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ON DOMAIN-LIKE OBJECTS IN THE CATEGORY OF UNITARY RINGS

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ABSTRACT. In the category $\mathcal{C}$ of unitary rings, Barr, Kennison and Raphael (2015) studied the limit closures of various classes of commutative integral domains; in particular the class of all domains and of integrally closed domains to form the reflective subcategories $\mathcal{K}_{\mathrm{dom}}$ and $\mathcal{K}_{\mathrm{ic}}$. This article looks at $\mathcal{K}_{\mathrm{dom}}(R)$ and $\mathcal{K}_{\mathrm{ic}}(R)$ – the same symbols are used for the subcategories and the reflection functors – for some rings R. The objects in $\mathcal{K}_{\mathrm{dom}}$ can be called "domain-like". Of particular interest is the ring $\mathrm{C}^1(\mathbb{R})$, the ring of real functions with continuous first derivative. The ring L is that of continuous functions that are C^1 on a dense open set of $\mathbb{R}$. Then $\mathcal{K}_{\mathrm{dom}}(\mathrm{C}^1(\mathbb{R})) \subset \mathcal{K}_{\mathrm{ic}}(\mathrm{C}^1(\mathbb{R})) \subset L$, with proper inclusions. Moreover, for $f \in \mathcal{K}_{\mathrm{dom}}(\mathrm{C}^1(\mathbb{R}))$, f has one-sided derivatives and those are one-sided continuous.

For commutative rings $R \subset S$ in $\mathcal{R}$ there is a subring of the integral closure of R in S introduced here, called the *split integral closure*, that is explored and turns out to be very useful.

1. Introduction. The category $\mathcal{C}$ of all unitary rings will be the context. Before outlining this paper, it will be necessary to present some historical background. The subject will revolve around the question of what is a *domain-like object* in $\mathcal{C}$.

The question of what is a *field-like object* in $\mathcal{C}$ was resolved much earlier by M. Hochster ([H]) in 1969. The subcategory $\mathcal{F}$, of fields in $\mathcal{C}$ is not algebraic because the defining axiom is for non-zero elements and, indeed, $\mathcal{F}$ is not closed under limits, in the categorical sense. The answer, in this case, is the subcategory of commutative (von Neumann) regular rings, $\mathcal{VN}$, in $\mathcal{R}$ defined by, in addition to the usual axioms for commutative rings, there is also: $\forall x, \exists y\,(x^2y = x)$. The functor making $\mathcal{VN}$ a reflective subcategory was defined in [H] using a new topology on $\operatorname{Spec} R$, called the *patch* or *constructible* topology. In fact, for each $R \in \mathcal{VN}$, $\operatorname{Spec} R$ already has the patch topology and R is the ring of sections of a sheaf of fields over $\operatorname{Spec} R$. Further properties of this functor were studied by R. Wiegand ([Wi]).

The history of domain-like objects is quite different. Once again the obstacle is that the defining axiom of domains is for non-zero elements. The topic was first examined by J. Kennison in [K] and J. Kennison and C.S. Ledbetter in [KL] in the 1970s. The starting point here is the paper [BKR] by M. Barr, J. Kennison and R. Raphael from 2015. Some of these results will be reviewed here. The first three sections of [BKR] build the tools required for the study of the domain-like objects but are much more general and

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are used to examine limit closures of other subcategories of domains in $\mathcal{C}$. Only two of these will come up in the sequel.

The limit closure of the subcategory, $\mathcal{D}$, of domains will contain the full rings of sections of sheaves whose stalks are domains. The question: "what are the domain-like objects in $\mathcal{C}$?" has an amazing multiple answer. Call the subcategory of, as yet to be defined, domain-like objects $\mathcal{K}_{\text{dom}}$. It is (1) the limit closure of $\mathcal{D}$, (2) an essentially algebraic subcategory of $\mathcal{C}$, (3) a reflective subcategory of $\mathcal{C}$ with a reflection functor defined by a transfinite inductive procedure, and (4) each ring in $\mathcal{D}$ is isomorphic to the full rings of sections of sheaves whose stalks are domains. (References will follow later in this section.) The objects in $\mathcal{K}_{\text{dom}}$ will be called *DL-closed*, for *domain-like*.

One other line of reasoning from [BKR] will come up. The same question is asked about the subcategory $\mathcal{D}_{\text{ic}}$ of integrally closed domains (integrally closed in their fields of fractions). The answer, $\mathcal{K}_{\text{ic}}$, has a development that often runs in parallel to that of $\mathcal{K}_{\text{dom}}$ and that will be reflected in the rest of the summary. Many details that can be found in [BKR] will be, of course, omitted in what follows.

The objects in the limit closure in $\mathcal{C}$ of a subcategory of domains are always commutative and without non-zero nilpotent elements. The development can thus be done in the category of commutative rings without non-zero nilpotent elements (commutative semiprime ring) that is denoted $\mathcal{R}$. This was the context used in [BKR].

For the objects in $\mathcal{K}_{\text{ic}}$, the term *(2,3)-closed* is used in [BKR]. The more usual terminology for these rings is *seminormal* (see [Sw] and [V]) and the *(2,3)-closure* was called the *seminormalization* in [Sw], where R. Swan first elaborated the theory. However, because of their descriptive value, the terms *(2,3)-closed* and *(2,3)-closure* will be retained in this paper.

1.1. Definition. *[BKR, Definition 4.1.1]*

1. If $R \in \mathcal{R}$, R is called (2,3)-closed *if whenever $a^3 = b^2 \in R$ there is $c \in R$ with $c^2 = a$ and $c^3 = b$.*

2. If $R \in \mathcal{R}$, R is called DL-closed *if whenever $a^3 = b^2 \in R$ and if, moreover, a is a square modulo each prime ideal of R, then there is $c \in R$ with $c^2 = a$ and $c^3 = b$.*

In the situation of Definition 1.1(1) and (2), the element c is unique ([BKR, Proposition 4.1.6]).

The condition in the definition of DL-closed rings about the squares is more manageable than at first glance by [BKR, Proposition 4.1.5]: $a \in R$ is a square modulo each prime of R if and only if there are $r_1, \dots, r_k \in R$, for some $k \in \mathbb{N}$, with $\prod_{i=1}^{k}(a - r_i^2) = 0$. This fact based on the compactness of the topology used in the study, will come up often.

1.2. Definition. *[BKR, Definition 2.2.20] For $R \in \mathcal{R}$ the* domain topology *is defined on Spec R and has as a subbase of open sets $\{V(r)\}_{r \in R}$, where $V(r) = \{\mathfrak{p} \in \text{Spec}\, R \mid r \in \mathfrak{p}\}$.*

The domain topology is coarser than the patch topology, that is known to be compact([H, Theorem 1]). Hence, the domain topology is compact. For $R \in \mathcal{R}$, the sheaf, called E_R, most relevant here for $\mathcal{K}_{\text{dom}}$ is that over $\text{Spec}\, R$ with the domain topology and stalks of the form $R/\mathfrak{p}$, $\mathfrak{p} \in \text{Spec}\, R$. The construction of the sheaf in [BKR, §3] is much more general than is needed here. The fact that the maps from R to the stalks are surjections simplifies the structure. The basic open sets in the *space étalé* are partial sections using elements of R and domain open sets in Spec R, i.e., for $r \in R$ and a domain open set U, $\{r + \mathfrak{p} \mid \mathfrak{p} \in U\}$ is a basic open set in $\bigcup_{\mathfrak{q} \in \operatorname{Spec} R} R/\mathfrak{q}$.

The next theorem is the major one of Section 4 of [BKR] (see also [K, Theorem A]).

1.3. THEOREM. *[BKR, Theorem 4.1.3] For $R \in \mathcal{R}$, the following are equivalent.*

(1) R is DL-closed,

(2) R is isomorphic to the ring of global sections of the sheaf E_R,

(3) R is isomorphic to the ring of global sections of some sheaf whose stalks are domains,

(4) R is in the limit closure of the subcategory of domains.

The subcategories $\mathcal{K}_{\text{dom}}$ and $\mathcal{K}_{\text{ic}}$ are reflective subcategories of $\mathcal{R}$. A few words need to be said about the reflection functors from $\mathcal{R}$ to $\mathcal{K}_{\text{dom}}$ and from $\mathcal{R}$ to $\mathcal{K}_{\text{ic}}$. (The same symbol is used for the subcategory and the functor.) These both follow from an inductive procedure for constructing the (2,3)-closure and the DL-closure of $R \in \mathcal{R}$. Note that if $\mathcal{K}_{\text{dom}}$ is applied to a general object R of $\mathcal{C}$, R is first made commutative by dividing out by the ideal generated by $\{xy - yx\}_{x,y \in R}$ to get $\overline{R}$, and then by the ideal of nilpotent elements $(\text{rad}(\overline{R}))$.

Note. The symbol $\subset$ will mean proper inclusion throughout.

1.4. DEFINITION. *[BKR, 4.2.1] 1. If $R \subset S$ are in $\mathcal{R}$, S is a* simple (2,3)-extension of R *if there is $s \in S$ such that $S = R[s]$ and $s^2, s^3 \in R$. More generally, S is a* (2,3)-extension of R *if there is an ordinal-indexed sequence $S_0 \subset S_1 \subset S_2 \subset \cdots S_\omega \subset \cdots \subset S_\alpha$ such that (a) $R = S_0$, (b) for all $\beta < \alpha$, $S_{\beta+1}$ is a simple (2,3)-extension of S_β, (c) if $\beta < \alpha$ is a limit ordinal then $S_\beta = \bigcup_{\gamma < \beta} S_\gamma$, and (d) $S = S_\alpha$.*

2. If $R \subset S$ are in $\mathcal{R}$, S is a simple DL-extension of R *if there is $s \in S$ such that $S = R[s]$ and $s^2, s^3 \in R$ and, moreover, s^2 is a square modulo each prime ideal of R. In addition, S is a DL-extension of R if there is a sequence as in part 1 where each simple extension is a simple DL-extension.*

Note that a misprint in [BKR, Definition 4.2.1(1)] has been corrected here. That is, s^2 is a square modulo each prime ideal of R, not s.

Any field is (2,3)-closed, as is a product of fields. (In fact any $R \in \mathcal{VN}$ is (2,3)-closed, by Theorem 1.3 (3).) Hence, any $R \in \mathcal{R}$ embeds in a (2,3)-closed and a DL-closed ring. It follows that when the process in Definition 1.4 stops, it is at a (2,3)-closed ring in part 1, and at a DL-closed ring in part 2. These rings are unique up to an isomorphism over R.

With this introduction to the framework of the paper, it is now possible to give an outline of the results.

Section 2 deals with some information about DL and (2,3)-extensions and closures. It also introduces the *split integral closure of a ring R in a ring S*. It is smaller than the integral closure but is more useful in this context. It is then shown (Proposition 2.15) that a ring R is DL-closed, respectively (2,3)-closed, if and only if the stalks of the Pierce sheaf are DL-closed, respectively (2,3)-closed. Section 3 takes a look at some simple examples of DL and (2,3)-closures.

Section 4, the core of the paper, looks at a specific example of a ring that is not DL-closed. It is the ring R of continuous real valued functions on $\mathbb{R}$ that have a continuous first derivative (the ring of C^1 functions). The DL and (2,3)-closures of the ring R turns out to be very rich in structure. Both $K = \mathcal{K}_{\mathrm{dom}}(R)$ and $S = \mathcal{K}_{\mathrm{ic}}(R)$ are seen to lie in the ring, L, of continuous functions that are C^1 on a dense open set D. The split integral closure of R in L is denoted T. The ring L is (2,3)-closed and it follows that $R \subset K \subset S \subset L$, and all the inclusions are strict. Moreover, $T \subset L$ and $K = T \cap S$.

Complete descriptions of the elements of K and S have not been found but a great deal can be said about them. However, a characterization is given of $f \in L$ such that $R[f]$ is a simple DL-extension (Proposition 4.32).

Much of Section 4 is devoted to establishing properties of elements of T. It is shown that if $f \in T$, then $\frac{d^+}{dx}f(x)$ and $\frac{d^-}{dx}f(x)$ exist for all $x \in \mathbb{R}$ and, moreover, $\frac{d^+}{dx}f$ is right continuous and $\frac{d^-}{dx}f$ is left continuous (Corollaries 4.36 and 4.37) (and a third property predicted in [K] in Proposition 4.38). A partial converse is established: If $f \in L$ is C^1 on a dense open set D and D^c is of finite Cantor-Bendixson index, then the existence of the two one-sided derivatives implies $f \in T$ (Theorem 4.40). However, Proposition 4.41 shows that there are elements $f \in K$ where D^c is not of finite Cantor-Bendixson index.

Section 5 continues the discussion of Section 4 with examples to show how the (2,3)-closure S and DL-closure K for R differ. In particular, there are elements of S that have vertical tangent lines and they cannot be in K (Corollary 5.43). Moreover, L is not integral over R, making L different from S (Proposition 5.48). For any dense open set $D \subset \mathbb{R}$ there is $f \in S \setminus K$ that has a derivative everywhere but the derivative is not continuous at any point of D^c (Proposition 5.50). The final example (Example 5.51) shows that there can be an element of $\mathrm{C}(\mathbb{R})$ that satisfies all the properties of elements of T but is not in L. Hence, those properties are not exhaustive. It resolves in the negative a conjecture by Kennison ([K, Example 7.5]) about the nature of the elements of K.

Section 6 is an addendum. It shows details of splines, called M-splines, used in Sections 4 and 5, and may not be familiar to the reader.

2. General remarks about the DL and (2,3)-closures.

The purpose of this section is to consider the DL and (2,3)-extensions and closures of general objects in $\mathcal{R}$. Later sections will deal with specific examples.

Notation. For $R \in \mathcal{R}$ the following notation will be used throughout. Any copy of the DL-closure of R will be written $\mathcal{K}_{\mathrm{dom}}(R)$; any copy of the (2,3)-closure of R will be written $\mathcal{K}_{\mathrm{ic}}(R)$. All such copies, in both cases, are isomorphic over R.

In a limit closed category, pullbacks exist. In particular, if K_1 and K_2 are DL-closed subrings of a ring T, then $K_1 \cap K_2$ is also DL-closed. Similarly, for (2,3)-closed rings. This means that if R is a subring of a DL-closed ring T then the DL-closure of R in T is the unique smallest DL-closed subring of T containing R. Similarly for (2,3)-closed.

The following results add information about the DL-closure and will come up in later parts of the paper.

2.5. PROPOSITION. *Let $R \in \mathcal{R}$ and R' a DL-extension of R. For $a \in R'$, there are $u_1, \ldots, u_k \in R$, for some $k \in \mathbb{N}$, such that $\prod_{i=1}^{k}(a - u_i) = 0$.*

PROOF. According to [BKR, Proposition 4.2.2], the spaces $\operatorname{Spec} R'$ and $\operatorname{Spec} R$ are homeomorphic in the domain topology and for each $\mathfrak{q} \in \operatorname{Spec} R'$ lying over $\mathfrak{p} \in \operatorname{Spec} R$, $R'/\mathfrak{q} \cong R/\mathfrak{p}$ via the inclusion of R into R'. For each $\mathfrak{q} \in \operatorname{Spec} R'$, there is $v(\mathfrak{q}) \in R$ such that $a \equiv v(\mathfrak{q}) \pmod{\mathfrak{q}}$. Then, there is a neighbourhood V, namely $V(a - v(\mathfrak{q}))$, in the domain topology, such that $a \equiv v(\mathfrak{q}) \pmod{\mathfrak{r}}$, for all $\mathfrak{r} \in V$. By the compactness of $\operatorname{Spec} R'$ in the domain topology, there is a finite subset $u_1, \ldots, u_k$ of $\{v(\mathfrak{q})\}_{\mathfrak{q} \in \operatorname{Spec} R'}$ such that $\prod_{i=1}^{k}(a - u_i) = 0$, using [BKR, Proposition 4.1.4]. ∎

Notice that, in Proposition 2.5, the equation $\prod_{i=1}^{k}(a - u_i) = 0$ implies also that $\prod_{i=1}^{k}(a^2 - u_i^2) = 0$. Moreover, if, in the proposition, a is a square modulo each prime in $\operatorname{Spec} S$, then the u_i may be taken to be squares of elements of R (see [BKR, Proposition 4.1.5]).

To continue with general statements, the following is also true. Since the (2,3)-closure of $R \in \mathcal{R}$ is DL-closed, the DL-closure of R, $\mathcal{K}_{\mathrm{dom}}(R)$, may be taken to lie in $\mathcal{K}_{\mathrm{ic}}(R)$.

2.6. PROPOSITION. *Let $R \in \mathcal{R}$, $S = \mathcal{K}_{ic}(R)$ and $K = \mathcal{K}_{dom}(R) \subseteq S$. Let a be in S. Then, $a \in K$ if and only if there exist $u_1, \ldots, u_k \in R$ with $\prod_{i=1}^{k}(a - u_i) = 0$.*

PROOF. One direction follows from Proposition 2.5; that is, if $a \in K$ then there exist $u_1, \ldots, u_k \in R$ with $\prod_{i=1}^{k}(a - u_i) = 0$.

Conversely, assume $\prod_{i=1}^{k}(a - u_i) = 0$, for some $u_1, \ldots, u_k \in R$. It must be shown that $a \in K$. [BKR, Proposition 4.2.2] is used. Since S is also DL-closed, it is isomorphic to the ring of sections of the sheaf based on $\operatorname{Spec} S$ with the domain topology. The base spaces $\operatorname{Spec} S$, $\operatorname{Spec} R$ and $\operatorname{Spec} K$ are all homeomorphic in the domain topologies via the natural maps. The stalks of the sheaf for S may be larger than those for the two other sheaves. Now look at the section of the sheaf for S defined by a. For each $\mathfrak{p} \in \operatorname{Spec} S$, there is some i, $1 \le i \le k$, with $a - u_i \in \mathfrak{p}$. Hence, a and u_i coincide modulo all the primes in the (domain) open set $V(a - u_i)$. These open sets cover $\operatorname{Spec} S$ and, since a is a section, the projections into the stalks are into $R/\mathfrak{q}$ for some prime $\mathfrak{q} \in \operatorname{Spec} S = \operatorname{Spec} R$. However, this is exactly how the elements of K are defined (see [BKR, Theorem 4.1.3(DL-2)]); they are the sections of the canonical sheaf, called E_R in [BKR, §3.2], (using the domain topology) for R. ∎

Notice that the condition in Proposition 2.6 is saying that for $a \in S$, $a \in K$ if and only if it satisfies a monic polynomial over R *which factors into monic linear factors*. Elements of S are integral over R ([BKR, Proposition 4.2.2]) but this condition is stronger.

These special integral elements will play a role in later sections. The next result will show that they can be used to define a subring of the integral closure.

2.7. Proposition. *Let $R \subseteq S$ be in $\mathcal{R}$. Then*

$$T = \left\{ s \in S \mid \textit{there exists for some } k \in \mathbb{N},\, u_1, \dots, u_k \in R \textit{ such that } \prod_{i=1}^{k}(s-u_i) = 0 \right\}$$

forms a subring of S.

Proof. Suppose $a, b \in T$ with $u_1, \dots, u_k, v_1, \dots, v_m \in R$ with $\prod_{i=1}^{k}(a-u_i) = 0$ and $\prod_{j=1}^{m}(b-v_j) = 0$. By the reasoning of [BKR, Proposition 4.1.5], for each $\mathfrak{p} \in \operatorname{Spec} S$, there is i with $a \equiv u_i \mod \mathfrak{p}$ and some j with $b \equiv v_j \mod \mathfrak{p}$. The i and j need not be unique. It follows that $\prod_{i,j}((a+b)-(u_i+v_j)) = 0$, since for any prime $\mathfrak{p}$, one of the factors is in $\mathfrak{p}$. The product ab is done in a similar manner. This shows that T is a subring. ∎

The expressions in the proof of Proposition 2.7 for $a+b$ and, implicitly, ab may be redundant but will always contain factors whose product is zero. The ring T in Proposition 2.7 will be useful and deserves a name.

2.8. Definition. *(1) If $R \subseteq S$ are rings and every $s \in S$ satisfies an equation $\prod_{i=1}^{k}(s-u_i) = 0$, where each $u_i \in R$, is said to be a* split integral extension *of R.*

(2) Let $R, S \in \mathcal{R}$, the subring $\{s \in S \mid$ there exist, for some $k \in \mathbb{N}$, $u_1, \dots, u_k \in R$ with $\prod_{i=1}^{k}(s-u_i) = 0\}$ is called the split integral closure of R *in S and denoted $\mathcal{SI}(R, S)$.*

2.9. Proposition. *If $R \subseteq S \subseteq T$ are rings and both $R \subseteq S$ and $S \subseteq T$ are split integral extensions, then $R \subseteq T$ is also a split integral extension.*

Proof. If $t \in T$ satisfies $\prod_{i=1}^{k}(t-s_i) = 0$ with $s_i \in S$ and each s_i satisfies $\prod_{j=1}^{k(i)}(s_i-u_{i,j}) = 0$ with each $u_{i,j} \in R$ then $\prod_{i=1}^{k}\prod_{j=1}^{k(i)}(t-u_{i,j}) = 0$. ∎

2.10. Proposition. *If $R \subseteq S$ are rings and S is DL-closed, then $T = \mathcal{SI}(R, S)$ is DL-closed.*

Proof. If $a \in S$ is such that a^2 is a square modulo the primes of R, then there are $t_1, \dots, t_k \in \mathcal{SI}(R, S)$ with $\prod_{i+1}^{k}(a^2-t_i^2) = 0 = \prod_{i=1}^{k}(a-t_i)(a+t_i)$. By the argument of Proposition 2.9, $a \in \mathcal{SI}(R, S)$. ∎

2.11. Corollary. *If, in Proposition 2.10, $S = \mathcal{K}_{ic}(R)$, then $T = \mathcal{K}_{dom}(R)$.*

Proof. This follows immediately from the Proposition 2.7 and Proposition 2.10. ∎

2.12. Corollary. *Let R be a ring and $S = \mathcal{K}_{ic}(R)$. If L is a DL-closed ring containing S then $\mathcal{SI}(R, L) \cap S = \mathcal{K}_{dom}(R)$.*

Proof. This is immediate from Proposition 2.10. ∎

Any ring $R \in \mathcal{R}$ can be embedded in a DL-closed ring W, for example

$$W = \prod_{\mathfrak{p} \in \operatorname{Spec} R} R/\mathfrak{p}.$$

2.13. PROPOSITION. *Let $R \in \mathcal{R}$ embedded in a DL-closed ring W, $T = \mathcal{SI}(R, W)$, and $\phi\colon \operatorname{Spec} T \to \operatorname{Spec} R$ given by intersection. If $\phi(\mathfrak{p}) = \mathfrak{q}$, then $R/\mathfrak{q} \cong T/\mathfrak{p}$, via the natural monomorphism.*

PROOF. For any $f \in T$, there are $u_1, \dots, u_k \in R$ with $\prod_{i=1}^{k}(f - u_i) = 0$. This means that for any $\mathfrak{p} \in \operatorname{Spec} T$, there is i such that $f \equiv u_i \pmod{\mathfrak{p}}$. This shows that the monomorphism $R/\mathfrak{q} \to T/\mathfrak{p}$ is a surjection. ∎

However, it can happen, in the situation of the proposition, that ϕ is not one-to-one. Example 3.23 shows this. However, in the above situation, the ring T is always DL-closed by Proposition 2.7. In addition, although $\phi\colon \operatorname{Spec} T \to \operatorname{Spec} R$ need not be one-to-one, all the stalks in $\phi^{-1}(\mathfrak{p})$ are isomorphic, for each $\mathfrak{p} \in \operatorname{Spec} R$.

If $R[s]$ is a simple (2,3)-extension of R then the inclusion $f\colon R \to R[s]$ is an epimorphism (in the categorical meaning of "epimorphism") in the category $\mathcal{R}$ (a special case of [BKR, Theorem 2.3.1(12)]). However, it is not an epimorphism in the category of all commutative (unitary) rings.

2.14. PROPOSITION. *Let $R \in \mathcal{R}$ and $R[s]$ a simple (2,3)-extension of R. Then, the inclusion of R in $R[s]$ is not an epimorphism in the category of all commutative rings.*

PROOF. Define $I = \{a \in R \mid as \in R\}$. This is a non-zero ideal since $s^2 \in I$. Let $J = I[s]$, an ideal of $R[s]$. Consider $T = R[s]/J$ and $\alpha\colon R[s] \to T$ given by $as + b \overset{\alpha}{\mapsto} \bar{b}$ and $\beta\colon R[s] \to T$ by $as + b \overset{\beta}{\mapsto} \overline{as + b}$. Notice that if $as + b \in R$ then $a \in I$. Hence, α and β coincide on R. However, $\alpha(s) \neq \beta(s)$. ∎

The next observation is to show that the property of being (2,3)-closed and the property of being DL-closed are Pierce properties. Recall from [P] (see also [BS]) the definition of the *Pierce sheaf*, here restricted to commutative rings. For a ring R, let $\mathbf{B}(R)$ be the boolean ring of idempotents of R and $\mathcal{X} = \operatorname{Spec} \mathbf{B}(R)$. For each $\mathbf{x} \in \mathcal{X}$ consider the ideal $\mathbf{x}R$ and the quotient $R_{\mathbf{x}} = R/\mathbf{x}R$. The rings $R_{\mathbf{x}}$ are the stalks of the Pierce sheaf for R over the boolean space $\mathcal{X}$. The stalks will have the discrete topology. The key property of this sheaf is that the ring of global sections is canonically isomorphic to the ring R ([P, Theorem 4.4]); sections and elements of R will be given the same symbol. Another property that will be used is that if two sections section $r, s \in R$ coincide in a stalk $R_{\mathbf{x}}$, i.e., $r_{\mathbf{x}} = s_{\mathbf{x}}$, then there is $e \in \mathbf{B}(R) \setminus \mathbf{x}$ such that $e(r - s) = 0$ (see [P, Lemma 4.3]). Finally, a property $\mathcal{P}$ of rings is called a *Pierce property* if R has property $\mathcal{P}$ if and only if each Pierce stalk has property $\mathcal{P}$. Note that being semiprime is a Pierce property. (If $\mathcal{P}$ also stands for the class of rings with property $\mathcal{P}$, then, in the terminology of [BS], $\mathcal{P}$ is a Pierce property if and only if $\{\mathcal{P}, \mathcal{P}\}$ is a *Pierce pair.*)

2.15. PROPOSITION. *The property of being (2,3)-closed and the property of being DL-closed are Pierce properties.*

Proof. Suppose first that R is (2,3)-closed or DL-closed. With the notation above, consider a Pierce stalk $R_{\mathbf{x}}$. This ring will be presented as a direct limit of direct ring summands of R, and, hence, of (2,3) and DL-closed rings. Consider the diagram of all Re, where $e \in \mathbf{B}(R) \setminus \mathbf{x}$ with maps $\phi_{e,e'} \colon Re \to Re'$ when $e, e' \notin \mathbf{x}$ and $ee' = e'$ and $\phi_{e,e'}(re) = re'$. Let the direct limit be A with maps $\phi_2 \colon Re \to A$.

There are also well-defined maps $\psi_e \colon Re \to R_{\mathbf{x}}$ defined by $\psi_e(re) = r + \mathbf{x}$. If $re = 0$ then $r = r(1-e)$ and $1 - e \in x$. These are compatible with the $\phi_{e,e'}$. Hence, there is a unique $\zeta \colon A \to R_x$, making the whole diagram commute. It is easy to check that ζ is an isomorphism.

(Note that this part of the proof works for any limit closed subcategory $\mathcal{L}$ of $\mathcal{R}$, i.e., if $R \in \mathcal{L}$ then each Pierce stalk is in $\mathcal{L}$.)

In the other direction the proof will be given for DL-closed; the proof for (2,3)-closed is similar and easier. Now suppose that R is such that, for each $\mathbf{x} \in \mathcal{X}$, that $R_{\mathbf{x}}$ is DL-closed. Suppose that $R[a]$ is a simple DL-extension of R. Then, $a^3 = r \in R$ and $a^2 = s \in R$, and there are $u_1, \ldots, u_k \in R$ with $\prod_{i=1}^k (a^2 - (u_i)^2) = 0$. Note that $r^2 = s^3$.

Fix $\mathbf{x} \in \mathcal{X}$. The equations $(r_{\mathbf{x}})^2 = (s_{\mathbf{x}})^3$ and $\prod_{i=1}^k (s_{\mathbf{x}} - (u_i^2)_{\mathbf{x}}) = 0$ hold in $R_{\mathbf{x}}$. According to [BKR, Definition 4.1.1(2)], there exists a *unique* $c_{\mathbf{x}}$, for some $c \in R$, such that $(c_{\mathbf{x}})^3 = r_{\mathbf{x}}$ and $(c_{\mathbf{x}})^2 = s_{\mathbf{x}}$. Then there is $e \in \mathbf{B}(R) \setminus \mathbf{x}$ such that $(r^2 - s^3)e = 0$, $\prod_{i=1}^k (s - u_i^2)e = 0$, $(c^3 - r)e = 0$ and $(c^2 - s)e = 0$. This can be done for each $\mathbf{x} \in \mathcal{X}$. The compactness of $\mathcal{X}$ and the uniqueness of the $c_{\mathbf{x}}$ mean that c is, in fact, a global section of the Pierce sheaf. By [BKR, Proposition 4.1.6], it follows that, in the semiprime ring $R[a]$, $a = c \in R$, contradicting the assumption that $R[a]$ was a simple DL-extension. ∎

An example of a DL-closed ring R would be one where every Pierce stalk is a domain. These rings were characterized in [NR] and mentioned in [BKR, Example 4.4.2]; these include all commutative von Neumann regular rings, i.e., those where the Pierce stalks are fields. In this context it might be mentioned that there are (2,3)-closed domains that are not integrally closed (see [BKR, Example 5.1.7]).

3. Some simple examples of DL-closures. In this short section examples are presented of rings that are not DL-closed along with their DL-closures. The starting point was an example pointed out to us by John Kennison, this is the ring $R_{2,2}$ in what follows (see [K, Example 7.6]).

Notation. For a prime integer p and $k \in \mathbb{N}$, $R_{p,k} = \{(a, b) \mid a, b \in \mathbb{Z}, a \equiv b \pmod{p^k}\}$.

3.16. Lemma. *For $k < l$ and any prime p, the extension $R_{p,l} \subset R_{p,k}$ is a DL-extension if and only if it is a (2,3)-extension.*

Proof. Only one direction needs to be proved. Suppose the extension is a (2,3)-extension. Then for any $(x, y) \in R_{p,k}$, $((x, y)^2 - (x, x)^2)((x, y)^2 - (y, y)^2) = (0, 0)$. This shows that (x, y) is in the DL-closure of $R_{p,l}$ by Proposition 2.6 . ∎

The following lemma will be used without mention in what follows.

3.17. Lemma. *For a prime integer p and $k \geq 1$, given $(x,y) \in R_{p,k} \setminus R_{p,k+1}$, then, for $m \geq 1$, $R_{p,k+m}[(x,y)] = R_{p,k}$.*

Proof. Put $R_{p,k+m}[(x,y)] = S$. Write $x = y + p^k z$, where $p \nmid z$. Then, $(x,y) = (y + p^k z, y)$ and subtracting (y,y) yields $(p^k z, 0) \in S$. Write $1 = ap^m + bz$, for some $a, b \in \mathbb{Z}$. From this, $p^k = p^{k+m}a + p^k bz$. Then, $(p^{k+m}a, 0) \in R_{p,k+m}$ and $(p^k bz, 0) \in S$ giving $(p^{k+m}a, 0) + (p^k bz, 0) = (p^k, 0) \in S$. For any $(u,v) \in R_{p,k}$, $u = v + p^k t$, for some $t \in \mathbb{Z}$. Then $(t,t)(p^k, 0) = (p^k t, 0) \in S$ and $(p^k t, 0) + (v,v) = (u,v) \in S$. ■

3.18. Proposition. *Let p be a prime integer and $k \in \mathbb{N}$. Then,*

(1) $R_{p,1}$ is (2,3)-closed and, hence, also DL-closed;

(2) for $k > 1$, $R_{p,k}$ is not DL-closed and there is $u \in R_{p,k-1}$ so that $R_{p,k}[u]$ is a simple DL-extension and, moreover, $R_{p,k}[u] = R_{p,k-1}$.

Proof. (1) Consider $r = (a,b)$ and $s = (c,d)$ with $r, s \in R_{p,1}$ and $r^3 = s^2$. An element (x,y) must be found with $x^2 = a, x^3 = c, y^2 = b$ and $y^3 = d$. Since $\mathbb{Z} \times \mathbb{Z}$ is (2,3)-closed, such integers can be found. It remains to show that $(x,y) \in R_{p,1}$. Since $p \mid x^2 - y^2$, $p \mid x - y$ or $p \mid x + y$. In the former case, $(x,y) \in R_{p,1}$ as required. If $p \nmid x - y$ then $p \mid x + y$ and since $p \mid x^3 - y^3$, $p \mid (x^2 + xy + y^2)$. But $x^2 + xy + y^2 = x(x+y) + y^2$ shows that $p \mid y^2$ and then $p \mid y$. From this, $p \mid y$ and $p \mid x + y$ yield $p \mid x$ and then $p \mid x - y$, a contradiction.

(2) For any prime p and $k > 1$, consider $u = (p^{k-1}, 0)$. Then $u^2, u^3 \in R_{p,k}$. By Lemma 3.16, this is a DL-extension. However, $u \notin R_{p,k}$ showing that $R_{p,k}[u]$ is a simple DL-extension. Moreover, $R_{p,k}[u] = R_{p,k-1}$, since if $(a,b) \in R_{p,k-1}$ then $a - b = mp^{k-1}$, for some $m \in \mathbb{Z}$. Then, $(a,b) = (a - mp^{k-1}, b) + mu = (b,b) + mu$, making $(a,b) \in R_{p,k}[u]$. ■

Notice that $R_{p,1}$ is also the (2,3)-closure of $R_{p,k}$.

The number of steps to get from $R_{p,k}$ to $R_{p,1}$ by the method in Proposition 3.18 is not minimal for any prime, as will be seen.

3.19. Proposition. *Let p be an odd prime and $k \in \mathbb{N}$. (1) For $1 \leq l \leq k$, $R_{p,2k+l} \subset R_{p,k}$ is a simple DL-extension. (2) $R_{p,3k+1} \subset R_{p,k}$ is not a simple DL-extension.*

Proof. (1) Consider $u = (x,y) = (p^k(p^l - 1)^3, p^k(p^l - 1)^2)$. Then $u \in R_{p,k} \setminus R_{p,k+1}$, showing that $x = y + p^k z$, for some z with $p \nmid z$ (and Lemma 3.17 applies). To show that $u^2 \in R_{p,2k+l}$, note that $p^{2k}(p^l - 1)^6 - p^{2k}(p^l - 1)^4 = p^{2k}(p^l - 1)^4((p^l - 1)^2 - 1) = p^{2k}(p^l - 1)^4(p^{2l} - 2p^l)$, which is divisible by p^{2k+l}. Moreover, it is required that $u^3 \in R_{p,2k+l}$. The difference between the two components of u^3 is $p^{3k}(p^l - 1)^6((p^l - 1)^3 - 1)$. But the last factor is prime to p. Hence, this is divisible by p^{2k+l} precisely when $1 \leq l \leq k$.

(2) Assume, on the contrary, that $R_{p,3k+1} \subset R_{p,k}$ is a simple DL-extension using $(x,y) \in R_{p,k} \setminus R_{p,k+1}$. It follows that $p^k \mid (x - y)$ but $p^{k+1} \nmid (x - y)$. Write $x = y + p^k z$, where $p \nmid z$. By assumption, $(x^2, y^2) \in R_{p,3k+1}$. Hence, $p^{2k+1} \mid (x + y)$. Write $x = -y + p^{2k+1}t$.

From the two equations for x, $0 = 2y + p^k z - p^{2k+1}t$ and $2x = -p^k z + p^{2k+1}t$. It follows that $p^k \mid y$ but $p^{k+1} \nmid y$, because $p \nmid z$. Similarly, $p^k \mid x$ but $p^{k+1} \nmid x$.

Since, $(x^3, y^3) \in R_{p,3k+1}$, $p^{3k+1} \mid (x^3 - y^3)$. From this, $p^{2k+1} \mid (x^2 + xy + y^2)$. Write $x^2 + xy + y^2 = x(x+y) + y^2$. The first term is divisible by $p^k p^{2k+1} = p^{3k+1}$, while the second is divisible by p^{2k} but no higher power of p. Their sum is divisible by p^{2k+1}, a contradiction. ■

The following will be used in Example 3.22.

3.20. Corollary. *Let p be an odd prime and $k \in \mathbb{N}$, then the number $\kappa(k)$ of simple DL-extensions needed to go from $R_{p,k}$ to $R_{p,1}$ grows without limit as $k \to \infty$.*

Proof. Suppose that $R_{p,k} \subset R_{p,k_1} \subset \cdots \subset R_{p,1}$ is a chain of simple DL-extensions. Put $k_0 = k$. Then, for any $i \geq 1$, according to 3.19, $k_i \geq k_{i-1}/3$. That is, the index can go down by a third, but no more. ■

The case $p = 2$ behaves somewhat differently.

3.21. Proposition. *(1) For any $k \geq 1$, $R_{2,k}$ is a simple DL-extension of $R_{2,2k}$. (2) For any $k > 1$, $R_{2,k}$ is not a simple DL-extension of $R_{2,3k-1}$.*

Proof. (1) Consider $R_{2,2k}[(2^k, 0)]$. Clearly $(2^k, 0) \in R_{2,k} \setminus R_{2,k+1}$ and $(2^{2k}, 0), (2^{3k}, 0) \in R_{2,2k}$. Then, as in the proof of Lemma 3.17, $R_{2,2k}[(2^k, 0)] = R_{2,k}$.

(2) Suppose $R_{2,3k-1}[(x, y)] = R_{2,k}$ is a simple DL-extension. Then, $2^k \mid x - y$ but $2^{k+1} \nmid x - y$. Write $x = y + 2^k z$, z odd. From $2^{3k-1} \mid x^2 - y^2$ and $2^{3k-1} \mid x^3 - y^3$, $2^{2k-1} \mid x + y$ and $2^{2k-1} \mid x^2 + xy + y^2$. As before, this implies that $2^{2k-1} \mid y^2$, showing that y is even. Write $y = 2^l u$, where u is odd. From this, $2l > 2k - 1$ and $2l \geq 2k$. Then, $x = y + 2^k z = 2^l u + 2^k z = 2^k(2^{l-k} u + z)$. From this, $x + y = 2^k(2^{l-k} u + z) + 2^l u = 2^{l+1} u + 2^k z = 2^k(2^{l-k+1} u + z)$; this expression is divisible by 2^k but not by any higher power of 2 since $2^{l-k+1} u + z$ is odd, a contradiction. ■

Proposition 3.21(1) can be improved when $k \geq 3$. In that case, $R_{2,2k+1}[(x, y)] = R_{2,k}$ is a simple DL-extension when $(x, y) = (2^{k-1} + 2^k, 2^{k-1})$.

Although the bounds given for $p = 2$ are different from those given for odd primes in 3.19, there is sufficient information to say that if $R_{2,m}[(x, y)] = R_{2,k}$ is a simple DL-extension with $m > k$, then $2k \leq m < 3k - 1$. This shows that Corollary 3.20 applies to $p = 2$, as well.

3.22. Example. *There is a ring R such that a countably infinite number of simple DL-extension steps are required to go from R to its DL-closure.*

Proof. Consider the ring T of sequences from $\mathbb{Z}$ that are eventually constant. Fix a prime p and consider the ring R, a subring T, with $(a_m)_{m\in\mathbb{N}} \in R$ if for any $n \in \mathbb{N}$, $p^n \mid a_{2n-1} - a_{2n}$. Note that Lemma 3.16 applies to the ring R. Moreover, the ring $S \subset T$ where $(b_m) \in S$ if for each $n \in \mathbb{N}$, $p \mid b_{2n-1} - b_{2n}$, is DL-closed. Then, $R \subset S \subset T$. For each $n \in \mathbb{N}$, let $R_n = \{(a_m) \in R \mid p^n \mid a_{2n-1} - a_{2n}$ and $a_m = 0$, $m \neq 2n-1, 2n\}$; R_n is an ideal of R; and, as a ring, is isomorphic to $R_{p,n}$. Similar notation, S_n, is used for S.

For $z \in \mathbb{Z}$, put $\mathbf{z}$ to be the sequence that is constantly z. Clearly $R = \bigoplus_{n\in\mathbb{N}} R_n + \{\mathbf{z} \mid z \in \mathbb{Z}\}$ and $S = \bigoplus_{n\in\mathbb{N}} S_n + \{\mathbf{z} \mid z \in \mathbb{Z}\}$. It must be shown that S is the DL-closure of R. This will be done by showing that S is the split integral closure of R in S (see Proposition 2.7). Pick $s \in S$. It has the form $s_1 + \cdots + s_m + \mathbf{z}$, for $s_i \in S_i$ and $z \in \mathbb{Z}$.

In $R_{p,i}$ there are $u_{i,1},\dots,u_{i,k_i}\in R_{p,i}$ with $\prod_{j=1}^{k_i}(s_i-u_{i,j})=0$. Put $k=\max\{k_1,\dots,k_m\}$. If $k_i<k$, put $u_{i,j}=0$, for $k_i<j\le k$. Then, define $\bar u_j$ to coincide with $u_{i,j}$ on the $2i-1$ and $2i$ components, $i=1,\dots,m$, and zero elsewhere. Clearly the $\bar u_j\in R$. Then, $\prod_{j=1}^{k}(s-\bar u_j)(s-\mathbf{z})=0$, as required.

Suppose now that S can be achieved from R by a sequence of m simple DL-extensions and take n such that $\kappa(n)>m$ (the notation is that of Corollary 3.20). Now look at $R_n\subset S_n$. The sequence of m simple DL-extensions, when restricted to the components $2n-1$ and $2n$, gives a sequence of m simple DL-extensions going from $R_{p,n}$ to $R_{p,1}$. This is impossible since $\kappa(n)>m$. ∎

The split integral closure of a ring R in a DL-closed ring W need not be the same as the DL-closure, as the following example shows.

3.23. Example. *For $k>1$, the split integral closure of $R_{p,k}$ in $\mathbb{Z}\times\mathbb{Z}$ is not the DL-closure of $R_{p,2}$.*

Proof. The DL-closure of $R_{p,k}$ is, as already shown, $R_{p,1}$. However, $\mathbb{Z}\times\mathbb{Z}$ is the split integral over $R_{p,2}$ since for any $(a,b)\in\mathbb{Z}\times\mathbb{Z}$, $((a,b)-(a,a))((a,b)-(b,b))=0$. ∎

For these rings, $\operatorname{Spec}R_{p,k}$ consists of three sorts of primes: (1) for a prime $q\neq p$, there is a pair of primes $\mathfrak{p}_{q,1}=(q\mathbb{Z}\times\mathbb{Z})\cap R_{p,k}$ and $\mathfrak{p}_{q,2}=(\mathbb{Z}\times q\mathbb{Z})\cap R_{p,k}$, (2) $\mathfrak{p}_p=(p\mathbb{Z}\times\mathbb{Z})\cap R_{p,k}=(\mathbb{Z}\times p\mathbb{Z})\cap R_{p,k}$, and (3) $\mathfrak{p}_1=(\mathbb{Z}\times\{0\})\cap R_{p,k}$ and $\mathfrak{p}_2=(\{0\}\times\mathbb{Z})\cap R_{p,k}$. In Example 3.23 there is only one pair in $\operatorname{Spec}(\mathbb{Z}\times\mathbb{Z})$ that restricts to one prime in $\operatorname{Spec}R_{p,k}$, namely, $p\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}\times p\mathbb{Z}$.

4. The ring of continuously differentiable real functions.

For any topological space X, which may be assumed to be completely regular ([GJ, Chapter 3]), the ring of continuous real valued functions on X, $\mathrm{C}(X)$, is (2,3)-closed ([BKR, Example 5.3.9.]) and, hence, DL-closed. However, unlike $\mathrm{C}(\mathbb{R})$, as already noted, the ring $\mathrm{C}^1(\mathbb{R})$ (the ring of functions with a continuous first derivative) is not DL-closed by [BKR, Example 4.4.5]. The purpose of this section is to examine the DL and (2,3)-closures of $\mathrm{C}^1(\mathbb{R})$. Before going farther to look at these rings, the cited example gives a template for some related rings (Remark 4.24).

Some notation and terminology. The ring of all continuous functions $\mathbb{R}\to\mathbb{R}$ is, as usual, denoted $\mathrm{C}(\mathbb{R})$. If $r\in\mathbb{R}$, the function with constant value r is written $\mathbf{r}$. For general sets, the complement of $A\subseteq X$ will be denoted A^c, when it is clear what X is. The rest of the notation will follows the classic text [GJ]. For $f\in\mathrm{C}(\mathbb{R})$, $\{x\in\mathbb{R}\mid f(x)\neq 0\}$ is denoted $\operatorname{coz}f$, the *cozero set* (an open set). Its complement is $\mathrm{z}(f)$, the *zero set*. In addition, for $f\in\mathrm{C}(\mathbb{R})$, $f^+(x)$ is $f(x)$ if $f(x)>0$ and zero otherwise; $f^-(x)$ is $f(x)$ if $f(x)<0$ and zero otherwise. An open set U in $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals. If (a,b) is one of those intervals it is said to *belong to U*. There are instances below when one-sided limits are used. When $f\in\mathrm{C}(\mathbb{R})$ and $x_0\in\mathbb{R}$ when the limit of f is computed with $x<x_0$ the notation is $\lim_{x\to x_0^-}f(x)$, and analogously $\lim_{x\to x_0^+}f(x)$. Similarly the notation $\frac{d^-}{dx}f(x_0)$ and $\frac{d^+}{dx}f(x_0)$ is used, where defined.

If X is a subset of a topological space Y, its closure is written $\operatorname{cl} X$, and if $x \in Y \setminus X$ and $x \in \operatorname{cl} X$, then x is called a *cluster point of* X (another term often used is *accumulation point*).

In analogy with [BKR, Example 4.4.5] (see also [K, Example 7.4]), there is the following.

4.24. REMARK. *For $n \in \mathbb{N}$, $C^n(\mathbb{R})$, the ring of real functions with n continuous derivatives, is not DL-closed.*

PROOF. For $n \in \mathbb{N}$ put

$$g_n(x) = \begin{cases} x^n & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Note that $g_n \in \mathrm{C}^{n-1}(\mathbb{R})$, but because its n^{th} derivative is $n!$ when $x > 0$ and 0 when $x < 0$, it is not in $\mathrm{C}^n(\mathbb{R})$. However, $g_n^2, g_n^3 \in \mathrm{C}^n(\mathbb{R})$ and $(g_n^2 - x^{2n})(g_n^2 - \mathbf{0}) = \mathbf{0}$, making $\mathrm{C}^n(\mathbb{R})[g_n]$ a simple DL-extension. ∎

However, [BKR, Example 4.4.6], $\mathrm{C}^\infty(\mathbb{R}) = \bigcap_{n \in \mathbb{N}} \mathrm{C}^n(\mathbb{R})$ is DL-closed.

Notation. From now on the focus will be on $\mathrm{C}^1(\mathbb{R})$. The ring $\mathrm{C}^1(\mathbb{R})$ will be called R in this section from now on. The following two symbols will also be used, $K = \mathcal{K}_{\text{dom}}(R)$ and $S = \mathcal{K}_{\text{ic}}(R)$, for the isomorphic copies of these closures lying in $\mathrm{C}(\mathbb{R})$. There are two other subrings of $\mathrm{C}(\mathbb{R})$ that will play a role in what is to follow, called L and T.

4.25. DEFINITION. *An element $f \in C(\mathbb{R})$ is said to be in L if there is a dense open subset $D \subseteq \mathbb{R}$ such that f is C^1 on D.*

Notice that if $f \in \mathrm{C}(\mathbb{R})$ is C^1 on open subsets $\{U_\alpha\}$, then it is C^1 on $\bigcup_\alpha U_\alpha$. Hence, for $f \in L$, it can be assumed that D is the unique maximal open set on which f is C^1. The **notation** used is $(f, D) \in L$ (or if the function needs to be specified, $(f, D_f) \in L$) and *it is assumed that D is the above-mentioned maximal dense open subset of* $\mathbb{R}$ on which f is C^1. (For $(f, D) \in L$, it is possible that $\frac{d}{dx} f(x)$ exists for $x \in D^c$, but it cannot be continuous in a neighbourhood of x: see Propositions 4.41 and 5.50 for illustrations.) The other ring to be studied is $T = \mathcal{SI}(R, L)$. From Corollary 2.11, $K = S \cap T$.

With this notation, it is possible to state the goals of the section. Analytic characterizations of K and S have not been found but many properties of the elements of K, S and T can be stated. In particular that all are in L. However, L is not integral over R, (see Proposition 5.48), and so S, which is integral over R ([BKR, Proposition 4.2.2(1)]) is a proper subring of L. Moreover, properties of elements of K show that it is a proper subring of S. Hence, S is not a subring of T (several classes of examples showing this are found in Section 5). It is unresolved whether or not $T = K$. Certainly Example 3.23 shows that $T \neq K$ is possible. However, it will be shown that if $(f, D) \in T$, then for all $x \in \mathbb{R}$, $\frac{d^+}{dx} f(x)$ and $\frac{d^-}{dx} f(x)$ exist; moreover, the function $\frac{d^+}{dx} f(x)$ is right continuous and $\frac{d^-}{dx} f(x)$ is left continuous (Corollaries 4.36 and 4.37). Partial converses, depending on the nature of D, are found (Theorem 4.40); more exactly, if $(f, D) \in L$, D^c is of finite

Cantor-Bendixon index, and for $x \in \mathbb{R}$, $\frac{d^-}{dx}f(x)$ and $\frac{d^+}{dx}f(x)$ exist, then $f \in T$.

Notation: When $f \in T$, there are, by definition, $u_1, \ldots, u_k \in R$ with $\prod_{i=1}^k (f - u_i) = \mathbf{0}$. In this case it is said that $f \in T$ *using* $u_1, \ldots, u_k$. The functions $u_1, \ldots, u_k \in R$ are not uniquely determined by f.

It first will be seen that L is, in fact, a ring, a lattice and is (2,3)-closed. It is clear that sums and products of elements of L are also in L, so that L is a subring of $\mathrm{C}(\mathbb{R})$. Some properties of L are listed in the next proposition.

4.26. Proposition. *Consider the ring L.*

(1) If $s \in C(\mathbb{R})$ and $s^3 \in L$ then $s \in L$.

(2) L is (2,3)-closed.

(3) L is a lattice.

(4) The module L_R is essential over R_R.

Proof. (1) Suppose $s^3 \in L$ for $s \in \mathrm{C}(\mathbb{R})$ and s^3 is C^1 on the dense open set D. Then, s is C^1 on the dense open set $(\operatorname{coz}(s^3) \cup \operatorname{Int} \mathrm{z}(s^3)) \cap D$.

(2) This is immediate from (1) and the fact that $\mathrm{C}(\mathbb{R})$ is (2,3)-closed ([BKR, Example 5.3.9]).

(3) It suffices to show that for $f \in L$, $|f| \in L$, by [GJ, §1.3]. This follows since if f is C^1 on the dense open set $D \subseteq \mathbb{R}$, then $|f|$ is C^1 on $(\operatorname{coz} a^+ \cup \operatorname{coz} a^- \cup \operatorname{Int} \mathrm{z}(a)) \cap D$, a dense open set.

(4) If $(f, D) \in L$ is non-zero then, $U = \operatorname{coz} f \cap D \neq \emptyset$. Let (r, s) be a non-empty interval in U and choose a C^1 function a whose cozero set is (r, s). Then, $\mathbf{0} \neq af \in R$. ■

Notice that Proposition 4.26(4) also says that L is a ring of quotients of R in the generalized meaning of rings of quotients, i.e., any ring properly containing R and in $Q(R)$. This says, in particular, that the complete rings of quotients of R and L coincide, i.e., $Q(L) = Q(R)$. See [Ste, Chapter XII, §2] (the complete ring of quotients is called the maximal ring of quotients in [Ste]), or [L, Chapter 2]. Because $Q(R)$ is a self-injective regular ring that is an essential L-module extension of L and, thus, must be its complete ring of quotients.

In addition, since L is (2,3)-closed, $S \subseteq L$; in fact, the inclusion is proper, as will be seen later.

In this context, it will be seen that T is also a lattice.

4.27. Proposition. *The ring T is a lattice.*

Proof. Let $f \in T$ using $u_1, \ldots, u_k$. It must be shown that $|f| \in T$. The function $|f| \in L$ by Proposition 4.26(4). However, $(|f| - f)(|f| + f) = \mathbf{0}$ showing that $|f| \in \mathcal{SI}(R, L)$, using the $\pm u_1, \ldots, \pm u_k$; more precisely, $\prod_{i=1}^k (|f| - u_i)(|f| + u_i) = \mathbf{0}$. ■

It can be shown that $\mathcal{SI}(R, \mathrm{C}(\mathbb{R})) = \mathcal{SI}(R, L)$, but this fact is not used below and the details are omitted.

The next steps are to look for elements of K and to examine properties of elements of T. Since derivatives will play an important role, the difference quotient in calculating the derivative of f at x_0 will appear often and the following notation will be used.
Notation: $\frac{f(x)-f(x_0)}{x-x_0} = \Delta_{f,x_0}(x)$, for $x \neq x_0$.

4.28. Lemma. *Let $(f, D) \in T$ using $u_1, \ldots, u_k \in R$. For any $y \in \mathbb{R}$ where $\frac{d}{dx}f(y)$ exists, there is i, $1 \leq i \leq k$, so that $f(y) = u_i(y)$ and $\frac{d}{dx}f(y) = \frac{d}{dx}u_i(y)$. This applies, in particular, to all $y \in D$.*

Proof. Note that i given in the statement need not be unique. Pick a sequence $\{x_n\}$ in $\mathbb{R}$ converging to y. For each n, there is some $i(n)$ such that $f(x_n) = u_{i(n)}(x_n)$. Since there are only k choices for $i(n)$, there is some i such that for all $n \in B \subseteq \mathbb{N}$, B infinite, where $i(n) = i$, for all $n \in B$. It follows that $f(y) = u_i(y)$. For each $n \in B$, $\Delta_{f,y}(x_n) = \Delta_{u_i,y}(x_n)$. The first expression converges to $\frac{d}{dx}f(y)$, while the second to $\frac{d}{dx}u_i(y)$. ∎

Before going on to find elements of K, the following finiteness condition will be crucial to much of what follows.

4.29. Proposition. *Let $(f, D) \in T$, using $u_1, \ldots, u_k \in R$, and $x_0 \in D^c$. The sets $\{\frac{d}{dx}f(x) \mid x \in (x_0, \infty) \cap D\}$ and $\{\frac{d}{dx}f(x) \mid x \in (-\infty, x_0) \cap D\}$ have cluster points at x_0 and at most finitely many such cluster points.*

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in D converging to x_0 from the right. By Lemma 4.28, there is there is i and an infinite subsequence $\{x_n\}_{n\in B_i}$ such that for all $n \in B_i$, $\frac{d}{dx}f(x_n) = \frac{d}{dx}u_i(x_n)$. By the continuity of $\frac{d}{dx}u_i$, the subsequence converges to $\frac{d}{dx}u_i(x_0)$. This is a cluster point. On the other hand the infinite subsequences B_i make up all but finitely many points of the original sequence, since k is finite. Thus, each cluster point of $\{\frac{d}{dx}f(x_n)\}_{n\in\mathbb{N}}$ is from the set $\{\frac{d}{dx}u_1(x_0), \ldots, \frac{d}{dx}u_k(x_0)\}$. Similarly to the left of x_0. ∎

Notation: In Proposition 4.29 the two kinds cluster points are called *right cluster points of f at x_0* and *left cluster points of f at x_0*, respectively. Much more will be said about these cluster points for elements of T. They will turn out to be unique and also one-sided derivatives. Notice an essential point here. The left and right cluster points are determined by the behaviour of $\frac{d}{dx}f$ on D. Yet they are computed using the functions $u_i \in R$ which express the fact that $f \in T$. The u_i are not uniquely determined, yet the cluster points computed from them are unique.

There are many consequences of Proposition 4.29. The next one will come up in Theorem 4.31 and in Proposition 5.50.

4.30. Proposition. *Let $(f, D) \in T$ using $u_1, \ldots, u_k \in R$. Suppose that there is an open interval I on which $\frac{d}{dx}f$ exists. Then $\frac{d}{dx}f$ is continuous on I.*

Proof. Pick $x_0 \in I$ and suppose that $\frac{d}{dx}f$ is not continuous at x_0. Set $\frac{d}{dx}f(x_0) = s$. A derivative does not have a removable singularity and so it may be assumed that $\lim_{x\to x_0}\frac{d}{dx}f(x)$ does not exist. In particular the limit is not s. Given $\varepsilon > 0$, for all $\delta > 0$ there is y, which may be taken in I, such that $|y - x_0| < \delta$ and $|\frac{d}{dx}f(y) - s| > \varepsilon$. Take $\delta = 1/n$, for $n \in \mathbb{N}$ and a corresponding y_n. Look at $P_1 = \{n \in \mathbb{N} \mid \frac{d}{dx}f(y_n) > s + \varepsilon\}$ and $P_2 = \{n \in \mathbb{N} \mid \frac{d}{dx}f(y_n) < s - \varepsilon\}$. One of these sets is infinite. Suppose P_1 is infinite. Pick any t with $s < t < s+\varepsilon$. By Darboux's Theorem, for $n \in P_1$, there is z_n, between x_0 and y_n with $\frac{d}{dx}f(z_n) = t$. There is some j, $1 \le j \le k$, with $\frac{d}{dx}f(z_n) = \frac{d}{dx}u_j(z_n)$, for infinitely many $n \in P_1$ (using Lemma 4.28). Let Q be this set of n. Then, $\lim_{n\in Q}\frac{d}{dx}f(z_n) = \lim_{n\in Q}\frac{d}{dx}u_j(z_n) = \frac{d}{dx}u_j(x_0) = t$. However, there are infinitely many such t and only finitely many $\frac{d}{dx}u_i(x_0)$. This is a contradiction, showing that $\frac{d}{dx}f$ is continuous at x_0. This applies to all $x_0 \in I$. ∎

The next result will yield many elements of K.

4.31. Theorem. *Suppose $(f, D) \in T$ using $u_1, \dots, u_k \in R$.*

(1) Assume that there is $r \in R$ such that $f|_{D^c} = r|_{D^c}$. Then, $f \in K$.

(2) If $\mathbb{R} \setminus D$ is discrete, then $f \in K$ (with no conditions on $f|_{D^c}$).

Proof. (1) Since $f \in T$, so is $f - r$. Hence, it suffices to show that $f - r \in K$. Thus, it can be assumed that $f|_{D^c} = \mathbf{0}$. Since $f \in T$, it is sufficient for $f \in K$ that $f \in S$ (Proposition 2.6). It will be seen that f^2 and f^3 are in R, showing $f \in K$.

The proof for f^3 will follow similar lines as that for f^2.

For $x_0 \in D^c$, $f^2(x_0) = 0$ and so:

$$(*)\ \Delta_{f^2,x_0}(x) = \frac{f^2(x)}{x - x_0}\ .$$

For any sequence $\sigma = \{h_n\}_{n\in\mathbb{N}}$ converging to x_0, put $B_i = \{n \in \mathbb{N} \mid f(h_n) = u_i(h_n)\}$. Let $B_{i_1}, \dots, B_{i_l}$ be those that are infinite. By continuity of f and of u_{i_j}, $f(x_0) = u_{i_j}(x_0) = 0$, for $j = 1, \dots, l$. Moreover, since $u_{i_j} \in R$, the quotient $(*)$ at h_n, $n \in B_{i_j}$ converges to $\frac{d}{dx}u^2_{i_j}(x_0) = 2u_{i_j}(x_0)\frac{d}{dx}u_{i_j}(x_0) = 0$. Given $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that for all $n > m$ and each $j = 1, \dots, l$, $\left|\frac{f^2(h_n)}{h_n - x_0}\right| = \left|\frac{u^2_{i_j}(h_n)}{h_n - x_0}\right| < \varepsilon$. From this $(*)$ goes to zero along any sequence converging to x_0. Hence, $\frac{d}{dx}f^2(x_0)$ exists and is 0.

Since $\frac{d}{dx}f^2(x)$ exists for $x \in D$ and it has just been shown that $\frac{d}{dx}f^2(x)$ also exists for $x \in D^c$, then, since $f^2 \in T$, Proposition 4.30 says that $f^2 \in R$. Similarly, $f^3 \in R$, showing $f \in K$.

(2) It was just shown that if $f|_{D^c} = \mathbf{0}$ then $f \in K$. The reverse implication is to be shown. Because the countable closed set D^c is discrete, the elements can be ordered with index set $B \subseteq \mathbb{Z}$. If there are elements of D^c that are ≥ 0, label the first one y_1 and continue in this manner. The negative elements of D^c are done in the same way. It will suffice to find an element $p \in R$ such that $f - p$ is zero on D^c.

For each $n \in B$, let $f(y_n) = z_n$. The function p needs to have values $p(y_n) = z_n$, for $n \in B$. For all $n \in B$, artificially assign the slope 0 to the point (y_n, z_n). If $n+1 \in B$ (the case where this is not true will be dealt with later), construct a cubic spline with the two data points and the two slopes (both 0). This will be p restricted to the interval $[y_n, y_{n+1}]$. If $n+1 \notin B$ then pick any $y_{n+1} \in (y_n, \infty)$ and let $p(y_{n+1}) = 1$ with slope 0. The function given by the cubic spline on $[y_n, y_{n+1}]$ can be extended with slope 0. The case where $n \in B$ is minimal is similarly dealt with. The process can be continued by upward induction, and an entirely similar process can be done to the left. When all these parts of p are assembled, there is a C^1 function with all the correct values.

From this, $f - p \in K$ by part (1). Hence, $f \in K$. ∎

Theorem 4.31 gives all that is necessary to characterize those elements of T that give rise to a simple DL-extension of R.

Notation: Put $\mathfrak{S} = \{s \in T \setminus R \mid R[s] \text{ is a simple DL-extension of } R\}$.

4.32. Proposition. *Let $(f, D) \in L$. Then, $f \in \mathfrak{S}$ if and only if $f \in T$ and $f|_{D^c} = \mathbf{0}$.*

Proof. (1) Assume $f \in \mathfrak{S}$. Then, $f^2, f^3 \in R$ and there are $u_i, \ldots, u_k \in R$ such that $\prod_{i=1}^{k}(f^2 - u_i^2) = \mathbf{0}$. This shows $f \in T$. Put $f^3 = r \in R$. Then, $f = r^{1/3}$ which is C^1 on $\operatorname{coz} f \cup \operatorname{Int} \mathrm{z}(f)$. Hence, if $x \notin D$, $x \in \mathrm{z}(f)$, as was to be proved.

(2) Assume $f|_{D^c} = \mathbf{0}$ and $f \in T$ using $u_1, \ldots, u_k \in R$. Then $\prod_{i=1}^{k}(f^2 - u_i^2) = \mathbf{0}$. The proof of Proposition 4.31(1) shows that $f^2, f^3 \in R$. Hence, $f \in \mathfrak{S}$. ∎

If $R[f]$ is a simple (2,3)-extension of R, then $r = f^3 \in R$. This makes $f = r^{1/3}$, and as above, $f|_{D^c} = \mathbf{0}$. However, it will be seen later that this latter condition is not sufficient for $R[f]$ to be a simple (2,3)-extension (Corollary 5.49).

It is now possible to present some elements of $\mathfrak{S}$.

For any $r \in R$, write $\operatorname{coz} r = \dot{\bigcup}_{n \in A \subseteq \mathbb{N}} (a_n, b_n)$, a disjoint union of intervals (one or two could be unbounded). Take a subset $B \subseteq A$ and define r_B by

$$r_B(x) = \begin{cases} r(x) & \text{if } x \in [a_n, b_n], n \in B \\ 0 & \text{otherwise .} \end{cases}$$

4.33. Examples. *Let $r \in R$ with $\operatorname{coz} r$ written $\operatorname{coz} r = \dot{\bigcup}_{n \in A} (a_n, b_n)$. For any $B \subseteq A$, $r_B \in \mathfrak{S}$, unless $r_B \in R$. In particular, $f_1 = r^+$, $f_2 = r^-$ and $f_3 = |r|$ are elements of $\mathfrak{S}$, unless they are in R.*

Proof. Note that r_B is continuous and is C^1 on $\operatorname{coz} r_B \cup \operatorname{Int} \mathrm{z}(r_B)$ which is dense open, and $(r_B - r)r_B = \mathbf{0}$, showing that $r_B \in T$. The only points where r_B may not have a continuous derivative are on the boundary of $\dot{\bigcup}_{n \in B} (a_n, b_n)$, which is in $\mathrm{z}(r_B)$. Proposition 4.32 then gives the result.

If B is chosen so that $\dot{\bigcup}_{n \in B} (a_n, b_n) = \operatorname{coz} r^+$, then $r_B = f_1 \in \mathfrak{S}$ (or is in R). Similarly for f_2. Finally, $f_3 = f_1 - f_2$ and $f_1 f_2 = \mathbf{0}$ showing that $f_3^2, f_3^3 \in R$ and f_3 satisfies the conditions of Proposition 4.32. ∎

Not all the elements of $(f, D) \in K$ satisfy the conditions of Theorem 4.31(1), and, hence, not all elements of K are in simple DL-extensions of R. See Example 5.45, below.

The next lemma will be the start of an examination of the cluster points of $(f, D) \in T$. It will also be used in Section 5. The first result is about elements of D^c that are endpoint of intervals belonging to D. It will turn out that the result holds for other types of elements of D^c, but more work will be required for that.

4.34. Lemma. *Let $(f, D) \in T$ using $u_1, \ldots, u_k \in R$. (1) Then, for any interval (a, b) belonging to D, $\lim_{x\to a^+} \frac{d}{dx} f(x)$ and $\lim_{x\to b^-} \frac{d}{dx} f(x)$ exist, whenever a or b is finite. (2) The one-sided derivatives of f, $\frac{d^+}{dx} f(a)$ and $\frac{d^-}{dx} f(b)$ exist, if a or b is finite; they coincide with the one-sided limits in (1).*

Proof. (1) The proof will be at a right endpoint, b. According to, Proposition 4.29 $\{\frac{d}{dx} f(x)\}_{x<b, x\in D}$ has finitely many left cluster points at b, and at least one. Suppose that there are more than one, say $r > s$, with no cluster points between them. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in D converging to b from the left such that for all $n \in \mathbb{N}$, $f(x_n) = u_i(x_n)$ and $\frac{d}{dx} f(x_n) = \frac{d}{dx} u_i(x_n)$, for some i and, a sequence $\{y_n\}_{n\in\mathbb{N}}$ in D converging to b from the left with $f(y_n) = u_j(y_n)$ and $\frac{d}{dx} f(y_n) = \frac{d}{dx} u_j(y_n)$, for some $j \neq i$, and, finally, $\frac{d}{dx} u_i(b) = r$ and $\frac{d}{dx} u_j(b) = s$. Put $t = r - s$. There is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|\frac{d}{dx} f(x_n) - r| < (1/4)t$ and $|\frac{d}{dx} f(y_n) - s| < (1/4)t$. This means that there is $z_n \in D$, z_n between x_n and y_n, such that $\frac{d}{dx} f(z_n) = s + (1/2)t$ (by Darboux's Theorem). This means that $s + (1/2)t$ is a right cluster point, a contradiction. Hence, there can only be one right cluster point. This means that $\lim_{x\to b^-} \frac{d}{dx} f(x)$ exists.

(2) Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in the interval converging to b from the left. It must be shown that $\Delta_{f,b}(x_n)$ converges. As usual, the sequence divides into a finite part and some infinite subsequences. Consider one such infinite subsequence, $\{x_{n_m}\}_{m\in\mathbb{N}}$, where for all $m \in \mathbb{N}$, $f(x_{n_m}) = u_i(x_{n_m})$ and $\frac{d}{dx} f(x_{n_m}) = \frac{d}{dx} u_i(x_{n_m})$. Along this subsequence, $\Delta_{f,b}(x_{n_m}) = \Delta_{u_i,b}(x_{n_m})$, and this converges to $\frac{d}{dx} u_i(b)$. On the other hand, by the continuity of $\frac{d}{dx} u_i(x)$, $\{\frac{d}{dx} f(x_{n_m})\} = \{\frac{d}{dx} u_i(x_{n_m})\}$ converges to $\frac{d}{dx} u_i(b)$, a cluster point. By part (1), there is only one such. Hence, all those infinite subsequences converge to the same value, $\frac{d}{dx} u_i(b)$. This is true of any sequence $\{x_n\}$ converging to b from the left, since a finite number of points can be ignored. Hence, f has a left derivative at b.

The last remark follows since $\frac{d}{dx} f(x)$ is continuous in (a, b) and has one-sided limits at the endpoints. ∎

As mentioned above, the left and right cluster points for an element of T can be shown to be unique. This will require the examination of points of D^c not endpoints of intervals belonging to D.

4.35. Theorem. *Let $(f, D) \in T$ using $u_1, \ldots, u_k \in R$. Then, for every point in $x_0 \in D^c$, the left and the right cluster points at x_0 are unique.*

Proof. The proof will look at a single point $x_0 \in D^c$ and at right cluster points. If x_0 is the left endpoint of an interval in D, then its right cluster point is unique by Lemma 4.34(2) so it will be assumed that x_0 is an accumulation point of $(x_0, \infty) \cap D^c$. The rest of the proof will be by contradiction with the assumption that $x_0 \in D^c$ has $l > 1$ distinct right cluster points, $r_1, \ldots, r_l$. Put $e = \min_{j \neq j'} |r_j - r_{j'}|$. Define $U_j = (r_j - e/4, r_j + e/4)$. Note that the U_j have disjoint closures. If the right cluster point r_j is attained using u_i, write $i \in \bar{j}$. The set $\bar{j}$ could have more than one element.

The first step will be to get close enough to x_0, say in (x_0, α_1), to make sure that if $x \in D$ and $f(x) = u_i(x)$ and $\frac{d}{dx} f(x) = \frac{d}{dx} u_i(x)$, then $i \in \bigcup_{j=1}^{l} \bar{j}$. If $i \notin \bigcup_{j \leq l} \bar{j}$ then (1) $f(x_0) \neq u_i(x_0)$, or (2) $f(x_0) = u_i(x_0)$ but there is no descending sequence $\{y_n\}_{n \in \mathbb{N}}$ in D converging to x_0, such that for all n, $f(y_n) = u_i(y_n)$ and $\frac{d}{dx} f(y_n) = \frac{d}{dx} u_i(y_n)$. In case (1), there is $\beta > x_0$ such that for (x_0, β), $f(x) \neq u_i(x)$. In case (2), there is $\gamma > x_0$, such that for $x \in (x_0, \gamma) \cap D$, if $f(x) = u_i(x)$ then $\frac{d}{dx} f(x) \neq \frac{d}{dx} u_i(x)$, since $\frac{d}{dx} u_i(x_0)$ is not a right cluster point of f at x_0. Choose β and γ small enough so that these properties hold for all $i \notin \bigcup_{j \leq l} \bar{j}$, this is possible since the number of such u_i is finite. Put $\alpha_1 = \min(\beta, \gamma)$.

The next stage is again to approach x_0 so that if $i \in \bar{j}$ then $\frac{d}{dx} u_i(x) \in U_j$, for $x \in (x_0, \alpha)$. There is also $\alpha > x_0$, chosen so that $x_0 < \alpha \leq \alpha_1$, such that for all $j \leq l$ and all $i \in \bar{j}$, $\frac{d}{dx} u_i(x) \in U_j$, for all $x \in (x_0, \alpha)$. This is possible since all the $\frac{d}{dx} u_i$ are continuous and $\frac{d}{dx} u_i(x_0) = r_j \in U_j$. Notice that by this choice of α, if $x \in (x_0, \alpha) \cap D^c$, then any cluster point (left or right) at x will be a limit of a sequence $\{\frac{d}{dx} u_i(y_n)\}$, with $\{y_n\}$ in $(x_0, \alpha) \cap D$ converging to x on the appropriate side with values in some unique U_j, where $i \in \bar{j}$, and, hence, will lie in $\operatorname{cl} U_j$. This uses the properties that define α and the fact that $\alpha \leq \alpha_1$.

With this choice of α, if (a, b) in an interval belonging to D contained in (x_0, α), then for some $y \in (a, b)$, $f(y) = u_i(y)$ and $\frac{d}{dx} f(y) = \frac{d}{dx} u_i(y)$, for some $i \in \bigcup_{j \leq l} \bar{j}$. Since $\frac{d}{dx} f$ is a continuous function on $(x_0, \alpha) \cap D$, $\frac{d}{dx} f((a, b))$ is connected. However, $\frac{d}{dx} f(y) \in U_j$. Hence, $\frac{d}{dx} f((a, b)) \subseteq U_j$. When this occurs, (a, b) is said to *be attached to* U_j.

Because of the nature of x_0, the point α may be chosen to be in D^c. With such a choice, $V = [x_0, \alpha] \cap D = (x_0, \alpha) \cap D$ is dense open in $[x_0, \alpha]$. If (a, b) is an interval belonging to D inside V, then (a, b) is attached to some U_j. For each $j = 1, \ldots, l$, put Σ_j to be the union of the intervals in V attached to U_j. Then, $V = \bigcup_{j=1}^{l} \Sigma_j$.

If $y \in \operatorname{cl} \Sigma_j \cap D^c$, the left and/or right cluster points of f at y are in $\operatorname{cl} U_j$. This is because these cluster points are limits of numbers in U_j. Note that, for each $j = 1, \ldots, l$, $\Sigma_j \neq \emptyset$ because of the l right cluster points at x_0. If the $\operatorname{cl} \Sigma_j$ were disjoint, $[x_0, \alpha]$ would be disconnected. Hence, there are j and j', $j \neq j'$, with $\operatorname{cl} \Sigma_j \cap \operatorname{cl} \Sigma_{j'} \neq \emptyset$. If y is in the intersection, then $y \in D^c$ and its cluster points are in $\operatorname{cl} U_j \cap \operatorname{cl} U_{j'}$, which is impossible. Hence, $l = 1$, as was to be proved. ∎

Theorem 4.35 has a corollary showing that for elements of $f \in T$, for all $x \in \mathbb{R}$, $\frac{d^+}{dx} f(x)$ and $\frac{d^-}{dx} f(x)$ exist.

4.36. Corollary. *Let $(f, D) \in T$ using $u_1, \ldots, u_k \in R$. Then, for each $x \in \mathbb{R}$, $\frac{d^+}{dx} f(x)$ and $\frac{d^-}{dx} f(x)$ exist. Moreover, if $x \in D^c$, then $\frac{d^+}{dx} f(x)$ is the unique right cluster point at x and $\frac{d^-}{dx} f(x)$ is the unique left cluster point at x.*

PROOF. Since $\frac{d}{dx}f$ exists on D, it suffices to look at $x_0 \in D^c$. The proof will be for the left derivative. The structure of the proof will be to show that $\lim_{x\to x_0^-} \Delta_{f,x_0}(x)$ exists and that it is the unique left cluster point at x_0.

Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_0 from the left. If infinitely many terms are in D, there is an infinite subsequence $\{x_n\}_{n\in B_1}$ and some i such that for all $n \in B_1$, $f(x_n) = u_i(x_n)$ and $\frac{d}{dx}f(x_n) = \frac{d}{dx}u_i(x_n)$. The sequence $\{\Delta_{f,x_0}(x_n)\}_{n\in B_1}$ converges to $\frac{d}{dx}u_i(x_0)$. Moreover, $\{\frac{d}{dx}u_i(x_n)\}_{n\in B_1}$ converges to $\frac{d}{dx}u_i(x_0)$, the unique left cluster point of x_0. If there is another infinite subsequence $\{x_n\}_{n\in B_2}$ with the same properties but with u_j, $j \neq i$, the limits would be the same because the left cluster point is unique. This suggests that $\frac{d^-}{dx}f(x_0)$ is the left cluster point at x_0. This will need to be verified. However, the sequence $\{x_n\}_{n\in\mathbb{N}}$ may have an infinite subsequence in D^c. It has been shown how any infinite subsequence in D behaves as predicted in the statement. So it may be assumed that $x_n \in D^c$, for all $n \in \mathbb{N}$.

Recall that each x_n, $n \in \mathbb{N}$, will have a unique *right* cluster point. For each n, fix a descending sequence from D converging to x_n from the right, say $\{y_{n,m(n)}\}_{m(n)\in\mathbb{N}}$, chosen from $(x_n, x_0) \cap D$, so that for some $j(n)$, $1 \leq j(n) \leq k$, $f(y_{n,m(n)}) = u_{j(n)}(y_{n,m(n)})$ and $\frac{d}{dx}f(y_{n,m(n)}) = \frac{d}{dx}u_{j(n)}(y_{n,m(n)})$; it follows that $\frac{d}{dx}u_{j(n)}(x_n)$ is the unique right cluster point at x_n. Since there are finitely many $j(n)$, there is at least one occurring infinitely often, call it j, say for all $n \in C \subseteq \mathbb{N}$.

One $m(n)$ will be chosen for each $n \in C$, but satisfying the constraints to be specified below. Throughout this discussion it will be assumed that $n \in C$. The idea is to show that $\frac{d}{dx}u_j(x_0)$ is the unique left cluster point at x_0.

Put the unique *right* cluster point of f at x_n to be M_n. The sequence $\{x_n\}$ can be chosen to be in some interval (α, x_0) and, hence, there is some $M > 0$ so that $|M_n| \leq M$ for all n, since the M_n are all from the set $\{\frac{d}{dx}u_j(x_n)\}$. Now look at $\Delta_{f,x_0}(x_n)$. It can be rewritten:

$$\frac{f(x_n) - f(y_{n,m(n)}) + f(y_{n,m(n)}) - f(x_0)}{x_n - x_0} =$$

$$(*)\ \frac{f(x_n) - f(y_{n,m(n)})}{x_n - y_{n,m(n)}} \cdot \frac{x_n - y_{n,m(n)}}{x_n - x_0} + \frac{f(y_{n,m(n)}) - f(x_0)}{y_{n,m(n)} - x_0} \cdot \frac{y_{n,m(n)} - x_0}{x_n - x_0}$$

Now choose $y_{n,m(n)}$ so that $x_n - y_{n,m(n)} < (x_0 - x_n)/n$,

$$x_n - y_{n,m(n)} < (x_n - x_0)^2 \text{ and } \left| \frac{f(x_n) - f(y_{n,m(n)})}{x_n - y_{n,m(n)}} - M_n \right| < 1/n\ .$$

Since the M_n are bounded, the first term in $(*)$ goes to 0 as $n \to \infty$. In the second term of $(*)$, the first factor is $\Delta_{f,x_0}(y_{n,m(n)}) = \Delta_{u_j,x_0}(y_{n,m(n)})$, and it converges to $\frac{d}{dx}u_j(x_0)$, the unique left cluster point of f at x_0, while the second factor goes to 1.

However, the calculations above depended on the choices of C and j. This is of no consequence since, with different choices, say C' and j', the conclusion would be $\frac{d}{dx}u_{j'}(x_0) = \frac{d}{dx}u_j(x_0)$, because of the uniqueness of the left cluster point at x_0.

It follows that $\{\Delta_{f,x_0}(x_n)\}_{n\in\mathbb{N}}$ converges to the left cluster point at x_0. But this is the definition of $\frac{d^-}{dx}f(x_0)$. ■

It has now been established that for $f \in T$, both $\frac{d^+}{dx}f(x)$ and $\frac{d^-}{dx}f(x)$ exist for all $x \in \mathbb{R}$. It will be seen that the function $\frac{d^+}{dx}f$ is right continuous and also that $\frac{d^-}{dx}f$ is left continuous. In [K, Example 7.5] a subring H of $C(\mathbb{R})$ is defined as the set of functions that have, for all $x \in \mathbb{R}$, a left and a right derivative. This is shown to be a subring and a lattice. Corollary 4.36 shows that $T \subseteq H$. However, this will later be refined.

4.37. Corollary. *Let $f \in T$ with the notation of Theorem 4.35. Then, the function $\frac{d^-}{dx}f$ is left continuous and $\frac{d^+}{dx}f$ is right continous.*

Proof. The left continuity of $\frac{d^-}{dx}f$ will be shown. As in Corollary 4.36, it was shown that for $x_0 \in \mathbb{R}$ there is i, $1 \le i \le k$, such that $f(x_0) = u_i(x_0)$ and $\frac{d^-}{dx}f(x_0) = \frac{d}{dx}u_i(x_0)$. Pick a sequence in $\mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}}$ converging to x_0 from the left. For each n, there is $j(n)$ such that $f(x_n) = u_{j(n)}(x_n)$ and $\frac{d^-}{dx}f(x_n) = \frac{d}{dx}u_{j(n)}(x_n)$.

Except for finitely many terms, the sequence is a union of infinite subsets where $j(n)$ is constant. Pick one such, say $\{x_n\}_{n\in B}$ with fixed $j(n) = j$. It follows that $f(x_0) = u_i(x_0) = u_j(x_0)$ and $\frac{d^-}{dx}f(x_0) = \frac{d}{dx}u_i(x_0) = \frac{d}{dx}u_j(x_0)$. Moreover, the proof of Corollary 4.36 shows that this common value is the unique left cluster point at x_0. The continuity of $\frac{d}{dx}u_j$ shows that $\{\frac{d}{dx}u_j(x_n)\}_{n\in B}$ converges to $\frac{d}{dx}u_j(x_0) = \frac{d}{dx}u_i(x_0)$. But this reasoning applies to all the infinite subsequences of $\{x_n\}$ where $j(n)$ is constant and the limit is always $\frac{d}{dx}u_i(x_0)$. Hence, for any sequence $\{x_n\}$ converging to x_0 from the left, $\{\frac{d^-}{dx}f(x_n)\}$ converges to $\frac{d^-}{dx}f(x_0)$. ∎

Note: If $f \in T \setminus R$, there must be $x \in D^c$ where $\frac{d^-}{dx}f(x) \ne \frac{d^+}{dx}f(x)$. More generally, in light of Proposition 4.30, if I is any open interval meeting D^c, there is $x \in I$ such that $\frac{d^+}{dx}f(x) \ne \frac{d^-}{dx}f(x)$.

This concludes a description of the elements of T. It is not known if $(f, D) \in L$ with all the properties of Corollaries 4.36 and 4.37 characterizes elements of T. However, these are very strong properties. And, in fact, there are conditions on D where Corollaries 4.36 and 4.37 have a converse, as will be seen (Theorem 4.40).

In [K, Example 7.5] a subring of $H_0 \subseteq H$ is defined that also is related to T. Recall that H is the subring of $C(\mathbb{R})$ of those functions having right and left derivatives for all $x \in \mathbb{R}$. The elements of H_0 have three additional properties: for $f \in H_0$, the function $\frac{d^+}{dx}f$ is right continuous and $\frac{d^-}{dx}f$ is left continuous; in addition for all $x_0 \in \mathbb{R}$, $\lim_{x\to x_0^+}\frac{d^-}{dx}f(x) = \frac{d^+}{dx}f(x_0)$, and similarly on the left. Corollaries 4.36 and 4.37 already show that elements of T satisfy most of the conditions to be in H_0. The next proposition supplies the one remaining one. The ring H_0 is shown in [K, Example 7.5] to be a subring and a lattice.

4.38. Proposition. *The ring T is a subring of H_0.*

Proof. The only item to be shown for $(f, D) \in T$ is the right convergence of the $\frac{d^-}{dx}f$. Fix $x_0 \in \mathbb{R}$. It is to be shown that $\lim_{x\to x_0^+}\frac{d^-}{dx}f(x) = \frac{d^+}{dx}f(x_0)$ (and, in the same manner, the corresponding left version). Pick any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_0 from the right. For each x_n there is $y_n \in D$ with $x_0 < y_n < x_n$, $|\frac{d^-}{dx}f(y_n) - \frac{d^-}{dx}f(x_n)| < 1/n$, and

$x_n - y_n < 1/n$. Then, $\{y_n\}$ converges to x_0 from the right. Since $y_n \in D$, $\frac{d^-}{dx}f(y_n) = \frac{d}{dx}f(y_n)$ and $\{\frac{d}{dx}f(y_n)\}$ converges to the unique right cluster point of f at x_0, hence, to $\frac{d^+}{dx}f(x_0)$.

For $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ that can be taken $\geq m$, such that $|\frac{d^+}{dx}f(x_0) - \frac{d}{dx}f(y_n)| < 1/m$. Then, $|\frac{d^+}{dx}f(x_0) - \frac{d^-}{dx}f(x_n)| < (1/m) + (1/n) \leq 2/m$, by the Triangle Inequality. This shows the convergence. ■

In [K, Example 7.5] it is conjectured that $H_0 = K$. By what was shown, $K \subseteq T \subseteq H_0$ and $T \subset L$. The results above do show the close connection between it and the existence and behaviour of the one-sided derivatives and the elements of $K \subseteq T$; certainly $T \subseteq H_0 \cap L$. Alan Dow has kindly supplied an example showing that the conjecture fails. See Example 5.51 and the discussion around it.

The first step in finding a partial converse Corollaries 4.36 and 4.37 is to determine a family of functions in K. The proposition will also be the first step in an induction but the class of function it defines are worth being given a name.

For the purposes of the next set of examples, a *piecewise C^1 function* $c \in \mathrm{C}(\mathbb{R})$ is defined as follows. There is a discrete closed subset F of $\mathbb{R}$ indexed by $B \subseteq \mathbb{Z}$, $\{y_n\}_{n\in B}$, where for $m, n \in B$ with $m < n$, $y_m < y_n$, and for each interval $I_n = [y_n, y_{n+1}]$, $n, n+1 \in B$, there is a C^1 function c_n on I_n, in the sense that c_n is C^1 on (y_n, y_{n+1}), c_n has a right derivative at y_n and a left derivative at y_{n+1}, and $\frac{d}{dx}c_n$ is continuous on I_n, and if $n, n+1 \in B$, $\frac{d^-}{dx}c_n(y_n) \neq \frac{d^+}{dx}c_{n+1}(y_n)$. There are suitable modifications: if B has a maximum element m, then c_m is defined on $[y_m, \infty)$, and similarly if B has a minimal element. Then let $c(x) = c_n(x)$ for $x \in I_n$.

An example would be a continuous piecewise linear function.

4.39. Proposition. *For an element $(f, D) \in L$ the following are equivalent:*

(1) f is piecewise C^1,

(2) $(f, D) \in L$, D^c is discrete and for all $x \in \mathbb{R}$, $\frac{d^+}{dx}f(x)$ and $\frac{d^-}{dx}f(x)$ exist.

When the two conditions above are satisfied for $(f, D) \in L$, then there is $r \in R$ such that $f - r \in \mathfrak{S}$, showing that $f \in K$.

Proof. (1) ⇒ (2): Clearly $(f, D) \in L$, where D is the union of the open intervals where f is C^1 and the points where f does not have a derivative form a discrete set, ordered by $B \subseteq \mathbb{Z}$. The functions $u_1, \ldots, u_k \in R$ must be found (here $k = 3$). If B is bounded above or below, it can be artificially extended to $\mathbb{Z}$ by taking integer points beyond the first or last elements of B, if any. For y_n and y_{n+2} in $\mathbb{Z}$, form a cubic spline, s_n, connecting $(y_n, f(y_n))$ and $(y_{n+2}, f(y_{n+2}))$ with the added data $\frac{d^-}{dx}f(y_n)$ and $\frac{d^+}{dx}f(y_{n+2})$ for the slopes at the endpoints. Also, let f_m be the function f restricted to $[y_m, y_{m+1}]$. The three functions in R are defined as follows:

$$u_1 = \begin{cases} f_m(x), & x \in [y_m, y_{m+1}], m = 3k+2 \\ s_n(x), & x \in [y_n, y_{n+2}], n = 3k \end{cases},$$

$$u_2 = \begin{cases} f_m(x), & x \in [y_m, y_{m+1}], m = 3k \\ s_n(x), & x \in [y_n, y_{n+2}], n = 3k+1 \end{cases},$$

$$u_3 = \begin{cases} f_m(x), & x \in [y_m, y_{m+1}], m = 3k+1 \\ s_n(x), & x \in [y_n, y_{n+2}], n = 3k+2 \end{cases}.$$

By construction $(f - u_1)(f - u_2)(f - u_3) = \mathbf{0}$.

(2) ⇒ (1): As usual, D is presented as a disjoint union of intervals. In each interval f is a C^1 function. In order to have a piecewise C^1 function, it is required that f have one-sided derivatives at the endpoints of the intervals. This is given by hypothesis, and, if $y \in D^c$, the left and right derivatives of f at y must be different.

For the last part, notice that when (f, D) satisfies the conditions (1) and (2), there is an element $r \in R$ such that r and f coincide on D^c; this is the method of Theorem 4.31(2). Then, as in Theorem 4.31(1), $f - r \in \mathfrak{S}$ (and, in addition, $f - r$ is also piecewise C^1). ∎

Recall the definition of the *Cantor-Bendixson index* (*C-B index*), here restricted to subsets of $\mathbb{R}$ (see, for example, [KR, Definitions 1.1 and 1.2], where the induction in *ibid* Definition 1.2 is finite). Let $X \subseteq \mathbb{R}$ and put A_1 to be the set of isolated points of X. Put A_2 to be the set of isolated points of $X \setminus A_1$. This process is continued: for $i > 1$, A_i is the set of isolated points of $X \setminus (A_1 \cup \cdots \cup A_{i-1})$. If, for some $m > 0$, $A_1 \cup \cdots \cup A_m = X$, X is said to be of C-B index m. Note that if X is closed of C-B index m, then A_m is closed. In any case the elements of A_i are called the elements of C-B index i.

With the above remark, (2) ⇒ (1) of Proposition 4.39 can be the first step in an induction where D^c is discrete, i.e., of C-B index 1. Since it may come up in the following proof, the only set of C-B index 0 is $\emptyset$, and if $(g, D) \in L$ and if D_g^c is of C-B index 0, then g is C^1.

4.40. THEOREM. *Suppose $(f, D) \in L$ and that D^c is of C-B index m. Assume, moreover, that f is such that, for all $x \in \mathbb{R}$, both $\frac{d^-}{dx} f(x)$ and $\frac{d^+}{dx} f(x)$ exist. Then, $f \in T$.*

PROOF. The case $m = 1$ follows from Proposition 4.39. The proof will be by induction on n. The last statement in 4.39 is not part of the induction assumption, since it is only asserted here that $f \in T$.

Fix $m \geq 1$ and assume that for all functions $(g, D_g) \in L$ satisfying the conditions of the theorem with D_g^c of C-B index $p \leq m$ that (1) $g \in T$ using $k(p)$ elements of R and where $k(p)$ is a non-decreasing function of p. Note that when $p = 1$, $k(1) \leq 3$ according to the proof of 4.39. Notice that when a function g satisfies the induction assumption for C-B index $\leq m$ the result that $g \in T$ can be taken to using *exactly* $k(m)$ functions from R, by duplicating some, if necessary.

Now assume that $(f, D) \in L$ satisfies the conditions of the statement and that D^c is of C-B index $m + 1$. Let A be the set of points of D^c of C-B index $m + 1$. It is a closed discrete set in $\mathbb{R}$. The elements of A can be ordered in the natural order $\{x_i\}_{i \in B \subseteq \mathbb{Z}}$. However, using the artificial method of (1) ⇒ (2) of 4.39, it may be assumed that $B = \mathbb{Z}$.

For any $i \in \mathbb{Z}$, consider the interval $I_i = (x_i, x_{i+1})$. All the points of $E_i = I_i \cap D^c$ are of C-B index $\leq m$. There is a C^1 order preserving homeomorphism $\phi_i \colon I_i \to \mathbb{R}$ with ϕ^{-1} also C^1. Then, $g_i = f|_{I_i} \phi_i^{-1}$ is in L with $D^c_{g_i} = \phi_i(E_i)$ and the induction hypothesis applies to g_i making $g_i \in T$. Hence, there are $u_{i,1}, \ldots, u_{i,k(m)} \in R$ with $\prod_{j=1}^{k(m)}(g_i - u_{i,j}) = \mathbf{0}$.

Now put $v_{i,j} = u_{i,j}\phi_i \colon I_i \to \mathbb{R}$. Note that $\prod_{j=1}^{k(m)}(f|_{I_i} - v_{i,j}) = \mathbf{0}$.

It will be necessary to have a C^1 spline connecting any x to any y, $x < y$, with prescribed slopes at the endpoints. Such splines exist; one such is called the M-spline – see Section 6: Addendum, for details. M-splines are, in fact, C^∞.

For later parts of the proof, it is required that each $v_{i,j}$ converge to $f(x_i)$ on the left and to $f(x_{i+1})$ on the right. However, for any sequence $\{y_n\}$ in I_i converging to x_i from the right, there are infinite subsequences where f and $v_{i,j}$ coincide, for some j. Hence, some of the $v_{i,j}$ converge to $f(x_i)$, say $v_{i,1}, \ldots, v_{i,l}$. Similarly at x_{i+1}. If $l < k(m)$, there is $a \in I_i$ so that $v_{i,l+1}, \ldots, v_{i,k(m)}$ are bounded away from $f(x_i)$ on (x_i, a). In these cases, connect $(x_i, f(x_i))$ and $(a, v_{i,j}(a))$ with an M-spline ζ_j with $\frac{d^+}{dx}\zeta_j(x_i) = \frac{d^+}{dx}f(x_i)$ and $\frac{d^-}{dx}\zeta_j(a) = \frac{d}{dx}v_{i,j}(a)$, for $j = l+1, \ldots, k(m)$. Similarly, at x_{i+1} perhaps with different $v_{i,j}$, with $b \in (x_i, x_{i+1})$, $a < b$. Redefine the $v_{i,j}$ on (x_i, a) and on (b, x_{i+1}), if necessary. The same symbols, $v_{i,j}$, are kept for the modified functions. Now, all the $v_{i,j}$ can be extended to $\operatorname{cl} I_i$ so that they coincide with f at the endpoints. Note the equation $\prod_{j=1}^{k(m)}(f|_{I_i} - v_{i,j}) = \mathbf{0}$ still holds.

It is required to find elements of R that put f into T. It turns out that $2k(m)$ such elements can be found.

Once again M-splines will be used. For $i \in \mathbb{Z}$, $j = 1, \ldots, k(m)$, define $\mu_{i,j} \colon [x_{i-1}, x_i] \to \mathbb{R}$ as an M-spline using the following data:

$\mu_{i,j}(x_{i-1}) = v_{i,j}(x_{i-1}) = f(x_{i-1})$, $\mu_{i,j}(x_i) = v_{i,j}(x_i) = f(x_i)$,

$\frac{d^+}{dx}\mu_{i,j}(x_{i-1}) = \frac{d^-}{dx}v_{i-1,j}(x_{i-1})$ and $\frac{d^-}{dx}\mu_{i,j}(x_i) = \frac{d^+}{dx}v_{i,j}(x_i)$. Recall that $\mu_{i,j}$ and $v_{i,j}$ are C^1 where defined.

From now on it will be necessary to consider two cases, i even and i odd. For i even, define $\nu_{e,j} \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\nu_{e,j}(x) = \begin{cases} v_{i,j}(x) \text{ if } x \in \operatorname{cl} I_i \,, \\ \mu_{i,j}(x) \text{ if } x \in \operatorname{cl} I_{i-1} \,. \end{cases}$$

Because of the slopes chosen for the endpoints of I_{i-1}, $\nu_{e,j} \in R$.

Now construct $\nu_{o,j}$ is the same manner, using i odd. Again $\nu_{o,j} \in R$. Because of the nature of the $v_{i,j}$, the following equation holds, $\prod_{j=1}^{k(m)}(f - \nu_{e,j}) \prod_{j=1}^{k(m)}(f - \nu_{o,j}) = \mathbf{0}$, showing that $f \in T$. ■

In the proof of Theorem 4.40 the one-sided continuity of the one-sided derivatives (true from Corollary 4.37) is not used but is a consequence of the theorem.

Theorem 4.40 gives a converse to Corollaries 4.36 and 4.37 in the case where D^c is of finite C-B index. However, there are functions in $(f, D) \in T$, in fact in K, where D^c is not of finite C-B index. The set F, in the next proposition, when compact, is homeomorphic to the Cantor set.

4.41. PROPOSITION. *Assume F is a nowhere dense closed subset of $\mathbb{R}$ such that for every $x \in F$ and every neighbourhood N of x, N contains an interval making up $\mathbb{R} \setminus F$ of the form (x_1, x_2) with $x_1, x_2 \in F$. Then there is $f \in K$ such that, with the usual notation, D^c contains F and $D^c \setminus F$ consists of isolated points.*

PROOF. The distance function d is used, where $d(x)$ is the distance from x to the closed set F. The function in the statement of the proposition is $f = d^2$. Notice that this is a C^1 function except at the points $\frac{x_1+x_2}{2}$ where (x_1, x_2) is a finite interval in $\mathbb{R} \setminus F$; call this set of midpoints A. At these points f does not have a derivative. However, by hypothesis, if $x \in F$, there is no neighbourhood of x where f is C^1.

The largest open set on which f is C^1, call it D, excludes the midpoints of the previous paragraph but also, as pointed out, every point of F. In other words, if $x \in F$, there is no open interval containing x where f is C^1. In other words, D is $\mathbb{R} \setminus (F \cup A)$.

It will turn out that Theorem 4.31 (1) can be used but it first must be shown that $f \in T$. To this end, let the set of bounded intervals among those disjoint open intervals making up $\mathbb{R} \setminus F$ be denoted $\mathfrak{F}$ and an element is written $(x, y) \in \mathfrak{F}$. Now consider the closed set $V = \bigcup_{(x,y)\in\mathfrak{F}} [x, \frac{x+y}{2}] \cup F \cup Z$, where Z is the closure of the infinite intervals in $\mathbb{R} \setminus F$ (if any). (Notice that if $z \notin V$, $z \in (\frac{x+y}{2}, y)$ for some $(x, y) \in \mathfrak{F}$, making V closed.) Since f has a left continuous left derivative at each $\frac{x+y}{2}$, $f|_V$ is C^1. By the Whitney Extension Theorem ([W, Theorem I]), there is $g \in R$ such that $g|_V = f|_V$. The same method can be used with the right hand intervals, $[\frac{x+y}{2}, y]$ to produce $h \in R$. Then, $(f - g)(f - h) = \mathbf{0}$, making $f \in T$.

In order to use Theorem 4.31 (1), it is necessary to find $l \in R$ so that $l|_{D^c} = f|_{D^c}$. To do this, it is only necessary to modify f in each $(x, y) \in \mathfrak{F}$. The method is to divide $(x, y) \in \mathfrak{F}$, say, into thirds via $x < a < b < y$ and to connect the points $(a, f(a))$ with $(\frac{x+y}{2}, f(\frac{x+y}{2}))$ using smooth spline with slopes $\frac{d}{dx} f(a)$ and 0, respectively. The same is done connecting $(\frac{x+y}{2}, f(\frac{x+y}{2}))$ and $(b, f(b))$, in the same manner. The resulting function l will be in R and $l|_{D^c} = f|_{D^c}$. This shows that $f \in K$. ∎

To have a familiar example, let F, in the above, be the standard Cantor set in $[0, 1]$. Then, A consists of the midpoints of the "middle thirds".

The following example uses Proposition 4.29. It is one of many illustrations of how S differs from K.

5.42. EXAMPLE. *Let f be the function $f(x) = x^2 \sin(1/x)$, for $x \neq 0$ and $f(0) = 0$. Then $f \in S \setminus T$, and, hence, $f \notin K$.*

PROOF. Notice that $f^2, f^3 \in R$, hence, f is in the (2,3)-closure, S, of R. If $f \in T$, there would be $u_1, \ldots, u_k \in R$ with $\prod_{i=1}^{k} (f - u_i) = \mathbf{0}$. However as $x \to 0$, $\frac{d}{dx} f(x)$ has infinitely many cluster points, which is impossible for elements of T. Notice that f has a derivative everywhere but it is not continuous at 0. Moreover, since $T \cap S = K$, f cannot be in K. ∎

This and similar examples show that the split integral closure can be strictly smaller than the integral closure; here of R in L.

The following is also a corollary of Proposition 4.29.

5.43. COROLLARY. *Let $(f, D) \in L$ be such that for some $x_0 \in D^c$, there is an interval $(r, x_0) \subset D$ such that f has a vertical tangent line as x_0 is approached from the left, or there is an interval $(x_0, s) \subset D$ such that f has a vertical tangent line as x_0 is approached from the right, then $f \notin T$.*

PROOF. Consider the first case and a sequence $\{x_n\}_{n \in \mathbb{N}}$ in (r, x_0) converging to x_0. The sequence $\{\frac{d}{dx} f(x_n)\}$ will grow indefinitely, contradicting Proposition 4.29. The other case is similar. ■

This corollary will be used to find additional sorts of elements in $S \setminus K$.

5.44. EXAMPLES. *For $m > 1$ an odd positive integer the function $f_m(x) = x^{1/m}$ is not in T. For m an even positive integer the function $g_m(x) = x^{1/m}$ for $x > 0$ and 0 if $x \leq 0$ is not in T. Both these types of functions are in S.*

PROOF. These are not in T by Corollary 5.43, and, as will be seen, since they are in S, they are not in K.

For odd $n \in \mathbb{N}$, $n > 1$, f_n^n is clearly in R. In addition $f_n^{n+1}(x) = x^{\frac{n+1}{n}}$ has a continuous derivative and is also in R. This means that f_n^n and f_n^{n+1} are in R showing, by [BKR, Lemma 5.1.5], that $f_n \in S$, since n and $n+1$ are relatively prime ((2,3)-closed is the same as $(n, n+1)$-closed).

For even n, g_n is similar except that $g_n^n \in K \subset S$ (by Examples 4.33) while $g_n^{n+1} \in R$. Again, $g_n \in S$. ■

The next example shows that not all elements of K are in simple DL-extensions of R.

5.45. EXAMPLE. *Suppose $(g, D) \in K$ is such that there is $x_0 \in D^c$ where there is a sequence $\{y_n\}$ in D^c converging to x_0 from the left and a sequence $\{z_n\}$ in D^c converging to x_0 from the right, and for all $n \in \mathbb{N}$, $g(y_n) = g(z_n) = 0$. For any $h \in K$ such that $D_h^c = \{x_0\}$, put $f = g + h$. Then, $f \in K$ is not in any simple DL-extension of R.*

PROOF. Note that $\frac{d^-}{dx} h(x_0) \neq \frac{d^+}{dx} h(x_0)$. Suppose that $s \in \mathfrak{S}$ and that $f = as + b$, $a, b \in R$. Then, $as \in R$, which is impossible, or $as \in \mathfrak{S}$. Suppose $y_n \in D_{as}$, then there is an open neighbourhood N of y_n on which as is C^1, Hence, in N, f would be C^1. This is not the case and, therefore, $y_n \in D_{as}^c$. It follows that $as(y_n) = 0$ (Proposition 4.32). Similarly for each z_n. Hence, for each $n \in \mathbb{N}$, $f(y_n) = h(y_n) = b(y_n)$ and $f(z_n) = h(z_n) = b(z_n)$. Calculating the derivative of b along $\{y_n\}$ yields $\frac{d}{dx} b(x_0)$ and similarly using $\{z_n\}$. But these same calculation give, on one side, $\frac{d^-}{dx} h(x_0)$ and, on the other, $\frac{d^+}{dx} h(x_0)$. Since these are unequal, there is a contradiction. ■

As an instance of this, take $l(x) = x^3 \sin(1/x)$ for $x \neq 0$ and $l(0) = 0$ and $g = |l|$. Any h as in the above will work, for example $h(x) = |x|$ and $x_0 = 0$.

The functions of Proposition 4.41 also produce illustrations. Take g to be one of these. For x_0, pick any element of F that is not an endpoint of one of the open intervals making up $\mathbb{R} \setminus F$. Then, it is possible to find sequences as required in Example 5.45. Any h as in the example will work. Then, $f = g + h$ is not in any simple DL-extension of R.

The next results show that L is not integral over R and, hence, that S is a proper subring of L, since S is integral over R.

5.46. PROPOSITION. *Let n be an odd integer ≥ 3 and $f_n(x) = x^{1/n}$. Then f_n is integral of degree n over R and n is the minimal degree.*

PROOF. Let $n > 1$ be an odd integer. Put $f_n(x) = x^{\frac{1}{n}}$. Certainly, $X^n - x = \mathbf{0}$ is satisfied by f_n.

Suppose that for $a_0, a_1, \ldots, a_{n-2} \in R$ that

$$X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_1 X + a_0 = 0 \,.$$

If this equation is satisfied by f_n then

$$x^{\frac{n-1}{n}} + a_{n-2}x^{\frac{n-2}{n}} + \cdots + a_1 x^{\frac{1}{n}} + a_0 = \mathbf{0} \,.$$

This expression can be differentiated when $x \neq 0$, to get

$$(1)\ \left(\frac{n-1}{n}x^{-\frac{1}{n}} + \frac{n-2}{n}a_{n-2}(x)x^{-\frac{2}{n}} + \cdots + x^{-\frac{n-1}{n}}a_1(x)x^{-\frac{n-1}{n}}\right) + \left(\frac{d}{dx}a_{n-2}(x)x^{\frac{n-2}{n}} + \cdots + \frac{d}{dx}a_1(x)x^{\frac{n-1}{n}} + \frac{d}{dx}a_0(x)\right) = \mathbf{0}$$

The second term in (1) is denoted $B(x)$. Notice that as $x \to 0$, $B(x) \to \frac{d}{dx}a_0(0)$. For x close to 0, the C^1 functions $a_i(x)$ can be approximated by linear functions, $a_i(x) \approx \frac{d}{dx}a_i(0)x + a_i(0)$. Then (1) becomes

$$(2)\ \mathbf{0} \approx \left(\frac{n-1}{n}x^{-\frac{1}{n}} + \frac{n-2}{n}a_{n-2}(0)x^{-\frac{2}{n}} + \cdots + \frac{1}{n}a_1(0)x^{-\frac{n-1}{n}}\right) + \left(\frac{n-2}{n}\frac{d}{dx}a_{n-2}(0)x^{\frac{n-2}{n}} + \cdots + \frac{1}{n}\frac{d}{dx}a_1(0)x^{\frac{1}{n}} + B(x)\right)$$

The second term of (2) goes to $\frac{d}{dx}a_0(0)$ as $x \to 0$. The first term, call it $A(x)$ begins with $\frac{n-1}{n}x^{-\frac{1}{n}}$ that is unbounded as $x \to 0$ since its coefficient is 1. The coefficients of the remaining terms of $A(x)$, since continuous, are all bounded as $x \to 0$ and, hence, the first term dominates the others showing, that $A(x) \to \infty$ as $x \to 0$. This is a contradiction. This shows that the original expression is not possible and f_n is integral over R of degree n but not of smaller degree. ∎

The following consequence of Proposition 5.46 is clear.

5.47. COROLLARY. *The ring S is not the result of a finite sequence of simple (2,3)-extensions starting at R.*

Corollary 5.47 could also be stated that S has elements of arbitrarily high integrality degree over R.

Notice that in Proposition 5.46 that the conclusion remains true if the functions are viewed as on an interval containing 0; this will be used below.

5.48. PROPOSITION. *The ring L is not integral over R. Hence, S is a proper subring of L.*

PROOF. From Proposition 5.46, for each odd $n \geq 3$ there is an element of L, namely $f_n(x) = x^{1/n}$, integral of degree exactly n over R. To simplify notation $f_1 \in R$ is also used. Define $F \in L$ as follows.

$$F(x) = \begin{cases} -\mathbf{1} & \text{if } x \leq -1 \\ f_n(x-2k) & \text{if } n = 2k+1, k \geq 0, k \text{ even}, \\ & \ 2k-1 \leq x \leq 2k+1 \\ -f_n(x-2k) & \text{if } n = 2k+1, k \geq 0, k \text{ odd}, \\ & \ 2k-1 \leq x \leq 2k+1\ . \end{cases}$$

Since the pieces making up F are designed so that F is continuous; it is clear that F is C^1 on a dense open subset of $\mathbb{R}$, i.e, $F \in L$. If F were integral over R then Proposition 5.46 would be contradicted. ■

In the remark after Proposition 4.32, it was shown that for $(f, D) \in L$, if $R[f]$ is a simple (2,3)-extension then $f|_{D^c} = \mathbf{0}$. This necessary condition is not sufficient.

5.49. COROLLARY. *There is $(f, D) \in L$ with $f|_{D^c} = \mathbf{0}$ but $f \notin S$.*

PROOF. The function F constructed in the proof of Proposition 5.48 is in $L \setminus S$ but does not have the property that $F|_{D^c} = \mathbf{0}$. However, F can be modified to create such an example, as follows. Consider horizontal lines through $1/2$ and $-1/2$. These will meet the graph of F at pairs of points. The idea is to connect adjacent intersections with M-splines to make the new function C^1 at the odd positive integers. A slight modification is needed at the point $(-1/2, -1/2)$; it can be connected to $(-3/2, -1)$. As an illustration, one M-spline connects $(1/2, 1/2)$ with $(2-1/8, 1/2)$ with slope at $(1/2, 1/2)$ equal to 1 and that at $(2-1/8, 1/2)$ equal to $-4/3$. Call the resulting function f and $D_f^c = \{2, 4, 6, \ldots\}$; f is not integral over R because it behaves like F in intervals around each even integer ≥ 2. ■

The next step is to reinforce the idea that S and K are of very different natures. The process will be to show that for any proper dense open set D of $\mathbb{R}$, there is $(f, D) \in S$ (in fact in a simple (2,3)-extension of R) that is not in K.

5.50. PROPOSITION. *Let D be a dense open set in $\mathbb{R}$ with $D \neq \mathbb{R}$. Then there exists an everywhere differentiable function $f \in S$ such that f is C^1 on D and D is the largest open set on which it is C^1. In addition, $f \notin T$.*

PROOF. As usual D is expressed as a countable (possibly finite) disjoint union of open intervals (a_n, b_n). These sets will be used to defined a countable family of functions and the desired function f will be their uniform limit.

The basis for the construction is the function

$$k(x) = x^2(1-x)^2 \sin(\frac{\pi}{8x(1-x)})$$

on $(0,1)$ and 0 elsewhere. It follows that $\frac{d}{dx}k$ exists everywhere but is discontinuous at 0 and at 1. Notice that k^2 and k^3 are C^1 functions.

Notice as well that k and $\frac{d}{dx}k$ can be seen to be bounded. Both of these hold because k and k' vanish outside of the interval $(0,1)$.

For a finite interval (a_n, b_n), write $g_n(x) = k(\frac{x-a_n}{b_n-a_n})$. For an infinite interval, say, (a_m, ∞), a slight modification is needed. Here, k is adapted to the interval (a_m, a_m+1) and at the midpoint, which is the maximum of the function with value $r > 0$, g_m continues to the right with the constant r. An infinite interval to the left is done in the same way. The function g_n has the same properties as k, namely it is differentiable everywhere, but not continuously so at a_n and b_n; its square and cube are C^1 and it and its derivative are bounded.

The functions g_n will be scaled by constants $\rho_n > 0$ so that both $|\rho_n g_n|$ and $|\rho_n \frac{d}{dx} g_n|$ are less than 2^{-n-1}. Furthermore the ρ_n will be chosen so that

$$|(\rho_n)^2 g_n^2|, |(\rho_n)^3 g_n^3|, |2(\rho_n)^2 g_n \frac{d}{dx} g_n| \text{ and } |3(\rho_n)^3 g_n^2 \frac{d}{dx} g_n|$$

are all less than 2^{-n-1}. These properties collectively are referred to as $(*)$.

This is clearly possible since g_n and its derivative are bounded. These conditions will allow us to apply the Weierstrass M-test to different sums.

Now let the function $f = \sum_{n \in B} \rho_n g_n$. When B is finite it is obvious that f has the required properties. Hence, assume that $B = \mathbb{N}$. By the M-test f is continuous on $\mathbb{R}$.

We first want f to be differentiable. It suffices to show differentiability for any point internal to an interval $[u, v], u < v$. Note that $[u, v]$ can, if necessary, be chosen large enough to contain a point in D^c where all the g_n vanish, so that $\sum g_n$ converges. The choice of the ρ_n implies that the $\sum \rho_n \frac{d}{dx} g_n$ converges uniformly on $[u, v]$.

By a result, often called the "third preservation theorem", (for example [DS, Theorem 8.7] or [T, Theorem 4.4.11]) $\sum \rho_n(g_n)$ converges uniformly to a differentiable function on all of [u,v] whose derivative is the $\sum \rho_n \frac{d}{dx} g_n$. This makes $\sum \rho_n g_n$ differentiable on all of $\mathbb{R}$.

Now we would like f to be in S. It follows from $(*)$ and the third preservation theorem that f^2 and f^3 are differentiable and that their derivatives are uniform limits of continuous functions, and hence continuous.

Lastly the claim that D is the largest subset on which f is C^1.

At each finite endpoint of an interval in D, the derivative is discontinuous by calculation. Moreover, if there is $x_0 \in D^c$ that is not such an endpoint, every interval around x_0 has points where f is zero, and then $\frac{d}{dx}f(x_0) = 0$. In addition, in any such interval there are points where $\frac{d}{dx}f$ is discontinuous. Thus f is not C^1 at such an x_0.

By Proposition 4.30, none of the functions constructed above can be in T. ■

The functions constructed in Proposition 5.50 are in the ring D^1, the ring of functions that have a derivative for all $x \in \mathbb{R}$, of [K, Example 7.5], but not in the ring H_0 defined there. This is because an everywhere differentiable function in H_0 would be C^1.

The next step is to show that there is $h \in H_0 \setminus L$. This demonstrates that $H_0 \neq K$, contrary to what was conjectured in [K, Example 7.5]. In Proposition 4.38, it was shown that $K \subseteq T \subseteq H_0$. However, as remarked at the start of Section 2, there is only one isomorphic copy over R of K in $\mathrm{C}(\mathbb{R})$, and it was shown to be in L.

Alan Dow has kindly contributed an example of a function, defined on $(0,1)$, that has all the properties of a function in H_0. It only needs to be "stretched" to be defined on $\mathbb{R}$ to yield the example as required.

5.51. Example. [Dow] *There is a function $h \in H_0 \setminus L$.*

Proof. The start is to define a function $f\colon (0,1) \to \mathbb{R}$. Let $\{q_n \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q} \cap (0,1)$. Define $f : (0,1) \to \mathbb{R}$ by

$$f(x) = \sum_{n\in\mathbb{N}} 2^{-n}|x - q_n| \ .$$

It will be shown that f has all the properties of an element of H_0 (although only defined on $(0,1)$).

If, for each $n \in \mathbb{N}$, $f_n(x) = 2^{-n}|x - q_n|$, then $f(x) = \sum_{n\in\mathbb{N}} f_n(x)$ is continuous by the M-test.

Let $x, t \in (0,1)$, $x < t$. Then

$$\begin{aligned} f(t) - f(x) &= \sum_{n\in\mathbb{N}} 2^{-n}(|t - q_n| - |x - q_n|) \\ &= \sum_{q_n \le x} 2^{-n}(t - x) + \sum_{x < q_n \le t} 2^{-n}((t - q_n) - (q_n - x)) \\ &\quad + \sum_{t < q_n} 2^{-n}(x - t) \ . \end{aligned}$$

In the above, now assume that $x \notin \mathbb{Q}$ and $x < t$. Consider $\Delta_{f,x}(t)$. By the above, this is the sum

$$\sum_{q_n < x} 2^{-n} - \sum_{t < q_n} 2^{-n} + \sum_{x < q_n \le t} 2^{-n}\frac{(t - q_n) - (q_n - x)}{t - x} \ .$$

However, in the numerator of the third term $t - q_n < t - x$ and $q_n - x < t - x$. Hence, the absolute value of the third term is at most $2 \cdot \sum_{x<q_n\le t} 2^{-n}$. By taking t sufficiently close to x, for any $n_0 \in \mathbb{N}$, it can be assumed that $x < q_n \le t$ means $n \ge n_0$. Hence, the third term goes to 0 as $t \to x^+$. When, $t < x$, the calculation is the same. Hence, $\frac{d}{dx}f(x)$ exists when $x \notin \mathbb{Q} \cap (0,1)$.

When $x \in \mathbb{Q}$, there is an extra term. Put $x = q_m$. Then $\Delta_{f,q_m}(t)$ is as above plus $2^{-m}|t-q_m|/(t-q_m)$. The conclusion is that $\frac{d^+}{dx}f(q_m) = 2^{-m} + \sum_{q_n<q_m} 2^{-n} - \sum_{q_m<q_n} 2^{-n}$ and $\frac{d^-}{dx}f(q_m) = -2^{-m} + \sum_{q_n<q_m} 2^{-n} - \sum_{q_m<q_n} 2^{-n}$.

Hence, one-sided derivatives exist for all $x \in (0,1)$.

There are proofs of one-sided continuity to be added.

1. The right continuity of $\frac{d}{dx}f(x)$ where $x \notin \mathbb{Q}$: for $x < t$

$$\frac{d^+}{dx}f(t) = \begin{cases} \sum_{q_n<t} 2^{-n} - \sum_{t<q_n} 2^{-n}\,, & t \notin \mathbb{Q} \\ 2^{-m} + \sum_{q_n<q_m} 2^{-n} - \sum_{q_m<q_n} 2^{-n}\,, & t = q_m\,. \end{cases}$$

In the second expression, as $t = q_m$ approaches x, the term 2^{-m} can be chosen so that $m \ge n_0$, for any $n_0 \in \mathbb{N}$. Hence, both expressions converge to $\sum_{q_n<x} 2^{-n} - \sum_{x<q_n} 2^{-n}$.

2. The right continuity of the right derivative of f at $x = q_m$. The right derivative at $t > q_m$, $t \notin \mathbb{Q}$, is

$$\sum_{q_n<t} 2^{-n} - \sum_{t<q_n} 2^{-n} = 2^{-m} + \sum_{q_n<t, n\ne m} 2^{-n} - \sum_{t<q_n} 2^{-n}\,.$$

Note that, since $t \notin \mathbb{Q}$, the first term could also be written $\sum_{q_n\le t, n\ne m} 2^{-n}$.

The right derivative of f at q_z with $q_m < q_z$ is

$$2^{-z} + \sum_{q_n<q_z} 2^{-n} - \sum_{q_z<q_n} 2^{-n}$$

$$= \sum_{q_n\le q_z} 2^{-n} - \sum_{q_z<q_n} 2^{-n} = 2^{-m} + \sum_{q_n\le q_z, n\ne m} 2^{-n} - \sum_{q_z<q_n} 2^{-n}\,.$$

Since these expressions are the same, the limit as $t \to q_m^+$ is the same whether t is rational or irrational. It is the right derivative of f at q_m.

3. The final step to show that f satisfies the remaining condition, i.e., that of the right limit of the left derivatives. It is to be shown that when $t \to x^+$, the left derivative at t converges to the right derivative at x.

Suppose first that $x \notin \mathbb{Q}$. When $t \notin \mathbb{Q}$, the left derivative at t is $\sum_{q_n<t} 2^{-n} - \sum_{t<q_n} 2^{-n}$. Note that since $t \notin \mathbb{Q}$, the second term can be written $-\sum_{t\le q_n} 2^{-n}$.

When $t = q_z$, the left derivative is

$$-2^{-z} + \sum_{q_n<q_z} 2^{-n} - \sum_{q_z<q_n} 2^{-n} = \sum_{q_n<q_z} 2^{-n} - \sum_{q_z\le q_n} 2^{-n}.$$

Both expressions converge to $\frac{d}{dx}f(x)$ as $t \to x^+$.

If $t > q_m$ and $t \notin \mathbb{Q}$, the left derivative at t is

$$\sum_{q_n<t} 2^{-n} - \sum_{t<q_n} 2^{-n} = 2^{-m} + \sum_{q_n<t, n\neq m} 2^{-n} - \sum_{t<q_n} 2^{-n}.$$

Again, the last term can be written $-\sum_{t\leq q_n} 2^{-n}$.

If $t = q_z > q_m$, the left derivative is

$$-2^{-z} + \sum_{q_n<q_z} 2^{-n} - \sum_{q_z<q_n} 2^{-n} = 2^{-m} + \sum_{q_n<q_z, n\neq m} 2^{-n} - \sum_{q_z\leq q_n} 2^{-n}.$$

This shows the convergence.

The final step is to get a function defined on all of $\mathbb{R}$. Let $\phi : (0,1) \to \mathbb{R}$ be an order preserving C^1 homeomorphism with ϕ^{-1} also C^1. Then, $h = f \cdot \phi^{-1}$ is a continuous function satisfying all the conditions of H_0. The usual formulas for the Chain Rule apply to one-sided derivatives. These and the fact that the homeomorphism ϕ^{-1} is both order preserving and C^1 show that all the properties of f also hold for h. ■

6. Addendum: M-splines. In the proofs of Theorem 4.40 and Corollary 5.49, it is required that, over an interval, a smooth function be found that has prescribed endpoints and prescribed one-sided slopes at these endpoints. The M-spline gives this. One starting point for the M-spline is found in [Ho, Lemma 1.2.3], although only dimension one is required here. The basis for the Mollifier-spline, or *M-spline* that is used above, is the Mollifier function:

$$\sigma(x) = \begin{cases} \exp \frac{-1}{1-x^2} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}.$$

Let $C = \int_{-1}^{1} \sigma(x)dx$. Put $\phi(x) = (1/C)\sigma(x)$. Thus $\int_{-\infty}^{\infty} \phi(x)dx$ exists and is equal to 1. Now define $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$. It follows that $\frac{d}{dx}\Phi(x) = \phi(x)$ and, from that, all the derivatives of $\Phi(x)$ are zero at ± 1, and, between -1 and 1, $\Phi(x)$ is a C^∞ function increasing from 0 to 1.

If, now, the idea is to have a C^∞ spline between the points (a, b) and (c, d) in $\mathbb{R}^2$, with $a < c$, first define $\Phi_{a,c}(x) = \Phi(\frac{2x-(a+c)}{c-a})$ and then $\gamma(x) = b + (d - b)\Phi_{a,c}(x)$ is defined on the interval $[a, c]$. The right derivatives of $\gamma(x)$ at a and the left derivative at c exist and are zero. If, for example, $b > d$, then $\gamma(x)$ decreases as x goes from a to c. (This part can also be found in [BR, Section 2a].)

M-splines will be modified again so that they do not have the same derivatives at the two endpoints of the interval. The schema used follows that suggested by Simone Brugiapaglia. The endpoints and the slopes will be specified after the general statement. Let the standard M-spline between the points (a, c) and (b, d) be written $\Phi_{a,b}^{c,d}(x)$. It is monotone between a and b and $\frac{d}{dx}\Phi_{a,b}^{c,d}(a) = \frac{d}{dx}\Phi_{a,b}^{c,d}(b) = 0$. Suppose the slopes at the endpoint of the spline are to be m_1 and m_2, respectively. Next define $\Psi(x) = \int_a^x \Phi_{a,b}^{m_1,m_2}(t)dt$. From this: $\Psi(a) = 0, \Psi(b) = \int_a^b \Phi_{a,b}^{m_1,m_2}(t)dt$, $\frac{d^+}{dx}\Psi(a) = \Phi_{a,b}^{m_1,m_2}(a) = m_1$, and $\frac{d^-}{dx}\Psi(b) = \Phi_{a,b}^{m_1,m_2}(b) = m_2$. Finally the function g is defined by $g(x) = \Phi_{a,b}^{c,d-\Psi(b)}(x) + \Psi(x)$. By the previous calculations, $g(a) = c + 0 = c$, $g(b) = d - \Psi(b) + \Psi(b) = d$, $\frac{d^+}{dx}g(a) = 0 + m_1 = m_1$, and $\frac{d^-}{dx}g(b) = 0 + m_2 = m_2$.

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