimes x^{\otimes p+1} \otimes 1)$$
$$= \; (-1)^{1p}(\psi \circ g_p \smile \varphi \circ g_1)(f_{p+1}(1 \otimes x^{p+1} \otimes 1)).$$

Evaluating in $1 \otimes y^2 x^p \otimes 1$ gives

$$(\psi g_p \smile \varphi g_1)(f_{p+1}(1 \otimes y^2 x^p \otimes 1))$$
$$= (\psi g_p \smile \varphi g_q)(1 \otimes y \otimes yx \otimes x^{\otimes p-1} \otimes 1$$
$$- 1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-1} \otimes 1$$
$$+ \sum_{i=0}^{p-2}(-1)^i \, 1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-2-i} \otimes 1)$$
$$- \varphi(g_1(1 \otimes y \otimes 1))\psi(g_p(1 \otimes yx \otimes x^{\otimes p-1} \otimes 1))$$
$$- \varphi(g_1(1 \otimes x \otimes \; 1))\psi(g_p(1 \otimes (y^2 + yx) \otimes x^{\otimes p-1} \otimes 1))$$
$$+ \sum_{i=0}^{p-2}(-1)^i \, \varphi(g_1(1 \otimes x \otimes 1))\psi(g_p(1 \otimes x^{\otimes 1+i} \otimes (y^2 + yx) \otimes x^{\otimes p-2-i} \otimes 1)).$$
$$= -\varphi(1 \otimes y^2 x^{p-1} \otimes 1)\psi(1 \otimes y^2 x^{p-1} \otimes 1), (\psi g_p \smile \varphi g_1)(c_{p,1}(f_{p+1}(1 \otimes y^2 x^p \otimes 1)))$$
$$= (\psi g_p \smile \varphi g_1)(c_{p,1}(1 \otimes y \otimes yx \otimes x^{\otimes p-1} \otimes 1$$
$$- 1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-1} \otimes 1$$
$$+ \sum_{i=0}^{p-2}(-1)^i \, 1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-2-i} \otimes 1))$$
$$= (\psi g_p \smile \varphi g_1)(1 \otimes t^{p+1} \cdot x \otimes y \otimes yx \otimes x^{\otimes p-2} \otimes 1$$
$$- 1 \otimes t^{p+1} \cdot x \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-2} \otimes 1$$
$$+ \sum_{i=0}^{p-3}(-1)^i \, 1 \otimes t^{p+1} \cdot x \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-3-i} \otimes 1$$
$$+ (-1)^{p-2} \, 1 \otimes t^p \cdot (y^2 + xy) \otimes x^{\otimes p} \otimes 1)$$

$$
\begin{aligned}
&= (\psi g_p \smile \varphi g_1)(1 \otimes (-1)^{p+1} \, x \otimes y \otimes yx \otimes x^{\otimes p-2} \otimes 1 \\
&\quad - 1 \otimes (-1)^{p+1} \, x \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-2} \otimes 1 \\
&\quad + \textstyle\sum_{i=0}^{p-3}(-1)^i \, 1 \otimes (-1)^{p+1} \, x \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-3-i} \otimes 1 \\
&\quad + (-1)^{p-2} \, 1 \otimes (y^2 - (p-1)yx - pxy) \otimes x^{\otimes p} \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{p+1}\varphi(g_1(1 \otimes x \otimes 1))\psi(g_p(1 \otimes y \otimes yx \otimes x^{\otimes p-2} \otimes 1)) \\
&= (-1)^{p+1}\varphi(1 \otimes y^2 x^{p-1} \otimes 1)\psi(1 \otimes y^2 x^{p-1} \otimes 1) \\
&= (-1)^p \, (\psi g_p \smile \varphi g_1)(f_{p+1}(1 \otimes y^2 x^p \otimes 1)).
\end{aligned}
$$

The last case to consider is $p, q \geq 2$, evaluating in $1 \otimes x^{p+q} \otimes 1$ gives

$$
\begin{aligned}
(\psi \circ g_p \smile \varphi \circ g_q)(f_{p+q}(1 \otimes x^{p+q} \otimes 1)) &= (\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes x^{\otimes p+q} \otimes 1) \\
&= \varphi(g_q(1 \otimes x^{\otimes q} \otimes 1))\psi(g_p(1 \otimes x^{\otimes p} \otimes 1)) \\
&= \varphi(1 \otimes x^q \otimes 1)\psi(1 \otimes x^p \otimes 1),
\end{aligned}
$$

$$
\begin{aligned}
(\psi \circ g_p \smile \varphi \circ g_q)(c_{p,q}(f_{p+q}(1 \otimes x^{p+q} \otimes 1))) &= (\psi \circ g_p \smile \varphi \circ g_q)(c_{p,q}(1 \otimes x^{\otimes p+q} \otimes 1)) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(c_{p,q}(1 \otimes x^{\otimes p} \otimes x^{\otimes q} \otimes 1)) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes t^p \cdot (x^{\otimes q}) \otimes x^{\otimes p} \otimes 1) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes (t^p \cdot x)^{\otimes q} \otimes x^{\otimes p} \otimes 1) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes ((-1)^p x)^{\otimes q} \otimes x^{\otimes p} \otimes 1) \\
&= (-1)^{pq}(\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes x^{\otimes p+q} \otimes 1) \\
&= (-1)^{pq}(\psi \circ g_p \smile \varphi \circ g_q)(f_{p+q}(1 \otimes x^{p+q} \otimes 1)),
\end{aligned}
$$

For $1 \otimes y^2 x^{p+q-1} \otimes 1$ we get

$$
\begin{aligned}
&(\psi g_p \smile \varphi g_q)(f_{q+p}(1 \otimes y^2 x^{q+p-1} \otimes 1)) \\
&= (\psi g_p \smile \varphi g_q)(1 \otimes y \otimes yx \otimes x^{\otimes q+p-2} \otimes 1 \\
&\quad - 1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes q+p-2} \otimes 1 \\
&\quad + \textstyle\sum_{i=0}^{q+p-3}(-1)^i \, 1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes q+p-3-i} \otimes 1) \\
&= \varphi g_q(1 \otimes y \otimes yx \otimes x^{\otimes q-2} \otimes 1)\psi g_p(1 \otimes x^{\otimes p} \otimes 1) \\
&\quad - \varphi g_q(1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes q-2} \otimes 1)\psi g_p(1 \otimes x^{\otimes p} \otimes 1) \\
&\quad + \textstyle\sum_{i=0}^{q-3}(-1)^i \, \varphi g_q(1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes q-3-i} \otimes 1)\psi g_p(1 \otimes x^{\otimes p} \otimes 1) \\
&\quad + \textstyle\sum_{i=q-2}^{q+p-3} \varphi g_q(1 \otimes x^{\otimes q} \otimes 1)\psi g_p(1 \otimes x^{\otimes 2+i-q} \otimes (y^2 + yx) \otimes x^{\otimes q+p-3-i} \otimes 1) \\
&= \varphi(1 \otimes y^2 x^{q-1} \otimes 1)\psi(1 \otimes x^p \otimes 1) + (-1)^{q-2}\varphi(1 \otimes x^q \otimes 1)\psi(1 \otimes y^2 x^{p-1} \otimes 1).
\end{aligned}
$$

Most of the terms in the above sums vanish because

$$
g_n(1 \otimes x^{\otimes j} \otimes y^2 \otimes x^{\otimes n-1-j} \otimes 1) = g_n(1 \otimes x^{\otimes j} \otimes yx \otimes x^{\otimes n-1-j} \otimes 1) = 0, \ \forall n \geq 2, j \geq 1.
$$

Regarding the unvanishing term, which corresponds to $i = q - 2$, we have that

$$
g_q(1 \otimes y^2 \otimes x^{\otimes q-1} \otimes 1) = 1 \otimes y^2 x^{q-1} \otimes 1, \ \text{and} \ g_q(1 \otimes yx \otimes x^{\otimes q-1} \otimes 1) = y \otimes x^q \otimes 1,
$$

which vanishes after applying the A-bimodule morphism ψ. Finally

$$
\begin{aligned}
&(\psi \circ g_p \smile \varphi \circ g_q)(c_{p,q}(f_{q+p}(1 \otimes y^2 x^{q+p-1} \otimes 1))) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(c_{p,q}(1 \otimes y \otimes yx \otimes x^{\otimes p-2+q} \otimes 1 \\
&\qquad - 1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-2+q} \otimes 1 \\
&\qquad + \textstyle\sum_{i=0}^{p-3}(-1)^i \, 1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-3-i+q} \otimes 1 \\
&\qquad + \textstyle\sum_{i=p-2}^{q+p-3}(-1)^i \, 1 \otimes x^{\otimes p} \otimes x^{\otimes 2+i-p} \otimes (y^2 + yx) \otimes x^{\otimes q+p-3-i} \otimes 1)) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes t^{p+1}(x^{\otimes q}) \otimes y \otimes yx \otimes x^{\otimes p-2} \otimes 1 \\
&\qquad - 1 \otimes t^{p+1}(x^{\otimes q}) \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-2} \otimes 1 \\
&\qquad + \textstyle\sum_{i=0}^{p-3}(-1)^i \, 1 \otimes t^{p+1}(x^{\otimes q}) \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-3-i} \otimes 1 \\
&\qquad + \textstyle\sum_{i=p-2}^{q+p-3}(-1)^i \, 1 \otimes t^p(x^{\otimes 2+i-p} \otimes (y^2 + yx) \otimes x^{\otimes q+p-3-i}) \otimes x^{\otimes p} \otimes 1) \\
&= (\psi \circ g_p \smile \varphi \circ g_q)(1 \otimes ((-1)^{p+1}x)^{\otimes q} \otimes y \otimes yx \otimes x^{\otimes p-2} \otimes 1 \\
&\qquad - 1 \otimes ((-1)^{p+1}x)^{\otimes q} \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-2} \otimes 1 \\
&\qquad + \textstyle\sum_{i=0}^{p-3}(-1)^i \, 1 \otimes ((-1)^{p+1}x)^{\otimes q} \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-3-i} \otimes 1 \\
&\qquad + \textstyle\sum_{i=p-2}^{q+p-3}(-1)^i \, 1 \otimes ((-1)^{p}x)^{\otimes 2+i-p} \otimes (y^2 - (p-1)yx - pxy) \\
&\qquad\qquad\qquad \otimes ((-1)^p x)^{\otimes q+p-3-i} \otimes x^{\otimes p} \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{(p+1)q}\varphi(g_q(1 \otimes x^{\otimes q} \otimes 1))\psi(g_p(1 \otimes y \otimes yx \otimes x^{\otimes p-2} \otimes 1)) \\
&\quad - (-1)^{(p+1)q}\varphi(g_q(1 \otimes x^{\otimes q} \otimes 1))\psi(g_p(1 \otimes x \otimes (y^2 + yx) \otimes x^{\otimes p-2} \otimes 1)) \\
&\quad + (-1)^{(p+1)q} \textstyle\sum_{i=0}^{p-3}(-1)^i \varphi(g_q(1 \otimes x^{\otimes q} \otimes 1)) \\
&\qquad\qquad\qquad \psi(g_p(1 \otimes x^{\otimes 2+i} \otimes (y^2 + yx) \otimes x^{\otimes p-3-i} \otimes 1)) \\
&\quad + (-1)^{p(q-1)} \textstyle\sum_{i=p-2}^{q+p-3}(-1)^i \varphi(g_q(1 \otimes x^{\otimes 2+i-p} \otimes (y^2 - (p-1)yx - pxy) \\
&\qquad\qquad\qquad \otimes x^{\otimes q+p-3-i} \otimes 1))\psi(g_p(1 \otimes x^{\otimes p} \otimes 1)) \\
&= (-1)^{(p+1)q}\varphi(1 \otimes x^q \otimes 1)\psi(1 \otimes y^2 x^{p-1} \otimes 1) \\
&\quad + (-1)^{p(q-1)}(-1)^{p-2}\varphi(1 \otimes y^2 x^{q-1} \otimes 1)\psi(1 \otimes x^p \otimes 1) \\
&= (-1)^{pq}\left((-1)^q\varphi(1 \otimes x^q \otimes 1)\psi(1 \otimes y^2 x^{p-1} \otimes 1) + \varphi(1 \otimes y^2 x^{q-1} \otimes 1)\psi(1 \otimes x^p \otimes 1)\right) \\
&= (-1)^{pq}(\psi \circ g_p \smile \varphi \circ g_q)(f_{q+p}(1 \otimes y^2 x^{q+p-1} \otimes 1)).
\end{aligned}
$$

The second term, as well as the terms in the sums corresponding to $i \leq p-3$, vanish after applying g_p in their second factor. Those corresponding to $i \geq p-2$ can be expressed as a sum of three terms by distributing over the expression in the tensor $y^2 - (p-1)yx - pxy$. The only non vanishing one corresponds to y^2.

References

[AM10] M. Aguiar and S. Mahajan, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010.

[AAPW22] N. Andruskiewitsch, I. Angiono, J. Pevtsova, and S. Witherspoon, "Cohomology rings of finite-dimensional pointed Hopf algebras over abelian groups," *Res. Math. Sci.*, vol. 9, no. 1, pp. 132, 2022.

[AnSch02] N. Andruskiewitsch and H. J. Schneider, "Pointed Hopf algebras," in *New directions in Hopf algebras*, Cambridge: Cambridge University Press, pp. 1–68, 2002. ISBN 0-521-81512-6.

[ArSch87] M. Artin and W. F. Schelter, "Graded algebras of global dimension 3," *Adv. Math.*, vol. 66, pp. 171–216, 1987.

[BK01] B. Bakalov and A. Kirillov, *Lectures on tensor categories and modular functors*, University Lecture Series, vol. 21, Providence, RI: American Mathematical Society, 2001.

[BW05] M. Baranov and A. W. Wiegmann, "Group and Hopf algebra structures in the cohomology of moduli spaces," *Proceedings of the American Mathematical Society*, vol. 133, no. 7, pp. 2131–2140, 2005.

[BM12] M. Batanin and M. Markl, "Centers and homotopy centers in enriched monoidal categories," *Adv. Math.*, vol. 230, no. 4-6, pp. 1811–1858, 2012.

[Bén65] J. Bénabou, "Catégories avec multiplication," *C.R Acad. Sci. Paris*, vol. 256, no. 9, pp. 1887, 1965.

[Ber98] F. Bergeron, *Combinatorial species and tree-like structures*, vol. 67, Cambridge University Press, 1998.

[CE99] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1999.

[C55] P. Cartier, "Cohomologie des coalgèbres," *Séminaire "Sophus Lie"*, vol. 2, pp. 1–18, 1955.

[Č37] E. Čech, "Multiplications on a complex," *Annals of Mathematics*, vol. 37, pp. 681–697, 1937.

[E63] C. Ehresmann, "Catégories structurées," in *Annales scientifiques de l'École Normale Supérieure*, vol. 80, no. 4, pp. 349–426, 1963.

[EM47] S. Eilenberg and S. MacLane, "Cohomology theory in abstract groups. I," *Annals of Mathematics*, pp. 51–78, 1947.

[EZ50] S. Eilenberg and J. A. Zilber, "Semi-simplicial complexes and singular homology," *Annals of Mathematics*, pp. 499–513, 1950.

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, Providence, RI: American Mathematical Society, 2015.

[FS04] M. A. Farinati and A. Solotar, "G-structure on the cohomology of Hopf algebras," *Proc. Am. Math. Soc.*, vol. 132, no. 10, pp. 2859–2865, 2004.

[GLF16] R. Garner and I. López Franco, "Commutativity," *J. Pure Appl. Algebra*, vol. 220, no. 5, pp. 1707–1751, 2016.

[G64] M. Gerstenhaber, "On the deformation of rings and algebras," *Ann. Math.*, vol. 79 no. 1, pp. 59–103, 1964.

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Her20] E. Herscovich, "An elementary computation of the cohomology of the Fomin-Kirillov algebra with 3 generators," *Homology Homotopy Appl.*, vol. 22, no. 2, pp. 367–386, 2020.

[J81] A. Joyal, "Une théorie combinatoire des séries formelles," *Advances in Mathematics*, vol. 42, no. 1, pp. 1–82, 1981.

[JS91] A. Joyal and R. Street, "The geometry of tensor calculus. I," *Adv. Math.*, vol. 88, no. 1, pp. 55–112, 1991.

[JS93] A. Joyal and R. Street, "Braided tensor categories," *Adv. Math.*, vol. 102, no. 1, pp. 20–78, 1993.

[KMOOW21] T. Karadağ, D. McPhate, P. S. Ocal, T. Oke, and S. J. Witherspoon, "Gerstenhaber brackets on Hochschild cohomology of general twisted tensor products," *J. Pure Appl. Algebra*, vol. 225, no. 6 pp. 106597, 2021.

[K64] G. M. Kelly, "On MacLane's conditions for coherence of natural associativities, commutativities, etc.," *J. Algebra*, vol. 1, pp. 397–402, 1964.

[K05] G. M. Kelly, "On the operads of JP May," *Repr. Theory Appl. Categ*, vol. 13, pp. 1–13, 2005.

[LS21] S. A. Lopes and A. Solotar, "Lie structure on the Hochschild cohomology of a family of subalgebras of the Weyl algebra," *J. Noncomm. Geom.*, vol. 15, no. 4, pp. 1373–1407, 2021.

[Mac63] S. Mac Lane, "Natural associativity and commutativity," *Rice Univ. Studies*, vol. 49, no. 4, pp. 28–46, 1963.

[Mac98] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, New York, NY: Springer, 1998.

[MPSW10] M. Mastnak, J. Pevtsova, P. Schauenburg, and S. Witherspoon, (2010). "Cohomology of finite-dimensional pointed Hopf algebras," *Proceedings of the London Mathematical Society*, vol. 100 no. 2, pp. 377–404.

[Mas99] A. Masuoka, "Extensions of Hopf algebras," *Trabajos de Matemática*, vol. 41, pp. 99, 1999.

[Mon93] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[RS18] S. Reca and A. Solotar, "Homological invariants relating the super Jordan plane to the Virasoro algebra," *J. Algebra*, vol. 507, pp. 120–185, 2018.

[ŞV16] D. Ştefan and C. Vay, "The cohomology ring of the 12-dimensional Fomin-Kirillov algebra," *Advances in Mathematics*, vol. 291, pp. 584–620, 2016.

[SÁ04] M. Suárez-Álvarez, "The Hilton-Eckmann argument for the anti-commutativity of cup products," *Proc. Am. Math. Soc.*, vol. 132, no. 8, pp. 2241–2246, 2004.

[T04] R. Taillefer, "Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product," *J. Algebra*, vol. 276, no. 1, pp. 259–279, 2004.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge: Cambridge University Press, 1994.

[Wit19] S. J. Witherspoon, *Hochschild cohomology for algebras*, Graduate Studies in Mathematics, vol. 204, American Mathematical Society, Providence, RI, 2019.

Instituto de Matemática y Estadística "Rafael Laguardia", Facultad de Ingeniería
Universidad de la República, Uruguay

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, Argentina

Email: `jcoppola@fing.edu.uy`
       `asolotar@dm.uba.ar`

This article may be accessed at `http://www.tac.mta.ca/tac/`

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.