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TOWARDS A NEW COHOMOLOGY THEORY FOR STRICT LIE 2-GROUPS

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Abstract. In this article, we introduce the first degrees of a cochain complex associated to a strict Lie 2-group whose cohomology is shown to extend the classical cohomology theory of Lie groups. In particular, we show that the second cohomology group classifies an appropriate type of extensions. We conclude putting forward evidence that this complex can be extended to arbitrary degrees.

1. Introduction

In [10] it is explained that the nerve of a strict Lie 2-group — regarded as a groupoid —, is a simplicial group and thus carries a bisimplicial structure. One can prove that the second cohomology group of the naturally associated double complex thereof classifies extensions by the unit Lie groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$ (see Subsection 2.7 below). The novel complex we present is an extension of this double complex though taking values on a 2-vector space.

This paper sprung from attempting to find a proof of the integrability of strict Lie 2-algebras to strict Lie 2-groups that still works in infinite dimensions (the finite-dimensional case was established in [18] using Lie algebra paths). Historically, such a proof for the integrability of Lie algebras was devised by van Est [20, 8] and was itself used to construct the first example of a non-integrable Lie algebra [21].

Van Est's proof is cohomological and can be roughly summarized as follows: Given a Lie algebra $\mathfrak{g}$, one uses its adjoint representation to recast it as an extension of the Lie subalgebra $\mathrm{ad}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ by the center $\mathfrak{z}(\mathfrak{g})$. There exists a Lie algebra cohomology class $[\omega] \in H^2(\mathrm{ad}(\mathfrak{g}), \mathfrak{z}(\mathfrak{g}))$ that classifies this extension. Since linear Lie algebras are integrable and one can always pick a 2-connected integration, the van Est Theorem says there is a unique group cohomology class whose associated extension is a Lie group integrating $\mathfrak{g}$. Implicitly then, the preliminary step to carry out van Est's strategy is to have cohomology theories that respectively classify abelian extensions of Lie groups and Lie algebras — or rather of the global and infinitesimal counterparts —, and more importantly, a van Est map relating them.

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As opposed to other complexes whose second cohomology classifies extensions (see, *e.g.*, [5, 16] for the Lie 2-group case – regarding them respectively as crossed modules and Lie 2-groupoids –, and [17] for the Lie 2-algebra case – regarding them as 2-term L_∞-algebras –), the complex in [10] allows for a simple Ansatz for a van Est map candidate. Indeed, the nerve of the underlying groupoid of the strict Lie 2-algebra of a Lie 2-groupoid is a simplicial Lie algebra whose dual co-simplicial dg-algebra can be turned into a double complex. Consequently, there is a natural map from the double complex of the strict Lie 2-group, whose columns are Lie group cochain complexes, to the double complex of its strict Lie 2-algebra, whose columns are Chevalley-Eilenberg complexes; namely, the one assembled columnwise by usual van Est maps.

This idea is successfully carried out in [2] to get the sought-after cohomological proof for the integrability of strict Lie 2-algebras. In the course of doing so, the above sketched double complex associated to a strict Lie 2-algebra needed to get enlarged to force its second cohomology to classify extensions by more general 2-vector spaces [1]. The result is a triple complex of sorts in which two of the differentials commute only up to homotopy. In the present article, we use this idea as a template to give part of a complex that serves as the global counterpart to the one in [1].

To explain in more detail, let $\mathcal{G}$ be a strict Lie 2-group together with a 2-representation on a 2-vector space $W \xrightarrow{\phi} V$. Assume $\mathcal{G}$ is presented as a groupoid and let $\mathcal{G}_\bullet$ be its nerve. Additionally, let $G \leq \mathcal{G}$ be the kernel of the source map. Set

$$C_r^{p,q}(\mathcal{G}, \phi) := C(\mathcal{G}_p^q \times G^r, W) \tag{1}$$

for $r \neq 0$, and

$$C_0^{p,q}(\mathcal{G}, \phi) := C(\mathcal{G}_p^q, V), \tag{2}$$

where $C(\cdot, A)$ is the vector space of A-valued smooth functions that vanish when evaluated at a tuple which has the identity element in one of its coordinates. One can enhance this three dimensional lattice of vector spaces to a *grid of complexes* by recognizing certain spaces of valued Lie groupoid cochains: those of a Lie group bundle along the r-direction, those of an action groupoid along the q direction, and those of the product of powers of $\mathcal{G}$ and powers of the unit groupoid $G \rightrightarrows G$ along the p-direction (see Section 3 for details on the groupoids and the representations).

Thus knit, this grid is not a triple complex because, for constant r, the differentials in the *r-page* do not commute. However, the differentials in the page $r = 0$ do commute up to isomorphism in V, and those in the successive r-pages do so up to homotopy (see Proposition 4.11). In so, the non-vanishing terms coming from the r-pages in the square of the the sum of the differentials can be canceled off by adding the homotopies Δ to the total differential. The homotopies, in order, do not commute with the differentials of the r-pages, but do so themselves respectively (at least in the lowest degrees), up to *higher homotopies* $\Delta_{2,1}$ and$\Delta_{1,2}$. We ultimately prove:

1.1. Theorem. *Let $\mathcal{G}$ be a Lie 2-group together with a representation on the 2-vector space $W \xrightarrow{\phi} V$. If*

$$C^n_{tot}(\mathcal{G},\phi) := \bigoplus_{p+q+r=n} C^{p,q}_r(\mathcal{G},\phi) \tag{3}$$

and

$$\nabla := (-1)^p\Big(\delta_{(1)} + \partial + \Delta + \Delta_{1,2} + (-1)^r(\delta + \Delta_{2,1})\Big), \tag{4}$$

where $\delta_{(1)}$, δ and ∂ are, respectively, the differentials of the complexes in the r, q and p directions, then ∇ squares to zero and there is a complex $(C^{\leq 3}_{tot}(\mathcal{G},\phi),\nabla)$.

The result in Theorem 1.1 can be improved to get a 6-term complex. We stop at degree 3 because it is enough to prove the main property we are after: that the second cohomology group classifies abelian extensions (see Theorem 3.21). We insist that the main advantage of the complex of Theorem 1.1 is that it is built out of complexes of Lie groupoids, thereby allowing one to directly get a van Est map candidate by assembling groupoid van Est maps [8] so as to land in the complex of [1] (see [2]). The drawback, however, is that it is not clear why the so-called higher homotopies exist in general, or from where their formula is derived (though we try and relate it back to some underlying structure, see Remarks 3.3 and 3.12). In each degree, we provide formulas and prove them to work, but since there is little indication as to why they work, our proofs end up being long and unenlightening computations.

This paper is organized as follows. In Section 2, we recall some basic facts and convene notation. We motivate the emergence of the complex and its differential by recalling the canonical double complex associated to a Lie 2-group whose second cohomology classifies extensions of a particular type. Then, we recall the definitions of the general linear Lie 2-group and of a representations, we provide examples and recall some associated constructions. We conclude the section by proving that the kind of actions induced by extensions are indeed among this type of representations (see Proposition 2.23). In Section 3, we carefully define the three dimensional grid and find out the homotopies together with which we get the $(4\times4\times4)$-tetrahedral complex of Lie 2-group cochains with values in a representation, and move on to study its cohomology. We show, in particular, that the equations that define a 2-cocycle are equivalent to the equations defining an abstract extension. In Section 4, we prove the general relations that the differentials in the background grid verify and heal the non-commuting part by introducing the general formula for the *difference maps*. We conclude by discussing what is needed to fully extend the grid to a complex. We include an appendix with the general formula for the *higher difference maps* needed to extend the complex of Theorem 1.1 to degree 5 by taking a $(6\times6\times6)$-tetrahedral slice of the grid of Section 4.

2. Preliminaries

In this section, we establish the notation conventions used throughout. As a motivation, we recall the complex of [10] and study its cohomology. We also recall the notions of

general linear Lie 2-group and strict representation.

2.1. Remark. From here on out, we make no distinction between a strict Lie 2-group and its associated crossed module. For future reference, we outline the equivalence between the category of Lie 2-groups and crossed modules of Lie groups at the level of objects (see [6] for details).

2.2. Definition. *A* strict Lie 2-group *is a groupoid object internal to the category of Lie groups.*

We write a generic strict Lie 2-group as

$$\begin{tikzcd} \mathcal{G} \times_H \mathcal{G} \arrow[r, "m"] & \mathcal{G} \arrow[loop below, "\iota"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H \arrow[r, "u"] & \mathcal{G}, \end{tikzcd}$$

and henceforth refer to it simply as a *Lie 2-groups*. In order to make clear the difference between the group operation and the groupoid operation in $\mathcal{G}$, we assume the following convention:

$$g_1 \boxtimes g_2 \qquad\qquad g_3 \bowtie g_4,$$

stand respectively for the group multiplication and the groupoid multiplication whenever $(g_1, g_2) \in \mathcal{G}^2$ and $(g_3, g_4) \in \mathcal{G} \times_H \mathcal{G}$. This notation intends to reflect that we think of the group multiplication as being "vertical", whereas the groupoid multiplication as being "horizontal".

2.3. Definition. *[22] A* crossed module of Lie groups *is a Lie group homomorphism* $G \xrightarrow{i} H$ *together with a right action of* H *on* G *by Lie group automorphisms satisfying*

$$i(g^h) = h^{-1} i(g) h,$$

$$g_1^{i(g_2)} = g_2^{-1} g_1 g_2,$$

for all $g, g_1, g_2 \in G$ *and* $h \in H$*, where we write* g^h *for* h *acting on* g*. Following the convention in the literature, we refer to these equations respectively as* **equivariance** *and* **Peiffer**.

Given a crossed module as in Definition 2.3, the space of arrows of its associated Lie 2-group $\mathcal{G}$ is defined to be the semi-direct product $G \rtimes H$ with respect to the H-action, whose product is explicitly given by

$$\begin{pmatrix} g_1 \\ h_1 \end{pmatrix} \boxtimes \begin{pmatrix} g_2 \\ h_2 \end{pmatrix} = \begin{pmatrix} g_1^{h_2} g_2 \\ h_1 h_2 \end{pmatrix}, \tag{5}$$

for $(g_1, h_1), (g_2, h_2) \in G \rtimes H$. Its structural maps are given by

$$s\begin{pmatrix} g \\ h \end{pmatrix} = h \qquad t\begin{pmatrix} g \\ h \end{pmatrix} = hi(g) \qquad \iota\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} g^{-1} \\ hi(g) \end{pmatrix} \qquad u(h) = \begin{pmatrix} 1 \\ h \end{pmatrix}$$

$$\begin{pmatrix} g' \\ hi(g) \end{pmatrix} \bowtie \begin{pmatrix} g \\ h \end{pmatrix} := \begin{pmatrix} gg' \\ h \end{pmatrix} \tag{6}$$

for $h \in H$ and $g, g' \in G$.

Conversely, given a Lie 2-group $\mathcal{G} \rightrightarrows H$, let G to be the Lie subgroup $\ker s \leq \mathcal{G}$. The associated crossed module is given by $t|_{\ker s} : G \longrightarrow H$ together with the right action given by conjugation by units in the group $\mathcal{G}$

$$g^h := u(h)^{-1} \between g \between u(h),$$

for $h \in H$ and $g \in G$. We stress that the -1 power stands for the inverse of the group multiplication $\between$.

Notice that the isomorphism of vector spaces $\mathcal{G} \cong G \times H$ is canonical because the unit map provides a natural splitting.

2.4. Lie groupoid cohomology. Let $G \rightrightarrows M$ be a Lie groupoid. There is a simplicial structure on the nerve of G whose maps are given by

$$\partial_k(g_0, ..., g_p) = \begin{cases} (g_1, ..., g_p) & \text{if } k = 0 \\ (g_0, ..., g_{k-1}g_k, ..., g_p) & \text{if } 0 < k \leq p \\ (g_0, ..., g_{p-1}) & \text{if } k = p+1, \end{cases} \tag{7}$$

for a given element $(g_0, ..., g_p) \in G^{(p+1)}$. With these, one builds the complex of Lie groupoid cochains $C^p(G) := C^\infty(G^{(p)})$ whose differential $\partial : C^\bullet(G) \longrightarrow C^{\bullet+1}(G)$ is defined by the formula

$$\partial\varphi = \sum_{k=0}^{p+1} (-1)^k \partial_k^* \varphi, \tag{8}$$

for $\varphi \in C^p(G)$. Note that in the case when M is a point, Eq. (8) is the usual differential of group cohomology.

Thus defined, $(C^\bullet(G), \partial)$ is referred to as the *groupoid complex* of G, and its cohomology is called *differentiable cohomology* of G [8].

2.5. Remark. Under the isomorphism of Remark 2.1, the space of p-composable arrows

$$\mathcal{G}_p = \mathcal{G} \times_H ... \times_H \mathcal{G} = \{(\gamma_1, ..., \gamma_p) \in \mathcal{G}^p : s(\gamma_j) = t(\gamma_{j+1}), \quad 1 \leq j < p\}$$

corresponds to $G^p \times H$; again, hereafter, we consider this isomorphism to be fixed and treat it as an equality, when necessary. For each coordinate γ_j of $\gamma \in \mathcal{G}_p$, there is a corresponding $(g_j, h_j) \in G \times H$. The defining relation for $\mathcal{G}_p$ then reads $h_j = h_{j+1} i(g_{j+1})$ (see Eq. (6)), thus making the map $\gamma \longmapsto (g_1, ..., g_p; h_p)$ an isomorphism with inverse $(g_1, ..., g_p; h) \longmapsto (g_1, hi(g_p...g_2); ...; g_{p-1}, hi(g_p); g_p, h)$. Under this isomorphism, we rewrite the face maps to be

$$\partial_k(g_0, ..., g_p; h) = \begin{cases} (g_1, ..., g_p; h) & \text{if } k = 0 \\ (g_0, ..., g_{k-2}, g_k g_{k-1}, g_{k+1}, ..., g_p; h) & \text{if } 0 < k \leq p \\ (g_0, ..., g_{p-1}; hi(g_p)) & \text{if } k = p+1. \end{cases} \tag{9}$$

2.5.1. Representations and cohomology with values. A *left representation* of $G \rightrightarrows M$ is a vector bundle E over M, together with a left action

$$\Delta : G_s \times_M E \longrightarrow E : (g, e) \longmapsto \Delta_g e$$

along the projection of the vector bundle. Let $t_p : G^{(p)} \longrightarrow M : (g_1, ...g_p) \longmapsto t(g_1)$ be the map that returns the final target of a p-tuple of composable arrows.

2.6. Remark. Under the isomorphism of Remark 2.5, in the case where the groupoid is a strict Lie 2-group, the final target map gets rewritten as

$$t_p : G^p \times H \longrightarrow H : (g_1, ..., g_p; h) \longmapsto hi\Big(\prod_{j=0}^{p-1} g_{p-j}\Big) = hi(g_p ... g_1).$$

Observe that the final target map is a composition of face maps and hence a group homomorphism.

The complex of Lie groupoid cochains with values in the left representation on E is defined by sections of the pull-back of E along the final target map

$$C^p(G; E) := \Gamma(t_p^* E)$$

together with the differential $\partial : C^\bullet(G; E) \longrightarrow C^{\bullet+1}(G; E)$ whose formula is essentially (8), though modifying the first term so that all terms lie on the same fibre and the sum can be performed. More specifically, for $\varphi \in C^p(G; E)$ and $(g_0, ..., g_p) \in G^{(p+1)}$,

$$(\partial\varphi)(g_0, ..., g_p) := \Delta_{g_0} \partial_0^* \varphi(g_0, ..., g_p) + \sum_{k=1}^{p+1} (-1)^k \partial_k^* \varphi(g_0, ..., g_p). \tag{10}$$

Similarly, a *right representation* of $G \rightrightarrows M$ is a vector bundle E over M, together with a right action

$$\Delta : E_M \times_t G \longrightarrow E : (e, g) \longmapsto \Delta_g e$$

along the projection of the vector bundle for which we use the same notation. Replacing each instance of the target map by the source map in the preceding discussion, one defines the complex of Lie groupoid cochains with values in the right representation on E, whose differential is

$$(\partial\varphi)(g_0, ..., g_p) := \sum_{k=0}^{p} (-1)^k \partial_k^* \varphi(g_0, ..., g_p) + (-1)^{p+1} \Delta_{g_p} \partial_{p+1}^* \varphi(g_0, ..., g_p). \tag{11}$$

Both left and right representations can be pulled-back along homomorphisms. If E is a left (resp. right) representation of $G \rightrightarrows M$ and

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[d, shift right] \arrow[d, shift left] & G \arrow[d, shift right] \arrow[d, shift left] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

is a Lie groupoid homomorphism, then there is a left (resp. right) action of $H \rightrightarrows N$ on the pull-back bundle $f^*E = N \times_M E$, where $h \in H$ acts on $e \in E_{f(s(h))}$ (resp. $E_{f(t(h))}$) by $\Delta_{\varphi(h)}e$.

2.7. The canonical simplicial object. - In this subsection, we recall that, as it is explained in [10], given a Lie 2-group $\mathcal{G} \rightrightarrows H$, its nerve $\mathcal{G}_\bullet$ is a simplicial group. For each p, $\mathcal{G}_p$ is a Lie subgroup of $\mathcal{G}^p$ and one can thus consider its nerve. Considering simultaneously the nerve of all $\mathcal{G}_p$'s yields a bisimplicial set $\mathcal{G}_\bullet^\bullet$. In particular, the face maps of the two simplicial structures always commute with one another; hence, *dualizing*, one gets the double complex:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C(H^3) \arrow[u] \arrow[r, "\partial"] & C(\mathcal{G}^3) \arrow[u] \arrow[r, "\partial"] & C(\mathcal{G}_2^3) \arrow[u] \arrow[r] & \cdots \\
C(H^2) \arrow[u, "\delta"] \arrow[r, "\partial"] & C(\mathcal{G}^2) \arrow[u, "\delta"] \arrow[r, "\partial"] & C(\mathcal{G}_2^2) \arrow[u, "\delta"] \arrow[r] & \cdots \\
C(H) \arrow[u, "\delta"] \arrow[r, "\partial"] & C(\mathcal{G}) \arrow[u, "\delta"] \arrow[r, "\partial"] & C(\mathcal{G}_2) \arrow[u, "\delta"] \arrow[r] & \cdots
\end{tikzcd}$$

whose columns are complexes of Lie group cochains, and whose qth row is the groupoid complex of $\mathcal{G}^q \rightrightarrows H^q$. The total complex of this double complex is

$$\Omega^k_{tot}(\mathcal{G}) = \bigoplus_{p+q=k} C(\mathcal{G}^q_p),$$

along with $d = (-1)^p(\partial + \delta)$.

In Section 4.2 we use a generalization of this construction to double Lie groupoids [9]. Recall that a double Lie groupoid is a groupoid object internal to the category of Lie groupoids. More explicitly, a *double Lie groupoid* consists of a square

$$\begin{tikzcd}
D \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & V \arrow[d, shift right] \arrow[d, shift left] \\
H \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

where each side is a Lie groupoid and the structural maps are smooth functors. The elements in D can be thought of as being squares whose vertical edges are arrows in V, whose horizontal edges are arrows in H and all of whose vertices are points in M. In

order to recognize the structural maps then, we adopt the following mnemonic device. We write $\underline{\overline{s}}$, $\underline{\overline{t}}$, etc. for the structural maps of the top groupoid; $|s|$, $|t|$, etc. for the left vertical groupoid; s_V, t_V, etc. for the right vertical groupoid; and concludingly, the usual s, t, etc. for the bottom groupoid. Thus, a given element $d \in D$, thought of as a square, has the following edges

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "|t|(d)"'] \\
\bullet \arrow[u, "\underline{\overline{t}}(d)"] & \bullet \arrow[l, "|s|(d)"] \arrow[u, "\underline{\overline{s}}(d)"']
\end{tikzcd}$$

Additionally, as we did for Lie 2-groups, we use the shorthand

$$d_1 \underline{\times} d_2 := |m|(d_1, d_2) \qquad\qquad d_3 \bowtie d_4 := \underline{\overline{m}}(d_3, d_4),$$

whenever $(d_1, d_2) \in D \times_H D$ and $(d_3, d_4) \in D \times_V D$ to reflect the fact that $d_1 \underline{\times} d_2$ and $d_3 \bowtie d_4$ represent respectively

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "|t|(d_1)"'] \\
\bullet \arrow[u, "\underline{\overline{t}}(d_1)"] & \bullet \arrow[l, "\frac{|s|(d_1)}{|t|(d_2)}"'] \arrow[u, "\underline{\overline{s}}(d_1)"'] \\
\bullet \arrow[u, "\underline{\overline{t}}(d_2)"] & \bullet \arrow[l, "|s|(d_2)"] \arrow[u, "\underline{\overline{s}}(d_2)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "|t|(d_3)"'] & \bullet \arrow[l, "|t|(d_4)"'] \\
\bullet \arrow[u, "\underline{\overline{t}}(d_3)"] & \bullet \arrow[l, "|s|(d_3)"] \arrow[u, "\underline{\overline{s}}(d_3)=\underline{\overline{t}}(d_4)"'] & \bullet \arrow[l, "|s|(d_4)"] \arrow[u, "\underline{\overline{s}}(d_4)"']
\end{tikzcd}$$

With this notation, the fact that the multiplication in either groupoid is a groupoid homomorphism yields the formula

$$(d_1 \bowtie d_2) \underline{\times} (d_3 \bowtie d_4) = (d_1 \underline{\times} d_3) \bowtie (d_2 \underline{\times} d_4),$$

whenever it makes sense. We call this formula the interchange law.

It is also customary to add the axiom that the double source map

$$\mathbb{S} := (|s|, \underline{\overline{s}}) : D \longrightarrow H_s \times_{s_V} V$$

is a submersion, though it is immaterial for our purposes.

If D is a double Lie groupoid, there are two simplicial structures for D given by its vertical and its horizontal groupoid structures. The commutativity of the structural maps of the square diagram representing a double Lie groupoid is but a shadow of the general interaction between these two simplicial structures. In what follows, let $\vec{d} = (d_1, ..., d_q) \in$

$M_{q\times p}(D)$ be represented by the matrix $\begin{pmatrix} d_{11} & d_{12} & \dots & d_{1p} \\ d_{21} & d_{22} & \dots & d_{2p} \\ \vdots & \vdots & \dots & \vdots \\ d_{q1} & d_{q2} & \dots & d_{qp} \end{pmatrix}$, where each $d_{mn} \in D$. In order to distinguish the vertical and the horizontal simplicial structures, we do as before and write ∂_k for the face maps associated to the horizontal groupoid and δ_j for the face maps associated to the vertical groupoid.

The complexes of groupoid complex of the horizontal and vertical groupoids fit into a double complex. Although this is an expected relation, we could not find a reference in the literature.

2.8. Proposition. *Given a double Lie groupoid D with our conventions,*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C(V^{(2)}) \arrow[r, "\partial"] \arrow[u] & C(D \times_H D) \arrow[r, "\partial"] \arrow[u] & C(D_2^2) \arrow[r] \arrow[u] & \cdots \\
C(V) \arrow[r, "\partial"] \arrow[u, "\delta"] & C(D) \arrow[r, "\partial"] \arrow[u, "\delta"] & C(D \times_V D) \arrow[r] \arrow[u, "\delta"] & \cdots \\
C(M) \arrow[r, "\partial"] \arrow[u, "\delta"] & C(H) \arrow[r, "\partial"] \arrow[u, "\delta"] & C(H^{(2)}) \arrow[r] \arrow[u, "\delta"] & \cdots
\end{tikzcd}$$

is a double complex, where

$$\begin{aligned} D_p^q := \{\vec{d} \in M_{q\times p}(D) : \quad & \underline{s}(d_{mn}) = \underline{t}(d_{mn+1}), \quad |s|(d_{mn}) = |t|(d_{m+1n}), \\ & \underline{t}(d_{mn}) = \underline{s}(d_{mn-1}), \quad |t|(d_{mn}) = |s|(d_{m-1n})\}. \end{aligned}$$

In the sequel, we refer to this object as the *double complex associated to D*, and the cohomology of its total complex $(C^\bullet_{tot}(D), d)$,

$$C^k_{tot}(D) = \bigoplus_{p+q=k} C(D_p^q) \qquad d = (-1)^p(\partial + \delta),$$

as the *double groupoid cohomology of D*.

2.8.1. Cohomology with trivial coefficients. We interpret $H^2_{tot}(\mathcal{G})$ for a Lie 2-group $\mathcal{G}$ as classifying certain type of extensions. Though this observation is barely remarkable, it does not follow from either [10, 5] because, in the extension, the space of objects is modified.

A 2-cocycle consists of a pair of functions $(F, f) \in C(H^2) \oplus C(\mathcal{G})$ such that:

1. $\delta F = 0$, i.e., $F(h_1, h_2) + F(h_0, h_1h_2) = F(h_0h_1, h_2) + F(h_0, h_1)$ for all triples $h_0, h_1, h_2 \in H$.

2. $\partial f = 0$, i.e., $f(\gamma_1 \bowtie \gamma_2) = f(\gamma_1) + f(\gamma_2)$ for all $(\gamma_1, \gamma_2) \in \mathcal{G}_2$. Using the isomorphism of Remark 2.1, this is equivalent to $f(g_2g_1, h) = f(g_2, h) + f(g_1, hi(g_2))$ for all $h \in H$ and $g_1, g_2 \in G$.

3. $\partial F - \delta f = 0$, i.e., $F(s(\gamma_0), s(\gamma_1)) - F(t(\gamma_0), t(\gamma_1)) = f(\gamma_1) - f(\gamma_0 \between \gamma_1) + f(\gamma_0)$ for all pairs $\gamma_0, \gamma_1 \in \mathcal{G}$. Again, under the isomorphism of Remark 2.1, this equation can be rewritten as

$$F(h_0, h_1) - F(h_0i(g_0), h_1i(g_1)) = f\begin{pmatrix} g_1 \\ h_1 \end{pmatrix} - f\begin{pmatrix} g_0^{h_1}g_1 \\ h_0h_1 \end{pmatrix} + f\begin{pmatrix} g_0 \\ h_0 \end{pmatrix}, \tag{12}$$

where $(g_1, h_1), (g_2, h_2) \in G \rtimes H$.

We point out that making $h_0 = h_1 = 1$ in Eq. (12) yields

$$f\begin{pmatrix} g_0g_1 \\ 1 \end{pmatrix} = f\begin{pmatrix} g_1 \\ 1 \end{pmatrix} + f\begin{pmatrix} g_0 \\ 1 \end{pmatrix} + F(i(g_0), i(g_1)). \tag{13}$$

Also, putting $g_2 = 1$, $\partial f = 0$ implies that $f(1, h) = 0$ for all $h \in H$.

Since $\delta F = 0$, F induces a (central) extension of H,

$$1 \longrightarrow \mathbb{R} \xrightarrow{\bar{1} \times I} H \ltimes^F \mathbb{R} \xrightarrow{pr_1} H \longrightarrow 1,$$

where $H \ltimes^F \mathbb{R}$ is the twisted semi-direct product, whose multiplication is given by the formula

$$(h_0, \lambda_0) \odot_F (h_1, \lambda_1) := (h_0h_1, \lambda_0 + \lambda_1 + F(h_0, h_1)),$$

where $(h_0, \lambda_0), (h_1, \lambda_1) \in H \times \mathbb{R}$.

2.9. Lemma. *If $d(F, f) = 0$, then*

$$\psi_f : G \longrightarrow H \ltimes^F \mathbb{R} \ : g \longmapsto (i(g), f(g, 1))$$

defines a crossed module for the action $g^{(h,\lambda)} := g^h$.

Proof. ψ_f is a Lie group homomorphism:

$$\begin{aligned} \psi_f(g_0) \odot_F \psi_f(g_1) &= (i(g_0), f(g_0, 1)) \odot_F (i(g_1), f(g_1, 1)) \\ &= (i(g_0)i(g_1), f(g_0, 1) + f(g_1, 1) + F(i(g_0), i(g_1))) \\ &= (i(g_0g_1), f(g_0g_1, 1)) = \psi_f(g_0g_1), \end{aligned}$$

where the third equality follows from Eq. (13). Due to the independence of the variable in $\mathbb{R}$, thus defined, the action is still a right action by automorphisms and verifies the Peiffer identity. As for the equivariance of ψ_f, on the one hand we have got

$$\psi_f(g^{(h,\lambda)}) = (i(g^h), f(g^h, 1)),$$

while on the other,

$$\begin{aligned} & (h,\lambda)^{-1} \odot_F \psi_f(g) \odot_F (h,\lambda) \\ = \; & (h^{-1}, -\lambda - F(h^{-1},h)) \odot_F (i(g), f(g,1)) \odot_F (h,\lambda) \\ = \; & (h^{-1}i(g), -\lambda - F(h^{-1},h) + f(g,1) + F(h^{-1}, i(g))) \odot_F (h,\lambda) \\ = \; & (h^{-1}i(g)h, -F(h^{-1},h) + f(g,1) + F(h^{-1},i(g)) + F(h^{-1}i(g),h)) \end{aligned}$$

The first entries coincide because i is the structural morphism of a crossed module. Evaluating $((1,h^{-1}),(g,1))$ and $((g,h^{-1}),(1,h))$ in Eq. (12), one gets respectively

$$f(g,1) + F(h^{-1}, i(g)) = f(g,h^{-1}),$$

and

$$f(g,h^{-1}) - F(h^{-1},h) + F(h^{-1}i(g),h) = f(g^h,1);$$

which combined, imply the result.

■

As a consequence Lemma 2.9, there is a short exact sequence of Lie 2-groups that we write using their associated crossed modules

$$\begin{tikzcd} 1 \arrow[r] & 1 \arrow[r] \arrow[d] & G \arrow[r, "Id_G"] \arrow[d, "\psi_f"'] & G \arrow[r] \arrow[d, "i"] & 1 \\ 1 \arrow[r] & \mathbb{R} \arrow[r] & H \ltimes^F \mathbb{R} \arrow[r, "pr_1"'] & H \arrow[r] & 1 \end{tikzcd}$$

2.10. Lemma. *Let $(F,f),(F',f') \in \Omega^2_{tot}(\mathcal{G})$ be a pair of cohomologous 2-cocycles. Then the induced extensions of Lemma 2.9 are isomorphic.*

Proof. Let $\phi \in \Omega^1_{tot}(\mathcal{G}) = C(H)$ be such that $(F,f) - (F',f') = d\phi = (\delta\phi, \partial\phi)$. In particular, F and F' are cohomologous cocycles; thus, the *object* extensions are isomorphic via

$$\alpha : H \ltimes^F \mathbb{R} \longrightarrow H \ltimes^{F'} \mathbb{R} \;\; : (h,\lambda) \longmapsto (h, \lambda + \phi(h)).$$

We claim that α, together with the identity of G induce the claimed isomorphism between the extensions. Indeed, using the notation of Lemma 2.9,

$$\begin{aligned} \alpha(\psi_f(g)) &= \alpha(i(g), f(g,1)) = (i(g), f(g,1) + \phi(i(g))) \\ &= (i(g), f'(g,1)) = \psi_{f'}(g). \end{aligned}$$

Also, trivially, $Id_G(g^{(h,\lambda)}) = Id_G(g)^{\alpha(h,\lambda)}$, thus finishing the proof.

■

2.11. Remark. Lemmas 2.9 and 2.10 should be taken as motivation to look for a complex whose cohomology classifies extensions starting from the bisimplicial structure naturally associated with a Lie 2-group. They should be interpreted as the "trivial coefficients" case, thus prompting us to define a representation of a Lie 2-group in hopes to classify extensions by more general 2**-vector spaces**, i.e., on flat abelian Lie 2-groups or, equivalently, on 2-term complexes of vector spaces.

2.12. Representations of Lie 2-groups. The General Linear Lie 2-group [19] is the Lie 2-group which plays the rôle of space of automorphisms of a 2-vector space. This 2-group can be traced back at least to Norrie's thesis [15], where it is called the *actor crossed module* of the 2-vector space regarded as an (abelian) 2-group. The domain of the crossed module of the actor is the space of regular derivations – referred to as the Whitehead group –, and the codomain is the space of automorphisms of the 2-group. The associated 2-group via Remark 2.1 can be identified with the category of linear invertible functors and natural homomorphic transformations, which together with the horizontal composition of natural transformations yields a 2-group. We recall its structure for reference: Let $W \xrightarrow{\phi} V$ be a 2-vector space. Then, the space of objects of its General Linear Lie 2-group is the subgroup of invertible self functors

$$GL(\phi)_0 = \{(F, f) \in GL(W) \times GL(V) : \phi \circ F = f \circ \phi\}.$$

The Whitehead group of $W \xrightarrow{\phi} V$ is given by

$$GL(\phi)_1 = \{A \in Hom(V, W) : (I + A\phi, I + \phi A) \in GL(W) \times GL(V)\},$$

endowed with the operation

$$A_1 \odot A_2 := A_1 + A_2 + A_1 \phi A_2, \tag{14}$$

for which the identity element is the 0 map, and inverses are given by either

$$A^\dagger = -A(I + \phi A)^{-1} = -(I + A\phi)^{-1} A.$$

We write $\dagger$ instead of -1 to avoid any possible overlap of notation with the actual inverse of a matrix. The crossed module map

$$GL(\phi)_1 \xrightarrow{\Delta} GL(\phi)_0.$$

is given by

$$\Delta A = (I + A\phi, I + \phi A)$$

for $A \in GL(\phi)_1$. This is well defined since by definition it takes values in $GL(W) \times GL(V)$, and

$$\phi(I + A\phi) = \phi + \phi A \phi = (I + \phi A)\phi.$$

Concluding, the right action of $GL(\phi)_0$ on $GL(\phi)_1$ is given by

$$A^{(F,f)} = F^{-1} A f. \tag{15}$$

2.13. Proposition. *[15, 19] Along with the group structure (14) and the action (15),*

$$GL(\phi)_1 \xrightarrow{\Delta} GL(\phi)_0$$

is a crossed module of Lie groups.

In the sequel, we write $GL(\phi)$ for the crossed module of Proposition 2.13 and its associated Lie 2-group as well.

2.14. Definition. *A* strict representation *of a Lie* 2*-group* $\mathcal{G}$ *on* $W \xrightarrow{\phi} V$ *is a morphism of Lie* 2*-groups*

$$\rho : \mathcal{G} \longrightarrow GL(\phi). \tag{16}$$

We remark that, by a morphism of Lie 2-groups, we mean a naïve smooth functor respecting the Lie group structures, as opposed to more general types of morphisms (such as bibundles of Lie groupoids). We refer shortly to a map (16) as a 2-representation.

A 2-representation taking values in either $A \longrightarrow 1$ or $1 \longrightarrow A$ coincides with the modules of [10, 5] whenever the abelian group A is simply connected. Definition 2.14 is also an instance of the actions of [15] in the particular case when the 2-group that is being acted on is abelian and simply connected. By definition, if $G \xrightarrow{i} H$ is the crossed module associated with $\mathcal{G}$ via Remark 2.1, the 2-representation (16) consists of a Lie group homomorphism

$$\rho_0 : H \longrightarrow GL(\phi)_0 \leq GL(W) \times GL(V),$$

at the level of objects, which amounts to two Lie group representations $\rho_0^1 : H \longrightarrow GL(W)$, $\rho_0^0 : H \longrightarrow GL(V)$ fitting in

$$\begin{tikzcd} W \arrow[r, "(\rho_0^1)_h"] \arrow[d, "\phi"'] & W \arrow[d, "\phi"] \\ V \arrow[r, "(\rho_0^0)_h"'] & V, \end{tikzcd}$$

for all $h \in H$, i.e.,

$$\rho_0^0(h) \circ \phi = \phi \circ \rho_0^1(h). \tag{17}$$

At the level of arrows,

$$\rho_1 : G \longrightarrow GL(\phi)_1 \leq Hom(V, W),$$

is a Lie group homomorphism if and only if

$$\rho_1(g_0 g_1) = \rho_1(g_0) + \rho_1(g_1) + \rho_1(g_0) \circ \phi \circ \rho_1(g_1) \tag{18}$$

for all $g_0, g_1 \in G$. The compatibility between the rest of the crossed module structures is encoded in the following relations:

$$\rho_0^0(i(g)) = I + \phi \circ \rho_1(g), \qquad \rho_0^1(i(g)) = I + \rho_1(g) \circ \phi \tag{19}$$

for all $g \in G$, and

$$\rho_1(g^h) = \rho_0^1(h)^{-1}\rho_1(g)\rho_0^0(h) \tag{20}$$

for all $h \in H$, $g \in G$.

2.15. Example. Trivial representations. If $(\rho_1, \rho_0) \equiv (0, I)$, the defining equations for a 2-representation are trivially satisfied.

2.16. Example. Usual Lie group representations. Letting $W = (0)$, a 2-representation is ultimately equivalent to a representation of $H/i(G)$ on V. More precisely, the 2-representation is defined by a single representation of H on V that vanishes along $i(G)$. In particular, a Lie group representation defines a 2-representation of a unit Lie 2-group on a unit 2-vector space.

Analogously, if $V = (0)$, a 2-representation is ultimately equivalent to a representation of the orbit space on W.

2.17. Remark. The class of examples in Example 2.16 are referred to in the literature (*e.g.*, [12]) as $\mathcal{G}$-*modules*, though generalized to allow other abelian groups besides vector spaces.

2.18. Example. The adjoint representation: Let $\mathfrak{g} \xrightarrow{\mu} \mathfrak{h}$ be the Lie 2-algebra of the Lie 2-group $G \xrightarrow{i} H$, then we have

$$\begin{tikzcd} G \arrow[r, "Ad_1"] \arrow[d] & GL(\mu)_1 \arrow[d] \\ H \arrow[r, "Ad_0"'] & GL(\mu)_0 \end{tikzcd}$$

where $(Ad_0)_h = ((-)^{h^{-1}}, Ad_{h^{-1}})$, and $(Ad_1)_g = d_e({}_g\wedge)$ with

$${}_g\wedge : H \longrightarrow G : h \longmapsto g(g^{-1})^{h^{-1}}.$$

2.19. Remark. Example 2.18 appears in [19] and is also the derivative of the *canonical* action of [15] which is a generalized conjugation.

Already in [15], it is explained that given a 2-representation ρ of $\mathcal{G}$ on $\mathbb{V}$, one can build a semi-direct product 2-group $\mathcal{G}_\rho \ltimes \mathbb{V}$. With the notation conventions of this section, the crossed module of the semi-direct product is

$$G_{\rho_0^1 \circ i} \ltimes W \xrightarrow{i \times \phi} H_{\rho_0^0} \ltimes V, \tag{21}$$

together with the right action given by

$$(g,w)^{(h,w)} = (g^h, \rho_0^1(h)^{-1}(w+\rho_1(g)v))$$

for $(h,w) \in H \times V$ and $(g,w) \in G \times W$.

The Lie group of arrows of $\mathcal{G}_\rho \ltimes \mathbb{V}$,

$$(G_{\rho_0^1 \circ i} \ltimes W) \ltimes (H_{\rho_0^0} \ltimes V),$$

is isomorphic to a semi-direct product of the Lie group $\mathcal{G}$ and $W \oplus V$ with respect to the honest representation that is the content of the following proposition.

2.20. Proposition. *Given a 2-representation $\rho : \mathcal{G} \longrightarrow GL(\phi)$, there is a Lie group representation*

$$\bar{\rho} : G \ltimes H \longrightarrow GL(W \oplus V) : (g,h) \longmapsto \begin{pmatrix} \rho_0^1(hi(g)) & \rho_0^1(h)\rho_1(g) \\ 0 & \rho_0^0(h) \end{pmatrix}$$

Proof. Consider the product

$$\begin{aligned}\bar{\rho}(g_0,h_0)\bar{\rho}(g_1,h_1) &= \begin{pmatrix} \rho_0^1(h_0 i(g_0)) & \rho_0^1(h_0)\rho_1(g_0) \\ 0 & \rho_0^0(h_0) \end{pmatrix} \begin{pmatrix} \rho_0^1(h_1 i(g_1)) & \rho_0^1(h_1)\rho_1(g_1) \\ 0 & \rho_0^0(h_1) \end{pmatrix} \\ &= \begin{pmatrix} \rho_0^1(h_0 i(g_0))\rho_0^1(h_1 i(g_1)) & \rho_0^1(h_0 i(g_0))\rho_0^1(h_1)\rho_1(g_1) + \rho_0^1(h_0)\rho_1(g_0)\rho_0^0(h_1) \\ 0 & \rho_0^0(h_0)\rho_0^0(h_1) \end{pmatrix}.\end{aligned}$$

The bottom row agrees with the bottom row of $\bar{\rho}((g_0,h_0) \between (g_1,h_1)) = \bar{\rho}(g_0^{h_1} g_1, h_0 h_1)$ because ρ_0^0 is a group homomorphism. The first entries of the top row coincide too, put simply, because the target is a group homomorphism as well:

$$\rho_0^1(h_0 h_1 i(g_0^{h_1} g_1)) = \rho_0^1(h_0 h_1 h_1^{-1} i(g_0) h_1 i(g_1)) = \rho_0^1(h_0 i(g_0))\rho_0^1(h_1 i(g_1)).$$

For the remaining entries, we use Eq.'s (18) and (20) to compute

$$\begin{aligned}\rho_0^1(h_0 h_1)\rho_1(g_0^{h_1} g_1) &= \rho_0^1(h_0 h_1)(\rho_1(g_0^{h_1})(I + \phi\rho_1(g_1)) + \rho_1(g_1)) \\ &= \rho_0^1(h_0 h_1)\rho_0^1(h_1)^{-1}\rho_1(g_0)\rho_0^0(h_1)(I + \phi\rho_1(g_1)) + \rho_0^1(h_0 h_1)\rho_1(g_1) \\ &= \rho_0^1(h_0)\rho_1(g_0)\rho_0^0(h_1) + \rho_0^1(h_0)\rho_1(g_0)\phi\rho_0^1(h_1)\rho_1(g_1) + \rho_0^1(h_0)\rho_0^1(h_1)\rho_1(g_1) \\ &= \rho_0^1(h_0)\rho_1(g_0)\rho_0^0(h_1) + \rho_0^1(h_0)(\rho_1(g_0)\phi + I)\rho_0^1(h_1)\rho_1(g_1),\end{aligned}$$

and the result follows from the second relation in Eq. (19).

∎

2.21. Remark. Forgetting the Lie group structure,

$$\begin{tikzcd} \mathcal{G}_{\bar\rho} \ltimes \mathbb{V} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & H_{\rho_0^0} \ltimes V \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & H \end{tikzcd}$$

has the structure of a VB-groupoid (see, *e.g.*, [7]). Thus, there is an associated representation up to homotopy [11]. In fact, since all vector bundles are trivial, there is an obvious splitting of the core sequence

$$(0) \longrightarrow \mathcal{G} \times W \longrightarrow \mathcal{G}_{\bar\rho} \ltimes \mathbb{V} \longrightarrow \mathcal{G} \times V \longrightarrow (0)$$

given by

$$\sigma : \mathcal{G} \times V \longrightarrow \mathcal{G}_{\bar\rho} \ltimes \mathbb{V} : (g, h; v) \longmapsto \sigma_{(g,h)}(h, v) := (g, h; 0, v),$$

which verifies $\sigma_{u(h)}(h, v) = \sigma_{(1,h)}(h, v) = (1, h; 0, v) = \hat{u}(h, v)$. Here, we use $\hat{\cdot}$ to refer to the structural maps of the top groupoid. σ defines a canonical representation up to homotopy $(\varrho, \Delta^V, \Delta^W, \Omega)$ associated to the 2-representation, where

$$\varrho : H \times W \longrightarrow H \times V : (h, w) \longmapsto \hat{t}(1, h; w, 0) = (h, \rho_0^0(h)\phi(w)),$$

the quasi-actions of $\mathcal{G} \cong G \rtimes H$ on $H \times V$ and $H \times W$ are respectively

$$\Delta^V_{(g,h)}(h, v) = \hat{t}(\sigma_{(g,h)}(h, v)) = (hi(g), v),$$

$$\Delta^W_{(g,h)}(h, w) = \sigma_{(g,h)}(\varrho(h, w))\hat{\bowtie}(1, h; w, 0)\hat{\bowtie}(g^{-1}, hi(g); 0, 0) = (hi(g); \rho_0^1(i(g))^{-1}w),$$

and the curvature form $\Omega \in \Gamma(s_2^*(H \times V^*) \otimes t_2^*(H \times W))$ at $(g_1, g_2, h) \in G^2 \times H \cong \mathcal{G}_2$ is

$$\Omega_{(g_1,g_2,h)}(v) = \Big(\sigma_{(g_2g_1,h)}(h, v) - \sigma_{(g_1,hi(g_2))}\big(\Delta^V_{(g_2,h)}(h, v)\big)\hat{\bowtie}\sigma_{(g_2,h)}(h, v)\Big)\hat{\bowtie}0_{(g_2g_1,h)^{-1}} = 0_{hi(g_2g_1)}.$$

Since Ω is identically zero, the quasi-actions define actual representations of the Lie 2-group $\mathcal{G}$ on the corresponding vector bundles.

We close this section by proving that splitting an abstract extension of Lie 2-groups induces a 2-representation in the sense of Definition 2.14.

2.22. Definition. *An* extension *of the Lie 2-group* $G \xrightarrow{i} H$ *by the 2-vector space* $W \xrightarrow{\phi} V$ *is a Lie 2-group* $E_1 \xrightarrow{\epsilon} E_0$ *that fits in*

$$\begin{tikzcd} 1 \arrow[r] & W \arrow[r, "j_1"] \arrow[d, "\phi"] & E_1 \arrow[r, "\pi_1"] \arrow[d, "\epsilon"] & G \arrow[r] \arrow[d, "i"] & 1 \\ 1 \arrow[r] & V \arrow[r, "j_0"'] & E_0 \arrow[r, "\pi_0"'] & H \arrow[r] & 1, \end{tikzcd}$$

i.e., where the top and bottom rows are short exact sequences and the squares are maps of Lie 2-groups.

2.23. Proposition. *Given an extension of the Lie 2-group $G \xrightarrow{i} H$ by the 2-vector space $W \xrightarrow{\phi} V$ and a smooth splitting σ,*

$$\begin{tikzcd}
1 \arrow[r] & W \arrow[r, "j_1"] \arrow[d, "\phi"'] & E_1 \arrow[r, "\pi_1"'] \arrow[d, "\epsilon"'] & G \arrow[l, bend right, "\sigma_1"'] \arrow[r] \arrow[d, "i"] & 1 \\
1 \arrow[r] & V \arrow[r, "j_0"'] & E_0 \arrow[r, "\pi_0"'] & H \arrow[l, bend right, "\sigma_0"'] \arrow[r] & 1,
\end{tikzcd}$$

there is an induced 2-representation $\rho^{\epsilon}_{\sigma} : \mathcal{G} \longrightarrow GL(\phi)$ given by

$$\rho^0_0(h)v = \sigma_0(h)v\sigma_0(h)^{-1} \qquad \rho^1_0(h)w = w^{\sigma_0(h)^{-1}}$$

$$\rho_1(g)v = \sigma_1(g)^v\sigma_1(g)^{-1},$$

for $h \in H$, $v \in V$, $w \in W$ and $g \in G$.

The proof of Proposition 2.23 follows easily after using the splitting to write E_1 and E_0 as semi-direct products. We thus postpone it to the end of the section in order to introduce the necessary notation.

We regard injective maps as inclusions; in so, $\phi = \epsilon|_W$. Given an extension as in the statement of Proposition 2.23, one uses the splitting to get the diffeomorphisms $H \times V \cong E_0$ and $G \times W \cong E_1$ given respectively by $(z, a) \longmapsto a\sigma_k(z)$ with inverse $e \longmapsto (\pi_k(e), e\sigma_k(\pi_k(e))^{-1})$, for $k \in \{0, 1\}$. We recall that one can use these diffeomorphisms to transfer the group structure, thus getting

$$(h_0, v_0) \cdot (h_1, v_1) := (h_0h_1, v_0 + \rho^0_0(h_0)v_1 + \omega_0(h_0, h_1)), \tag{22}$$

where $(h_0, v_0), (h_1, v_1) \in H \times V$, ρ^0_0 is defined as in Proposition 2.23 and $\omega_0(h_0, h_1) := \sigma_0(h_0)\sigma_0(h_1)\sigma_0(h_0h_1)^{-1} \in V$. This is the usual twisted semi-direct product $H_{\rho^0_0} \ltimes^{\omega_0} V$ from the theory of Lie group extensions. Conversely, the operation defined by Eq. (22) with $\omega_0 \in C(H^2, V)$ is an associative product if and only if ω_0 is a 2-cocycle for the Lie group cohomology of H with values in ρ^0_0 (see Eq. (10)).

An identical reasoning implies there is an isomorphism of Lie groups $E_1 \cong G_{\rho^1_0 \circ i} \ltimes^{\omega_1} W$, with ρ^1_0 defined as in Proposition 2.23 and $\omega_1(g_0, g_1) = \sigma_1(g_0)\sigma_1(g_1)\sigma_1(g_0g_1)^{-1}$.

We rewrite the rest of the crossed module structure of the extension using these trivializations. The homomorphism ϵ gets rewritten as

$$G \times W \longrightarrow H \times V : (g, w) \longmapsto (i(x), \phi(w) + \varphi(g)),$$

where $\varphi(g) := \epsilon(\sigma_1(g))\sigma_0(i(g))^{-1}$, and the action of $(h, v) \in H \times V$ on $(g, w) \in G \times W$ as

$$(g, w)^{(h,v)} := (g^h, \rho^1_0(h)^{-1}(w + \rho_1(g)v) + \alpha(h; g)),$$

where ρ^1_0 and ρ_1 are defined as in Proposition 2.23, and $\alpha(h; g) := \sigma_1(g)^{\sigma_0(h)}\sigma_1\big(g^h\big)^{-1}$.

2.24. Remark. Notice that, in case one can take the splitting σ in Proposition 2.23 to be a crossed module map, ω_0, ω_1, φ and α vanish identically and one recovers the semi-direct product structure (2.19).

Proof of Proposition 2.23. We make the necessary computations to prove that ρ^ϵ_σ is a 2-representation.

- Well-defined: We use the exactness of the sequences to see that the maps land where they are supposed to. Let $h \in H$, $v \in V$, $w \in W$ and $g \in G$, then

$$\pi_0\big(\sigma_0(h)v\sigma_0(h)^{-1}\big) = \pi_0(\sigma_0(h))\pi_0(v)\pi_0\big(\sigma_0(h)^{-1}\big) = h1h^{-1} = 1 \Longrightarrow \rho^0_0(h)v \in V,$$

$$\pi_1\big(w^{\sigma_0(h)^{-1}}\big) = \pi_1(w)^{\pi_0\left(\sigma_0(h)^{-1}\right)} = 1^{h^{-1}} = 1 \Longrightarrow \rho^1_0(h)w \in W,$$

$$\pi_1\big(\sigma_1(g)^v\sigma_1(g)^{-1}\big) = \pi_1(\sigma_1(g))^{\pi_1(v)}\pi_1\big(\sigma_1(g)^{-1}\big) = g^1g^{-1} = 1 \Longrightarrow \rho_1(g)v \in W.$$

These components are smooth and linear. Further, thus defined, $\rho^0_0(h)\circ\phi = \phi\circ\rho^1_0(h)$ for each $h \in H$. Indeed, let $w \in W$, then

$$\rho^0_0(h)(\phi(w)) = \sigma_0(h)\epsilon(w)\sigma_0(h)^{-1} = \epsilon\big(w^{\sigma_0(h)^{-1}}\big) = \phi(\rho^1_0(h)w).$$

Thus, for all $h \in H$, $\rho_0(h) := (\rho^0_0(h), \rho^1_0(h)) \in GL(\phi)_0$.

- ρ^0_0 is a Lie group representation: Let $h_1, h_2 \in H$ and $v \in V$, then

$$\begin{aligned}\rho^0_0(h_1)\rho^0_0(h_2)v &= \sigma_0(h_1)\sigma_0(h_2)v\sigma_0(h_2)^{-1}\sigma_0(h_1)^{-1}\\ &= \omega_0(h_1,h_2)(\rho^0_0(h_1h_2)v)\omega_0(h_1,h_2)^{-1}.\end{aligned}$$

Since both $\omega_0(h_1,h_2), \rho^0_0(h_1h_2)v \in V$, the conjugation is trivial and the claim follows.

- ρ^1_0 is a Lie group representation: Let $h_1, h_2 \in H$ and $w \in W$, then

$$\begin{aligned}\rho^1_0(h_1)\rho^1_0(h_2)w &= w^{\sigma_0(h_2)^{-1}\sigma_0(h_1)^{-1}}\\ &= (\rho^1_0(h_1h_2)w)^{\omega_0(h_1,h_2)^{-1}}.\end{aligned}$$

Since $\omega_0(h_1,h_2) \in V$ and $\rho^1_0(h_1h_2)w \in W$, the action is trivial and the claim follows.

- ρ_1 is a Lie group homomorphism: Let $g_1, g_2 \in G$ and $v \in V$, then

$$\begin{aligned}\rho_1(g_1)\odot\rho_1(g_2)v &= (\rho_1(g_1)+\rho_1(g_2)+\rho_1(g_1)\phi\rho_1(g_2))v\\ &= \sigma_1(g_1)^v\sigma_1(g_1)^{-1}\sigma_1(g_2)^v\sigma_1(g_2)^{-1}\rho_1(g_1)\epsilon(\sigma_1(g_2)^v\sigma_1(g_2)^{-1})\\ &= \sigma_1(g_1)^v\sigma_1(g_1)^{-1}\sigma_1(g_2)^v\sigma_1(g_2)^{-1}\sigma_1(g_1)^{\epsilon\left(\sigma_1(g_2)^v\sigma_1(g_2)^{-1}\right)}\sigma_1(g_1)^{-1}.\end{aligned}$$

Using the Peiffer equation, we get

$$\sigma_1(g_1)^{\epsilon\left(\sigma_1(g_2)^v\sigma_1(g_2)^{-1}\right)} = \big(\sigma_1(g_2)^v\sigma_1(g_2)^{-1}\big)^{-1}\sigma_1(g_1)\big(\sigma_1(g_2)^v\sigma_1(g_2)^{-1}\big);$$

thus yielding,

$$\begin{aligned}
& \rho_1(g_1) \odot \rho_1(g_2)v \\
= \; & \sigma_1(g_1)^v \sigma_1(g_2)^v \sigma_1(g_2)^{-1} \sigma_1(g_1)^{-1} \\
= \; & \sigma_1(g_1)^v \sigma_1(g_2)^v \big(\sigma_1(g_1g_2)^v\big)^{-1} \sigma_1(g_1g_2)^v \sigma_1(g_1g_2)^{-1} \sigma_1(g_1g_2)\sigma_1(g_2)^{-1}\sigma_1(g_1)^{-1} \\
= \; & (\omega_1(g_1,g_2))^v (\rho_1(g_1g_2)v)\omega_1(g_1,g_2)^{-1}.
\end{aligned}$$

Since both $\omega_1(g_1,g_2), \rho_1(g_1g_2)v \in W$, both the action and the conjugation are trivial and the claim follows.

- $\rho_0 \circ i = \Delta \circ \rho_1$: For $g \in G$, this equation breaks into two components,

$$\rho_0^0(i(g)) = I + \phi \circ \rho_1(g) \in GL(V) \qquad \text{and} \qquad \rho_0^1(i(g)) = I + \rho_1(g) \circ \phi \in GL(W).$$

Let $v \in V$ and $w \in W$, then

$$\begin{aligned}
(I + \phi\rho_1(g))v &= v\epsilon\big(\sigma_1(g)^v \sigma_1(g)^{-1}\big) = \epsilon(\sigma_1(g))v\epsilon(\sigma_1(g))^{-1} \\
&= \varphi(g)(\rho_0^0(i(g))v)\varphi(g)^{-1} = \rho_0^0(i(g))v,
\end{aligned}$$

and

$$\begin{aligned}
(I + \rho_1(g)\phi)w &= w\sigma_1(g)^{\epsilon(w)}\sigma_1(g)^{-1} = \sigma_1(g)w\sigma_1(g)^{-1} \\
&= w^{\epsilon\left(\sigma_1(g)^{-1}\right)} = (\rho_0^1(i(g))w)^{\varphi(g)^{-1}} = \rho_0^1(i(g))w,
\end{aligned}$$

where the last equality in each sequence follows from $\varphi(g) \in V$.

- ρ_1 respects the actions: Let $g \in G$, $h \in H$ and $v \in V$, then

$$\begin{aligned}
\rho_1(g)^{\rho_0(h)}v &= \rho_0^1(h)^{-1}\rho_1(g)\rho_0^0(h)v = \big(\sigma_1(g)^{\sigma_0(h)v\sigma_0(h)^{-1}}\sigma_1(g)^{-1}\big)^{\sigma_0(h)} \\
&= \big(\sigma_1(g)^{\sigma_0(h)}\sigma_1(g^h)^{-1}\big)^v \sigma_1\big(g^h\big)^v \big(\sigma_1(g)^{\sigma_0(h)}\big)^{-1} \\
&= \alpha(h;g)^v\big(\rho_1(g^h)v\big)\alpha(h;g)^{-1}.
\end{aligned}$$

Since both $\alpha(h;g), \rho_1(g^h)v \in W$, both the action and the conjugation are trivial and the claim follows.

■

3. The grid, the snapshot complex and its cohomology

In this section, inspired by the triple complex associated to a Lie 2-algebra [1] and by the concurrence of the double cohomology with the cohomology of [10], we introduce the *grid* of complexes of Lie 2-group cochains with values in a 2-representation. We establish notation and specify all groupoids and representations appearing in the three dimensional

grid. Since we define the complex in the r-direction to start differently in 0th degree, we prove that this fits into a complex accordingly. We expose how this grid fails to be a triple complex. Specifically, we study the square of the total differential degree by degree and define the *difference maps* via the commutator of the non-commuting differentials, thus producing a complex up to degree 3. We study the cohomology of the resulting snapshot complex and show that its second cohomology classifies abelian extensions.

3.1. The background grid. Throughout, let $\mathcal{G}$ be a Lie 2-group with associated crossed module $G \xrightarrow{i} H$ and let ρ be a 2-representation of $\mathcal{G}$ on the 2-vector space $W \xrightarrow{\phi} V$. Using the notation conventions laid down in Subection 2.12, we think of ρ as a triple $(\rho_0^0, \rho_0^1; \rho_1)$. Also, we take the isomorphisms of Remarks 2.1 and 2.5 to be fixed and we abuse notation and often treat them as equalities.

3.2. Remark. In the sequel, we define the grid using a series of groupoids and groupoid representations that involve taking pull-backs along the final target map (see Remark 2.6). One could also define an equivalent structure by pulling-back along the "initial source" map $s_p : \mathcal{G}_p \longrightarrow H : (\gamma_1, ..., \gamma_p) \longmapsto s(\gamma_p)$. There is no economy in working with either one, one necessarily pays a computational price somewhere.

Let $C_r^{p,q}(\mathcal{G}, \phi)$ be defined by Eq.'s (1) and (2). This three dimensional lattice of vector spaces comes together with a grid of complexes of groupoid cochains (see (10) and (11)).

3.2.1. The p-direction. When $q = 0$, one has got the trivial complexes

$$C_0^{0,0}(\mathcal{G}, \phi) = V \xrightarrow{\partial=0} C_0^{1,0}(\mathcal{G}, \phi) = V \xrightarrow{\partial=Id_V} C_0^{2,0}(\mathcal{G}, \phi) = V \xrightarrow{\partial=0} C_0^{3,0}(\mathcal{G}, \phi) = V \longrightarrow \cdots$$

when $r = 0$, and

$$C(G^r, W) \xrightarrow{\partial=0} C(G^r, W) \xrightarrow{\partial=Id_{C(G^r,W)}} C(G^r, W) \xrightarrow{\partial=0} C(G^r, W) \longrightarrow \cdots$$

otherwise. When $q \neq 0$ and $r = 0$, the complex

$$C(H^q, V) \xrightarrow{\partial} C(\mathcal{G}^q, V) \xrightarrow{\partial} C(\mathcal{G}_2^q, V) \xrightarrow{\partial} C(\mathcal{G}_3^q, V) \longrightarrow \cdots$$

is the cochain complex of the product groupoid

$$\mathcal{G}^q \rightrightarrows H^q$$

with respect to the trivial representation on the vector bundle $H^q \times V \xrightarrow{pr_1} H^q$. For any other value of r, the complex

$$C(H^q \times G^r, W) \xrightarrow{\partial} C(\mathcal{G}^q \times G^r, W) \xrightarrow{\partial} C(\mathcal{G}_2^q \times G^r, W) \xrightarrow{\partial} C(\mathcal{G}_3^q \times G^r, W) \longrightarrow \cdots$$

is the cochain complex of the product groupoid

$$\mathcal{G}^q \times G^r \rightrightarrows H^q \times G^r$$

with respect to the left representation on the trivial bundle $H^q \times G^r \times W \xrightarrow{pr_1} H^q \times G^r$,

$$(\gamma_1, ..., \gamma_q; \vec{f}) \cdot (h_1, ..., h_q; \vec{f}, w) := (h_1 i(g_1), ..., h_q i(g_q); \vec{f}, \rho_0^1(i(pr_G(\gamma_1 \barwedge ... \barwedge \gamma_q)))^{-1} w), \quad (23)$$

where $\gamma_k = (g_k, h_k) \in \mathcal{G}$ and $\vec{f} \in G^r$.

3.3. Remark. Observe that for $q = 1$, the representations defining the complexes in the p-direction are those of the representation up to homotopy induced by the 2-representation (see Remark 2.21). Indeed, for $r = 0$, the representation coincides with Δ^V, and for $r > 0$, the representation coincides with the pull-back of Δ^W along the projection onto $\mathcal{G} \rightrightarrows H$.

3.4. Lemma. *Eq. (23) defines a representation.*

Notice that this lemma is not straightforward, because the projection pr_G is not in general a group homomorphism. In order to make cleaner computations, we introduce the following auxiliary straightforward lemma.

3.5. Lemma. *Let $\gamma_1, ..., \gamma_q \in \mathcal{G}$. If $\gamma_k = (g_k, h_k) \in G \rtimes H$, then*

$$\gamma_1 \barwedge ... \barwedge \gamma_q = (g_1^{h_2...h_q} g_2^{h_3...h_q} ... g_{q-2}^{h_{q-1}h_q} g_{q-1}^{h_q} g_q, h_1...h_q).$$

Proof. By induction on q, for $q = 2$ the formula is nothing but the definition of the product in $\mathcal{G} \cong G \rtimes H$. Now, suppose the equation holds for $q - 1$ elements, then

$$\begin{aligned}\gamma_1 \barwedge ... \barwedge \gamma_q &= (\gamma_1 \barwedge ... \barwedge \gamma_{q-1}) \barwedge \gamma_q \\ &= (g_1^{h_2...h_{q-1}} g_2^{h_3...h_{q-1}} ... g_{q-3}^{h_{q-2}h_{q-1}} g_{q-2}^{h_{q-1}} g_{q-1}, h_1...h_{q-1}) \barwedge (g_q, h_q) \\ &= ((g_1^{h_2...h_{q-1}} g_2^{h_3...h_{q-1}} ... g_{q-3}^{h_{q-2}h_{q-1}} g_{q-2}^{h_{q-1}} g_{q-1})^{h_q} g_q, h_1...h_{q-1}h_q),\end{aligned}$$

and the result follows since the action of H is by automorphisms.

■

Proof of Lemma 3.4. Observe that the first coordinates in the right hand side of Eq. (23) are given by the target; hence, we just need to focus on the W coordinate.

Units act trivially: Let $w \in W$ and $\gamma_1, ..., \gamma_q \in \mathcal{G}$ be such that $\gamma_k = (1, h_k) \in G \rtimes H$ for each k. Then, the W coordinate in the right hand side of Eq. (23) reads

$$\begin{aligned}\rho_0^1(i(pr_G(\gamma_1 \barwedge ... \barwedge \gamma_q)))^{-1} w &= \rho_0^1(i(1^{h_2...h_q} 1^{h_3...h_q} ... 1^{h_{q-1}h_q} 1^{h_q} 1))^{-1} w \\ &= \rho_0^1(i(1...1))^{-1} w = w.\end{aligned}$$

Groupoid multiplication: The arrows $(\gamma_1', ..., \gamma_q'; (\vec{f})'), (\gamma_1, ..., \gamma_q; \vec{f}) \in \mathcal{G}^q \times G^r$ are composable if and only if $(\vec{f})' = \vec{f}$ and $(\gamma_k', \gamma_k) \in \mathcal{G}^{(2)}$ for each k, or equivalently, if $\gamma_k' = (g_k', h_k i(g_k)) \in G \rtimes H$. For such a pair, the groupoid multiplication is given by

$$(\gamma_1', ..., \gamma_q'; \vec{f}) \cdot (\gamma_1, ..., \gamma_q; \vec{f}) := \left(\begin{pmatrix} g_1 g_1' & ... & g_q g_q' \\ h_1 & ... & h_q \end{pmatrix}; \vec{f} \right).$$

The compatibility follows from the formula

$$\begin{aligned}&(g_1 g_1')^{h_2...h_q} (g_2 g_2')^{h_3...h_q} ... (g_{q-1} g_{q-1}')^{h_q} g_q g_q' = \\ &\qquad g_1^{h_2...h_q} g_2^{h_3...h_q} ... g_{q-1}^{h_q} g_q (g_1')^{h_2 i(g_2)...h_q i(g_q)} (g_2')^{h_3 i(g_3)...h_q i(g_q)} ... (g_{q-1}')^{h_q i(g_q)} g_q'.\end{aligned}$$

We proceed by induction on q. For $q=2$,

$$(g_1g_1')^{h_2}g_2g_2' = g_1^{h_2}(g_1')^{h_2}g_2g_2' = g_1^{h_2}g_2g_2^{-1}(g_1')^{h_2}g_2g_2' = g_1^{h_2}g_2(g_1')^{h_2 i(g_2)}g_2'.$$

Suppose now that the equation holds for $q-1$, then

$$\begin{aligned}
\prod_{k=1}^{q}(g_kg_k')^{\prod_{j=k+1}^{q}h_j} &= \left(\prod_{k=1}^{q-1}(g_kg_k')^{\left(\prod_{j=k+1}^{q-1}h_j\right)h_q}\right)g_qg_q' \\
&= \left(\prod_{k=1}^{q-1}(g_kg_k')^{\prod_{j=k+1}^{q-1}h_j}\right)^{h_q} g_qg_q' \\
&=^{(I.H.)} \left(\prod_{k=1}^{q-1}g_k^{\prod_{j=k+1}^{q-1}h_j}\prod_{k=1}^{q-1}(g_k')^{\prod_{j=k+1}^{q-1}h_j i(g_j)}\right)^{h_q} g_qg_q' \\
&= \left(\prod_{k=1}^{q-1}g_k^{\prod_{j=k+1}^{q-1}h_j}\right)^{h_q} g_qg_q^{-1}\left(\prod_{k=1}^{q-1}(g_k')^{\prod_{j=k+1}^{q-1}h_j i(g_j)}\right)^{h_q} g_qg_q' \\
&= \prod_{k=1}^{q}g_k^{\prod_{j=k+1}^{q}h_j}\left(\prod_{k=1}^{q-1}(g_k')^{\prod_{j=k+1}^{q-1}h_j i(g_j)}\right)^{h_q i(g_q)} g_q',
\end{aligned}$$

which is precisely what we wanted.

■

3.5.1. The q direction. When $r=0$, the complex

$$V \xrightarrow{\delta} C(\mathcal{G}_p, V) \xrightarrow{\delta} C(\mathcal{G}_p^2, V) \xrightarrow{\delta} C(\mathcal{G}_p^3, V) \longrightarrow \cdots$$

is the group complex of $\mathcal{G}_p$ with values in the pull-back of the representation ρ_0^0 along the final target map t_p; when $r \neq 0$, the complex

$$C(G^r, W) \xrightarrow{\delta} C(\mathcal{G}_p \times G^r, W) \xrightarrow{\delta} C(\mathcal{G}_p^2 \times G^r, W) \xrightarrow{\delta} C(\mathcal{G}_p^3 \times G^r, W) \longrightarrow \cdots$$

is the cochain complex of the (right!) transformation groupoid

$$\mathcal{G}_p \ltimes G^r \rightrightarrows G^r$$

with respect to the right representation

$$(g_1, ..., g_r; w) \cdot (\gamma; g_1, ..., g_r) := (g_1^{t_p(\gamma)}, ..., g_r^{t_p(\gamma)}; \rho_0^1(t_p(\gamma))^{-1}w) \tag{24}$$

on the trivial vector bundle $G^r \times W \xrightarrow{pr_1} G^r$, where $g_1, ..., g_r \in G$, $\gamma \in \mathcal{G}_p$ and $w \in W$. It is obvious that Eq. (24) defines a representation, as, on the one hand, is defined explicitly using the source map of the transformation groupoid and, on the other, it is given fibre-wise by the pull-back of a representation along a homomorphism.

When writing the groupoid differential, we use the shorthand $\rho^r_{\mathcal{G}_p}(\gamma; g)w$ instead of the lengthier Eq. (24).

3.5.2. The r-direction. When $q = 0$, the complex

$$V \xrightarrow{\delta'} C(G, W) \xrightarrow{\delta_{(1)}} C(G^2, W) \xrightarrow{\delta_{(1)}} C(G^3, W) \longrightarrow \cdots$$

is the group complex of G with values in the pull-back of the representation ρ_0^1 along the crossed module homomorphism i, but for the 0th degree; when $q \neq 0$, the complex

$$C(\mathcal{G}_p^q, V) \xrightarrow{\delta'} C(\mathcal{G}_p^q \times G, W) \xrightarrow{\delta_{(1)}} C(\mathcal{G}_p^q \times G^2, W) \xrightarrow{\delta_{(1)}} C(\mathcal{G}_p^q \times G^3, W) \longrightarrow \cdots$$

is, again except for the 0th degree, the cochain complex of the Lie group bundle

$$\mathcal{G}_p^q \times G \rightrightarrows \mathcal{G}_p^q$$

with respect to the left representation

$$(\gamma_1, ..., \gamma_q; g) \cdot (\gamma_1, ..., \gamma_q; w) := (\gamma_1, ..., \gamma_q; \rho_0^1(i(g^{t_p(\gamma_1)...t_p(\gamma_q)}))w) \tag{25}$$

on the trivial vector bundle $G_p^q \times W \xrightarrow{pr_1} G_p^q$, where $\gamma_1, ..., \gamma_q \in \mathcal{G}_p$, $g \in G_p$ and $w \in W$. Eq. (25) clearly defines a representation, as it is given fibre-wise by the pull-back of a representation along a homomorphism, namely the composition of the crossed module homomorphism i with the crossed module action. Notice that right and left representations of a Lie group bundle coincide; hence, though Eq. (25) could be taken as a right representation, we emphasize that it is a left representation as the formula for the differential of right and left representations differ (cf. (10) and (11)).

The missing maps $\delta' : V \longrightarrow C(G, W)$ and $\delta' : C(\mathcal{G}_p^q, V) \longrightarrow C(\mathcal{G}_p^q \times G, W)$ are defined respectively by

$$(\delta' v)(g) := \rho_1(g)v, \tag{26}$$

for $v \in V$ and $g \in G$, and by

$$\delta'\omega(\gamma_1, ..., \gamma_q; g) = \rho_0^1(t_p(\gamma_1)...t_p(\gamma_q))^{-1}\rho_1(g)\omega(\gamma_1, ..., \gamma_q), \tag{27}$$

for $\omega \in C(\mathcal{G}_p^q, V)$, $\gamma_1, ..., \gamma_q \in \mathcal{G}_p$ and $g \in G$.

The next two lemmas justify how, in spite of the replacements in 0th degree, the complexes in the r-direction remain complexes.

3.6. Lemma.

$$V \xrightarrow{\delta'} C(G, W) \xrightarrow{\delta_{(1)}} C(G^2, W),$$

where δ' is defined by Eq. (26), is a complex.

Proof. We prove that, for $v \in V$, $\delta_{(1)}\delta' v = 0$. Let $g_0, g_1 \in G$, then

$$\begin{aligned}
\delta_{(1)}(\delta' v)(g_0, g_1) &= \rho_0^1(i(g_0))(\delta' v)(g_1) - (\delta' v)(g_0 g_1) + (\delta' v)(g_0) \\
&= \rho_0^1(i(g_0))\rho_1(g_1)v - \rho_1(g_0 g_1)v + \rho_1(g_0)v \\
&= (I + \rho_1(g_0)\phi)\rho_1(g_1)v - \rho_1(g_0 g_1)v + \rho_1(g_0)v,
\end{aligned}$$

which is zero due to Eq. (18).

∎

3.7. Lemma.

$$C(\mathcal{G}_p^q, V) \xrightarrow{\delta'} C(\mathcal{G}_p^q \times G, W) \xrightarrow{\delta_{(1)}} C(\mathcal{G}_p^q \times G^2, W),$$

where δ' is defined by Eq. (27), is a complex.

Proof. We prove that, for $\omega \in C(\mathcal{G}_p^q, V)$, $\delta_{(1)}\delta'\omega = 0$. Let $\gamma_1, ..., \gamma_q \in \mathcal{G}_p$ and $g_0, g_1 \in G$, then

$$\begin{aligned}
&\delta_{(1)}\delta'\omega(\gamma_1, ..., \gamma_q; g_0, g_1) \\
&\quad = (\gamma_1, ..., \gamma_q; g_0) \cdot \delta'\omega(\gamma_1, ..., \gamma_q; g_1) - \delta'\omega(\gamma_1, ..., \gamma_q; g_0 g_1) + \delta'\omega(\gamma_1, ..., \gamma_q; g_0) \\
&\quad = \rho_0^1(i(g_0^{t_p(\gamma_1)...t_p(\gamma_q)}))\rho_0^1(t_p(\gamma_1)...t_p(\gamma_q))^{-1}\rho_1(g_1)\omega(\gamma_1, ..., \gamma_q) + \\
&\qquad - \rho_0^1(t_p(\gamma_1)...t_p(\gamma_q))^{-1}\rho_1(g_0 g_1)\omega(\gamma_1, ..., \gamma_q) + \rho_0^1(t_p(\gamma_1)...t_p(\gamma_q))^{-1}\rho_1(g_0)\omega(\gamma_1, ..., \gamma_q) \\
&\quad = \rho_0^1(t_p(\gamma_1)...t_p(\gamma_q))^{-1}\Big(\rho_0^1(i(g_0))\rho_1(g_1)\omega(\gamma_1, ..., \gamma_q) - \rho_1(g_0 g_1)\omega(\gamma_1, ..., \gamma_q) + \rho_1(g_0)\omega(\gamma_1, ..., \gamma_q)\Big),
\end{aligned}$$

which is zero due to Eq.'s (18) and (19).

■

Consider $C^\bullet_{tot}(\mathcal{G}, \phi)$ as in (3). For expository purposes, we preliminarily define the total differential

$$\nabla := (-1)^p(\delta_{(1)} + \partial + (-1)^r\delta), \tag{28}$$

as though the grid were a triple complex. In the course of the remainder of this section, we study ∇^2 degree by degree and conclude that, despite ∂ and δ fail to commute, one can add corrections to (28) in order to have a complex

$$C^0_{tot}(\mathcal{G}, \phi) \xrightarrow{\nabla} C^1_{tot}(\mathcal{G}, \phi) \xrightarrow{\nabla} C^2_{tot}(\mathcal{G}, \phi) \xrightarrow{\nabla} C^3_{tot}(\mathcal{G}, \phi). \tag{29}$$

Then, we move on to study the cohomology of (29). We postpone a more general study of the relations among the differentials in the grid until the next section.

3.8. Degree 0. By definition $(C^n_{tot}(\mathcal{G}, \phi), \nabla)$ is concentrated in nonnegative degrees, thereby defining a complex in degree 0. Let $v \in C^0_{tot}(\mathcal{G}, \phi) = C_0^{0,0}(\mathcal{G}, \phi) = V$ and consider its differential

$$\nabla v = (\cancelto{0}{\partial v}, \delta v, \delta' v) \in C^1(\mathcal{G}, \phi) = C_0^{1,0}(\mathcal{G}, \phi) \oplus C_0^{0,1}(\mathcal{G}, \phi) \oplus C_1^{0,0}(\mathcal{G}, \phi).$$

If v is a 0-cocycle, then

$$\rho_0^0(h)v = v \qquad \text{and} \qquad \rho_1(g)v = 0$$

for all $h \in H$ and all $g \in G$; therefore,

$$H^0_\nabla(\mathcal{G}, \phi) = V^{\mathcal{G}} := \{v \in V : \bar{\rho}_{(g,h)}(0, v) = (0, v), \quad \forall (g, h) \in G \rtimes H \cong \mathcal{G}\},$$

where $\bar{\rho}$ is the honest representation of Proposition 2.20.

3.9. Degree 1. A 1-cochain λ is a triple $(v, \lambda_0, \lambda_1) \in C^1_{tot}(\mathcal{G}, \phi) = V \oplus C(H, V) \oplus C(G, W)$ whose differential has six entries. Adopting the convention that $(\nabla\lambda)^{p,q}_r \in C^{p,q}_r(\mathcal{G}, \phi)$,

$$(\nabla\lambda)^{0,2}_0 = \delta\lambda_0$$

$$(\nabla\lambda)^{1,1}_0 = \partial\lambda_0 - \delta v \qquad (\nabla\lambda)^{0,1}_1 = \delta'\lambda_0 - \delta\lambda_1$$

$$(\nabla\lambda)^{2,0}_0 = -\partial v = -v \qquad (\nabla\lambda)^{1,0}_1 = \partial\lambda_1 - \delta' v = -\delta' v \qquad (\nabla\lambda)^{0,0}_2 = \delta_{(1)}\lambda_1.$$

Let $v \in V$ and put $\lambda = \nabla v$. With the exception $(\nabla^2 v)^{0,1}_1$ and $(\nabla^2 v)^{1,1}_0$, all components of $\nabla^2 v$ vanish by definition. In the next lemma, we prove that $(\nabla^2 v)^{0,1}_1$ also vanishes.

3.10. Lemma.

$$\begin{tikzcd} C(H,V) \arrow[r, "\delta'"] & C(H \times G, W) \\ V \arrow[u, "\delta"] \arrow[r, "\delta'"'] & C(G,W), \arrow[u, "\delta"'] \end{tikzcd}$$

commutes.

Proof. Let $v \in V$ and $(h; g) \in H \times G$, then

$$\begin{aligned}(\delta\delta' v)(h; g) &= (\delta' v)(g^h) - \rho^1_H(h; g)(\delta' v)(g) = \rho_1(g^h)v - \rho^1_0(h)^{-1}\rho_1(g)v \\ &= \rho^1_0(h)^{-1}\rho_1(g)(\rho^0_0(h)v - v) = \rho^1_0(h)^{-1}\rho_1(g)(\delta v)(h) = (\delta'\delta v)(h; g).\end{aligned}$$

■

In fact, since for $p > 0$ the action of $\mathcal{G}_p$ on G and the right representation $\rho^1_{\mathcal{G}_p}$ are respectively pull-backs along t_p of the action of H on G and the right representation ρ^1_H, the proof of Lemma 3.10 implies the following corollary.

3.11. Corollary.

$$\begin{tikzcd} C(\mathcal{G}_p,V) \arrow[r, "\delta'"] & C(\mathcal{G}_p \times G, W) \\ V \arrow[u, "\delta"] \arrow[r, "\delta'"'] & C(G,W), \arrow[u, "\delta"'] \end{tikzcd}$$

commutes.

$(\nabla^2 v)^{1,1}_0$, as it is, does not vanish. Let $\gamma \in \mathcal{G}$ and let $(g, h) \in G \rtimes H$ be its image under the isomorphism of Remark 2.1, then

$$\begin{aligned}(\nabla^2 v)^{1,1}_0(\gamma) &= (\partial\delta v)(\gamma) = (\delta v)(h) - (\delta v)(hi(g)) \\ &= \rho^0_0(h)(v - \rho^0_0(i(g))v) = -\rho^0_0(h)\big(\phi \circ \rho_1(g)v\big),\end{aligned}$$

where the last equality is the first part of Eq. (19). Let

$$\Delta : C_1^{0,0}(\mathcal{G},\phi) \longrightarrow C_0^{1,1}(\mathcal{G},\phi)$$

be defined by

$$\Delta\omega(\gamma) := \rho_0^0(h)\circ\phi(\omega(g)), \tag{30}$$

for $\omega \in C(G,W)$ and $\gamma \in \mathcal{G}$. Here, $(g,h) \in G\rtimes H$ is the image of γ under the isomorphism of Remark 2.1.

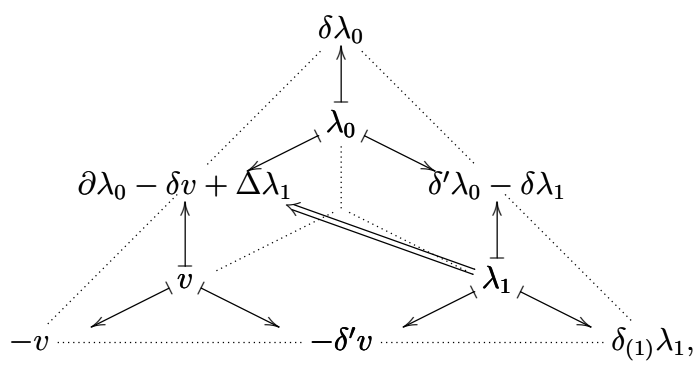
3.12. Remark. Observe that Δ is related to the structural map ϱ of the representation up to homotopy induced by the 2-representation (see Remark 2.21): $\omega \in C(G,W)$ defines the bundle map $\bar{\omega} : H\times G \longrightarrow H\times W$, $\bar{\omega}(h;g) = (h;\omega(g))$. Then, correctly interpreted, $\Delta\omega = \varrho\circ\bar{\omega}$.

Adding Δ to Eq. (28), makes $(\nabla^2 v)_0^{1,1} = 0$ as

$$(\Delta\delta' v)(\gamma) = \rho_0^0(h)\circ\phi((\delta' v)(g)) = \rho_0^0(h)\big(\phi\circ\rho_1(g)v\big),$$

for all $\gamma = (g,h) \in G\rtimes H \cong \mathcal{G}$.

Schematically, the updated differential of the 1-cochain λ is



where the solid arrows represent the grid differentials and the double arrow represents Δ.

If $(v,\lambda_0,\lambda_1) \in C^1_{tot}(\mathcal{G},\phi)$ is a 1-cocylce, then $v = 0$, λ_0 is a crossed homomorphism of H into V with respect to ρ_0^0, and λ_1 is a crossed homomorphisms of G into W with respect to $\rho_0^1\circ i$. In symbols,

$$\lambda_0(h_0h_1) = \lambda_0(h_0) + \rho_0^0(h_0)\lambda_0(h_1), \qquad \forall h_0,h_1 \in H, \tag{31}$$

$$\lambda_1(g_0g_1) = \lambda_1(g_0) + \rho_0^1(i(g_0))\lambda_1(g_1), \qquad \forall g_0,g_1 \in G. \tag{32}$$

Additionally, the following relations hold for every $\gamma \in \mathcal{G}$ and all $(h;g) \in H\times G$:

$$(\partial\lambda_0 + \Delta\lambda_1)(\gamma) = 0, \qquad (\delta'\lambda_0 - \delta\lambda_1)(h;g) = 0.$$

If $(g,h) \in G\rtimes H$ is the image of γ under the isomorphism of Remark 2.1, these relations are respectively

$$\lambda_0(h) + \rho_0^0(h)\circ\phi(\lambda_1(g)) = \lambda_0(hi(g)), \tag{33}$$

and

$$\rho_0^1(h)^{-1}\rho_1(g)\lambda_0(h) + \rho_0^1(h)^{-1}\lambda_1(g) = \lambda_1(g^h). \tag{34}$$

Eq. (33) implies that the map

$$\bar{\lambda} : \mathcal{G} \longrightarrow W \oplus V : (g,h) \longrightarrow (\rho_0^1(h)\lambda_1(g), \lambda_0(h))$$

respects both the source and the target; indeed, when $h = 1$, it implies the commutativity of

$$\begin{tikzcd} G \arrow[r, "\lambda_1"] \arrow[d, "i"'] & W \arrow[d, "\phi"] \\ H \arrow[r, "\lambda_0"'] & V. \end{tikzcd}$$

In fact, $\bar{\lambda}$ is a functor. Let $(\gamma_0, \gamma_1) \in \mathcal{G}_2$ and let $(g_0, hi(g_1)), (g_1, h) \in G \rtimes H$ be their respective images under the isomorphism of Remark 2.1. $\bar{\lambda}(\gamma_0)$ and $\bar{\lambda}(\gamma_1)$ are composable in the 2-vector space; indeed, combining Eq.'s (33) and (17), one gets

$$\lambda_0(hi(g)) = \lambda_0(h) + \rho_0^0(h)\phi(\lambda_1(g)) = \lambda_0(h) + \phi\big(\rho_0^1(h)\lambda_1(g)\big).$$

Using Eq. (32), one computes the composition of $\bar{\lambda}(\gamma_0)$ and $\bar{\lambda}(\gamma_1)$ to be

$$\begin{aligned} \bar{\lambda}(\gamma_0)\bar{\lambda}(\gamma_1) &= \big(\rho_0^1(h)\big(\lambda_1(g_1) + \rho_0^1(i(g_1))\lambda_1(g_0)\big), \lambda_0(h)\big) \\ &= (\rho_0^1(h)\lambda_1(g_1g_0), \lambda_0(h)) = \bar{\lambda}(g_1g_0, h) = \bar{\lambda}(\gamma_0 \bowtie \gamma_1), \end{aligned}$$

yielding the claim. Further using Eq. (34), one shows that $\bar{\lambda}$ is a crossed homomorphism into $W \oplus V$ with respect to $\bar{\rho}$:

$$\begin{aligned} \bar{\lambda}(\gamma_0 \between \gamma_1) &= \bar{\lambda}(g_0^{h_1}g_1, h_0h_1) = (\rho_0^1(h_0h_1)\lambda_1(g_0^{h_1}g_1), \lambda_0(h_0h_1)) \\ &= \big(\rho_0^1(h_0h_1)\big(\lambda_1(g_0^{h_1}) + \rho_0^1(i(g_0^{h_1}))\lambda_1(g_1)\big), \lambda_0(h_0) + \rho_0^0(h_0)\lambda_0(h_1)\big) \\ &= \big(\rho_0^1(h_0)\big(\rho_1(g_0)\lambda_0(h_1) + \lambda_1(g_0) + \rho_0^1(i(g_0)h_1)\lambda_1(g_1)\big), \lambda_0(h_0) + \rho_0^0(h_0)\lambda_0(h_1)\big) \\ &= \bar{\lambda}(\gamma_0) + \bar{\rho}_{(g_0,h_0)}\bar{\lambda}(\gamma_1) \end{aligned}$$

A coboundary ∇v, will induce a crossed homomorphism-functor that can also be seen as $\bar{\rho}_{(g,h)}(0, v)$. We could not find a terminology for these in the literature; hence, we introduce the following definitions by analogy. We use the notation conventions of this section.

3.13. Definition. *The space of crossed functors of a Lie 2-group $\mathcal{G}$ with respect to a 2-representation ρ on the 2-vector $\mathbb{V} = W \xrightarrow{\phi} V$ is defined to be*

$$Cr\mathrm{Hom}(\mathcal{G}, \phi) := \{\bar{\lambda} \in \mathrm{Hom}_{Gpd}(\mathcal{G}, \mathbb{V}) : \bar{\lambda}(g,h) = (\rho_0^1(h)\lambda_1(g), \lambda_0(h))$$

$$\text{is a crossed homomorphism with respect to } \bar{\rho}\}.$$

The space of principal crossed functors is defined to be

$$PCr\mathrm{Hom}(\mathcal{G},\phi) := \{\bar{\lambda} \in CrHom(\mathcal{G},\phi) : \bar{\lambda}(g,h) = \bar{\rho}_{(g,h)}(0,v) \text{ for some } v \in V\}.$$

With these definitions,

$$H^1_\nabla(\mathcal{G},\phi) = Cr\mathrm{Hom}(\mathcal{G},\phi)/PCr\mathrm{Hom}(\mathcal{G},\phi).$$

3.14. Degree 2. A 2-cochain $\vec{\omega}$ is a 6-tuple $(v,\varphi,\omega_0,\lambda,\alpha,\omega_1)$, where

$$\omega_0 \in C(H^2,V)$$

$$\varphi \in C(\mathcal{G},V) \qquad \alpha \in C(H\times G,W)$$

$$v \in V \qquad \lambda \in C(G,W) \qquad \omega_1 \in C(G^2,W).$$

The coordinates of the differential $\nabla\vec{\omega}$ are

$$(\nabla\vec{\omega})^{3,0}_0 = \partial v = 0 \qquad (\nabla\vec{\omega})^{0,3}_0 = \delta\omega_0 \qquad (\nabla\vec{\omega})^{0,0}_3 = \delta_{(1)}\omega_1$$

$$(\nabla\vec{\omega})^{1,2}_0 = \partial\omega_0 - \delta\varphi \qquad (\nabla\vec{\omega})^{0,2}_1 = \delta'\omega_0 - \delta\alpha$$

$$(\nabla\vec{\omega})^{2,1}_0 = \delta v - \partial\varphi \quad (\nabla\vec{\omega})^{1,1}_1 = \partial\alpha + \delta\lambda - \delta'\varphi \quad (\nabla\vec{\omega})^{0,1}_2 = \delta_{(1)}\alpha + \delta\omega_1$$

$$(\nabla\vec{\omega})^{2,0}_1 = \delta' v - \cancelto{\lambda}{\partial\lambda} \qquad (\nabla\vec{\omega})^{1,0}_2 = \cancelto{0}{\partial\omega_1} - \delta_{(1)}\lambda.$$

Let $\lambda = (v,\lambda_0,\lambda_1) \in C^1_{tot}(\mathcal{G},\phi)$. We study ∇^2 applied to each coordinate of λ separately:

In $\nabla^2 v$, $(\nabla^2 v)^{3,0}_0 = (\nabla^2 v)^{1,2}_0 = 0$, because ∂ and δ are differentials. Moreover, $(\nabla^2 v)^{1,0}_2 = 0$ due to Lemma 3.6, and $(\nabla^2 v)^{1,1}_1 = 0$ due to Corollary 3.11. By definition, $\partial : C^{1,0}_r(\mathcal{G},\phi) \longrightarrow C^{2,0}_r(\mathcal{G},\phi)$ is the identity; hence, $(\nabla^2 v)^{2,0}_1 = 0$ trivially. $(\nabla^2 v)^{2,1}_0$, as it is, does not vanish. Let $(g_1,g_2,h) \in G^2\times H \cong \mathcal{G}_2$, then

$$\begin{aligned}(\nabla^2 v)^{2,1}_0(g_1,g_2,h) &= (\partial\delta v - \delta\partial v)(g_1,g_2,h) = \rho^0_0(hi(g_2))v - v - \big(\rho^0_0(hi(g_2g_1))v - v\big)\\ &= \rho^0_0(hi(g_2))\big(v - \rho^0_0(i(g_1))v\big) = -\rho^0_0(hi(g_2))\big(\phi\circ\rho_1(g_1)v\big),\end{aligned}$$

where the last equality is the first part of Eq. (19). Let

$$\Delta : C^{1,0}_1(\mathcal{G},\phi) \longrightarrow C^{2,1}_0(\mathcal{G},\phi)$$

be defined by

$$\Delta\omega(g_1,g_2,h) := \rho^0_0(hi(g_2))\circ\phi(\omega(g_1)), \tag{35}$$

for $\omega \in C(G,W)$ and $(g_1,g_2,h) \in G^2\times H \cong \mathcal{G}_2$ under the isomorphism of Remark 2.5. Subtracting Δ from Eq. (28), makes $(\nabla^2 v)^{2,1}_0 = 0$ as

$$-\Delta(-\delta' v)(g_1,g_2,h) = \rho^0_0(hi(g_2))\circ\phi((\delta' v)(g)) = \rho^0_0(hi(g_2))\big(\phi\circ\rho_1(g_1)v\big),$$

for all $(g_1, g_2, h) \in \mathcal{G}_2$. All other components vanish from the onset.

In $\nabla^2\lambda_0$, $(\nabla^2\lambda_0)_0^{0,3} = (\nabla^2\lambda_0)_0^{2,1} = 0$, because δ and ∂ are differentials. Moreover, $(\nabla^2\lambda_0)_2^{0,1} = 0$ due to Lemma 3.6. Let $(h_1, h_2; g) \in H^2 \times G$, then

$$\begin{aligned}(\delta\delta'\lambda_0)(h_1, h_2; g) &= (\delta'\lambda_0)(h_2; g^{h_1}) - (\delta'\lambda_0)(h_1h_2; g) + \rho_0^1(h_2)^{-1}(\delta'\lambda_0)(h_1; g)\\ &= \rho_0^1(h_2)^{-1}\rho_1(g^{h_1})\lambda_0(h_2) + \rho_0^1(h_1h_2)^{-1}\rho_1(g)\big(-\lambda_0(h_1h_2) + \lambda_0(h_1)\big)\\ &= \rho_0^1(h_1h_2)^{-1}\rho_1(g)\big(\rho_0^0(h_1)\lambda_0(h_2) - \lambda_0(h_1h_2) + \lambda_0(h_1)\big) = (\delta'\delta\lambda_0)(h_1, h_2; g)\end{aligned}$$

and $(\nabla^2\lambda_0)_1^{0,2} = 0$. This computation can be easily generalized to prove the following:

3.15. Lemma.

$$\begin{tikzcd} C(\mathcal{G}_p^{q+1}, V) \arrow[r, "\delta'"] & C(\mathcal{G}_p^{q+1} \times G, W) \\ C(\mathcal{G}_p^{q}, V) \arrow[u, "\delta"] \arrow[r, "\delta'"'] & C(\mathcal{G}_p^{q} \times G, W), \arrow[u, "\delta"'] \end{tikzcd}$$

commutes for all $q > 0$.

Proof. Let $\omega \in C(\mathcal{G}_p^q, V)$, $\vec{\gamma} = (\gamma_0, ..., \gamma_q)^T \in \mathcal{G}_p^{q+1}$ and $g \in G$, then

$$\begin{aligned}\delta'\delta\omega(\vec{\gamma}; g) &= \rho_0^1(t_p(\gamma_0)...t_p(\gamma_q))^{-1}\rho_1(g)\delta\omega(\vec{\gamma})\\ &= \rho_0^1(t_p(\gamma_0)...t_p(\gamma_q))^{-1}\rho_1(g)\Big(\rho_0^0(t_p(\gamma_0))\omega(\delta_0\vec{\gamma}) + \sum_{j=1}^{q+1}(-1)^j\omega(\delta_j\vec{\gamma})\Big),\end{aligned}$$

and

$$\begin{aligned}\delta\delta'\omega(\vec{\gamma}; g) &= \delta'\omega(\delta_0\vec{\gamma}; g^{t_p(\gamma_0)}) + \sum_{j=1}^{q}(-1)^j\delta'\omega(\delta_j\vec{\gamma}; g) + (-1)^{q+1}\rho_{\mathcal{G}_p}^1(\gamma_q; g)\delta'\omega(\delta_{q+1}\vec{\gamma}; g)\\ &= \rho_0^1(t_p(\gamma_1)...t_p(\gamma_q))^{-1}\rho_1(g^{t_p(\gamma_0)})\omega(\delta_0\vec{\gamma}) +\\ &\quad + \sum_{j=1}^{q}(-1)^j\rho_0^1(t_p(\gamma_0)...t_p(\gamma_{j-1} \boxtimes \gamma_j)...t_p(\gamma_q))^{-1}\rho_1(g)\omega(\delta_j\vec{\gamma}) +\\ &\quad + (-1)^{q+1}\rho_0^1(t_p(\gamma_q))^{-1}\rho_0^1(t_p(\gamma_0)...t_p(\gamma_{q-1}))^{-1}\rho_1(g)\omega(\delta_{q+1}\vec{\gamma})\\ &= \rho_0^1(t_p(\gamma_0)...t_p(\gamma_q))^{-1}\rho_1(g)\Big(\rho_0^0(t_p(\gamma_0))\omega(\delta_0\vec{\gamma}) + \sum_{j=1}^{q+1}(-1)^j\omega(\delta_j\vec{\gamma})\Big).\end{aligned}$$

■

Let $(\gamma; f) \in \mathcal{G} \times G$ and let $(g, h) \in G \rtimes H$ be the image of γ under the isomorphism of Remark 2.1, then

$$\begin{aligned}(\partial\delta'\lambda_0)(\gamma; f) &= \rho_0^1(i(g))^{-1}(\delta'\lambda_0)(h; f) - (\delta'\lambda_0)(hi(g); f)\\ &= \rho_0^1(i(g))^{-1}\rho_0^1(h)^{-1}\rho_1(f)\lambda_0(h) - \rho_0^1(hi(g))^{-1}\rho_1(f)\lambda_0(hi(g))\\ &= \rho_0^1(hi(g))^{-1}\rho_1(f)\big(\lambda_0(h) - \lambda_0(hi(g))\big) = (\delta'\partial\lambda_0)(\gamma; f)\end{aligned}$$

and $(\nabla^2\lambda_0)_1^{1,1} = 0$. This computation can be easily generalized to prove the following:

3.16. Lemma.

$$\begin{tikzcd} C(\mathcal{G}_p^q \times G, W) \arrow[r, "\partial"] & C(\mathcal{G}_{p+1}^q \times G, W) \\ C(\mathcal{G}_p^q, V) \arrow[u, "\delta'"] \arrow[r, "\partial"'] & C(\mathcal{G}_{p+1}^q, V), \arrow[u, "\delta'"'] \end{tikzcd}$$

commutes for all $q > 1$.

Proof. We adopt the convention that, for $\gamma_a = (\,\gamma_{a0} \quad \cdots \quad \gamma_{ap}\,) \in \mathcal{G}_{p+1}$ and $\begin{pmatrix} g_{ab} \\ h_{ab} \end{pmatrix}$ is the image of γ_{ab} under the isomorphism of Remark 2.1.

Let $\omega \in C(\mathcal{G}_p^q, V)$, $\vec{\gamma} = (\gamma_1, ..., \gamma_q)^T \in \mathcal{G}_{p+1}^q$ and $f \in G$, then

$$\begin{aligned}\partial\delta'\omega(\vec{\gamma}; f) &= \rho_0^1(i(pr_G(\gamma_{10} \barwedge ... \barwedge \gamma_{q0})))^{-1}\delta'\omega(\partial_0\vec{\gamma}; f) + \sum_{j=1}^{p+1}(-1)^j\delta'\omega(\partial_j\vec{\gamma}; f)\\ &= \rho_0^1(i(pr_G(\gamma_{10} \barwedge ... \barwedge \gamma_{q0})))^{-1}\rho_0^1(t_p(\partial_0\gamma_1)...t_p(\partial_0\gamma_q))^{-1}\rho_1(f)\omega(\partial_0\vec{\gamma}) +\\ &\quad + \sum_{j=1}^{p+1}(-1)^j\rho_0^1(t_p(\partial_j\gamma_1)...t_p(\partial_j\gamma_q))^{-1}\rho_1(f)\omega(\partial_j\vec{\gamma}).\end{aligned}$$

If $j > 0$, $t_p(\partial_j\gamma_a) = t(\gamma_{a0})$; otherwise, $t_p(\partial_0\gamma_a) = s(\gamma_{a0})$. Hence, since s and t are group homomorphisms,

$$\begin{aligned}\partial\delta'\omega(\vec{\gamma}; f) &= \rho_0^1(t(\gamma_{10})...t_p(\gamma_{q0}))^{-1}\rho_1(f)\sum_{j=0}^{p+1}(-1)^j\omega(\partial_j\vec{\gamma})\\ &= \rho_0^1(t_{p+1}(\gamma_1)...t_{p+1}(\gamma_q))^{-1}\rho_1(f)\partial\omega(\vec{\gamma}) = \delta'\partial\omega(\vec{\gamma}; f).\end{aligned}$$

■

$(\nabla^2\lambda_0)^{1,2}_0$, as it is, does not vanish. Let $(\gamma_1,\gamma_2)\in\mathcal{G}^2$ and let $(g_k,h_k)\in G\rtimes H$ be the image of γ_k under the isomorphism of Remark 2.1 for $k\in\{1,2\}$, then

$$\begin{aligned}(\nabla^2\lambda_0)^{1,2}_0(\gamma_1,\gamma_2) &= (\partial\delta\lambda_0-\delta\partial\lambda_0)(\gamma_1,\gamma_2)\\ &= \rho^0_0(h_1)\big(\lambda_0(h_2)-\rho^0_0(i(g_1))\lambda_0(h_2)\big) = -\rho^0_0(h_1)\big(\phi\circ\rho_1(g_1)\lambda_0(h_2)\big),\end{aligned}$$

where the last equality is the first part of Eq. (19). Let

$$\Delta : C^{0,1}_1(\mathcal{G},\phi)\longrightarrow C^{1,2}_0(\mathcal{G},\phi)$$

be defined by

$$\Delta\omega(\gamma_1,\gamma_2) := \rho^0_0(h_1h_2)\circ\phi(\omega(h_2;g_1)), \tag{36}$$

for $\omega\in C(H\times G,W)$ and $(\gamma_1,\gamma_2)\in\mathcal{G}^2$. Here, $(g_k,h_k)\in G\rtimes H$ is the image of γ_k under the isomorphism of Remark 2.1 for $k\in\{1,2\}$. Adding Δ to Eq. (28), makes $(\nabla^2\lambda_0)^{1,2}_0=0$ as a consequence of Eq. (17) and that for all $(\gamma_1,\gamma_2)\in\mathcal{G}^2$

$$(\Delta\delta'\lambda_0)(\gamma_1,\gamma_2) = \rho^0_0(h_1h_2)\circ\phi((\delta'\lambda_0)(h_2;g_1)) = \rho^0_0(h_1h_2)\circ\phi\big(\rho^1_0(h_2)^{-1}\rho_1(g_1)\lambda_0(h_2)\big).$$

All other components vanish from the onset.

In $\nabla^2\lambda_1$, $(\nabla^2\lambda_1)^{0,0}_3=(\nabla^2\lambda_1)^{2,0}_1=(\nabla^2\lambda_1)^{0,2}_1=0$, because $\delta_{(1)}$, ∂ and δ are differentials. Since by definition, $\partial : C^{0,0}_r(\mathcal{G},\phi)\longrightarrow C^{1,0}_r(\mathcal{G},\phi)$ is zero, $(\nabla^2\lambda_1)^{1,0}_2=0$ trivially. Moreover, $\delta_{(1)}$ commutes with δ; hence, $(\nabla^2\lambda_1)^{0,1}_2=0$. Indeed, let $(h;g_1,g_2)\in H\times G^2$, then

$$\begin{aligned}(\delta\delta_{(1)}\lambda_1)(h;g_1,g_2) &= (\delta_{(1)}\lambda_1)(g_1^h,g_2^h)-\rho^1_0(h)^{-1}(\delta_{(1)}\lambda_1)(g_1,g_2)\\ &= \rho^1_0(i(g_1^h))\lambda_1(g_2^h)-\lambda_1((g_1g_2)^h)+\lambda_1(g_1^h)+\\ &\qquad -\rho^1_0(h)^{-1}\big(\rho^1_0(i(g_1))\lambda_1(g_2)-\lambda_1(g_1g_2)+\lambda_1(g_1)\big)\\ &= \rho^1_0(i(g_1^h))\big(\lambda_1(g_2^h)-\rho^1_0(h)^{-1}\lambda_1(g_2)\big)-(\delta\lambda_1)(h;g_1g_2)+(\delta\lambda_1)(h;g_1)\\ &= (\delta_{(1)}\delta\lambda_1)(h;g_1,g_2).\end{aligned}$$

$(\nabla^2\lambda_1)^{1,1}_1$, $(\nabla^2\lambda_1)^{2,1}_0$ and $(\nabla^2\lambda_1)^{1,2}_0$, as they are, do not vanish. Let $(\gamma;f)\in\mathcal{G}\times G$ and let $(g,h)\in G\rtimes H$ be the image of γ under the isomorphism of Remark 2.1, then

$$\begin{aligned}(\nabla^2\lambda_1)^{1,1}_1(\gamma;f) &= (\delta\cancelto{0}{\partial\lambda_1}-\partial\delta\lambda_1-\delta'\Delta\lambda_1)(\gamma;f)\\ &= \lambda_1(f^{hi(g)})-\rho^1_0(i(g))^{-1}\lambda_1(f^h)-\rho^1_0(hi(g))^{-1}\rho_1(f)\rho^0_0(h)\phi\lambda_1(g)\\ &= \lambda_1(f^{hi(g)})-\rho^1_0(i(g))^{-1}\big(\lambda_1(f^h)+(\rho^1_0(i(f^h))-I)\lambda_1(g)\big)\end{aligned}$$

where the last equality follows from Eq. (20) combined with the second part of Eq. (19). Let

$$\Delta : C^{0,0}_2(\mathcal{G},\phi)\longrightarrow C^{1,1}_1(\mathcal{G},\phi)$$

be defined by

$$\Delta\omega(\gamma;f) := \rho_0^1(i(g)))^{-1}\omega(f^h,g) + \omega(g^{-1},f^h g) - \omega(g^{-1},g), \tag{37}$$

for $\omega \in C(G^2,W)$ and $(\gamma;f) \in \mathcal{G}\times G$. Here, $(g,h) \in G\rtimes H$ is the image of γ under the isomorphism of Remark 2.1. Adding Δ to Eq. (28), makes $(\nabla^2\lambda_1)_1^{1,1} = 0$ as, using the cocycle equation $\delta^2_{(1)}\lambda_1 = 0$,

$$\begin{aligned}(\Delta\delta_{(1)}\lambda_1)(\gamma;f) &= \rho_0^1(i(g)))^{-1}(\delta_{(1)}\lambda_1)(f^h,g) + (\delta_{(1)}\lambda_1)(g^{-1},f^h g) - (\delta_{(1)}\lambda_1)(g^{-1},g)\\ &= (\delta_{(1)}\lambda_1)(g^{-1},f^h) + (\delta_{(1)}\lambda_1)(g^{-1}f^h,g) - (\delta_{(1)}\lambda_1)(g^{-1},g)\\ &= \rho_0^1(i(g))^{-1}\lambda_1(f^h) + \rho_0^1(i(g^{-1}f^h))\lambda_1(g) - \lambda_1(f^{hi(g)}) - \rho_0^1(i(g))^{-1}\lambda_1(g)\end{aligned}$$

for all $(\gamma;f) \in \mathcal{G}\times G$.

Let $(g_1,g_2,h) \in G^2\times H \cong \mathcal{G}_2$, then

$$\begin{aligned}(\nabla^2\lambda_1)_0^{2,1}(g_1,g_2,h) &= -(\partial\Delta\lambda_1 + \Delta\cancelto{0}{\partial\lambda_1})(g_1,g_2,h)\\ &= -\Delta\lambda_1(g_2,h) + \Delta\lambda_1(g_2g_1,h) - \Delta\lambda_1(g_1,hi(g_2))\\ &= -\rho_0^0(h)\circ\phi\big(\lambda_1(g_2) - \lambda_1(g_2g_1) + \rho_0^1(i(g_2))\lambda_1(g_1)\big).\end{aligned}$$

Let

$$\Delta_{2,1} : C_2^{0,0}(\mathcal{G},\phi) \longrightarrow C_0^{2,1}(\mathcal{G},\phi)$$

be defined by

$$\Delta_{2,1}\omega(g_1,g_2,h) := \rho_0^0(h)\circ\phi(\omega(g_2,g_1)), \tag{38}$$

for $\omega \in C(G^2,W)$ and $(g_1,g_2,h) \in G^2\times H \cong \mathcal{G}_2$. Adding $\Delta_{2,1}$ to Eq. (28), makes $(\nabla^2\lambda_1)_0^{2,1} = 0$ as

$$(\Delta_{2,1}\delta_{(1)}\lambda_1)(g_1,g_2,h) = \rho_0^0(h)\circ\phi\big(\delta_{(1)}\lambda_1(g_2,g_1)\big)$$

for all $(g_1,g_2,h) \in G^2\times H \cong \mathcal{G}_2$.

Let $(\gamma_1,\gamma_2) \in \mathcal{G}^2$ and let $(g_k,h_k) \in G\rtimes H$ be the image of γ_k under the isomorphism of Remark 2.1 for $k \in \{1,2\}$, then

$$\begin{aligned}(\nabla^2\lambda_1)_0^{1,2}(\gamma_1,\gamma_2) &= -(\delta\Delta\lambda_1 + \Delta\delta\lambda_1)(\gamma_1,\gamma_2)\\ &= -\rho_0^0(h_1h_2)\circ\phi\big(\rho_0^1(i(g_1^{h_2}))\lambda_1(g_2) - \lambda_1(g_1^{h_2}g_2) + \lambda_1(g_1^{h_2})\big)\end{aligned}$$

Let

$$\Delta_{1,2} : C_2^{0,0}(\mathcal{G},\phi) \longrightarrow C_0^{1,2}(\mathcal{G},\phi)$$

be defined by

$$\Delta_{1,2}\omega(\gamma_1,\gamma_2) := \rho_0^0(h_1h_2)\circ\phi(\omega(g_1^{h_2},g_2)), \tag{39}$$

for $\omega \in C(G^2, W)$ and $(\gamma_1, \gamma_2) \in \mathcal{G}^2$. Here, $(g_k, h_k) \in G \times H$ is the image of γ_k under the isomorphism of Remark 2.1 for $k \in \{1, 2\}$. Adding $\Delta_{1,2}$ to Eq. (28), makes $(\nabla^2 \lambda_1)_0^{1,2} = 0$ as

$$(\Delta_{1,2}\delta_{(1)}\lambda_1)(\gamma_1, \gamma_2) = \rho_0^0(h_1 h_2) \circ \phi\big(\delta_{(1)}\lambda_1(g_1^{h_2}, g_1)\big)$$

for all $(\gamma_1, \gamma_2) \in \mathcal{G}^2$. The remaining two components vanish from the onset.

Ultimately, the updated differential ∇ is the one appearing in Eq. (4). ∇ squares to zero in the snapshot complex (29) whose building blocks we represent schematically:

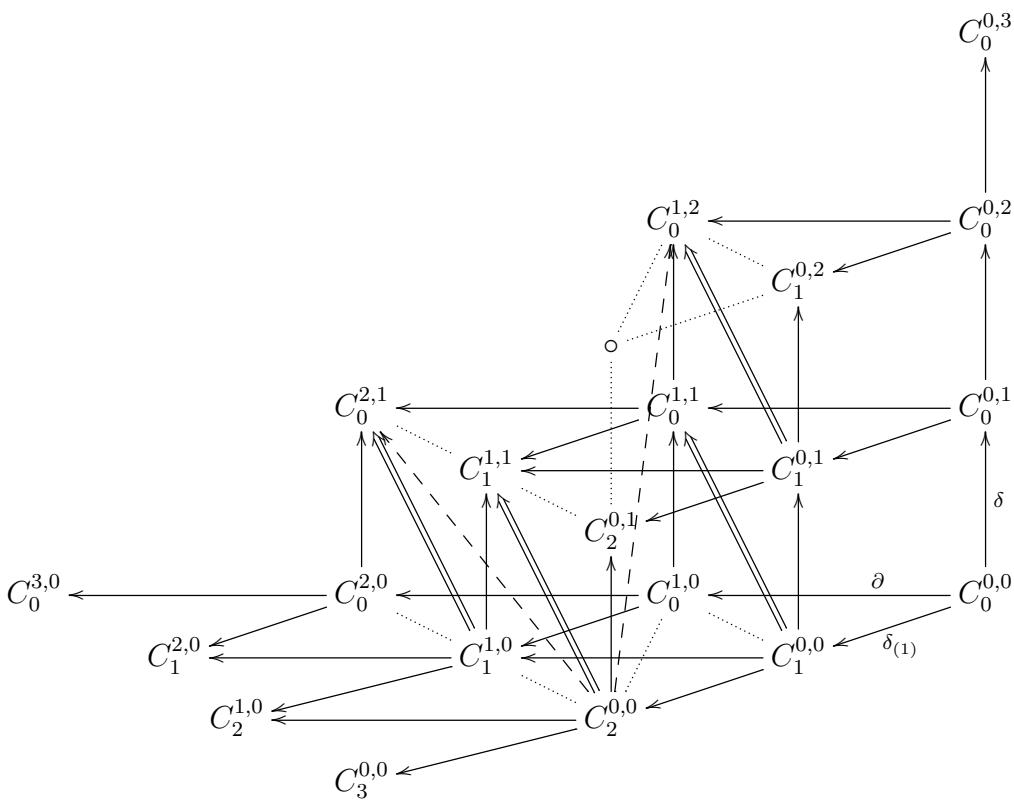


Fig. (40)

In this diagram, the double arrows and the dashed arrows represent respectively the first difference maps

$$\Delta : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_{r-1}^{p+1,q+1}(\mathcal{G}, \phi)$$

and, for $(a, b) \in \{(1, 2), (2, 1)\}$, the second difference maps

$$\Delta_{a,b} : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_{r-2}^{p+a,q+b}(\mathcal{G}, \phi),$$

which owe the ordinal in their names to the degree difference in the r-direction.

3.16.1. The cocycle equations. The second cohomology group $H^2_\nabla(\mathcal{G},\phi)$ is in one-to-one correspondence with the equivalence classes of split extensions of the Lie 2-group $\mathcal{G}$ by the 2-vector space $W \xrightarrow{\phi} V$. To make this patent, we express an extension in terms of simpler data – as in the proof of Proposition 2.23 –, and determine the equations these need to satisfy to build the extension back up.

3.17. Proposition. *Let ρ be a 2-representation of $G \xrightarrow{i} H$ on $W \xrightarrow{\phi} V$ and $(\check{\varphi},\omega_0,\alpha,\omega_1) \in C(G,V)\oplus C(H^2,V)\oplus C(H\times G,W)\oplus C(G^2,W)$. Let $\mathcal{E}_{(\check{\varphi},\omega_0,\alpha,\omega_1)}$ be the space in the middle of the sequence*

$$\begin{tikzcd}
1 \arrow[r] & W \arrow[r, hook, "j_1"] \arrow[d, "\phi"'] & G_{\rho_0^1\circ i} \ltimes^{\omega_1} W \arrow[r, "pr_1"] \arrow[d, "\epsilon"'] & G \arrow[r] \arrow[d, "i"] & 1 \\
1 \arrow[r] & V \arrow[r, hook, "j_0"'] & H_{\rho_0^0} \ltimes^{\omega_0} V \arrow[r, "pr_1"'] & H \arrow[r] & 1,
\end{tikzcd}$$

with

$$\epsilon(g,w) := (i(g), \phi(w) + \check{\varphi}(g)), \tag{41}$$

and

$$(g,w)^{(h,v)} := (g^h, \rho_0^1(h)^{-1}(w+\rho_1(g)v) + \alpha(h;g)), \qquad \text{for } (h,v)\in H\times V. \tag{42}$$

Then,

i) the product on $H_{\rho_0^0} \ltimes^{\omega_0} V$ (cf. Eq. (22)) is associative if and only if ω_0 is a group cocycle with respect to ρ_0^0;

ii) the product on $G_{\rho_0^1\circ i} \ltimes^{\omega_1} W$ is associative if and only if ω_1 is a group cocycle with respect to $\rho_0^1\circ i$;

iii) Eq. (41) defines a Lie group homomorphism if and only if for all $g_1,g_2\in G$,

$$\phi(\omega_1(g_1,g_2)) - \omega_0(i(g_1),i(g_2)) = \rho_0^0(i(g_1))\check{\varphi}(g_2) - \check{\varphi}(g_1g_2) + \check{\varphi}(g_1); \tag{43}$$

iv) Eq. (42) defines a right action if for all $h_1,h_2\in H$ and all $g\in G$,

$$\rho_0^1(h_1h_2)^{-1}\rho_1(g)\omega_0(h_1,h_2) = \rho_0^1(h_2)^{-1}\alpha(h_1;g) - \alpha(h_1h_2;g) + \alpha(h_2;g^{h_1}); \tag{44}$$

v) Eq. (41) defines an equivariant map if and only if for all $(h;g)\in H\times G$

$$\check{\varphi}(g^h) - \rho_0^0(h^{-1})\check{\varphi}(g) + \phi(\alpha(h;g)) = \rho_0^0(h^{-1})\omega_0(i(g),h) + \omega_0(h^{-1},i(g)h) - \omega_0(h^{-1},h), \tag{45}$$

vi) Peiffer equation holds if and only if for all $g_1, g_2 \in G$,

$$\rho_0^1(i(g_2))^{-1}\rho_1(g_1)\check{\varphi}(g_2)+\alpha(i(g_2);g_1) = \rho_0^1(i(g_2))^{-1}\omega_1(g_1,g_2)+\omega_1(g_2^{-1},g_1g_2)-\omega_1(g_2^{-1},g_2), \tag{46}$$

and

vii) Eq. (42) defines an action by automorphisms if for all $h \in H$ and all $g_1, g_2 \in G$,

$$\rho_0^1(h)^{-1}\omega_1(g_1,g_2) - \omega_1(g_1^h, g_2^h) = \rho_0^1(i(g_1^h))\alpha(h;g_2) - \alpha(h;g_1g_2) + \alpha(h;g_1). \tag{47}$$

Therefore, $\mathcal{E}_{(\check{\varphi},\omega_0,\alpha,\omega_1)}$ is a Lie 2-group extension if and only if i)-vii) hold.

3.18. Remark. As a consistency check, we prove that, when applied to the trivial 2-representation on $(0) \longrightarrow \mathbb{R}$, Proposition 3.17 is equivalent to Lemma 2.9. For this case, the only non-trivial equations in Proposition 3.17 are

i) $\delta\omega_0 = 0$,

ii) $-\omega_0(i(g_1), i(g_2)) = \check{\varphi}(g_2) - \check{\varphi}(g_1g_2) + \check{\varphi}(g_1)$, and

iii) $\check{\varphi}(g^h) - \check{\varphi}(g) = \omega_0(i(g), h) + \omega_0(h^{-1}, i(g)h) - \omega_0(h^{-1}, h)$.

Let $(F, f) \in \Omega^2(\mathcal{G})$ be a 2-cocycle, $\omega_0 := F$ and $\check{\varphi}(g) := f(g, 1)$. Then Eq. i) holds right away and Eq. ii) coincides with Eq. (13). Eq. iii) follows from evaluating $\delta f - \partial F = 0$ at

$$\begin{pmatrix} g & 1 \\ h^{-1} & h \end{pmatrix} : \cancel{f\begin{pmatrix} 1 \\ h \end{pmatrix}} - f\begin{pmatrix} g^h \\ 1 \end{pmatrix} + f\begin{pmatrix} g \\ h^{-1} \end{pmatrix} = F(h^{-1}, h) - F(h^{-1}i(g), h),$$

$$\begin{pmatrix} 1 & g \\ h^{-1} & 1 \end{pmatrix} : f\begin{pmatrix} g \\ 1 \end{pmatrix} - f\begin{pmatrix} g \\ h^{-1} \end{pmatrix} + \cancel{f\begin{pmatrix} 1 \\ h^{-1} \end{pmatrix}} = \cancel{F(h^{-1}, 1)} - F(h^{-1}, i(g)).$$

Adding these expressions together yields

$$f\begin{pmatrix} g \\ 1 \end{pmatrix} - f\begin{pmatrix} g^h \\ 1 \end{pmatrix} = F(h^{-1}, h) - F(h^{-1}i(g), h) - F(h^{-1}, i(g))$$

and, since $\delta F = 0$,

$$F(h^{-1}i(g), h) + F(h^{-1}, i(g)) = F(i(g), h) + F(h^{-1}, i(g)h),$$

and the claim follows. Conversely, suppose ω_0 and $\check{\varphi}$ verify Eq.'s i)-iii), and let $F := \omega_0$ and $f(g, h) := \omega_0(h, i(g)) + \check{\varphi}(g)$. Naturally, $\delta F = 0$ holds right away. Using successively Eq.'s ii) and i),

$$f\begin{pmatrix} g_2g_1 \\ h \end{pmatrix} = \omega_0(h, i(g_2g_1)) + \check{\varphi}(g_2g_1) = \omega_0(h, i(g_2g_1)) + \omega_0(i(g_2), i(g_1)) + \check{\varphi}(g_1) + \check{\varphi}(g_2)$$

$$= \omega_0(hi(g_2), i(g_1)) + \omega_0(h, i(g_2)) + \check{\varphi}(g_1) + \check{\varphi}(g_2) = f\begin{pmatrix} g_1 \\ hi(g_2) \end{pmatrix} + f\begin{pmatrix} g_2 \\ h \end{pmatrix};$$

hence, $\partial f = 0$. As a consequence of $\partial f = 0$ and Eq. ii),

$$\begin{aligned} f\begin{pmatrix} g_1^{h_2} g_2 \\ h_1 h_2 \end{pmatrix} &= \omega_0(h_1 h_2 i(g_1^{h_2}), i(g_2)) + \check{\varphi}(g_2) + \omega_0(h_1 h_2, i(g_1^{h_2})) + \check{\varphi}(g_1^{h_2}) \\ &= \omega_0(h_1 h_2 i(g_1^{h_2}), i(g_2)) + \check{\varphi}(g_2) + \omega_0(h_1 h_2, i(g_1^{h_2})) + \check{\varphi}(g_1) + \\ &\quad + \omega_0(i(g_1), h_2) + \omega_0(h_2^{-1}, i(g_1) h_2) - \omega_0(h_2^{-1}, h_2). \end{aligned}$$

Subtracting this expression from

$$f\begin{pmatrix} g_2 \\ h_2 \end{pmatrix} + f\begin{pmatrix} g_1 \\ h_1 \end{pmatrix} = \omega_0(h_2, i(g_2)) + \check{\varphi}(g_2) + \omega_0(h_1, i(g_1)) + \check{\varphi}(g_1),$$

one gets

$$\begin{aligned} \delta f\begin{pmatrix} g_1 & g_2 \\ h_1 & h_2 \end{pmatrix} &= \omega_0(h_2, i(g_2)) + \omega_0(h_1, i(g_1)) - \omega_0(h_1 i(g_1) h_2, i(g_2)) - \omega_0(h_1 h_2, i(g_1^{h_2})) + \\ &\quad - \omega_0(i(g_1), h_2) - \omega_0(h_2^{-1}, i(g_1) h_2) + \omega_0(h_2^{-1}, h_2). \end{aligned}$$

Since $\delta\omega_0(h_1 i(g_1), h_2, i(g_2)) - \delta\omega_0(h_1, i(g_1), h_2) = 0$,

$$\omega_0(h_2, i(g_2)) + \omega_0(h_1, i(g_1)) - \omega_0(h_1 i(g_1) h_2, i(g_2)) - \omega_0(i(g_1), h_2) = \omega_0(h_1, i(g_1) h_2) - \omega_0(h_1 i(g_1), h_2 i(g_2))$$

and since $\delta\omega_0(h_1 h_2, h_2^{-1}, i(g_2) h_2) - \delta\omega_0(h_1, h_2, h_2^{-1}) = 0$,

$$\omega_0(h_1, i(g_1) h_2) - \omega_0(h_1 h_2, i(g_1^{h_2})) - \omega_0(h_2^{-1}, i(g_1) h_2) = \omega_0(h_1, h_2) - \omega_0(h_2, h_2^{-1}).$$

Therefore,

$$\delta f\begin{pmatrix} g_1 & g_2 \\ h_1 & h_2 \end{pmatrix} = \omega_0(h_1, h_2) - \omega_0(h_2, h_2^{-1}) - \omega_0(h_1 i(g_1), h_2 i(g_2)) + \omega_0(h_2^{-1}, h_2) = \partial\omega_0\begin{pmatrix} g_1 & g_2 \\ h_1 & h_2 \end{pmatrix},$$

where the last equality follows from $\delta\omega_0(h_2^{-1}, h_2, h_2^{-1}) = \omega_0(h_2, h_2^{-1}) - \omega_0(h_2^{-1}, h_2) = 0$.

Next, we consider equivalent extensions $\mathcal{E}$ and $\mathcal{F}$ as in

$$\begin{tikzcd}
 & & E_1 \arrow[dd, "\epsilon"] \arrow[dr, dotted, "\psi_1"] \arrow[drr] & & \\
1 \arrow[r] & W \arrow[ur] \arrow[dd] \arrow[drr] & & F_1 \arrow[r] \arrow[dd, "f"] & G \arrow[r] \arrow[dd] & 1 \\
 & & E_0 \arrow[dr, dotted, "\psi_0"] \arrow[drr] & & & \\
1 \arrow[r] & V \arrow[ur] \arrow[drr] & & F_0 \arrow[r] & H \arrow[r] & 1.
\end{tikzcd}$$

Supposing that the extensions split, one can apply the decomposition of the proof of Proposition 2.23. Picking a splitting of either extension and composing it with the

isomorphism yields a splitting for the other. In this manner, the induced 2-representations of Proposition 2.23 are identical, and we identify both $\mathcal{E}$ and $\mathcal{F}$ with a corresponding (twisted) semi-direct product. We write the functor ψ in these coordinates as

$$\psi_k(z,a) = (\psi_k^{Gp}(z,a), \psi_k^{Vec}(z,a)),$$

for $k \in \{0,1\}$. Both ψ_0 and ψ_1 respect inclusions and projections; hence, $\psi_k(1,a) = (1,a)$ and $\psi_k^{Gp}(z,a) = z$. Further using that both ψ_k's are group homomorphisms together with the factorization $(z,a) = (1,a) \boxtimes (z,0)$,

$$\begin{aligned}\psi_k(z,a) &= \psi_k((1,a) \boxtimes (z,0)) = \psi_k(1,a) \boxtimes \psi_k(z,0) \\ &= (1,a) \boxtimes (z, \psi_k^{Vec}(z,0)) = (z, a + \cancel{\rho_k(1)}(\psi_k^{Vec}(z,0)) + \cancel{\omega_k(1,z)}),\end{aligned}$$

where ρ_k stands for ρ_0^0 when $k=0$ and for $\rho_0^1 \circ i$ when $k=1$. As a consequence,

$$\psi_k(z,a) = (z, a + \lambda_k(z)),$$

where the maps $\lambda_0 : H \longrightarrow V$ and $\lambda_1 : G \longrightarrow W$ are defined by $\psi_k^{Vec}(z,0)$ for $k=0$ and $k=1$ respectively.

3.19. Proposition. *Let ρ be a 2-representation of $G \xrightarrow{i} H$ on $W \xrightarrow{\phi} V$ and let $(\check{\varphi}, \omega_0, \alpha, \omega_1)$, $(\check{\varphi}', \omega_0', \alpha', \omega_1')$ be a pair of 4-tuples verifying the equations of Proposition 3.17. Then the extensions $\mathcal{E}_{(\check{\varphi},\omega_0,\alpha,\omega_1)}$ and $\mathcal{E}_{(\check{\varphi}',\omega_0',\alpha',\omega_1')}$ are isomorphic if and only if there are maps $\lambda_0 \in C(H,V)$ and $\lambda_1 \in C(G,W)$ verifying*

i) $\omega_0 - \omega_0' = \delta\lambda_0$. Explicitly, for all pairs $h_0, h_1 \in H$,

$$\omega_0(h_0,h_1) - \omega_0'(h_0,h_1) = \rho_0^0(h_0)\lambda_0(h_1) - \lambda_0(h_0h_1) + \lambda_0(h_0).$$

ii) $\omega_1 - \omega_1' = \delta\lambda_1$. Explicitly, for all pairs $g_0, g_1 \in G$,

$$\omega_1(g_0,g_1) - \omega_1'(g_0,g_1) = \rho_0^1(i(g_0))\lambda_1(g_1) - \lambda_1(g_0g_1) + \lambda_1(g_0).$$

iii) For all $g \in G$,

$$\check{\varphi}(g) - \check{\varphi}'(g) = \phi(\lambda_1(g)) - \lambda_0(i(g)). \tag{48}$$

iv) For all $h \in H$ and all $g \in G$,

$$\alpha(h;g) - \alpha'(h;g) = \rho_0^1(h)^{-1}(\lambda_1(g) + \rho_1(g)\lambda_0(h)) - \lambda_1(g^h). \tag{49}$$

Proof. Items $i)$ and $ii)$ are the usual relations identifying isomorphic extensions with cohomologous cocycles of Lie groups. Define $\psi_k(z,a) := (z, a+\lambda_k(z))$ for $k \in \{0,1\}$. Then,

$$\begin{tikzcd} G_{\rho_0^1\circ i} \ltimes^{\omega_1} W \arrow[r, "\psi_1"] \arrow[d] & G_{\rho_0^1\circ i} \ltimes^{\omega_1'} W \arrow[d] \\ H_{\rho_0^0} \ltimes^{\omega_0} V \arrow[r, "\psi_0"'] & H_{\rho_0^0} \ltimes^{\omega_0'} V \end{tikzcd}$$

commutes if and only if

$$\big(i(g), \phi(w+\lambda_1(g)) + \check{\varphi}'(g)\big) = \big(i(g), \phi(w) + \check{\varphi}(g) + \lambda_0(i(g))\big),$$

which is equivalent to Eq. (48). On the other hand, the expressions

$$\begin{aligned} \psi_1\big((g,w)^{(h,v)}\big) &= \psi_1\big(g^h, \rho_0^1(h)^{-1}(w+\rho_1(g)v) + \alpha(h;g)\big) \\ &= \big(g^h, \rho_0^1(h)^{-1}(w+\rho_1(g)v) + \alpha(h;g) + \lambda_1(g^h)\big), \end{aligned}$$

and

$$\begin{aligned} \psi_1(g,w)^{\psi_0(h,v)} &= \big(g, w+\lambda_1(g)\big)^{(h,v+\lambda_0(h))} \\ &= \big(g^h, \rho_0^1(h)^{-1}\big(w+\lambda_1(g)+\rho_1(g)(v+\lambda_0(h))\big) + \alpha'(h;g)\big) \end{aligned}$$

coincide if and only if Eq. (49) holds. ∎

3.20. Remark. Continuing Remark 3.18, one can also prove that, when appropriately restricted to the trivial 2-representation on $(0) \longrightarrow \mathbb{R}$, Proposition 3.19 is equivalent to Lemma 2.10.

If $\vec{\omega} = (0, \varphi, \omega_0, \lambda, \alpha, \omega_1) \in C^2_{tot}(\mathcal{G}, \phi)$ is a 2-cocycle, then $(\nabla\vec{\omega})_1^{2,0} = 0$ implies $\lambda = 0$. Furthermore, $(\nabla\vec{\omega})_0^{2,1} = 0$ reads

$$\Delta_{2,1}\omega_1 = \partial\varphi;$$

which, evaluated at an arbitrary element $(g_0, g_1, h) \in G^2 \times H \cong \mathcal{G}_2$, yields

$$\varphi\begin{pmatrix} g_1 g_0 \\ h \end{pmatrix} = \varphi\begin{pmatrix} g_1 \\ h \end{pmatrix} + \varphi\begin{pmatrix} g_0 \\ h i(g_1) \end{pmatrix} - \phi\big(\rho_0^1(h)\omega_1(g_1, g_0)\big).$$

In particular, making $g_0 = 1$, one sees that φ vanishes identically on the space of units H. Defining

$$\check{\varphi}(g) := \varphi\begin{pmatrix} g \\ 1 \end{pmatrix}, \tag{50}$$

one is left with a 4-tuple $(\check{\varphi}, \omega_0, \alpha, \omega_1)$ as in the statement of Proposition 3.17. In the sequel, we show that this 4-tuple verifies the equations of Proposition 3.17 if and only if $\nabla\vec{\omega} = 0$, and thus defines an extension.

- $(\nabla\vec{\omega})_0^{0,3} = \delta\omega_0 = 0$; therefore, ω_0 is a 2-cocycle with respect to ρ_0^0.
- $(\nabla\vec{\omega})_3^{0,0} = \delta_{(1)}\omega_1 = 0$; therefore, ω_1 is a 2-cocycle with respect to $\rho_0^1 \circ i$.
- Evaluating $(\nabla\vec{\omega})_0^{1,2} = 0$ at $\begin{pmatrix} g_1 & g_2 \\ 1 & 1 \end{pmatrix} \in \mathcal{G}^2$ yields Eq. (43).
- $(\nabla\vec{\omega})_1^{0,2} = 0$ is exactly Eq. (44).
- Evaluate $(\nabla\vec{\omega})_0^{1,2} = 0$ at

$$\begin{pmatrix} g & 1 \\ h^{-1} & h \end{pmatrix} : \omega_0(h^{-1}, h) - \omega_0(h^{-1}i(g), h) + \check{\varphi}(g^h) - \varphi\begin{pmatrix} g \\ h^{-1} \end{pmatrix} + \phi(\alpha(h; g)) = 0,$$

and

$$\begin{pmatrix} 1 & g \\ h^{-1} & 1 \end{pmatrix} : -\omega_0(h^{-1}, i(g)) - \rho_0^0(h)^{-1}\check{\varphi}(g) + \varphi\begin{pmatrix} g \\ h^{-1} \end{pmatrix} = 0,$$

and consider their sum

$$-\omega_0(h^{-1}, h) + \omega_0(h^{-1}i(g), h) + \omega_0(h^{-1}, i(g)) = \check{\varphi}(g^h) - \rho_0^0(h)^{-1}\check{\varphi}(g) + \phi(\alpha(h; g)).$$

Since $\delta\omega_0 = 0$,

$$\rho_0^0(h)^{-1}\omega_0(i(g), h) + \omega_0(h^{-1}, i(g)h) = \omega_0(h^{-1}i(g), h) + \omega_0(h^{-1}, i(g));$$

hence, by substituting, one gets Eq. (45).

- Evaluating $(\nabla\vec{\omega})_1^{1,1} = 0$ at $(\gamma; g_1) \in \mathcal{G} \times G$, where $\gamma = \begin{pmatrix} g_2 \\ 1 \end{pmatrix} \in \mathcal{G}$ yields Eq. (46).
- $(\nabla\vec{\omega})_2^{0,1} = 0$ is exactly Eq. (47).

Conversely, let $(\check{\varphi}, \omega_0, \alpha, \omega_1)$ be a 4-tuple as in the statement of Proposition 3.17 verifying the equations therein. Define

$$\varphi\begin{pmatrix} g \\ h \end{pmatrix} := \omega_0(h, i(g)) + \rho_0^0(h)\check{\varphi}(g) \tag{51}$$

and set $\vec{\omega} = (0, \varphi, \omega_0, 0, \alpha, \omega_1)$. We proceed to show that $\vec{\omega}$ is a 2-cocycle. From the previous discussion, it suffices to check that $(\nabla\vec{\omega})_0^{2,1}$, $(\nabla\vec{\omega})_1^{1,1}$ and $(\nabla\vec{\omega})_0^{1,2}$ vanish. Indeed, let $(g_1, g_2, h) \in G^2 \times H$, then from the cocycle equation $\delta\omega_0 = 0$ and Eq. (43),

$$\omega_0(h, i(g_2g_1)) = -\rho_0^0(h)\omega_0(i(g_2), i(g_1)) + \omega_0(hi(g_2), i(g_1)) + \omega_0(h, i(g_2));$$

and

$$\begin{aligned}\varphi\begin{pmatrix} g_2g_1 \\ h\end{pmatrix} &= \omega_0(h, i(g_2g_1)) + \rho_0^0(h)\check{\varphi}(g_2g_1)\\ &= \omega_0(hi(g_2), i(g_1)) + \omega_0(h, i(g_2)) + \rho_0^0(h)\big[\rho_0^1(i(g_2))\check{\varphi}(g_1) + \check{\varphi}(g_2) - \phi(\omega_1(g_2, g_1))\big]\\ &= \varphi\begin{pmatrix} g_1 \\ hi(g_2)\end{pmatrix} + \varphi\begin{pmatrix} g_2 \\ h\end{pmatrix} - \rho_0^0(h)\circ\phi(\omega_1(g_2, g_1));\end{aligned}$$

hence, $(\nabla\vec{\omega})_0^{2,1} = 0$. Next, let $(\gamma; f) \in \mathcal{G}\times G$ and $(g, h) \in G\rtimes H$ the image of γ under the isomorphism of Remark 2.1. Then,

$$\begin{aligned}\delta'\varphi(\gamma; f) &= \rho_0^1(hi(g))^{-1}\rho_1(f)\varphi(\gamma) = \rho_0^1(hi(g))^{-1}\rho_1(f)\big(\omega_0(h, i(g)) + \rho_0^0(h)\check{\varphi}(g)\big)\\ &= \rho_0^1(hi(g))^{-1}\rho_1(f)\omega_0(h, i(g)) + \rho_0^1(i(g))^{-1}\rho_1(f^h)\check{\varphi}(g).\end{aligned}$$

Using Eq.'s (44) and (46), one gets

$$\begin{aligned}\delta'\varphi(\gamma; f) = \rho_0^1(i(g))^{-1}\alpha(h; f) - \alpha(hi(g); f) + \alpha(i(g); f^h)+\\ - \alpha(i(g), f^h) + \rho_0^1(i(g))^{-1}\omega_1(f^h, g) + \omega_1(g^{-1}, f^hg) - \omega_1(g^{-1}, g);\end{aligned}$$

hence, $(\nabla\vec{\omega})_1^{1,1} = 0$. Concludingly, let $(\gamma_1, \gamma_2) \in \mathcal{G}^2$ and let $(g_k, h_k) \in G\rtimes H$ be the image of γ_k under the isomorphism of Remark 2.1 for $k \in \{1, 2\}$. Then,

$$\begin{aligned}\delta\varphi\begin{pmatrix} g_1 & g_2 \\ h_1 & h_2\end{pmatrix} &= \rho_0^0(h_1i(g_1))\varphi\begin{pmatrix} g_2 \\ h_2\end{pmatrix} - \varphi\begin{pmatrix} g_1^{h_2}g_2 \\ h_1h_2\end{pmatrix} + \varphi\begin{pmatrix} g_1 \\ h_1\end{pmatrix}\\ &= \rho_0^0(h_1i(g_1))\big[\omega_0(h_2, i(g_2)) + \rho_0^0(h_2)\check{\varphi}(g_2)\big]+\\ &\quad - \big(\omega_0(h_1h_2, i(g_1^{h_2}g_2)) + \rho_0^0(h_1h_2)\check{\varphi}(g_1^{h_2}g_2)\big) + \omega_0(h_1, i(g_1)) + \rho_0^0(h_1)\varphi(g_1)\end{aligned}$$

Using successively Eq.'s (43) and (45), one computes

$$\begin{aligned}\check{\varphi}(g_1^{h_2}g_2) &= \rho_0^0(i(g_1^{h_2}))\check{\varphi}(g_2) + \check{\varphi}(g_1^{h_2}) + \omega_0(i(g_1^{h_2}), i(g_2)) - \phi(\omega_1(g_1^{h_2}, g_2))\\ &= \rho_0^0(i(g_1^{h_2}))\check{\varphi}(g_2) + \rho_0^0(h_2)^{-1}\omega_0(i(g_1), h_2) + \omega_0(h_2^{-1}, i(g_1)h_2) - \omega_0(h_2^{-1}, h_2)+\\ &\quad + \rho_0^0(h_2)^{-1}\check{\varphi}(g_1) - \phi(\alpha(h_2; g_1)) + \omega_0(i(g_1^{h_2}), i(g_2)) - \phi(\omega_1(g_1^{h_2}, g_2)).\end{aligned}$$

Substituting,

$$\begin{aligned}\delta\varphi\begin{pmatrix} g_1 & g_2 \\ h_1 & h_2\end{pmatrix} &= \rho_0^0(h_1h_2)\phi(\alpha(h_2; g_1) + \omega_1(g_1^{h_2}, g_2)) + R(\omega_0)\\ &= \Delta\alpha\begin{pmatrix} g_0 & g_1 \\ h_0 & h_1\end{pmatrix} + \Delta_{1,2}\omega_1\begin{pmatrix} g_0 & g_1 \\ h_0 & h_1\end{pmatrix} + R(\omega_0),\end{aligned}$$

where

$$R(\omega_0) = \rho_0^0(h_1 i(g_1))\omega_0(h_2, i(g_2)) - \omega_0(h_1h_2, i(g_1^{h_2}g_2)) + \omega_0(h_1, i(g_1)) - \rho_0^0(h_1)\omega_0(i(g_1), h_2) + \\ - \rho_0^0(h_1h_2)\big(\omega_0(h_2^{-1}, i(g_1)h_2) - \omega_0(h_2^{-1}, h_2) + \omega_0(i(g_1^{h_2}), i(g_2))\big).$$

We claim that $R(\omega_0) = \partial\omega_0 \begin{pmatrix} g_1 & g_2 \\ h_1 & h_2 \end{pmatrix}$, and consequently $(\nabla\vec{\omega})_0^{1,2} = 0$. Indeed, this claim follows from repeatedly applying the cocycle equation to rewrite $R(\omega_0)$. Using $\delta\omega_0 = 0$ evaluated at $(h_2^{-1}, i(g_1)h_2, i(g_2))$ yields

$$R(\omega_0) = \rho_0^0(h_1 i(g_1))\omega_0(h_2, i(g_2)) - \omega_0(h_1h_2, i(g_1^{h_2}g_2)) + \omega_0(h_1, i(g_1)) - \rho_0^0(h_1)\omega_0(i(g_1), h_2) + \\ - \rho_0^0(h_1)\omega_0(i(g_1)h_2, i(g_2)) - \rho_0^0(h_1h_2)\big(\omega_0(h_2^{-1}, i(g_1)h_2 i(g_2)) - \omega_0(h_2^{-1}, h_2)\big).$$

Successively, using $\delta\omega_0 = 0$ evaluated at $(h_1h_2, h_2^{-1}, i(g_1)h_2, i(g_2))$, $(h_1, i(g_1)h_2, i(g_2))$ and $(h_1, i(g_1), h_2)$ yields

$$\begin{aligned} R(\omega_0) &= \rho_0^0(h_1 i(g_1))\omega_0(h_2, i(g_2)) - \omega_0(h_1, i(g_1)h_2 i(g_2)) - \omega_0(h_1h_2, h_2^{-1}) + \omega_0(h_1, i(g_1)) + \\ &\quad - \rho_0^0(h_1)\big(\omega_0(i(g_1), h_2) + \omega_0(i(g_1)h_2, i(g_2))\big) + \rho_0^0(h_1h_2)\omega_0(h_2^{-1}, h_2) \\ &= \rho_0^0(h_1 i(g_1))\omega_0(h_2, i(g_2)) - \omega_0(h_1 i(g_1)h_2, i(g_2)) - \omega_0(h_1h_2, h_2^{-1}) + \omega_0(h_1, i(g_1)) + \\ &\quad - \omega_0(h_1, i(g_1)h_2) - \rho_0^0(h_1)\omega_0(i(g_1), h_2) + \rho_0^0(h_1h_2)\omega_0(h_2^{-1}, h_2) \\ &= \rho_0^0(h_1 i(g_1))\omega_0(h_2, i(g_2)) - \omega_0(h_1 i(g_1)h_2, i(g_2)) - \omega_0(h_1h_2, h_2^{-1}) + \\ &\quad - \omega_0(h_1 i(g_1), h_2) + \rho_0^0(h_1h_2)\omega_0(h_2^{-1}, h_2). \end{aligned}$$

Thus, the claim $R(\omega_0) = -\omega_0(h_1 i(g_1), h_2 i(g_2)) + \omega_0(h_1, h_2)$ follows from $\delta\omega_0(h_1 i(g_1), h_2, i(g_2)) = \delta\omega_0(h_1h_2, h_2^{-1}, h_2) = 0$.

Furthermore, if

$$(\vec{\omega})^k = (0, \varphi^k, \omega_0^k, \lambda^k, \alpha^k, \omega_1^k), \qquad k \in \{1, 2\}$$

are a pair of cohomologous of 2-cocycles, say

$$(\vec{\omega})^2 - (\vec{\omega})^1 = \nabla(v, \lambda_0, \lambda_1),$$

then coordinate-wise one has got

$$\omega_0^2 - \omega_0^1 = \delta\lambda_0$$

$$\varphi^2 - \varphi^1 = \partial\lambda_0 - \delta v + \Delta\lambda_1 \qquad \alpha^2 - \alpha^1 = \delta'\lambda_0 - \delta\lambda_1$$

$$0 = -\partial v = -v \qquad 0 = -\delta' v \qquad \omega_1^2 - \omega_1^1 = \delta_{(1)}\lambda_1.$$

Explicitly, for $\gamma = (g,h) \in G \rtimes H \cong \mathcal{G}$ and $(h;f) \in H \times G$

$$(\varphi^2 - \varphi^1)(\gamma) = \lambda_0(h) - \lambda_0(hi(g)) + \phi(\rho_0^1(h)\lambda_1(g)) \tag{52}$$

$$(\alpha^2 - \alpha^1)(h;f) = \rho_0^1(h)^{-1}\rho_1(f)\lambda_0(h) - \big(\lambda_1(f^h) - \rho_0^1(h)^{-1}\lambda_1(f)\big). \tag{53}$$

Eq. (52) evaluated at $\gamma = (g,1)$ yields Eq. (48) with $\check{\varphi}^k$ defined by Eq. (50), and Eq. (53) is exactly Eq. (49). Conversely, given two 4-tuples as in the statement of Proposition 3.19 whose difference verifies the equations therein and defining a corresponding pair of φ^k's by Eq. (51),

$$\begin{aligned}(\varphi^2 - \varphi^1)(\gamma) &= \omega_0^2(h,i(g)) + \rho_0^0(h)\varphi^2(g) - \omega_0^1(h,i(g)) - \rho_0^0(h)\varphi^1(g)\\ &= \delta\lambda_0(h,i(g)) + \rho_0^0(h)(\phi(\lambda_1(g)) - \lambda_0(i(g)))\\ &= \rho_0^0(h)\lambda_0(i(g)) - \lambda_0(hi(g)) + \lambda_0(h) + \phi(\rho_0^1(h)\lambda_1(g)) - \rho_0^0(h)\lambda_0(i(g))\\ &= \partial\lambda_0(\gamma) + \Delta\lambda_1(\gamma).\end{aligned}$$

Summing this discussion up, we have shown the following:

3.21. Theorem. *$H^2_\nabla(\mathcal{G},\phi)$, the second degree cohomology of the subcomplex of $C_{tot}(\mathcal{G},\phi)$ that has trivial $\binom{p,0}{0}$-coordinate for $p > 0$, is in one-to-one correspondence with split extensions of the Lie 2-group $\mathcal{G}$ by the 2-vector space $W \xrightarrow{\phi} V$.*

3.22. Remark. Because of the application that we have in mind (see the Introduction), Theorem 3.21 is stated and proved to classify extensions of $\mathcal{G}$ by 2-vector spaces. However, do notice that the results of this section can be effortlessly extended to more general abelian 2-groups: Replacing the 2-representation by an action of $\mathcal{G}$ on an abelian 2-group (see [15]), one forms a snapshot complex using the same formulae. The second cohomology of the complex with values in the abelian 2-group $\mathbb{A}$ is seen to classify extensions by $\mathbb{A}$ simply reinterpreting the arguments we presented.

4. Inkling of a larger complex

In this section we study the grid of Section 3. We prove that for either q or p constant, there is a double complex. More precisely, we prove that ∂ and $\delta_{(1)}$ do commute, yielding a sequence of double complexes that we call the q-pages, and show that δ and $\delta_{(1)}$ also commute, thus yielding a sequence of double complexes that we call the p-pages. We prove that there is a general formula for the *difference maps* Δ that measure the difference between $\partial\delta$ and $\delta\partial$. In fact, using Δ we show that the two dimensional grid $r = 0$ commutes up to isomorphism in the 2-vector space (cf. Proposition 4.11), and the successive *r-pages* commute up to homotopy (cf. Proposition 4.16). Since ∂ and δ fail to commute on the nose, the total differential ∇ of Eq. (28) does not, in general, square to zero. In Section 3, it is explained how this defect is partially solved by adding the difference maps;

however, the updated ∇ still fails to square to zero. In a similar fashion, the coordinates that fail to vanish, do so because the homotopies do not commute with some differentials; however, they do commute up to new homotopies. Accordingly, we call these homotopies *higher difference maps* and note them $\Delta_{a,b}$ (cf. Eq.'s (38) and (39)). We provide explicit formulas for some families of higher difference maps and explain what the grid is lacking to form a complex.

Ultimately, lacking additional structure such as a product or a more insightful definition of the higher difference maps, the proofs of the relations we present boil down to unfortunately long and rather unenlightening computations.

4.1. Remark. In this section, we collect evidence suggesting there is a full triple complex analog to that in [1]. Eventually, one could collapse one dimension in the triple complex, by taking, *e.g.*, the total complex of the p-pages; however, as opposed to the case of [1], the remaining object is not a double complex because of the higher difference maps $\Delta_{a,b}$ with $a > 1$.

Throughout, let $\mathcal{G}$ be a Lie 2-group with associated crossed module $G \xrightarrow{i} H$ and let ρ be a 2-representation of $\mathcal{G}$ on the 2-vector space $W \xrightarrow{\phi} V$. Using the notation conventions laid down in Subsection 2.12, we think of ρ as a triple $(\rho_0^0, \rho_0^1; \rho_1)$. Also, we take the isomorphisms of Remarks 2.1 and 2.5 to be fixed and we abuse notation and often treat them as equalities. Recall the notation from Section 3: We write

$$\partial : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_r^{p+1,q}(\mathcal{G}, \phi)$$

for the differential in the p-direction (cf. 3.2.1),

$$\delta : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_r^{p,q+1}(\mathcal{G}, \phi)$$

for the differential in the q-direction (cf. 3.5.1), and

$$\delta_{(1)} : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_{r+1}^{p,q}(\mathcal{G}, \phi)$$

for the differential in the r-direction (cf. 3.5.2).

4.2. Commuting differentials. In this subsection, we prove that for constant q the two dimensional grid of maps is a double complex that we call the *q-page*. Analogously, for constant p, there is a double complex that we call the *p-page*.

q-pages. When $q = 0$, there is a double complex all of whose columns are equal and the intertwining maps are either zero or the identity, thus commuting trivially. For $q > 0$, there is a double complex as well.

4.3. Proposition. *For each $q > 0$,*

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & C(H^q \times G^2, W) \arrow[r, "\partial"] \arrow[u] & C(\mathcal{G}^q \times G^2, W) \arrow[r, "\partial"] \arrow[u] & C(\mathcal{G}_2^q \times G^2, W) \arrow[r] \arrow[u] & \cdots \\
 & C(H^q \times G, W) \arrow[r, "\partial"] \arrow[u, "\delta_{(1)}"] & C(\mathcal{G}^q \times G, W) \arrow[r, "\partial"] \arrow[u, "\delta_{(1)}"] & C(\mathcal{G}_2^q \times G, W) \arrow[r] \arrow[u, "\delta_{(1)}"] & \cdots \\
 & C(H^q, V) \arrow[r, "\partial"] \arrow[u, "\delta'"] & C(\mathcal{G}^q, V) \arrow[r, "\partial"] \arrow[u, "\delta'"] & C(\mathcal{G}_2^q, V) \arrow[r] \arrow[u, "\delta'"] & \cdots
\end{tikzcd}$$

is a double complex.

Proof. Each row is a complex and due to Lemma 3.7, so is each column. Moreover, due to Lemma 3.16, we just need to prove that ∂ commutes with $\delta_{(1)}$.

Let $\omega \in C(\mathcal{G}_p^q \times G^r, V)$, $\vec{\gamma} = (\gamma_1, ..., \gamma_q)^T \in \mathcal{G}_{p+1}^q$ and $\vec{f} = (f_0, ..., f_r) \in G^{r+1}$. Adopting the convention of Lemma 3.16 for $\gamma_b \in \mathcal{G}_{p+1}$,

$$\delta_{(1)}\partial\omega(\vec{\gamma}; \vec{f}) = \rho_0^1(i(f_0^{t_{p+1}(\gamma_1)\dots t_{p+1}(\gamma_q)}))\partial\omega(\vec{\gamma}; \delta_0\vec{f}) + \sum_{k=1}^{r+1}(-1)^k\partial\omega(\vec{\gamma}; \delta_k\vec{f})$$

$$= \rho_0^1(i(f_0^{t_{p+1}(\gamma_1)\dots t_{p+1}(\gamma_q)}))\Big(\rho_0^1(i(pr_G(\gamma_{10} \barwedge \dots \barwedge \gamma_{q0})))^{-1}\omega(\partial_0\vec{\gamma}; \delta_0\vec{f}) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\vec{\gamma}; \delta_0\vec{f})\Big) +$$

$$+ \sum_{k=1}^{r+1}(-1)^k\Big(\rho_0^1(i(pr_G(\gamma_{10} \barwedge \dots \barwedge \gamma_{q0})))^{-1}\omega(\partial_0\vec{\gamma}; \delta_k\vec{f}) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\vec{\gamma}; \delta_k\vec{f})\Big).$$

Now, given that

$$\rho_0^1\Big(i\big(f_0^{t_{p+1}(\gamma_1)\dots t_{p+1}(\gamma_q)}\big)\Big) = \rho_0^1\Big(i\big(f_0^{t(\gamma_{10})\dots t(\gamma_{q0})}\big)\Big) = \rho_0^1\Big(i\big(f_0^{t(\gamma_{10} \barwedge \dots \barwedge \gamma_{q0})}\big)\Big)$$

$$= \rho_0^1\Big(i\big(f_0^{h_{10}\dots h_{q0} i(pr_G(\gamma_{10} \barwedge \dots \barwedge \gamma_{q0}))}\big)\Big)$$

$$= \rho_0^1(i(pr_G(\gamma_{10} \barwedge \dots \barwedge \gamma_{q0})))^{-1}\rho_0^1(i(f_0^{h_{10}\dots h_{q0}}))\rho_0^1(i(pr_G(\gamma_{10} \barwedge \dots \barwedge \gamma_{q0}))),$$

one can group terms to get

$$\delta_{(1)}\partial\omega(\vec{\gamma};\vec{f}) = \rho_0^1(i(pr_G(\gamma_{10}\barwedge\ldots\barwedge\gamma_{q0})))^{-1}\Big(\rho_0^1(i(f_0^{h_{10}\ldots h_{q0}}))\omega(\partial_0\vec{\gamma};\delta_0\vec{f}) + \sum_{k=1}^{r+1}(-1)^k\omega(\partial_0\vec{\gamma};\delta_k\vec{f})\Big)$$

$$+\sum_{j=1}^{p+1}(-1)^j\Big(\rho_0^1(i(f_0^{t_{p+1}(\gamma_1)\ldots t_{p+1}(\gamma_q)}))\omega(\partial_j\vec{\gamma};\delta_0\vec{f}) + \sum_{k=1}^{r+1}(-1)^k\omega(\partial_j\vec{\gamma};\delta_k\vec{f})\Big).$$

If $j>0$, $t_p(\partial_j\gamma_b) = t(\gamma_{b0}) = t_{p+1}(\gamma_b)$; otherwise, $t_p(\partial_0\gamma_b) = t(\gamma_{b1}) = s(\gamma_{b0}) = h_{b0}$. Hence,

$$\delta_{(1)}\partial\omega(\vec{\gamma};\vec{f}) = \rho_0^1(i(pr_G(\gamma_{10}\barwedge\ldots\barwedge\gamma_{q0})))^{-1}\delta_{(1)}\omega(\partial_0\vec{\gamma};\vec{f}) + \sum_{j=1}^{p+1}(-1)^j\delta_{(1)}\omega(\partial_j\vec{\gamma};\vec{f}) = \partial\delta_{(1)}\omega(\vec{\gamma};\vec{f}).$$

■

p-PAGES. We start by outlining some general facts about actions of Lie groups and representations.

4.4. LEMMA. *Let X and Y be Lie groups, and let Y act on the right on X by Lie group automorphisms. Then*

$$\begin{tikzcd} Y \ltimes X \arrow[r, shift left, shift left] \arrow[r, shift right, shift right] \arrow[d, shift right] \arrow[d, shift left] & Y \arrow[d, shift right] \arrow[d, shift left] \\ X \arrow[r, shift left, shift left] \arrow[r, shift right, shift right] & *, \end{tikzcd} \tag{54}$$

where the top groupoid is a bundle of Lie groups and the left groupoid is the (right!) action groupoid for the action of Y on X, i.e,

$$|s|(y;x) = x^y,$$

for $x\in X$ and $y\in Y$, is a double Lie groupoid.

4.5. REMARK. A word on the odd choice of notation: we opted out of calling our groups G and H, so to avoid any concurrence and to help the reader stray away from believing that there is a Lie 2-group somewhere in this diagram. There is none. In fact, though the action of Y on X is by automorphisms, there is no crossed module around.

PROOF OF LEMMA 4.4. Given each of the sides of the square (54) is clearly a Lie groupoid, to prove that the array (54) is a double Lie groupoid, we show that the horizontal structural maps are groupoid morphisms. First, the square commutes as all compositions land in $*$. Both $\underline{\overline{s}} = \underline{\overline{t}}$ are functors: for $x\in X$ and $y_1,y_2\in Y$,

$$\underline{\overline{s}}(1;x) = 1 \quad\text{and}\quad \underline{\overline{s}}((y_1;x)\barwedge(y_2;x^{y_1})) = \underline{\overline{s}}(y_1y_2;x) = y_1y_2 = \underline{\overline{s}}(y_1;x)\underline{\overline{s}}(y_2;x^{y_1}).$$

$\underline{\overline{u}}$ is a functor as well: for $y \in Y$,

$$|s|(\underline{\overline{u}}(y)) = |s|(y;1) = 1^y = 1 = u(*) \qquad \text{and} \qquad |t|(\underline{\overline{u}}(y)) = |t|(y;1) = 1 = u(*),$$

so it is well-defined, and it respects units and the multiplication:

$$\underline{\overline{u}}(1) = (1;1) \quad \text{and} \quad \underline{\overline{u}}(y_1y_2) = (y_1y_2;1) = (y_1;1) \between (y_2;1^{y_1}) = \underline{\overline{u}}(y_1) \between \underline{\overline{u}}(y_2).$$

To conclude, $(Y \ltimes X)_{\underline{\overline{s}}} \times_{\underline{\overline{t}}} (Y \ltimes X) \cong Y \ltimes X^2 \rightrightarrows X^2$ is the right action groupoid for the diagonal action, we prove that $\underline{\overline{m}}$ is a Lie groupoid homomorphism: for $x_1, x_2 \in X$ and $y \in Y$,

$$|s|(\underline{\overline{m}}(y;x_1,x_2)) = |s|(y;x_1x_2) = (x_1x_2)^y = x_1^y x_2^y = m(|s|^2(y;x_1,x_2))$$

and

$$|t|(\underline{\overline{m}}(y;x_1,x_2)) = |t|(y;x_1x_2) = x_1x_2 = m(|t|^2(y;x_1,x_2)).$$

Furthermore, $\underline{\overline{m}}$ is also compatible with units and the multiplication, as $\underline{\overline{m}}(1;x_1,x_2) = (1;x_1x_2)$ and $((y_1;x_1) \between (y_2;x_1^{y_1})) \bowtie ((y_1;x_2) \between (y_2;x_2^{y_1}))$ equals

$$\begin{aligned}(y_1y_2;x_1) \bowtie (y_1y_2;x_2) &= \underline{\overline{m}}(y_1y_2;x_1,x_2) = (y_1y_2;x_1x_2) \\ &= (y_1;x_1x_2) \between (y_2;(x_1x_2)^{y_1}) = \underline{\overline{m}}(y_1;x_1,x_2) \between \underline{\overline{m}}(y_2;x_1^{y_1},x_2^{y_1}).\end{aligned}$$

■

4.6. Example. Let W be a vector space and let $X = Y = GL(W)$ along with the right action by conjugation on itself. Then

$$\begin{tikzcd} GL(W) \ltimes GL(W) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & GL(W) \arrow[d, shift right] \arrow[d, shift left] \\ GL(W) \arrow[r, shift left] \arrow[r, shift right] & * \end{tikzcd}$$

is a double Lie groupoid.

4.7. Lemma. *Let X and Y be as in the statement of Lemma 4.4 and let*

$$Y \ltimes X \longrightarrow GL(W) \ltimes GL(W) : (y;x) \longmapsto (\rho_Y(y), \rho_X(x)) \tag{55}$$

be a map of double Lie groupoids from the double Lie groupoid of Lemma 4.4 to the one in Example 4.6. Then, for every pair of integers $q, r \geq 0$, there are representations of the Lie group bundles

$$Y^q \times X \rightrightarrows Y^q \tag{56}$$

on $Y^q \times W \longrightarrow Y^q$ and of the right transformation groupoids

$$Y \ltimes X^r \rightrightarrows X^r \tag{57}$$

on $X^r \times W \longrightarrow X^r$ that make the grid

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C(Y^2, W) \arrow[r, "\delta'"] \arrow[u] & C(Y^2 \times X, W) \arrow[r, "\delta'"] \arrow[u] & C(Y^2 \times X^2, W) \arrow[r] \arrow[u] & \cdots \\
C(Y, W) \arrow[r, "\delta'"] \arrow[u, "\delta"] & C(Y \times X, W) \arrow[r, "\delta'"] \arrow[u, "\delta"] & C(Y \times X^2, W) \arrow[r] \arrow[u, "\delta"] & \cdots \\
W \arrow[r, "\delta'"] \arrow[u, "\delta"] & C(X, W) \arrow[r, "\delta'"] \arrow[u, "\delta"] & C(X^2, W) \arrow[r] \arrow[u, "\delta"] & \cdots
\end{tikzcd}$$

whose rows and columns are Lie groupoid cochain complexes taking values in these representations into a double complex.

Proof. Since (55) is a map of double Lie groupoids, its restrictions to the bottom and right groupoids give respectively representations ρ_X of X and ρ_Y of Y, both on W. These are the representations for the group bundle over a point $X \rightrightarrows *$ and for the trivial transformation groupoid $Y \ltimes * \rightrightarrows *$. For each $q > 0$, the representation of the group bundle (56) is the pull-back of ρ_X along the groupoid homomorphism $|s|_q$. In symbols, define the representation ρ_X^q of the group bundle (56) on $(s_Y)_q^* W = Y^q \times W \longrightarrow Y^q$ by

$$\rho_X^q(y_1, ..., y_q; x) := |s|_q^* \rho_X(y_1, ..., y_q; x) = \rho_X(x^{y_1 \cdots y_q})$$

for $(y_1, ..., y_q; x) \in Y^q \times X \cong (Y \ltimes X)^{(q)}$. Analogously, for each $r > 0$, the representation of the transformation groupoid (57) is the pull-back of ρ_Y along the groupoid homomorphisms $\underline{\bar{t}}_r$. In symbols, define the right representation ρ_Y^r of the transformation groupoid (57) on $t_r^* W = X^r \times W \longrightarrow X^r$ by

$$\rho_Y^r(y; x_1, ..., x_r) := (\iota_Y \circ \underline{\bar{t}}_r)^* \rho_Y(y; x_1, ..., x_r) = \rho_Y(y)^{-1}$$

for $(y; x_1, ..., x_r) \in Y \times X^r \cong (Y \times X)^{(r)}$.

If $y \in Y$ and $\vec{x} = (x_1, ..., x_r) \in X^r$, we adopt the convention that $(\vec{x})^y := (x_1^y, ..., x_r^y)$.

We proceed to check that, for fixed (q, r), the spaces of cochains of the groupoids (56) and (57) with respect to ρ_X^q and ρ_Y^r concur. On the one hand,

$$(Y^q \times X)^{(r)} = \{(\vec{y}_1, x_1; ...; \vec{y}_r, x_r) \in (Y^q \times X)^r : \underline{\bar{s}}(\vec{y}_j, x_j) = \vec{y}_j = \vec{y}_{j+1} = \underline{\bar{t}}(\vec{y}_{j+1}, x_{j+1})\};$$

therefore, $(Y^q \times X)^{(r)} \cong Y^q \times X^r$, where the diffeomorphism is obviously given by $(\vec{y}, x_1; ...; \vec{y}, x_r) \longmapsto (\vec{y}; x_1, ..., x_r)$. On the other hand,

$$(Y \ltimes X^r)^{(q)} = \{(y_1, \vec{x}_1; ...; y_q, \vec{x}_q) \in (Y \ltimes X^r)^q : |s|(y_j, \vec{x}_j) = (\vec{x}_j)^{y_j} = \vec{x}_{j+1} = |t|(y_{j+1}, \vec{x}_{j+1})\};$$

therefore, $(Y \ltimes X^r)^{(q)} \cong Y^q \times X^r$, where the diffeomorphism is naturally given by $(y_1, \vec{x}; y_2, (\vec{x})^{y_1}; ...; y_q, (\vec{x})^{y_1 \cdots y_{q-1}}) \longmapsto (y_1, ..., y_q; \vec{x})$. Since the representations ρ_X and ρ_Y take values on a trivial vector bundle, the pull-back bundle along any groupoid homomorphism remains trivial and its sections are but smooth functions to W; thus,

$$C^r(Y^q \times X; Y^q \times W) = C(Y^q \times X^r, W) = C^q(Y \ltimes X^r; X^r \times W).$$

We use the diffeomorphisms of the latter discussion and the face maps δ_j of the Lie group Y and δ'_k of the Lie group X to rewrite the face maps of the groupoids (56) and (57): For $\vec{y} = (y_0, ..., y_q) \in Y^{q+1}$ and $\vec{x} = (x_0, ..., x_r) \in X^{r+1}$,

$$\delta_j(\vec{y}; \vec{x}) = \begin{cases} (\delta_0 \vec{y}; (\vec{x})^{y_0}) & \text{if } j = 0 \\ (\delta_j \vec{y}; \vec{x}) & \text{otherwise} \end{cases} \quad \text{and} \quad \delta'_k(\vec{y}; \vec{x}) = (\vec{y}; \delta'_k \vec{x}).$$

We are left to prove that the generic square

$$\begin{tikzcd} C(Y^{q+1} \times X^r, W) \arrow[r, "\delta'"] & C(Y^{q+1} \times X^{r+1}, W) \\ C(Y^q \times X^r, W) \arrow[u, "\delta"] \arrow[r, "\delta'"'] & C(Y^q \times X^{r+1}, W) \arrow[u, "\delta"'] \end{tikzcd} \tag{58}$$

commutes. Indeed, let $\omega \in C(Y^q \times X^r, W)$ and $\vec{y}$ and $\vec{x}$ be as above. Then,

$$\delta' \delta \omega(\vec{y}; \vec{x}) = \rho_X^{q+1}(\vec{y}; x_0) \delta \omega(\vec{y}; \delta'_0 \vec{x}) + \sum_{k=1}^{r+1} (-1)^k \delta \omega(\vec{y}; \delta'_k \vec{x}),$$

while

$$\delta \delta' \omega(\vec{y}; \vec{x}) = \delta' \omega(\delta_0 \vec{y}; (\vec{x})^{y_0}) + \sum_{j=1}^{q} (-1)^j \delta' \omega(\delta_j \vec{y}; \vec{x}) + (-1)^{q+1} \rho_Y^{r+1}(y_q; \vec{x}) \delta' \omega(\delta_{q+1} \vec{y}; \vec{x}).$$

We expand further to make the common terms evident:

$$\delta' \delta \omega(\vec{y}; \vec{x}) = \rho_X^{q+1}(\vec{y}; x_0) \Big[\omega(\delta_0 \vec{y}; (\delta'_0 \vec{x})^{y_0}) + \sum_{j=1}^{q} (-1)^j \omega(\delta_j \vec{y}; \delta'_0 \vec{x}) + (-1)^{q+1} \rho_Y^r(y_q; \delta'_0 \vec{x}) \omega(\delta_{q+1} \vec{y}; \delta'_0 \vec{x}) \Big] +$$

$$+ \sum_{k=1}^{r+1} (-1)^k \Big[\omega(\delta_0 \vec{y}; (\delta'_k \vec{x})^{y_0}) + \sum_{j=1}^{q} (-1)^j \omega(\delta_j \vec{y}; \delta'_k \vec{x}) + (-1)^{q+1} \rho_Y^r(y_q; \delta'_k \vec{x}) \omega(\delta_{q+1} \vec{y}; \delta'_k \vec{x}) \Big]$$

and

$$\delta\delta'\omega(\vec{y};\vec{x}) = \rho_X^q(\delta_0\vec{y};x_0^{y_0})\omega(\delta_0\vec{y};\delta_0'(\vec{x})^{y_0}) + \sum_{k=1}^{r+1}(-1)^k\omega(\delta_0\vec{y};\delta_k'(\vec{x})^{y_0}) +$$

$$+\sum_{j=1}^{q}(-1)^j\Big[\rho_X^q(\delta_j\vec{y};x_0)\omega(\delta_j\vec{y};\delta_0'\vec{x}) + \sum_{k=1}^{r+1}(-1)^k\omega(\delta_j\vec{y};\delta_k'\vec{x})\Big] +$$

$$+(-1)^{q+1}\rho_Y^{r+1}(y_q;\vec{x})\Big[\rho_X^q(\delta_{q+1}\vec{y};x_0)\omega(\delta_{q+1}\vec{y};\delta_0'\vec{x}) + \sum_{k=1}^{r+1}(-1)^k\omega(\delta_{q+1}\vec{y};\delta_k'\vec{x})\Big].$$

The equality follows from the following identities: Firstly, one obviously has

$$(\delta_k'\vec{x})^{y_0} = \delta_k'(\vec{x})^{y_0}.$$

Secondly,

$$\rho_Y^r(y_q;\delta_k'\vec{x}) = \rho_Y(y_q)^{-1} = \rho_Y^{r+1}(y_q;\vec{x})$$

and

$$\rho_X^{q+1}(\vec{y};x_0) = \rho_X(x_0^{y_0\cdots y_q}) = \rho_X((x_0^{y_0})^{y_1\cdots y_q}) = \rho_X^q(\delta_0\vec{y};x_0^{y_0}).$$

Also, for all values $0 < j \leq q$,

$$\rho_X^{q+1}(\vec{y};x_0) = \rho_X(x_0^{y_0\cdots y_q}) = \rho_X(x_0^{y_0\cdots(y_{j-1}y_j)\cdots y_q}) = \rho_X^q(\delta_j\vec{y};x_0).$$

Finally, as $\rho_Y \times \rho_X$ is a double Lie groupoid map

$$\rho_X(x^y) = \rho_X(|s|(y;x)) = |s|(\rho_Y(y),\rho_X(x)) = \rho_Y(y)^{-1}\rho_X(x)\rho_Y(y);$$

thereby implying,

$$\rho_X^{q+1}(\vec{y};x_0) = \rho_X(x_0^{y_0\cdots y_q}) = \rho_X((x_0^{y_0\cdots y_{q-1}})^{y_q}) = \rho_Y(y_q)^{-1}\rho_X(x_0^{y_0\cdots y_{q-1}})\rho_Y(y_q),$$

which is

$$\rho_X^{q+1}(\vec{y};x_0)\rho_Y^r(y_q;\delta_0'\vec{x}) = \rho_Y^{r+1}(y_q;\vec{x})\rho_X^q(\delta_{q+1}\vec{y};x_0)$$

and so, the commutativity of the square (58) follows.

■

We use Lemma 4.7 to help proving that the p-pages are double complexes. For any given p, the right action of $\mathcal{G}_p$ on G is by automorphisms, as it is defined by the pull-back of the action of H along t_p, that is, for $g \in G$ and $\gamma \in \mathcal{G}_p$,

$$g^\gamma := g^{t_p(\gamma)}.$$

Furthermore, the map

$$(\gamma;g) \longmapsto (\rho_{\mathcal{G}_p}(\gamma),\rho_G(g)) := (\rho_0^1(t_p(\gamma)),\rho_0^1(i(g)))$$

whose components are pull-back representations, verifies the hypothesis of Lemma 4.7. Indeed, after looking at the compatibility with the whole structure, one sees that it suffices to prove

$$\rho_G(|s|(\gamma; g)) = |s|(\rho_{\mathcal{G}_p}(\gamma), \rho_G(g)).$$

This follows easily from the equivariance of the crossed module map i,

$$\rho_G(|s|(\gamma; g)) = \rho_0^1(i(g^{t_p(\gamma)})) = \rho_0^1(t_p(\gamma)^{-1} i(g) t_p(\gamma)) = \rho_{\mathcal{G}_p}(\gamma)^{-1} \rho_G(g) \rho_{\mathcal{G}_p}(\gamma).$$

In fact, the p-pages coincide by definition with the outcome of Lemma 4.7, but for one caveat; the first column of the p-pages consists of the 0th degrees of the complexes in the r-direction, which are modified (see Lemmas 3.6 and 3.7).

4.8. Proposition. *For each p,*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C(\mathcal{G}_p^2, V) \arrow[u] \arrow[r, "\delta'"] & C(\mathcal{G}_p^2 \times G, W) \arrow[u] \arrow[r, "\delta_{(1)}"] & C(\mathcal{G}_p^2 \times G^2, W) \arrow[u] \arrow[r] & \cdots \\
C(\mathcal{G}_p, V) \arrow[u, "\delta"] \arrow[r, "\delta'"] & C(\mathcal{G}_p \times G, W) \arrow[u, "\delta"] \arrow[r, "\delta_{(1)}"] & C(\mathcal{G}_p \times G^2, W) \arrow[u, "\delta"] \arrow[r] & \cdots \\
V \arrow[u, "\delta"] \arrow[r, "\delta'"] & C(G, W) \arrow[u, "\delta"] \arrow[r, "\delta_{(1)}"] & C(G^2, W) \arrow[u, "\delta"] \arrow[r] & \cdots
\end{tikzcd}$$

is a double complex.

Proof. Due to Lemmas 3.6 and 3.7, each row is a complex, and clearly so is each column. Lemma 4.7 implies that, disregarding the first column of squares, we have got a double complex. In order to finish the proof, one needs to check that the generic square in the first column commutes. This is precisely the content of Lemma 3.10, Corollary 3.11 and Lemma 3.15.

■

4.9. Difference maps. In contrast with Subsection 4.2, when r is left constant, the resulting r-page fails to be a double complex. Nonetheless, as it is briefly mentioned at the beginning of this section, the *front page*, i.e., the r-page for $r = 0$ commutes up to isomorphism in the 2-vector space, and, when $r > 0$, there is a commutation relation up to homotopy. In this subsection, we make these comments precise and prove them.

THE FRONT PAGE. Let $\omega \in C(\mathcal{G}_p^q, V)$ and $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$, then $\delta\partial\omega(\vec{\gamma}), \partial\delta\omega(\vec{\gamma}) \in V$. Recall that in the 2-vector space $W \xrightarrow{\phi} V$, $v_1, v_2 \in V$ belong to the same orbit if there exists a $w \in W$ such that $v_2 = v_1 + \phi(w)$. When we say that the front page commutes up to isomorphism, we mean that $\delta\partial\omega(\vec{\gamma})$ and $\partial\delta\omega(\vec{\gamma})$ belong to the same orbit of V. The element in W that realizes the isomorphism, is coherently defined to be the composition of the map δ' from the complex in the r-direction and certain maps that we note Δ and refer to as the *difference maps*.

We start by setting notation and defining the difference maps. Throughout this subsection, $\vec{\gamma} \in \mathcal{G}_p^q$ has components

$$\vec{\gamma} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_q \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1p} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2p} \\ \vdots & \vdots & & \vdots \\ \gamma_{q1} & \gamma_{q2} & \cdots & \gamma_{qp} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1p} & h_1 \\ g_{21} & g_{22} & \cdots & g_{2p} & h_2 \\ \vdots & \vdots & & \vdots & \vdots \\ g_{q1} & g_{q2} & \cdots & g_{qp} & h_q \end{pmatrix}, \tag{59}$$

where the last equality is a notation abuse corresponding to the row-wise isomorphism $\mathcal{G}_p \cong G^p \times H$ from Remark 2.5. Here, (g_{ab}, h_{ab}) is the image of γ_{ab} under the isomorphism of Remark 2.1 for all values of a and b. We abbreviate $\partial_0\delta_0\vec{\gamma}$ by regarding $\vec{\gamma}$ as a matrix and using "minor" notation, i.e.,

$$\vec{\gamma}_{1,1} := \partial_0\delta_0\vec{\gamma} = \begin{pmatrix} \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2q} \\ \gamma_{32} & \gamma_{33} & \cdots & \gamma_{3q} \\ \vdots & \vdots & & \vdots \\ \gamma_{p2} & \gamma_{p3} & \cdots & \gamma_{pq} \end{pmatrix}.$$

The difference maps

$$\Delta : C(\mathcal{G}_p^q \times G, W) \longrightarrow C(\mathcal{G}_{p+1}^{q+1}, V)$$

are defined by

$$\Delta\omega(\vec{\gamma}) := \rho_0^1(t_p(\partial_0\gamma_1)...t_p(\partial_0\gamma_{q+1})) \circ \phi\big(\omega(\vec{\gamma}_{1,1}; g_{11})\big),$$

for $\omega \in C(\mathcal{G}_p^q \times G, W)$ and $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$. Clearly, when $q = 0$, one drops the $\vec{\gamma}_{1,1}$-entry (compare with Eq.'s (30), (35) and (36)).

4.10. LEMMA. *Let $v \in C_0^{p,0}(\mathcal{G}, \phi) = V$, then*

$$\delta\partial v = \partial\delta v + \Delta\delta' v \in C(\mathcal{G}_{p+1}, V). \tag{60}$$

PROOF. First, we compute the first term on the right hand side of Eq. (60). Let $\gamma \in \mathcal{G}_{p+1}$ and let $(g_0, ..., g_p; h) \in G^{p+1} \times H$ be its corresponding image under the isomorphism of

Remark 2.5. Then,

$$(\partial\delta v)(\gamma) = (\delta v)(\partial_0\gamma) + \sum_{j=1}^{p}(-1)^{j+1}(\delta v)(\partial_j\gamma) + (-1)^{p+1}(\delta v)(\partial_{p+1}\gamma)$$

$$= \rho_0^0(hi(g_p...g_1))v - v + \sum_{j=1}^{p}(-1)^{j+1}(\rho_0^0(hi(g_p...g_0))v - v)+$$

$$+ (-1)^{p+1}(\rho_0^0(hi(g_p)i(g_{p-1}...g_0))v - v),$$

and

$$(\partial\delta v)(\gamma) = \begin{cases} \rho_0^0(hi(g_p...g_1))v - v & \text{if } p \text{ is odd} \\ \rho_0^0(hi(g_p...g_1))(v - \rho_0^0(i(g_0))v) & \text{otherwise.} \end{cases}$$

On the other hand,

$$(\delta\partial v)(\gamma) = \begin{cases} \rho_0^0(hi(g_p...g_0))v - v & \text{if } p \text{ is odd} \\ 0 & \text{otherwise;} \end{cases}$$

hence, either way,

$$(\delta\partial v - \partial\delta v)(\gamma) = \rho_0^0(hi(g_p...g_1))(\rho_0^0(i(g_0))v - v)$$

and the result follows from the first part of Eq. (19).

■

4.11. Proposition. *Let $\omega \in C(\mathcal{G}_p^q, V)$, then*

$$\delta\partial\omega = \partial\delta\omega + \Delta\delta'\omega \in C(\mathcal{G}_{p+1}^{q+1}, V). \tag{61}$$

Proof. Eq. (61) holds essentially due to the commutativity of δ_j and ∂_k for all values of (j,k), one just need to take care of the representations that appear at $(0,0)$. Let $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$, then

$$\delta\partial\omega(\vec{\gamma}) = \rho_0^0(t_{p+1}(\gamma_1))\partial\omega(\delta_0\vec{\gamma}) + \sum_{j=1}^{q+1}(-1)^j\partial\omega(\delta_j\vec{\gamma})$$

$$= \rho_0^0(t_{p+1}(\gamma_1))\sum_{k=0}^{p+1}(-1)^k\omega(\partial_k\delta_0\vec{\gamma}) + \sum_{j=1}^{q+1}\sum_{k=0}^{p+1}(-1)^{j+k}\omega(\partial_k\delta_j\vec{\gamma}).$$

On the other hand,

$$\partial\delta\omega(\vec{\gamma}) = \sum_{k=0}^{p+1}(-1)^k\delta\omega(\partial_k\vec{\gamma})$$

$$= \sum_{k=0}^{p+1}(-1)^k\Big(\rho_0^0(t_p((\partial_k\vec{\gamma})_1))\omega(\delta_0\partial_k\vec{\gamma}) + \sum_{j=1}^{q+1}(-1)^j\omega(\delta_j\partial_k\vec{\gamma})\Big).$$

As stated, the double sums in the above expressions coincide. By definition, $(\partial_k\vec{\gamma})_1 = \partial_k\gamma_1$. In accordance with Remark 2.5, this corresponds to $(g_{11}, ..., g_{1k-2}, g_{1k}g_{1k-1}, g_{1k+1}, ..., g_{1p+1}, h_1)$, when $0 < k \leq p$, and to $(g_{11}, ..., g_{1p}, h_0 i(g_{0p+1}))$, when $k = p+1$. Hence, for all $0 < k \leq p+1$,

$$t_p((\partial_k\vec{\gamma})_1) = h_1 i(g_{1p+1}...g_{11}) = t_{p+1}(\gamma_1).$$

Thus, using the first part of Eq. (19), one computes

$$\begin{aligned}(\delta\partial\omega - \partial\delta\omega)(\vec{\gamma}) &= \rho_0^0(t_{p+1}(\gamma_1))\omega(\vec{\gamma}_{1,1}) - \rho_0^0(t_p(\partial_0\gamma_1))\omega(\vec{\gamma}_{1,1}) \\ &= \rho_0^0(h_1 i(g_{1p+1}...g_{12}))\Big[\rho_0^0(i(g_{11})) - I\Big]\omega(\vec{\gamma}_{1,1}) = \rho_0^0(t_p(\partial_0\gamma_1)) \circ \phi\Big(\rho_1(g_{11})\omega(\vec{\gamma}_{1,1})\Big).\end{aligned}$$

■

4.12. Remark. As claimed, Lemma 4.10 and Proposition 4.11 are interpreted as saying that, if $\omega \in C(\mathcal{G}_p^q, V)$ and $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$, $\delta\partial\omega(\vec{\gamma})$ and $\partial\delta\omega(\vec{\gamma})$ lie on the same orbit of V. Indeed, their difference lies in the image of the difference map Δ, which, using Eq. (17), lies in the image of the structural map ϕ of the 2-vector space.

r-pages. If $\omega \in C(\mathcal{G}_p^q \times G^r, W)$ and $(\vec{\gamma}; \vec{f}) \in \mathcal{G}_{p+1}^{q+1} \times G^r$, then $\delta\partial\omega(\vec{\gamma}; \vec{f}), \partial\delta\omega(\vec{\gamma}; \vec{f}) \in W$. One cannot expect results analogous to Lemma 4.10 and Proposition 4.11, because there are no orbits in W. We prove instead that the compositions $\delta \circ \partial$ and $\partial \circ \delta$ are homotopic when regarded as maps between r-complexes. In what seems an overlap of notation, we call the homotopies Δ and refer to them as difference maps too.

We need to introduce further notation conventions to define the difference maps. Let $\vec{f} = (f_1, ..., f_r) \in G^r$, then for any pair of integers $1 \leq a < b \leq r$, define

$$\vec{f}_{[a,b]} := (f_a, f_{a+1}, ..., f_{b-1}, f_b) \qquad \text{and} \qquad \vec{f}_{[a,b)} := (f_a, f_{a+1}, ..., f_{b-2}, f_{b-1}).$$

With this shorthand, for $r > 1$ and $0 < n < r$, we define

$$c_{2n-1}, c_{2n} : G^{r-1} \times \mathcal{G} \longrightarrow G^r$$

by

$$c_{2n-1}(\vec{f}; \gamma) := \Big(\big(\vec{f}_{[1,r-n)}\big)^{hi(g)}, g^{-1}, \big(\vec{f}_{[r-n,r-2]}\big)^h, f_{r-1}^h g\Big) \tag{62}$$

and

$$c_{2n}(\vec{f}; \gamma) := \Big(\big(\vec{f}_{[1,r-n)}\big)^{hi(g)}, g^{-1}, \big(\vec{f}_{[r-n,r-2]}\big)^h, g\Big), \tag{63}$$

where $\vec{f} = (f_1, ..., f_{r-1}) \in G^{r-1}$, $\gamma \in \mathcal{G}$ and $(g, h) \in G \rtimes H$ is the image of γ under the isomorphism of Remark 2.1.

Let $p \geq 0$ and $q = 0$, then the difference maps

$$\Delta : C(G^r, W) \longrightarrow C(\mathcal{G}_{p+1} \times G^{r-1}, W)$$

are defined by

$$\Delta\omega(\gamma;\vec{f}) = \rho_0^1(i(g_0))^{-1}\omega((\vec{f})^{h_0}, g_0) + \sum_{n=1}^{r-1}(-1)^{n+1}\big(\omega(c_{2n-1}(\vec{f};\gamma_0)) - \omega(c_{2n}(\vec{f};\gamma_0))\big),$$

for $\omega \in C(G^r, W)$, $\vec{f} = (f_1, ..., f_{r-1}) \in G^{r-1}$ and $\gamma = (\gamma_0, ..., \gamma_p) \in \mathcal{G}_{p+1}$. Here, $(g_0, h_0) \in G\times H$ is the image of γ_0 under the isomorphism of Remark 2.1.

When $q > 0$, the difference maps

$$\Delta : C(\mathcal{G}_p^q \times G^r, W) \longrightarrow C(\mathcal{G}_{p+1}^{q+1} \times G^{r-1}, W)$$

are defined by

$$\Delta\omega(\vec{\gamma};\vec{f}) = \rho_0^1(i(pr_G(\gamma_{21} \barwedge ... \barwedge \gamma_{(q+1)1})))^{-1}\Big[\rho_0^1(i(g_{11}^{h_{21}...h_{(q+1)1}}))^{-1}\omega(\vec{\gamma}_{1,1}; (\vec{f})^{h_{11}}, g_{11}) +$$

$$+ \sum_{n=1}^{r-1}(-1)^{n+1}\big(\omega(\vec{\gamma}_{1,1}; c_{2n-1}(\vec{f};\gamma_{11})) - \omega(\vec{\gamma}_{1,1}; c_{2n}(\vec{f};\gamma_{11}))\big)\Big],$$

for $\omega \in C(\mathcal{G}_p^q \times G^{r+1}, W)$, $\vec{f} \in G^{r-1}$ and $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$ is as in Eq. (59).

Lemma 4.13 and Proposition 4.14 below justify the choice of notation.

4.13. Lemma. *Let $\omega \in C_1^{p,0}(\mathcal{G}, \phi) = C(G, W)$, then*

$$(\partial\delta - \delta\partial)\omega = (\Delta\delta_{(1)} - \delta'\Delta)\omega \in C(\mathcal{G}_{p+1} \times G, W). \tag{64}$$

Proof. Let $f \in G$, $\gamma \in \mathcal{G}_{p+1}$ and $(g_0, ..., g_p; h) \in G^{p+1} \times H$ its corresponding image under the isomorphism of Remark 2.5. Then,

$$\delta\partial\omega(\gamma; f) = \begin{cases} \omega(f^{t_{p+1}(\gamma)}) - \rho_0^1(t_{p+1}(\gamma))^{-1}\omega(f) & \text{if } p \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\partial\delta\omega(\gamma; f) = \rho_0^1(i(g_0))^{-1}\delta\omega(\partial_0\gamma; f) + \sum_{j=1}^{p+1}(-1)^j\delta\omega(\partial_j\gamma; f)$$

$$= \rho_0^1(i(g_0))^{-1}\Big[\omega(f^{t_p(\partial_0\gamma)}) - \rho_0^1(t_p(\partial_0\gamma))^{-1}\omega(f)\Big] +$$

$$+ \sum_{j=1}^{p+1}(-1)^j\Big[\omega(f^{t_p(\partial_j\gamma)}) - \rho_0^1(t_p(\partial_j\gamma))^{-1}\omega(f)\Big].$$

As in the proof of Proposition 4.11, $t_p(\partial_0\gamma) = hi(g_p...g_1)$ and, for $0 < j \leq p+1$, $t_p(\partial_j\gamma) = hi(g_p...g_0) = t_{p+1}(\gamma)$; therefore,

$$\partial\delta\omega(\gamma; f) = \begin{cases} \rho_0^1(i(g_0))^{-1}\omega(f^{t_p(\partial_0\gamma)}) - \rho_0^1(t_{p+1}(\gamma))^{-1}\omega(f) & \text{if } p \text{ is odd} \\ \rho_0^1(i(g_0))^{-1}\omega(f^{t_p(\partial_0\gamma)}) - \omega(f^{t_{p+1}(\gamma)}) & \text{otherwise,} \end{cases}$$

and, either way,

$$(\partial\delta - \delta\partial)\omega(\gamma; f) = \rho_0^1(i(g_0))^{-1}\omega(f^{t_p(\partial_0\gamma)}) - \omega(f^{t_{p+1}(\gamma)}).$$

On the other hand,

$$\begin{aligned}\delta'\Delta\omega(\gamma; f) &= \rho_0^1(t_{p+1}(\gamma))^{-1}\rho_1(f)\Delta\omega(\gamma)\\ &= \rho_0^1(t_p(\partial_0\gamma)i(g_0))^{-1}\rho_1(f)\rho_0^0(t_p(\partial_0\gamma))\phi\big(\omega(g_0)\big) = \rho_0^1(i(g_0))^{-1}\rho_1(f^{t_p(\partial_0\gamma)})\phi\big(\omega(g_0)\big)\end{aligned}$$

and

$$\begin{aligned}\Delta\delta_{(1)}\omega(\gamma; f) &= \rho_0^1(i(g_0))^{-1}\delta_{(1)}\omega(f^{h_0}, g_0) + \delta_{(1)}\omega(g_0^{-1}, f^{h_0}g_0) - \delta_{(1)}\omega(g_0^{-1}, g_0)\\ &= \rho_0^1(i(g_0))^{-1}\Big[\rho_0^1(i(f^{h_0}))\omega(g_0) - \omega(f^{h_0}g_0) + \omega(f^{h_0})\Big] +\\ &\quad + \Big[\rho_0^1(i(g_0))^{-1}\omega(f^{h_0}g_0) - \omega(g_0^{-1}f^{h_0}g_0) + \omega(g_0^{-1})\Big]\\ &\quad - \Big[\rho_0^1(i(g_0))^{-1}\omega(g_0) - \cancel{\omega(g_0^{-1}g_0)} + \omega(g_0^{-1})\Big]\\ &= \rho_0^1(i(g_0))^{-1}\Big[\rho_0^1(i(f^{h_0}))\omega(g_0) + \omega(f^{h_0}) - \omega(g_0)\Big] - \omega(f^{h_0 i(g_0)})\\ &= \rho_0^1(i(g_0))^{-1}\Big[\rho_1(f^{h_0})\phi\big(\omega(g_0)\big) + \omega(f^{h_0})\Big] - \omega(f^{h_0 i(g_0)}),\end{aligned}$$

where the last equality follows from the second half of Eq. (19). In so, given that $t_p(\partial_0\gamma) = h_0$ and $t_{p+1}(\gamma) = h_0 i(g_0)$, one computes the difference to be

$$(\Delta\delta_{(1)} - \delta'\Delta)\omega(\gamma; f) = \rho_0^1(i(g_0))^{-1}\omega(f^{t_p(\partial_0\gamma)}) - \omega(f^{t_{p+1}(\gamma)}),$$

and the result follows.

■

4.14. Proposition. *Let $\omega \in C(\mathcal{G}_p^q \times G, W)$, then*

$$(\partial\delta - \delta\partial)\omega = (\Delta\delta_{(1)} - \delta'\Delta)\omega \in C(\mathcal{G}_{p+1}^{q+1} \times G, W). \tag{65}$$

Proof. Let $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$ be as in Eq. (59) and $f \in G$. To ease up notation, we introduce the shorthand

$$\rho^q(\vec{\gamma}_{\bullet 1}) := \rho_0^1(i(pr_G(\gamma_{11} \boxtimes \ldots \boxtimes \gamma_{q1})))^{-1}. \tag{66}$$

We compute the left hand side of Eq. (65). On the one hand,

$$\partial\delta\omega(\vec{\gamma};f) = \rho^{q+1}(\vec{\gamma}_{\bullet 1})\delta\omega(\partial_0\vec{\gamma};f) + \sum_{j=1}^{p+1}(-1)^j\delta\omega(\partial_j\vec{\gamma};f)$$

$$= \rho^{q+1}(\vec{\gamma}_{\bullet 1})\Big(\omega(\vec{\gamma}_{1,1};f^{t_p(\partial_0\gamma_1)}) + \sum_{k=1}^{q}(-1)^k\omega(\delta_k\partial_0\vec{\gamma};f)+$$

$$+ (-1)^{q+1}\rho_0^1(t_p(\partial_0\gamma_{q+1}))^{-1}\omega(\delta_{q+1}\partial_0\vec{\gamma};f)\Big)+$$

$$+\sum_{j=1}^{p+1}(-1)^j\Big(\omega(\delta_0\partial_j\vec{\gamma};f^{t_p(\partial_j\gamma_1)}) + \sum_{k=1}^{q}(-1)^k\omega(\delta_k\partial_j\vec{\gamma};f)+$$

$$+ (-1)^{q+1}\rho_0^1(t_p(\partial_j\gamma_{q+1}))^{-1}\omega(\delta_{q+1}\partial_j\vec{\gamma};f)\Big);$$

while on the other,

$$\delta\partial\omega(\vec{\gamma};f) = \partial\omega(\delta_0\vec{\gamma};f^{t_{p+1}(\gamma_1)}) + \sum_{k=1}^{q}(-1)^k\partial\omega(\delta_k\vec{\gamma};f) + (-1)^{q+1}\rho_0^1(t_{p+1}(\gamma_{q+1}))^{-1}\omega(\delta_{q+1}\vec{\gamma};f)$$

$$= \rho^q((\delta_0\vec{\gamma})_{\bullet 1})\omega(\vec{\gamma}_{1,1};f^{t_{p+1}(\gamma_1)}) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\delta_0\vec{\gamma};f^{t_{p+1}(\gamma_1)})+$$

$$+\sum_{k=1}^{q}(-1)^k\Big(\rho^q((\delta_k\vec{\gamma})_{\bullet 1})\omega(\partial_0\delta_k\vec{\gamma};f) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\delta_k\vec{\gamma};f)\Big)+$$

$$+(-1)^{q+1}\rho_0^1(t_{p+1}(\gamma_{q+1}))^{-1}\Big(\rho^q((\delta_{q+1}\vec{\gamma})_{\bullet 1})\omega(\partial_0\delta_{q+1}\vec{\gamma};f) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\delta_{q+1}\vec{\gamma};f)\Big).$$

We claim that taking the difference yields

$$(\partial\delta - \delta\partial)\omega(\vec{\gamma};f) = \rho^{q+1}(\vec{\gamma}_{\bullet 1})\omega(\vec{\gamma}_{1,1};f^{t_p(\partial_0\gamma_1)}) - \rho^q((\delta_0\vec{\gamma})_{\bullet 1})\omega(\vec{\gamma}_{1,1};f^{t_{p+1}(\gamma_1)}).$$

This follows from the commutativity of all simplicial maps $\delta_k\partial_j = \partial_j\delta_k$, together with the following identities:

- $\rho^{q+1}(\vec{\gamma}_{\bullet 1}) = \rho^q((\delta_k\vec{\gamma})_{\bullet 1})$ for $1 \leq k \leq q$. Indeed, for the ranging values of k,

$$\rho^q((\delta_k\vec{\gamma})_{\bullet 1}) = \rho_0^1(i(pr_G(\gamma_{11}\barwedge\ldots\barwedge(\gamma_{(k-1)1}\barwedge\gamma_{k1})\barwedge\ldots\barwedge\gamma_{(q+1)1})))^{-1} = \rho^{q+1}(\vec{\gamma}_{\bullet 1}).$$

- $\rho^{q+1}(\vec{\gamma}_{\bullet 1})\rho_0^1(t_p(\partial_0\gamma_{q+1}))^{-1} = \rho_0^1(t_{p+1}(\gamma_{q+1}))^{-1}\rho^q((\delta_{q+1}\vec{\gamma})_{\bullet 1})$. Indeed, using Lemma 3.5, we write

$$\begin{aligned}\rho^{q+1}(\vec{\gamma}_{\bullet 1}) &= \rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}g_{21}^{h_{31}\dots h_{(q+1)1}}\dots g_{(q-1)1}^{h_{q1}h_{(q+1)1}}g_{q1}^{h_{(q+1)1}}g_{(q+1)1}))^{-1}\\ &= \rho_0^1(i(g_{(q+1)1}))^{-1}\rho_0^1\big(i\big((g_{11}^{h_{21}\dots h_{q1}}g_{21}^{h_{31}\dots h_{q1}}\dots g_{(q-1)1}^{h_{q1}}g_{q1})^{h_{(q+1)1}}\big)\big)^{-1}\\ &= \rho_0^1(h_{(q+1)1}i(g_{(q+1)1}))^{-1}\rho_0^1\big(i(g_{11}^{h_{21}\dots h_{q1}}g_{21}^{h_{31}\dots h_{q1}}\dots g_{(q-1)1}^{h_{q1}}g_{q1})\big)^{-1}\rho_0^1(h_{(q+1)1}),\end{aligned}$$

  and by definition $t_p(\partial_0\gamma_{q+1}) = t(\gamma_{(q+1)2}) = s(\gamma_{(q+1)1}) = h_{(q+1)1}$.

- For $1 \le j \le q+1$, $t_p(\partial_j\gamma_b) = t_{p+1}(\gamma_b)$.

Using Lemma 3.5 once more,

$$\begin{aligned}\rho^{q+1}(\vec{\gamma}_{\bullet 1}) &= \rho_0^1(i(g_{21}^{h_{31}\dots h_{(q+1)1}}\dots g_{(q-1)1}^{h_{q1}h_{(q+1)1}}g_{q1}^{h_{(q+1)1}}g_{(q+1)1}))^{-1}\rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}))^{-1}\\ &= \rho^q((\delta_0\vec{\gamma})_{\bullet 1})\rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}))^{-1},\end{aligned}$$

and we rewrite the difference as

$$(\partial\delta - \delta\partial)\omega(\vec{\gamma}; f) = \rho^q((\delta_0\vec{\gamma})_{\bullet 1})\big[\rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}))^{-1}\omega(\vec{\gamma}_{1,1}; f^{h_{11}}) - \omega(\vec{\gamma}_{1,1}; f^{h_{11}i(g_{11})})\big].$$

We turn to compute the right hand side of Eq. (65),

$$\begin{aligned}\delta'\Delta\omega(\vec{\gamma}; f) &= \rho_0^1(t_{p+1}(\gamma_1)\dots t_{p+1}(\gamma_{q+1}))^{-1}\rho_1(f)\Delta\omega(\vec{\gamma})\\ &= \rho_0^1(t(\gamma_{11}\between\dots\between\gamma_{(q+1)1}))^{-1}\rho_1(f)\rho_0^0(t_p(\partial_0\gamma_1)\dots t_p(\partial_0\gamma_{q+1}))\phi(\omega(\vec{\gamma}_{1,1}; g_{11}))\\ &= \rho_0^1(h_{11}\dots h_{(q+1)1}i(pr_G(\gamma_{11}\between\dots\between\gamma_{(q+1)1})))^{-1}\rho_1(f)\rho_0^0(h_{11}\dots h_{(q+1)1})\phi(\omega(\vec{\gamma}_{1,1}; g_{11}))\\ &= \rho^{q+1}(\vec{\gamma}_{\bullet 1})\rho_1(f^{h_{11}\dots h_{(q+1)1}})\phi(\omega(\vec{\gamma}_{1,1}; g_{11})).\end{aligned}$$

Adding all terms, factoring $\rho^{q+1}(\vec{\gamma}_{\bullet 1})$ as before and using the second half of Eq. (19) yields

$$\begin{aligned}(\delta\partial - \partial\delta - \delta'\Delta)\omega(\vec{\gamma}; f) = \rho^q((\delta_0\vec{\gamma})_{\bullet 1})\Big[&\omega(\vec{\gamma}_{1,1}; f^{h_{11}i(g_{11})}) +\\ &- \rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}))^{-1}\big(\omega(\vec{\gamma}_{1,1}; f^{h_{11}}) + [\rho_0^1(i(f^{h_{11}\dots h_{(q+1)1}})) - I]\omega(\vec{\gamma}_{1,1}; g_{11})\big)\Big].\end{aligned}$$

Adding and subtracting $\rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}))^{-1}\omega(\vec{\gamma}_{1,1}; f^{h_{11}}g_{11})$ and $\omega(\vec{\gamma}_{1,1}; g_{11}^{-1})$,

$$\begin{aligned}(\delta\partial - \partial\delta - \delta'\Delta)\omega(\vec{\gamma}; f) =&\rho^q((\delta_0\vec{\gamma})_{\bullet 1})\Big[\delta_{(1)}\omega(\vec{\gamma}_{1,1}; g_{11}^{-1}, g_{11}) - \delta_{(1)}\omega(\vec{\gamma}_{1,1}; g_{11}^{-1}, f^{h_{11}}g_{11}) +\\ &- \rho_0^1(i(g_{11}^{h_{21}\dots h_{(q+1)1}}))^{-1}\delta_{(1)}\omega(\vec{\gamma}_{1,1}; f^{h_{11}}, g_{11})\Big] = -\Delta\delta_{(1)}\omega(\vec{\gamma}; f).\end{aligned}$$

■

4.15. Proposition. *Let $r > 1$ and $\omega \in C_r^{p,0}(\mathcal{G}, \phi) = C(G^r, W)$, then*

$$(-1)^r(\delta\partial - \partial\delta)\omega = (\Delta\delta_{(1)} - \delta_{(1)}\Delta)\omega \in C(\mathcal{G}_{p+1} \times G^r, W). \tag{67}$$

Proof. We compute the left hand side of Eq. (67). Let $\vec{f} = (f_1, ..., f_r) \in G^r$, $\gamma = (\gamma_0, ..., \gamma_p) \in \mathcal{G}_{p+1}$ and $(g_0, h_0) \in G \rtimes H$ be the image of γ_0 under the isomorphism of Remark 2.1. Then,

$$\delta\partial\omega(\gamma; \vec{f}) = \begin{cases} \omega\big((\vec{f})^{h_0 i(g_0)}\big) - \rho_0^1(h_0 i(g_0))^{-1}\omega(\vec{f}) & \text{if } p \text{ is odd} \\ 0 & \text{otherwise} \end{cases},$$

and

$$\begin{aligned} \partial\delta\omega(\gamma; \vec{f}) &= \rho_0^1(i(g_0))^{-1}\delta\omega(\partial_0\gamma; \vec{f}) + \sum_{j=1}^{p+1}(-1)^j\delta\omega(\partial_j\gamma; \vec{f}) \\ &= \rho_0^1(i(g_0))^{-1}\Big[\omega\big((\vec{f})^{h_0}\big) - \rho_0^1(h_0)^{-1}\omega(\vec{f})\Big] + \\ &\qquad + \sum_{j=1}^{p+1}(-1)^j\Big[\omega\big((\vec{f})^{h_0 i(g_0)}\big) - \rho_0^1(h_0 i(g_0))^{-1}\omega(\vec{f})\Big]. \end{aligned}$$

As in the proof of Lemma 4.13, one concludes

$$\partial\delta\omega(\gamma; \vec{f}) = \begin{cases} \rho_0^1(i(g_0))^{-1}\Big[\omega\big((\vec{f})^{h_0}\big) - \rho_0^1(h_0)^{-1}\omega(\vec{f})\Big] & \text{if } p \text{ is odd} \\ \rho_0^1(i(g_0))^{-1}\omega\big((\vec{f})^{h_0}\big) - \omega\big((\vec{f})^{h_0 i(g_0)}\big) & \text{otherwise} \end{cases},$$

and in both cases

$$(\delta\partial - \partial\delta)\omega(\gamma; \vec{f}) = \omega\big((\vec{f})^{h_0 i(g_0)}\big) - \rho_0^1(i(g_0))^{-1}\omega\big((\vec{f})^{h_0}\big). \tag{68}$$

Turning to the left hand side of Eq. (67), on the one hand,

$$\Delta\delta_{(1)}\omega(\gamma; \vec{f}) = \rho_0^1(i(g_0))^{-1}\delta_{(1)}\omega((\vec{f})^{h_0}, g_0) + \sum_{n=1}^{r}(-1)^{n+1}T_n,$$

where

$$T_n := \delta_{(1)}\omega(c_{2n-1}(\vec{f}; \gamma_0)) - \delta_{(1)}\omega(c_{2n}(\vec{f}; \gamma_0)),$$

and, on the other,

$$\delta_{(1)}\Delta\omega(\gamma; \vec{f}) = \rho_0^1(i(f_1^{h_0 i(g_0)}))\Delta\omega(\gamma; \delta_0\vec{f}) + \sum_{k=1}^{r}(-1)^k\Delta\omega(\gamma; \delta_k\vec{f}). \tag{69}$$

Rearranging Eq. (69),

$$\delta_{(1)}\Delta\omega(\gamma; \vec{f}) = S_0 + \sum_{n=1}^{r-1}(-1)^{n+1}S_n,$$

where

$$S_0 := \rho_0^1(i(f_1^{h_0 i(g_0)}))\rho_0^1(i(g_0))^{-1}\omega((\delta_0\vec{f})^{h_0}, g_0) + \sum_{k=1}^{r}(-1)^k\rho_0^1(i(g_0))^{-1}\omega((\delta_k\vec{f})^{h_0}, g_0)$$

and

$$S_n := \rho_0^1(i(f_1^{h_0 i(g_0)}))\big(\omega(c_{2n-1}(\delta_0\vec{f};\gamma_0)) - \omega(c_{2n}(\delta_0\vec{f};\gamma_0)\big) + \\ + \sum_{k=1}^{r}(-1)^k\big(\omega(c_{2n-1}(\delta_k\vec{f};\gamma_0)) - \omega(c_{2n}(\delta_k\vec{f};\gamma_0)\big).$$

Expanding further the first term of $\Delta\delta_{(1)}\omega$,

$$\rho_0^1(i(g_0))^{-1}\delta_{(1)}\omega((\vec{f})^{h_0}, g_0) = \rho_0^1(i(g_0))^{-1}\Big(\rho_0^1(i(f_1^{h_0}))\omega(\delta_0(\vec{f})^{h_0}, g_0) + \\ + \sum_{k=1}^{r-1}(-1)^k\omega(\delta_k(\vec{f})^{h_0}, g_0) + (-1)^r\omega(\delta_r(\vec{f})^{h_0}, f_r^{h_0}g_0) + (-1)^{r+1}\omega((\vec{f})^{h_0})\Big),$$

it is made patent that, if $\epsilon_0 := \rho_0^1(i(g_0))^{-1}\delta_{(1)}\omega((\vec{f})^{h_0}, g_0) - S_0$, then

$$\epsilon_0 = (-1)^r\rho_0^1(i(g_0))^{-1}\Big(\omega(\delta_r(\vec{f})^{h_0}, f_r^{h_0}g_0) - \omega((\vec{f})^{h_0}) + \omega((\delta_r\vec{f})^{h_0}, g_0)\Big),$$

as $(\delta_k\vec{f})^{h_0} = \delta_k(\vec{f})^{h_0}$ and $g_0^{-1}f_1^{h_0} = f_1^{h_0 i(g_0)}g_0^{-1}$.

Since one can equivalently write

$$c_{2r-1}(\vec{f};\gamma_0) = (g_0^{-1}, (\delta_r\vec{f})^{h_0}, f_r^{h_0}g_0) \quad \text{and} \quad c_{2r}(\vec{f};\gamma_0) = (g_0^{-1}, (\delta_r\vec{f})^{h_0}, g_0),$$

T_r gets expanded as

$$\begin{aligned} T_r =& \rho_0^1(i(g_0))^{-1}\big(\omega((\delta_r\vec{f})^{h_0}, f_r^{h_0}g_0) - \omega((\delta_r\vec{f})^{h_0}, g_0)\big) + \\ &- \big(\omega(g_0^{-1}f_1^{h_0}, \delta_0(\delta_r\vec{f})^{h_0}, f_r^{h_0}g_0) - \omega(g_0^{-1}f_1^{h_0}, \delta_0(\delta_r\vec{f})^{h_0}, g_0)\big) + \\ &+ \sum_{k=1}^{r-2}(-1)^{k+1}\big(\omega(g_0^{-1}, \delta_k(\delta_r\vec{f})^{h_0}, f_r^{h_0}g_0) - \omega(g_0^{-1}, \delta_k(\delta_r\vec{f})^{h_0}, g_0)\big) + \\ &+ (-1)^r\big(\omega(g_0^{-1}, \delta_{r-1}(\delta_r\vec{f})^{h_0}, (f_{r-1}f_r)^{h_0}g_0) - \omega(g_0^{-1}, \delta_{r-1}(\delta_r\vec{f})^{h_0}, f_{r-1}^{h_0}g_0)\big) + \\ &+ (-1)^{r+1}\big(\cancel{\omega(g_0^{-1}, (\delta_r\vec{f})^{h_0})} - \cancel{\omega(g_0^{-1}, (\delta_r\vec{f})^{h_0})}\big). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
S_{r-1} =&\rho_0^1(i(f_1^{h_0 i(g_0)}))^{-1}\big(\omega(g_0^{-1},\delta_{r-1}(\delta_0\vec{f})^{h_0},f_r^{h_0}g_0)-\omega(g_0^{-1},\delta_{r-1}(\delta_0\vec{f})^{h_0},g_0)\big)+\\
&+\sum_{k=1}^{r-2}(-1)^k\big(\omega(g_0^{-1},\delta_{r-1}(\delta_k\vec{f})^{h_0},f_r^{h_0}g_0)-\omega(g_0^{-1},\delta_{r-1}(\delta_k\vec{f})^{h_0},g_0)\big)+\\
&+(-1)^{r-1}\big(\omega(g_0^{-1},\delta_{r-1}(\delta_{r-1}\vec{f})^{h_0},(f_{r-1}f_r)^{h_0}g_0)-\cancel{\omega(g_0^{-1},\delta_{r-1}(\delta_{r-1}\vec{f})^{h_0},g_0)}\big)\\
&+(-1)^{r}\big(\omega(g_0^{-1},\delta_{r-1}(\delta_r\vec{f})^{h_0},f_{r-1}^{h_0}g_0)-\cancel{\omega(g_0^{-1},\delta_{r-1}(\delta_r\vec{f})^{h_0},g_0)}\big);
\end{aligned}
$$

hence, updating the difference $\epsilon_1 := \epsilon_0 + (-1)^{r+1}T_r - (-1)^r S_{r-1}$ becomes

$$
\begin{aligned}
\epsilon_1 =&(-1)^{r+1}\Big(\rho_0^1(i(g_0))^{-1}\omega((\vec{f})^{h_0})-\omega(g_0^{-1}f_1^{h_0},\delta_0(\delta_r\vec{f})^{h_0},f_r^{h_0}g_0)+\omega(g_0^{-1}f_1^{h_0},\delta_0(\delta_r\vec{f})^{h_0},g_0)+\\
&+\rho_0^1(i(f_1^{h_0 i(g_0)}))^{-1}\big(\omega(g_0^{-1},\delta_{r-1}(\delta_0\vec{f})^{h_0},f_r^{h_0}g_0)-\omega(g_0^{-1},\delta_{r-1}(\delta_0\vec{f})^{h_0},g_0)\big)\Big).
\end{aligned}
$$

In general, for $2 \le n < r$,

$$
\begin{aligned}
T_n =&\rho_0^1(i(f_1^{h_0 i(g_0)}))^{-1}\big(\omega((\vec{f}_{[2,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r)})^{h_0},f_r^{h_0}g_0)+\\
&\qquad-\omega((\vec{f}_{[2,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r)})^{h_0},g_0)\big)+\\
&+\sum_{k=1}^{r-(n+1)}(-1)^k\big(\omega(\delta_k(\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r)})^{h_0},f_r^{h_0}g_0)+\\
&\qquad-\omega(\delta_k(\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r)})^{h_0},g_0)\big)+\\
&+(-1)^{r-n}\big(\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)},g_0^{-1}f_{r-n}^{h_0},(\vec{f}_{[r-n+1,r)})^{h_0},f_r^{h_0}g_0)+\\
&\qquad-\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)},g_0^{-1}f_{r-n}^{h_0},(\vec{f}_{[r-n+1,r)})^{h_0},g_0)\big)+\\
&+(-1)^{r-n+1}\big(\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1}f_{r-n+1}^{h_0},(\vec{f}_{[r-n+2,r)})^{h_0},f_r^{h_0}g_0)+\\
&\qquad-\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1}f_{r-n+1}^{h_0},(\vec{f}_{[r-n+2,r)})^{h_0},g_0)\big)+\\
&+\sum_{k=1}^{n-2}(-1)^{r-n+1+k}\big(\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},\delta_k(\vec{f}_{[r-n+1,r)})^{h_0},f_r^{h_0}g_0)+\\
&\qquad-\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},\delta_k(\vec{f}_{[r-n+1,r)})^{h_0},g_0)\big)+\\
&+(-1)^{r}\big(\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r-1)})^{h_0},(f_{r-1}f_r)^{h_0}g_0)+\\
&\qquad-\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r-1)})^{h_0},f_{r-1}^{h_0}g_0)\big)+\\
&+(-1)^{r+1}\big(\cancel{\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r)})^{h_0})}-\cancel{\omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)},g_0^{-1},(\vec{f}_{[r-n+1,r)})^{h_0})}\big),
\end{aligned}
$$

and, for $1 \leq n \leq r-2$,

$$\begin{aligned}
S_n =& \rho_0^1(i(f_1^{h_0 i(g_0)}))^{-1}\big(\omega((\vec{f}_{[2,r-n]})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n+1,r)})^{h_0}, f_r^{h_0} g_0) + \\
& \qquad - \omega((\vec{f}_{[2,r-n]})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n+1,r)})^{h_0}, g_0)\big) + \\
& + \sum_{k=1}^{r-(n+1)} (-1)^k \big(\omega(\delta_k(\vec{f}_{[1,r-n]})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n+1,r)})^{h_0}, f_r^{h_0} g_0) + \\
& \qquad - \omega(\delta_k(\vec{f}_{[1,r-n]})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n+1,r)})^{h_0}, g_0)\big) + \\
& + \sum_{k=r-n}^{r-2} (-1)^k \big(\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)}, g_0^{-1}, (\delta_k \vec{f}_{[r-n,r)})^{h_0}, f_r^{h_0} g_0) + \\
& \qquad - \omega((\vec{f}_{[1,r-n]})^{h_0 i(g_0)}, g_0^{-1}, (\delta_k \vec{f}_{[r-n,r)})^{h_0}, g_0)\big) + \\
& + (-1)^{r-1}\big(\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n,r-1)})^{h_0}, (f_{r-1} f_r)^{h_0} g_0) + \\
& \qquad - \cancel{\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n,r-1)})^{h_0}, g_0)}\big) + \\
& + (-1)^{r}\big(\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n,r-1)})^{h_0}, f_{r-1}^{h_0} g_0) + \\
& \qquad - \cancel{\omega((\vec{f}_{[1,r-n)})^{h_0 i(g_0)}, g_0^{-1}, (\vec{f}_{[r-n,r-1)})^{h_0}, g_0)}\big).
\end{aligned}$$

Thus, defining inductively $\epsilon_{n+1} := \epsilon_n + (-1)^{r+1-n}(T_{r-n} + S_{r-(n+1)})$,

$$\begin{aligned}
\epsilon_{r-1} =& (-1)^{r+1}\rho_0^1(i(g_0))^{-1}\omega((\vec{f})^{h_0}) + \\
& - \Big(\rho_0^1(i(f_1^{h_0 i(g_0)}))^{-1}\big(\omega((\vec{f}_{[2,r)})^{h_0 i(g_0)}, g_0^{-1}, f_r^{h_0} g_0) - \omega((\vec{f}_{[2,r)})^{h_0 i(g_0)}, g_0^{-1}, g_0)\big) + \\
& + \sum_{k=1}^{r-2}(-1)^k\big(\omega(\delta_k(\vec{f}_{[1,r)})^{h_0 i(g_0)}, g_0^{-1}, f_r^{h_0} g_0) - \omega(\delta_k(\vec{f}_{[1,r)})^{h_0 i(g_0)}, g_0^{-1}, g_0)\big) + \\
& + (-1)^{r-1}\big(\omega((\vec{f}_{[1,r-1)})^{h_0 i(g_0)}, g_0^{-1} f_{r-1}^{h_0}, f_r^{h_0} g_0) - \omega((\vec{f}_{[1,r-1)})^{h_0 i(g_0)}, g_0^{-1} f_{r-1}^{h_0}, g_0)\big)\Big).
\end{aligned}$$

Naturally, $(\Delta\delta_{(1)} - \delta'\Delta)\omega(\gamma; \vec{f}) = \epsilon_{r-1} + T_1$; therefore,

$$\begin{aligned}
(\Delta\delta_{(1)} - \delta'\Delta)\omega(\gamma; \vec{f}) =& (-1)^{r+1}\rho_0^1(i(g_0))^{-1}\omega((\vec{f})^{h_0}) + \\
& + (-1)^r\Big(\omega((\delta_r \vec{f})^{h_0 i(g_0)}, g_0^{-1} f_r^{h_0} g_0) - \cancelto{0}{\omega((\delta_r \vec{f})^{h_0 i(g_0)}, g_0^{-1} g_0)}\Big)
\end{aligned}$$

and the result follows.

■

4.16. Proposition. *Let $r > 1$ and $\omega \in C(\mathcal{G}_p^q \times G^r, W)$, then*

$$(-1)^r(\delta\partial - \partial\delta)\omega = (\Delta\delta_{(1)} - \delta_{(1)}\Delta)\omega \in C(\mathcal{G}_{p+1}^{q+1} \times G^r, W) \tag{70}$$

Proof. Eq. (70) follows from an argument analog to the one in Proposition 4.15 after noticing that the left hand side takes the form of Eq. (68) (cf. Eq. (71)). Let $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+1}$ be as in Eq. (59) and $\vec{f} = (f_1, ..., f_r) \in G^r$, then

$$\delta\partial\omega(\vec{\gamma}; \vec{f}) = \partial\omega(\delta_0\vec{\gamma}; (\vec{f})^{h_{11}i(g_{11})}) + \\ + \sum_{k=1}^{q}(-1)^k\partial\omega(\delta_k\vec{\gamma}; \vec{f}) + (-1)^{q+1}\rho_0^1(h_{(q+1)1}i(g_{(q+1)1}))^{-1}\partial\omega(\delta_{q+1}\vec{\gamma}; \vec{f}).$$

Assuming the convention of Eq. (66),

$$\delta\partial\omega(\vec{\gamma}; \vec{f}) = \rho^q((\delta_0\vec{\gamma})_{\bullet 1})\omega(\vec{\gamma}_{1,1}; (\vec{f})^{h_{11}i(g_{11})}) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\delta_0\vec{\gamma}; (\vec{f})^{h_{11}i(g_{11})}) + \\ + \sum_{k=1}^{q}(-1)^k\Big[\rho^{q+1}(\vec{\gamma}_{\bullet 1})\omega(\partial_0\delta_k\vec{\gamma}; \vec{f}) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\delta_k\vec{\gamma}; \vec{f})\Big] + \\ +(-1)^{q+1}\rho_0^1(h_{(q+1)1}i(g_{(q+1)1}))^{-1}\Big[\rho^q((\delta_{q+1}\vec{\gamma})_{\bullet 1})\omega(\vec{\gamma}_{0,q}; \vec{f}) + \sum_{j=1}^{p+1}(-1)^j\omega(\partial_j\delta_{q+1}\vec{\gamma}; \vec{f})\Big],$$

and

$$\partial\delta\omega(\vec{\gamma}; \vec{f}) = \rho^{q+1}(\vec{\gamma}_{\bullet 1})\delta\omega(\partial_0\vec{\gamma}; \vec{f}) + \sum_{j=1}^{p+1}(-1)^j\delta\omega(\partial_j\vec{\gamma}; \vec{f}) \\ = \rho^{q+1}(\vec{\gamma}_{\bullet 1})\Big[\omega(\vec{\gamma}_{1,1}; (\vec{f})^{h_{11}}) + \sum_{k=1}^{q}(-1)^k\omega(\delta_k\partial_0\vec{\gamma}; \vec{f}) + (-1)^{q+1}\rho_0^1(h_{(q+1)1})^{-1}\omega(\vec{\gamma}_{q+1,1}; \vec{f})\Big] + \\ + \sum_{j=1}^{p+1}(-1)^j\Big[\omega(\delta_0\partial_j\vec{\gamma}; (\vec{f})^{h_{11}i(g_{11})}) + \\ + \sum_{k=1}^{q}(-1)^k\omega(\delta_k\partial_j\vec{\gamma}; \vec{f}) + (-1)^{q+1}\rho_0^1(h_{(q+1)1}i(g_{(q+1)1}))^{-1}\omega(\delta_{q+1}\partial_j\vec{\gamma}; \vec{f})\Big].$$

Aside from the visible terms in common, using the following identity for $y \in H$ and $x, z \in G$:

$$\rho_0^1(yi(x))^{-1}\rho_0^1(i(z))^{-1} = \rho_0^1(i(z)yi(x))^{-1} = \rho_0^1(yi(z^y)i(x)))^{-1} = \rho_0^1(i(z^yx))^{-1}\rho_0^1(y)^{-1},$$

applied to $y = h_{(q+1)1}$, $x = g_{(q+1)1}$ and $z = pr_G(\gamma_{11} \barwedge ... \barwedge \gamma_{q1}) = g_{11}^{h_{21}...h_{q1}} g_{21}^{h_{31}...h_{q1}} ... g_{q-1}^{h_{q1}} g_{q1}$ (cf. Lemma 3.5), the difference becomes

$$(\delta\partial - \partial\delta)\omega(\vec{\gamma}; \vec{f}) = \rho^{q+1}((\delta_0\vec{\gamma})_{\bullet 1})\Big(\omega(\vec{\gamma}_{1,1}; (\vec{f})^{h_{11}i(g_{11})}) - \rho_0^1(i(g_{11}^{h_{21}...h_{q1}}))^{-1}\omega(\vec{\gamma}_{1,1}; (\vec{f})^{h_{11}})\Big). \tag{71}$$

■

4.17. Higher Difference maps and outlook. Let $\omega \in C_r^{p,q}(\mathcal{G}, \phi)$. In Section 3, it is explained that if the total differential is defined by Eq. (28), $\nabla^2\omega$ is not necessarily zero; more specifically, in general,

$$(\nabla^2\omega)_r^{p+1,q+1} = \partial\delta\omega - \delta\partial\omega \neq 0.$$

We can sum up Subsection 4.9 as saying that, if the total differential is redefined by

$$\nabla = (-1)^p\Big(\delta_{(1)} + \partial + \Delta + (-1)^r\delta\Big), \tag{72}$$

$(\nabla^2\omega)_r^{p+1,q+1}$ is ensured to vanish. However, adding the difference maps does not yet imply that $\nabla^2\omega$ vanishes; indeed, $(\nabla^2\omega)_{r-1}^{p+2,q+1}$ and $(\nabla^2\omega)_{r-1}^{p+1,q+2}$ need not be zero, as Δ does not necessarily commute with neither ∂ nor δ. In Subsection 3.14, these non-vanishing coordinates of $\nabla^2\omega$ are studied in the case $p+q+r = 1$, and it is explained that one can ensure again $\nabla^2\omega = 0$ by adding the second difference maps $\Delta_{2,1}$ and $\Delta_{1,2}$ (cf. Eq.'s (38) and (39)). Thus, redefining the total differential by Eq. (4) forces $(\nabla^2\omega)_{r-1}^{p+2,q+1}$ and $(\nabla^2\omega)_{r-1}^{p+1,q+2}$ to be zero, but, in general, as with the first difference maps, the ∇ of Eq. (4) does not yet square to zero in general. Adding higher difference maps creates new non-vanishing coordinates, which, in turn, can be made to zero by adding further higher difference maps. Ultimately, if the total differential is defined by

$$\nabla = (-1)^p\Big(\delta_{(1)} + \sum_{a+b>0} (-1)^{(a+1)(r+b+1)}\Delta_{a,b}\Big) \tag{73}$$

for some

$$\Delta_{a,b} : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_{r+1-(a+b)}^{p+a,q+b}(\mathcal{G}, \phi)$$

where we set $\Delta_{1,0} := \partial$, $\Delta_{0,1} := \delta$, $\Delta_{1,1} := \Delta$ and $\Delta_{a,0} = \Delta_{0,b} = 0$ whenever $a, b > 1$, the following is a rephrasing of the relation $\nabla^2 = 0$:

4.18. Theorem. *Let $\omega \in C_r^{p,q}(\mathcal{G}, \phi)$ and ∇ be defined by (73). Then, $\nabla^2\omega = 0$ if and only if*

i) $\delta_{(1)}^2\omega = 0$;

ii) for all $0 \leq n \leq r+1$ and $0 \leq m \leq n$,

$$\sum_{0<i+j<n} (-1)^{i(i-n)+(i+1)(j+1)}\Delta_{n-m-i,m-j} \circ \Delta_{i,j}\omega = (-1)^r[\delta_{(1)}, \Delta_{n-m,m}]\omega, \tag{74}$$

where $[\cdot,\cdot]$ stands for the commutator of operators; and

iii) for all $0 < m \leq r+1$,

$$\sum_{0<i+j<r+2} (-1)^{i(i-r)+(i+1)(j+1)} \Delta_{r+2-m-i,m-j} \circ \Delta_{i,j}\omega = (-1)^{r+1} \Delta_{r+2-m,m} \circ \delta_{(1)}\omega. \quad (75)$$

Here, we assume the convention that $\Delta_{a,b} = 0$ *whenever* $a < 0$ *or* $b < 0$.

As it is briefly mentioned above, in Section 3, the necessary higher difference maps for the cases $p+q+r \leq 1$ are defined and they are shown to verify the relations of Theorem 4.18; moreover, that $\delta_{(1)}$ squares to zero is established (see Lemmas 3.6 and 3.7), as well as the relations of Eq. (74) in the cases when $n \leq 2$ (see Subsections 3.2.1, 3.5.1, 4.2, 4.9). Although as of the writing of this paper, we are unable to provide a general formula for the higher difference maps $\Delta_{a,b}$, we present evidence that suggests that one can find such maps ultimately turning $(C_{tot}(\mathcal{G},\phi),\nabla)$ into a complex. In particular, with the formulas we include in the appendix, the complex (29) is extended up to degree 5 (see Theorem 4.23 below).

We devote the remainder of this section to define the necessary higher difference maps to prove that Eq. (75) holds in general for $m \in \{1, r+1\}$.

Let

$$\Delta_{r,1} : C_r^{p,q}(\mathcal{G},\phi) \longrightarrow C_0^{p+r,q+1}(\mathcal{G},\phi)$$

be defined by

$$\Delta_{r,1}\omega(\vec{\gamma}) := \rho_0^0(t_p(\partial_0^r\gamma_1)...t_p(\partial_0^r\gamma_{q+1})) \circ \phi\big(\omega(\partial_0^r\delta_0\vec{\gamma}; g_{1r}, g_{1r-1}, ..., g_{12}, g_{11})\big), \quad (76)$$

for $\omega \in C(\mathcal{G}_p^q \times G^r, W)$ and $\vec{\gamma} \in \mathcal{G}_{p+r}^{q+1}$ as in Eq. (59).

4.19. Proposition. *Let* $\omega \in C(\mathcal{G}_p^q \times G^r, W)$, *then*

$$((-1)^{r+1}\Delta_{r,1} \circ \partial + \partial \circ \Delta_{r,1})\omega = (-1)^{r+1}\Delta_{r+1,1} \circ \delta_{(1)}\omega. \quad (77)$$

Proof. To prove Eq. (77), one needs to consider two separate cases: $q = 0$ and $q > 0$.

For $q = 0$, let $\gamma \in \mathcal{G}_{p+r+1}$ and let $(g_0, ..., g_{p+r}; h) \in G^{p+r+1} \times H$ be its image under the isomorphism of Remark 2.5, then

$$\partial\Delta_{r,1}\omega(\gamma) = \sum_{j=0}^{p+r+1} (-1)^j \Delta_{r,1}\omega(\partial_j\gamma).$$

Using Eq. (9), we compute

$$\partial_0^r\partial_j\gamma = \begin{cases} (g_{r+1}, ..., g_{p+r}; h) & \text{if } 0 \leq j \leq r \\ (g_r, ..., g_j g_{j-1}, ..., g_{p+r}; h) & \text{if } r < j \leq r+p \\ (g_r, ..., g_{p+r-1}; hi(g_{p+r})) & \text{if } j = r+p+1, \end{cases} \quad (78)$$

thus yielding

$$\partial\Delta_{r,1}\omega(\gamma) = \phi\Bigg[\rho_0^1(hi(g_{r+p}...g_{r+1}))\Big(\omega(g_r,...,g_1) - \omega(g_r,...,g_2,g_1g_0) + ...+$$

$$+ (-1)^{r-1}\omega(g_r, g_{r-2}g_{r-1},...,g_1,g_0) + (-1)^r\omega(g_rg_{r-1}, g_{r-2}...,g_1,g_0)\Big)+$$

$$+ \sum_{j=r+1}^{r+p}(-1)^j\rho_0^1(hi(g_{r+p}...g_r))\omega(g_{r-1},...,g_0)+$$

$$+ (-1)^{r+p+1}\rho_0^1(hi(g_{r+p})i(g_{r+p-1}...g_r))\omega(g_{r-1},...,g_0)\Bigg].$$

Since the terms in the sum have got no dependence on j and are equal to the last term,

$$\partial\Delta_{r,1}\omega(\gamma) = \begin{cases} (-1)^{r+1}\phi\Big[\rho_0^1(hi(g_{p+r}...g_{r+1}))\Big(\delta_{(1)}\omega(g_r,...,g_0) - \rho_0^1(i(g_r))\omega(g_{r-1},...,g_0)\Big)\Big] & \text{if } p \text{ is odd,} \\ (-1)^{r+1}\phi\Big(\rho_0^1(hi(g_{p+r}...g_{r+1}))\delta_{(1)}\omega(g_r,...,g_0)\Big) & \text{otherwise.} \end{cases}$$

On the other hand,

$$\Delta_{r,1}\partial\omega(\gamma) = \begin{cases} \rho_0^0(hi(g_{p+r}...g_r)) \circ \phi(\omega(g_{r-1},...,g_0)) & \text{if } p \text{ is odd,} \\ 0 & \text{otherwise;} \end{cases}$$

thus, in either case,

$$(\partial \circ \Delta_{r,1}\omega + (-1)^{r+1}\Delta_{r,1} \circ \partial\omega)(\gamma) = (-1)^{r+1}(\Delta_{r+1,1} \circ \delta_{(1)}\omega)(\gamma). \tag{79}$$

For $q \geq 1$, let $\vec{\gamma} \in \mathcal{G}_{p+r+1}^{q+1}$ be as in Eq. (59), then

$$\partial\Delta_{r,1}\omega(\vec{\gamma}) = \sum_{j=0}^{p+r+1}(-1)^j\Delta_{r,1}\omega(\partial_j\vec{\gamma})$$

$$= \phi\Big[\rho_0^1(t_p(\partial_0^{r+1}\gamma_1)...t_p(\partial_0^{r+1}\gamma_{q+1}))\omega(\partial_0^r\delta_0(\partial_0\vec{\gamma}); g_{1r+1},...,g_{12})+$$

$$- \rho_0^1(t_p(\partial_0^r\partial_1\gamma_1)...t_p(\partial_0^r\partial_1\gamma_{q+1}))\omega(\partial_0^r\delta_0(\partial_1\vec{\gamma}); g_{1r+1},...,g_{13},g_{12}g_{11}) + ...+$$

$$+(-1)^{r-1}\rho_0^1(t_p(\partial_0^r\partial_{r-1}\gamma_1)...t_p(\partial_0^r\partial_{r-1}\gamma_{q+1}))\omega(\partial_0^r\delta_0(\partial_{r-1}\vec{\gamma}); g_{1r+1}, g_{1r}g_{1r-1},...,g_{12},g_{11})+$$

$$+ (-1)^r\rho_0^1(t_p(\partial_0^r\partial_r\gamma_1)...t_p(\partial_0^r\partial_r\gamma_{q+1}))\omega(\partial_0^r\delta_0(\partial_r\vec{\gamma}); g_{1r+1}g_{1r}, g_{1r-1}...,g_{12},g_{11})+$$

$$+ \sum_{j=r+1}^{p+r+1}(-1)^j\rho_0^1(t_p(\partial_0^r\partial_j\gamma_1)...t_p(\partial_0^r\partial_j\gamma_{q+1}))\omega(\partial_0^r\delta_0(\partial_j\vec{\gamma})_{0,0}; g_{1r},...,g_{11})\Big].$$

Using Eq. (78) along with $h_{ab} = h_{ab+1} i(g_{ab+1}) = h_{ap+r} i(g_{ap+r} ... g_{ab+1})$, one gets

$$\partial \Delta_{r,1} \omega(\vec{\gamma}) = (-1)^{r+1} \phi \Bigg[\rho_0^1 (h_{1r+1} ... h_{(q+1)(r+1)}) \Big(\delta_{(1)} \omega(\partial_0^{r+1} \delta_0 \vec{\gamma}; g_{1r+1}, ..., g_{11}) +$$

$$- \rho_0^1 (i(g_{1r+1}^{h_{2r+1} ... h_{(q+1)(r+1)}})) \omega(\partial_0^{r+1} \delta_0 \vec{\gamma}; g_{1r}, ..., g_{11}) \Big) +$$

$$- \sum_{j=r+1}^{p+r+1} (-1)^{j-r} \rho_0^1 (h_{1r} ... h_{(q+1)r}) \omega(\partial_0^r \delta_0 (\partial_j \vec{\gamma}); g_{1r}, ..., g_{11}) \Bigg].$$

Notice that, as opposed to the case $q = 0$, due to the explicit dependence on j, the terms in the sum do not cancel one another.

On the other hand,

$$\Delta_{r,1} \partial \omega(\vec{\gamma}) = \rho_0^0 (t_{p+1}(\partial_0^r \gamma_1) ... t_{p+1}(\partial_0^r \gamma_{q+1})) \circ \phi \Big(\partial \omega(\partial_0^r \delta_0 \vec{\gamma}; g_{1r}, g_{1r-1}, ..., g_{12}, g_{11}) \Big)$$

$$= \rho_0^0 (h_{1r} ... h_{(q+1)r}) \circ \phi \Big(\rho_0^1 (i(pr_G(\gamma_{2r+1} \barwedge ... \barwedge \gamma_{(q+1)(r+1)}))))^{-1} \omega(\partial_0^{r+1} \delta_0 \vec{\gamma}; g_{1r}, ..., g_{11}) +$$

$$+ \sum_{j=1}^{p+1} (-1)^j \omega(\partial_j \partial_0^r \delta_0 \vec{\gamma}; g_{1r}, g_{1r-1}, ..., g_{12}, g_{11}) \Big) \Big);$$

thus, by means of $\partial_0^r \partial_j = \partial_{j-r} \partial_0^r$ for $j \geq r+1$ and

$$h_{1r} ... h_{(q+1)r} = h_{1r+1} ... h_{(q+1)(r+1)} i(pr_G(\gamma_{1r+1} \barwedge ... \barwedge \gamma_{(q+1)(r+1)})),$$

multiplying by the factor of $(-1)^{r+1}$, it follows that

$$(\partial \Delta_{r,1} \omega + (-1)^{r+1} \Delta_{r,1} \partial \omega)(\vec{\gamma}) = (-1)^{r+1} \rho_0^0 (h_{1r+1} ... h_{(q+1)(r+1)}) \circ \phi \Big(\delta_{(1)} \omega(\partial_0^{r+1} \delta_0 \vec{\gamma}; g_{1r+1}, ..., g_{11}) \Big)$$

$$= (-1)^{r+1} (\Delta_{r+1,1} \circ \delta_{(1)} \omega)(\vec{\gamma}),$$

as desired.

■

Let

$$\Delta_{1,r} : C_r^{p,q}(\mathcal{G}, \phi) \longrightarrow C_0^{p+1,q+r}(\mathcal{G}, \phi)$$

be defined by

$$\Delta_{1,r} \omega(\vec{\gamma}) := \rho_0^0 (t_p(\partial_0 \gamma_1) ... t_p(\partial_0 \gamma_{q+r})) \circ \phi \big(\omega(\partial_0 \delta_0^r \vec{\gamma}; g_{11}^{h_{21} ... h_{r1}}, g_{21}^{h_{31} ... h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1}) \big), \quad (80)$$

for $\omega \in C(\mathcal{G}_p^q \times G^r, W)$ and $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+r}$ as in Eq. (59).

4.20. Proposition. *Let $\omega \in C(\mathcal{G}_p^q \times G^r, W)$, then*

$$(\Delta_{1,r} \circ \delta + (-1)^{r+1} \delta \circ \Delta_{1,r}) \omega = (-1)^{r+1} \Delta_{1,r+1} \circ \delta_{(1)} \omega. \quad (81)$$

PROOF. Let $\vec{\gamma} \in \mathcal{G}_{p+1}^{q+r+1}$ be as in Eq. (59). Then,

$$\delta\Delta_{1,r}\omega(\vec{\gamma}) = \rho_0^0(t_{p+1}(\gamma_1))\Delta_{1,r}\omega(\delta_0\vec{\gamma}) + \sum_{j=1}^{q+r+1}(-1)^j\Delta_{1,r}\omega(\delta_j\vec{\gamma})$$

$$= \rho_0^0(h_{11}i(g_{11}))\phi\Big[\rho_0^1(t_p(\partial_0\gamma_2)...t_p(\partial_0\gamma_{q+r+1}))\omega(\partial_0\delta_0^r(\delta_0\vec{\gamma}); g_{21}^{h_{31}...h_{(r+1)1}}, ..., g_{r1}^{h_{(r+1)1}}, g_{(r+1)1})+$$

$$- \rho_0^1(t_p(\partial_0(\gamma_1 \barwedge \gamma_2))...t_p(\partial_0\gamma_{q+r+1}))\omega(\partial_0\delta_0^r(\delta_1\vec{\gamma}); (g_{11}^{h_{21}}g_{21})^{h_{31}...h_{(r+1)1}}, ..., g_{r1}^{h_{(r+1)1}}, g_{(r+1)1})+$$

$$+\sum_{j=2}^{r}(-1)^j\rho_0^1(t_p(\partial_0\gamma_1)...t_p(\partial_0(\gamma_j \barwedge \gamma_{j+1}))...t_p(\partial_0\gamma_{q+r+1}))\omega(\partial_0\delta_0^r(\delta_j\vec{\gamma}); ..., (g_{j1}^{h_{(j+1)1}}g_{(j+1)1})^{h_{(j+2)1}...h_{(r+1)1}}, ...)+$$

$$+\sum_{j=r+1}^{r+q}(-1)^j\rho_0^1\rho_0^1(t_p(\partial_0\gamma_1)...t_p(\partial_0(\gamma_j \barwedge \gamma_{j+1}))...t_p(\partial_0\gamma_{q+r+1}))\omega(\partial_0\delta_0^r(\delta_j\vec{\gamma}); g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1})+$$

$$+ (-1)^{r+q+1}\rho_0^1(t_p(\partial_0\gamma_1)...t_p(\partial_0\gamma_{q+r}))\omega(\partial_0\delta_0^r(\delta_{r+q+1}\vec{\gamma}); g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1})\Big]$$

$$= \rho_0^0(h_{11}...h_{(q+r+1)1})\phi\Big[\rho_0^1(i(g_{11}^{h_{21}...h_{(q+r+1)1}}))\omega(\partial_0\delta_0^{r+1}\vec{\gamma}; g_{21}^{h_{31}...h_{(r+1)1}}, ..., g_{r1}^{h_{(r+1)1}}, g_{(r+1)1})+$$

$$+\sum_{k=1}^{r}(-1)^k\omega(\partial_0\delta_0^{r+1}\vec{\gamma}; \delta_k(g_{11}^{h_{21}...h_{(r+1)1}}, ..., g_{r1}^{h_{(r+1)1}}, g_{(r+1)1}))+$$

$$+\sum_{j=r+1}^{r+q}(-1)^j\omega(\partial_0\delta_0^r(\delta_j\vec{\gamma}); g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1}))$$

$$+ (-1)^{r+q+1}\rho_0^1(h_{(q+r+1)1})^{-1}\omega(\partial_0\delta_0^r(\delta_{r+q+1}\vec{\gamma}); g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1})\Big]$$

and

$$\Delta_{1,r}\delta\omega(\vec{\gamma}) = \rho_0^1(t_p(\partial_0\gamma_1)...t_p(\partial_0\gamma_{q+r+1}))\circ\phi\Big(\delta\omega(\partial_0\delta_0^r\vec{\gamma}; g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1})\Big)$$

$$= \rho_0^0(h_{11}...h_{(q+r+1)1})\phi\Big[\omega(\partial_0\delta_0^{r+1}\vec{\gamma}; (g_{11}^{h_{21}...h_{r1}})^{h_{(r+1)1}}, ..., (g_{(r-1)1}^{h_{r1}})^{h_{(r+1)1}}, g_{r1}^{h_{(r+1)1}}))+$$

$$+\sum_{j=1}^{q}(-1)^j\omega(\delta_j\partial_0\delta_0^r\vec{\gamma}; g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1})+$$

$$+ (-1)^{q+1}\rho_0^1(t_p(\partial_0\gamma_{q+r+1}))^{-1}\omega(\delta_{q+1}\partial_0\delta_0^r\vec{\gamma}; g_{11}^{h_{21}...h_{r1}}, ..., g_{(r-1)1}^{h_{r1}}, g_{r1})\Big)\Big].$$

Since $\delta_0^r\delta_j = \delta_{j-r}\delta_0^r$ for $j \geq r+1$ and $t_p(\partial_0\gamma_{q+r+1}) = h_{(q+r+1)1}$, it follows that

$$(\delta\Delta_{1,r} + (-1)^{r+1}\Delta_{1,r}\delta)\omega(\vec{\gamma}) = \rho_0^0(h_{11}...h_{(q+r+1)1}) \circ \phi\Big(\delta_{(1)}\omega(\partial_0\delta_0^{r+1}\vec{\gamma}; g_{11}^{h_{21}...h_{(r+1)1}}, ..., g_{r1}^{h_{(r+1)1}}, g_{(r+1)1})\Big)$$
$$= (\Delta_{1,r+1} \circ \delta_{(1)}\omega)(\vec{\gamma}),$$

as desired.

■

4.21. Remark. Propositions 4.19 and 4.20 are the simplest of relations in Theorem 4.18, both due to the definition of the higher difference maps involved and to the number of terms in Eq. (75). Continuing Remark 4.12, these results prove that though the difference maps do not commute in general with neither ∂ nor δ, they do so up to isomorphism in the 2-vector space.

Appendix

We devote this appendix to give the rather cumbersome formulas for some families of higher difference maps and all maps necessary to extend (29) up to degree 5.

We introduce further notation in order to abbreviate the formulas. Associated to $\vec{\gamma} \in \mathcal{G}_{p+a}^{q+b}$ as in Eq. (59) for $a, b > 0$, define the sub-matrices

- $\vec{\gamma}_{\sqcap}^{a,b} := (\gamma_{ij})_{1\leq i\leq b; 1\leq j\leq a} \in \mathcal{G}_a^b$, and
- $\vec{\gamma}_{\sqcup}^{a,b} := (\gamma_{ij})_{b+1\leq i\leq q+b; 1\leq j\leq a} \in \mathcal{G}_a^q$.

Also, define $\| \vec{\gamma} \|$ to be the *full product* of the coordinates of $\vec{\gamma}$, i.e.,

$$\| \vec{\gamma} \| := \partial_1^{p+a-1}\delta_1^{q+b-1}\vec{\gamma} = (\gamma_{11} \bowtie ... \bowtie \gamma_{1p+a}) \barwedge ... \barwedge (\gamma_{(q+b)1} \bowtie ... \bowtie \gamma_{(q+b)(p+a)}),$$

and $\| \vec{\gamma} \|_G := pr_G(\| \vec{\gamma} \|)$.

Front page. Each difference map landing on the front page

$$\Delta_{a,b} : C_{a+b-1}^{p,q}(\mathcal{G}, \phi) \longrightarrow C_0^{p+a,q+b}(\mathcal{G}, \phi)$$

is defined by

$$\Delta_{a,b}\omega(\vec{\gamma}) = \rho_0^0\Big(s(\| \vec{\gamma}_{\sqcap}^{a,b} \| \barwedge \| \vec{\gamma}_{\sqcup}^{a,b} \|)\Big) \circ \phi\Bigg[\sum_{\alpha\in I_{a,b}} \zeta(\alpha)\omega\Big(\partial_0^a\delta_0^b\vec{\gamma}; c_{a,b}^{\alpha}(\vec{\gamma}_{\sqcap}^{a,b})\Big)\Bigg],$$

for $\omega \in C(\mathcal{G}_p^q \times G^{a+b-1}, W)$ and $\vec{\gamma} \in \mathcal{G}_{p+a}^{q+b}$. Here, $I_{a,b}$ is a set of indices, $\zeta(\alpha)$ is a sign and $\{c_{a,b}^{\alpha}\}_{\alpha\in I_{a,b}}$ is a collection of maps

$$c_{a,b}^{\alpha} : \mathcal{G}_a^b \longrightarrow G^{a+b-1}.$$

Otherwise. Each difference map landing off the front page

$$\Delta_{a,b} : C_r^{p,q} \longrightarrow C_{r+1-(a+b)}^{p+a,q+b}$$

with $a+b<r+1$ is defined by

$$\Delta_{a,b}\omega(\vec{\gamma};\vec{f}) = \rho_0^1(i(\| \vec{\gamma}_{\sqcup}^{a,b} \|_G))^{-1}\Bigg[\sum_{\beta\in J_{a,b}(r)} \zeta(\beta)\omega\Big(\partial_0^a\delta_0^b\vec{\gamma}; c_\beta^{a,b}(r)\big(\vec{f};\vec{\gamma}_{\sqcap}^{a,b}\big)\Big)+$$

$$+\rho_0^1(i(\| \vec{\gamma}_{\sqcap}^{a,b} \|_G^{s(\|\vec{\gamma}_{\sqcup}^{a,b}\|)}))^{-1}\sum_{\alpha\in I_{a,b}} \zeta(\alpha)\omega\Big(\partial_0^a\delta_0^b\vec{\gamma}; (\vec{f})^{s(\|\vec{\gamma}_{\sqcap}^{a,b}\|)}, c_{a,b}^{\alpha}\big(\vec{\gamma}_{\sqcap}^{a,b}\big)\Big)\Bigg],$$

for $\omega\in C(\mathcal{G}_p^q\times G^r, W)$, $\vec{\gamma}\in\mathcal{G}_{p+a}^{q+b}$ and $\vec{f}\in G^{r+1-(a+b)}$. Again, $J_{a,b}(r)$ is a set of indices, $\zeta(\beta)$ stands for the sign of the index β and $\{c_\beta^{a,b}(r)\}_{\beta\in J_{a,b}(r)}$ is a set of maps

$$c_\beta^{a,b}(r) : G^{r+1-(a+b)}\times\mathcal{G}_a^b \longrightarrow G^r.$$

We limit ourselves to specifying the index set $I_{a,b}$, the values of $c_{a,b}^{\alpha}$ and the signs $\zeta(\alpha)$ by writing them as the formal polynomial $p_{a,b}=\sum_{\alpha\in I_{a,b}}\zeta(\alpha)c_{a,b}^{\alpha}$. Analogously, we use the formal polynomial $p_{a,b}^{(r)}=\sum_{\beta\in J_{a,b}(r)}\zeta(\beta)c_\beta^{a,b}(r)$ to indicate $J_{a,b}(r)$, $c_\beta^{a,b}(r)$ and $\zeta(\beta)$.

$$p_{2,2}\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix} = (g_{12}^{h_{22}}g_{22}, g_{11}^{h_{21}}, g_{21}) - ((g_{12}g_{11})^{h_{22}}, g_{22}, g_{21}) + (g_{12}^{h_{22}}, g_{11}^{h_{22}}, g_{22}) +$$

$$+ (g_{12}^{h_{22}}g_{22}, g_{22}^{-1}, g_{11}^{h_{22}}g_{22}) - (g_{12}^{h_{22}}g_{22}, g_{22}^{-1}, g_{22})$$

Using these coordinates to define $\Delta_{2,2}$ implies

$$(\Delta_{2,1}\circ\delta - \Delta_{1,2}\circ\partial - \Delta\circ\Delta - \partial\circ\Delta_{1,2} + \delta\circ\Delta_{2,1})\omega = -\Delta_{2,2}\circ\delta_{(1)}\omega \tag{82}$$

for all $\omega\in C_2^{p,q}(\mathcal{G},\phi)$.

We point out that there is a certain recurrence in $p_{2,2}$. Indeed, one can use $p_{1,2}$, $p_{2,1}$ and the coordinates of the first difference map Δ, $c_{2n-1}^{1,1}(2), c_{2n}^{1,1}(2)$ (cf. Eq.'s (62) and (63)) to recast $p_{2,2}$ as

$$p_{2,2}(\vec{\gamma}) = \Big(\| \partial_0\vec{\gamma} \|_G, c_{1,2}(\partial_2\vec{\gamma})\Big) - \Big(\| \vec{\gamma}_{\sqcap}^{2,1} \|_G^{s(\|\delta_0\vec{\gamma}\|)}, c_{2,1}(\delta_0\vec{\gamma})\Big) +$$

$$+\Big(\big(c_{2,1}(\vec{\gamma}_{\sqcap}^{1,1})\big)^{s(\partial_0\delta_0\vec{\gamma})}, c_{1,1}(\partial_0\delta_0\vec{\gamma})\Big) + \Big(\| \partial_0\vec{\gamma} \|_G, p_{1,1}^{(2)}(c_{1,1}(\vec{\gamma}_{\sqcap}^{1,1}); \partial_0\delta_0\vec{\gamma})\Big),$$

where $\vec{\gamma}\in\mathcal{G}_2^2$ is as in Eq. (59). Using this scheme, one can write more easily the seventeen terms in $p_{3,2}$ and the fifteen terms in $p_{2,3}$; however, in turn, these are written in part using $p_{2,1}^{(3)}$ and $p_{1,2}^{(3)}$.

$$p^{(3)}_{2,1}\big(f;(\,\gamma_{11}\quad\gamma_{12}\,)\big)=((g_{12}g_{11})^{-1},f^{h_{12}}g_{12},g_{11})-((g_{12}g_{11})^{-1},g_{12},g_{11})+$$
$$-(g_{11}^{-1},g_{12}^{-1},f^{h_{12}}g_{12})+(g_{11}^{-1},g_{12}^{-1},g_{12})-(g_{11}^{-1},f^{h_{11}},g_{11}).$$

$$p^{(3)}_{1,2}\big(f;\begin{pmatrix}\gamma_{11}\\ \gamma_{21}\end{pmatrix}\big)=((g_{11}^{h_{21}}g_{21})^{-1},(f^{h_{11}}g_{11})^{h_{21}},g_{21})-((g_{11}^{h_{21}}g_{21})^{-1},g_{11}^{h_{21}},g_{21})+$$
$$-(g_{21}^{-1},(g_{11}^{h_{21}})^{-1},(f^{h_{11}}g_{11})^{h_{21}})+(g_{21}^{-1},(g_{11}^{h_{21}})^{-1},g_{11}^{h_{21}})-(g_{21}^{-1},f^{h_{11}i(g_{11})h_{21}},g_{21}).$$

Inductively, for $\vec{f}=(f_1,...,f_{r-1})\in G$ and $\vec{\gamma}\in\mathcal{G}_2$ as in Eq. (59),

$$p^{(r+1)}_{2,1}(\vec{f};\vec{\gamma})=\Big(f_1^{h_{11}i(g_{11})},p^{(r)}_{2,1}(\delta_0\vec{f};\vec{\gamma})\Big)+(-1)^{r+1}\Big[\Big(g_{11}^{-1},p^{(r)}_{1,1}(\vec{f};\vec{\gamma})\Big)+\Big(g_{11}^{-1},(\vec{f})^{h_{11}},g_{11}\Big)\Big],$$

and for $\vec{\gamma}\in\mathcal{G}^2$ as in Eq. (59),

$$p^{(r+1)}_{1,2}(\vec{f};\vec{\gamma})=\Big(f_1^{h_{11}i(g_{11})h_{21}i(g_{21})},p^{(r)}_{1,2}(\delta_0\vec{f};\vec{\gamma})\Big)+$$
$$+(-1)^{r+1}\Big[\Big(g_{21}^{-1},\big(p^{(r)}_{1,1}(\vec{f};\vec{\gamma})\big)^{h_{21}}\Big)+\Big(g_{21}^{-1},(\vec{f})^{h_{11}i(g_{11})h_{21}},g_{21}\Big)\Big],$$

Using these coordinates to define $\Delta_{2,1}$ and $\Delta_{1,2}$ implies that for all $\omega\in C^{p,q}_r(\mathcal{G},\phi)$,

$$(\Delta\circ\partial+\partial\circ\Delta)\omega=(-1)^r(\delta_{(1)}\circ\Delta_{2,1}-\Delta_{2,1}\circ\delta_{(1)})\omega, \tag{83}$$

and

$$(\Delta\circ\delta+\delta\circ\Delta)\omega=(-1)^r(\delta_{(1)}\circ\Delta_{1,2}-\Delta_{1,2}\circ\delta_{(1)})\omega. \tag{84}$$

Eq.'s (82), (83) and (84) were the missing relations to imply the following result:

4.22. Theorem. *The composition*

$$C^2_{tot}(\mathcal{G},\phi)\xrightarrow{\nabla}C^3_{tot}(\mathcal{G},\phi)\xrightarrow{\nabla}C^4_{tot}(\mathcal{G},\phi) \tag{85}$$

is identically zero.

For $\vec{\gamma}\in\mathcal{G}^2_3$ as in Eq. (59),

$$p_{3,2}(\vec{\gamma})=\Big(\parallel\partial_0^2\vec{\gamma}\parallel_G,c_{2,2}(\vec{\gamma}^{2,2}_{\sqcap})\Big)+\Big(\parallel\vec{\gamma}^{3,1}_{\sqcap}\parallel_G^{s(\|\delta_0\vec{\gamma}\|)},c_{3,1}(\delta_0\vec{\gamma})\Big)+$$
$$+\Big(\big(c_{3,1}(\vec{\gamma}^{3,1}_{\sqcap})\big)^{s(\partial_0^2\delta_0\vec{\gamma})},c_{1,1}(\partial_0^2\delta_0\vec{\gamma})\Big)+\Big(\parallel\partial_0^2\vec{\gamma}\parallel_G,p^{(3)}_{1,1}(c_{2,1}(\vec{\gamma}^{2,1}_{\sqcap});\partial_0^2\delta_0\vec{\gamma})\Big)+$$
$$+\Big(\delta_0\big(c_{3,1}(\vec{\gamma}^{3,1}_{\sqcap})\big)^{s(\partial_0\delta_0\vec{\gamma})},c_{2,1}(\partial_0\delta_0\vec{\gamma})\Big)+\Big(\parallel\partial_0\vec{\gamma}\parallel_G,p^{(3)}_{2,1}(c_{1,1}(\vec{\gamma}^{1,1}_{\sqcap});\partial_0\delta_0\vec{\gamma})\Big).$$

For $\vec{\gamma} \in \mathcal{G}_2^3$ as in Eq. (59),

$$\begin{aligned} p_{2,3}(\vec{\gamma}) =& \Big(\parallel \partial_0\vec{\gamma} \parallel_G, c_{1,3}(\vec{\gamma}_{\sqcap}^{1,3}) \Big) - \Big(\parallel \vec{\gamma}_{\sqcap}^{2,1} \parallel_G^{s(\|\delta_0\vec{\gamma}\|)}, c_{2,2}(\delta_0\vec{\gamma}) \Big) + \\ &+ \Big(\big(c_{2,2}^1(\vec{\gamma}_{\sqcap}^{2,2})\big)^{s(\partial_0\delta_0^2\vec{\gamma})}, c_{1,1}(\partial_0\delta_0^2\vec{\gamma}) \Big) + \Big(\parallel \partial_0\vec{\gamma} \parallel_G, p_{1,1}^{(3)}(c_{1,2}(\vec{\gamma}_{\sqcap}^{1,2}); \partial_0\delta_0^2\vec{\gamma}) \Big) + \\ &+ \Big(\big(c_{2,1}(\vec{\gamma}_{\sqcap}^{2,1})\big)^{s(\partial_0\delta_0\vec{\gamma})}, c_{1,2}(\partial_0\delta_0\vec{\gamma}) \Big) + \Big(\parallel \partial_0\vec{\gamma} \parallel_G, p_{2,1}^{(3)}(c_{1,1}(\vec{\gamma}_{\sqcap}^{1,1}); \partial_0\delta_0\vec{\gamma}) \Big), \end{aligned}$$

where, for clarity, $c_{2,2}^1(\vec{\gamma}_{\sqcap}^{2,2}) = (g_{12}^{h_{22}} g_{22}, g_{11}^{h_{21}}, g_{21})$.

Using these coordinates to define $\Delta_{3,2}$ and $\Delta_{2,3}$ implies that for all $\omega \in C_3^{p,q}(\mathcal{G}, \phi)$,

$$(\Delta_{3,1} \circ \delta + \Delta_{2,2} \circ \partial + \Delta_{2,1} \circ \Delta + \Delta \circ \Delta_{2,1} - \partial \circ \Delta_{2,2} + \delta \circ \Delta_{3,1})\omega = \Delta_{3,2} \circ \delta_{(1)}\omega \tag{86}$$

and

$$(\Delta_{2,2} \circ \delta + \Delta_{1,3} \circ \partial + \Delta_{1,2} \circ \Delta + \Delta \circ \Delta_{1,2} + \partial \circ \Delta_{1,3} - \delta \circ \Delta_{2,2})\omega = \Delta_{2,3} \circ \delta_{(1)}\omega \tag{87}$$

We conclude defining the necessary higher difference maps to extend (85) to degree 5.

$$\begin{aligned} p_{3,1}^{(4)}\big(f; (\,\gamma_{11} \quad \gamma_{12} \quad \gamma_{13}\,)\big) =& \Big(g_{11}^{-1}, p_{2,1}^{(3)}\big(f; (\,\gamma_{12} \quad \gamma_{13}\,)\big) \Big) - ((g_{12}g_{11})^{-1}, f^{h_{12}}, g_{12}, g_{11}) + \\ &+ ((g_{13}g_{12}g_{11})^{-1}, f^{h_{13}}g_{13}, g_{12}, g_{11}) - ((g_{13}g_{12}g_{11})^{-1}, g_{13}, g_{12}, g_{11}). \end{aligned}$$

$$\begin{aligned} p_{1,3}^{(4)}\big(f; \begin{pmatrix} \gamma_{11} \\ \gamma_{21} \\ \gamma_{31} \end{pmatrix}\big) =& \Big(g_{31}^{-1}, \big(p_{1,2}^{(3)}(f; \begin{pmatrix} \gamma_{12} \\ \gamma_{13} \end{pmatrix})\big)^{h_{31}} \Big) - ((g_{21}^{h_{31}} g_{31})^{-1}, f^{h_{11} i(g_{11}) h_{21} h_{31}}, g_{21}^{h_{31}}, g_{31}) + \\ &+ ((g_{11}^{h_{21}h_{31}} g_{21}^{h_{31}} g_{31})^{-1}, (f^{h_{11}} g_{11})^{h_{21}h_{31}}, g_{21}^{h_{31}}, g_{31}) - ((g_{11}^{h_{21}h_{31}} g_{21}^{h_{31}} g_{31})^{-1}, g_{11}^{h_{21}h_{31}}, g_{21}^{h_{31}}, g_{31}). \end{aligned}$$

Using these coordinates to define $\Delta_{3,1}$ and $\Delta_{1,3}$ implies

$$(\partial \circ \Delta_{2,1} - \Delta_{2,1} \circ \partial)\omega = (\Delta_{3,1} \circ \delta_{(1)} - \delta' \circ \Delta_{3,1})\omega \tag{88}$$

and

$$(\Delta_{1,2} \circ \delta - \delta \circ \Delta_{1,2})\omega = (\Delta_{1,3} \circ \delta_{(1)} - \delta' \circ \Delta_{1,3})\omega, \tag{89}$$

for all $\omega \in C_3^{p,q}(\mathcal{G}, \phi)$.

Finally,

$$
\begin{aligned}
p^{(4)}_{2,2}(f;\vec{\gamma}) = &\Big(\parallel \vec{\gamma}^{1,2}_{\sqcap} \parallel^{-1}_{G}, f^{h_{11}h_{21}}, c_{1,2}(\vec{\gamma}^{1,2}_{\sqcap})\Big) - \Big(\parallel \delta_0\vec{\gamma} \parallel^{-1}_{G}, f^{t(\gamma_{11})h_{21}}, c_{2,1}(\delta_0\vec{\gamma})\Big) + \\
&- \Big(\parallel \vec{\gamma}^{1,2}_{\sqcap} \parallel^{-1}_{G}, p^{(3)}_{1,2}(f;\partial_0\vec{\gamma})\Big) + \Big(\parallel \delta_0\vec{\gamma} \parallel^{-1}_{G}, \big(p^{(3)}_{2,1}(\vec{\gamma}^{2,1}_{\sqcap})\big)^{s(\|\vec{\gamma}^{1,2}_{\sqcap}\|_G)}\Big) + \\
&- \Big(\parallel \vec{\gamma}^{1,2}_{\sqcap} \parallel^{-1}_{G}, p^{(3)}_{1,1}(f^{h_{11}}, g_{11};\partial_0\delta_0\vec{\gamma})\Big) - \Big(\parallel \delta_0\vec{\gamma} \parallel^{-1}_{G}, \big(p^{(2)}_{1,1}(f;\gamma_{11})\big)^{h_{22}}, g_{22}\Big) + \\
&+ \Big(\parallel \vec{\gamma} \parallel^{-1}_{G}, \delta_1\big(f^{h_{12}}, p_{2,2}(\vec{\gamma})\big)\Big) - \Big(\parallel \vec{\gamma} \parallel^{-1}_{G}, p_{2,2}(\vec{\gamma})\Big) - \big((g_{22}g_{11}^{h_{21}}g_{21})^{-1}, f^{h_{11}h_{22}}, g_{11}^{h_{22}}, g_{22}\big) + \\
&+ \big(g_{21}^{-1}, g_{22}^{-1}, (g_{11}^{h_{22}})^{-1}, f^{h_{11}h_{22}}g_{11}^{h_{22}}g_{22}\big) - \big(g_{21}^{-1}, g_{22}^{-1}, (g_{11}^{h_{22}})^{-1}, g_{11}^{h_{22}}g_{22}\big) + \\
&- \big(g_{21}^{-1}, (g_{11}^{h_{21}})^{-1}, g_{21}^{-1}, f^{h_{11}h_{22}}g_{11}^{h_{22}}g_{22}\big) + \big(g_{21}^{-1}, (g_{11}^{h_{21}})^{-1}, g_{21}^{-1}, g_{11}^{h_{22}}g_{22}\big)
\end{aligned}
$$

for $f \in G$ and $\vec{\gamma} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in \mathcal{G}^2_2$.

Using the latter to define $\Delta_{2,2}$ implies that for all $\omega \in C^{p,q}_3(\mathcal{G},\phi)$,

$$
(\Delta_{2,1}\circ\delta - \Delta_{1,2}\circ\partial - \Delta\circ\Delta - \partial\circ\Delta_{1,2} + \delta\circ\Delta_{2,1})\omega = (\Delta_{2,2}\circ\delta_{(1)} - \delta'\circ\Delta_{2,2})\omega; \tag{90}
$$

thus, together with Eq.'s (86), (87), (88) and (89), the following holds:

4.23. Theorem. *The composition*

$$
C^3_{tot}(\mathcal{G},\phi) \xrightarrow{\nabla} C^4_{tot}(\mathcal{G},\phi) \xrightarrow{\nabla} C^5_{tot}(\mathcal{G},\phi) \tag{91}
$$

is identically zero.

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