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CATEGORICAL GENERALISATIONS OF QUANTUM DOUBLE MODELS

ANNA-KATHARINA HIRMER AND CATHERINE MEUSBURGER

Abstract. We show that every involutive Hopf monoid in a complete and finitely cocomplete symmetric monoidal category gives rise to invariants of oriented surfaces defined in terms of ribbon graphs. For every ribbon graph this yields an object in the category, defined up to isomorphism, that depends only on the homeomorphism class of the associated surface. This object is constructed via (co)equalisers and images and equipped with a mapping class group action. It can be viewed as a categorical generalisation of the ground state of Kitaev's quantum double model or of a representation variety for a surface. We apply the construction to group objects in cartesian monoidal categories, in particular to simplicial groups as group objects in SSet and to crossed modules as group objects in Cat. The former yields a simplicial set consisting of representation varieties, the latter a groupoid whose sets of objects and morphisms are obtained from representation varieties.

1. Introduction

Constructions that assign algebraic or geometric objects with mapping class group actions to oriented surfaces are of interest in many contexts. They arise in 3d topological quantum field theories (TQFTs) of Turaev–Viro–Barrett–Westbury or Reshetikhin–Turaev type [TV, BrW, RT] and are encoded in the weaker notion of a modular functor, see [BK, Def. 5.1.1]. For a recent construction of modular functors from finite tensor categories, see Fuchs, Schweigert and Schaumann [FSSa], for a classification via factorisation homology, Brochier and Woike [BrW].

They also arise in Hamiltonian quantisation formalisms for representation varieties. The associated mapping class group actions were first discovered in the combinatorial quantisation formalism by Alekseev, Grosse, Schomerus [AGSa, AGSb, AS] and Buffenoir and Roche [BR95, BR96], subsequently related to factorisation homology by Ben-Zvi, Brochier and Jordan [BBJa, BBJb] and studied by Faitg [Fa18, Fa19].

Objects with mapping class group actions also arise from correlators in conformal field theories, see the work by Fuchs, Schweigert and Stigner [FS, FSSb]. They are also present in models from condensed matter physics and topological quantum computing such as Levin–Wen models [LW] and Kitaev's quantum double model [Ki]. Due to the work of

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Lyubashenko [Lya, Lyb, Lyc] it is well understood how to construct projective mapping class group representations from Hopf algebras in abelian ribbon categories.

Many of these constructions are based on assignments of algebraic data to certain graphs on surfaces, and they require linear categories with duals, often also abelian, finite or semisimple. These restrictions are of course well motivated from the context of TQFTs, in the quantisation of gauge theories or in condensed matter physics. However, it is also desirable to go beyond them.

In this article we show that any involutive Hopf monoid H in a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$ yields an invariant of oriented surfaces. We compute this invariant for examples, such as simplicial groups as Hopf monoids in SSet and crossed modules as Hopf monoids in Cat.

The construction is based on the choice of a ribbon graph. It determines an object in $\mathcal{C}$, defined up to isomorphisms, that depends only on the homeomorphism class of the surface obtained by attaching discs to the faces of the graph. As Hopf monoids are categorical generalisations of Hopf algebras, it can be viewed as a categorical generalisation of Kitaev's quantum double model or of representation varieties or moduli spaces of flat bundles on surfaces.

More precisely, we consider for a ribbon graph the $|E|$-fold tensor product $H^{\otimes E}$, where E is the edge set of the graph. We use the structure morphisms of the Hopf monoid to associate H-module structures to its vertices and H-comodule structures to its faces. This requires a choice of a marking for each vertex or face, and each marking defines a Yetter–Drinfeld module structure over H. The object assigned to the graph is obtained by equalising the H-comodule structures, by coequalising the H-module structures and by combining them via a categorical image. It generalises the protected space or ground state of Kitaev's quantum double model, and we therefore call it the protected object. We then show

Theorem. (Theorem 5.25) The isomorphism class of the protected object for a ribbon graph depends only on the homeomorphism class of the associated surface.

For a finite-dimensional semisimple complex Hopf algebra H the protected space of Kitaev's quantum double model on a surface Σ coincides with the vector space a Turaev–Viro TQFT for its representation category $H-$Mod assigns to Σ, see Balsam and Kirillov [BaK]. It was also shown by Meusburger [M] that in this situation the endomorphism algebra of the protected space coincides with the quantum moduli algebra from [AGSa, AGSb, AS] and [BR95, BR96] for the Drinfeld double $D(H)$. More generally, our construction is of Turaev–Viro type, as it makes use of Yetter–Drinfeld modules over the Hopf monoid H in $\mathcal{C}$. This generalises modules over the Drinfeld double of a Hopf algebra and the centre of its representation category.

Essentially the same construction was used by Meusburger and Voß in [MV] to construct mapping class group actions from pivotal Hopf monoids in symmetric monoidal categories. The involutive Hopf monoids considered here are a special case of pivotal ones, where the pivotal structure is given by the unit of the Hopf monoid. These mapping class group actions are obtained from graphs with a single vertex and face and act on the associated protected object. However, it was not established in [MV] that this object is independent of the graph. By combining our results with the ones from [MV] we obtain

THEOREM. (Theorem 8.1) The protected object for a Hopf monoid H and a surface Σ of genus $g \geq 1$ is equipped with an action of the mapping class group $\mathrm{Map}(\Sigma)$ by automorphisms.

The remainder of this article is dedicated to the study of examples. For a finite-dimensional semisimple Hopf algebra H as a Hopf monoid in $\mathrm{Vect}_{\mathbb{C}}$ the protected object assigned to a surface coincides with the protected space of the associated quantum double model, as defined by Kitaev and by Buerschaper et al. in [Ki, BMCA]. However, our model is also defined in the non-semisimple case. For instance, for a group algebra $k[G]$ over a commutative ring k as a Hopf monoid in $k-\mathrm{Mod}$ and a surface Σ of genus $g \geq 1$, the protected object is the free k-module generated by the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

A large class of examples of involutive Hopf monoids are group objects in cartesian monoidal categories, where the tensor product is a categorical product. In all our examples of this type, the protected object is given in terms of representation varieties. For a group H as a Hopf monoid in Set, the object assigned to a connected surface Σ is the representation variety $\mathrm{Hom}(\pi_1(\Sigma), H)/H$. For a simplicial group $H = (H_n)_{n\in\mathbb{Z}}$ as a Hopf monoid in SSet, it is a simplicial set given by the representation varieties for the groups H_n and post-composition with the face maps and degeneracies. In this sense the construction can be viewed as a generalisation of representation varieties from groups to group objects. The module structures at vertices correspond to the group action and the comodule structures at faces of the graph to moment maps.

Our main example is the case where the underlying symmetric monoidal category is the category Cat of small categories and functors between them. Group objects in Cat are precisely crossed modules. They are given by a group homomorphism $\partial : A \to B$ and an action $\blacktriangleright : B \times A \to A$ by group automorphisms, subject to some consistency conditions. As Cat is complete and cocomplete, the associated protected objects exist, but are difficult to determine concretely. We relate them to simplicial groups via the nerve functor and its left adjoint, which yields an explicit description of the protected objects and their mapping class group actions.

THEOREM. (Theorem 7.23, Corollary 8.3) The protected object for a crossed module $(B, A, \blacktriangleright, \partial)$ as a group object in Cat and a surface Σ of genus $g \geq 1$ is a groupoid $\mathcal{G}$ with $\mathrm{Ob}\mathcal{G} = \mathrm{Hom}(\pi_1(\Sigma), B)/B$ and with equivalence classes of group homomorphisms $\tau : \pi_1(\Sigma) \to A \rtimes B$ as morphisms. The action of the mapping class group $\mathrm{Map}(\Sigma)$ is induced by its action on $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$.

The equivalence classes of morphisms are given by the equivalence relation $\tau_1 \circ \tau_2 \sim \tau_1' \circ \tau_2'$ on the set of group homomorphisms $\tau : F_{2g} \to A \rtimes B$, whenever the composites exist and τ_1, τ_1' and τ_2',τ_2 are conjugate. We compute the protected object and the associated mapping class group action explicitly for some simple examples of crossed modules.

To our knowledge this construction is new and differs substantially from the constructions with crossed modules in higher gauge theory settings such as the work of Martins and Picken [MPa, MPb] on higher holonomies, the work [BC+a, BC+b] by Bullivant et. al. on higher lattices and topological phases, the work [KMM] by Koppen, Martins and Martin on topological phases from crossed modules of Hopf algebras and the recent work [SV] by Sozer and Virelizier on 3d homotopy quantum field theory. In those settings the structure maps of the crossed module often encode higher categorical structures in the topological data, such as homotopies between paths, or relate data in triangulations or cell decompositions. In our approach they enter the formalism as data – a specific example of a group object – but the crossed module structure is not required to encode the topology or geometry.

The article is structured as follows. Section 2 introduces the algebraic background for the article. In Section 2.1 we summarise the background on Hopf monoids in symmetric monoidal categories. In Section 2.8 we discuss their (co)modules and the construction of their (co)invariants via (co)equalisers and images in complete and finitely cocomplete symmetric monoidal categories. Section 3 contains the required background on ribbon graphs and surfaces.

In Section 4 we formulate the categorical counterpart of Kitaev's quantum double model for an involutive Hopf monoid H in a complete and finitely cocomplete symmetric monoidal category. This is a simple generalisation of the formulation in [BMCA] for finite-dimensional semisimple complex Hopf algebras, and an almost identical construction was used in [MV].

For each ribbon graph we consider the tensor product $H^{\otimes E}$, where E is the edge set of the graph. We assign to each marked vertex a H-module structure and to each marked face an H-comodule structure on $H^{\otimes E}$. The protected object is constructed by (co)equalising these (co)module structures and taking an image.

In Section 5 we show that the protected object defined by a ribbon graph depends only on the homeomorphism class of the associated oriented surface Σ. We first demonstrate that moving the markings for the (co)module structures and edge reversals yield isomorphic protected objects. We then consider a number of graph transformations that are sufficient to reduce every connected ribbon graph to a standard graph and prove that these induce isomorphisms of the protected object.

These sections are necessarily rather technical. The reader primarily interested in the results may skip to the main theorem in Section 5.24, where we also treat some examples. In particular, we show that the protected object for a group H as a Hopf monoid in Set and a connected surface Σ is the representation variety $\mathrm{Hom}(\pi_1(\Sigma), H)/H$. We then consider group algebras $H = k[G]$ and their duals $k[G]^*$ for a commutative ring k as Hopf monoids in $k-\mathrm{Mod}$. The associated protected objects are the free k-module $\langle \mathrm{Hom}(\pi_1(\Sigma), G)/G\rangle_k$ and the set of maps $\mathrm{Hom}(\pi_1(\Sigma), G)/G \to k$.

Section 6 treats the example of a simplicial group $H = (H_n)_{n\in\mathbb{N}_0}$ as a Hopf monoid in SSet. In this case, the protected object is a simplicial set given by the representation varieties $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ and by post-composition with the face maps and degeneracies of H. This result is required for the construction of the protected object of a crossed module as a group object in Cat.

The construction of the protected object for group objects in Cat is more involved and treated in Section 7. We start by summarising the required background on crossed modules in Section 7.1. In Section 7.6 we discuss equalisers and coequalisers in Cat and summarise how the latter can be constructed via the nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$ and its left adjoint. In Section 7.13 we apply these results to (co)equalise the (co)module structures over Hopf monoids in Cat. In Section 7.18 we apply this to the (co)module structures associated with a ribbon graph and determine the protected object for the associated surface. We describe it explicitly and treat a simple example.

In Section 8 we describe the mapping class group action on the protected object. By combining our results with the ones from [MV], we obtain that the mapping class group of an oriented surface Σ acts on the associated protected objects. We show that in the case of a simplicial group $H = (H_n)_{n\in\mathbb{N}_0}$ this action is the one induced by its action on the representation varieties $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$. For the case of a crossed module $(B, A, \blacktriangleright, \partial)$ as a group object in Cat we obtain a mapping class group action by invertible endofunctors on the associated groupoid, which is induced by the action on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$.

2. Algebraic background

2.1. Involutive Hopf monoids. Throughout the article $\mathcal{C}$ is a symmetric monoidal category with unit object e and braidings $\tau_{X,Y} : X \otimes Y \to Y \otimes X$. We also suppose that $\mathcal{C}$ is complete and finitely cocomplete. In formulas, we suppress associators and unit constraints and coherence data of monoidal functors.

2.2. Definition.

1. *A* **Hopf monoid** *in $\mathcal{C}$ is an object H in $\mathcal{C}$ together with morphisms $m : H\otimes H \to H$, $\eta : e \to H$, $\Delta : H \to H \otimes H$, $\epsilon : H \to e$ and $S : H \to H$, the multiplication, unit, comultiplication, counit and antipode, such that*
   - *the (co)multiplication satisfies the (co)associativity and (co)unitality conditions*

$$m \circ (m \otimes 1_H) = m \circ (1_H \otimes m), \quad m \circ (\eta \otimes 1_H) = m \circ (1_H \otimes \eta) = 1_H, \quad (1)$$
$$(\Delta \otimes 1_H) \circ \Delta = (1_H \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes 1_H) \circ \Delta = (1_H \otimes \epsilon) \circ \Delta = 1_H,$$

   - *comultiplication and counit are monoid morphisms*

$$\Delta \circ \eta = \eta \otimes \eta, \quad \Delta \circ m = (m \otimes m) \circ (1_H \otimes \tau_{H,H} \otimes 1_H) \circ (\Delta \otimes \Delta) \quad (2)$$
$$\epsilon \circ \eta = 1_e, \quad \epsilon \circ m = \epsilon \otimes \epsilon,$$

   - *S satisfies the antipode condition*

$$m \circ (S \otimes 1_H) \circ \Delta = m \circ (1_H \otimes S) \circ \Delta = \eta \circ \epsilon. \quad (3)$$

It is called **involutive** *if* $S \circ S = 1_H$.

2. *A* **morphism of Hopf monoids** *in* $\mathcal{C}$ *is a morphism* $f : H \to H'$ *in* $\mathcal{C}$ *with*

$$f \circ m = m' \circ (f \otimes f), \quad f \circ \eta = \eta', \quad (f \otimes f) \circ \Delta = \Delta' \circ f, \quad \epsilon' \circ f = \epsilon. \tag{4}$$

We denote by $\mathrm{Hopf}(\mathcal{C})$ *the category of Hopf monoids and morphisms of Hopf monoids in* $\mathcal{C}$.

The antipode of a Hopf monoid is unique, and it is an anti-monoid and anti-comonoid morphism

$$S \circ m = m^{\mathrm{op}} \circ (S \otimes S), \quad S \circ \eta = \eta, \quad (S \otimes S) \circ \Delta = \Delta^{\mathrm{op}} \circ S, \quad \epsilon \circ S = \epsilon, \tag{5}$$

see for instance Porst [Po, Prop. 36]. If H is involutive, the antipode satisfies the additional identities

$$m^{\mathrm{op}} \circ (S \otimes 1_H) \circ \Delta = m^{\mathrm{op}} \circ (1_H \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{6}$$

Every morphism of Hopf monoids $f : H \to H'$ satisfies $f \circ S = S' \circ f$. This follows as for Hopf algebras by considering the convolution monoid $\mathrm{Hom}_{\mathcal{C}}(H, H)$ with the product $f \star g = m \circ (f \otimes g) \circ \Delta$.

In the following, we use generalised Sweedler notation for the coproduct in a Hopf monoid and write $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $(\Delta \otimes 1_H) \circ \Delta(h) = (1_H \otimes \Delta) \circ \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ etc. This is analogous to Sweedler notation for a Hopf algebra. It can be viewed as a shorthand notation for a diagram that describes a morphism in a symmetric monoidal category, see [MV] for examples. We also write $m^{(n)} : H^{\otimes(n+1)} \to H$ and $\Delta^{(n)} : H \to H^{\otimes(n+1)}$ for n-fold products and coproducts.

2.3. Example.

1. For any commutative ring k a Hopf monoid in $k-\mathrm{Mod}$ is a Hopf algebra over k. In particular, for any field $\mathbb{F}$ a Hopf monoid in $\mathrm{Vect}_{\mathbb{F}}$ is a Hopf algebra over $\mathbb{F}$.
2. For any finite group G and commutative ring k, the group algebra $k[G]$ and its dual $k[G]^*$ are Hopf monoids in $k - \mathrm{Mod}$.
3. The tensor product of two Hopf monoids in $\mathcal{C}$ has a Hopf monoid structure given by the tensor product of (co)units, (co)multiplications and antipodes and the braiding morphisms. Any tensor product of Hopf monoid morphisms is a morphism of Hopf monoids.
4. Every Hopf monoid $H = (H, m, \eta, \Delta, \epsilon, S)$ in a symmetric monoidal category $\mathcal{C}$ defines a Hopf monoid $H^* = (H, \Delta, \epsilon, m, \eta, S)$ in the symmetric monoidal category $\mathcal{C}^{\mathrm{op}}$. This generalises the dual Hopf algebra in $\mathrm{Vect}_{\mathbb{F}}$.

The following example yields many subexamples, which are a focus in this article.

2.4. Example. Let $(\mathcal{C}, \times)$ be a cartesian monoidal category with terminal object $\bullet$. Let $\epsilon_X : X \to \bullet$ be the terminal morphism and $\Delta_X : X \to X \times X$ the diagonal morphism for an object X.

A Hopf monoid in $\mathcal{C}$ is a **group object** in $\mathcal{C}$: an object H together with morphisms $m : H \times H \to H$, $\eta : \bullet \to H$ and $I : H \to H$ such that the following diagrams commute

$$\begin{CD} H \times H \times H @>{1_H \times m}>> H \times H \\ @V{m \times 1_H}VV @VV{m}V \\ H \times H @>>{m}> H \end{CD} \qquad \begin{array}{ccccc} H \cong \bullet \times H & \xrightarrow{\eta \times 1_H} & H \times H & \xleftarrow{1_H \times \eta} & H \times \bullet \cong H \\ & {\scriptstyle 1_H} \searrow & \downarrow {\scriptstyle m} & \swarrow {\scriptstyle 1_H} & \\ & & H & & \end{array} \tag{7}$$

$$\begin{CD} H \times H @>{I \times 1_H}>> H \times H \\ @A{\Delta_H}AA @VV{m}V \\ H @>{\epsilon_H}>> \bullet @>{\eta}>> H \end{CD} \qquad \begin{CD} H @>{\epsilon_H}>> \bullet @>{\eta}>> H \\ @V{\Delta_H}VV @AA{m}A \\ H \times H @>{1_H \times I}>> H \times H. \end{CD}$$

A morphism of Hopf monoids is a **morphism of group objects**: a morphism $F : H \to H'$ with

$$F \circ m = m' \circ (F \times F). \tag{8}$$

Note that this implies $F \circ \eta = \eta'$ and $I' \circ F = F \circ I$.

In a cartesian monoidal category, every object has a unique comonoid structure given by the diagonal map. This follows from the counitality axiom and the fact that ϵ is the terminal map. Hence, every Hopf monoid in a cartesian monoidal category is of this type.

2.5. Example.

1. A group object in the cartesian monoidal category $(\mathrm{Set}, \times)$ is a group.
2. A group object in the cartesian monoidal category $(\mathrm{Top}, \times)$ is a topological group.
3. A group object in the cartesian monoidal category $(\mathrm{Cat}, \times)$ of small categories and functors between them is a crossed module (cf. Definition 7.3).
4. Let G be a group and $G - \mathrm{Set} = \mathrm{Set}^{BG}$ the cartesian monoidal category of G-sets and G-equivariant maps. A group object in $G - \mathrm{Set}$ is a group with a G-action by automorphisms.
5. A group object in the cartesian monoidal category $\mathrm{SSet} = \mathrm{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets and simplicial maps is a simplicial group (cf. Definition 6.1).

The last two examples in Example 2.5 have counterparts for any functor category $\mathcal{C}^{\mathcal{D}}$, where $\mathcal{D}$ is small and $\mathcal{C}$ symmetric monoidal. In this case the functor category $\mathcal{C}^{\mathcal{D}}$ inherits a symmetric monoidal structure from $\mathcal{C}$, and we have

2.6. Lemma. *For any symmetric monoidal category $\mathcal{C}$ and a small category $\mathcal{D}$ the monoidal categories $\mathrm{Hopf}(\mathcal{C}^{\mathcal{D}})$ and $\mathrm{Hopf}(\mathcal{C})^{\mathcal{D}}$ are symmetric monoidally equivalent.*

PROOF. The equivalence is given by the functor $R : \mathrm{Hopf}(\mathcal{C}^{\mathcal{D}}) \to \mathrm{Hopf}(\mathcal{C})^{\mathcal{D}}$ that sends a Hopf monoid $(H, m, \eta, \Delta, \epsilon, S)$ to the functor $K : \mathcal{D} \to \mathrm{Hopf}(\mathcal{C})$ with $K(D) = H(D)$ and the component morphisms m_D, η_D, Δ_D, ϵ_D, S_D for $D \in \mathrm{Ob}(\mathcal{D})$ and with $K(f) = H(f)$ for a morphism f in $\mathcal{D}$. Hopf monoid morphisms in $\mathcal{C}^{\mathcal{D}}$ are sent to themselves. The functor R has an obvious inverse, and both functors are symmetric monoidal. ■

Further examples are obtained by taking the images of Hopf monoids under symmetric monoidal functors. If both of the categories are cartesian monoidal, it is sufficient that the functor preserves finite products, which holds in particular for any right adjoint functor.

2.7. EXAMPLE.

1. Let $F : \mathcal{C} \to \mathcal{C}'$ be a symmetric monoidal functor. Then for every Hopf monoid H in $\mathcal{C}$ the image $F(H)$ has a canonical Hopf monoid structure.
2. If $\mathcal{C}, \mathcal{C}'$ are cartesian monoidal categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor that preserves finite products, then F is symmetric monoidal, and for every group object H in $\mathcal{C}$ the image $F(H)$ is a group object in $\mathcal{C}'$.

2.8. (CO)MODULES AND THEIR (CO)INVARIANTS. As their definitions involve only structure maps, (co)modules over Hopf monoids in symmetric monoidal categories are defined analogously to (co)modules over Hopf algebras. The only difference is that linear maps are replaced by morphisms.

2.9. DEFINITION. *Let H be a Hopf monoid in $\mathcal{C}$.*

1. *An H-**module** in $\mathcal{C}$ is an object M in $\mathcal{C}$ with a morphism $\rhd : H \otimes M \to M$ satisfying*

$$\rhd \circ (m \otimes 1_M) = \rhd \circ (1_H \otimes \rhd), \qquad \rhd \circ (\eta \otimes 1_M) = 1_M. \tag{9}$$

   *A **morphism of H-modules** is a morphism $f : M \to M'$ in $\mathcal{C}$ with $\rhd' \circ (1_H \otimes f) = f \circ \rhd$.*

2. *An H-**comodule** in $\mathcal{C}$ is an object M in $\mathcal{C}$ with a morphism $\delta : M \to H \otimes M$ satisfying*

$$(\Delta \otimes 1_M) \circ \delta = (1_H \otimes \delta) \circ \delta, \qquad (\epsilon \otimes 1_M) \circ \delta = 1_M. \tag{10}$$

   *A **morphism of H-comodules** is a morphism $f : M \to M'$ in $\mathcal{C}$ with $(1_H \otimes f) \circ \delta = \delta' \circ f$.*

There are analogous notions of right (co)modules and bi(co)modules and morphisms between them. Just as in the case of a Hopf algebra, there are also various compatibility conditions that can be imposed between modules and comodule structures. The most important one in the following is the one for Yetter–Drinfeld modules.

2.10. DEFINITION. *Let H be a Hopf monoid in $\mathcal{C}$.*

1. *A* **Yetter–Drinfeld module** *over H is a triple $(M, \triangleright, \delta)$ such that $(M, \triangleright)$ is an H-module, (M, δ) is an H-comodule and*

$$\delta \circ \triangleright = (m^{(2)} \otimes \triangleright) \circ (1_{H^{\otimes 2}} \circ \tau_{H,H} \otimes 1_M) \circ (1_{H^{\otimes 3}} \otimes S \otimes 1_M) \\ \circ (1_H \otimes \tau_{H^{\otimes 2},H} \otimes 1_M) \circ (\Delta^{(2)} \otimes \delta).$$

2. *A* **morphism of Yetter–Drinfeld modules** *is a morphism $f : M \to M'$ that is a module and a comodule morphism.*

In Sweedler notation with the conventions $\delta(m) = m_{(0)} \otimes m_{(1)}$ and $\Delta(h) = h_{(1)} \otimes h_{(2)}$ the Yetter–Drinfeld module condition in Definition 2.10 reads

$$(h \triangleright m)_{(0)} \otimes (h \triangleright m)_{(1)} = h_{(1)} m_{(0)} S(h_{(3)}) \otimes (h_{(2)} \triangleright m_{(1)}). \tag{11}$$

Yetter–Drinfeld modules over group objects in cartesian monoidal categories are especially simple to describe. In this case, composing the coaction morphism $\delta : M \to H \times M$ with the projection morphism $\pi_1 : H \times M \to H$ yields a morphism $F = \pi_1 \circ \delta : M \to H$ reminiscent of a moment map. The Yetter–Drinfeld module condition states that this morphism intertwines the H-module structure on M and the conjugation action of H on itself.

2.11. Example. Let H be a group object in a cartesian monoidal category, $(M, \triangleright)$ a module and (M, δ) a comodule over H. Then $(M, \triangleright, \delta)$ is a Yetter–Drinfeld module over H iff the morphism $F := \pi_1 \circ \delta : M \to H$ satisfies

$$F \circ \triangleright = m^{(2)} \circ (1_H \times \tau_{H,F(M)}) \circ (1_H \times I \times 1_{F(M)}) \circ (\Delta_H \times F). \tag{12}$$

If the objects of $\mathcal{C}$ are sets, condition (12) reads $F(h \triangleright m) = hF(m)h^{-1}$ for all $h \in H$, $m \in M$. By an abuse of notation, we sometimes write such formulas for the general case to keep notation simple.

By Example 2.7 the images of Hopf monoids under symmetric monoidal functors are Hopf monoids. Analogous statements hold for their (co)modules.

2.12. Example.

1. If $F : \mathcal{C} \to \mathcal{C}'$ is a symmetric monoidal functor and M a (co)module over a Hopf monoid H in $\mathcal{C}$, then $F(M)$ is a (co)module over the Hopf monoid $F(H)$.
2. Let $\mathcal{C}, \mathcal{C}'$ be cartesian monoidal categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor that preserves finite products. Then for every (co)module M over a group object H in $\mathcal{C}$ the image $F(M)$ is a (co)module over the group object $F(H)$.

(Co)invariants of (co)modules cannot be generalised directly from Hopf algebras over fields to Hopf monoids in symmetric monoidal categories. To obtain generalised notions of (co)invariants, we require that the symmetric monoidal category $\mathcal{C}$ has all equalisers and coequalisers.

2.13. Definition. [MV, Def. 2.6] *Let $\mathcal{C}$ be a symmetric monoidal category that has all equalisers and coequalisers, H a Hopf monoid in $\mathcal{C}$.*

1. *The* **invariants** *of an H-module $(M, \rhd)$ are the coequaliser (M^H, π) of $\rhd$ and $\epsilon\otimes 1_M$:*

$$H \otimes M \underset{\epsilon\otimes 1_M}{\overset{\rhd}{\rightrightarrows}} M \xrightarrow{\pi} M^H.$$

2. *The* **coinvariants** *of an H-comodule (M, δ) are the equaliser $(M^{\mathrm{co}H}, \iota)$ of δ and $\eta \otimes 1_M$:*

$$M^{\mathrm{co}H} \xrightarrow{\iota} M \underset{\eta\otimes 1_M}{\overset{\delta}{\rightrightarrows}} H \otimes M.$$

As expected, H-(co)module morphisms induce morphisms between the (co)invariants. This follows directly from the universal properties of the (co)equalisers.

2.14. Lemma. [MV, Lemma 2.7] *Suppose that $\mathcal{C}$ has all equalisers and coequalisers and H is a Hopf monoid in $\mathcal{C}$. Then for every H-module morphism $f : (M, \rhd) \to (M', \rhd')$ there is a unique morphism $f^H : M^H \to M'^H$ with $f^H \circ \pi = \pi' \circ f$. Likewise, for every H-comodule morphism $f : (M, \delta) \to (M', \delta')$ there is a unique morphism $f^{\mathrm{co}H} : M^{\mathrm{co}H} \to M'^{\mathrm{co}H}$ with $\iota' \circ f^{\mathrm{co}H} = f \circ \iota$.*

Note that all definitions in this section are symmetric with respect to a Hopf monoid H in $\mathcal{C}$ and the dual Hopf monoid H^* in $\mathcal{C}^{\mathrm{op}}$ from Example 2.3. Modules and comodules over H in $\mathcal{C}$ correspond to comodules and modules over H^* in $\mathcal{C}^{\mathrm{op}}$, respectively, and the same holds for their (co)invariants. It is also directly apparent from the formula in Definition 2.10 that Yetter–Drinfeld modules over H correspond to Yetter–Drinfeld modules over H^*.

For objects in a symmetric monoidal category $\mathcal{C}$ that are both, modules and comodules over certain Hopf monoids in $\mathcal{C}$, we combine the notion of invariants and coinvariants and impose both conditions. This requires that the category $\mathcal{C}$ is equipped with *images*. We work with a general non-abelian notion of image, see Mitchell [Mi, Sec. I.10] and Pareigis [Pa, Sec. 1.13]. There is an analogous notion of a *coimage*, which is the image of the corresponding morphism in $\mathcal{C}^{\mathrm{op}}$, see [Mi, Sec. I.10].

An *image* of a morphism $f : C \to C'$ in $\mathcal{C}$ is an object $\mathrm{im}(f)$ together with a pair (P, I) of a monomorphism $I : \mathrm{im}(f) \to C'$ and a morphism $P : C \to \mathrm{im}(f)$ with $I \circ P = f$ and the following universal property: for any pair (Q, J) of a monomorphism $J : X \to C'$ and a morphism $Q : C \to X$ with $J \circ Q = f$ there is a unique morphism $v : \mathrm{im}(f) \to X$ with $I = J \circ v$.

Images are unique up to unique isomorphism. If $\mathcal{C}$ has all equalisers, then $P : C \to \mathrm{im}(f)$ is an epimorphism [Mi, Prop. 10.1, Sec. I.10]. In an abelian category $\mathcal{C}$ this notion of image coincides with the usual definition of an image as the kernel of the cokernel [Pa, Lemma 3, Sec. 4.2]. If $\mathcal{C}$ is complete, then all images exist, as any complete category has intersections [Mi, Prop. 2.3, Sec. II.2]. This implies the existence of all images [Mi, Sec. I.10].

2.15. Definition. [MV, Def. 2.8][1] *Let $\mathcal{C}$ be a complete and finitely cocomplete symmetric monoidal category and H, K Hopf monoids in $\mathcal{C}$. The* **biinvariants** *of an H-module and K-comodule M are the image of the morphism $\pi \circ \iota : M^{\mathrm{co}K} \to M^H$*

$$\begin{array}{ccccc} M^{\mathrm{co}K} & \xrightarrow{\iota} & M & \xrightarrow{\pi} & M^H \\ & {\scriptstyle P}\searrow & & \nearrow{\scriptstyle I} & \\ & & M_{\mathrm{inv}} := \mathrm{im}(\pi \circ \iota). & & \end{array} \tag{13}$$

Requiring $\mathcal{C}$ to be complete and finitely cocomplete ensures the existence of invariants, coinvariants and biinvariants. Examples of such categories are Set, Top, Grp, $\mathrm{Vect}_{\mathbb{F}}$, Cat, $k-\mathrm{Mod}$ and the category $\mathrm{Ch}_{k-\mathrm{Mod}}$ of chain complexes of k-modules. For a small category $\mathcal{D}$ and a complete and finitely cocomplete category $\mathcal{C}$ the category $\mathcal{C}^{\mathcal{D}}$ is also complete and finitely cocomplete, see for instance Pareigis [Pa, Th. 1, Sec. 2.7]. Hence, $G-\mathrm{Set}$ and SSet also satisfy the requirement.

As discussed in [MV, Rem. 2.9] one could also consider the *coimage* of the morphism $\pi \circ \iota$ instead of its *image*. This amounts to passing from modules and comodules over the Hopf monoids H, K in $\mathcal{C}$ to comodules and modules over the Hopf monoids H^*, K^* in $\mathcal{C}^{\mathrm{op}}$ from Example 2.3.

We illustrate (co)invariants and biinvariants with a few simple examples. (Co)invariants of (co)modules over Hopf monoids in SSet and Cat and the associated biinvariants for Yetter–Drinfeld modules are treated in Sections 6 and 7.13, respectively.

2.16. Example.

1. A Hopf monoid H in $\mathcal{C} = \mathrm{Set}$ (in $\mathcal{C} = \mathrm{Top}$) is a (topological) group H and
   - an H-module is a (continuous) $H-\mathrm{Set}$ $\rhd : H \times M \to M$,
   - $M^H = \{H \rhd m \mid m \in M\}$ with $\pi : M \to M^H$, $m \mapsto H \rhd m$ (and the quotient topology),
   - an H-comodule is given by a (continuous) map $F : M \to H$,
   - $M^{\mathrm{co}H} = F^{-1}(1)$ with the inclusion $\iota : F^{-1}(1) \to M$ (and the subspace topology),
   - $M_{\mathrm{inv}} = \pi(F^{-1}(1)) = \{H \rhd m \mid F(m) = 1\}$ (with the final topology induced by π).

   An H-module and H-comodule $(M, \rhd, F)$ is a Yetter–Drinfeld module iff $F(h \rhd m) = hF(m)h^{-1}$ for all $m \in M$, $h \in H$.

2. Let G be a group and H a group with a G-action by automorphisms, viewed as a Hopf monoid in $G-\mathrm{Set} = \mathrm{Set}^{BG}$. Then H-modules are $H \rtimes G$-sets, H-comodules are G-sets M with G-equivariant maps $F : M \to H$ and
   - $M^H = \{H \rhd m \mid m \in M\}$ is the orbit space for H with the induced G-action and G-equivariant canonical surjection $\pi : M \to M^H$,
   - $M^{\mathrm{co}H} = F^{-1}(1)$ with the induced G-action and G-equivariant inclusion $\iota : F^{-1}(1) \to M$,

[1] Def. 2.8 in [MV] considers only the case $H = K$, as that is the only one required there.

- $M_{\text{inv}} = \pi(F^{-1}(1))$ with the induced G-action.

3. For a Hopf algebra H over a commutative ring k as a Hopf monoid in $k-\text{Mod}$, H-(co)modules and Yetter–Drinfeld modules are (co)modules and Yetter–Drinfeld modules over H in the usual sense. Their (co)invariants and biinvariants are
   - $M^H = M/\langle\{h \rhd m - \epsilon(h)m \mid h \in H, m \in M\}\rangle$,
   - $M^{\text{co}H} = \{m \in M \mid \delta(m) = 1 \otimes m\}$,
   - $M_{\text{inv}} = \pi(M^{\text{co}H})$.

While the coinvariants in Example 2.16, 3. coincide with the usual coinvariants for comodules over a Hopf algebra, the invariants form a quotient rather than a subset. This distinction is irrelevant in the case of semisimple Hopf algebras, but not in general. As our definition is symmetric with respect to Hopf monoids in a symmetric monoidal category $\mathcal{C}$ and the dual Hopf monoids in $\mathcal{C}^{\text{op}}$, it is more natural in our setting. The following example illustrates this.

2.17. Example. For a finite group G and a commutative ring k the group algebra $k[G]$ and its dual $k[G]^*$ are Hopf monoids in $k-\text{Mod}$.

For the group algebra $H = k[G]$
- the invariants of a H-module $(M, \rhd)$ are $M^H = M/\langle\{g \rhd m - m \mid m \in M, g \in G\}\rangle$,
- comodules are G-graded k-modules $M = \oplus_{g\in G} M_g$ with $\delta(m) = g \otimes m$ for all $m \in M_g$,
- their coinvariants are $M^{\text{co}H} = M_1$.

A $k[G]$-module and comodule $(M, \rhd, \delta)$ is a Yetter–Drinfeld module iff $g \rhd M_h = M_{ghg^{-1}}$ for all $g, h \in G$, and in this case $M_{\text{inv}} \cong H_0(G, M_1)$.

For the dual Hopf monoid $H = k[G]^*$
- modules are G-graded k-modules $M = \oplus_{g\in G} M_g$ with $\delta_g \rhd m = \delta_g(h)m$ for $m \in M_h$,
- their invariants are $M^H = M/(\oplus_{g\in G, g\neq 1} M_g) \cong M_1$,
- comodules are $k[G]$-right modules $(M, \lhd)$ with $\delta(m) = \sum_{g\in G} \delta_g \otimes (m \lhd g)$,
- their coinvariants are $M^{\text{co}H} = \{m \in M \mid m \lhd g = m\, \forall g \in G\}$.

A $k[G]^*$-module and comodule $(M, \rhd, \delta)$ is a Yetter–Drinfeld module iff $M_h \lhd g = M_{ghg^{-1}}$ for all $g, h \in G$, and in this case $M_{\text{inv}} \cong H^0(G, M_1)$.

By Lemma 2.14 morphisms of (co)modules over a Hopf monoid H induce morphisms between their (co)invariants. The question if morphisms of both, modules and comodules, induce morphisms between the associated biinvariants is more subtle in general. It is shown in [MV, Lemma 2.10] that this always holds for *isomorphisms*. As a direct generalisation we have in the notation of (13)

2.18. Lemma. *Let $\mathcal{C}$ be complete and finitely cocomplete, H, K Hopf monoids in $\mathcal{C}$ and $\Phi : M \to M'$ an isomorphism of H-modules and K-comodules. There is a unique morphism $\Phi_{\text{inv}} : M_{\text{inv}} \to M'_{\text{inv}}$ with $\pi' \circ \Phi \circ \iota = I' \circ \Phi_{\text{inv}} \circ P$, and Φ_{inv} is an isomorphism.*

3. Ribbon graphs and surfaces

In this section we summarise the background on *ribbon graphs*, also called *fat graphs* or *embedded graphs*, for more details we refer to the textbooks of Lando et al. [L+] and Ellis-Monaghan and Moffatt [EM]. Throughout this article, all graphs are *directed* graphs with a finite number of vertices and edges. In contrast to [MV] we do not require that the graphs are connected and allow **isolated vertices** with no incident edges.

3.1. Definition. *A* **ribbon graph** *is a graph with a cyclic ordering of the edge ends at each vertex.*

The cyclic ordering of edge ends at the vertices of a ribbon graph allows one to thicken its edges to strips or ribbons and defines the faces of the ribbon graph. One says that a path in a ribbon graph **turns maximally left at a vertex** if it enters the vertex along an edge end and leaves it along an edge end that comes directly before it with respect to the cyclic ordering. A **face** of a ribbon graph is defined as a cyclic equivalence class of closed paths that turn maximally left at each vertex and traverse each edge at most once in each direction. Each isolated vertex is also viewed as a face, and such a face is called an **isolated face**.

In the following we denote by V, E, F the sets of vertices, edges and faces of a ribbon graph and by $s(\alpha), t(\alpha)$ the starting and target vertex of an edge α. We say that two edge ends incident at a vertex $v \in V$ are **neighbours** or **neighbouring** if one of them comes directly before or after the other with respect to the cyclic ordering at v. An edge α with $s(\alpha) = t(\alpha)$ is called a **loop**. A loop at v whose starting and target end are neighbours is called an **isolated loop**. When drawing a ribbon graph we take the cyclic ordering of edge ends at vertices as the one in the drawing.

Ribbon graphs are directly related to embedded graphs on oriented surfaces. Every graph Γ embedded into an oriented surface Σ inherits a cyclic ordering of the edge ends at each vertex and hence a ribbon graph structure. Attaching discs to the faces of the ribbon graph Γ yields an oriented surface Σ_Γ such that the connected components of $\Sigma_\Gamma \setminus \Gamma$ are discs and in bijection with faces of Γ, see Figure 1. If Γ is embedded into an oriented surface Σ, the surface Σ_Γ is homeomorphic to Σ iff each connected component of $\Sigma \setminus \Gamma$ is a disc. In this case, we call Γ **properly embedded** in Σ. Note that this implies a bijection between connected components of Γ and of Σ, and connected components of Σ containing an isolated vertex are spheres. The genus g of a connected component of Σ is then determined by the Euler characteristic $2 - 2g = |V| - |E| + |F|$, where $|V|, |E|, |F|$ are the number of vertices, edges and faces of the associated connected component of Γ.

Note that each ribbon graph or embedded graph has a Poincaré dual obtained by replacing each vertex (face) with a face (vertex) and each edge with a dual edge. This transforms the paths that characterise faces into paths that go counterclockwise around a vertex and vice versa. Edge ends correspond to edge sides of the dual graph and their cyclic ordering at a vertex to the cyclic ordering of the edge sides in the dual face.

In the following we sometimes require a *linear* ordering of the edge ends at a vertex or of the edge sides in a face. This is achieved by inserting a marking, the *cilium*, that

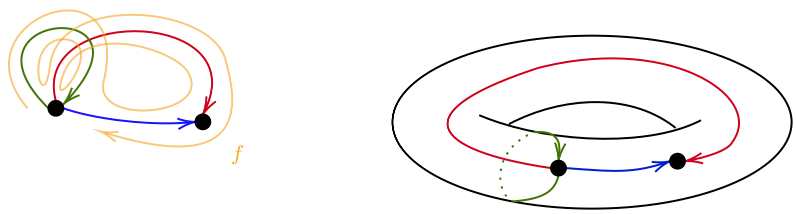


Figure 1: Attaching a disc to the face f yields a torus.

separates the edge ends or edge sides of minimal and maximal order, see for instance Figure 2, Definition 3.3 or Example 4.4. For faces this corresponds to the choice of a starting vertex for the associated cyclic equivalence class of paths.

3.2. Definition.

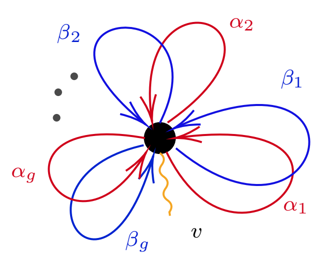
1. *A* **ciliated vertex** *in a ribbon graph is a vertex with a choice of linear ordering of the incident edge ends that is compatible with their cyclic ordering.*
2. *A* **ciliated face** *in a ribbon graph is a closed path that turns maximally left at each vertex, including the starting vertex, and traverses each edge at most once in each direction.*

A **ciliated ribbon graph** *is a ribbon graph in which each face and vertex is assigned a cilium. Isolated vertices and faces are trivially ciliated.*

For a closed surface Σ of genus $g \geq 0$ we often work with a ciliated ribbon graph with a single vertex and a single face that is given by a set of generators of the fundamental group

$$\pi_1(\Sigma) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid [\beta_g^{-1}, \alpha_g] \cdots [\beta_1^{-1}, \alpha_1] = 1 \rangle. \quad (14)$$

3.3. Definition. *The* **standard graph** *of an oriented surface Σ of genus $g \geq 1$ is the graph*



(15)

with the face $f = [\beta_g^{-1}, \alpha_g] \cdots [\beta_1^{-1}, \alpha_1]$ and the ordering of edge ends at v given by $s(\alpha_1) < s(\beta_1) < t(\alpha_1) < t(\beta_1) < \ldots < s(\alpha_g) < s(\beta_g) < t(\alpha_g) < t(\beta_g)$. In particular, the standard graph for S^2 consists of a single isolated vertex and the associated isolated face.

In the following we use certain graph transformations to relate properly embedded ribbon graphs in a connected surface Σ to its standard graph.

3.4. Definition. *Let Γ be a ribbon graph with edge set E and vertex set V.*

1. *The* **edge reversal** *reverses the orientation of an edge $\beta \in E$.*
2. *The* **contraction** *of an edge $\alpha \in E$ that is not a loop removes $\alpha \in E$ and fuses the vertices $s(\alpha)$ and $t(\alpha)$.*
3. *The* **edge slide** *slides an end of $\beta \in E$ that is a neighbour of an end of $\alpha \in E$ along α.*
4. *The* **loop deletion** *removes an isolated loop $\beta \in E$ from Γ.*

In all cases except 2. the resulting ribbon graph inherits all cilia from Γ. In 2. one erases either the cilium of $t(\alpha)$ or of $s(\alpha)$ and speaks of contracting α towards $t(\alpha)$ and $s(\alpha)$, respectively.

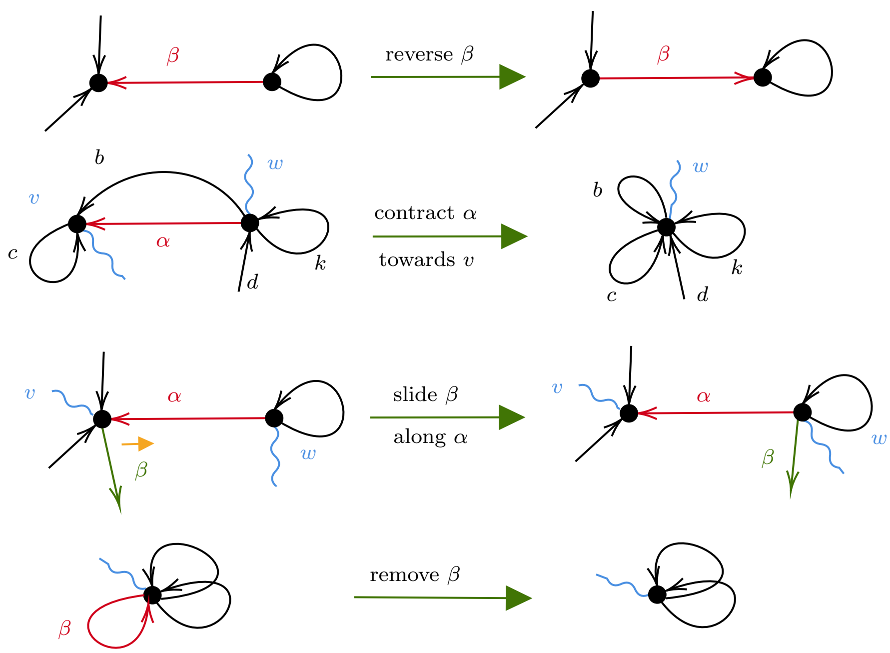


Figure 2: Examples of graph transformations

These graph transformations are illustrated in Figure 2. Note that they are not independent. Contracting an edge α towards $t(\alpha)$ is the same as first sliding some edge ends along α and then contracting α towards $t(\alpha)$. Contracting an edge α towards $t(\alpha)$ is also the same as first reversing α, then contracting α towards $s(\alpha)$ and then reversing α. By reversing α and β before and after a slide, one can reduce all edge slides to the ones that slide the target end of β along the left of α.

There are of course other possible graph transformations such as deleting edges, which is dual to edge contractions. However, the graph transformations in Definition 3.4 are sufficient to transform any connected ribbon graph into a standard graph. This is well known and appears implicitly in many publications. We summarise the argument for the convenience of the reader.

3.5. Proposition. *Every connected ribbon graph can be transformed into the standard graph* (15) *by edge reversals, edge slides, edge contractions and loop deletions.*

Proof. Selecting a maximal tree in Γ and contracting all edges in the tree transforms Γ into a graph Γ' with a single vertex. By applying edge slides one can transform Γ' into a graph Γ'' that coincides with (15) up to edge orientation and up to the presence a number of isolated loops between the cilium and the starting end of α_1. This follows from an analogous statement for chord diagrams, which correspond to ribbon graphs with a single vertex, see for instance Chmutov, Duzhin and Mostovoy [CDM, Sec. 4.8.6]. Deleting the isolated loops and reversing edges in Γ'' then yields the standard graph (15). ■

4. (Co)modules from Hopf monoids and ribbon graphs

In this section we use involutive Hopf monoids in symmetric monoidal categories to assign (co)modules over Hopf monoids to ciliated ribbon graphs. In Section 5 we then show that their biinvariants are topological invariants: their isomorphism classes depend only on the genus of the surface obtained by attaching discs to the faces of the graph. In Sections 6 and 7 we determine these biinvariants for simplicial groups as Hopf monoids in SSet and for crossed modules as group objects in Cat.

The construction generalises Kitaev's quantum double model and the toric code from [Ki], which was first formulated for the group algebra of a finite group over $\mathbb{C}$ and then generalised by Buerschaper et al. in [BMCA] to finite-dimensional semisimple C^*-Hopf algebras. A very similar construction to the one in this article is used in [MV] to obtain mapping class group actions from pivotal Hopf monoids in symmetric monoidal categories. The work [MV] considers the biinvariants of a Yetter–Drinfeld module structure assigned to the standard graph (15), but it does not establish that the biinvariants are graph-independent.

The construction of the (co)module structures from an involutive Hopf monoid and a ciliated ribbon graph in this section is directly analogous to the one in [MV], which in turn is a straightforward generalisation of [Ki, BMCA]. The only difference is that H^*-modules in [BMCA] are replaced by H-comodules and $D(H)$-modules by Yetter–Drinfeld modules over H.

What differs substantially from [Ki, BMCA] are the notions of (co)invariants, biinvariants and the construction of the topological invariant. The works in [Ki, BMCA] rely on the normalised Haar integral of a finite-dimensional semisimple complex Hopf algebra, which is not available in our setting. Our construction is more general, as the only assumptions are that the underlying symmetric monoidal category is complete and finitely cocomplete and the Hopf monoid involutive. The article [MV] also allows pivotal Hopf monoids. The involutive Hopf monoids in this article are examples of pivotal Hopf monoids, with their unit as pivotal structure.

Let H be an involutive Hopf monoid in a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$ and Γ a ciliated ribbon graph with vertex set V, edge set E and face set F.

We consider the $|E|$-fold tensor product of H with itself, together with an assignment of the copies of H in this tensor product to the edges of Γ, which we emphasise by writing $H^{\otimes E}$. If $E = \emptyset$, we set $H^{\otimes E} = e$. The object $H^{\otimes E}$ can be viewed as the counterpart of the Hilbert space of Kitaev's quantum double model in [Ki, BMCA].

We assign to each edge $\alpha \in E$ two H-module structures $\rhd_{\alpha\pm} : H \otimes H^{\otimes E} \to H^{\otimes E}$ and H-comodule structures $\delta_{\alpha\pm} : H^{\otimes E} \to H \otimes H^{\otimes E}$. The H-module structures $\rhd_{\alpha+}$ and $\rhd_{\alpha-}$ are assigned to the target and starting end of α and the H-comodule structures to its left and right side, respectively. They are induced by the standard H-(co)module structures on H via left (co)multiplication.

This requires some notation. Given a morphism $f : H \to K$ in $\mathcal{C}$ and an edge $\alpha \in E$ we write f_α for the morphism that applies f to the copy of H in $H^{\otimes E}$ that belongs to α and the identity morphism to the other copies. We write $\tau_\alpha : H^{\otimes E} \to H^{\otimes E}$ or $\tau_\alpha : H \otimes H^{\otimes E} \to H \otimes H^{\otimes E}$ for the composite of braidings that moves the copy of H for α to the leftmost position in $H^{\otimes E}$. We denote by $m_\alpha : H \otimes H^{\otimes E} \to H^{\otimes E}$ the morphism that moves the first copy of H to the left of the one for α and then applies m to them.

4.1. Definition. *The H-module structures $\rhd_{\alpha\pm} : H \otimes H^{\otimes E} \to H^{\otimes E}$ and H-comodule structures $\delta_{\alpha\pm} : H^{\otimes E} \to H \otimes H^{\otimes E}$ for an edge $\alpha \in E$ are*

$$\rhd_{\alpha+} := m_\alpha, \;\; \rhd_{\alpha-} := S_\alpha \circ \rhd_{\alpha+} \circ (1_H \otimes S_\alpha), \;\; \delta_{\alpha+} := \tau_\alpha \circ \Delta_\alpha, \;\; \delta_{\alpha-} := (1_H \otimes S_\alpha) \circ \delta_{\alpha+} \circ S_\alpha.$$

By definition, the (co)module structures assigned to different edges of a graph commute, since they (co)act on different copies of H in the tensor product $H^{\otimes E}$. A direct computation using (1) and (5) shows that the two H-(co)module structures assigned to a given edge commute as well. The proof is directly analogous to the ones for Hopf algebras in [BMCA].

4.2. Lemma. [MV, Lemma 5.2, 2.] *For any edge $\alpha \in E$ the H-module structures $\rhd_{\alpha\pm}$ and the H-comodule structures $\delta_{\alpha\pm}$ commute:*

$$\rhd_{\alpha-} \circ (1_H \otimes \rhd_{\alpha+}) = \rhd_{\alpha+} \circ (1_H \otimes \rhd_{\alpha-}) \circ (\tau_{H,H} \otimes 1_{H^{\otimes E}}),$$
$$(1_H \otimes \delta_{\alpha-}) \circ \delta_{\alpha+} = (\tau_{H,H} \otimes 1_{H^{\otimes E}}) \circ (1_H \otimes \delta_{\alpha+}) \circ \delta_{\alpha-}.$$

The (co)module structures from Definition 4.1 define an H-module structure on $H^{\otimes E}$ for each ciliated vertex v and an H-comodule structure on $H^{\otimes E}$ for each ciliated face f of Γ. The former applies the comultiplication to H, distributes the resulting copies of H to the edge ends at v according to their ordering and acts on them with $\rhd_{\alpha\pm}$ according to their orientation. Dually, the coaction applies the H-coaction $\delta_{\alpha\pm}$ to each edge α in f, depending on its orientation relative to f, and multiplies the resulting copies of H according to the order of the edge sides in f.

4.3. Definition. [MV, Def. 5.3]

1. *The H-module structure $\rhd_v : H \otimes H^{\otimes E} \to H^{\otimes E}$ for a ciliated vertex v with incident edge ends $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ is*

$$\rhd_v = \rhd_{\alpha_1} \circ (1_H \otimes \rhd_{\alpha_2}) \circ \ldots \circ (1_{H^{\otimes(n-1)}} \otimes \rhd_{\alpha_n}) \circ (\Delta^{(n-1)} \otimes 1_{H^{\otimes E}}), \qquad (16)$$

where $\rhd_\alpha = \rhd_{e(\alpha)+}$ if α is incoming, $\rhd_\alpha = \rhd_{e(\alpha)-}$ if α is outgoing and $e(\alpha)$ is the edge of α.

2. *The H-comodule structure $\delta_f : H^{\otimes E} \to H \otimes H^{\otimes E}$ for a ciliated face f that traverses the edges $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ in this order is*

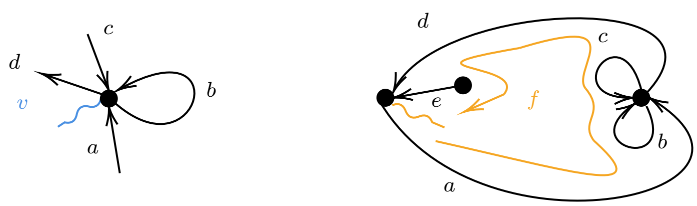
$$\delta_f = (m^{(n-1)} \otimes 1_{H^{\otimes E}}) \circ (1_{H^{\otimes(n-1)}} \otimes \delta_{\alpha_r}) \circ \ldots \circ (1_H \otimes \delta_{\alpha_2}) \circ \delta_{\alpha_1}, \qquad (17)$$

where $\delta_\alpha = \delta_{e(\alpha)+}$ if α is traversed with, $\delta_\alpha = \delta_{e(\alpha)-}$ if α is traversed against its orientation and $e(\alpha)$ is the edge of α.

To an isolated vertex and face we assign the (co)module structures $\rhd_v = \epsilon \otimes 1_{H^{\otimes E}}$ and $\delta_f = \eta \otimes 1_{H^{\otimes E}}$.

To avoid heavy notation we use Sweedler notation and describe these (co)module structures by labelling edges of a graph with letters representing the associated copies of H.

4.4. Example.



The H-module structure $\rhd_v$ for the ciliated vertex v with incident edge ends $t(a) < s(b) < t(b) < t(c) < s(d)$ and the H-comodule structure δ_f for the ciliated face $f = e \circ e^{-1} \circ d \circ c^{-1} \circ b \circ a$ are

$$h \rhd_v (a \otimes b \otimes c \otimes d) = h_{(1)}a \otimes h_{(3)}bS(h_{(2)}) \otimes h_{(4)}c \otimes dS(h_{(5)}),$$
$$\delta_f (a \otimes b \otimes c \otimes d \otimes e) = e_{(1)}S(e_{(3)})d_{(1)}S(c_{(2)})b_{(1)}a_{(1)} \otimes a_{(2)} \otimes b_{(2)} \otimes c_{(1)} \otimes d_{(2)} \otimes e_{(2)}.$$

The interaction of the H-module and H-comodule structures assigned to ciliated vertices and faces of the graph is investigated in [Ki, BMCA, MV]. They are local in the sense that the H-(co)module structure for a vertex (face) affects only those copies of H that belong to their incident edges. As the action $\rhd_{\alpha+}$ for an edge $\alpha \in E$ acts by left- and $\rhd_{\alpha-}$ by right-multiplication, the H-module structures for different vertices commute. The same holds for the H-comodule structures at different faces. Moreover, H-module structures commute with H-comodule structures unless their cilia share a vertex or a face. The H-module and H-comodule structure for each cilium define a Yetter–Drinfeld module structure.

4.5. Lemma. [MV, Lemma 5.5]

1. *The H-left module structures for distinct vertices $v \neq v' \in V$ and the H-left comodule structures for distinct faces $f \neq f' \in F$ commute for all choices of cilia:*

$$\rhd_{v'} \circ (1_H \otimes \rhd_v) = \rhd_v \circ (1_H \otimes \rhd_{v'}) \circ (\tau_{H,H} \otimes 1_{H^{\otimes E}}), \tag{18}$$
$$(1_H \otimes \delta_{f'}) \circ \delta_f = (\tau_{H,H} \otimes 1_{H^{\otimes E}}) \circ (1_H \otimes \delta_f) \circ \delta_{f'}. \tag{19}$$

2. *If two cilia are at distinct vertices and distinct faces, the H-module structure for one of them commutes with the H-comodule structure for the other:*

$$\delta_f \circ \rhd_v = (1_H \otimes \rhd_v) \circ (\tau_{H,H} \otimes 1_{H^{\otimes E}}) \circ (1_H \otimes \delta_f). \tag{20}$$

3. *If $v \in V$ and $f \in F$ share a cilium, then $(H^{\otimes E}, \rhd_v, \delta_f)$ is a Yetter–Drinfeld module over H.*

4.6. Example. Let H be an involutive Hopf monoid in $\mathcal{C}$ and Γ the standard graph (15) on a surface Σ of genus $g \geq 1$. Then the associated Yetter–Drinfeld module structure on $H^{\otimes E}$ is

$$\begin{aligned} & h \rhd (a^1 \otimes b^1 \otimes \ldots \otimes a^g \otimes b^g) \\ & \quad = h_{(3)} a^1 S(h_{(1)}) \otimes h_{(4)} b^1 S(h_{(2)}) \otimes \ldots \otimes h_{(4g-1)} a^g S(h_{(4g-3)}) \otimes h_{(4g)} b^g S(h_{(4g-2)}) \\ & \delta(a^1 \otimes b^1 \otimes \ldots \otimes a^g \otimes b^g) \\ & \quad = S(b^g_{(3)}) a^g_{(1)} b^g_{(1)} S(a^g_{(3)}) \cdots S(b^1_{(3)}) a^1_{(1)} b^1_{(1)} S(a^1_{(3)}) \otimes a^1_{(2)} \otimes b^1_{(2)} \otimes \ldots \otimes a^g_{(2)} \otimes b^g_{(2)}. \end{aligned} \tag{21}$$

If H is a group object in a cartesian monoidal category, this reduces to

$$\begin{aligned} h \rhd (a_1, b_1, \ldots, a_g, b_g) &= (h a_1 h^{-1}, h b_1 h^{-1}, \ldots, h a_g h^{-1}, h b_g h^{-1}) \\ \delta(a_1, b_1, \ldots, a_g, b_g) &= ([b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1], a_1, b_1, \ldots, a_g, b_g). \end{aligned} \tag{22}$$

If each vertex and face of Γ is equipped with a cilium, then Definition 4.3 assigns an H-(co)module structure on $H^{\otimes E}$ to each vertex (face) of Γ. By Lemma 4.5 these (co)module structures commute and hence combine into $H^{\otimes E}$-module and $H^{\otimes F}$-comodule structures on $H^{\otimes E}$.

4.7. Definition. *The $H^{\otimes n}$-module structure for a subset $\emptyset \neq \mathcal{V} := \{v_1, \ldots, v_n\} \subset V$ and $H^{\otimes m}$-comodule structure for a subset $\emptyset \neq \mathcal{F} := \{f_1, \ldots, f_m\} \subset F$ are*

$$\begin{aligned} \rhd_{\mathcal{V}} &:= \rhd_{v_1} \circ (1_H \otimes \rhd_{v_2}) \circ \cdots \circ (1_{H^{\otimes(n-2)}} \otimes \rhd_{v_{n-1}}) \circ (1_{H^{\otimes(n-1)}} \otimes \rhd_{v_n}) : \\ & \quad H^{\otimes n} \otimes H^{\otimes E} \to H^{\otimes E}, \\ \delta_{\mathcal{F}} &:= (1_{H^{\otimes(m-1)}} \otimes \delta_{f_m}) \circ (1_{H^{\otimes(m-2)}} \otimes \delta_{f_{m-1}}) \circ \cdots \circ (1_H \otimes \delta_{f_2}) \circ \delta_{f_1} : \\ & \quad H^{\otimes E} \to H^{\otimes m} \otimes H^{\otimes E}. \end{aligned} \tag{23}$$

Equations (18) and (19) ensure that the (co)actions do not depend on the numbering of vertices or faces in Definition 4.7. That (23) defines an $H^{\otimes n}$-module structure follows from the identity

$$\rhd_{\mathcal{V}'} \circ (1_{H^{\otimes|\mathcal{V}'|}} \otimes \rhd_v) \circ (\tau_{H,H^{\otimes|\mathcal{V}'|}} \otimes 1_{H^{\otimes E}}) = \rhd_v \circ (1_H \otimes \rhd_{\mathcal{V}'}),$$

valid for any subset $\emptyset \neq \mathcal{V}' \subset V$, $v \in V \backslash \mathcal{V}'$. The dual statement for $\delta_{\mathcal{F}}$ follows analogously.

The module and comodule structure from Definition 4.7 define the categorical counterpart of the *protected space* or *ground state* in Kitaev's quantum double model. In the models based on a finite-dimensional semisimple complex Hopf algebras in [Ki, BMCA] the ground state is an eigenspace of a Hamiltonian that combines these H-(co)module structures. The normalised Haar integral defines a projector on the ground state. In our setting these structures are not available. Instead, we consider the biinvariants from Definition 2.15 for the action and coaction from (23).

4.8. Definition. *The* **protected object** *for an involutive Hopf monoid H and a ciliated ribbon graph Γ are the biinvariants $M_{\text{inv}} = \text{Im}(\pi \circ \iota)$ of $H^{\otimes E}$ with the module structure $\rhd_V$ and comodule structure δ_F from (23).*

In the quantum double models for a finite-dimensional semisimple complex Hopf algebra it is directly apparent that imposing (co)invariance under all individual (co)actions at the vertices (faces) of a graph is the same as imposing (co)invariance under the combined action in Definition 4.7. In this setting the (co)invariants for the individual (co)actions are linear subspaces of $H^{\otimes E}$ and the (co)invariants of the combined (co)actions their intersections. In our setting an analogous statement follows from the universal properties of the coequaliser $\pi_{\mathcal{V}} : H^{\otimes E} \to M^H_{\mathcal{V}}$ for the action $\rhd_{\mathcal{V}}$ and the equaliser $\iota_{\mathcal{F}} : M^{coH}_{\mathcal{F}} \to H^{\otimes E}$ for the coaction $\delta_{\mathcal{F}}$, as given in Definition 2.13.

4.9. Lemma. *Let $\emptyset \neq \mathcal{V} \subset V$, $\emptyset \neq \mathcal{F} \subset F$ be subsets.*

1. *For any subset $\emptyset \neq \mathcal{V}' \subset \mathcal{V}$ the morphism $\pi_{\mathcal{V}} : H^{\otimes E} \to M^H_{\mathcal{V}}$ satisfies*
$$\pi_{\mathcal{V}} \circ \rhd_{\mathcal{V}'} = \pi_{\mathcal{V}} \circ (\epsilon^{|\mathcal{V}'|} \otimes 1_{H^{\otimes E}}). \tag{24}$$
There is a unique morphism $\chi_{\mathcal{V}',\mathcal{V}} : M^H_{\mathcal{V}'} \to M^H_{\mathcal{V}}$ with $\chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'} = \pi_{\mathcal{V}}$. It is an epimorphism.

2. *For any subset $\emptyset \neq \mathcal{F}' \subset \mathcal{F}$ the morphism $\iota_{\mathcal{F}} : M^{coH}_{\mathcal{F}} \to H^{\otimes E}$ satisfies*
$$\delta_{\mathcal{F}'} \circ \iota_{\mathcal{F}} = (\eta^{|\mathcal{F}'|} \otimes 1_{H^{\otimes E}}) \circ \iota_{\mathcal{F}}. \tag{25}$$
There is a unique morphism $\xi_{\mathcal{F}',\mathcal{F}} : M^{coH}_{\mathcal{F}} \to M^{coH}_{\mathcal{F}'}$ with $\iota_{\mathcal{F}'} \circ \xi_{\mathcal{F}',\mathcal{F}} = \iota_{\mathcal{F}}$. It is a monomorphism.

Proof. We prove 1., as 2. is the dual statement. It suffices to verify (24) for $\mathcal{V} = \{v_1, \ldots, v_n\}$, $\mathcal{V}' = \{v_j\}$, and the claim follows by induction over $|\mathcal{V}'|$. For this note first that Definition 4.7 implies
$$\rhd_{\mathcal{V}} \circ (\eta^{\otimes(j-1)} \otimes 1_H \otimes \eta^{\otimes(n-j)} \otimes 1_{H^{\otimes E}}) = \rhd_{v_j} \qquad \forall j \in \{1, \ldots, n\}. \tag{26}$$
As $\pi_{\mathcal{V}}$ is the coequaliser of $\rhd_{\mathcal{V}}$ and $\epsilon^{\otimes n} \otimes 1_{H^{\otimes E}}$ one obtains
$$\begin{aligned}\pi_{\mathcal{V}} \circ \rhd_{v_j} &\overset{(26)}{=} \pi_{\mathcal{V}} \circ \rhd_{\mathcal{V}} \circ (\eta^{\otimes(j-1)} \otimes 1_H \otimes \eta^{\otimes(n-j)} \otimes 1_{H^{\otimes E}})\\ &= \pi_{\mathcal{V}} \circ (\epsilon^{\otimes n} \otimes 1_{H^{\otimes E}}) \circ (\eta^{\otimes(j-1)} \otimes 1_H \otimes \eta^{\otimes(n-j)} \otimes 1_{H^{\otimes E}}) \;=\; \pi_{\mathcal{V}} \circ (\epsilon \otimes 1_{H^{\otimes E}}).\end{aligned}$$

Equation (24) and the universal property of the coequaliser $\pi_{\mathcal{V}'}$ imply the existence of a unique morphism $\chi_{\mathcal{V}',\mathcal{V}} : M^H_{\mathcal{V}'} \to M^H_{\mathcal{V}}$ with $\chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'} = \pi_{\mathcal{V}}$. For any two morphisms $q_1, q_2 : M^H_{\mathcal{V}} \to X$ with $q_1 \circ \chi_{\mathcal{V}',\mathcal{V}} = q_2 \circ \chi_{\mathcal{V}',\mathcal{V}}$ one has $q_1 \circ \chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'} = q_1 \circ \pi_{\mathcal{V}} = q_2 \circ \pi_{\mathcal{V}} = q_2 \circ \chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'}$. As $\pi_{\mathcal{V}}$ is a coequaliser and hence an epimorphism, this implies $q_1 = q_2$, and $\chi_{\mathcal{V}',\mathcal{V}}$ is an epimorphism. ■

It is also directly apparent from Definition 4.7 that (co)module morphisms with respect to all individual (co)module structures at vertices and faces in $\mathcal{V}$ and $\mathcal{F}$ are also (co)module morphisms with respect to the (co)actions $\rhd_{\mathcal{V}}$ and $\delta_{\mathcal{F}}$. More precisely, for ciliated ribbon graphs Γ, Γ', subsets $\emptyset \neq \mathcal{V} \subset V$, $\emptyset \neq \mathcal{V}' \subset V'$ and a bijection $\varphi : \mathcal{V} \to \mathcal{V}'$, $v \mapsto v'$, any morphism $g : H^{\otimes E} \to H^{\otimes E'}$ that is a module morphism with respect to $\rhd_v$ and $\rhd_{v'}$ for all $v \in V$ is also a module morphism with respect to $\rhd_{\mathcal{V}}$ and $\rhd_{\mathcal{V}'}$. An analogous statement holds for $\delta_{\mathcal{F}}$ and comodule morphisms.

5. Graph independence

In this section we show that the protected object from Definition 4.8 is a topological invariant: Although its definition requires a ciliated ribbon graph Γ, its isomorphism class depends only on the homeomorphism class of the surface obtained by attaching discs to the faces of Γ.

To prove this, we show first in Section 5.1 that the (co)invariants associated to the (co)module structures at the vertices (faces) of Γ depend neither on the edge orientation nor on the choices of the cilia. Reversing the orientation of edges and different choices of cilia yield isomorphisms between these (co)invariants and hence also between the bi-invariants. We then show in Section 5.6 and 5.19 that the other graph transformations from Definition 3.4 induce isomorphisms between the protected objects, although not necessarily between the (co)invariants. In Section 5.24 we combine these results to obtain topological invariance and treat some simple examples.

As in Section 4 we consider a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$, an involutive Hopf monoid H in $\mathcal{C}$ and a ciliated ribbon graph Γ.

5.1. Edge orientation reversal and moving the cilium. As edge orientation reversal switches the start and target and the left and right side of an edge $\alpha \in E$, it exchanges the associated actions $\rhd_{\alpha\pm}$ and coactions $\delta_{\alpha\pm}$ from Definition 4.1. It is directly apparent from their definitions that this is achieved by applying the antipode.

5.2. Definition. *The automorphism of $H^{\otimes E}$ associated to the* **reversal** *of an edge $\alpha \in E$ is $S_\alpha : H^{\otimes E} \to H^{\otimes E}$.*

5.3. Lemma. *For any ciliated vertex $v \in V$, ciliated face $f \in F$ and edge $\beta \in E$ the edge reversal S_β is an isomorphism of H-modules and H-comodules with respect to $\rhd_v$ and δ_f.*

Proof. We denote by $\rhd'_v$ and δ'_f the module and comodule structure in the graph where the orientation of β is reversed and verify that $\rhd'_v \circ (1_H \otimes S_\beta) = S_\beta \circ \rhd_v$ and $\delta'_f \circ S_\beta = (1_H \otimes S_\beta) \circ \delta_f$. If β is not incident at v and f, the copy of H in $H^{\otimes E}$ assigned to β is not affected by $\rhd_v, \rhd'_v$ and δ_f, δ'_f, and the identity follows directly. If β is incident at v or f, it follows from the expressions for the (co)actions in Definitions 4.1 and 4.3. ∎

As a direct consequence of Lemma 5.3, Lemma 2.14 and Lemma 2.18 one has

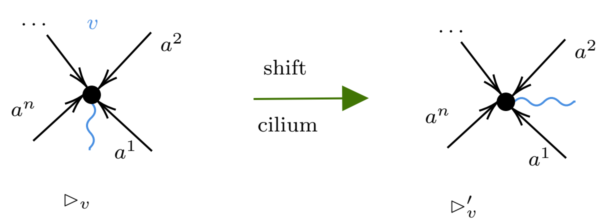
5.4. Corollary. *Reversing the orientation of an edge in Γ to obtain Γ' induces isomorphisms between the invariants, coinvariants and protected objects of Γ and Γ'.*

5.5. Lemma. *The (co)invariants for the H-(co)module structure at a given vertex (face) do not depend on the choice of cilia: moving the position of the cilium yields isomorphic (co)invariants. This induces isomorphisms of the protected objects.*

Proof. We focus on the H-module structure and its invariants. We consider a fixed position of the cilium at a vertex v with associated vertex action $\rhd_v$ and coequaliser $\pi_v : H^{\otimes E} \to M_v^H$ and compare it to the action $\rhd'_v$ and coequaliser $\pi'_v : H^{\otimes E} \to M_v'^H$ obtained by rotating the cilium counterclockwise by one position. We first show that the coequaliser $\pi_v : H^{\otimes E} \to M_v^H$ satisfies

$$\pi_v \circ \rhd'_v = \pi_v \circ (\epsilon \otimes 1_{H^{\otimes E}}). \tag{27}$$

By definition of the H-module structure $\rhd_v$ and by Lemma 5.3 it is sufficient to prove this for a vertex with n incoming edges. The computations for vertices with incident loops are analogous.



For a vertex with n incoming edges we have

$$\begin{aligned}
&\pi_v \circ \rhd'_v (h \otimes a^1 \otimes a^2 \otimes \ldots \otimes a^n) \;=\; \pi_v (h_{(n)} a^1 \otimes h_{(1)} a^2 \otimes \ldots \otimes h_{(n-1)} a^n)\\
&= \pi_v (h_{(2)(1)} S(h_{(1)}) h_{(3)} a^1 \otimes h_{(2)(2)} a^2 \otimes \ldots \otimes h_{(2)(n)} a^n)\\
&= \pi_v \circ \rhd_v (h_{(2)} \otimes S(h_{(1)}) h_{(3)} a^1 \otimes a^2 \otimes \ldots \otimes a^n)\\
&= \pi_v (\epsilon(h_{(2)}) \otimes S(h_{(1)}) h_{(3)} a^1 \otimes a^2 \otimes \ldots \otimes a^n)\\
&= \pi_v \circ (\epsilon \otimes 1_{H^{\otimes n}}) (h \otimes a^1 \otimes a^2 \otimes \ldots \otimes a^n),
\end{aligned}$$

where we used first the definition of $\rhd'_v$, then the defining property of the antipode and that $S \circ S = 1_H$, then the definition of $\rhd_v$, the fact that π_v coequalises $\rhd_v$ and $\epsilon \otimes 1_{H^{\otimes E}}$ and then again the defining properties of the antipode and the counitality of H.

Inductively, we obtain (27) for all positions of the cilium at v and the same identity with $\pi_v, \rhd_v$ and $\pi'_v, \rhd'_v$ swapped. With the universal property of the coequalisers π_v, π'_v this yields unique morphisms $\phi : M_v^H \to M_v'^H$, $\phi' : M_v'^H \to M_v^H$ with $\phi \circ \pi_v = \pi'_v$ and $\phi' \circ \pi'_v = \pi_v$. As π_v, π'_v are epimorphisms, this implies $\phi' = \phi^{-1}$.

The dual claim for the comodule structure and its coinvariants follows analogously. For all positions of the cilium at f with associated coaction δ'_f, there is a unique morphism $\psi : M^{\mathrm{co}H} \to M'^{\mathrm{co}H}$ with $\iota'_f \circ \psi = \iota_f$, and ψ is an isomorphism. Combining these statements for the (co)invariants of all vertices (faces) and using Lemmas 2.18 and 4.9 yields isomorphisms of the protected objects. ∎

5.6. Edge slides and edge contractions. We now consider the edge slides and edge contractions from Definition 3.4. Edge slides were already investigated in [MV], where it was shown that they define mapping class group actions. They yield automorphisms of the object $H^{\otimes E}$ that are morphisms of H-modules and H-comodules as long as no edge ends slide over cilia.

5.7. Definition. [MV, Def. 6.1]
Let $\alpha \neq \beta$ be edges of Γ with the starting end of α directly before the target end of β in the ordering at $s(\alpha) = t(\beta)$. The **edge slide** *of the target end of β along α corresponds to the isomorphism*

$$S_{\alpha,\beta} := \rhd_{\beta+} \circ \delta_{\alpha+} : H^{\otimes E} \to H^{\otimes E} \text{ with } S^{-1}_{\alpha,\beta} = \rhd_{\beta+} \circ (S \otimes 1_{H^{\otimes E}}) \circ \delta_{\alpha+} : H^{\otimes E} \to H^{\otimes E}.$$

Edge slides for other edge orientations are defined by reversing edge orientations with the antipode.

5.8. Example. The isomorphisms induced by the edge slides

are obtained by (a) applying Definition 5.7 and (b) first reversing the orientation of α, applying the inverse edge slide from Definition 5.7 and then reversing the orientation of α. This yields

$$(a) \qquad S_{\alpha,\beta}(\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \mu \otimes \nu) = \alpha_{(2)} \otimes \alpha_{(1)}\beta \otimes \gamma \otimes \delta \otimes \mu \otimes \nu,$$
$$(b) \qquad S_{\alpha,\beta}(\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \mu \otimes \nu) = \alpha_{(1)} \otimes \beta \otimes \gamma \otimes \alpha_{(2)}\delta \otimes \mu \otimes \nu.$$

By construction, edge slides affect only the two copies of H in $H^{\otimes E}$ of the edges involved in the slide and commute with edge orientation reversals. Moreover, they respect the module and comodule structures at vertices and faces and hence induce isomorphisms between the protected objects.

5.9. Proposition. [MV, Prop. 6.2] *Let v and f be a ciliated vertex and face in a ribbon graph Γ with associated H-module structure $\rhd_v$ and H-comodule structure δ_f. Any edge slide that does not slide edge ends over their cilia is an isomorphism of H-left modules and H-left comodules with respect to $\rhd_v$ and δ_f.*

5.10. Corollary. *Edge slides from a ribbon graph* Γ *to a ribbon graph* Γ' *induce isomorphisms between the invariants, coinvariants and protected objects of* Γ *and* Γ'.

Proof. For edge slides that do not slide edge ends over cilia, this follows directly from Lemmas 2.14, 2.18 and Proposition 5.9. If an edge end slides over a cilium, we can apply Lemma 5.5 to move the cilium and obtain the same result. ∎

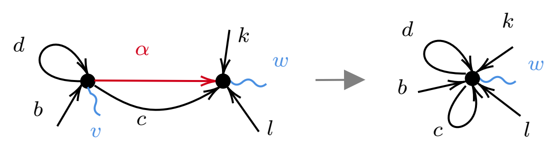
We now consider edge contractions. Recall from Definition 3.4 that an edge $\alpha \in E$ may only be contracted if its starting and target vertex differ and that contracting α towards $v \in \{s(\alpha), t(\alpha)\}$ erases the cilium at v, while the cilium at the other vertex is preserved.

5.11. Definition. *The morphism* $c_{\alpha,v} : H^{\otimes E} \to H^{\otimes(E-1)}$ *induced by an* **edge contraction** *of an edge* α *towards* $v \in \{s(\alpha), t(\alpha)\}$ *is*

$$c_{\alpha,v} = \begin{cases} \rhd_{v,\alpha} \circ \tau_\alpha \circ S_\alpha & \text{if } v = t(\alpha) \\ \rhd_{v,\alpha} \circ \tau_\alpha & \text{if } v = s(\alpha) \end{cases}$$

where $\rhd_{v,\alpha} : H^{\otimes E} \to H^{\otimes(E-1)}$ *denotes the* H*-module structure from Definition 4.3 at* v, *where* α *is replaced by a cilium and* τ_α *is given before Definition 4.1. If* v *is univalent, then* $c_{\alpha,v} = \epsilon_\alpha$.

5.12. Example. Contracting the edge α towards v in



gives the morphism $c_{\alpha,v}$ with

$$c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l) = \alpha_{(3)} b \otimes c S(\alpha_{(4)}) \otimes \alpha_{(1)} d S(\alpha_{(2)}) \otimes k \otimes l.$$

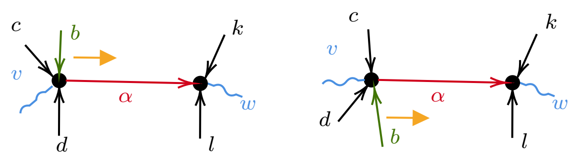
It follows directly from Definition 5.11 that first reversing the orientation of an edge β and then contracting it is the same as just contracting β. It also follows from Definitions 4.1 and 4.3 that reversing the orientation of an edge β commutes with contractions of all edges $\alpha \neq \beta$. The contraction of an edge α also commutes with edge slides along α, which allows one to express any edge contraction as a composite of edge slides and an edge contraction towards a univalent vertex.

5.13. Lemma. *Let* Γ' *be obtained by reversing an edge* β *in* Γ. *Then*

$$c'_{\beta,v} \circ S_\beta = c_{\beta,v} \qquad\qquad c'_{\alpha,v} \circ S_\beta = S_\beta \circ c_{\alpha,v} \quad \text{for } \alpha \neq \beta. \tag{28}$$

5.14. Lemma. *Contracting an edge* α *gives the same morphism as first sliding edge ends along* α *and then contracting* α.

PROOF. It suffices to slide a single edge end along α, as the statement follows inductively. We denote by $c_{\alpha,v}$ the contraction of α in Γ and by $c'_{\alpha,v}$ the contraction of α in the graph Γ' obtained by sliding an edge b along α. Suppose that there are no loops incident at $s(\alpha)$ and $t(\alpha)$ in Γ and Γ'. As edge slides and edge contractions commute with edge reversals by Definition 5.7 and Lemma 5.13, respectively, we can assume $v = s(\alpha)$ and all other edge ends at v and $w = t(\alpha)$ are incoming. It is then sufficient to consider an edge slide of b along the left and right of α:



(29)

Omitting the copies of H for edges not incident at v, w we compute for the edge slides in (29)

$$\begin{aligned}
c'_{\alpha,v} \circ S_{\alpha,b}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l) &= c'_{\alpha,v}(\alpha_{(2)} \otimes \alpha_{(1)} b \otimes c \otimes d \otimes k \otimes l) \\
&= \alpha_{(1)} b \otimes \alpha_{(2)} c \otimes \alpha_{(3)} d \otimes k \otimes l \\
&= c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l), \\
c'_{\alpha,v} \circ S_{\alpha,b}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l) &= c'_{\alpha,v}(\alpha_{(1)} \otimes \alpha_{(2)} b \otimes c \otimes d \otimes k \otimes l) \\
&= \alpha_{(3)} b \otimes \alpha_{(1)} c \otimes \alpha_{(2)} d \otimes k \otimes l \\
&= c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l).
\end{aligned}$$

As edge slides from w to v are the inverses of edge slides from v to w, the corresponding identities for those follow by pre-composing with the inverses. The proof for vertices with different numbers of incident edge ends or incident loops is analogous. ■

Next, we consider the interaction of edge contractions with the (co)module structures for the vertices (faces) of the graph. For this, note that the contraction of an edge α towards $v \in \{s(\alpha), t(\alpha)\}$ defines a bijection between the sets F, F' of faces before and after the contraction and likewise a bijection between the sets $V \setminus \{v\}$ and V'. If faces and vertices are identified via these bijections, the edge contraction becomes a (co)module morphism. In contrast, the module structure $\rhd_v$ is coequalised.

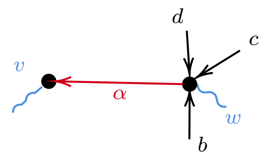
5.15. LEMMA. *The contraction of an edge α towards a ciliated vertex v coequalises $\rhd_v$ and $\epsilon \otimes 1_{H^{\otimes E}}$ and is a (co)module morphism with respect to the (co)actions $\rhd_z$ and δ_f for all ciliated vertices $z \neq v$ and ciliated faces $f \in F$ that do not start at v:*

$$c_{\alpha,v} \circ \rhd_v = c_{\alpha,v} \circ (\epsilon \otimes 1_{H^{\otimes E}}), \tag{30}$$

$$c_{\alpha,v} \circ \rhd_z = \rhd'_z \circ (1_H \otimes c_{\alpha,v}), \tag{31}$$

$$\delta'_f \circ c_{\alpha,v} = (1_H \otimes c_{\alpha,v}) \circ \delta_f. \tag{32}$$

Proof. As edge slides along α are module and comodule isomorphisms by Proposition 5.9 and commute with the contraction of α by Lemma 5.14, we can assume that v is univalent. With Lemma 5.13 we can assume that $v = t(\alpha)$ and that all edge ends at $w = s(\alpha)$ are incoming:



(33)

For the vertices v and w in (33) we compute

$$\begin{aligned} c_{\alpha,v} \circ \rhd_v(h \otimes \alpha \otimes b \otimes c \otimes d) &= c_{\alpha,v}(h\alpha \otimes b \otimes c \otimes d) = \epsilon(h\alpha) b \otimes c \otimes d \\ &= c_{\alpha,v}(\epsilon(h)\alpha \otimes b \otimes c \otimes d) \\ &= c_{\alpha,v} \circ (\epsilon \otimes 1_{H^{\otimes E}})(h \otimes \alpha \otimes b \otimes c \otimes d) \\ c_{\alpha,v} \circ \rhd_w(h \otimes \alpha \otimes b \otimes c \otimes d) &= c_{\alpha,v}(\alpha S(h_{(3)}) \otimes h_{(4)} b \otimes h_{(1)} c \otimes h_{(2)} d) \\ &= \epsilon(\alpha) h_{(3)} b \otimes h_{(1)} c \otimes h_{(2)} d \\ &= \rhd'_w \circ (1_H \otimes c_{\alpha,v})(h \otimes \alpha \otimes b \otimes c \otimes d). \end{aligned}$$

The computations for graphs with a different number of edge ends or loops incident at w are analogous. For vertices $z \in V \setminus \{v, w\}$ the action $\rhd_z$ does not affect the copy of H for α and commutes with $\rhd_{v,\alpha}$ and hence with $c_{v,\alpha}$. This proves (30) and (31).

If f is a face that contains α, but does not start at v, then the associated coaction is of the form

$$\delta_f(\alpha \otimes b \otimes c \otimes d \otimes \ldots) = (\cdots S(d_{(2)}) S(\alpha_{(3)}) \alpha_{(1)} b_{(1)} \cdots) \otimes \alpha_{(2)} \otimes b_{(2)} \otimes c \otimes d_{(1)} \otimes \ldots,$$

where the dots stand for contributions of parts of Γ that are not drawn in (33). This yields

$$\begin{aligned} &(1_H \otimes c_{\alpha,v}) \circ \delta_f(\alpha \otimes b \otimes c \otimes d \otimes \ldots) \\ &= \epsilon(\alpha_{(2)})(\cdots S(d_{(2)}) S(\alpha_{(3)}) \alpha_{(1)} b_{(1)} \cdots) \otimes b_{(2)} \otimes c \otimes d_{(1)} \otimes \ldots \\ &\overset{(6)}{=} \epsilon(\alpha)(\cdots S(d_{(2)}) b_{(1)} \cdots) \otimes b_{(2)} \otimes c \otimes d_{(1)} \otimes \ldots = \delta'_f \circ c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes \ldots). \end{aligned}$$

If f does not contain α, the edge α does not contribute to the coaction δ_f, which proves (32). ■

With these results, we investigate how edge contractions interact with the (co)invariants of the H-(co)module structures at ciliated vertices and faces of Γ. For subsets $\emptyset \neq \mathcal{V} \subset V$ and $\emptyset \neq \mathcal{F} \subset F$ we denote by $\rhd_{\mathcal{V}}$ and $\delta_{\mathcal{F}}$ the associated $H^{\otimes \mathcal{V}}$-module structure and $H^{\otimes \mathcal{F}}$-comodule structure from (23) and by $\pi_{\mathcal{V}}$ and $\iota_{\mathcal{F}}$ their invariants and coinvariants from Definition 2.13.

We then find that edge contractions send coinvariants for $\delta_{\mathcal{F}}$ to coinvariants for the corresponding face set in the contracted graph. The same holds for the invariants of the action $\rhd_{\mathcal{V}}$, as long as $\mathcal{V}$ contains the starting and target vertex of the contracted edge. The morphism η_α that creates a copy of H assigned to α by applying the unit of H is right inverse to the edge contraction $c_{v,\alpha}$ and a left inverse on the coinvariants. This corresponds to the following technical lemma.

5.16. LEMMA. *Let Γ' be obtained from Γ by contracting an edge α incident at $v, w \in V$. Then $\eta_\alpha : H^{\otimes(E-1)} \to H^{\otimes E}$ is right inverse to the edge contraction $c_{\alpha,v} : H^{\otimes E} \to H^{\otimes(E-1)}$, and for all subsets $\{v, w\} \subset \mathcal{V} \subset V$, $\emptyset \neq \mathcal{F} \subset F$ one has*

$$\delta'_{\mathcal{F}} \circ c_{\alpha,v} \circ \iota_{\mathcal{F}} = (\eta^{\otimes|\mathcal{F}|} \otimes c_{\alpha,v}) \circ \iota_{\mathcal{F}} \tag{34}$$

$$\delta_{\mathcal{F}} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} = (\eta^{\otimes|\mathcal{F}|} \otimes \eta_\alpha) \circ \iota'_{\mathcal{F}} \tag{35}$$

$$\pi_{\mathcal{V}} \circ \eta_\alpha \circ c_{\alpha,v} = \pi_{\mathcal{V}} \tag{36}$$

$$\pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \rhd_{\mathcal{V}} = \pi'_{\mathcal{V}} \circ (\epsilon^{\otimes|\mathcal{V}|} \otimes c_{\alpha,v}) \tag{37}$$

$$\pi_{\mathcal{V}} \circ \eta_\alpha \circ \rhd'_{\mathcal{V}} = \pi_{\mathcal{V}} \circ (\epsilon^{\otimes|\mathcal{V}|-1} \otimes \eta_\alpha). \tag{38}$$

PROOF. 1. It follows directly from Definition 5.11 that the morphism η_α is a right inverse to $c_{\alpha,v}$. From the formula for the (co)action in Definition 4.3 it is apparent that η_α is a comodule morphism for the coactions δ_f at all ciliated faces and a module morphism with respect to the actions $\rhd_z$ at all vertices $z \in V \setminus \{v, w\}$. Moreover, it is clear from Definition 5.7 that sliding edge ends over α after applying η_α yields a morphism η''_α which splits the vertex w in a different way. Thus, we have

$$c_{\alpha,v} \circ \eta_\alpha = 1_{H^{\otimes E}}, \quad \delta_f \circ \eta_\alpha = \eta_\alpha \circ \delta'_f, \quad \eta_\alpha \circ \rhd'_z = \rhd_z \circ \eta_\alpha, \quad S_{\alpha,\beta} \circ \eta_\alpha = \eta''_\alpha \tag{39}$$

for all vertices $z \in V \setminus \{v, w\}$ and faces $f \in F$ and edge slides $S_{\alpha,\beta}$ along α. We can therefore assume that the vertex $v = t(\alpha)$ is univalent, all edge ends at $w = s(\alpha)$ are incoming, the graph Γ is locally given by (33) and the edge contraction by $c_{\alpha,v} = \epsilon_\alpha$, as in the proof of Lemma 5.15.

2. We prove the auxiliary identities

$$\pi_v \circ \eta_\alpha \circ c_{\alpha,v} = \pi_v, \tag{40}$$

$$\pi_{\{v,w\}} \circ \eta_\alpha \circ (\epsilon \otimes 1_{H^{\otimes(E-1)}}) = \pi_{\{v,w\}} \circ \eta_\alpha \circ \rhd'_w, \tag{41}$$

$$\delta'_f \circ c_{\alpha,v} \circ \iota_f = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ c_{\alpha,v} \circ \iota_f \qquad \forall f \in F. \tag{42}$$

Omitting all copies of H in $H^{\otimes E}$ except the one for α, we verify (40)

$$\begin{aligned}\pi_v(\alpha \otimes \ldots) &= \pi_v \circ (\alpha \rhd_v)(1 \otimes \ldots) = \pi_v(\epsilon(\alpha)\, 1 \otimes \ldots) = \pi_v \circ \eta_\alpha \circ \epsilon_\alpha(1 \otimes \ldots) \\ &= \pi_v \circ \eta_\alpha \circ c_{\alpha,v}(1 \otimes \ldots).\end{aligned}$$

To show (41), we consider the graph (33) and compute with Lemma 4.9

$$\begin{aligned}&\pi_{\{v,w\}} \circ \eta_\alpha \circ \rhd'_w(h \otimes b \otimes c \otimes d) = \pi_{\{v,w\}} \circ \eta_\alpha(h_{(3)}b \otimes h_{(1)}c \otimes h_{(2)}d)\\ &= \pi_{\{v,w\}}(1 \otimes h_{(3)}b \otimes h_{(1)}c \otimes h_{(2)}d) = \pi_{\{v,w\}}(h_{(3)}S(h_{(4)}) \otimes h_{(5)}b \otimes h_{(1)}c \otimes h_{(2)}d)\\ &= \pi_{\{v,w\}} \circ h_{(3)} \rhd_v (S(h_{(4)}) \otimes h_{(5)}b \otimes h_{(1)}c \otimes h_{(2)}d)\\ &= \pi_{\{v,w\}}(\epsilon(h_{(3)})\, S(h_{(4)}) \otimes h_{(5)}b \otimes h_{(1)}c \otimes h_{(2)}d) = \pi_{\{v,w\}}(S(h_{(3)}) \otimes h_{(4)}b \otimes h_{(1)}c \otimes h_{(2)}d)\\ &= \pi_{\{v,w\}} \circ h \rhd_w (1 \otimes b \otimes c \otimes d) = \pi_{\{v,w\}}(\epsilon(h)\, 1 \otimes b \otimes c \otimes d)\\ &= \pi_{\{v,w\}} \circ \eta_\alpha \circ (\epsilon \otimes 1_{H^{\otimes(E-1)}})(b \otimes c \otimes d).\end{aligned}$$

Identity (42) follows from identity (32) in Lemma 5.15 for all faces $f \in F$ that do not start at v. If f starts at v one has for the graph in (33)

$$\begin{aligned}&\delta'_f \circ c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d) = \epsilon(\alpha)\delta'_f(b \otimes c \otimes d) = \epsilon(\alpha)b_{(1)} \cdots S(d_{(2)}) \otimes b_{(2)} \otimes c \otimes d_{(1)}\\ &= S(\alpha_{(2)})\alpha_{(1)}b_{(1)} \cdots S(d_{(2)})S(\alpha_{(4)})\alpha_{(3)} \otimes b_{(2)} \otimes c \otimes d_{(1)}\\ &= (\lhd_{ad} \otimes 1_{H^{\otimes(E-1)}}) \circ (1_H \otimes \tau_\alpha) \circ \delta_f(\alpha \otimes b \otimes c \otimes d),\end{aligned}$$

where $\lhd_{ad} : H \otimes H \to H$, $h \otimes \alpha \mapsto S(\alpha_{(1)})h\alpha_{(2)}$. In this case, contracting α deletes the cilium of f, but Lemma 5.5 allows one to place a new cilium for f in any position. As $\lhd_{ad} \circ (\eta \otimes 1_H) = \eta \circ \epsilon : H \to H$ this yields

$$\begin{aligned}&\delta'_f \circ c_{\alpha,v} \circ \iota_f = (\lhd_{ad} \otimes 1_{H^{\otimes(E-1)}}) \circ (1_H \otimes \tau_\alpha) \circ \delta_f \circ \iota_f\\ &= (\lhd_{ad} \otimes 1_{H^{\otimes(E-1)}}) \circ (1_H \otimes \tau_\alpha) \circ (\eta \otimes 1_{H^{\otimes E}}) \circ \iota_f = (\eta \circ \epsilon \otimes 1_{H^{\otimes(E-1)}}) \circ \tau_\alpha \circ \iota_f\\ &= (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ \epsilon_\alpha \circ \iota_f = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ c_{\alpha,v} \circ \iota_f.\end{aligned}$$

3. We prove the identities in the Lemma. Identity (34) follows by pre-composing (42) with the morphism $\xi_{f,\mathcal{F}} := \xi_{\{f\},\mathcal{F}}$ from Lemma 4.9 and inductively applying this equation for all $f \in \mathcal{F}$. Likewise, identity (35) follows by applying the identity $\delta_f \circ \eta_\alpha \circ \iota'_f = (\eta \otimes 1_{H^{\otimes E}}) \circ \eta_\alpha \circ \iota'_f$ obtained from the second identity in (39) and pre-composing it with $\xi'_{f,\mathcal{F}}$. Post-composing (40) with the morphism $\chi_{v,\mathcal{V}} := \chi_{\{v\},\mathcal{V}}$ from Lemma 4.9 yields (36). From (31), we obtain for all $z \in V \setminus \{v, w\}$

$$\pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \rhd_z = \chi'_{z,\mathcal{V}} \circ \pi'_z \circ c_{\alpha,v} \circ \rhd_z = \chi'_{z,\mathcal{V}} \circ \pi'_z \circ \rhd'_z \circ (1_H \otimes c_{\alpha,v}) = \pi'_{\mathcal{V}} \circ (\epsilon \otimes c_{\alpha,v}). \tag{43}$$

Together with the identity $\pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \rhd_w \circ (1_H \otimes \rhd_v) = \pi'_{\mathcal{V}} \circ (\epsilon^{\otimes 2} \otimes c_{\alpha,v})$, which follows from (30) and (31) with $z = w$ and the identity $\pi'_{\mathcal{V}} = \chi'_{w,\mathcal{V}} \circ \pi'_w$, this yields (37). Identity (38) follows by post-composing (41) with $\chi_{\{v,w\},\mathcal{V}}$ and the third identity in (39) with $\pi_{\mathcal{V}} = \chi_{z,\mathcal{V}} \circ \pi_z$. ■

We now apply Lemma 5.16 to show that edge contractions induce morphisms between the coinvariants for $\emptyset \neq \mathcal{F} \subset F$. If $\mathcal{V}$ contains the starting and target vertex of the contracted edge, they also induce isomorphisms between the invariants and isomorphisms between the protected objects.

For this, we consider a ciliated ribbon graph Γ and the graph Γ' obtained by contracting an edge α in Γ. We denote by $\mathcal{M}^{\mathrm{co}H}$, $\mathcal{M}^H$, $\mathcal{M}_{\mathrm{inv}}$ the coinvariants, invariants and biinvariants of $\delta_{\mathcal{F}}$, $\rhd_{\mathcal{V}}$ for Γ and by $\mathcal{M}'^{\mathrm{co}H}$, $\mathcal{M}'^H$, $\mathcal{M}'_{\mathrm{inv}}$ the corresponding quantities for Γ'. As in Lemma 4.9 we write $\iota_{\mathcal{F}}$ and $\pi_{\mathcal{V}}$ for the associated equaliser and coequaliser and $I : \mathcal{M}_{\mathrm{inv}} \to \mathcal{M}^H$ and $P : \mathcal{M}^{\mathrm{co}H} \to \mathcal{M}_{\mathrm{inv}}$ for the monomorphism and epimorphism that characterise $\mathcal{M}_{\mathrm{inv}}$ as the image of $\pi_{\mathcal{V}} \circ \iota_{\mathcal{F}}$. The corresponding morphisms for Γ' are denoted $\iota'_{\mathcal{F}}$, $\pi'_{\mathcal{V}}$, I' and P'.

5.17. Proposition. *Let Γ' be obtained from a ciliated ribbon graph Γ by contracting an edge α incident at v, w towards v. Then for all $\{v, w\} \subset \mathcal{V} \subset V$, $\emptyset \neq \mathcal{F} \subset F$ the contraction of α induces*

- *a morphism $u : M^{\mathrm{co}H}_{\mathcal{F}} \to M'^{\mathrm{co}H}_{\mathcal{F}}$ with a right inverse that satisfies $\iota'_{\mathcal{F}} \circ u = c_{\alpha,v} \circ \iota_{\mathcal{F}}$,*
- *an isomorphism $r : M^H_{\mathcal{V}} \to M'^H_{\mathcal{V}}$ that satisfies $r \circ \pi_{\mathcal{V}} = \pi'_{\mathcal{V}} \circ c_{\alpha,v}$,*
- *an isomorphism $\phi_{\mathrm{inv}} : M_{\mathrm{inv}} \to M'_{\mathrm{inv}}$ with $I = r^{-1} \circ I' \circ \phi_{\mathrm{inv}}$.*

Proof. Using equation (34) together with the universal property of the equaliser $\iota'_{\mathcal{F}}$ yields a unique morphism $u : M^{\mathrm{co}H}_{\mathcal{F}} \to M'^{\mathrm{co}H}_{\mathcal{F}}$ with $\iota'_{\mathcal{F}} \circ u = c_{\alpha,v} \circ \iota_{\mathcal{F}}$. Equation (35) and the equaliser $\iota_{\mathcal{F}}$ yield a unique morphism $u^{-1} : M'^{\mathrm{co}H}_{\mathcal{F}} \to M^{\mathrm{co}H}_{\mathcal{F}}$ with $\iota_{\mathcal{F}} \circ u^{-1} = \eta_\alpha \circ \iota'_{\mathcal{F}}$. To show that u^{-1} is a right inverse of u note that $\iota'_{\mathcal{F}} \circ u \circ u^{-1} = c_{\alpha,v} \circ \iota_{\mathcal{F}} \circ u^{-1} = c_{\alpha,v} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} = \iota'_{\mathcal{F}}$, since η_α is right inverse to $c_{\alpha,v}$. As $\iota'_{\mathcal{F}}$ is a monomorphism, this implies $u \circ u^{-1} = 1_{M'^{\mathrm{co}H}_{\mathcal{F}}}$.

Analogously, (37) and the universal property of the coequaliser $\pi_{\mathcal{V}}$ define a unique morphism $r : M^H_{\mathcal{V}} \to M'^H_{\mathcal{V}}$ with $r \circ \pi_{\mathcal{V}} = \pi'_{\mathcal{V}} \circ c_{\alpha,v}$. The coequaliser $\pi'_{\mathcal{V}}$ together with (38) yields a unique morphism $r^{-1} : M'^H_{\mathcal{V}} \to M^H_{\mathcal{V}}$ with $r^{-1} \circ \pi'_{\mathcal{V}} = \pi_{\mathcal{V}} \circ \eta_\alpha$. The morphisms r and r^{-1} are mutually inverse isomorphisms, since $\pi'_{\mathcal{V}}$, $\pi_{\mathcal{V}}$ are epimorphisms with

$$r \circ r^{-1} \circ \pi'_{\mathcal{V}} = r \circ \pi_{\mathcal{V}} \circ \eta_\alpha = \pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \eta_\alpha = \pi'_{\mathcal{V}},$$
$$r^{-1} \circ r \circ \pi_{\mathcal{V}} = r^{-1} \circ \pi'_{\mathcal{V}} \circ c_{\alpha,v} = \pi_{\mathcal{V}} \circ \eta_\alpha \circ c_{\alpha,v} \overset{(36)}{=} \pi_{\mathcal{V}}.$$

Hence, we constructed commuting diagrams

$$\begin{array}{ccccc} M^{\mathrm{co}H}_{\mathcal{F}} & \xrightarrow{\iota_{\mathcal{F}}} & M & \xrightarrow{\pi_{\mathcal{V}}} & M^H_{\mathcal{V}} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle c_{\alpha,v}} & & \downarrow{\scriptstyle r} \\ M'^{\mathrm{co}H}_{\mathcal{F}} & \xrightarrow[\iota'_{\mathcal{F}}]{} & M' & \xrightarrow[\pi'_{\mathcal{V}}]{} & M'^H_{\mathcal{V}}. \end{array} \qquad \begin{array}{ccccc} M'^{\mathrm{co}H}_{\mathcal{F}} & \xrightarrow{\iota'_{\mathcal{F}}} & M' & \xrightarrow{\pi'_{\mathcal{V}}} & M'^H_{\mathcal{V}} \\ \downarrow{\scriptstyle u^{-1}} & & \downarrow{\scriptstyle \eta_\alpha} & & \downarrow{\scriptstyle r^{-1}} \\ M^{\mathrm{co}H}_{\mathcal{F}} & \xrightarrow[\iota_{\mathcal{F}}]{} & M & \xrightarrow[\pi_{\mathcal{V}}]{} & M^H_{\mathcal{V}} \end{array}$$

To construct the isomorphism ϕ_{inv}, we set $j := r^{-1} \circ I' : M'_{\mathrm{inv}} \to M^H_{\mathcal{V}}$ and $q := P' \circ u : M^{\mathrm{co}H}_{\mathcal{F}} \to M'_{\mathrm{inv}}$. As r^{-1} is an isomorphism and I' a monomorphism, the morphism j is a monomorphism. The composite $j \circ q$ satisfies

$$\begin{aligned} j \circ q &= r^{-1} \circ I' \circ P' \circ u = r^{-1} \circ \pi'_{\mathcal{V}} \circ \iota'_{\mathcal{F}} \circ u = \pi_{\mathcal{V}} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} \circ u = \pi_{\mathcal{V}} \circ \eta_\alpha \circ c_{\alpha,v} \circ \iota_{\mathcal{F}} \\ &\overset{(36)}{=} \pi_{\mathcal{V}} \circ \iota_{\mathcal{F}}. \end{aligned}$$

The universal property of the image M_{inv} then yields a unique morphism $\phi_{\mathrm{inv}} : M_{\mathrm{inv}} \to M'_{\mathrm{inv}}$ with $I = j \circ \phi_{\mathrm{inv}} = r^{-1} \circ I' \circ \phi_{\mathrm{inv}}$. To construct its inverse we set $j' := r \circ I : M_{\mathrm{inv}} \to$

$M'^{H}_{\mathcal{V}}$ and $q' := P \circ u^{-1} : M'^{\text{co}H}_{\mathcal{F}} \to M_{\text{inv}}$. As r is an isomorphism and I a monomorphism, j' is a monomorphism, and we have

$$j' \circ q' = r \circ I \circ P \circ u^{-1} = r \circ \pi_{\mathcal{V}} \circ \iota_{\mathcal{F}} \circ u^{-1} = \pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \iota_{\mathcal{F}} \circ u^{-1} = \pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \eta_\alpha \circ \iota'_{\mathcal{F}}$$
$$= \pi'_{\mathcal{V}} \circ \iota'_{\mathcal{F}},$$

where we used that η_α is right inverse to $c_{\alpha,v}$ in the last step. By the universal property of the image M'_{inv} there is a unique morphism $\phi^{-1}_{\text{inv}} : M'_{\text{inv}} \to M_{\text{inv}}$ with $I' = j' \circ \phi^{-1}_{\text{inv}} = r \circ I \circ \phi^{-1}_{\text{inv}}$ and

$$I \circ \phi^{-1}_{\text{inv}} \circ \phi_{\text{inv}} = r^{-1} \circ r \circ I \circ \phi^{-1}_{\text{inv}} \circ \phi_{\text{inv}} = r^{-1} \circ I' \circ \phi_{\text{inv}} = I,$$
$$I' \circ \phi_{\text{inv}} \circ \phi^{-1}_{\text{inv}} = r \circ r^{-1} \circ I' \circ \phi_{\text{inv}} \circ \phi^{-1}_{\text{inv}} = r \circ I \circ \phi^{-1}_{\text{inv}} = I'.$$

As I, I' are monomorphisms, it follows that ϕ_{inv} and ϕ^{-1}_{inv} are mutually inverse isomorphisms. ∎

5.18. Corollary. *Edge contractions induce isomorphisms between protected objects.*

5.19. Deleting isolated loops. We now consider the last graph transformation from Definition 3.4, the deletion of isolated loops. The morphism associated to the deletion of an isolated loop α applies the counit to the corresponding copy of the Hopf monoid H. Just as edge contractions, this is in general not an isomorphism in $\mathcal{C}$. The morphism η_α that creates a copy of H for α by applying the unit is a right inverse and corresponds to inserting a loop.

5.20. Definition. *The morphism induced by* **deleting an isolated loop** α *is* $\epsilon_\alpha : H^{\otimes E} \to H^{\otimes E \setminus \{\alpha\}}$.

As for edge contractions we investigate how these morphisms interact with the coinvariants for the $H^{\otimes \mathcal{F}}$-comodule structure $\delta_{\mathcal{F}}$ and the $H^{\otimes \mathcal{V}}$-module structure $\rhd_{\mathcal{V}}$ from Definition 4.7 for subsets $\emptyset \neq \mathcal{F} \subset F$ and $\emptyset \neq \mathcal{V} \subset V$. We find that loop deletions send the invariants for $\rhd_{\mathcal{V}}$ to invariants for the corresponding vertex set of the graph with the loop removed. The same holds for coinvariants of $\delta_{\mathcal{F}}$, as long as the two faces incident to the loop are contained in $\mathcal{F}$. Analogous statements hold for the right inverse η_α, and on the coinvariants η_α is also a left inverse. This is a consequence of the following technical lemma.

5.21. Lemma. *Let Γ^+ be obtained from a ciliated ribbon graph Γ by removing an isolated loop α with adjacent faces f_1, f_2 at a vertex v. Then for all subsets $\emptyset \neq \mathcal{V} \subset V$ and $\{f_1, f_2\} \subset \mathcal{F} \subset F$*

$$\pi^+_{\mathcal{V}} \circ \epsilon_\alpha \circ \rhd_{\mathcal{V}} = \pi^+_{\mathcal{V}} \circ (\epsilon^{\otimes|\mathcal{V}|} \otimes \epsilon_\alpha) \tag{44}$$
$$\delta^+_{\mathcal{F}} \circ \epsilon_\alpha \circ \iota_{\mathcal{F}} = (\eta^{\otimes|\mathcal{F}|-1} \otimes \epsilon_\alpha) \circ \iota_{\mathcal{F}} \tag{45}$$
$$\eta_\alpha \circ \epsilon_\alpha \circ \iota_{\mathcal{F}} = \iota_{\mathcal{F}} \tag{46}$$
$$\pi_{\mathcal{V}} \circ \eta_\alpha \circ \rhd^+_{\mathcal{V}} = \pi_{\mathcal{V}} \circ (\epsilon^{\otimes|\mathcal{V}|} \otimes \eta_\alpha) \tag{47}$$
$$\delta_{\mathcal{F}} \circ \eta_\alpha \circ \iota^+_{\mathcal{F}} = (\eta^{\otimes|\mathcal{F}|} \otimes \eta_\alpha) \circ \iota^+_{\mathcal{F}}. \tag{48}$$

PROOF. 1. We first prove some auxiliary identities for the interaction of the morphisms ϵ_α and η_α with the module and comodules structures at the vertices and faces.

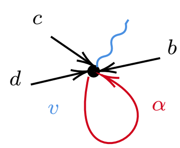
1.(a) As η_α and ϵ_α affect only the copy of H for α, we have for any vertex $z \neq v$ and any ciliated face f that does not contain α

$$\epsilon_\alpha \circ \rhd_z = \rhd_z^+ \circ (1_H \otimes \epsilon_\alpha) \qquad \eta_\alpha \circ \rhd_z^+ = \rhd_z \circ (1_H \otimes \eta_\alpha) \tag{49}$$
$$(1_H \otimes \epsilon_\alpha) \circ \delta_f = \delta_f^+ \circ \epsilon_\alpha \qquad (1_H \otimes \eta_\alpha) \circ \delta_f^+ = \delta_f \circ \eta_\alpha. \tag{50}$$

1.(b) For the H-module structure at the vertex v, we show that

$$\pi_v^+ \circ \epsilon_\alpha \circ \rhd_v = \pi_v^+ \circ (\epsilon \otimes \epsilon_\alpha) \tag{51}$$
$$\pi_v \circ \eta_\alpha \circ \rhd_v^+ = \pi_v \circ (\epsilon \otimes \eta_\alpha). \tag{52}$$

As reversing edge orientations commutes with $\epsilon_\alpha, \eta_\alpha$ and $\rhd_v$, we can assume that all edges $\beta \neq \alpha$ at v are incoming and that $s(\alpha)$ is directly before $t(\alpha)$ with respect to the cyclic ordering at v:



For this graph we compute

$$\begin{aligned}
&\pi_v^+ \circ \epsilon_\alpha \circ \rhd_v (h \otimes \alpha \otimes b \otimes c \otimes d) = \pi_v^+ \circ \epsilon_\alpha (h_{(4)} \alpha S(h_{(3)}) \otimes h_{(5)} b \otimes h_{(1)} c \otimes h_{(2)} d) \\
&= \pi_v^+ (\epsilon(\alpha) \otimes h_{(3)} b \otimes h_{(1)} c \otimes h_{(2)} d) = \pi_v^+ \circ (\epsilon \otimes \rhd_v^+) (\alpha \otimes h \otimes b \otimes c \otimes d) \\
&= \pi_v^+ \circ (\epsilon \otimes \epsilon \otimes 1_{H^{\otimes 3}}) (h \otimes \alpha \otimes b \otimes c \otimes d) = \pi_v^+ \circ (\epsilon \otimes \epsilon_\alpha) (h \otimes \alpha \otimes b \otimes c \otimes d),
\end{aligned}$$

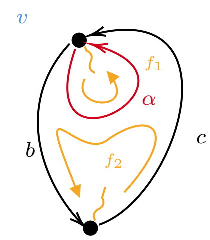
$$\begin{aligned}
&\pi_v \circ \eta_\alpha \circ \rhd_v^+ (h \otimes b \otimes c \otimes d) = \pi_v \circ \eta_\alpha (h_{(3)} b \otimes h_{(1)} c \otimes h_{(2)} d) \\
&= \pi_v (h_{(4)} S(h_{(3)}) \otimes h_{(5)} b \otimes h_{(1)} c \otimes h_{(2)} d) = \pi_v \circ \rhd_v (h \otimes 1 \otimes b \otimes c \otimes d) \\
&= \pi_v \circ (\epsilon \otimes 1_{H^{\otimes 4}}) (h \otimes 1 \otimes b \otimes c \otimes d) = \pi_v \circ (\epsilon \otimes \eta_\alpha) (h \otimes b \otimes c \otimes d),
\end{aligned}$$

which proves (51) and (52). The computation for graphs with a different number of edge ends at v are analogous. The claim for the case where the cilium is between the edge ends of α follows, because the invariants of the H-module structure at v do not depend on the choice of the cilium by Lemma 5.5. It can also be verified directly by analogous computations.

1.(c) We consider the H-comodule structures at the faces f_1, f_2. Under the assumption that the starting end of α comes directly before its target end with respect to the cyclic ordering at v, one of these faces coincides with α, and we assume it is $f_1 = \alpha$. We then have

$$\delta_{f_1} \circ \eta_\alpha = \eta \otimes \eta_\alpha \tag{53}$$
$$\delta_{f_2} \circ \eta_\alpha = (1_H \otimes \eta_\alpha) \circ \delta_{f_2}^+ \qquad \delta_{f_2} \circ \eta_\alpha \circ \iota_{f_2}^+ = (\eta \otimes \eta_\alpha) \circ \iota_{f_2}^+. \tag{54}$$

Equation (53) is obvious, and to prove (54), we can assume that Γ is locally given by



(55)

as edge reversals commute with the module and comodule structures at the vertices and faces and with the morphisms η_α and ϵ_α. We then compute

$$\delta_{f_2} \circ \eta_\alpha(b \otimes c) = \delta_{f_2}(1 \otimes b \otimes c) = b_{(1)}c_{(1)} \otimes 1 \otimes b_{(2)} \otimes c_{(2)} = (1_H \otimes \eta_\alpha) \circ \delta^+_{f_2}(b \otimes c).$$

The computations for graphs with different numbers of edges in f_2 are analogous, and with the identity $\delta^+_{f_2} \circ \iota^+_{f_2} = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ \iota^+_{f_2}$ we obtain the second identity in (54).

2. We prove the identities (44) to (48). To show (46) it is sufficient to consider the graph (55) with

$$\delta_{f_1}(\alpha \otimes b \otimes c) = \alpha_{(1)} \otimes \alpha_{(2)} \otimes b \otimes c$$
$$\delta_{f_2}(\alpha \otimes b \otimes c) = b_{(1)}S(\alpha_{(2)})c_{(1)} \otimes \alpha_{(1)} \otimes b_{(2)} \otimes c_{(2)}.$$

As $f_1, f_2 \in \mathcal{F}$, this yields

$$\iota_\mathcal{F} = ((\epsilon \circ \eta) \otimes 1_{H^{\otimes E}}) \circ \iota_\mathcal{F} = (\epsilon \otimes 1_{H^{\otimes E}}) \circ \delta_{f_2} \circ \iota_\mathcal{F} \overset{(*)}{=} \eta_\alpha \circ \epsilon_\alpha \circ \iota_\mathcal{F},$$

where we apply in $(*)$ the coinvariance under δ_{f_1}.

Identity (44) follows inductively from the identity $\pi^+_\mathcal{V} \circ \epsilon_\alpha \circ \rhd_z = \pi^+_\mathcal{V} \circ (\epsilon \otimes \epsilon_\alpha)$ for all vertices $z \in V$, which is obtained for $z \neq v$ by post-composing the first identity in (49) with $\pi^+_\mathcal{V} = \chi^+_{z,\mathcal{V}} \circ \pi^+_z$ and for $z = v$ by post-composing (51) with $\chi_{v,\mathcal{V}}$.

Identity (45) follows from (46) and the first identities in (50), (54), which yield for all $f \in \mathcal{F} \setminus \{f_1\}$

$$\delta^+_f \circ \epsilon_\alpha \circ \iota_\mathcal{F} = (1_H \otimes (\epsilon_\alpha \circ \eta_\alpha)) \circ \delta^+_f \circ \epsilon_\alpha \circ \iota_\mathcal{F} \overset{(50),(54)}{=} (1_H \otimes \epsilon_\alpha) \circ \delta_f \circ \eta_\alpha \circ \epsilon_\alpha \circ \iota_\mathcal{F}$$
$$\overset{(46)}{=} (1_H \otimes \epsilon_\alpha) \circ \delta_f \circ \iota_\mathcal{F} = (1_H \otimes \epsilon_\alpha) \circ \delta_f \circ \iota_f \circ \xi_{f,\mathcal{F}} = (\eta \otimes \epsilon_\alpha) \circ \iota_\mathcal{F}.$$

Identity (47) follows inductively by applying the identity $\pi_\mathcal{V} \circ \eta_\alpha \circ \rhd^+_z = \pi_\mathcal{V} \circ (\epsilon \otimes \eta_\alpha)$ for $z \in V$, obtained by post-composing (49) with $\pi_\mathcal{V} = \chi_{z,\mathcal{V}} \circ \pi_z$ for $z \neq v$ and (52) with $\chi_{v,\mathcal{V}}$ for $z = v$.

Analogously, (48) follows from the identity $\delta_f \circ \eta_\alpha \circ \iota^+_\mathcal{F} = (\eta \otimes \eta_\alpha) \circ \iota^+_\mathcal{F}$ for $f \in F$, which is obtained for $f = f_1$ by pre-composing (53) with $\iota^+_\mathcal{F}$, for $f = f_2$ by pre-composing (54) with $\xi^+_{f_2,\mathcal{F}}$ and for $f \notin \{f_1, f_2\}$ by pre-composing (50) with $\iota^+_\mathcal{F} = \iota^+_f \circ \xi^+_{f,\mathcal{F}}$. ∎

We now apply Lemma 5.21 to show that loop deletions induce morphisms between the invariants of $\rhd_{\mathcal{V}}$ for subsets $\emptyset \neq \mathcal{V} \subset V$. If $\mathcal{F}$ contains the two faces adjacent to the loop, they also induce isomorphisms between the coinvariants of $\delta_{\mathcal{F}}$ and isomorphisms between the protected objects.

For this we denote by Γ^+ the graph obtained by deleting a loop α in Γ. For Γ we use the notation from Proposition 5.17. For Γ^+ we denote by $\mathcal{M}^{+coH}$, $\mathcal{M}^{+H}$, $\mathcal{M}^+_{\text{inv}}$ the coinvariants, invariants and biinvariants of $\delta^+_{\mathcal{F}}$, $\rhd^+_{\mathcal{V}}$, by $\iota^+_{\mathcal{F}}$ and $\pi^+_{\mathcal{V}}$ the associated equaliser and coequaliser and by $I^+ : \mathcal{M}^+_{\text{inv}} \to \mathcal{M}^{+H}$ and $P^+ : \mathcal{M}^{+coH} \to \mathcal{M}^+_{\text{inv}}$ the monomorphism and epimorphism that characterise $\mathcal{M}^+_{\text{inv}}$ as the image of $\pi^+_{\mathcal{V}} \circ \iota^+_{\mathcal{F}}$.

5.22. Proposition. *Let Γ^+ be obtained from Γ by removing an isolated loop α with incident faces f_1, f_2. Then for all subsets $\emptyset \neq \mathcal{V} \subset V$, $\{f_1, f_2\} \subset \mathcal{F} \subset F$ the loop removal induces*

- *an isomorphism $y : M^{\text{co}H}_{\mathcal{F}} \to M^{+coH}_{\mathcal{F}}$ with $\iota^+_{\mathcal{F}} \circ y = \epsilon_\alpha \circ \iota_{\mathcal{F}}$,*
- *a morphism $t : M^H_{\mathcal{V}} \to M^{+H}_{\mathcal{V}}$ with a right inverse and $t \circ \pi_{\mathcal{V}} = \pi^+_{\mathcal{V}} \circ \epsilon_\alpha$*
- *an isomorphism $\psi_{\text{inv}} : M_{\text{inv}} \to M^+_{\text{inv}}$ with $I = t^{-1} \circ I^+ \circ \psi_{\text{inv}}$.*

Proof. Equation (45) and the universal property of the equaliser $\iota^+_{\mathcal{F}}$ yield a unique morphism $y : M^{\text{co}H}_{\mathcal{F}} \to M^{+coH}_{\mathcal{F}}$ with $\iota^+_{\mathcal{F}} \circ y = \epsilon_\alpha \circ \iota_{\mathcal{F}}$. The universal property of the equaliser $\iota_{\mathcal{F}}$ and (48) provide a unique morphism $y^{-1} : M^{+coH}_{\mathcal{F}} \to M^{\text{co}H}_{\mathcal{F}}$ with $\iota_{\mathcal{F}} \circ y^{-1} = \eta_\alpha \circ \iota^+_{\mathcal{F}}$. The two morphisms are inverse to each other, as $\iota_{\mathcal{F}}$, $\iota^+_{\mathcal{F}}$ are monomorphisms and

$$\iota^+_{\mathcal{F}} \circ y \circ y^{-1} = \epsilon_\alpha \circ \iota_{\mathcal{F}} \circ y^{-1} = \epsilon_\alpha \circ \eta_\alpha \circ \iota^+_{\mathcal{F}} = \iota^+_{\mathcal{F}}$$
$$\iota_{\mathcal{F}} \circ y^{-1} \circ y = \eta_\alpha \circ \iota^+_{\mathcal{F}} \circ y = \eta_\alpha \circ \epsilon_\alpha \circ \iota_{\mathcal{F}} \overset{(46)}{=} \iota_{\mathcal{F}}.$$

Similarly, equation (44) and the universal property of the coequaliser $\pi_{\mathcal{V}}$ yield a unique morphism $t : M^H_{\mathcal{V}} \to M^{+H}_{\mathcal{V}}$ with $t \circ \pi_{\mathcal{V}} = \pi^+_{\mathcal{V}} \circ \epsilon_\alpha$ and equation (47) with the universal property of the coequaliser $\pi^+_{\mathcal{V}}$ a unique morphism $t^{-1} : M^{+H}_{\mathcal{V}} \to M^H_{\mathcal{V}}$ with $t^{-1} \circ \pi^+_{\mathcal{V}} = \pi_{\mathcal{V}} \circ \eta_\alpha$. The morphism t^{-1} is a right inverse of t, since $\pi^+_{\mathcal{V}}$ is an epimorphism and

$$t \circ t^{-1} \circ \pi^+_{\mathcal{V}} = t \circ \pi_{\mathcal{V}} \circ \eta_\alpha = \pi^+_{\mathcal{V}} \circ \epsilon_\alpha \circ \eta_\alpha = \pi^+_{\mathcal{V}}.$$

We have constructed commuting diagrams

$$\begin{array}{ccccc} M^{\text{co}H}_{\mathcal{F}} & \xrightarrow{\iota_{\mathcal{F}}} & M & \xrightarrow{\pi_{\mathcal{V}}} & M^H_{\mathcal{V}} \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle \epsilon_\alpha} & & \downarrow{\scriptstyle t} \\ M^{+coH}_{\mathcal{F}} & \xrightarrow[\iota^+_{\mathcal{F}}]{} & M^+ & \xrightarrow[\pi^+_{\mathcal{V}}]{} & M^{+H}_{\mathcal{V}}. \end{array} \qquad \begin{array}{ccccc} M^{+coH}_{\mathcal{F}} & \xrightarrow{\iota^+_{\mathcal{F}}} & M^+ & \xrightarrow{\pi^+_{\mathcal{V}}} & M^{+H}_{\mathcal{V}} \\ \downarrow{\scriptstyle y^{-1}} & & \downarrow{\scriptstyle \eta_\alpha} & & \downarrow{\scriptstyle t^{-1}} \\ M^{\text{co}H}_{\mathcal{F}} & \xrightarrow[\iota_{\mathcal{F}}]{} & M & \xrightarrow[\pi_{\mathcal{V}}]{} & M^H_{\mathcal{V}} \end{array}$$

The construction of $\psi_{\text{inv}} : M_{\text{inv}} \to M^+_{\text{inv}}$ and its inverse is as in the proof of Proposition 5.17 with the monomorphisms $j = t^{-1} \circ I^+$, $j^+ = t \circ I$ and epimorphisms $q = P^+ \circ y$, $q^+ = P \circ y^{-1}$. ∎

5.23. COROLLARY. *Deletions of isolated loops induce isomorphisms between the protected objects.*

5.24. PROTECTED OBJECTS. Combining the results from Section 5.1 to 5.19 one has that ciliated ribbon graphs related by moving cilia, edge reversals, edge contractions and deletions of isolated loops have isomorphic protected objects. As these are sufficient to relate any connected ribbon graph to the standard graph from (15), the protected object of a ciliated ribbon graph is determined up to isomorphisms by the genera of the connected components of the associated surface.

5.25. THEOREM. *The isomorphism class of the protected object for an involutive Hopf monoid H and a ciliated ribbon graph Γ depends only on H and the homeomorphism class of the surface for Γ.*

PROOF. By Lemma 5.5 the invariants, coinvariants and hence the protected object of a ciliated ribbon graph are independent of the choice of the cilia. By Proposition 3.5 every ribbon graph can be transformed into a disjoint union of standard graphs by edge reversals, edge contractions, edge slides and removing isolated loops. In each step the cilia can be arranged in such a way that no edge ends slide over cilia. By Corollaries 5.4, 5.10, 5.18 and 5.23 these graph transformations induce isomorphisms between the protected objects. ∎

As the protected object is a topological invariant, one can use any embedded graph whose complement is a disjoint union of discs to compute the protected object. For a sphere, the simplest such graph consists of a single isolated vertex. This is associated with the trivial H-(co)module structure on e given by the (co)unit of H and yields the tensor unit as protected object.

5.26. EXAMPLE. The protected object for a sphere S^2 is the tensor unit of H: $\mathcal{M}_{\text{inv}} = e$.

We now focus on oriented surfaces Σ of genus $g \geq 1$ and use the standard graphs (15) to determine their protected objects. The associated module and comodule structures are given in Example 4.6 and form a Yetter–Drinfeld module.

5.27. EXAMPLE. For a group H as a Hopf monoid in $\mathcal{C} = \text{Set}$ the coinvariants are the set of group homomorphisms from $\pi_1(\Sigma)$ to H

$$M^{\text{co}H} = \{(a_1, b_1, \ldots, a_g, b_g) \in H^{\times 2g} : [b_g^{-1}, a_g] \cdot \ldots \cdot [b_1^{-1}, a_1] = 1\} \cong \text{Hom}(\pi_1(\Sigma), H). \tag{56}$$

The invariants are the set of orbits for the conjugation action $\triangleright$ from (22) on $H^{\times 2g}$, and the protected object is the *representation variety* or *moduli space of flat H-bundles* $M_{\text{inv}} \cong \text{Hom}(\pi_1(\Sigma), H)/H$.

5.28. EXAMPLE. For a topological group H as a Hopf monoid in $\mathcal{C} = \mathrm{Top}$ the protected object is $M_{\mathrm{inv}} \cong \mathrm{Hom}(\pi_1(\Sigma), H)/H$ as a set by Example 2.16. It is equipped with the quotient topology induced by the canonical surjection $\pi : \mathrm{Hom}(\pi_1(\Sigma), H) \to \mathrm{Hom}(\pi_1(\Sigma), H)/H$ and the compact-open topology on $\mathrm{Hom}_{\mathrm{Top}}(\pi_1(\Sigma), H)$ for the discrete topology on $\pi_1(\Sigma)$.

5.29. EXAMPLE. For a Hopf monoid H in $\mathcal{C} = G - \mathrm{Set} = \mathrm{Set}^{\mathrm{B}G}$ the coinvariants for the comodule structure δ from Example 4.6 are the set (56) with the diagonal G-action. The invariants for the module structure $\rhd$ are the associated orbit space. By Example 2.16, 2. the protected object is the representation variety $M_{\mathrm{inv}} \cong \mathrm{Hom}(\pi_1(\Sigma), H)/H$ with the induced $G - \mathrm{Set}$ structure.

5.30. EXAMPLE. Let k be a commutative ring, $\mathcal{C} = k - \mathrm{Mod}$ and G a finite group.

For the group algebra $H = k[G]$ as a Hopf monoid in $\mathcal{C}$ and the standard graph in (15) one has $M = k[G]^{2g} \cong k[G^{\times 2g}]$. The Yetter–Drinfeld module structure of M is given by (22) on a basis. The coinvariants and invariants are

$$\begin{aligned} M^{\mathrm{co}H} &= \langle \{(a_1, b_1, \ldots, a_g, b_g) \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = 1\} \rangle_k \cong \langle \mathrm{Hom}(\pi_1(\Sigma), G) \rangle_k \\ M^H &= k[G^{\times 2g}]/\langle \{(a_1, \ldots, b_g) - (ha_1h^{-1}, \ldots, hb_gh^{-1}) \mid a_1, b_1, \ldots, a_g, b_g, h \in G\} \rangle, \end{aligned}$$

and the protected object is the free k-module generated by the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$

$$M_{\mathrm{inv}} = \langle \mathrm{Hom}(\pi_1(\Sigma), G)/G \rangle_k. \tag{57}$$

For the dual Hopf monoid $H = k[G]^* = \mathrm{Map}(G, k)$ of maps from G to k with Hopf monoid structure

$$\delta_g \cdot \delta_h = \delta_g(h)\delta_g, \quad 1 = \sum_{g \in G} \delta_g, \quad \Delta(\delta_g) = \sum_{x,y \in G, xy = g} \delta_x \otimes \delta_y, \quad \epsilon(\delta_g) = \delta_g(e), \quad S(\delta_g) = \delta_{g^{-1}} \tag{58}$$

one has $M = \mathrm{Map}(G, k)^{\otimes 2g} \cong \mathrm{Map}(G^{\times 2g}, k)$ with the Yetter–Drinfeld module structure

$$\delta_h \rhd (\delta_{a_1} \otimes \delta_{b_1} \otimes \ldots \otimes \delta_{a_g} \otimes \delta_{b_g}) = \delta_h([b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1])\, \delta_{a_1} \otimes \delta_{b_1} \otimes \ldots \otimes \delta_{a_g} \otimes \delta_{b_g} \tag{59}$$

$$\delta(\delta_{a_1} \otimes \delta_{b_1} \otimes \ldots \otimes \delta_{a_g} \otimes \delta_{b_g}) = \Sigma_{h \in G} \delta_{h^{-1}} \otimes \delta_{ha_1h^{-1}} \otimes \delta_{hb_1h^{-1}} \otimes \ldots \otimes \delta_{ha_gh^{-1}} \otimes \delta_{hb_gh^{-1}}$$

computed from (21) and (58). It follows that the coinvariants and invariants are given by

$$\begin{aligned} M^{\mathrm{co}H} &= \mathrm{Map}(G^{\times 2g}, k)^G \\ M^H &= \{f : G^{\times 2g} \to k \mid \mathrm{supp}(f) \subseteq \{(a_1, \ldots, b_g) \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = 1\}\}, \end{aligned} \tag{60}$$

and the protected object is the set of functions

$$M_{\mathrm{inv}} = \mathrm{Map}(\mathrm{Hom}(\pi_1(\Sigma), G)/G, k). \tag{61}$$

Example 5.30 shows that the protected object in this article indeed generalises the protected space of Kitaev's quantum double models, If one sets $k = \mathbb{C}$ in Example 5.30 one obtains precisely the protected space for Kitaev's quantum double model for the group algebra $\mathbb{C}[G]$ and its dual, see [Ki, Sec. 4]. This coincides with the vector space a Turaev–Viro TQFT for the category Vec_G of G-graded vector spaces assigns to the surface. However, Example 5.30 also yields an analogous result for any commutative ring k, for which the usual quantum double models are not defined.

6. Protected objects in SSet

In this section, we investigate protected objects for group objects in the category SSet. We denote by Δ the simplex category with finite ordinals $[n] = \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}_0$ as objects and weakly monotonic maps $\alpha : [m] \to [n]$ as morphisms from $[m]$ to $[n]$.

Objects in $\mathrm{SSet} = \mathrm{Set}^{\Delta^{\mathrm{op}}}$ are simplicial sets, functors $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ that are specified by sets X_n, face maps $d_i : X_{n+1} \to X_n$ and degeneracies $s_i : X_n \to X_{n+1}$ for $n \in \mathbb{N}_0$ and $i \in \{0, \ldots, n\}$ that satisfy the simplicial relations

$$\begin{aligned} & d_j \circ d_i = d_i \circ d_{j+1} \text{ if } i \leq j, \quad s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j, \\ & d_i \circ s_j = s_{j-1} \circ d_i \text{ if } i < j, \quad d_i \circ s_j = \mathrm{id} \text{ if } i \in \{j, j+1\}, \quad d_i \circ s_j = s_j \circ d_{i-1} \text{ if } i > j+1. \end{aligned} \tag{62}$$

Morphisms in SSet are simplicial maps, natural transformations $f : X \to Y$ specified by component maps $f_n : X_n \to Y_n$ satisfying $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$ for $n \in \mathbb{N}_0$ and admissible i. The category SSet is cartesian monoidal with the objectwise product induced by the product in Set.

Unpacking the definition of a group object in a cartesian monoidal category from Example 2.4 yields

6.1. Definition.

1. *A group object in* SSet *is a* **simplicial group***: a simplicial set* $H : \Delta^{\mathrm{op}} \to \mathrm{Set}$ *with group structures on the sets* H_n *such that all face maps and degeneracies are group homomorphisms.*
2. *A morphism of group objects in* SSet *is a* **morphism of simplicial groups***: a simplicial map* $f : H \to H'$ *such that all maps* $f_n : H_n \to H'_n$ *are group homomorphisms.*

For examples of simplicial groups, see Section 7.6, in particular Corollary 7.11 and Example 7.12. Modules, comodules and Yetter–Drinfeld modules over simplicial groups are given by Example 2.11.

6.2. Lemma. *Let* $H : \Delta^{\mathrm{op}} \to \mathrm{Set}$ *be a simplicial group.*

1. *A* **module** *over* H *is a simplicial set* $M : \Delta^{\mathrm{op}} \to \mathrm{Set}$ *together with a collection of* H_n*-actions* $\rhd_n : H_n \times M_n \to M_n$ *that define a simplicial map* $\rhd : H \times M \to M$.
2. *A* **comodule** *over* H *is a simplicial set* $M : \Delta^{\mathrm{op}} \to \mathrm{Set}$ *with a simplicial map* $F : M \to H$.

3. *If $(M, \rhd)$ is a module and (M, F) a comodule over H, then $(M, \rhd, F)$ is a* **Yetter–Drinfeld module** *over H iff $F_n(g \rhd_n m) = g \cdot F_n(m) \cdot g^{-1}$ for all $m \in M_n$, $g \in H_n$ and $n \in \mathbb{N}_0$.*

As (co)limits in SSet are objectwise, see for instance Riehl [R, Prop. 3.3.9] or Leinster [L, Th. 6.2.5], (co)invariants of a (co)module over a group object in SSet are obtained from (co)equalisers in Set. It is also straightforward to compute the biinvariants of a Yetter–Drinfeld module.

6.3. Proposition. *Let H be a simplicial group.*

1. *The coinvariants $\mathcal{M}^{\mathrm{co}H}$ of a H-comodule M defined by a simplicial map $F : M \to H$ are given by the sets $M_n^{\mathrm{co}H} = \{m \in M_n \mid F_n(m) = e\}$ and the induced face maps and degeneracies.*
2. *The invariants M^H of a H-module $(M, \rhd)$ are given by the sets $M_n^H = \{H_n \rhd_n m \mid m \in M_n\}$ and the induced face maps and degeneracies.*
3. *The biinvariants M_{inv} of a Yetter–Drinfeld module $(M, \rhd, F)$ over H are given by the sets $(M_{\mathrm{inv}})_n = \{H_n \rhd_n m \mid m \in M_n, F_n(m) = e\}$ and the induced face maps and degeneracies.*

Proof. 1. The coinvariant object of a H-comodule (M, F) is the equaliser of the simplicial maps $\delta = F \times \mathrm{id} : M \to H \times M$ and $\eta \times \mathrm{id} : M \to H \times M$. As limits in SSet are objectwise, this is the simplicial set $M^{\mathrm{co}H} : \Delta^{\mathrm{op}} \to$ Set that assigns to an ordinal $[n]$ the equaliser in Set of the maps $F_n \times \mathrm{id} : M_n \to H_n \times M_n$ and $\eta_n \times \mathrm{id} : M_n \to H_n \times M_n$, which is $M_n^{\mathrm{co}H} = \{m \in M_n \mid F_n(m) = e\}$. The face maps and degeneracies are induced by the ones of M, and the simplicial map $\iota : M^{\mathrm{co}H} \to M$ is given by the maps $\iota_n : M_n^{\mathrm{co}H} \to M_n$, $m \mapsto m$.

2. Analogously to 1., the invariant object of $(M, \rhd)$ is the simplicial set $M^H : \Delta^{\mathrm{op}} \to$ Set that assigns to the ordinal $[n]$ the coequaliser in Set of the maps $\rhd_n : H_n \times M_n \to M_n$ and $\epsilon_n \times \mathrm{id} : H_n \times M_n \to M_n$. This is the set $M_n^H = M_n/\sim_n$ with $m \sim_n m'$ iff there is a $g \in H_n$ with $m' = g \rhd_n m$. The simplicial map $\pi : M \to M^H$ is given by the maps $\pi_n : M_n \to M_n^H$, $m \mapsto H_n \rhd_n m$.

3. The simplicial maps $I : M_{\mathrm{inv}} \to M^H$ and $P : M^{\mathrm{co}H} \to M_{\mathrm{inv}}$ with $\pi \circ \iota = I \circ P$ that characterise M_{inv} with $(M_{\mathrm{inv}})_n = \{H_n \rhd_n m \mid m \in M_n^{\mathrm{co}H}\}$ as the image of $\pi \circ \iota$ are given by

$$I_n : (M_{\mathrm{inv}})_n \to M_n^H, H_n \rhd_n m \mapsto H_n \rhd_n m, \qquad P_n : M_n^{\mathrm{co}H} \to (M_{\mathrm{inv}})_n, m \mapsto H_n \rhd_n m.$$

As monomorphisms and epimorphisms in SSet are those simplicial maps whose component morphisms are injective and surjective, see for instance [L, Ex. 6.2.20], it follows directly that I is a monomorphism and P an epimorphism in SSet. Every pair (J, Q) of a monomorphism $J : X \to M^H$ and morphism $Q : M^{\mathrm{co}H} \to X$ in SSet with $J \circ Q = \pi \circ \iota$ defines injective maps $J_n : X_n \to M_n^H$ and thus identifies $Q(M_n^{\mathrm{co}H})$ with a subset of M_n^H. As J_n is a monomorphism and due to the identity $J_n \circ Q_n(g \rhd_n m) = \pi_n \circ \iota_n(g \rhd_n m) = \pi_n \circ \iota_n(m) = J_n \circ Q_n(m)$, we have $Q_n(g \rhd_n m) = Q_n(m)$ for all $m \in M_n^{\mathrm{co}H}$ and $g \in H_n$.

The maps $V_n : (M_{\text{inv}})_n \to X_n$, $H_n \rhd_n m \mapsto Q_n(m)$ define a simplicial map $V : M_{\text{inv}} \to X$ with $I = J \circ V$. ■

We now determine the coinvariants, invariants and the protected objects for Kitaev models on oriented surfaces Σ of genus $g \geq 1$ and for a simplicial group H as a Hopf monoid in SSet.

6.4. Proposition. *Let H be a simplicial group and Σ an oriented surface of genus $g \geq 1$. The associated protected object is the simplicial set $X : \Delta^{\text{op}} \to \text{Set}$ with $X_n = \text{Hom}(\pi_1(\Sigma), H_n)/H_n$, where the quotient is with respect to conjugation by H_n, and face maps and degeneracies given by*

$$d_i : X_n \to X_{n-1},\ [\rho] \mapsto [d_i \circ \rho], \qquad s_i : X_n \to X_{n+1},\ [\rho] \mapsto [s_i \circ \rho].$$

Proof. By Theorem 5.25 the protected object of Σ can be computed from the standard graph in (15). This yields a Yetter–Drinfeld module $(M, \rhd, F)$ over H given by formula (22) in Example 4.6. Hence, we have $M_n = H_n^{\times 2g}$ for all $n \in \mathbb{N}_0$ with the face maps and degeneracies of H applied to each component simultaneously. The Yetter–Drinfeld module structure is given by

$$F_n : H_n^{\times 2g} \to H_n,\ (a_1, b_1, \ldots, a_g, b_g) \mapsto [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1]$$
$$\rhd_n : H_n \times H_n^{\times 2g} \to H_n^{\times 2g},\ (h, a_1, b_1, \ldots, a_g, b_g) \mapsto (ha_1h^{-1}, hb_1h^{-1}, \ldots, ha_gh^{-1}, hb_gh^{-1}).$$

By Proposition 6.3 the associated protected object is the simplicial set M_{inv} with

$$\begin{aligned}(M_{\text{inv}})_n &= \{H_n \rhd_n (a_1, b_1, \ldots, a_g, b_g) \in H_n^{\times 2g} \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = e\} \\ &\cong \text{Hom}(\pi_1(\Sigma), H_n)/H_n.\end{aligned}$$

Face maps and degeneracies are given by post-composing group homomorphisms $\rho : \pi_1(\Sigma) \to H_n$ with the face maps $d_i : H_n \to H_{n-1}$ and degeneracies $s_i : H_n \to H_{n+1}$. ■

7. Protected objects in Cat

7.1. Crossed modules as group objects in Cat. We consider the category Cat of small categories and functors between them as a cartesian monoidal category with terminal object $\{\cdot\}$. For a finite product $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ of small categories, we denote by $\pi_i : \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{C}_i$ the associated projection functors. For a small category $\mathcal{C}$ we denote by $\text{Ob}(\mathcal{C})$ the set of objects and by $\mathcal{C}^{(1)} = \bigcup_{X,Y \in \text{Ob}(\mathcal{C})} \text{Hom}_{\mathcal{C}}(X, Y)$ the set of all morphisms in $\mathcal{C}$.

7.2. Definition.

1. *A **group object** in Cat is a small category H together with functors $m : H \times H \to H$, $\eta : \{\cdot\} \to H$ and $I : H \to H$ such that the diagrams (7) commute.*
2. *A **morphism** $F : (H, m, \eta, I) \to (H', m', \eta', I')$ **of group objects** is a functor $F : H \to H'$ that satisfies (8).*

We denote by $\mathcal{G}(\mathrm{Cat})$ the category of group objects and morphisms of group objects in Cat and write $e := \eta(\cdot)$, $f^{-1} = I(f)$, $g \cdot f = m(g, f)$ and likewise for multiple products.

Brown and Spencer [BS] showed that group objects in Cat correspond to crossed modules. We summarise this correspondence for the convenience of the reader.

7.3. Definition. *A* ***crossed module*** *is a quadruple $(B, A, \blacktriangleright, \partial)$ of groups A and B, a group homomorphism $\partial : A \to B$ and a group action $\blacktriangleright: B \times A \to A$ by automorphisms that satisfy the Peiffer identities*

$$\partial(b \blacktriangleright a) = b\partial(a)b^{-1}, \qquad \partial(a) \blacktriangleright a' = aa'a^{-1} \qquad \forall a, a' \in A, b \in B. \tag{63}$$

A ***morphism of crossed modules*** *$f = (f_1, f_2) : (B, A, \blacktriangleright, \partial) \to (B', A', \blacktriangleright', \partial')$ is a pair of group homomorphisms $f_1 : B \to B'$, $f_2 : A \to A'$ such that*

$$\partial' \circ f_2 = f_1 \circ \partial, \qquad \blacktriangleright' \circ (f_1 \times f_2) = f_2 \circ \blacktriangleright .$$

We denote by $\mathcal{CM}$ the category of crossed modules and morphisms between them.

7.4. Example.

1. A normal subgroup $A \subset B$ defines a crossed module with the inclusion $\partial : A \to B$, $a \mapsto a$ and the conjugation action $\blacktriangleright: B \times A \to A$, $b \blacktriangleright a = bab^{-1}$.
2. Any crossed module $(A, B, \blacktriangleright, \partial)$ yields a crossed module $(A/\ker\partial, B, \blacktriangleright', \partial')$ with injective ∂'. This identifies $A/\ker\partial$ with a normal subgroup of B and hence yields 1.
3. Any group action $\blacktriangleright: B \times A \to A$ by automorphisms of an abelian group A yields a crossed module with $\partial \equiv e_B$.
4. Any group A defines a crossed module with $B = \mathrm{Aut}(A)$, $\blacktriangleright: \mathrm{Aut}(A) \times A \to A$, $\phi \blacktriangleright a = \phi(a)$ and $\partial : A \to \mathrm{Aut}(A)$, $g \mapsto C_g$, where $C_g(x) = gxg^{-1}$.
5. Every extension of a group G by a group X

$$1 \longrightarrow X \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G \longrightarrow 1$$

   defines a crossed module with $A := X$, $B := E$, $\partial := \iota$ and $b \blacktriangleright a := \iota^{-1}(b\iota(a)b^{-1})$ for all $b \in B$, $a \in A$. The group extension is central iff $\iota(X) \subset Z(E)$, which is equivalent to $\blacktriangleright$ trivial.
6. Conversely, any crossed module $(B, A, \blacktriangleright, \partial)$ gives an extension of $B/\partial(A)$ by $A/\ker(\partial)$:

$$1 \longrightarrow A/\ker(\partial) \overset{\partial'}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} B/\partial(A) \longrightarrow 1$$

   with $\partial'(a\ker(\partial)) := \partial(a)$. For crossed modules with surjective ∂, the group A is also an extension of B by $\ker(\partial)$:

$$1 \longrightarrow \ker(\partial) \overset{\iota}{\hookrightarrow} A \overset{\partial}{\twoheadrightarrow} B \longrightarrow 1.$$

7.5. Theorem. [BS, Th. 1] *The following functors* $\triangle : \mathcal{G}(\mathrm{Cat}) \to \mathcal{CM}$ *and* $\triangledown : \mathcal{CM} \to \mathcal{G}(\mathrm{Cat})$ *form an equivalence of categories.*

The functor $\triangle$ *sends a group object* (H, m, η, I) *to the crossed module* $(B, A, \blacktriangleright, \partial)$ *with*

- $A = \mathrm{Cost}_e := \bigcup_{X \in \mathrm{Ob}(H)} \mathrm{Hom}_H(X, e)$ *with multiplication* $m_A : A \times A \to A$, $(a, a') \mapsto m(a, a')$,
- $B = \mathrm{Ob}(H)$ *with multiplication* $m_B : B \times B \to B$, $(b, b') \mapsto m(b, b')$,
- $\partial : A \to B$, $(f : X \to e) \mapsto X$,
- $\blacktriangleright : B \times A \to A$, $b \blacktriangleright a = 1_b \cdot a \cdot 1_b^{-1}$,

and a morphism of group objects $F : (H, m, \eta, I) \to (H', m', \eta', I')$ *to the pair of group homomorphisms* $f_1 := F : \mathrm{Ob}(H) \to \mathrm{Ob}(H')$ *and* $f_2 := F : \mathrm{Cost}_e \to \mathrm{Cost}'_{e'}$.

The functor $\triangledown$ *sends a crossed module* $(B, A, \blacktriangleright, \partial)$ *to the group object* (H, m, η, I) *with*

- $\mathrm{Ob}(H) = B$,
- $\mathrm{Hom}_H(b, b') = \{(a, b) \in A \times B : \partial(a)b = b'\}$ *with composition* $(a', \partial(a)b) \circ (a, b) = (a'a, b)$,
- $m : H \times H \to H$ *with* $m(b, b') = bb'$ *and* $m((a', b'), (a, b)) = (a'(b' \blacktriangleright a), b'b)$,
- $\eta : \{\cdot\} \to H$, $\eta(\cdot) = e_B$, $\eta(1.) = (e_A, e_B)$,
- $I : H \to H$, $I(b) = b^{-1}$, $I((a, b)) = (b^{-1} \blacktriangleright a^{-1}, b^{-1})$,

and a morphism $(f_1, f_2) : (B, A, \blacktriangleright, \partial) \to (B', A', \blacktriangleright', \partial')$ *of crossed modules to the functor* $F : H \to H'$ *with* $F(b) = f_1(b)$ *for all* $b \in B = \mathrm{Ob}(H)$ *and* $F((a, b)) = (f_2(a), f_1(b))$.

By Theorem 7.5 the *group structure* on the set $H^{(1)}$ of morphisms of a group object H is the semidirect product $A \rtimes B$ for the group action $\blacktriangleright : B \times A \to A$. As a *category* H is the action groupoid for the group action $\blacktriangleright' : A \times B \to B$, $a \blacktriangleright' b = \partial(a)b$.

7.6. Equalisers and coequalisers in Cat. To determine the coinvariants, invariants and the protected object for a group object in Cat, we require equalisers, coequalisers and images in Cat. It is well known that Cat is complete and cocomplete, see for instance [R, Prop. 3.5.6, Cor. 4.5.16]. The following result on equalisers is standard, see for example Schubert [Sch, Sec. 7.2].

7.7. Lemma. *The equaliser of two functors* $F, K : \mathcal{C} \to \mathcal{D}$ *between small categories is the subcategory* $\mathcal{E} \subset \mathcal{C}$ *with*

- $\mathrm{Ob}(\mathcal{E}) = \{C \in \mathrm{Ob}(\mathcal{C}) \mid F(C) = K(C)\}$,
- $\mathrm{Hom}_{\mathcal{E}}(C, C') = \{f \in \mathrm{Hom}_{\mathcal{C}}(C, C') \mid F(f) = K(f)\}$.

To describe coequalisers in Cat we use that Cat is a reflective subcategory of SSet with the inclusion given by the nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$, which is full and faithful. Its left adjoint is the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$, and the composite $hN : \mathrm{Cat} \to \mathrm{Cat}$ is naturally isomorphic to the identity functor via the counit of the adjunction, see Riehl [R, Ex. 4.5.14 (vi)] or Lurie [Lu, Sec. 1.2]. As a left adjoint, h preserves colimits. This allows one to compute colimits in Cat by applying the homotopy functor h to the associated colimits in SSet, see for instance [R, Prop. 4.5.15].

7.8. Lemma. *The coequaliser of two functors* $F, K : \mathcal{C} \to \mathcal{D}$ *between small categories is the functor* $h(\pi) : hN(\mathcal{D}) \to h(X)$*, where* $\pi : N(\mathcal{D}) \to X$ *is the coequaliser of* $N(F), N(K)$ *in* SSet*.*

To compute such coequalisers, we require an explicit description of the nerve and the homotopy functor. We summarise the details from [R, Ex. 4.5.14 (vi)] and [Lu, Sec. 1.2]. For $n \in \mathbb{N}_0$ we denote by $[n]$ the ordinals in Δ as well as the associated categories with objects $0, 1, \ldots, n$ and a single morphism from i to j if $i \leq j$. As every weakly monotonic map $\alpha : [m] \to [n]$ defines a functor $\alpha : [m] \to [n]$, this defines an embedding $\iota : \Delta \to \mathrm{Cat}$.

7.9. Definition. *The* **nerve** $N : \mathrm{Cat} \to \mathrm{SSet}$ *is the functor that sends a small category* $\mathcal{C}$ *to the simplicial set* $N(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Set}$ *with*

- $N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$*,*
- $N(\mathcal{C})(\alpha) : N(\mathcal{C})_n \to N(\mathcal{C})_m$*,* $F \mapsto F \circ \alpha$ *for every weakly monotonic* $\alpha : [m] \to [n]$*,*

and a functor $F : \mathcal{C} \to \mathcal{D}$ *to the simplicial map* $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ *that post-composes with* F*.*

By definition, $N(\mathcal{C})_0 = \mathrm{Ob}\,\mathcal{C}$ and $N(\mathcal{C})_n$ is the set of sequences $(f_1, \ldots, f_n) : C_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} C_n$ of composable morphisms in $\mathcal{C}$ for $n \in \mathbb{N}$. The simplicial set structure is given by the face maps $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ and degeneracies $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ for $i \in \{0, ..., n\}$. The face maps act on a sequence $(f_1, \ldots, f_n)$ by removing f_1 and f_n for $i = 0$ and $i = n$, respectively, and by replacing $(\ldots, f_i, f_{i+1}, \ldots)$ with $(\ldots, f_{i+1} \circ f_i, \ldots)$ for $1 \leq i \leq n-1$. For $n = 1$ and $f_1 : C_0 \to C_1$ one has $d_0(f_1) = C_1$ and $d_1(f_1) = C_0$. The degeneracies act on $(f_1, \ldots, f_n)$ by inserting the identity morphism 1_{C_i}. In particular, for $n = 0$ one has $s_0(C) = 1_C$ for every $C \in \mathrm{Ob}\,\mathcal{C}$. The simplicial map $N(F)$ for a functor $F : \mathcal{C} \to \mathcal{D}$ applies F to all morphisms in $(f_1, \ldots, f_n)$.

The left adjoint of the nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ is the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$. It is the left Kan extension along the Yoneda embedding $y : \Delta \to \mathrm{SSet}$ of the embedding functor $\iota : \Delta \to \mathrm{Cat}$. Concretely, it is given as follows.

7.10. Definition. *The* **homotopy functor** $h : \mathrm{SSet} \to \mathrm{Cat}$ *sends a simplicial set* X *to the category* hX *with* $\mathrm{Ob}\, hX = X_0$*, generating morphisms* $\sigma : d_1(\sigma) \to d_0(\sigma)$ *for* $\sigma \in X_1$ *and relations*

$$s_0(x) = 1_x \text{ for } x \in X_0, \qquad d_1(\sigma) = d_0(\sigma) \circ d_2(\sigma) \text{ for } \sigma \in X_2. \tag{64}$$

It sends a simplicial map $f : X \to Y$ *to the functor* $hf : hX \to hY$ *given by* f *on the generators.*

The simplicial relations imply that for elements of X_2 that are in the image of a degeneracy map, the second relation in (64) is satisfied trivially. In this case one of the two morphisms on the right is an identity and the other coincides with the morphism on the left. Only non-degenerate elements of X_2 give rise to non-trivial relations in hX.

In general, morphisms in the homotopy category of a simplicial set X are finite sequences of composable elements of X_1. However, if the simplicial set X is an ∞-category, which is always the case if $X = N(\mathcal{C})$ for some category $\mathcal{C}$, every morphism in hX is represented by a single element in X_1, see for instance [Lu, Sec. 1.2.5]. Most of the simplicial sets considered in the following are even Kan complexes, as they are nerves of groupoids.

As a right adjoint, the nerve preserves limits, and as a left adjoint, the homotopy functor preserves colimits. It follows directly from its definition that the nerve also preserves coproducts, and the homotopy functor preserves finite products, see for instance Joyal [Jo, Prop. 1.3]. This implies with Examples 2.7 and 2.12

7.11. Corollary. *The nerve* $N : \mathrm{Cat} \to \mathrm{SSet}$ *and the homotopy functor* $h : \mathrm{SSet} \to \mathrm{Cat}$ *are symmetric monoidal with respect to the cartesian monoidal category structures of* Cat *and* SSet. *In particular:*

1. *The nerve of a crossed module is a simplicial group.*
2. *The homotopy category of a simplicial group is a crossed module.*
3. *The nerve of a (co)module over a crossed module is a (co)module over its nerve.*
4. *The homotopy category of a (co)module over a simplicial group is a (co)module over its homotopy category.*

Concretely, the nerve of a crossed module $(B, A, \blacktriangleright, \partial)$ is the simplicial group H with $H_n = A^{\times n} \times B$ for $n \in \mathbb{N}_0$ with group multiplication

$$(a_1, ..., a_n, b) \cdot (a'_1, ..., a'_n, b') = (a_1(b \blacktriangleright a'_1), a_2(\partial(a_1)b \blacktriangleright a'_2), ..., a_n(\partial(a_{n-1} \cdots a_1)b \blacktriangleright a'_n), bb') \tag{65}$$

and face maps and degeneracies

$$d_i : H_n \to H_{n-1}, \quad (a_1, ..., a_n, b) \mapsto \begin{cases} (a_2, ..., a_n, \partial(a_1)b) & i = 0 \\ (a_1, ..., a_{i+1}a_i, ..., a_n, b) & 1 \leq i \leq n-1 \\ (a_1, ..., a_{n-1}, b) & i = n \end{cases} \tag{66}$$

$$s_i : H_n \to H_{n+1}, \quad (a_1, ..., a_n, b) \mapsto (a_1, \ldots, a_i, 1, a_{i+1}, ..., a_n, b) \quad 0 \leq i \leq n.$$

The relation between crossed modules and simplicial groups is well-known, see for instance Conduché [Co].

7.12. Example.

1. A group action $\blacktriangleright: B \times A \to A$ by automorphisms on an abelian group A yields a simplicial group with $H_n = A^{\times n} \rtimes' B$, where B acts diagonally via $\blacktriangleright$, and with the face maps and degeneracies (66) for $\partial \equiv 1$.
2. Every injective group homomorphism $\partial : A \to B$ from an abelian group A into the centre of a group B yields a simplicial group, where $H_n = A^{\times n} \times B$ is the direct product, and the face maps and degeneracies are given by (66).
3. Every abelian group A is a simplicial group with $H_n = A^{\times n}$, the group multiplication of $A^{\times n}$, the face maps and degeneracies (66) for $B = \{e\}$ and $\partial \equiv 1$.
4. Any normal subgroup $A \subset B$ determines a simplicial group with $H_n = A^{\times n} \times B$ and group multiplication (65), face maps and degeneracies (66), where $\partial : A \to B$ is the inclusion and $\blacktriangleright: B \times A \to A$ the conjugation action.

7.13. (Co)invariants of (co)modules over group objects in Cat. The coinvariants of a comodule $(\mathcal{M},\delta)$ over a group object (H,m,η,I) in Cat are given as the equaliser of $\delta=(F\times 1_{\mathcal{M}})\circ\Delta:\mathcal{M}\to H\times\mathcal{M}$ and $\eta\times 1_{\mathcal{M}}:\mathcal{M}\to H\times\mathcal{M}$. This is the subcategory on which δ and $\eta\times 1_{\mathcal{M}}$ coincide, together with its inclusion functor, see Lemma 7.7. In terms of the associated functor $F:\mathcal{M}\to H$ from Example 2.11 we have

7.14. Lemma. *Let $(\mathcal{M},\delta)$ be a comodule over a group object (H,m,η,I) in* Cat*. Then the coinvariants are given by the subcategory $\mathcal{M}^{\mathrm{co}H}\subset\mathcal{M}$ with*

- $\mathrm{Ob}(\mathcal{M}^{\mathrm{co}H})=\{A\in\mathrm{Ob}(\mathcal{M})\mid F(A)=e\}$,
- $\mathrm{Hom}_{\mathcal{M}^{\mathrm{co}H}}(A,A')=\{f\in\mathrm{Hom}_{\mathcal{M}}(A,A')\mid F(f)=1_e\}$,

and the inclusion functor $\iota:\mathcal{M}^{\mathrm{co}H}\to\mathcal{M}$.

The invariants of a module $(\mathcal{M},\rhd)$ over a group object H in Cat are the coequaliser of the functors $\rhd,\pi_2:H\times\mathcal{M}\to\mathcal{M}$. They are computed with Lemma 7.8.

7.15. Proposition. *Let $(\mathcal{M},\rhd)$ be a module over a group object $H=\bigtriangledown(B,A,\blacktriangleright,\partial)$ in* Cat*. Then its invariants are the category $\mathcal{M}^H$, whose*

- *objects are orbits of the B-action on* $\mathrm{Ob}(\mathcal{M})$,
- *morphisms are generated by orbits of the $A\rtimes B$-action on $\mathcal{M}^{(1)}$ subject to the relations $[f_2]\circ[f_1]=[f_2\circ f_1]$ for all $A\rtimes B$-orbits $[f_1]$, $[f_2]$ of composable morphisms f_1,f_2 in $\mathcal{M}$.*

We denote by $\pi:\mathcal{M}\to\mathcal{M}^H$ the projection functor that sends each object of $\mathcal{M}$ to its B-orbit and each morphism in $\mathcal{M}$ to the equivalence class of its $A\rtimes B$-orbit.

Proof. By Corollary 7.11, applying the nerve to the group object H in Cat and to a module $(\mathcal{M},\rhd)$ over H yields a simplicial group $N(H)$ and a module $N(\mathcal{M})$ over $N(H)$ in SSet. By Lemma 7.8 the coequaliser of the morphisms $\rhd,\pi_2:H\times\mathcal{M}\to\mathcal{M}$ in Cat is obtained by applying the homotopy functor to the coequaliser of $\rhd'=N(\rhd),\pi_2'=N(\pi_2):N(H)\times N(\mathcal{M})\to N(\mathcal{M})$ in SSet.

As colimits in SSet are computed objectwise, see for instance [R, Prop. 3.3.9], the coequaliser of $\rhd',\pi_2'$ is the simplicial set $N(\mathcal{M})^H$ with $N(\mathcal{M})^H_n=N(\mathcal{M})_n/\sim_n$, where $\sim_n$ is the equivalence relation on $N(\mathcal{M})_n$ defined by the $N(H)$-action: $m\sim_n m'$ iff there is a $g\in N(H)_n$ with $m'=g\rhd' m$. The face maps and degeneracies of $N(\mathcal{M})^H$ are induced by the ones of $N(\mathcal{M})$.

As $N(H)_0=\mathrm{Ob}\,H=B$ and $N(\mathcal{M})_0=\mathrm{Ob}\,\mathcal{M}$, the elements of $N(\mathcal{M})^H_0$ are the orbits of the B-action on $\mathrm{Ob}\,\mathcal{M}$. As $N(H)_1=H^{(1)}=A\rtimes B$, the set $N(\mathcal{M})^H_1$ contains the orbits of the $A\rtimes B$-action on $\mathcal{M}^{(1)}$. Elements of $N(\mathcal{M})_2$ and $N(H)_2$ are pairs of composable morphisms in $\mathcal{M}$ and H. Thus, the set $N(\mathcal{M})^H_2$ consists of equivalence classes of pairs (f_1,f_2) of composable morphisms in $\mathcal{M}$ with $(f_1,f_2)\sim(f_1',f_2')$ if there are $(a_1,b_1),(a_2,b_2)\in A\rtimes B$ with $\partial(a_1)b_1=b_2$ such that $f_1'=(a_1,b_1)\rhd' f_1$ and $f_2'=(a_2,b_2)\rhd' f_2$.

For any composable pair $(f_1,f_2)\in N(\mathcal{M})_2$, one has $d_0(f_1,f_2)=f_2$, $d_1(f_1,f_2)=f_2\circ f_1$ and $d_2(f_1,f_2)=f_1$. This implies $d_0[(f_1,f_2)]=[f_2]$, $d_1[(f_1,f_2)]=[f_2\circ f_1]$ and $d_2[(f_1,f_2)]=[f_1]$ for their equivalence classes in $N(\mathcal{M})^H_2$ and $N(\mathcal{M})^H_1$.

Applying the homotopy functor from Definition 7.10 thus yields a category $\mathcal{M}^H$ with objects $\mathrm{Ob}\,\mathcal{M}^H = N(\mathcal{M})_0^H = \mathrm{Ob}\,\mathcal{M}/B$. Its generating morphisms are $A \rtimes B$-orbits of morphisms in $\mathcal{M}$, and the second relation in (64) translates into the relation $[f_2] \circ [f_1] = [f_2 \circ f_1]$ for the $A \rtimes B$-orbits of composable pairs (f_1, f_2) of morphisms in $\mathcal{M}$. ∎

We now restrict attention to Yetter–Drinfeld modules $(\mathcal{M}, \rhd, \delta)$ over group objects H in Cat and determine their biinvariants. We denote again by $F : \mathcal{M} \to H$ the functor defined by δ from Example 2.11, by $\iota : \mathcal{M}^{\mathrm{co}H} \to \mathcal{M}$ the inclusion functor from Lemma 7.14 and by $\pi : \mathcal{M} \to \mathcal{M}^H$ the projection functor from Proposition 7.15.

7.16. PROPOSITION. *Let $(\mathcal{M}, \rhd, F)$ be a Yetter–Drinfeld module over a group object H in* Cat. *Then $\mathcal{M}_{\mathrm{inv}}$ is given by*

$$\mathrm{Ob}\mathcal{M}_{\mathrm{inv}} = \{\pi(M) \mid M \in \mathrm{Ob}\mathcal{M} \text{ with } F(M) = e\},$$
$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{inv}}}(\pi(M_1), \pi(M_2)) = \{\pi(f) \mid f \in \mathcal{M}^{(1)} \text{ with } \pi(s(f)) = \pi(M_1), \pi(t(f)) = \pi(M_2), F(f) = 1_e\}.$$

PROOF. 1. We verify that $\mathcal{M}_{\mathrm{inv}}$ is a category. If $F(M) = e$ for an object M in $\mathcal{M}$, then $F(g \rhd M) = g \cdot F(M) \cdot g^{-1} = e$ for all objects g in H by the Yetter–Drinfeld module condition in Example 2.11. Likewise, if f is a morphism in $\mathcal{M}$ with $F(f) = 1_e$, then $F(g \rhd f) = g \cdot F(f) \cdot g^{-1} = 1_e$ for all $g \in H^{(1)}$. This shows that for every object M and morphism f of $\mathcal{M}^{\mathrm{co}H}$ the entire $\mathrm{Ob}\,H$-orbit of M and $H^{(1)}$-orbit of f is contained in $\mathcal{M}^{\mathrm{co}H}$. Any identity morphism on an object $M \in \mathrm{Ob}\,\mathcal{M}^{\mathrm{co}H}$ satisfies $F(1_M) = 1_e$ and hence is contained in $\mathcal{M}^{\mathrm{co}H}$. If (f_1, f_2) is a pair of composable morphisms in $\mathcal{M}^{\mathrm{co}H}$, then $F(f_2 \circ f_1) = F(f_2) \circ F(f_1) = 1_e$ and hence $f_2 \circ f_1 \in \mathcal{M}^{\mathrm{co}H}$ as well.

Suppose now that $f_1 : M_0 \to M_1$ and $f_2 : M_1' \to M_2$ are morphisms in $\mathcal{M}^{\mathrm{co}H}$ such that $\pi(f_1)$ and $\pi(f_2)$ are composable in $\mathcal{M}^H$. Then there is a $g \in \mathrm{Ob}\,H$ with $M_1' = g \rhd M_1$, and the morphisms f_1 and $g^{-1} \rhd f_2$ are composable in $\mathcal{M}^{\mathrm{co}H}$. With the relations of $\mathcal{M}_{\mathrm{inv}}$ one obtains $\pi(f_2) \circ \pi(f_1) = \pi(g^{-1} \rhd f_2) \circ \pi(f_1) = \pi((g^{-1} \rhd f_2) \circ f_1)$ with $(g^{-1} \rhd f_2) \circ f_1 \in \mathcal{M}^{\mathrm{co}H}$.

2. We show that $\mathcal{M}_{\mathrm{inv}}$ has the universal property of the image in Cat. The inclusion functor $I : \mathcal{M}_{\mathrm{inv}} \to \mathcal{M}^H$ is a monomorphism in Cat and satisfies $IP = \pi\iota$, where $P : \mathcal{M}^{\mathrm{co}H} \to \mathcal{M}_{\mathrm{inv}}$ is the functor that sends an object M in $\mathcal{M}^{\mathrm{co}H}$ to $\pi(M)$ and a morphism f in $\mathcal{M}^{\mathrm{co}H}$ to $\pi(f)$. If (J, Q) is a pair of a monomorphism $J : \mathcal{C} \to \mathcal{M}^H$ and a functor $Q : \mathcal{M}^{\mathrm{co}H} \to \mathcal{C}$ with $JQ = \pi\iota$, then J is a monomorphism in Cat, which allows one to identify $\mathcal{C}$ with a subcategory of $\mathcal{M}^H$ and J with its inclusion functor. As $JQ = \pi\iota$, the subcategory $\mathcal{C} \subset \mathcal{M}^H$ contains $\mathcal{M}_{\mathrm{inv}}$ as a subcategory $\mathcal{M}_{\mathrm{inv}} \subset \mathcal{C}$ and hence there is a unique functor, the inclusion $V : \mathcal{M}_{\mathrm{inv}} \to \mathcal{C}$, with $I = JV$. ∎

7.17. REMARK. Coequalisers in Cat can also be determined via the construction of Bednarczyk, Borzyszkowski and Pawlowski [BBP], using *generalised congruences* and associated *quotient categories*. For a summary of this construction, see also Bruckner [Br] and Haucourt [Ha]. For a module $(\mathcal{M}, \rhd)$ over a group object in Cat the associated quotient category gives the invariants of $(\mathcal{M}, \rhd)$ as in Proposition 7.15. For a triple $(\mathcal{M}, \rhd, \delta)$ the generalised congruence $(\sim_0, \sim_m)$ restricts to a generalised congruence on $\mathcal{M}^{\mathrm{co}H}$ whose quotient category are the biinvariants of $(\mathcal{M}, \rhd, \delta)$.

7.18. Protected objects for group objects in Cat. We now give a concrete description of the coinvariants and the protected objects for oriented surfaces Σ of genus $g \geq 1$ and group objects $H = \bigtriangledown(B, A, \blacktriangleright, \partial)$ in Cat. We start by considering the Yetter–Drinfeld module and the coinvariants for the standard graph from (15) and show that they are given by group homomorphisms $\rho : F_{2g} \to A \rtimes B$ and $\rho : \pi_1(\Sigma) \to A \rtimes B$, respectively. To describe their category structure, we consider group-valued 1-cocycles.

7.19. Definition. *Let K, A be groups and $\blacktriangleright: K \times A \to A$ a group action of K on A by automorphisms.*

1. *A 1-**cocycle** is a map $\phi : K \to A$ with $\phi(\lambda\mu) = \phi(\lambda) \cdot (\lambda \blacktriangleright \phi(\mu))$ for all $\lambda, \mu \in K$.*
2. *A 1-**coboundary** is a map $\eta_a : K \to A$, $\lambda \mapsto a(\lambda \blacktriangleright a^{-1})$ for some $a \in A$.*
3. *$\phi, \psi : K \to A$ are **related by a coboundary** if $\psi(\lambda) = a \cdot \phi(\lambda) \cdot (\lambda \blacktriangleright a^{-1})$ for some $a \in A$.*

If A is abelian, 1-cocycles form a group $Z^1(K, A, \blacktriangleright)$ with pointwise multiplication and coboundaries a subgroup $B^1(K, A, \blacktriangleright)$. The factor group is the first cohomology group $H^1(K, A, \blacktriangleright)$. More generally, 1-cocycles with values in a (not necessarily abelian) group A arise from group homomorphisms into a semidirect product $A \rtimes B$.

7.20. Lemma. *Let $\blacktriangleright: B \times A \to A$ a group action by automorphisms and $A \rtimes B$ the associated semidirect product.*

1. *Group homomorphisms $\sigma : K \to A \rtimes B$ correspond to pairs (ϕ, ρ) of a group homomorphism $\rho : K \to B$ and a 1-cocycle $\phi : K \to A$ for the action $\rho^* \blacktriangleright: K \times A \to A$, $(\lambda, a) \mapsto \rho(\lambda) \blacktriangleright a$.*
2. *Two 1-cocycles $\phi, \phi' : K \to A$ for $\rho^* \blacktriangleright$ are related by a coboundary iff the group homomorphisms $(\phi, \rho), (\phi', \rho) : K \to A \rtimes B$ are related by conjugation with $A \subset A \rtimes B$.*

If the semidirect product in Lemma 7.20 arises from a crossed module $(B, A, \blacktriangleright, \partial)$, the group homomorphism $\partial : A \to B$, allows one to organise the group homomorphisms $\rho : K \to B$ and 1-cocycles $\phi : K \to A$ from Lemma 7.20 into a groupoid. Denoting by $\phi \cdot \psi$ and ϕ^{-1} the pointwise product and inverse of maps $\phi, \psi : K \to A$ we have

7.21. Lemma. *Any group K and crossed module $(B, A, \blacktriangleright, \partial)$ defines a groupoid* $\mathrm{Hom}(K, B \blacktriangleright A)$ *with*

- *group homomorphisms $\rho : K \to B$ as objects,*
- $\mathrm{Hom}(\rho, \rho') = \{(\phi, \rho) \mid \phi : K \to A$ *1-cocycle for* $\rho^* \blacktriangleright$ *with* $(\partial \circ \phi) \cdot \rho = \rho'\}$,
- *composition of morphisms: $(\psi, (\partial \circ \phi) \cdot \rho) \circ (\phi, \rho) = (\psi \cdot \phi, \rho)$,*
- *inverse morphisms: $(\phi, \rho)^{-1} = (\phi^{-1}, (\partial \circ \phi) \cdot \rho)$.*

Proof. A direct computation using (63) shows that for any pair (ϕ, ρ) of a group homomorphism $\rho : K \to B$ and a 1-cocycle $\phi : K \to A$ for $\rho^* \blacktriangleright$, the map $(\partial \circ \phi) \cdot \rho : K \to B$ is another group homomorphism. Similarly, if ϕ is a 1-cocycle for $\rho^* \blacktriangleright$ and ψ a 1-cocycle for $((\partial \circ \phi) \cdot \rho)^* \blacktriangleright$, then $\psi \cdot \phi$ is another 1-cocycle for $\rho^* \blacktriangleright$. The formula for the inverse morphism follows directly. ∎

By applying this lemma to Example 4.6, we obtain a groupoid that describes the Yetter–Drinfeld module for a group object $H = \triangledown(B, A, \blacktriangleright, \partial)$ in Cat and the standard graph (15), if we set $K = F_{2g}$ and identify the generators of F_{2g} with the edges of the graph. An analogous result holds for the associated coinvariants for $K = \pi_1(\Sigma)$ and any properly embedded graph with a single vertex.

7.22. Proposition. *Let* Γ *be a properly embedded graph with a single vertex on a surface* Σ *of genus* $g \geq 1$ *and* $H = \triangledown(B, A, \blacktriangleright, \partial)$ *a group object in* Cat*. Then the associated coinvariants are the groupoid from Lemma 7.21 for* $K = \pi_1(\Sigma)$*.*

Proof. By Theorem 5.25 it suffices to consider the graph in (15).

By Example 4.6 the coinvariants are the equaliser of the morphisms $\eta\,\epsilon, F : H^{\times 2g} \to H$ in Cat, where $\epsilon : H^{\times 2g} \to \{\cdot\}$ is the terminal morphism, $\eta : \{\cdot\} \to H$ is as in Definition 7.2 and $F : H^{\times 2g} \to H$ is given by $F(a_1, b_1, \ldots, a_g, b_g) = [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1]$. By Lemma 7.7, this equaliser is the subcategory $\mathcal{E} \subset H^{\times 2g}$ consisting of objects C and morphisms f with $F(C) = e$ and $F(f) = 1_e$. For $H = \triangledown(B, A, \blacktriangleright, \partial)$, this yields with Theorem 7.5

$$\begin{aligned} \mathrm{Ob}(\mathcal{E}) &= \{(a_1, b_1, \ldots, a_g, b_g) \in B^{\times 2g} \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = 1\} \\ \mathcal{E}^{(1)} &= \{(a_1, b_1, \ldots, a_g, b_g) \in (A \rtimes B)^{2g} \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = 1\}. \end{aligned}$$

Thus, every object $\rho \in \mathrm{Ob}(\mathcal{E})$ corresponds to a group homomorphism $\rho : \pi_1(\Sigma) \to B$ and every morphism $\sigma \in \mathcal{E}^{(1)}$ to a group homomorphism $\sigma : \pi_1(\Sigma) \to A \rtimes B$. By Lemma 7.20 the latter defines a pair $\sigma = (\phi, \rho)$ of a group homomorphism $\rho : \pi_1(\Sigma) \to B$ and a 1-cocycle ϕ for $\rho^* \blacktriangleright$. ∎

We now use the description of the coinvariants in Proposition 7.22 and the description of the image object in Cat from Proposition 7.16 to compute the protected object for a surface Σ of genus $g \geq 1$ and a crossed module $(B, A, \blacktriangleright, \partial)$.

7.23. Theorem. *The protected object for a group object* $H = \triangledown(B, A, \blacktriangleright, \partial)$ *in* Cat *and a surface* Σ *of genus* $g \geq 1$ *is a groupoid* $\mathcal{M}_{H,\Sigma}$ *with*

- *conjugacy classes of group homomorphisms* $\rho : \pi_1(\Sigma) \to B$ *as objects,*
- *equivalence classes of group homomorphisms* $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ *as morphisms from* $[\rho]$ *to* $[(\partial \circ \phi) \cdot \rho]$*.*

The equivalence relation is generated by $\tau_2 \circ \tau_1 \sim \tau_2' \circ \tau_1'$ *for all composable pairs* (τ_1, τ_2) *and* (τ_1', τ_2') *of group homomorphisms* $\tau_i, \tau_i' : F_{2g} \to A \rtimes B$ *such that* τ_i, τ_i' *are conjugate and* $\tau_2 \circ \tau_1, \tau_2' \circ \tau_1'$ *define group homomorphisms* $\pi_1(\Sigma) \to A \rtimes B$*.*

Proof. By Theorem 5.25 the protected object of Σ is a topological invariant and can be computed from the standard graph in (15). This yields a Yetter–Drinfeld module $(\mathcal{M}, \triangleright, \delta)$ over $\triangledown(B, A, \blacktriangleright, \partial)$ given by formula (22). Hence, we have $\mathcal{M}^{(1)} = (A \rtimes B)^{2g} \cong \mathrm{Hom}(F_{2g}, A \rtimes B)$ with the module structure given by conjugation and the comodule structure by the defining relation of $\pi_1(\Sigma)$.

By Proposition 7.22 the associated coinvariants form a groupoid $\mathcal{M}^{\mathrm{co}H}$ with group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects and group homomorphisms $\tau = (\phi, \rho)$:

$\pi_1(\Sigma) \to A \rtimes B$ as morphisms from ρ to $(\partial \circ \phi) \cdot \rho$. By Propositions 7.15 and 7.16 the associated image object is the groupoid, whose objects are orbits of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ under the conjugation action of B and whose morphisms are the images of group homomorphisms $\tau : \pi_1(\Sigma) \to A \rtimes B$ under the projection functor $\pi : \mathcal{M} \to \mathcal{M}^H$. The latter is given by the equivalence relation in the theorem. ∎

There are a number of cases in which the protected object has a particularly simple form. They correspond to crossed modules in which part of the data is trivial. The first corresponds to the case, where the Moore complex of the crossed module has trivial non-abelian homologies, namely $\ker(\partial) = \{1\}$ and $B/\partial(A) = 1$. The second is the case where the action of B on A is trivial.

7.24. Example. Let Σ be a surface of genus $g \geq 1$ and $(B, A, \blacktriangleright, \partial)$ a crossed module, where ∂ is an isomorphism. Then the protected object has

- conjugacy classes of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects,
- exactly one morphism between any two objects.

Proof. All morphism sets in the groupoid from Lemma 7.21 contain exactly one morphism, since $\mathrm{Hom}(\rho, \sigma) = \{(\partial^{-1}(\sigma \cdot \rho^{-1}), \rho)\}$ for all group homomorphisms $\rho, \sigma : \pi_1(\Sigma) \to B$. Conjugating a morphism in $\mathrm{Hom}(\rho, \sigma)$ with an element of $(a, b) \in A \rtimes B$ yields the unique morphism from $b\rho b^{-1}$ to $(\partial(a)b)\, \sigma\, (\partial(a)b)^{-1}$. This shows that all morphisms from conjugates of a group homomorphism $\rho : \pi_1(\Sigma) \to B$ to a conjugate of a group homomorphism $\sigma : \pi_1(\Sigma) \to B$ are conjugated and hence identified in $\mathcal{M}^H$ and in $\mathcal{M}_{\mathrm{inv}}$. ∎

7.25. Example. Let Σ be a surface of genus $g \geq 1$ and $(B, A, \blacktriangleright, \partial)$ a crossed module with a trivial group action $\blacktriangleright$. Then the protected object is $\mathrm{Hom}(\pi_1(\Sigma), A \times B)/A \times B$ with

- conjugacy classes of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects,
- group homomorphisms $\phi : \pi_1(\Sigma) \to A$ as morphisms from $[\rho]$ to $[(\partial \circ \phi) \cdot \rho]$.

Proof. If $\blacktriangleright : B \times A \to A$ is trivial, then conditions (63) imply that A is abelian with $\partial(A) \subset Z(B)$. As A is abelian and $\blacktriangleright$ trivial, the 1-cocycles from Definition 7.19 are simply group homomorphisms $\phi : \pi_1(\Sigma) \to A$ and any 1-coboundary is trivial. The groupoid $\mathcal{M}^{\mathrm{co}H}$ from Lemma 7.21 thus has as objects group homomorphisms $\rho : \pi_1(\Sigma) \to B$ and as morphisms $\tau = (\phi, \rho) : \rho \to (\partial \circ \phi) \cdot \rho$ group homomorphisms $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \times B$.

As A is abelian and $\blacktriangleright$ trivial, two group homomorphisms $\tau = (\phi, \rho), \tau' = (\phi', \rho') : \pi_1(\Sigma) \to A \rtimes B$ are conjugate iff $\phi' = \phi$ and $\rho' = b\rho b^{-1}$ for some $b \in B$. Thus, the relation on morphisms in Theorem 7.23 identifies τ and τ' iff $\phi = \phi'$ and $[\rho] = [\rho']$. ∎

In the case of a trivial group homomorphism $\partial : A \to B$ all morphisms in $\mathcal{M}$, $\mathcal{M}^{\mathrm{co}H}$, $\mathcal{M}^H$ and $\mathcal{M}_{\mathrm{inv}}$ are automorphisms. This yields

7.26. Example. Let Σ be a surface of genus $g \geq 1$ and $H = \triangledown(B, A, \blacktriangleright, \partial)$ with A abelian and a trivial group homomorphism $\partial \equiv 1$. Then the associated protected object is

$$\mathcal{M}_{H,\Sigma} = \amalg_{[\rho] \in \mathrm{Hom}(\pi_1(\Sigma), B)/B}\; G_{[\rho]},$$

where $G_{[\rho]}$ is a factor group of $H^1(\pi_1(\Sigma), A, \rho^* \blacktriangleright)$.

PROOF. If ∂ is trivial and A abelian, then every 1-cocycle $\phi : \pi_1(\Sigma) \to A$ for $\rho^* \blacktriangleright$ defines an automorphism of ρ in $\mathcal{M}^{\mathrm{co}H}$, which implies $\mathcal{M}^{\mathrm{co}H} = \amalg_{\rho \in \mathrm{Hom}(\pi_1(\Sigma), B)} Z^1(\pi_1(\Sigma), A, \rho^* \blacktriangleright)$.

As all morphisms in $\mathcal{M}^{\mathrm{co}H}$ are automorphisms, two morphisms given by group homomorphisms $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ and $\tau' = (\phi', \rho') : \pi_1(\Sigma) \to A \rtimes B$ are composable iff $\rho = \rho'$. By Lemma 7.20, 2. two group homomorphisms $(\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ and $(\phi', \rho) : \pi_1(\Sigma) \to A \rtimes B$ are related by conjugation with $A \subset A \rtimes B$ iff ϕ, ϕ' are related by a 1-coboundary. Thus for a group homomorphism $\rho : \pi_1(\Sigma) \to B$, the automorphism group of $[\rho]$ in $\mathcal{M}_{\mathrm{inv}}$ is a factor group of $H^1(\pi_1(\Sigma), A, \rho^* \blacktriangleright)$. ∎

By Theorem 7.23 group homomorphisms $\tau, \tau' : \pi_1(\Sigma) \to A \rtimes B$ that are conjugated define the same morphism in $\mathcal{M}_{\mathrm{inv}}$. This implies in particular that the morphism in $\mathcal{M}_{H,\Sigma}$ defined by a group homomorphism $\sigma = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ depends on ϕ only up to coboundaries. Modifying ϕ with a coboundary yields a group homomorphism $\sigma' = (\phi', \rho)$ conjugated to σ by Lemma 7.20, 2.

However, except for the situation in Examples 7.24 and 7.25, it is difficult to describe the category $\mathcal{M}_{H,\Sigma}$ explicitly, even for genus $g = 1$ and crossed modules given by normal subgroups. This is due to the fact that the equivalence relation in Theorem 7.23 also identifies morphisms in $\mathcal{M}^{\mathrm{co}H}$ in different $A \rtimes B$-orbits. This is illustrated by the following two examples.

7.27. EXAMPLE. Let Σ be the torus with $\pi_1(\Sigma) = \mathbb{Z} \times \mathbb{Z}$ and consider the crossed module $(S_3, A_3, \blacktriangleright, \iota)$, where $\iota : A_3 \to S_3$ is the inclusion and $\blacktriangleright: S_3 \times A_3 \to A_3$, $b \blacktriangleright a = bab^{-1}$ the conjugation action.

We specify group homomorphisms $\rho : \mathbb{Z} \times \mathbb{Z} \to S_3$ and 1-cocycles $\phi : \mathbb{Z} \times \mathbb{Z} \to A_3$ by the images of $(1, 0)$ and $(0, 1)$ and write $\rho = (\rho(1, 0), \rho(0, 1))$ for the former and $\phi = \langle \phi(1, 0), \phi(0, 1) \rangle$ for the latter. Then the conjugacy classes of group homomorphisms $\rho : \mathbb{Z} \times \mathbb{Z} \to S_3$ are given by

$C_1 = \{(\mathrm{id}, \mathrm{id})\}$	
$C_2 = \{(\mathrm{id}, c) \mid c \in A_3 \setminus \{\mathrm{id}\}\}$	$C_2' = \{(c, \mathrm{id}) \mid c \in A_3 \setminus \{\mathrm{id}\}\}$
$C_3 = \{(c, c) \mid c \in A_3 \setminus \{\mathrm{id}\}\}$	
$C_4 = \{(c, c') \mid c \neq c' \in A_3 \setminus \{\mathrm{id}\}\}$	
$C_5 = \{(\mathrm{id}, \sigma) \mid \sigma \in S_3 \setminus A_3\}$	$C_5' = \{(\sigma, \mathrm{id}) \mid \sigma \in S_3 \setminus A_3\}$
$C_6 = \{(\sigma, \sigma) \mid \sigma \in S_3 \setminus A_3\}$	

If $\rho(\mathbb{Z} \times \mathbb{Z}) \subset A_3$, then 1-cocycles for $\rho^* \blacktriangleright$ are simply group homomorphisms $\phi : \mathbb{Z} \times \mathbb{Z} \to A_3$. If $\rho \in C_5$, $\rho \in C_5'$ or $\rho \in C_6$ then the 1-cocycles for $\rho^* \blacktriangleright$ are given by

$$\langle \mathrm{id}, c \rangle(k, l) = \begin{cases} c & l \text{ odd} \\ \mathrm{id} & l \text{ even} \end{cases}, \ \langle c, \mathrm{id} \rangle(k, l) = \begin{cases} c & k \text{ odd} \\ \mathrm{id} & k \text{ even} \end{cases}, \ \langle c, c \rangle(k, l) = \begin{cases} c & k + l \text{ odd} \\ \mathrm{id} & k + l \text{ even} \end{cases},$$

respectively, where $c \in A_3$, $k, l \in \mathbb{Z}$.

The protected object $\mathcal{M}_{\text{inv}}$ has the objects C_1, C_2, C_2', C_3, C_4, C_5, C_5', C_6. Its morphisms are equivalence classes of morphisms in $\mathcal{M}^{\text{co}H}$, that is, of 1-cocycles. The morphisms in $\mathcal{M}^{\text{co}H}$ starting in $\rho \in C_1$ are the trivial 1-cocycle $\phi \equiv \text{id}$ as identity morphism and the conjugate pairs

$$\begin{aligned}
&\langle \text{id}, (123)\rangle : (\text{id}, \text{id}) \to (\text{id}, (123)) &&\sim \langle \text{id}, (132)\rangle : (\text{id}, \text{id}) \to (\text{id}, (132))\\
&\langle (123), \text{id}\rangle : (\text{id}, \text{id}) \to ((123), \text{id}) &&\sim \langle (132), \text{id}\rangle : (\text{id}, \text{id}) \to ((132), \text{id})\\
&\langle (123), (123)\rangle : (\text{id}, \text{id}) \to ((123), (123)) &&\sim \langle (132), (132)\rangle : (\text{id}, \text{id}) \to ((132), (132))\\
&\langle (123), (132)\rangle : (\text{id}, \text{id}) \to ((123), (132)) &&\sim \langle (132), (123)\rangle : (\text{id}, \text{id}) \to ((132), (123)),
\end{aligned}$$

where we use cycle notation for elements of S_3. As each of these pairs defines a single morphism in $\mathcal{M}_{\text{inv}}$, there is exactly one morphism from C_1 to each of the conjugacy classes C_2, C_2', C_3, C_4. As $\mathcal{M}_{\text{inv}}$ is a groupoid, there is exactly one morphism between any two of these conjugacy classes.

Each of the 1-cocycles $\langle \text{id}, c\rangle$, $\langle c, \text{id}\rangle$, $\langle c, c\rangle$ with $c \in A_3$ defines a morphism in $\mathcal{M}^{\text{co}H}$ within the conjugacy classes C_5, C_5', C_6. The morphisms between objects in C_5 in $\mathcal{M}^{\text{co}H}$ are

$$\begin{aligned}
&\langle \text{id}, \text{id}\rangle : (\text{id}, (12)) \to (\text{id}, (12)) &&\langle \text{id}, \text{id}\rangle : (\text{id}, (13)) \to (\text{id}, (13))\\
&\langle \text{id}, (123)\rangle : (\text{id}, (12)) \to (\text{id}, (13)) &&\langle \text{id}, (123)\rangle : (\text{id}, (13)) \to (\text{id}, (23))\\
&\langle \text{id}, (132)\rangle : (\text{id}, (12)) \to (\text{id}, (23)) &&\langle \text{id}, (132)\rangle : (\text{id}, (13)) \to (\text{id}, (12))\\
&\langle \text{id}, \text{id}\rangle : (\text{id}, (23)) \to (\text{id}, (23))\\
&\langle \text{id}, (123)\rangle : (\text{id}, (23)) \to (\text{id}, (12))\\
&\langle \text{id}, (132)\rangle : (\text{id}, (23)) \to (\text{id}, (13)).
\end{aligned}$$

All morphisms in the first and fourth line are conjugate. The first morphism in the second line is conjugate to the morphisms in the second, third, fifth and sixth line via cyclic permutations and transpositions. As

$$\begin{aligned}
\langle \text{id}, \text{id}\rangle &= \langle \text{id}, (123)\rangle \circ \langle \text{id}, (132)\rangle : (\text{id}, (12)) \to (\text{id}, (12))\\
\langle \text{id}, (123)\rangle &= \langle \text{id}, (132)\rangle \circ \langle \text{id}, (132)\rangle : (\text{id}, (12)) \to (\text{id}, (13)).
\end{aligned}$$

with $\langle \text{id}, (132)\rangle \sim \langle \text{id}, (123)\rangle$, all morphisms are identified by the relation in Theorem 7.23 and define a single morphism in $\mathcal{M}_{\text{inv}}$. Hence, the identity morphism is the only automorphism of C_5 in $\mathcal{M}_{\text{inv}}$ and likewise for C_5' and C_6. Thus $\mathcal{M}_{\text{inv}}$ is the groupoid in Figure 3.

7.28. Example. Let Σ be the torus and consider the crossed module $(S_3, A_3, \blacktriangleright, \partial)$ with the trivial group homomorphism $\partial : A_3 \to S_3$, $a \mapsto \text{id}$ and $\blacktriangleright: S_3 \times A_3 \to A_3$, $b \blacktriangleright a = bab^{-1}$.

Then the protected object $\mathcal{M}_{\text{inv}}$ has the same objects as in Example 7.27. As ∂ is trivial, all morphisms in $\mathcal{M}^{\text{co}H}$ and $\mathcal{M}_{\text{inv}}$ are automorphisms. The object (id, id) in $\mathcal{M}^{\text{co}H}$

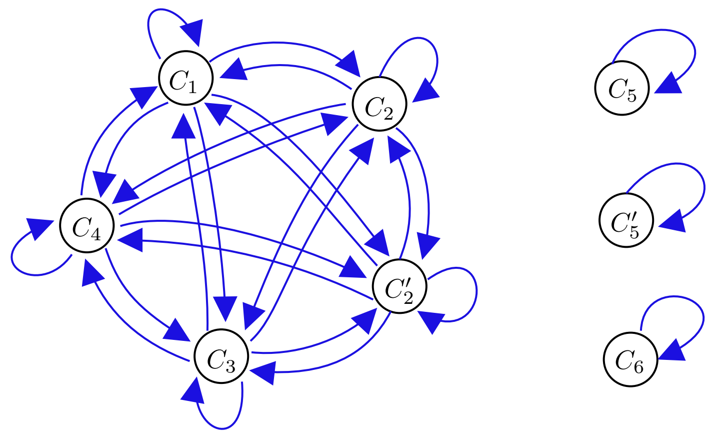


Figure 3: The groupoid $\mathcal{M}_{\mathrm{inv}}$ from Example 7.27

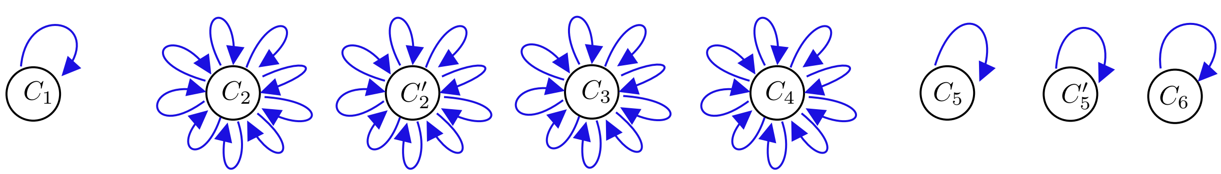


Figure 4: The groupoid $\mathcal{M}_{\mathrm{inv}}$ from Example 7.28

has the identity morphism $\langle \mathrm{id}, \mathrm{id}\rangle$ and the following four conjugate pairs of automorphisms

$$\langle \mathrm{id}, (123)\rangle \sim \langle \mathrm{id}, (132)\rangle, \qquad \langle (123), \mathrm{id}\rangle \sim \langle (132), \mathrm{id}\rangle,$$
$$\langle (123), (123)\rangle \sim \langle (132), (132)\rangle, \qquad \langle (123), (132)\rangle \sim \langle (132), (123)\rangle.$$

The relation for morphisms in Theorem 7.23 then implies

$$\langle \mathrm{id}, \mathrm{id}\rangle = \langle \mathrm{id}, (123)\rangle \circ \langle \mathrm{id}, (132)\rangle \sim \langle \mathrm{id}, (132)\rangle \circ \langle \mathrm{id}, (132)\rangle = \langle \mathrm{id}, (123)\rangle$$
$$\langle \mathrm{id}, \mathrm{id}\rangle = \langle (123), \mathrm{id}\rangle \circ \langle (132), \mathrm{id}\rangle \sim \langle (132), \mathrm{id}\rangle \circ \langle (132), \mathrm{id}\rangle = \langle (123), \mathrm{id}\rangle$$
$$\langle \mathrm{id}, \mathrm{id}\rangle = \langle (123), (123)\rangle \circ \langle (132), (132)\rangle \sim \langle (132), (132)\rangle \circ \langle (132), (132)\rangle = \langle (123), (123)\rangle$$
$$\langle \mathrm{id}, \mathrm{id}\rangle = \langle (123), (132)\rangle \circ \langle (132), (123)\rangle \sim \langle (132), (123)\rangle \circ \langle (132), (123)\rangle = \langle (123), (132)\rangle.$$

As all automorphisms of $(\mathrm{id}, \mathrm{id})$ in $\mathcal{M}^{\mathrm{co}H}$ are identified, C_1 has a single automorphism in $\mathcal{M}_{\mathrm{inv}}$. As in Example 7.27, all morphisms between objects in C_5 are identified and likewise for C_5', C_6.

In contrast, the automorphism group of each element of C_2, C_2', C_3, C_4 in $\mathcal{M}$ and $\mathcal{M}^{\mathrm{co}H}$ is $A_3 \times A_3$. Automorphisms of these objects in $\mathcal{M}$ coincide with their automorphisms in $\mathcal{M}^{\mathrm{co}H}$. Each automorphism of an object in one of these conjugacy classes is conjugate only to itself and to automorphisms of different objects in the same conjugacy class. As any composable sequence of morphisms in $\mathcal{M}$ involves only automorphisms of the same object, the automorphism groups of these conjugacy classes in $\mathcal{M}_{\mathrm{inv}}$ are given by $A_3 \times A_3$. Thus, the groupoid $\mathcal{M}_{\mathrm{inv}}$ is as in Figure 4.

8. Mapping class group actions

In this section, we describe the mapping class group actions on the protected objects for connected closed surfaces. In the following Σ is a surface of genus $g \geq 1$ and $\Sigma \setminus D$ the associated surface with a disc removed and fundamental group $\pi_1(\Sigma \setminus D) = F_{2g}$.

The mapping class group of Σ is the quotient of the group $\mathrm{Homeo}_+(\Sigma)$ of orientation preserving homeomorphisms of Σ by the normal subgroup $\mathrm{Homeo}_0(\Sigma)$ of homeomorphisms homotopic to the identity. It is isomorphic to the group of outer automorphisms of the fundamental group $\pi_1(\Sigma)$

$$\mathrm{Map}(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma) \cong \mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma)).$$

The mapping class group of $\Sigma \setminus D$ is defined analogously with the additional condition that all homeomorphisms fix the boundary of D pointwise, see Farb and Margalit [FM, Sec. 2.1]. The mapping class groups $\mathrm{Map}(\Sigma)$ and $\mathrm{Map}(\Sigma \setminus D)$ can be presented with the same generators but with additional relations for $\mathrm{Map}(\Sigma)$, see for instance the presentation by Gervais [Ge].

Mapping class group actions associated with protected objects for involutive and, more generally, pivotal Hopf monoids in a finitely complete and cocomplete symmetric monoidal category are constructed in [MV]. It is shown in [MV, Th. 9.2] that the mapping class group $\mathrm{Map}(\Sigma \setminus D)$ acts on the Yetter–Drinfeld module in Example 4.6 by automorphisms. By [MV, Th. 9.5] this induces an action of $\mathrm{Map}(\Sigma)$ by automorphisms of its biinvariants.

The mapping class group actions in [MV] are obtained from a concrete presentation of the mapping class groups $\mathrm{Map}(\Sigma)$ and $\mathrm{Map}(\Sigma \setminus D)$ in terms of generating Dehn twists and relations. They associate to each generating Dehn twist a finite sequence of edge slides and prove that resulting automorphisms of $H^{\otimes E}$ from Definition 5.7 satisfy the relations for $\mathrm{Map}(\Sigma \setminus D)$ in [Ge]. The induced automorphisms of the protected object then satisfy the additional relations of $\mathrm{Map}(\Sigma)$ in [Ge].

As we established in Theorem 5.25 that the protected object is independent of the choice of the underlying graph, we can reformulate [MV, Th. 9.5] as follows.

8.1. THEOREM. *Let H be an involutive Hopf monoid in $\mathcal{C}$ and Σ an oriented surface of genus $g \geq 1$. Then the edge slides from Definition 5.7 induce an action of the mapping class group* $\mathrm{Map}(\Sigma)$ *by automorphisms of the protected object.*

For group objects in cartesian monoidal categories such as simplicial groups and crossed modules this mapping class group action admits a concrete description in terms of mapping class group actions on representation varieties. For this, recall that for any group G the group $\mathrm{Aut}(\pi_1(\Sigma))$ acts on the set of group homomorphisms $\rho : \pi_1(\Sigma) \to G$ via $(\phi \rhd \rho)(\lambda) = \rho(\phi^{-1}(\lambda))$ for all $\lambda \in \pi_1(\Sigma)$ and $\phi \in \mathrm{Aut}(\pi_1(\Sigma))$. This induces an action of $\mathrm{Map}(\Sigma) = \mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma))$ on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

To relate this to the mapping class group actions from [MV, Th. 9.5] note that for a group object H in a cartesian monoidal category the formulas for the edge slides in Definition 5.7 and Example 5.8 reduce to left and right multiplication with H, sometimes composed with inversions.

It follows that any finite sequence of edge slides from the standard graph to itself induces an automorphism of $H^{\otimes 2g}$ that arises from an automorphism of $F_{2g} = \pi_1(\Sigma \setminus D)$. As it preserves the Yetter–Drinfeld module structure in Example 4.6, it induces automorphisms of $\mathcal{M}^{\mathrm{co}H}$, $\mathcal{M}^H$ and $\mathcal{M}_{\mathrm{inv}}$. Inner automorphism of $\pi_1(\Sigma)$ induce trivial automorphisms of $\mathcal{M}_{\mathrm{inv}}$. For a group H as a group object in Set it is then directly apparent that the induced action of $\mathrm{Map}(\Sigma)$ on $\mathcal{M}_{\mathrm{inv}}$ is the one on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), H)/H$, see also Examples 9.6 and 9.7 in [MV]. This result can be applied to determine the mapping class group action for a simplicial group.

8.2. COROLLARY. *Let $H = (H_n)_{n \in \mathbb{N}_0}$ be a simplicial group as a Hopf monoid in* SSet. *Then the action of* $\mathrm{Map}(\Sigma)$ *on the representation varieties* $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ *induces an action of* $\mathrm{Map}(\Sigma)$ *on* $\mathcal{M}_{\mathrm{inv}}$ *by simplicial maps, and this coincides with the action in [MV, Th. 9.5].*

PROOF. The induced $\mathrm{Map}(\Sigma)$-action on $\mathcal{M}_{\mathrm{inv}}$ is by simplicial maps, because the face maps and degeneracies of $\mathcal{M}_{\mathrm{inv}}$ act elements of the representation varieties $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ by post-composition with the face maps and degeneracies $d_i : H_n \to H_{n-1}$ and $s_i : H_n \to H_{n+1}$, whereas $\mathrm{Map}(\Sigma)$ acts by pre-composition. This coincides with the action from [MV, Th. 9.5], because the latter reduces to the $\mathrm{Map}(\Sigma)$-action on $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ for the group H_n as an involutive Hopf monoid in Set, and all (co)limits, images and (co)actions in SSet are degreewise. ■

In the case of a crossed module as a Hopf monoid in Cat, the mapping class group action on the protected object is induced by the mapping class group action on the representation variety for the associated semidirect product group.

8.3. Corollary. *Let* $H = (B, A, \blacktriangleright, \partial)$ *be a crossed module. Then the* $\mathrm{Map}(\Sigma)$*-action on* $\mathcal{M}_{\mathrm{inv}}$ *from Theorem 8.1 is induced by the* $\mathrm{Map}(\Sigma)$*-action on* $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$.

Proof. As the group structure of H as a group object in Cat is the one of the semidirect product $A \rtimes B$, the $\mathrm{Map}(\Sigma \setminus D)$-action on $\mathcal{M} = H^{\times 2g}$ for the standard graph (15) can be identified with the $\mathrm{Map}(\Sigma \setminus D)$-action on $\mathcal{M} = (A \rtimes B)^{\times 2g}$ one for the group $A \rtimes B$ as a group object in Set. The crossed module structure ensures that this $\mathrm{Map}(\Sigma \setminus D)$-action respects the category structure of $(A \rtimes B)^{\times 2g}$ and defines a $\mathrm{Map}(\Sigma \setminus D)$-action by invertible endofunctors.

The $\mathrm{Map}(\Sigma \setminus D)$-action on $\mathcal{M}$ induces the $\mathrm{Map}(\Sigma)$-action on the protected object $\mathcal{M}_{\mathrm{inv}}$ for both, the group $A \rtimes B$ as a group object in Set and for H as a group object in Cat. The former is the action on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$. As the protected object $\mathcal{M}_{\mathrm{inv}}$ is a quotient of this representation variety by Theorem 7.23, its $\mathrm{Map}(\Sigma)$-action is induced by the $\mathrm{Map}(\Sigma)$-action on the representation variety. ■

8.4. Example. We consider the mapping class group action on the groupoids $\mathcal{M}_{\mathrm{inv}}$ from Example 7.27, 7.28 for the crossed module $(S_3, A_3, \blacktriangleright, \partial)$ and the torus.

The mapping class group of the torus T is the group

$$\mathrm{Map}(T) = \mathrm{SL}(2, \mathbb{Z}) = \langle D_a, D_b \mid D_a D_b D_a = D_b D_a D_b, (D_a D_b D_a)^4 = 1 \rangle. \tag{67}$$

It is generated by the Dehn twists D_a, D_b along the a- and b-cycle, which act on $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ by

$$D_a : a \mapsto a, b \mapsto b - a \qquad\qquad D_b : a \mapsto a + b, b \mapsto b. \tag{68}$$

In both, Example 7.27 and 7.28, the $\mathrm{SL}(2, \mathbb{Z})$-action on the objects of $\mathcal{M}_{\mathrm{inv}}$ is the $\mathrm{SL}(2, \mathbb{Z})$-action on the representation variety $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, S_3)/S_3$ with orbits $\{C_5, C_5', C_6\}$, $\{C_1\}$ and $\{C_2, C_2', C_3, C_4\}$.

In Example 7.27 the $\mathrm{SL}(2, \mathbb{Z})$-action on $\mathcal{M}_{\mathrm{inv}}$ is determined uniquely by the action on the objects. This follows, because for all choices of objects $s, t \in \mathrm{Ob}\, \mathcal{M}_{\mathrm{inv}}$ the groupoid $\mathcal{M}_{\mathrm{inv}}$ has at most one morphism $f : s \to t$. In Example 7.28 an analogous statement holds for morphisms between the objects C_1, C_5, C_5', C_6, since all of them are identity morphisms.

In contrast, the $\mathrm{SL}(2, \mathbb{Z})$-action on the automorphisms of C_2, C_2', C_3, C_4 in Example 7.28 is non-trivial and can be identified with an orbit of the $\mathrm{SL}(2, \mathbb{Z})$-action on $\mathrm{Mat}(2 \times$

$2, \mathbb{Z}_3)$ by left multiplication. In this action, D_a and D_b correspond to left-multiplication with the generators

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Automorphisms of C_2, C_2', C_3, C_4 in $\mathcal{M}_{\text{inv}}$ are given by group homomorphisms $\tau : \mathbb{Z} \times \mathbb{Z} \to A_3^{\times 2} \cong \mathbb{Z}_3^{\times 2}$, which are determined by the images $\tau(1,0), \tau(0,1) \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Interpreting an element $(c,d) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ as an automorphism $c : d \to d$ and taking $\tau(0,1)$ as the first and $\tau(1,0)$ as the second row of a matrix, we find that the $\mathrm{SL}(2,\mathbb{Z})$-action induced by (68) coincides with the $\mathrm{SL}(2,\mathbb{Z})$-orbit containing those matrices whose second column is non-trivial.

As our construction yields objects equipped with mapping class group actions and assigns the tensor unit to the sphere S^2, it is natural to ask if the protected objects satisfy the axioms of a modular functor from [BK, Def 5.1.1]. Although the latter are formulated for categories of vector spaces, they have obvious generalisations to other symmetric monoidal categories.

However, the assignment of protected objects to surfaces can in general not be expected to satisfy these axioms. The problem is axiom (iii) in [BK, Def 5.1.1], which requires that the object assigned to a disjoint union $\Sigma_1 \amalg \Sigma_2$ of surfaces is the tensor product of the objects assigned to Σ_1, Σ_2. This does in general not hold for the protected objects from Definition 4.8, as they are constructed by taking equalisers and coequalisers. The tensor product of two (co)equalisers in a symmetric monoidal category $\mathcal{C}$ is in general not a (co)equaliser of their tensor product. This is already apparent in the symmetric monoidal category $\mathrm{Ab} = \mathbb{Z} - \mathrm{Mod}$ with the usual tensor product that does not preserve equalisers. Nevertheless, the construction satisfies this axiom, if the underlying symmetric monoidal category is Set, SSet or Cat.

8.5. PROPOSITION. *Let H be a group object in $\mathcal{C} = $ Set, SSet or Cat and Σ an oriented surface with connected components $\Sigma_1, \ldots, \Sigma_k$. Then the protected object for Σ is the product of the protected objects for $\Sigma_1, \ldots, \Sigma_k$.*

PROOF. The claim follows by induction over k, and it is sufficient to consider $k = 2$. By Theorem 5.25 we can compute the protected object for Σ by choosing a standard graph Γ_i from (15) on each connected component Σ_i. We denote by E_i the edge set of Γ_i and by $E = E_1 \cup E_2$ the edge set of Γ.

The (co)actions $\rhd$ and δ for Γ are then given by formula (23), and it follows directly that they are the products of the (co)actions $\rhd_i$, δ_i for Γ_i, up to braidings. Lemma 4.5 and some simple computations imply that $(H^{\times E}, \rhd, \delta)$ is a Yetter–Drinfeld module over $H \times H$.

We denote by $F_\Gamma : H^{\times E} \to H \times H$ the morphism from Example 2.11 for Γ and by $F_{\Gamma_i} : H^{\times E_i} \to H$ the corresponding morphisms for Γ_i.

- $\mathcal{C} = \text{Set}$: As in Example 2.16 we obtain

$$M^{\mathrm{co}H} = F_\Gamma^{-1}(1) = \{(x,y) \in H^{\times E_1} \times H^{\times E_2} : (F_{\Gamma_1}(x), F_{\Gamma_2}(y)) = (1,1)\} \;=\; M_1^{\mathrm{co}H} \times M_2^{\mathrm{co}H},$$
$$M^H = \{H^{\times 2} \rhd (m_1, m_2) : m_1 \in H^{\times E_1}, m_2 \in H^{\times E_2}\} = M_1^H \times M_2^H$$

with inclusions $\iota = (\iota_1, \iota_2) : M^{\mathrm{co}H} \to H^{\times E}$ and canonical surjections $\pi = (\pi_1, \pi_2) : H^{\times E} \to M^H$. As the image of a morphism $f : A \to B$ in Set is the usual image of a map, we have $\mathrm{im}((f_1, f_2)) = (\mathrm{im}(f_1), \mathrm{im}(f_2))$ and $M_{\mathrm{inv}} = M_{\mathrm{inv},1} \times M_{\mathrm{inv},2}$.

- $\mathcal{C} = \text{SSet}$: By Proposition 6.3 the coinvariants are given by the sets

$$M_n^{\mathrm{co}H} = \{(m_1, m_2) \in (H^{\times E_1} \times H^{\times E_2})_n : (F_{\Gamma_{1,n}}(m_1), F_{\Gamma_{2,n}}(m_2)) = (1,1)\} \;=\; M_{1,n}^{\mathrm{co}H} \times M_{2,n}^{\mathrm{co}H}.$$

Face maps and degeneracies are induced by the ones of $H^{\times E_1} \times H^{\times E_2}$ and the simplicial map $\iota : M^{\mathrm{co}H} \to H^{\times E}$ is given by the maps $\iota_n = (\iota_{1,n}, \iota_{2,n})$. As the product in SSet is objectwise, this yields $M^{\mathrm{co}H} = M_1^{\mathrm{co}H} \times M_2^{\mathrm{co}H}$. An analogous argument shows that the sets M_n^H, $(M_{\mathrm{inv}})_n$ from Proposition 6.3 are given by $M_n^H = M_{1,n}^H \times M_{2,n}^H$ and $(M_{\mathrm{inv}})_n = (M_{\mathrm{inv},1})_n \times (M_{\mathrm{inv},2})_n$.

- $\mathcal{C} = \text{Cat}$: By Lemma 7.14 the coinvariants $M^{\mathrm{co}H}$ for the comodule $M = H^{\times E} \cong H^{E_1} \times H^{\times E_2}$ are the subcategory with objects

$$\mathrm{Ob}(M^{\mathrm{co}H}) = \{(A_1, A_2) \mid A_i \in \mathrm{Ob}(H^{\times E_i}), F_{\Gamma_i}(A_i) = e\}$$
$$\mathrm{Hom}_{M^{\mathrm{co}H}}((A_1, A_2), (A_1', A_2')) = \{(f_1, f_2) \mid f_i \in \mathrm{Hom}_M(A_i, A_i'), F_{\Gamma_i}(f_i) = 1_e\}$$

and hence $M^{\mathrm{co}H}$ is the product category $M_1^{\mathrm{co}H} \times M_2^{\mathrm{co}H}$. For the invariants we can apply Lemma 7.8 and the results from SSet. As a right adjoint, the nerve N preserves products, and hence $N(\rhd) = N(\rhd_1) \times N(\rhd_2)$, up to braidings, and the same holds for the trivial actions $\epsilon \otimes 1_{H^{\otimes E}}$ and $\epsilon \otimes 1_{H^{\otimes E_i}}$. It follows that the coequaliser of $N(\rhd)$ and $N(\epsilon^{\times 2} \times 1_{H^{\times E}})$ is the product of the coequalisers of $N(\rhd_i)$ and $N(\epsilon \times 1_{H^{\times E_i}})$. As the homotopy functor preserves finite products, see for instance [Jo, Prop. 1.3], this yields $M^H = M_1^H \times M_2^H$ and $M_{\mathrm{inv}} = M_{\mathrm{inv},1} \times M_{\mathrm{inv},2}$. ∎

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Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg
Cauerstraße 11, 91058 Erlangen, Germany

Email: anna-katharina.hirmer@fau.de
catherine.meusburger@math.uni-erlangen.de

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