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FROM SPECKER ℓ-GROUPS TO BOOLEAN ALGEBRAS VIA Γ

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ABSTRACT. The author constructed in 1986 an equivalence Γ between abelian ℓ-groups with a strong unit and C.C. Chang MV-algebras. In 1958 Chang proved that boolean algebras coincide with MV-algebras satisfying the equation $x \oplus x = x$. In this paper it is proved that Γ yields, by restriction, an equivalence between the category S of Specker ℓ-groups whose distinguished unit is singular, and the category of boolean algebras. As a consequence, Grothendieck's K_0 functor yields an equivalence between abelian Bratteli AF-algebras and the countable fragment of S. An equivalence in the opposite direction is obtained by a combination of Γ with the Stone and Gelfand dualities.

to Ernst Paul Specker, in memoriam

1. Introduction

... and one may say that the invention of functors is one of the main goals of modern mathematicians, and one which usually yields the most startling results.

J. Dieudonné, [12, p.236]

By an *ℓ-group* G we mean a lattice-ordered abelian group. By a *unital* ℓ-group we mean a pair (G, u) where G is an ℓ-group and u is a strong order-unit of G, [4, 2.2.12]. To avoid trivialities, $u \neq 0$. Throughout this paper, by a "unit" in an ℓ-group H we will mean a strong order-unit. When an ℓ-homomorphism $\psi: G \to G'$ satisfies $\psi(u) = u'$ we say that ψ is *unital* (or *unit-preserving*), and we write $\psi: (G, u) \to (G', u')$.

We refer to [4] for ℓ-groups and their spectral spaces. An element s in an ℓ-group G is said to be *singular* if $s \geq 0$ and $t \wedge (s - t) = 0$ for each $t \in G$ with $0 \leq t \leq s$. An ℓ-group is a *Specker ℓ-group* if it is generated, as a group, by its singular elements. For Specker ℓ-groups we refer to [3, 9, 10, 16].

An *MV-algebra* is a structure $A = (A, 0, \neg, \oplus)$ satisfying the following equations:

$$\begin{aligned} x \oplus (y \oplus z) &= (x \oplus y) \oplus z \\ x \oplus 0 &= x \\ \neg\neg x &= x \end{aligned}$$

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$$\begin{aligned} x \oplus \neg 0 &= \neg 0 \\ \neg(\neg x \oplus y) \oplus y &= \neg(\neg y \oplus x) \oplus x. \end{aligned}$$

A straightforward verification with the help of [8, 1.7] shows that Chang's original axioms [7] for MV-algebras are equivalent to the list of six axioms in [8, Definition 1.1.1], given by the present axiomatization together with the commutativity axiom $x \oplus y = y \oplus x$. In [17] it is shown that commutativity follows from the five equations listed here.

Boolean algebras coincide with *idempotent* MV-algebras, i.e., MV-algebras satisfying the equation $x \oplus x = x$. Furthermore, in any boolean algebra the MV-algebraic operation $\oplus$ coincides with the underlying lattice operation $\vee$. See [7, 1.16-1.17] or [8, 1.5.3 (i)$\leftrightarrow$(vi)]. Since any homomorphism of two boolean algebras *qua* MV-algebras is automatically a boolean homomorphism, boolean algebras are a full subcategory of MV-algebras. To avoid trivialities, in all MV-algebras in this paper $0 \neq \neg 0$.

In [19, §3] an equivalence Γ is constructed between the category of unital ℓ-groups with unital ℓ-homomorphisms and the category of MV-algebras with their homomorphisms. See Theorems 2.1 and 2.3 for the main properties of Γ used in this paper. A functor Ξ providing a converse categorical equivalence between MV-algebras and unital ℓ-groups is described in Theorem 2.2 following [8, §7.1].

In view of Theorem 2.1 one may ask which (presumably relevant in the literature) category of unital ℓ-groups is equivalent, via Γ, to boolean algebras, *qua* idempotent MV-algebras. The answer given by Theorem 3.2 is that the Γ functor restricts to a categorical equivalence between

> the category S of unital Specker ℓ-groups whose distinguished unit is singular, with their unit-preserving ℓ-homomorphisms,

and

> the category BA of boolean algebras, with their homomorphisms.

In Corollary 3.4 we prove that (G, u) is a unital Specker ℓ-group whose distinguished unit u is singular if and only if (G, u) is unitally ℓ-isomorphic to the ℓ-group of all continuous $\mathbb{Z}$-valued functions over some compact Hausdorff space X, with the distinguished unit given by the constant function 1 over X. The latter space, in turn, is homeomorphic to the maximal spectral space $\mu(G)$.

The specification of the underlying categorical equivalence Γ and of the associated homeomorphism $X \cong \mu(G)$ refines the representation in [3, Corollary 2.12].

As a further refinement, in the final section of this paper we apply our results to abelian AF-algebras. As originally defined by Bratteli in [5], an *AF-algebra* (which is short for "approximately finite-dimensional C*-algebra") is the norm-closure of an ascending sequence of finite-dimensional C*-algebras, all with the same unit. In Theorem 4.5 it is proved that the category of abelian AF-algebras is equivalent to the fragment of the category S given by countable Specker ℓ-groups whose distinguished unit is singular.

There are two parallel ways to implement the equivalence between these two categories: The first one involves composition of the following three categorical equivalences:

$\mathcal{G}$, from abelian C*-algebras to the opposite of the category of compact Hausdorff spaces. This is a part of the (Stone-Naimark)-Gelfand duality, [1, 3.39(6), p.38], [18].

$\mathcal{S}$, from the opposite of boolean spaces to boolean algebras. This is part of Stone duality.

Ξ, from boolean algebras to S.

The second way makes direct use of Grothendieck K_0 in the framework of Elliott classification, [13], [14, §5], [22, 7.3.4], [11, §IV].

Over the category of abelian AF-algebras and their *-algebra homomorphisms, our final Theorem 4.5 can be summarized by writing

$$\boxed{K_0 \;\cong\; \Xi \circ \mathcal{S} \circ \mathcal{G}.}$$

2. Equivalences

For background in category theory we refer to [1]. Readers familiar with the Γ functor may jump to Theorem 3.2.

2.1. THEOREM. ([19, §3]) *For any unital ℓ-group (G, u), let $\Gamma(G, u)$ be the unit interval $[0, u]$ equipped with the operations $\neg x = u - x$ and $x \oplus y = u \wedge (x + y)$. For any unital ℓ-group (H, v) and unital ℓ-homomorphism $\eta\colon (G, u) \to (H, v)$ let Γ restrict η to $[0, u]$. Then Γ is a categorical equivalence between unital ℓ-groups and MV-algebras.*

Proposition 3.36(1) in [1] then ensures the existence of a converse equivalence between MV-algebras and unital ℓ-groups. A concrete example Ξ of such equivalence is constructed in [8, §7], where it is proved:

2.2. THEOREM. (See [8, 7.1.2] for details)

(i) The composite functor $\Gamma \circ \Xi$ is naturally equivalent to the identity functor of the category of MV-algebras. In symbols, $\Gamma \circ \Xi \cong \mathbf{1}_{\text{MV-algebras}}$.

(ii) (See [8, 7.1.7] for details) *The composite functor $\Xi \circ \Gamma$ is naturally equivalent to the identity functor of the category of unital ℓ-groups, $\Xi \circ \Gamma \cong \mathbf{1}_{\text{unital } \ell\text{-groups}}$.*

2.3. THEOREM. ([8, §7.2]) *For any unital ℓ-group (G, u) let the MV-algebra A be defined by $A = \Gamma(G, u)$. Let $\mathcal{I}(G)$ (resp., $\mathcal{I}(A)$) denote the lattice of ℓ-ideals of G (resp., of A) ordered by inclusion.*

(i) The correspondence $\phi\colon J \mapsto \phi(J) = \{x \in G \mid |x| \wedge u \in J\}$ is an order-isomorphism from $\mathcal{I}(A)$ onto $\mathcal{I}(G)$. The inverse order-isomorphism ψ is given by $I \in \mathcal{I}(G) \mapsto \psi(I) = I \cap [0, u] \in \mathcal{I}(A)$.

(ii) *The map* $\theta: J \mapsto J \cap [0,u]$ *is an order-isomorphism between the set of prime* ℓ*-ideals of* G *and the set of prime ideals of* $\Gamma(G,u)$*, both sets being equipped with the inclusion ordering. Thus in particular,* θ *restricts to an injection of the* set *of maximal* ℓ*-ideals of* G *onto the* set *of maximal ideals of* $\Gamma(G,u)$*.*

(iii) *For every* ℓ*-ideal* J *of* G *we have the* ℓ*-isomorphism*

$$\Gamma(G/J, u/J) \cong \Gamma(G,u)/(J \cap [0,u]).$$

2.4. Theorem. Γ *determines a homeomorphism of the compact space* $\mu(G)$ *of maximal* ℓ*-ideals of any unital* ℓ*-group* (G,u)*, onto the space* $\mu(A)$ *of maximal ideals of the MV-algebra* $\Gamma(G,u)$*, where both spectral spaces are equipped with their hull-kernel (Zariski-Jacobson) topologies.*

Proof. Combine Theorem 2.3(ii)-(iii) with [4, 13.2.6] and [20, §4]. ∎

The following result is an exercise in ℓ-group theory. To help the reader we give the elementary proof.

2.5. Lemma. *For any unital* ℓ*-group* (G,u) *the* set $[0,u] = \{x \in G \mid 0 \leq x \leq u\}$ *generates* G *as a group.*

Proof. With $G^+ = \{x \in G \mid x \geq 0\}$ the positive cone of G, we have $G = G^+ - G^+$. Therefore, it is enough to show that every $x \in G^+$ is a sum of elements in $[0,u]$. Let $a \in G$ with $0 \leq a \leq nu$ for some integer $n \geq 1$. Recall that $a = a^+ - a^-$, where $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$, so $-a^- = a \wedge 0$. Because $a = (a \wedge nu) - (a \wedge 0)$, we have

$$a = \sum_{m=0}^{n-1} [(a \wedge (m+1)u) - (a \wedge mu)].$$

Using the ℓ-group identity $(a \wedge b) - c = (a-c) \wedge (b-c)$ and distributivity of the lattice reduct, we can write for each $m = 0, \ldots, n-1$:

$$\begin{aligned}
(a \wedge (m+1)u) - (a \wedge mu) &= [(a \wedge (m+1)u) \text{-} mu] - [(a \wedge mu)\text{-}mu] \\
&= [(a - mu) \wedge u] - [(a - mu) \wedge 0] \\
&= [(a - mu) \wedge u] - [(a - mu) \wedge u \wedge 0] \\
&= [(a - mu) \wedge u] + [(a - mu) \wedge u]^- \\
&= [(a - mu) \wedge u]^+ \\
&= [(a - mu) \wedge u] \vee 0 \\
&= [(a - mu) \vee 0] \wedge (u \vee 0) \\
&= (a - mu)^+ \wedge u.
\end{aligned}$$

Therefore,

$$a = \sum_{m=0}^{n-1}[(a \wedge (m+1)u) - (a \wedge mu)] = \sum_{m=0}^{n-1}[(a - mu)^{+} \wedge u],$$

which shows that a is a sum of elements in $[0, u]$. ∎

2.6. DEFINITION. ([3, Definition 2.2][1]) *Let G be an ℓ-group. An element $s \in G$ is said to be singular if $s \geq 0$ and $t \wedge (s - t) = 0$ for each $0 \leq t \leq s$.*

A moment's reflection shows that if s is singular and $0 \leq t \leq s$, then t is singular. Furthermore,

$$\text{if } r \text{ and } s \text{ are singular, then so is } r \vee s. \tag{1}$$

For details see, e.g., [4, 11.2.9-11.2.10].

2.7. LEMMA. *Let (G, u) be a unital ℓ-group. If the unit u is singular then u is the largest singular element of G, and $[0, u]$ is the set of singular elements of G.*

PROOF. By way of contradiction, assume some singular $s \in G$ satisfies $s \not\leq u$. By (1), $s^* = u \vee s$ is singular and $s^* > u$. It follows that $d = s^* - u$ is singular > 0, and $s^* \wedge d = 0 = u \wedge d$. By [4, 1.2.24], $mu \wedge d = 0$ for all $0 \leq m \in \mathbb{Z}$. Since u is a unit of G, $\overline{m}u \geq d$ for some $\overline{m} = 2, 3, \ldots$. Hence, $\overline{m}u \wedge d = d > 0$, a contradiction. ∎

3. Specker lattice-ordered groups whose distinguished unit is singular

3.1. DEFINITION. ([3, Definition 2.2(2)], [9, Definition of S-group, p. 206]).[2] *An ℓ-group is a Specker ℓ-group if it is generated, as a group, by its singular elements.*

Throughout this paper, by a "unit" in an ℓ-group H we will mean a strong order-unit, [4, 2.2.12]. By [3, Lemma 2.11(2)], strong and "weak" order-units are the same in every Specker ℓ-group.

In what follows,

> The symbol S will denote the category of Specker ℓ-groups (G, u) with a (necessarily unique by Lemma 2.7) singular unit u, and their unital ℓ-homomorphisms.

By Lemma 2.5, an ℓ-group is a Specker ℓ-group if and only if it is generated, *as an ℓ-group*, by its singular elements.

3.2. THEOREM. *Let Γ' be the restriction to S of the Γ functor. Then Γ' is a categorical equivalence between S and the category of boolean algebras.*

[1] The present definition of a singular element goes back to Iwasawa [15, top page 784]. In the definition of a singular element s given in [9, p.207] and [4, 11.2.7] it is assumed $s > 0$. This has no effect on the results of this paper.

[2] For the origins of this terminology see [23] and [3, §2].

Proof. Let $(G, u) \in \mathsf{S}$ be a unital Specker ℓ-group. By Lemma 2.7, the unit interval $[0, u]$ of G coincides with the set of all singular elements of G. We first prove:

$$\text{For all } s, t \in [0, u],\ s \oplus t = s \vee t. \tag{2}$$

By [8, Corollary 1.2.14], the intersection of all prime ideals of the MV-algebra $A = \Gamma(G, u)$ is the zero ideal $\{0\}$. By Theorem 2.3(ii), the intersection of all prime ℓ-ideals of G is the zero ideal of G. Stated otherwise, if $0 \neq y \in G$ then $0/P \neq y/P \in G/P$ for some prime ideal P of G. Furthermore, by [8, Lemma 1.2.3(v)], for every prime ideal P of A the quotient MV-algebra A/P is totally ordered, and by Theorem 2.3(iii), for every prime ℓ-ideal Q of G the quotient G/Q is totally ordered. It follows that the unit u/Q of G/Q is an *atom* (i.e., a nonzero minimal element) of G^+/Q. Therefore, u/Q is the only nonzero singular element of G/Q, and $[0/Q, u/Q] = \{0/Q, u/Q\}$. Arguing now by cases, it is easy to see that for all $s, t \in [0, u]$, $(s \oplus t)/Q = (s \vee t)/Q$. This settles (2).

By definition, $\Gamma'(G, u) = \Gamma(G, u)$ is the MV-algebra $\{[0, u], 0, \neg, \oplus\}$ equipped with the operations

$$\neg x = u - x \text{ and } x \oplus y = u \wedge (x + y).$$

By (2) and [8, 1.5.3(i)$\leftrightarrow$(iv)], the MV-algebra $\Gamma'(G, u)$ is a boolean algebra, and necessarily the $\vee$ operation coincides with the $\oplus$ operation of $\Gamma(G, u)$. In symbols,

$$\Gamma'(G, u) = \{[0, u], 0, \neg, \vee\}.$$

Since every unital ℓ-homomorphism of (G, u) is restricted by Γ' to the unit interval $[0, u] = \Gamma'(G, u)$, we have proved that Γ' is a *functor* from S into boolean algebras.

To prove that Γ' is an *equivalence* between these two categories, according to [1, Definition 3.33] we have to prove that Γ' is full, faithful, and (isomorphism-)dense. To this purpose, let us first recall that MV-algebraic homomorphisms between boolean algebras *qua* MV-algebras are the same as boolean homomorphisms, [8, 1.5.3]. Then BA is a full subcategory of MV. Since by [19, 3.5-3.4] Γ is full and faithful, then so is Γ'.

There remains to prove that Γ' is dense. Let B be a boolean algebra, with the intent of finding a Specker ℓ-group H with a singular unit v such that $\Gamma'(H, v) \cong B$. To this purpose we argue as follows: Since Γ is dense ([19, Theorem 3.8]), there is a *unital ℓ-group* (H, v) and an *MV-algebra* B' such that

$$\Gamma(H, v) = B' \cong B. \tag{3}$$

As an isomorphic copy of B, B' is a *boolean algebra*.

Claim: H is a Specker ℓ-group with a singular unit v

As a matter of fact, by Theorem 2.1, $\Gamma(H, v)$ is the unit interval $[0, v]$ equipped with the operations of negation $\neg x = v - x$ and truncated addition $x \oplus y = (x + y) \wedge v$. Therefore, B' is a boolean algebra defined over the interval $[0, v]$, and the $\vee$ operation of H restricted to $[0, v]$ coincides with the $\oplus$ operation of B', as well as with the derived $\vee$ operation of

B' as an MV-algebra. By (3), every element of $[0, v] \subseteq H$ is singular. By Lemma 2.7, the unit v is the largest singular element of H. By Lemma 2.5, H is generated, as a group, by its singular elements. Hence,

$$(H, v) \in \mathsf{S}, \tag{4}$$

as required to settle our claim.

From (3)-(4) it follows that $\Gamma(H, v) = \Gamma'(H, v) \cong B$, which shows that Γ' is dense. This completes the proof. ∎

3.3. Remark. An alternative proof of Theorem 3.2 can be obtained by piecing together some existing results in the literature as follows: By [9, §4.7] a Specker ℓ-group S with a strong order-unit has a unique multiplication such that singular elements of S are precisely its idempotents. Therefore, S becomes a Specker $\mathbb{Z}$-algebra, and hence the categories of Specker ℓ-groups with a strong order-unit and Specker $\mathbb{Z}$-algebras are isomorphic. Since Γ applied to a Specker ℓ-group, viewed as a Specker $\mathbb{Z}$-algebra, is precisely the idempotent functor, an application of [2, Theorem 3.8] yields Theorem 3.2.

Representation of Specker ℓ-groups with a singular unit. A *boolean space* is a totally disconnected compact Hausdorff space X. To avoid trivialities, throughout we assume $X \neq \emptyset$. Equivalently, X is the Stone space of a nontrivial boolean algebra.

3.4. Corollary. (For (i)-(ii) see [3, Corollary 2.12]. Also see [4, 13.5.4] for a related result. For (iii) see Remark 3.5.) *For X an arbitrary boolean space, let $(C(X, \mathbb{Z}), 1)$ denote the unital ℓ-group of all integer-valued continuous*[3] *functions on X, with the constant function 1 as the distinguished unit. We then have:*

(i) $(C(X, \mathbb{Z}), 1)$ is a Specker ℓ-group with the singular unit 1.

(ii) Letting X range over all boolean spaces, up to unital ℓ-isomorphism in the category S, $(C(X, \mathbb{Z}), 1)$ ranges over all Specker ℓ-groups with a distinguished singular unit.

(iii) Arbitrarily fix $(G, u) \in \mathsf{S}$. In view of (ii), let X be a boolean space such that $(G, u) \cong (C(X, \mathbb{Z}), 1)$. Then X is homeomorphic to the maximal spectral space $\mu(G)$.

Proof. (i) The MV-algebra $\Gamma(C(X, \mathbb{Z}), 1)$ is the boolean algebra

$$C(X, \{0, 1\})$$

of all continuous $\{0, 1\}$-valued functions on X. Theorem 3.2, yields a Specker ℓ-group G with singular unit w such that

$$\Gamma(G, w) \cong C(X, \{0, 1\}) = \Gamma(C(X, \mathbb{Z}), 1).$$

It follows that $(G, w) \cong (C(X, \mathbb{Z}), 1)$, whence $(C(X, \mathbb{Z}), 1)$ is a Specker ℓ-group with unit 1.

[3] Both the ℓ-group of integers $\mathbb{Z}$, and the set $\{0, 1\}$ are equipped with the discrete topology.

(ii) By Stone duality, up to isomorphism,

$$\Gamma(C(X,\mathbb{Z}),1) = C(X,\{0,1\})$$

ranges over all boolean algebras. By (i) and Theorem 3.2, up to unital ℓ-isomorphism, $(C(X,\mathbb{Z}),1)$ ranges over all Specker ℓ-groups with a singular unit.

(iii) The existence of a boolean space X such that

$$(G,u) \cong (C(X,\mathbb{Z}),1)$$

is guaranteed by (ii). For all $x,y \in X$ with $x \neq y$ there is a clopen W of X such that $x \in W$ and $y \notin W$. Correspondingly, there is a function in the unit interval of $(C(X,\mathbb{Z}),1)$ taking value 1 at x and value 0 at y. In other words, the functions in the MV-algebra $\Gamma(C(X,\mathbb{Z}),1)$ *separate points*. By [8, §3.6], $\Gamma(C(X,\mathbb{Z}),1)$ is *semisimple*, in the sense that the intersection of its maximal ideals is the zero ideal. Equivalently, by [8, 3.6.4], $\Gamma(C(X,\mathbb{Z}),1)$ has no infinitesimals. By Theorem 2.3(ii), the intersection of the (automatically nonempty) set of maximal ℓ-ideals of the ℓ-group $C(X,\mathbb{Z})$ is the zero ℓ-ideal $\{0\}$. Letting $\mu(C(X,\mathbb{Z}))$ denote the maximal ℓ-ideal space of $C(X,\mathbb{Z})$, by Theorem 2.4 and [20, Theorem 4.16(iv)] we have (canonical) homeomorphisms

$$\mu(G) \cong \mu(C(X,\mathbb{Z})) \cong \mu(\Gamma(C(X,\mathbb{Z}),1)) \cong X.$$

The proof is complete. ∎

3.5. Remark. An alternative (shorter and more ring-theoretic) proof of Corollary 3.4(iii) may be obtained as follows: If $S = C(X,\mathbb{Z})$ with X a Stone space, then $\mu(S)$ is the Yosida space of S, which is homeomorphic to the Stone space of the boolean algebra I of idempotents of S. For details see [2]. Now I is isomorphic to the boolean algebra of clopen subsets of X, and hence its Stone space is homeomorphic to X.

In general, Specker ℓ-groups need not have a strong (or weak) order-unit, in which case they correspond to generalized Boolean algebras. Thus, their maximal spectra spaces are only locally compact. See [3].

4. Abelian AF-algebras, Stone/Gelfand dualities, Grothendieck group

The reader of this final section is assumed to be acquainted with the Gelfand representation of abelian unital C*-algebras, [22, Theorem 1.2.3], [6, Proposition 3.1], and with Elliott classification, [13, 14], [11], [22]

Following Bratteli's original definition [5], an *AF-algebra* is the norm closure $\overline{\bigcup_i \mathfrak{A}_i}$ of the union of an ascending sequence of finite-dimensional C^*-algebras $\mathfrak{A}_i$, all with the same unit. AF denotes the category of AF-algebras with *-algebra homomorphisms. We also let

$$\mathsf{AF}_{ab}$$

denote the category of abelian AF-algebras and their *-algebra homomorphisms.

As the reader will recall, the functor $\mathcal{G}$ assigns to any $\mathfrak{A} \in \mathsf{AF}_{ab}$ its maximal spectral space $\mu(\mathfrak{A})$. Further, $\mathcal{S}$ assigns to every boolean space the boolean algebra of its clopen sets. Finally, Ξ assigns to every boolean algebra B a unital Specker ℓ-group (G, u) such that $\Gamma(G, u) = B$. See Theorem 2.2 for details. Since $\mathcal{S} \circ \mathcal{G}$ is a categorical equivalence, then so is $\Xi \circ \mathcal{S} \circ \mathcal{G}$.

Let p be a *projection* in an AF-algebra $\mathfrak{A} \in \mathsf{AF}$, i.e., $p = p^2 = p^*$. For all projections $p, q \in \mathfrak{A}$ we write $p \sim q$ if there is an element $x \in \mathfrak{A}$ such that $p = x^*x$ and $q = xx^*$. It turns out that $\sim$ is an equivalence relation on the projections of $\mathfrak{A}$. A projection s is a *subprojection* of p (in symbols, $s \leq p$) if $ps = sp = s$. If p is equivalent to a subprojection of q we write $p \preceq q$. The partial order $\preceq$ on the set of equivalence classes of projections in any AF-algebra $\mathfrak{A} \in \mathsf{AF}$ is called the *Murray-von Neumann order.* Letting $[p]$ denote the equivalence class of a projection p we (trivially) have:

4.1. Proposition. *For any abelian AF-algebra $\mathfrak{A}$ and projections $p, q \in \mathfrak{A}$, $[p] = \{p\}$. Also, $q \preceq p$ if and only if p is a subprojection of q.*

When $\mathfrak{A} \in \mathsf{AF}_{ab}$, combining Gelfand's representation [22, Theorem 1.2.3] with Bratteli's [6, Proposition 3.1], we have

4.2. Proposition. *Any abelian AF-algebra $\mathfrak{A}$ is *-isomorphic to the C*-algebra $C(X, \mathbb{C})$ of all complex-valued continuous functions on a homeomorphic copy X of the boolean space $\mu(\mathfrak{A})$. The projections of $C(X, \mathbb{C})$ are precisely the $\{0, 1\}$-valued functions of $C(X, \mathbb{C})$. Moreover, p is a subprojection of q if and only if $p \leq q$ in the pointwise order of real-valued functions over X.*

Elliott classification and K_0 in AF_{ab}. For any $\mathfrak{A} \in \mathsf{AF}$, *Elliott's partial addition* is defined by setting $[p]+[q] = [p+q]$ whenever projections p and q are orthogonal. The set of equivalence classes of projections of $\mathfrak{A}$ then has the structure of a countable partially ordered "local" semigroup, denoted $D(\mathfrak{A})$. Elliott's partially defined addition $+$ is monotone with respect to the $\preceq$-order.

In particular, when $\mathfrak{A}$ is abelian, from $p \preceq q \Leftrightarrow p \leq q$ and Proposition 4.2 we immediately have:

4.3. Proposition. *For any $\mathfrak{A} \in \mathsf{AF}_{ab}$ the Murray-von Neumann order of $D(\mathfrak{A})$ is a lattice.*

As noted by Elliott in [14, p.33], D is a functor that preserves finite direct sums and inductive limits of sequences. D is known as *Elliott's classifier* because of the following fundamental result, proved in [13]:

$$D(\mathfrak{A}_1) \cong D(\mathfrak{A}_2) \quad \text{iff} \quad \mathfrak{A}_1 \cong \mathfrak{A}_2.$$

For any AF algebra $\mathfrak{A}$, $D(\mathfrak{A})$ may be embedded in a unique way as a generating subset of a group which turns out to coincide with the group $K_0(\mathfrak{A})$ of algebraic K-theory. For details see [22, §6.3, Theorem 7.3.4] or [11, IV.1.5-1.6]. Then the group $K_0(\mathfrak{A})$ is torsion-free, and the semigroup $K_0(\mathfrak{A})^+$ generated by $D(\mathfrak{A})$ has zero intersection with its negative; so with this semigroup as positive cone, the group becomes a partially ordered abelian group, also denoted $K_0(\mathfrak{A})$. From [22, Proposition 5.1.7] or [11, IV.2.2(iii)] it follows that the K_0-image $\{1_{\mathfrak{A}}\}$ of the unit element $1_{\mathfrak{A}}$ is a strong unit of $K_0(\mathfrak{A})$. Whenever there is no danger of confusion we will write $K_0(\mathfrak{A})$ instead of $(K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\})$ or $(K_0(\mathfrak{A}), K_0(\mathfrak{A})^+, \{1_{\mathfrak{A}}\})$.

Assume $\mathfrak{A}$ is an abelian AF-algebra. Since by Proposition 4.3 the underlying order of the Elliott classifier $D(\mathfrak{A})$ is a lattice, we may apply [21, Theorem 2.1](i). This provides a unique associative commutative monotone extension $\oplus\colon D(\mathfrak{A})^2 \to D(\mathfrak{A})$ of the partial addition $+$ in $D(\mathfrak{A})$, such that for each projection $p \in \mathfrak{A}$, $[1_{\mathfrak{A}} - p]$ is the $\preceq$-smallest equivalence class $[q] = \{q\} \in E(\mathfrak{A})$ satisfying $[p] \oplus [q] = [1_{\mathfrak{A}}]$. Equivalently, $1_{\mathfrak{A}} - p$ is the $\leq$-smallest projection q of $\mathfrak{A}$ such that $p \vee q = 1_{\mathfrak{A}}$.

For our purposes in this paper, using Proposition 4.2 and Theorem 3.2, we may conveniently identify (via Gelfand representation) any $\mathfrak{A} \in \mathsf{AF}_{ab}$ with $C(X, \mathbb{C})$, where X is a homeomorphic copy of its maximal spectral space $\mu(\mathfrak{A})$. The above general construction of $K_0(\mathfrak{A})$ then takes the following simpler form:

4.4. PROPOSITION. *Let* $\mathfrak{A} \in \mathsf{AF}_{ab}$. *For* $X \cong \mu(\mathfrak{A})$ *let us identify* $\mathfrak{A}$ *with* $C(X, \mathbb{C})$.

(i) The uniquely determined operation $\oplus\colon D(\mathfrak{A})^2 \to D(\mathfrak{A})$ *provided by [21, Theorem 2.1](i) coincides with the* $\vee$ *operation of the idempotent MV-algebra (i.e., the boolean algebra)* $C(X, \{0,1\})$. X *is a separable boolean space. The countable unital* ℓ*-group* $K_0(\mathfrak{A})$ *is generated, as a group, by the* $\{0,1\}$*-valued functions of* $C(X, \mathbb{C})$. *Thus* $K_0(\mathfrak{A})$ *is the countable unital Specker* ℓ*-group* $C(X, \mathbb{Z})$ *equipped with the distinguished unit* $\{1_{\mathfrak{A}}\}$ *given by its largest singular element. In symbols,* $(K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\}) = (C(X, \mathbb{Z}), 1)$.

(ii) As $\mathfrak{A}$ *ranges over all abelian AF-algebras,* $(K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\})$ *yields unitally* ℓ*-isomorphic copies of all countable unital Specker* ℓ*-groups* (G, u) *whose distinguished unit* u *is singular. Furthermore, up to homeomorphism,* $\mu(\mathfrak{A})$ *ranges over all separable boolean spaces.*

4.5. THEOREM. *Let us restrict Grothendieck's functor* K_0 *to the category* AF_{ab} *of abelian AF-algebras and their *-algebra homomorphisms.*

(i) K_0 *is a categorical equivalence between* AF_{ab} *and the fragment* S_ω *of* S *given by countable unital Specker* ℓ*-groups* (G, u) *whose distinguished unit* u *is singular.*

(ii) For any abelian AF algebra $\mathfrak{A}$, $K_0(\mathfrak{A}) \;\cong\; \Xi \circ \mathcal{S} \circ \mathcal{G}(\mathfrak{A})$.

PROOF. By [6, Proposition 3.1],

$$\text{the linear span of projections of } \mathfrak{A} \text{ is norm-dense in } \mathfrak{A}. \tag{5}$$

(i) K_0 *is a functor.* Let $\mathfrak{A}, \mathfrak{A}' \in \mathsf{AF}_{ab}$. By Proposition 4.4, $K_0(\mathfrak{A})$ and $K_0(\mathfrak{A}')$ are countable Specker ℓ-groups whose distinguished units $\{1_{\mathfrak{A}}\}$ and $\{1_{\mathfrak{A}'}\}$ are singular. By (5), every *-algebra homomorphism $\phi\colon \mathfrak{A} \to \mathfrak{A}'$ is uniquely determined by its restriction $\phi^\natural$ to the set of projections of $\mathfrak{A}$. Likewise, by Propositions 4.1 and 4.4, $\phi^\natural$ uniquely extends to a unital

ℓ-homomorphism $K_0(\phi)\colon (K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\}) \to (K_0(\mathfrak{A}'), \{1_{\mathfrak{A}'}\})$. We have thus shown that K_0 is a functor from AF_{ab} to S_ω.

K_0 *is dense.* By Proposition 4.4(ii).

K_0 *is full.* Any unital ℓ-homomorphism

$$\rho\colon (K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\}) \to (K_0(\mathfrak{A}'), \{1_{\mathfrak{A}'}\})$$

is uniquely determined by its restriction ς to the set of singular elements of $K_0(\mathfrak{A})$, i.e., the projections of $\mathfrak{A}$. In turn, ς uniquely determines a *-morphism $\sigma\colon \mathfrak{A} \to \mathfrak{A}'$. This again follows by combining Proposition 4.1 with (5). Evidently, $\rho = K_0(\sigma)$.

K_0 *is faithful.* Let τ and χ be different *-algebra homomorphisms of $\mathfrak{A}$ into $\mathfrak{A}'$. Again by Proposition 4.1 and (5), τ differs from χ at some projection of $\mathfrak{A}$. Correspondingly, the unital ℓ-homomorphisms $K_0(\tau)$ and $K_0(\chi)$ will assign different values to some singular element of $K_0(\mathfrak{A})$.

We have just proved that K_0 is a categorical equivalence between AF_{ab} and S_ω.

(ii) Let X be the maximal spectral space of $\mathfrak{A}$. By [21, Theorem 2.1](v), the idempotent MV-algebra (=boolean algebra) $C(X, \{0,1\})$ of Proposition 4.4 is isomorphic to $\Gamma(K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\})$. In symbols,

$$C(X, \{0,1\}) \cong \Gamma(K_0(\mathfrak{A}), \{1_{\mathfrak{A}}\}).$$

On the other hand, by definition of the functors $\mathcal{G}$ and $\mathcal{S}$, from the homeomorphism $X \cong \mu(C(X, \{0,1\}))$ we get

$$C(X, \{0,1\}) \cong \mathcal{S}(\mathcal{G}(\mathfrak{A})).$$

As a consequence, $\Gamma \circ K_0(\mathfrak{A}) \cong \mathcal{S} \circ \mathcal{G}(\mathfrak{A})$, whence

$$\Xi \circ \Gamma \circ K_0(\mathfrak{A}) \cong \Xi \circ \mathcal{S} \circ \mathcal{G}(\mathfrak{A}).$$

By Theorem 2.2 we can now write

$$K_0(\mathfrak{A}) \cong \Xi \circ \mathcal{S} \circ \mathcal{G}(\mathfrak{A}),$$

which completes the proof. ∎

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