EXISTENCE OF GROUPOID MODELS FOR DIAGRAMS OF GROUPOID CORRESPONDENCES

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ABSTRACT. This article continues the study of diagrams in the bicategory of étale groupoid correspondences. We prove that any such diagram has a groupoid model and that the groupoid model is a locally compact étale groupoid if the diagram is locally compact and proper. A key tool for this is the relative Stone–Čech compactification for spaces over a locally compact Hausdorff space.

1. Introduction

Many interesting C^*-algebras may be realised as C^*-algebras of étale, locally compact groupoids. Examples are the C^*-algebras associated to group actions on spaces, (higher-rank) graphs, and self-similar groups. These examples of C^*-algebras are defined by some combinatorial or dynamical data. This data is interpreted in [1, 5] as a diagram in a certain bicategory, whose objects are étale groupoids and whose arrows are called groupoid correspondences. A groupoid correspondence is a space with commuting actions of the two groupoids, subject to some conditions. In favourable cases, the C^*-algebra associated to such a diagram is a groupoid C^*-algebra of a certain étale groupoid built from the diagram. A candidate for this groupoid is proposed in [5], where it is called the groupoid model of the diagram.

Here we prove two important results about groupoid models. First, any diagram of groupoid correspondences has a groupoid model. Secondly, the groupoid model is a locally compact groupoid provided the diagram is proper and consists of locally compact groupoid correspondences. The latter is crucial because the groupoid C^*-algebra of an étale groupoid is only defined if it is locally compact.

By the results in [5], the groupoid model exists if and only if the category of actions of the diagram on spaces defined in [5] has a terminal object, and then it is unique up to isomorphism. To show that such a terminal diagram action exists, we prove that the category of actions is cocomplete and has a coseparating set of objects; this criterion is also used to prove the Special Adjoint Functor Theorem.

Proving that the groupoid model is locally compact is more challenging. The key ingredient here is the relative Stone–Čech compactification. This is defined for a space Y

with a continuous map to a locally compact Hausdorff "base space" B, and produces another space over B that is proper in the sense that the map to B is proper and its underlying space is Hausdorff. If B is a point, then the relative Stone–Čech compactification becomes the usual Stone–Čech compactification. An action of a diagram on a space Y contains a map $Y \to \mathcal{G}^0$ for a certain space $\mathcal{G}^0$, which is locally compact and Hausdorff if and only if the diagram is locally compact. If the diagram is proper, then the action on Y extends uniquely to an action on the relative Stone–Čech compactification. Since the relative Stone–Čech compactification is a Hausdorff space with a proper map to the locally compact Hausdorff space $\mathcal{G}^0$, it is itself a locally compact Hausdorff space. Then a standard categorical argument shows that the relative Stone–Čech compactification of a universal action must be homeomorphic to the universal action. This shows that the universal action lives on a locally compact Hausdorff space that is proper over $\mathcal{G}^0$. As a consequence, this space is compact if $\mathcal{G}^0$ is compact.

The main result in this article answers an important, but technical question in the previous article [5]. Therefore, we assume that the reader has already seen [5] and we do not attempt to make this article self-contained. In Section 2, we only recall the most crucial results from [1,5]. In Section 3, we prove that any diagram has a groupoid model – not necessarily Hausdorff or locally compact. In Section 4, we introduce the relative Stone–Čech compactification and prove some properties that we are going to need. In Section 5, we prove that an action of an étale groupoid or of a diagram of proper, locally compact étale groupoid correspondences extends canonically to the relative Stone–Čech compactification. In Section 6, we use this to prove that the universal action of such a diagram lives on a space that is Hausdorff, locally compact, and proper over $\mathcal{G}^0$. To conclude, we discuss two examples. One of them shows that the groupoid model may fail to be locally compact if the groupoid correspondences in the underlying diagram are not proper.

2. Preparations

In this section, we briefly recall the definition of the bicategory of groupoid correspondences, diagrams of groupoid correspondences, their actions on spaces, and the universal action of a diagram. We describe actions of diagrams through slices. More details may be found in [1,5].

We describe a topological groupoid $\mathcal{G}$ by topological spaces $\mathcal{G}$ and $\mathcal{G}^0 \subseteq \mathcal{G}$ of arrows and objects with continuous range and source maps $r, s \colon \mathcal{G} \rightrightarrows \mathcal{G}^0$, a continuous multiplication map $\mathcal{G} \times_{s,\mathcal{G}^0,r} \mathcal{G} \to \mathcal{G}$, $(g, h) \mapsto g \cdot h$, such that each object has a unit arrow and each arrow has an inverse with the usual algebraic properties and the unit map and the inversion are continuous as well. We tacitly assume all groupoids to be *étale*, that is, s and r are local homeomorphisms. This implies that each arrow $g \in \mathcal{G}$ has an open neighbourhood $\mathcal{U} \subseteq \mathcal{G}$ such that $s|_{\mathcal{U}}$ and $r|_{\mathcal{U}}$ are homeomorphisms onto open subsets of $\mathcal{G}^0$. Such an open subset is called a *slice*.

2.1. DEFINITION. An (étale) groupoid $\mathcal{G}$ is called *locally compact* if its object space $\mathcal{G}^0$

is Hausdorff and locally compact.

If $\mathcal{G}$ is a locally compact groupoid, then its arrow space $\mathcal{G}$ is locally compact and locally Hausdorff, but it need not be Hausdorff. We only know that each slice $\mathcal{U} \subseteq \mathcal{G}$ is Hausdorff locally compact because it is homeomorphic to an open subset in $\mathcal{G}^0$. As in [5], we allow groupoids that are not locally compact. We need this for the general existence result for groupoid models.

2.2. DEFINITION. [5, Definitions 2.7–9] Let $\mathcal{H}$ and $\mathcal{G}$ be (étale) groupoids. An (étale) *groupoid correspondence* from $\mathcal{G}$ to $\mathcal{H}$, denoted $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$, is a space $\mathcal{X}$ with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right, such that the right anchor map $s\colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism and the right $\mathcal{G}$-action is basic. A correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is *proper* if the map $r_*\colon \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$ induced by r is proper. A groupoid correspondence $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ is *locally compact* if $\mathcal{H}$ and $\mathcal{G}$ are locally compact and $\mathcal{X}/\mathcal{G}$ is Hausdorff.

The "groupoids" and "groupoid correspondences" as defined in [1] are the "locally compact groupoids" and the "locally compact groupoid correspondences" in the notation of this article.

2.3. DEFINITION. [1, Definition 7.2] Let $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. A *slice* of $\mathcal{X}$ is an open subset $\mathcal{U} \subseteq \mathcal{X}$ such that both $s\colon \mathcal{X} \to \mathcal{G}^0$ and the orbit space projection $\mathsf{p}\colon \mathcal{X} \twoheadrightarrow \mathcal{X}/\mathcal{G}$ are injective on $\mathcal{U}$. Let $\mathcal{S}(\mathcal{X})$ be the set of all slices of $\mathcal{X}$.

Let $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. Then the slices of $\mathcal{X}$ form a basis for the topology of $\mathcal{X}$.

Groupoid correspondences may be composed, and this gives rise to a bicategory $\mathfrak{Gr}$ (see [1]). We only need this structure to talk about bicategory homomorphisms into $\mathfrak{Gr}$. Such a homomorphism is described more concretely in [5]:

2.4. PROPOSITION. [5, Proposition 3.1] *Let $\mathcal{C}$ be a category. A $\mathcal{C}$-shaped diagram of groupoid correspondences $F\colon \mathcal{C} \to \mathfrak{Gr}$ is given by*

(1) *groupoids $\mathcal{G}_x$ for all objects x of $\mathcal{C}$;*

(2) *correspondences $\mathcal{X}_g\colon \mathcal{G}_x \leftarrow \mathcal{G}_y$ for all arrows $g\colon x \leftarrow y$ in $\mathcal{C}$;*

(3) *isomorphisms of correspondences $\mu_{g,h}\colon \mathcal{X}_g \circ_{\mathcal{G}_y} \mathcal{X}_h \xrightarrow{\sim} \mathcal{X}_{gh}$ for all pairs of composable arrows $g\colon z \leftarrow y$, $h\colon y \leftarrow x$ in $\mathcal{C}$;*

such that

(2.4.1) *$\mathcal{X}_x$ for an object x of $\mathcal{C}$ is the identity correspondence $\mathcal{G}_x$ on $\mathcal{G}_x$;*

(2.4.2) *$\mu_{g,y}\colon \mathcal{X}_g \circ_{\mathcal{G}_y} \mathcal{G}_y \xrightarrow{\sim} \mathcal{X}_g$ and $\mu_{x,g}\colon \mathcal{G}_x \circ_{\mathcal{G}_x} \mathcal{X}_g \xrightarrow{\sim} \mathcal{X}_g$ for an arrow $g\colon x \leftarrow y$ in $\mathcal{C}$ are the canonical isomorphisms;*

(2.4.3) *for all composable arrows $g_{01}\colon x_0 \leftarrow x_1$, $g_{12}\colon x_1 \leftarrow x_2$, $g_{23}\colon x_2 \leftarrow x_3$ in $\mathcal{C}$, the following diagram commutes:*

$$
\begin{array}{c}
\xymatrix{
& \mathcal{X}_{g_{02}} \circ_{\mathcal{G}_{x_2}} \mathcal{X}_{g_{23}} \\
(\mathcal{X}_{g_{01}} \circ_{\mathcal{G}_{x_1}} \mathcal{X}_{g_{12}}) \circ_{\mathcal{G}_{x_2}} \mathcal{X}_{g_{23}} & \\
\cong \Big\downarrow \text{ associator} & \mathcal{X}_{g_{03}} \\
\mathcal{X}_{g_{01}} \circ_{\mathcal{G}_{x_1}} (\mathcal{X}_{g_{12}} \circ_{\mathcal{G}_{x_2}} \mathcal{X}_{g_{23}}) & \\
& \mathcal{X}_{g_{01}} \circ_{\mathcal{G}_{x_1}} \mathcal{X}_{g_{13}}
}
\end{array}
\tag{1}
$$

with maps $\mu_{g_{01},g_{12}} \circ_{\mathcal{G}_{x_2}} \mathrm{id}_{\mathcal{X}_{g_{23}}}$, $\mu_{g_{02},g_{23}}$, $\mu_{g_{01},g_{13}}$, $\mathrm{id}_{\mathcal{X}_{g_{01}}} \circ_{\mathcal{G}_{x_1}} \mu_{g_{12},g_{23}}$

here $g_{02} := g_{01} \circ g_{12}$, $g_{13} := g_{12} \circ g_{23}$, and $g_{03} := g_{01} \circ g_{12} \circ g_{23}$.

2.5. DEFINITION. [5, Definition 3.8] Let $\mathcal{C}$ be a category. A diagram of groupoid correspondences $F\colon \mathcal{C} \to \mathfrak{Gr}$ described by the data $(\mathcal{G}_x, \mathcal{X}_g, \mu_{g,h})$ is *proper* if all the groupoid correspondences $\mathcal{X}_g$ are proper. It is *locally compact* if all the groupoids $\mathcal{G}_x$ and the correspondences $\mathcal{X}_g$ are locally compact.

2.6. DEFINITION. [5, Definition 4.5] An *F-action* on a space Y consists of

- a partition $Y = \bigsqcup_{x \in \mathcal{C}^0} Y_x$ into clopen subsets;

- continuous maps $r\colon Y_x \to \mathcal{G}_x^0$;

- open, continuous, surjective maps $\alpha_g\colon \mathcal{X}_g \times_{s,\mathcal{G}_x^0,r} Y_x \to Y_{x'}$ for arrows $g\colon x' \leftarrow x$ in $\mathcal{C}$, denoted multiplicatively as $\alpha_g(\gamma, y) = \gamma \cdot y$;

such that

(2.6.1) $r(\gamma_2 \cdot y) = r(\gamma_2)$ and $\gamma_1 \cdot (\gamma_2 \cdot y) = (\gamma_1 \cdot \gamma_2) \cdot y$ for composable arrows g_1, g_2 in $\mathcal{C}$, $\gamma_1 \in \mathcal{X}_{g_1}$, $\gamma_2 \in \mathcal{X}_{g_2}$, and $y \in Y_{s(g_2)}$ with $s(\gamma_1) = r(\gamma_2)$, $s(\gamma_2) = r(y)$;

(2.6.2) if $\gamma \cdot y = \gamma' \cdot y'$ for $\gamma, \gamma' \in \mathcal{X}_g$, $y, y' \in Y_{s(g)}$, there is $\eta \in \mathcal{G}_{s(g)}$ with $\gamma' = \gamma \cdot \eta$ and $y = \eta \cdot y'$.

2.7. DEFINITION. [5, Definition 4.13] An *F*-action Ω is *universal* if for any *F*-action Y, there is a unique *F*-equivariant map $Y \to \Omega$.

2.8. DEFINITION. [5, Definition 4.13] A *groupoid model for F-actions* is an étale groupoid $\mathcal{U}$ with natural bijections between the sets of $\mathcal{U}$-actions and *F*-actions on Y for all spaces Y.

It follows from [5, Proposition 5.12] that a diagram has a groupoid model if and only if it has a universal *F*-action. By definition, an *F*-action is universal if and only if it is terminal in the category of *F*-actions. Our first goal below will be to prove that any diagram of groupoid correspondences has a universal *F*-action and hence also a groupoid

model. The universal action and the groupoid model of a diagram are unique up to canonical isomorphism if they exist (see [5, Proposition 4.16]).

A key point in our construction of the universal F-action is an alternative description of an F-action, which uses partial homeomorphisms associated to slices of the groupoid correspondences in the diagram.

Let $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence and let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ be slices. Recall that $\langle x, y \rangle$ for $x, y \in \mathcal{X}$ with $\mathsf{p}(x) = \mathsf{p}(y)$ is the unique arrow in $\mathcal{G}$ with $x \cdot \langle x, y \rangle = y$. The subset

$$\langle \mathcal{U}, \mathcal{V} \rangle := \{ \langle x, y \rangle \in \mathcal{G} : x \in \mathcal{U}, \ y \in \mathcal{V}, \ \mathsf{p}(x) = \mathsf{p}(y) \}$$

is a slice in the groupoid $\mathcal{G}$ by [1, Lemma 7.7]. Next, let $\mathcal{X}\colon \mathcal{H} \leftarrow \mathcal{G}$ and $\mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$ be groupoid correspondences and let $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$ be slices. Then

$$\mathcal{U} \cdot \mathcal{V} := \{ [x, y] \in \mathcal{X} \circ_{\mathcal{G}} \mathcal{Y} : x \in \mathcal{U}, \ y \in \mathcal{V}, \ s(x) = r(y) \}$$

is a slice in the composite groupoid correspondence $\mathcal{X} \circ_{\mathcal{G}} \mathcal{Y}$ by [1, Lemma 7.14].

Let F be a diagram of groupoid correspondences. Let $\mathcal{S}(F)$ be the set of all slices of the correspondences $\mathcal{X}_g$ for all arrows $g \in \mathcal{C}$, modulo the relation that we identify the empty slices of $\mathcal{S}(\mathcal{X}_g)$ for all $g \in \mathcal{C}$. Given composable arrows $g, h \in \mathcal{C}$ and slices $\mathcal{U} \subseteq \mathcal{X}_g$, $\mathcal{V} \subseteq \mathcal{X}_h$, then $\mathcal{U}\mathcal{V} := \mu_{g,h}(\mathcal{U} \cdot \mathcal{V})$ is a slice in $\mathcal{X}_{gh}$. If g, h are not composable, then we let $\mathcal{U}\mathcal{V}$ be the empty slice $\emptyset$. This turns $\mathcal{S}(F)$ into a semigroup with zero element $\emptyset$.

2.9. DEFINITION. Let Y be a topological space. A *partial homeomorphism* of Y is a homeomorphism between two open subsets of Y. These are composed by the obvious formula: if f, g are partial homeomorphisms of Y, then fg is the partial homeomorphism of Y that is defined on $y \in Y$ if and only if $g(y)$ and $f(g(y))$ are defined, and then $(fg)(y) := f(g(y))$. If f is a partial homeomorphism of Y, we let f^* be its "partial inverse", defined on the image of f by $f^*(f(y)) = y$ for all y in the domain of f.

Let Y with the partition $Y = \bigsqcup_{x \in \mathcal{C}^0} Y_x$ be an F-action. Then slices in $\mathcal{S}(F)$ act on Y by partial homeomorphisms. For an arrow $g\colon x \leftarrow x'$ in $\mathcal{C}$, a slice $\mathcal{U} \subseteq \mathcal{X}_g$ acts on Y by the partial homeomorphism

$$\vartheta(\mathcal{U})\colon Y_{x'} \supseteq r^{-1}(s(\mathcal{U})) \to Y_x$$

that maps $y \in Y_{x'}$ with $r(y) \in s(\mathcal{U})$ to $\gamma \cdot y$ for the unique $\gamma \in \mathcal{U}$ with $s(\gamma) = r(y)$. The following lemmas describe F-actions and F-equivariant maps through these partial homeomorphisms.

2.10. LEMMA. [5, Lemma 5.3] *Let Y be a space and let $r\colon Y \to \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}_x^0$ and $\vartheta\colon \mathcal{S}(F) \to I(Y)$ be maps. These come from an F-action on Y if and only if*

(2.10.1) $\vartheta(\mathcal{U}\mathcal{V}) = \vartheta(\mathcal{U})\vartheta(\mathcal{V})$ *for all* $\mathcal{U}, \mathcal{V} \in \mathcal{S}(F)$;

(2.10.2) $\vartheta(\mathcal{U}_1)^*\vartheta(\mathcal{U}_2) = \vartheta(\langle \mathcal{U}_1, \mathcal{U}_2 \rangle)$ *for all* $g \in \mathcal{C}$, $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{S}(\mathcal{X}_g)$;

(2.10.3) *the images of $\vartheta(\mathcal{U})$ for $\mathcal{U} \in \mathcal{X}_g$ cover* $Y_{r(g)} := r^{-1}(\mathcal{G}_{r(g)}^0)$ *for each* $g \in \mathcal{C}$;

(2.10.4) $r \circ \vartheta(\mathcal{U}) = \mathcal{U}_\dagger \circ r$ as partial maps $Y \to \mathcal{G}^0$ for any $\mathcal{U} \in \mathcal{S}(F)$.

The corresponding F-action on Y is unique if it exists, and it satisfies

(2.10.5) for $U \subseteq \mathcal{G}_x^0$ open, $\vartheta(U)$ is the identity map on $r^{-1}(U)$;

(2.10.6) for any $\mathcal{U} \in \mathcal{S}(F)$, the domain of $\vartheta(\mathcal{U})$ is $r^{-1}(s(\mathcal{U}))$.

2.11. LEMMA. [5, Lemma 5.4] Let Y and Y' be F-actions. A continuous map $\varphi \colon Y \to Y'$ is F-equivariant if and only if $r' \circ \varphi = r$ and $\vartheta'(\mathcal{U}) \circ \varphi = \varphi \circ \vartheta(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{S}(F)$.

3. General existence of a groupoid model

Our next goal is to prove that any diagram of groupoid correspondences has a groupoid model. By the results of [5] mentioned above, it suffices to show that its category of actions has a terminal object. Our proof will use the following criterion for this:

3.1. LEMMA. Let $\mathcal{D}$ be a cocomplete, locally small category. Assume that there is a set of objects $\Phi \subseteq \mathcal{D}$ such that for any object $x \in \mathcal{D}^0$ there is a $y \in \Phi$ and an arrow $x \to y$. Then $\mathcal{D}$ has a terminal object.

PROOF. This is dual to [7, Lemma 4.6.5], which characterises the existence of an initial object in a complete, locally small category.   ∎

3.2. THEOREM. Any diagram of groupoid correspondences $F \colon \mathcal{C} \to \mathfrak{Gr}$ has a universal F-action and a groupoid model.

PROOF. By the discussion above, it suffices to prove that the category of F-actions satisfies the assumptions in Lemma 3.1. We first exhibit the set of objects Φ.

Let Y be any space with an F-action. Equip Y with the canonical action of the inverse semigroup $\mathcal{I}(F)$. Call an open subset of Y necessary if it is the domain of some element of $\mathcal{I}(F)$. Let τ' be the topology on Y that is generated by the necessary open subsets, and let Y' be Y with the topology τ'. Let Y'' be the quotient of Y' by the equivalence relation where two points y_1, y_2 are identified if

$$\{U \in \tau' : y_1 \in U\} = \{U \in \tau' : y_2 \in U\}$$

and $r(y_1) = r(y_2)$ for the canonical continuous map $r \colon Y \to \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}_x^0$. The continuous map $r \colon Y \to \mathcal{G}^0 := \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}_x^0$ descends to a map on Y'', which is continuous because the subsets $r^{-1}(U)$ for open subsets $U \subseteq \mathcal{G}_x^0$, $x \in \mathcal{C}^0$ are "necessary" by (2.10.5). The $\mathcal{I}(F)$-action on Y descends to an $\mathcal{I}(F)$-action on Y'' because all the domains of elements of $\mathcal{I}(F)$ are in τ'. Then Lemma 2.10 implies that the F-action on Y descends to an F-action on Y''. The quotient map $Y \twoheadrightarrow Y''$ is a continuous F-equivariant map.

Next, we control the cardinality of the set Y''. By construction, finite intersections of necessary open subsets form a basis of the topology τ'. A point in Y'' is determined by its image in $\mathcal{G}^0$ and the set of basic open subsets that contain it. This defines an injective

map from Y'' to the product of $\mathcal{G}^0$ and the power set $\mathcal{P}(\Upsilon)$ for the set Υ of finite subsets of $\mathcal{I}(F)$. We may use this injective map to transfer the F-action on Y'' to an isomorphic F-action on a subset of $\mathcal{G}^0 \times \mathcal{P}(\Upsilon)$, equipped with some topology. Let Φ be the set of all F-actions on subsets of $\mathcal{G}^0 \times \mathcal{P}(\Upsilon)$, equipped with some topology. This is indeed a set, not a class. The argument above shows that any F-action admits a continuous F-equivariant map to an F-action in Φ, as required.

The category of F-actions is clearly locally small. It remains to prove that it is cocomplete. It suffices to prove that it has all small coproducts and coequalisers (see [7, Theorem 3.4.12]). Coproducts are easy: if $(Y_i)_{i\in I}$ is a set of F-actions, then the disjoint union $\bigsqcup_{i\in I} Y_i$ with the canonical topology carries a unique F-action for which the inclusions $Y_i \to \bigsqcup_{i\in I} Y_i$ are all F-equivariant, and this is a coproduct in the category of F-actions. Now let Y_1 and Y_2 be two spaces with F-actions and let $f, g\colon Y_1 \rightrightarrows Y_2$ be two F-equivariant continuous maps. Equip Y_2 with the equivalence relation $\sim$ that is generated by $f(y) \sim g(y)$ for all $y \in Y_1$ and let Y be $Y_2/\sim$ with the quotient topology. This is the coequaliser of f, g in the category of topological spaces. We claim that there is a unique F-action on Y so that the quotient map is F-equivariant. And this F-action turns Y into a coequaliser of f, g in the category of F-actions. We use Lemma 2.10 to build the F-action on Y. Since f, g are F-equivariant, the continuous map $r\colon Y_2 \to \mathcal{G}^0$ equalises f, g. Then r descends to a continuous map $r\colon Y \to \mathcal{G}^0$. Let $t \in \mathcal{I}(F)$. The domain of t is closed under $\sim$ because f, g are $\mathcal{I}(F)$-equivariant, and $y_1 \sim y_2$ implies $\vartheta(t)(y_1) \sim \vartheta(t)(y_2)$. Therefore, the image of the domain of $\vartheta(t)$ in Y is open in the quotient topology and $\vartheta(t)$ descends to a partial homeomorphism of Y. This defines an action of $\mathcal{I}(F)$ on Y. All conditions in Lemma 2.10 pass from Y_2 to Y. We have found an F-action on Y. Any continuous map $h\colon Y_2 \to Z$ with $h \circ f = h \circ g$ descends uniquely to a continuous map $h^\flat\colon Y \to Z$. If h is F-equivariant, then so is $h^\flat$ by Lemma 2.11. Thus Y is a coequaliser of f, g. This finishes the proof that the category of F-actions is cocomplete. And then the existence of a final object follows. ∎

Theorem 3.2 has the merit that it works for any diagram of groupoid correspondences. Applications to C*-algebras, however, need the groupoid model to be locally compact. Equivalently, the underlying space Ω of the universal action should be locally compact and Hausdorff. Example 6.6 shows that Ω may fail to be locally compact in rather simple examples. In the following sections, we are going to prove that Ω is locally compact and Hausdorff whenever F is a diagram of proper, locally compact groupoid correspondences. Like the proof of Theorem 3.2, our proof of this statement will not be constructive. The key tool is a relative form of the Stone–Čech compactification, which we will use to show that any F-action maps to an F-action on a locally compact Hausdorff space.
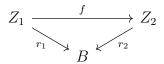
4. The relative Stone–Čech compactification

We begin by recalling some well known definitions.

4.1. PROPOSITION. [3, I.10.1, I.10.3 Proposition 7] *Let X and Y be topological spaces. A map $f\colon X \to Y$ is* proper *if and only if $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is closed for every topological space Z.*

If X is Hausdorff and Y is Hausdorff, locally compact, then $f\colon X \to Y$ is proper if and only if preimages of compact subsets are compact.

4.2. DEFINITION. [4] Let B be a topological space. A *B-space* is a topological space Z with a continuous map $r\colon Z \to B$, called anchor map. It is called *proper* if r is a proper map. Let (Z_1, r_1) and (Z_2, r_2) be two B-spaces. A *B-map* is a continuous map $f\colon Z_1 \to Z_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Z_1 & \xrightarrow{f} & Z_2 \\
 & {}_{r_1}\searrow \quad \swarrow_{r_2} & \\
 & B &
\end{array}
$$

Let B-Space be the category of B-spaces, which has B-spaces as its objects and B-maps as its morphisms, with the usual composition of maps. Let B-$\mathsf{Space}_{\mathrm{proper}} \subseteq B$-Space be the full subcategory of those B-spaces (Z, r) where the space Z is Hausdorff and the map r is proper.

4.3. REMARK. If B is Hausdorff, locally compact and (Z, r) is a proper B-space, then Z is locally compact by Proposition 4.1. This is how we are going to prove that the underlying space of a universal action is locally compact.

For a topological space X, its Stone–Čech compactification is a compact Hausdorff space βX with a continuous map $\iota_X\colon X \to \beta X$, such that any continuous map from X to a compact Hausdorff space factors uniquely through ι_X. In other words, the Stone–Čech compactification β is left adjoint to the inclusion of the full subcategory of compact Hausdorff spaces into the category of all topological spaces. If B is the one-point space, then a B-space is just a space, and B-maps are just continuous maps. A proper, Hausdorff B-space is just a compact Hausdorff space. Thus the Stone–Čech compactification is a left adjoint for the inclusion B-$\mathsf{Space}_{\mathrm{proper}} \subseteq B$-Space in the case where B is a point. The *relative* Stone–Čech compactification generalises this to all Hausdorff, locally compact spaces B.

For a topological space X, let $\mathrm{C_b}(X)$ be the C*-algebra of all bounded, continuous functions $X \to \mathbb{C}$. A continuous map $f\colon X \to Y$ induces a *-homomorphism $f^*\colon \mathrm{C_b}(Y) \to \mathrm{C_b}(X)$, $h \mapsto h \circ f$. If X is Hausdorff, locally compact, then we let $\mathrm{C_0}(X) \subseteq \mathrm{C_b}(X)$ be the ideal of all continuous functions $X \to \mathbb{C}$ that vanish at ∞. If X and Y are Hausdorff, locally compact spaces and $f\colon X \to Y$ is a continuous map, then the restriction of $f^*\colon \mathrm{C_b}(X) \to \mathrm{C_b}(Y)$ to $\mathrm{C_0}(X)$ is *nondegenerate*, that is,

$$f^*(\mathrm{C_0}(X)) \cdot \mathrm{C_0}(Y) = \mathrm{C_0}(Y).$$

Conversely, any nondegenerate *-homomorphism is of this form for a unique continuous map f. The range of f^* is contained in $\mathrm{C_0}(Y)$ if and only if f is proper.

4.4. DEFINITION. Let B be a locally compact Hausdorff space and let (X, r) be a B-space. The *relative Stone–Čech compactification $\beta_B X$* of X over B is defined as the spectrum of the C^*-subalgebra

$$H_X := \mathrm{C_b}(X) \cdot r^*(\mathrm{C_0}(B)) \subseteq \mathrm{C_b}(X).$$

We show that the relative Stone–Čech compactification is indeed the reflector (left adjoint) of the inclusion $B\text{-Space}_{\mathrm{proper}} \hookrightarrow B\text{-Space}$.

In the following, we let B be a locally compact Hausdorff space, (X, r) an object in $B\text{-Space}$ and (X', r') an object in $B\text{-Space}_{\mathrm{proper}}$. Then X' is Hausdorff by the definition of $B\text{-Space}_{\mathrm{proper}}$ and locally compact by Remark 4.3.

The inclusion $i^* \colon H_X \hookrightarrow \mathrm{C_b}(X)$ is a *-homomorphism. For each $x \in X$, denote by ev_x the evaluation map at x. Then $\mathrm{ev}_x \circ i^* \colon H_X \to \mathbb{C}$ is a character on H_X. It is nonzero on H_X because $\mathrm{ev}_x \circ i^*(1 \cdot r^*(h)) \neq 0$ if $h \in \mathrm{C_0}(B)$ satisfies $h(r(x)) \neq 0$. Thus $\mathrm{ev}_x \circ i^*$ is a point in the spectrum $\beta_B X$ of H_X. This defines a map $i \colon X \to \beta_B X$. The map i is continuous because $h \circ i$ is continuous for all $h \in H_X = \mathrm{C_0}(\beta_B X)$.

4.5. LEMMA. *Let $f, g \colon X \rightrightarrows X'$. If $f \neq g$, then $f^* \neq g^* \colon \mathrm{C_0}(X') \to \mathrm{C_b}(X)$.*

PROOF. By assumption, there is $x \in X$ with $f(x) \neq g(x)$ in X'. Since X' is Hausdorff and locally compact, we may separate $f(x)$ and $g(x)$ by relatively compact, open neighbourhoods U_f and U_g. Urysohn's Lemma gives a continuous function $h \colon \overline{U_f} \to [0,1]$ with $h(f(x)) = 1$ and $h|_{\partial U_f} = 0$. Extend h by 0 to a function $\tilde{h}$ on X'. This belongs to $\mathrm{C_0}(X')$ because $\tilde{h}|_{X \setminus U_f} = 0$ and $\overline{U_f}$ is compact, and $\tilde{h}(g(x)) = 0$. Thus $f^*(\tilde{h}) \neq g^*(\tilde{h})$. ∎
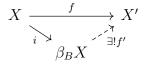
4.6. LEMMA. *Let S be a subset of a locally compact Hausdorff space X'. If the restriction map from $\mathrm{C_0}(X')$ to $\mathrm{C_b}(S)$ is injective, then S is dense in X'.*

PROOF. We prove the contrapositive statement. Suppose that S is not dense in X'. Then $\overline{S} \neq X'$. As in the proof of Lemma 4.5, there is a nonzero continuous function $h \in \mathrm{C_0}(X' \setminus \overline{S})$. Extending h by zero gives a nonzero function in $\mathrm{C_0}(X')$ that vanishes on S. ∎

4.7. LEMMA. *The image of X in $\beta_B X$ is dense.*

PROOF. Lemma 4.6 shows this because $i^* \colon \mathrm{C_0}(\beta_B X) \cong H_X \to \mathrm{C_b}(X)$ is injective. ∎

4.8. PROPOSITION. *Let $f \colon X \to X'$ be a morphism in $B\text{-Space}$. Assume X' to be a Hausdorff proper B-space. Then there is a unique continuous map $f' \colon \beta_B X \to X'$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & X' \\
& \searrow^{i} \quad \nearrow_{\exists! f'} & \\
& \beta_B X &
\end{array}
$$

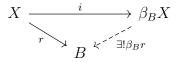*The map f' is automatically proper.*

PROOF. Let $f^*\colon C_0(X') \to C_b(X)$ be the dual map of f and let $i^*\colon H_X \hookrightarrow C_b(X)$ be the inclusion map. Since r' is proper, it induces a nondegenerate *-homomorphism $(r')^*\colon C_0(B) \to C_0(X')$. We use this to show that $f^*(C_0(X')) \subseteq H_X$:

$$f^*(C_0(X')) = f^*(r'^*(C_0(B)) \cdot C_0(X'))$$
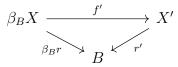$$= f^*(r'^*(C_0(B))) \cdot f^*(C_0(X')) = r^*(C_0(B)) \cdot f^*(C_0(X')) \subseteq H_X.$$

Let $(f^*)'$ be f^* viewed as a *-homomorphism $C_0(X') \to H_X$. We claim that $(f^*)'$ is nondegenerate. The proof uses that a *-homomorphism is nondegenerate if and only if it maps an approximate unit again to an approximate unit; this well known result goes back at least to [6, Proposition 3.4]. Let $(e_i)_{i\in I}$ be an approximate unit in $C_0(B)$. Then $r'^*(e_i)$ is an approximate unit in $C_0(X')$. Now $(f^*)'(r'^*(e_i)) = r^*(e_i) = i^*(\beta_B r)^*(e_i)$. For any $\varphi_1 \in C_b(X)$ and $\varphi_2 \in C_0(B)$, $\|\varphi_1 \cdot r^*(\varphi_2)r^*(e_i) - \varphi_1 \cdot r^*(\varphi_2)\| \leq \|\varphi_1\| \cdot \|r^*(\varphi_2 e_i) - r^*(\varphi_2)\| \to 0$, as r^* is continuous. Hence $r^*(e_i)$ is an approximate unit in H_X. We let $f'\colon \beta_B X \to X'$ be the dual of $(f^*)'$. This is a proper continuous map. Since X' is Hausdorff, two continuous maps to X' that are equal on a dense subset are equal everywhere. Therefore, f' is unique by Lemma 4.7. ∎

4.9. COROLLARY. *The anchor map* $r\colon X \to B$ *extends uniquely to a proper continuous map* $\beta_B r\colon \beta_B X \to B$, *such that the following diagram commutes:*

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;i\;\;} & \beta_B X \\
 & \searrow{\scriptstyle r} \quad \nwarrow{\scriptstyle \exists!\beta_B r} & \\
 & B &
\end{array}
$$

PROOF. Since the identity map $B \to B$ is proper, B is an object in $B\text{-Space}_{\text{proper}}$. Now apply Proposition 4.8 in the case where $X' = B$ and $f = r\colon X \to B$. ∎

4.10. PROPOSITION. *In the above setting, the following diagram commutes:*

$$
\begin{array}{ccc}
\beta_B X & \xrightarrow{\;\;f'\;\;} & X' \\
 & \searrow{\scriptstyle \beta_B r} \quad \swarrow{\scriptstyle r'} & \\
 & B &
\end{array}
$$

PROOF. We get $i^* \circ (f^*)' \circ (r')^* = i^* \circ (\beta_B r)^*$ by construction. Since i^* is a monomorphism, this implies $(f^*)' \circ (r')^* = (\beta_B r)^*$. Then use Lemma 4.5. ∎

4.11. THEOREM. β_B *is a reflector or, equivalently, it is left adjoint to the inclusion functor* $I\colon B\text{-Space}_{\text{proper}} \hookrightarrow B\text{-Space}$.

PROOF. The two propositions above tell us that β_B is left adjoint to I. ∎

4.12. LEMMA. *Let X be a topological space, and let Y and Z be locally compact Hausdorff spaces. Let $f_1\colon X \to Y$ be continuous and let $f_2\colon Y \to Z$ be proper and continuous. Then $\beta_Y(X, f_1) \cong \beta_Z(X, f_2 \circ f_1)$.*
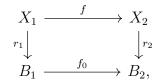
PROOF. It suffices to show that $C_b(X) \cdot f_1{}^*(C_0(Y)) = C_b(X) \cdot (f_2 f_1)^*(C_0(Z))$. Since f_2 is proper, $f_2^*\colon C_0(Z) \to C_0(Y)$ is nondegenerate. In particular, $f_1^*(f_2^*(C_0(Z)) \subseteq f_1^*(C_0(Y)))$, giving the inclusion "$\supseteq$". Since $f_2^*(C_0(Z)) \cdot C_0(Y) = C_0(Y)$, we compute

$$f_1^*(C_0(Y)) = f_1^*(f_2^*(C_0(Z) \cdot C_0(Y))) = (f_2 f_1)^*(C_0(Z)) \cdot f_1^*(C_0(Y))$$

and then

$$C_b(X) \cdot f_1^*(C_0(Y)) = C_b(X) \cdot (f_2 f_1)^*(C_0(Z)) \cdot f_1^*(C_0(Y)) \subseteq C_b(X) \cdot (f_2 f_1)^*(C_0(Z)).$$

∎

4.13. LEMMA. *In a commuting diagram of topological spaces and continuous maps*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ f\ } & X_2 \\
{\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_2} \\
B_1 & \xrightarrow{\ f_0\ } & B_2,
\end{array}
$$

assume B_1 and B_2 to be locally compact Hausdorff and f_0 to be proper. Then there is a unique continuous map $\tilde{f}\colon \beta_{B_1}X_1 \to \beta_{B_2}X_2$ that makes the following diagram commute:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ f\ } & X_2 \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\
\beta_{B_1}X_1 & \dashrightarrow{\ \tilde{f}\ } & \beta_{B_2}X_2 \\
{\scriptstyle \beta_{B_1} r_1}\downarrow & & \downarrow{\scriptstyle \beta_{B_2} r_2} \\
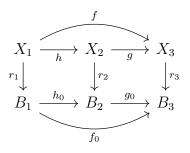B_1 & \xrightarrow{\ f_0\ } & B_2
\end{array}
$$

PROOF. Since f_0 is proper, B_1 is an object in the category of Hausdorff proper B_2-spaces. Then Theorem 4.11 implies $\beta_{B_2}B_1 \cong B_1$ and gives a commuting diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ f\ } & X_2 \\
{\scriptstyle i_1^2}\downarrow & & \downarrow{\scriptstyle i_2} \\
\beta_{B_2}X_1 & \xrightarrow{\ \beta_{B_2} f\ } & \beta_{B_2}X_2 \\
{\scriptstyle \beta_{B_2} r_1}\downarrow & & \downarrow{\scriptstyle \beta_{B_2} r_2} \\
B_1 & \xrightarrow{\ f_0\ } & B_2.
\end{array}
$$

Lemma 4.12 gives $\beta_{B_1}X_1 = \beta_{B_2}X_1$, and this turns the diagram above into what we need. The map $\tilde{f} = \beta_{B_2} f$ is unique because $\beta_{B_2}X_2$ is Hausdorff and the image of X_1 in $\beta_{B_2}X_1$ is dense by Lemma 4.7.

∎

4.14. LEMMA. *Given a commuting diagram of continuous maps*

with locally compact Hausdorff spaces B_j and proper h_0 and g_0, the maps constructed in Lemma 4.13 satisfy $\tilde{f} = \tilde{g} \circ \tilde{h}$.

PROOF. The map $\tilde{g} \circ \tilde{h}$ also has the properties that uniquely characterise $\tilde{f}$. ∎

4.15. LEMMA. *If the map f in Lemma 4.13 is a homeomorphism, then so is $\tilde{f}$.*

PROOF. Apply Lemma 4.14 to the compositions $f \circ f^{-1}$ and $f^{-1} \circ f$. ∎

The following results will be used in the next section to extend an action of a diagram to the relative Stone–Čech compactification.

4.16. LEMMA. *Let I be a set, let B_i for $i \in I$ be locally compact Hausdorff spaces, and let $r_i \colon Y_i \to B_i$ be topological spaces over B_i. Let $B = \bigsqcup_{i \in I} B_i$ and $Y = \bigsqcup_{i \in I} Y_i$ with the induced map $r \colon Y \to B$. Then $\beta_B Y \cong \bigsqcup_{i \in I} \beta_{B_i} Y_i$.*

PROOF. The map that takes the family (Y_i, r_i) of spaces over B_i to (Y, r) as a space over B is an equivalence of categories from the product of categories $\prod_{i \in I} B_i$-Space to the category B-Space. A space over B is Hausdorff and proper if and only if its pieces over B_i are Hausdorff and proper for all $i \in I$. That is, the isomorphism of categories above identifies the subcategory B-Space$_{\text{proper}}$ of Hausdorff and proper B-spaces with the product of the subcategories B_i-Space$_{\text{proper}}$. The product of the reflectors $\beta_{B_i} \colon B_i$-Space $\to B_i$-Space$_{\text{proper}}$ is a reflector $\prod_{i \in I} B_i$-Space $\to \prod_{i \in I} B_i$-Space$_{\text{proper}}$. Under the equivalence above, this becomes the reflector β_B. Both reflectors must be naturally isomorphic. ∎

4.17. LEMMA. *Let V be an open subset of B, and let $(Z, r \colon Z \to B)$ be a locally compact Hausdorff space over B. Then $C_0(r^{-1}(V)) \cong r^*(C_0(V)) \cdot C_0(Z)$.*

PROOF. Let $J := r^*(C_0(V)) \cdot C_0(Z)$. This is an ideal in $C_0(Z)$. So its spectrum $\hat{J}$ is an open subset of Z. Namely, it consists of those $z \in Z$ for which there is $f \in J$ with $f(z) \neq 0$. There is always $h \in C_0(Z)$ with $h(z) \neq 0$. Therefore, $z \in \hat{J}$ if and only if there is $g \in C_0(V)$ with $r^*(g)(z) \neq 0$. Since $r^*(g)(z) = g(r(z))$, such a g exists if and only if $r(z) \in V$. Thus $\hat{J} = r^{-1}(V)$. ∎

4.18. LEMMA. *Let B be a locally compact Hausdorff space and let $V \subseteq B$ be an open subset. Let (Y, r_Y) be a space over B and let $(\beta_B Y, r_{\beta_B Y}) \in B$-Space be its relative Stone–Čech compactification. Then $r_Y^{-1}(V) \subseteq Y$ is a space over V, so that $\beta_V(r_Y^{-1}(V))$ is defined, and $\beta_V(r_Y^{-1}(V)) \cong (r_{\beta_B Y})^{-1}(V) \subseteq \beta_B Y$.*

PROOF. We proceed in terms of their C*-algebras. By definition of the relative Stone–Čech compactification, $\beta_V(r_Y^{-1}(V))$ is the spectrum of the commutative C*-algebra $C_b(r_Y^{-1}(V)) \cdot r_Y^*(C_0(V))$. By Lemma 4.17, $(\beta_{\mathcal{G}^0} r_Y)^{-1}(V)$ is the spectrum of $C_0(\beta_{\mathcal{G}^0} Y) \cdot r_Y^*(C_0(V))$. Since $C_0(\mathcal{G}^0) \cdot C_0(V) = C_0(V)$, we compute

$$C_0(\beta_{\mathcal{G}^0} Y) \cdot r_Y^*(C_0(V)) = C_b(Y) \cdot r_Y^*(C_0(\mathcal{G}^0)) \cdot r_Y^*(C_0(V)) = C_b(Y) \cdot r_Y^*(C_0(V)).$$

Therefore, it suffices to show that $C_b(r_Y^{-1}(V)) \cdot r_Y^*(C_0(V)) \cong C_b(Y) \cdot r_Y^*(C_0(V))$.

Bounded functions on Y restrict to bounded functions on $r_Y^{-1}(V)$, and this restriction map is injective on the subalgebra $C_b(Y) \cdot r_Y^*(C_0(V))$ because functions in this subalgebra vanish outside $r_Y^{-1}(V)$. Therefore, there is an inclusion

$$\varrho \colon C_b(Y) \cdot r_Y^*(C_0(V)) \hookrightarrow C_b(r_Y^{-1}(V)) \cdot r_Y^*(C_0(V)).$$

We must prove that it is surjective. Any element of $C_b(r_Y^{-1}(V)) \cdot r_Y^*(C_0(V))$ is of the form $f \cdot h$ with $f \in C_b(r_Y^{-1}(V))$ and $h \in r_Y^*(C_0(V))$. The Cohen–Hewitt Factorisation Theorem gives $h_1, h_2 \in r_Y^*(C_0(V))$ with $h = h_1 \cdot h_2$. Let φ be the extension of $f \cdot h_1 \colon r_Y^{-1}(V) \to \mathbb{C}$ by zero. We are going to show that φ is continuous on Y. Since $\varrho(\varphi \cdot h_2) = f \cdot h$, it follows that ϱ is surjective.

It remains to prove that φ is continuous. The only points where this is unclear are the boundary points of $r_Y^{-1}(V)$. Let $(y_n)_{n \in N}$ be a net that converges towards such a boundary point. We claim that $\varphi(y_n)$ converges to 0. This proves the claim. If $y_n \notin r_Y^{-1}(V)$, then $\varphi(y_n) = 0$ by construction. So it is no loss of generality to assume $y_n \in r_Y^{-1}(V)$ for all $n \in N$. Then $r_Y(y_n)$ is a net in V that converges towards ∞. Therefore, $\lim h_1'(r_Y(y_n)) = 0$ for all $h_1' \in C_0(V)$. This implies $\lim h_1(y_n) = 0$. Since f is bounded, this implies $\lim \varphi(y_n) = 0$. ∎

5. Extending actions to the relative Stone–Čech compactification

The aim of this section is to extend an action of a diagram on a topological space Y with the anchor map $r_Y \colon Y \to \mathcal{G}^0$ to $\beta_{\mathcal{G}^0} Y$. Actions of étale groupoids are a special case of such diagram actions, and this special case is a bit easier. Therefore, we first treat only actions of groupoids. Since our aim is to generalise to diagram actions, we do not complete the proof in this case, however. We only prove a more technical result about the action of slices of the groupoid.

Let $\mathcal{G}$ be a locally compact étale groupoid acting on a topological space Y with the anchor map $r_Y \colon Y \to \mathcal{G}^0$. The action of $\mathcal{G}$ on Y may be encoded as in Lemma 2.10 by the anchor map $r_Y \colon Y \to \mathcal{G}^0$ and partial homeomorphisms $\vartheta_Y(\mathcal{U})$ of Y for all slices $\mathcal{U} \in \mathcal{S}(\mathcal{G})$ on Y, subject to some conditions. In fact, in this case the conditions simplify

quite a bit, but we do not go into this here. The anchor map r_Y extends to a continuous map $\beta_{\mathcal{G}^0} r_Y \colon \beta_{\mathcal{G}^0} Y \to \mathcal{G}^0$ by construction. The following lemma describes the canonical extension of the partial homeomorphisms $\vartheta_Y(\mathcal{U})$:

5.1. PROPOSITION. *Let $\mathcal{U} \in \mathcal{S}(\mathcal{G})$. The partial homeomorphism $\vartheta_Y(\mathcal{U})$ of Y extends uniquely to a partial homeomorphism*

$$\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \colon (\beta_{\mathcal{G}^0} r_Y)^{-1}(s(\mathcal{U})) \xrightarrow{\sim} (\beta_{\mathcal{G}^0} r_Y)^{-1}(r(\mathcal{U})).$$

Here "extends" means $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \circ i_Y = i_Y \circ \vartheta_Y(\mathcal{U})$ for the canonical map $i_Y \colon Y \to \beta_{\mathcal{G}^0} Y$.

PROOF. The anchor map $r_Y \colon Y \to \mathcal{G}^0$ is $\mathcal{G}$-equivariant when we let $\mathcal{G}$ act on $\mathcal{G}^0$ in the usual way. The slice $\mathcal{U}$ acts both on Y and on $\mathcal{G}^0$, where the latter action is the composite homeomorphism $r|_{\mathcal{U}} \circ (s|_U)^{-1} \colon s(\mathcal{U}) \xrightarrow{\sim} \mathcal{U} \xrightarrow{\sim} r(\mathcal{U})$. The naturality of the construction of ϑ shows that the following diagram commutes:

$$
\begin{array}{ccc}
r_Y^{-1}(s(\mathcal{U})) & \xrightarrow[\cong]{\vartheta_Y(\mathcal{U})} & r_Y^{-1}(r(\mathcal{U})) \\
{\scriptstyle r_Y} \downarrow & & \downarrow {\scriptstyle r_Y} \\
s(\mathcal{U}) & \xrightarrow[\cong]{\vartheta_{\mathcal{G}^0}(\mathcal{U})} & r(\mathcal{U})
\end{array}
$$

Now Lemma 4.13 with $B_1 = s(\mathcal{U})$ and $B_2 = r(\mathcal{U})$ gives a map

$$\widetilde{\vartheta_Y(\mathcal{U})} \colon \beta_{s(\mathcal{U})}(r_Y^{-1}(s(\mathcal{U}))) \to \beta_{r(\mathcal{U})}(r_Y^{-1}(r(\mathcal{U}))).$$

It is a homeomorphism by Lemma 4.15. Lemma 4.18 identifies the domain and codomain of $\widetilde{\vartheta_Y(\mathcal{U})}$ with $(\beta_{\mathcal{G}^0} r_Y)^{-1}(s(\mathcal{U}))$ and $(\beta_{\mathcal{G}^0} r_Y)^{-1}(r(\mathcal{U}))$ as spaces over $s(\mathcal{U})$ and $r(\mathcal{U})$, respectively. So we get a partial homeomorphism $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})$ of $\beta_{\mathcal{G}^0} Y$ that makes the following diagram commute:

$$
\begin{array}{ccccc}
r_Y^{-1}(s(\mathcal{U})) & \xrightarrow[\cong]{\vartheta_Y(\mathcal{U})} & r_Y^{-1}(r(\mathcal{U})) & & Y \\
{\scriptstyle i_Y} \downarrow & & \downarrow {\scriptstyle i_Y} & & \\
r_{\beta_{\mathcal{G}^0} Y}^{-1}(s(\mathcal{U})) & \xrightarrow[\cong]{\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})} & r_{\beta_{\mathcal{G}^0} Y}^{-1}(r(\mathcal{U})) & & \beta_{\mathcal{G}^0} Y \\
{\scriptstyle r_{\beta_{\mathcal{G}^0} Y}} \downarrow & & \downarrow {\scriptstyle r_{\beta_{\mathcal{G}^0} Y}} & & \\
s(\mathcal{U}) & \xrightarrow[\cong]{\vartheta_{\mathcal{G}^0}(\mathcal{U})} & r(\mathcal{U}) & & \mathcal{G}^0
\end{array}
\qquad (2)
$$

In particular, $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \circ i_Y = i_Y \circ \vartheta_Y(\mathcal{U})$. The argument above also shows that the image of $r_Y^{-1}(s(\mathcal{U}))$ in $(\beta_{\mathcal{G}^0} r_Y)^{-1}(r(\mathcal{U}))$ is dense. Since the space $\beta_{\mathcal{G}^0} Y$ is Hausdorff, it follows that the condition $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \circ i_Y = i_Y \circ \vartheta_Y(\mathcal{U})$ determines $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})$ uniquely. ∎

To show that the $\mathcal{G}$-action on Y extends uniquely to a $\mathcal{G}$-action on $\beta_{\mathcal{G}^0}Y$, it would remain to prove that the partial homeomorphisms $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ for slices $\mathcal{U} \in \mathcal{S}(\mathcal{G})$ satisfy the conditions in Lemma 2.10. We will prove this in the more general case of diagram actions.

Before we continue to this more general case, we rewrite the diagram (2) in a way useful for the generalisation to F-actions below. We claim that (2) commutes if and only if the following diagram commutes, where the dashed arrows are partial homeomorphisms and the usual arrows are globally defined continuous maps:

$$
\begin{array}{ccccc}
Y & \dashrightarrow{\ \vartheta_Y(\mathcal{U})\ } & Y & \dashrightarrow{\ \vartheta_Y(\mathcal{U})^*\ } & Y \\
\downarrow{i_Y} & & \downarrow{i_Y} & & \downarrow{i_Y} \\
\beta_{\mathcal{G}^0}Y & \dashrightarrow{\ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})\ } & \beta_{\mathcal{G}^0}Y & \dashrightarrow{\ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})^*\ } & \beta_{\mathcal{G}^0}Y \\
\downarrow{r_{\beta_{\mathcal{G}^0}Y}} & & \downarrow{r_{\beta_{\mathcal{G}^0}Y}} & & \downarrow{r_{\beta_{\mathcal{G}^0}Y}} \\
\mathcal{G}^0 & \dashrightarrow{\ \vartheta_{\mathcal{G}^0}(\mathcal{U})\ } & \mathcal{G}^0 & \dashrightarrow{\ \vartheta_{\mathcal{G}^0}(\mathcal{U})^*\ } & \mathcal{G}^0
\end{array}
\tag{3}
$$

A diagram of partial maps commutes if and only if any two parallel partial maps in the diagram are equal, and this includes an equality of their domains. The domain of $r_{\beta_{\mathcal{G}^0}Y} \circ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ is equal to the domain of $\vartheta_{\mathcal{G}^0}(\mathcal{U})$, whereas the domain of $\vartheta_{\mathcal{G}^0}(\mathcal{U}) \circ r_{\beta_{\mathcal{G}^0}Y}$ is $r_{\beta_{\mathcal{G}^0}Y}^{-1}(s(\mathcal{U}))$ because $\vartheta_{\mathcal{G}^0}(\mathcal{U})$ has domain $s(\mathcal{U})$. Thus the bottom left square implies that $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ has the domain $r_{\beta_{\mathcal{G}^0}Y}^{-1}(s(\mathcal{U}))$. Similarly, the bottom right square implies that $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})^*$ has the domain $r_{\beta_{\mathcal{G}^0}Y}^{-1}(r(\mathcal{U}))$. Equivalently, $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ has the image $r_{\beta_{\mathcal{G}^0}Y}^{-1}(r(\mathcal{U}))$. In the top row, the domain and image of $\vartheta_Y(\mathcal{U})$ must be $r_Y^{-1}(s(\mathcal{U}))$ and $r_Y^{-1}(r(\mathcal{U}))$ for the diagram to commute. In addition, the diagram commutes as a diagram of ordinary maps when we replace each entry by the domain of the partial maps that start there. This gives exactly (2). So the diagram (3) encodes both the commutativity of (2) and the domains and images of the partial maps in that diagram.

Now let $\mathcal{C}$ be a category and let $(\mathcal{G}_x, \mathcal{X}_g, \mu_{g,h})$ describe a $\mathcal{C}$-shaped diagram $F \colon \mathcal{C} \to \mathfrak{Gr}_{\mathrm{lc,proper}}$. Hence, each $\mathcal{G}_x$ for $x \in \mathcal{C}^0$ is a locally compact, étale groupoid, each $\mathcal{X}_g$ for $g \in \mathcal{C}(x,x')$ is a proper, locally compact, étale groupoid correspondence $\mathcal{X}_g \colon \mathcal{G}_{x'} \leftarrow \mathcal{G}_x$, and each $\mu_{g,h}$ for $g, h \in \mathcal{C}$ with $s(g) = r(h)$ is a homeomorphism $\mu_{g,h} \colon \mathcal{X}_g \circ_{\mathcal{G}_{s(g)}} \mathcal{X}_h \xrightarrow{\sim} \mathcal{X}_{gh}$, subject to the conditions in Proposition 2.4. Let Y be a topological space with an action of F. The action contains a disjoint union decomposition $Y = \bigsqcup_{x \in \mathcal{C}^0} Y_x$ and continuous maps $r_x \colon Y_x \to \mathcal{G}_x^0$, which we assemble into a single continuous map $r \colon Y \to \mathcal{G}^0$ with $\mathcal{G}^0 := \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}_x^0$. This makes Y a space over Y and allows us to define the Stone–Čech compactification $\beta_{\mathcal{G}^0}Y$ of Y relative to $\mathcal{G}^0$. We are going to extend the action of F on Y to an action on $\beta_{\mathcal{G}^0}Y$.

The key is the description of F-actions in Lemma 2.10. The space $\beta_{\mathcal{G}^0}Y$ comes with a canonical map $r_{\beta_{\mathcal{G}^0}Y} \colon \beta_{\mathcal{G}^0}Y \to \mathcal{G}^0$, which is one piece of data assumed in Lemma 2.10. We are going to construct partial homeomorphisms $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{S}(F)$ and then

check the conditions in Lemma 2.10. Before we start, we notice that, by Lemma 4.16,

$$\beta_{\mathcal{G}^0} Y = \bigsqcup_{x \in \mathcal{C}^0} \beta_{\mathcal{G}_x^0} Y_x.$$

5.2. LEMMA. *Let* $x, x' \in \mathcal{C}^0$, $g \in \mathcal{C}(x, x')$ *and* $\mathcal{U} \in \mathcal{S}(\mathcal{X}_g)$. *There is a commuting diagram*

(4)

where continuous maps are drawn as usual arrows, partial homeomorphisms as dashed arrows, and one partial map is drawn as a dotted arrow. The partial homeomorphism $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ *is uniquely determined by the commutativity of the upper left square.*

PROOF. We first recall how the arrows $\mathcal{U}_*$ and r_* in (4) are defined. Since $\mathcal{U}$ is a slice, $s|_{\mathcal{U}}\colon \mathcal{U} \xrightarrow{\sim} s(\mathcal{U}) \subseteq \mathcal{G}_x^0$ and $\mathsf{p}|_{\mathcal{U}}\colon \mathcal{U} \xrightarrow{\sim} \mathsf{p}(\mathcal{U}) \subseteq \mathcal{X}_g/\mathcal{G}_x$ are homeomomorphisms onto open subsets. This yields the partial homeomomorphism $\mathcal{U}_* := \mathsf{p}|_{\mathcal{U}} \circ (s|_{\mathcal{U}})^{-1}\colon s(\mathcal{U}) \xrightarrow{\sim} \mathsf{p}(\mathcal{U})$. The map $r_*\colon \mathcal{X}_g/\mathcal{G}_x \to \mathcal{G}_{x'}^0$ in (4) is induced by the anchor map $r\colon \mathcal{X}_g \to \mathcal{G}_{x'}^0$. By definition, $\mathcal{U}_\dagger := r_* \circ \mathcal{U}_*\colon s(\mathcal{U}) \to \mathsf{p}(\mathcal{U}) \to r(\mathcal{U}) \subseteq \mathcal{G}_{x'}^0$, so that the triangle in (4) commutes.

The vertical maps in the first and third column of diagram (4) and the maps $i_{Y_{x'}}$ and $r_{\beta_{\mathcal{G}_{x'}^0}Y_{x'}}$ in the second column are part of the construction of the relative Stone–Čech compactification. Next we construct a map $\pi\colon \beta_{\mathcal{G}_{x'}^0}Y_{x'} \to \mathcal{X}_g/\mathcal{G}_x$ with $r_* \circ \pi = r_{\beta_{\mathcal{G}_{x'}^0}Y_{x'}}$. There is a canonical map $\pi_Y\colon Y_{x'} \xrightarrow{\sim} \mathcal{X}_g \circ Y_x \to \mathcal{X}_g/\mathcal{G}_x$ that maps $\gamma \cdot y$ for $\gamma \in \mathcal{X}_g$, $y \in Y_x$ with $s(\gamma) = r_Y(y)$ to the right $\mathcal{G}_x$-orbit of γ; this is well defined because $\gamma \cdot y = \gamma_2 \cdot y_2$ implies $\gamma_2 = \gamma \cdot \eta$ and $y_2 = \eta^{-1} \cdot y$ for some $\eta \in \mathcal{G}$ with $r(\eta) = s(\gamma)$. We compute $r_* \circ \pi_Y = r_Y\colon Y_{x'} \to \mathcal{G}_{x'}^0$ because $r_* \circ \pi_Y(\gamma \cdot y) = r(\gamma) = r_Y(\gamma \cdot y)$ for all $\gamma \in \mathcal{X}_g$, $y \in Y_x$ with $s(\gamma) = r_Y(y)$. So π_Y is a map over $\mathcal{G}_{x'}^0$. By assumption, the space $\mathcal{X}_g/\mathcal{G}_x$ is Hausdorff and the map $r_*\colon \mathcal{X}_g/\mathcal{G}_x \to \mathcal{G}_{x'}^0$ is proper. By Theorem 4.11, π_Y factors uniquely through a proper, continuous map $\pi\colon \beta_{\mathcal{G}^0}Y \to \mathcal{X}_g/\mathcal{G}_x$ over $\mathcal{G}_{x'}^0$. That this is a map over $\mathcal{G}_{x'}^0$ means that $r_* \circ \pi = r_{\beta_{\mathcal{G}^0}Y}\colon \beta_{\mathcal{G}_{x'}^0}Y_{x'} \to \mathcal{G}_{x'}^0$.

Next we recall the construction of the partial homeomorphism $\vartheta_Y(\mathcal{U})$ from Y_x to $Y_{x'}$. By construction, $\vartheta_Y(\mathcal{U})$ has the domain $r_Y^{-1}(s(\mathcal{U}))$ and is defined by $\vartheta_Y(\mathcal{U})(y) := \gamma \cdot y$ if $y \in r_Y^{-1}(s(\mathcal{U}))$ and $\gamma \in \mathcal{U}$ is the unique element with $s(\gamma) = r_Y(y)$. As a consequence, $\pi_Y(\vartheta_Y(\mathcal{U})(y)) = \mathsf{p}(\gamma) = \mathcal{U}_*(s(\gamma)) = \mathcal{U}_*(r_Y(y))$. Since the partial maps $\pi_Y \circ \vartheta_Y(\mathcal{U})$ and

$\mathcal{U}_* \circ r_Y$ both have the domain $r_Y^{-1}(s(\mathcal{U}))$, we conclude that $\pi_Y \circ \vartheta_Y(\mathcal{U}) = \mathcal{U}_* \circ r_Y$ as partial maps from Y_x to $\mathcal{X}_g/\mathcal{G}_x$.

Next, we prove the equality of partial maps

$$(\mathcal{U}_*)^* \circ \pi_Y = r_Y \circ \vartheta_Y(\mathcal{U})^*. \tag{5}$$

The domain of $(\mathcal{U}_*)^* \circ \pi_Y$ is $\pi_Y^{-1}(\mathsf{p}(\mathcal{U}))$ because the image of $\mathcal{U}_*$ is $\mathsf{p}(\mathcal{U})$, and the domain of $r_Y \circ \vartheta_Y(\mathcal{U})^*$ is the image of $\vartheta_Y(\mathcal{U})$. Therefore, we must show that the image of $\vartheta_Y(\mathcal{U})$ is equal to $\pi_Y^{-1}(\mathsf{p}(\mathcal{U})) \subseteq Y_{x'}$. It is clear that π_Y maps the image of $\vartheta_Y(\mathcal{U})$ into $\mathsf{p}(\mathcal{U})$. Conversely, let $z \in \pi_Y^{-1}(\mathsf{p}(\mathcal{U})) \subseteq Y_{x'}$. There are $\gamma \in \mathcal{X}_g$, $y \in Y_{x'}$ with $s(\gamma) = r(y)$ and $z = \gamma \cdot y$. Then $\pi_Y(z) := \mathsf{p}(\gamma)$, and this belongs to $\mathsf{p}(\mathcal{U})$ by assumption. Therefore, there is a unique $\eta \in \mathcal{G}$ with $s(\gamma) = r(\eta)$ and $\gamma \cdot \eta \in \mathcal{U}$. Then $z = (\gamma\eta) \cdot (\eta^{-1}y) = \vartheta_Y(\mathcal{U})(\eta^{-1}y)$. So z belongs to the image of $\vartheta_Y(\mathcal{U})$. In addition, we get

$$r_Y(\vartheta_Y(\mathcal{U})^*(z)) = r_Y(\eta^{-1}y) = r(\eta^{-1}) = s(\eta) = s(\gamma \cdot \eta) = (\mathcal{U}_*)^*\mathsf{p}(\gamma) = (\mathcal{U}_*)^*\pi_Y(z).$$

This finishes the proof of (5).

As in the proof of Proposition 5.1, we now apply Lemma 4.13 and Lemma 4.15 with $B_1 = s(\mathcal{U})$ and $B_2 = \mathsf{p}(\mathcal{U})$ to get a unique homeomorphism

$$\widetilde{\vartheta_Y(\mathcal{U})} \colon \beta_{s(\mathcal{U})}(r_Y^{-1}(s(\mathcal{U}))) \xrightarrow{\sim} \beta_{\mathsf{p}(\mathcal{U})}(\pi_Y^{-1}(\mathsf{p}(\mathcal{U})))$$

with $i_Y\vartheta_Y(\mathcal{U}) = \widetilde{\vartheta_Y(\mathcal{U})}i_Y$ on $r_Y^{-1}(s(\mathcal{U})) \subseteq Y_x$. Then Lemma 4.18 identifies the domain and codomain of $\widetilde{\vartheta_Y(\mathcal{U})}$:

$$\beta_{s(\mathcal{U})}(r_Y^{-1}(s(\mathcal{U}))) \cong (\beta_{\mathcal{G}^0}r_Y)^{-1}(s(\mathcal{U})) \subseteq \beta_{\mathcal{G}^0}Y_x,$$
$$\beta_{\mathsf{p}(\mathcal{U})}(\pi_Y^{-1}(\mathsf{p}(\mathcal{U}))) \cong \pi^{-1}(\mathsf{p}(\mathcal{U})) \subseteq \beta_{\mathcal{X}_g/\mathcal{G}_x}Y_{x'}.$$

Lemma 4.18 identifies $\beta_{\mathcal{X}_g/\mathcal{G}_x}Y_{x'}$ with the Stone–Čech compactification of $Y_{x'}$ relative to $\mathcal{G}^0_{x'}$ because $r_* \colon \mathcal{X}_g/\mathcal{G}_x \to \mathcal{G}^0_{x'}$ is proper. Composing $\widetilde{\vartheta_Y(\mathcal{U})}$ with these homeomorphisms gives a partial homeomorphism $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ of $\beta_{\mathcal{G}^0}Y$ that makes the diagram (4) commute. It is unique because the target space is Hausdorff and i_Y maps $r_Y^{-1}(s(\mathcal{U}))$ to a dense subset of its domain, where the top left square in (4) determines $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$. ∎

5.3. THEOREM. *Let $F \colon \mathcal{C} \to \mathfrak{Gr}_{\mathrm{lc,proper}}$ be a diagram of proper, locally compact groupoid correspondences. Let Y be a topological space with an F-action. There is a unique F-action on $\beta_{\mathcal{G}^0}Y$ such that the canonical map $i_Y \colon Y \to \beta_{\mathcal{G}^0}Y$ is F-equivariant.*

PROOF. The Stone–Čech compactification relative to $\mathcal{G}^0 := \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}^0_x$ is well defined because $\bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}^0_x$ is locally compact and Hausdorff. There is a canonical map $\beta_{\mathcal{G}^0}r \colon \beta_{\mathcal{G}^0}Y \to \mathcal{G}^0$. It is the unique map with $\beta_{\mathcal{G}^0}r \circ i_Y = r \colon Y \to \mathcal{G}^0$. Hence this is the only choice for an anchor map if we want i to be F-equivariant. Lemma 5.2 provides partial homeomorphisms $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ of $\beta_{\mathcal{G}^0}Y$ for all slices $\mathcal{U} \in \mathcal{S}(F)$. We claim that these satisfy the conditions in Lemma 2.10.

We first check (2.10.1). Let $\mathcal{U} \in \mathcal{S}(\mathcal{X}_g)$, $\mathcal{V} \in \mathcal{S}(\mathcal{X}_h)$ for $g \in \mathcal{C}(x, x')$, $h \in \mathcal{C}(x'', x)$ for $x, x', x'' \in \mathcal{C}^0$. The diagram in (4) describes the domain and the codomain of the maps $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})$ as the preimages of $s(\mathcal{U})$ and $\mathsf{p}(\mathcal{U})$, respectively. The domain of the map $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V})$ is the set of $y \in Y_{x''}$ with $r_{\beta_{\mathcal{G}^0} Y}(y) \in s(\mathcal{V})$ and $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V})(y) \in r_{\beta_{\mathcal{G}^0} Y}^{-1}(s(\mathcal{U}))$. Since $r_{\beta_{\mathcal{G}^0} Y} \circ \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V}) = \mathcal{V}_* \circ r_{\beta_{\mathcal{G}^0} Y}$, the second condition on y is equivalent to $\mathcal{V}_*(r_{\beta_{\mathcal{G}^0} Y}(y)) \in s(\mathcal{U})$. As a consequence, $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V})$ and $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}\mathcal{V})$ have the same domain. The diagram in (4) also implies $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V}) i_Y = \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}\mathcal{V}) i_Y$. Since the target space $\beta_{\mathcal{G}^0} Y$ of $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V})$ and $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}\mathcal{V})$ is Hausdorff and $i_Y(r_Y^{-1}(s(\mathcal{U}\mathcal{V})))$ is dense in the domain $r_{\beta_{\mathcal{G}^0} Y}^{-1}(s(\mathcal{U}\mathcal{V}))$ of our two partial maps, we get $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}) \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{V}) = \vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U}\mathcal{V})$.

The proof of (2.10.2) is similar, using also the right half of (4). To prove condition (2.10.3), we use that the range of $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})$ is $\pi^{-1}(\mathsf{p}(\mathcal{U}))$. These open subsets for slices $\mathcal{U}$ of $\mathcal{X}_g$ cover $\beta_{\mathcal{G}^0} Y_{x'}$ because the open subsets $\mathsf{p}(\mathcal{U}) \subseteq \mathcal{X}_g/\mathcal{G}_x$ for slices $\mathcal{U}$ cover $\mathcal{X}_g/\mathcal{G}_x$. Finally, condition (2.10.4) is already contained in (4).

Now Lemma 2.10 shows that the map $r_{\beta_{\mathcal{G}^0} Y}$ and the partial homeomorphisms $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})$ give a unique F-action on $\beta_{\mathcal{G}^0} Y$. By Lemma 2.11, the top part of (4) says that the map i_Y is F-equivariant. In addition, since this determines the partial homeomorphisms $\vartheta_{\beta_{\mathcal{G}^0} Y}(\mathcal{U})$ uniquely, the F-action on $\beta_{\mathcal{G}^0} Y$ is unique as asserted.   ∎

6. Locally compact groupoid models for proper diagrams

In this subsection, we prove the main result of this article, namely, that the universal action of a diagram of proper, locally compact groupoid correspondences takes place on a Hausdorff proper $\mathcal{G}^0$-space Ω. Since $\mathcal{G}^0$ is Hausdorff, locally compact, it follows that Ω is Hausdorff, locally compact. The key point is the following proposition:

6.1. PROPOSITION. *Let $F\colon \mathcal{C} \to \mathfrak{Gr}_{\mathrm{lc,proper}}$ be a diagram of proper, locally compact groupoid correspondences. The full subcategory of F-actions on Hausdorff proper $\mathcal{G}^0$-spaces is a reflective subcategory of the category of all F-actions. The left adjoint to the inclusion maps an F-action on a space Y to the induced F-action on the Stone–Čech compactification of Y relative to $\mathcal{G}^0 := \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}_x^0$.*

PROOF. Theorem 4.11 says that the full subcategory of Hausdorff proper $\mathcal{G}^0$-spaces is a reflective subcategory of the category of all $\mathcal{G}^0$-spaces, with the relative Stone–Čech compactification $\beta_{\mathcal{G}^0}$ as the left adjoint functor of the inclusion.

Let Y and Y' be topological spaces with an action of F and let $\varphi\colon Y \to Y'$ be an F-equivariant map. Assume that Y' is Hausdorff and that its anchor map $r'\colon Y' \to \mathcal{G}^0$ is proper. By Theorem 5.3, there is a unique F-action on the relative Stone–Čech compactification $\beta_{\mathcal{G}^0} Y$ that makes the inclusion map $i_Y\colon Y \to \beta_{\mathcal{G}^0} Y$ F-equivariant. By Proposition 4.8, there is a unique $\mathcal{G}^0$-map $\tilde{\varphi}\colon \beta_{\mathcal{G}^0} Y \to Y'$ with $\tilde{\varphi} i_Y = \varphi$. Any F-equivariant map is also a $\mathcal{G}^0$-map by Lemma 2.11. Therefore, $\tilde{\varphi}$ is the only map $\beta_{\mathcal{G}^0} Y \to Y'$ with $\tilde{\varphi} i_Y = \varphi$ that has a chance to be F-equivariant. To complete the proof, we must show

that $\tilde{\varphi}$ is indeed F-equivariant. We describe an F-action on a space Y as in Lemma 2.10 through a continuous map $r_Y \colon Y \to \mathcal{G}^0$ and partial homeomorphisms $\vartheta_Y(\mathcal{U})$ for all slices $\mathcal{U} \in \mathcal{S}(F)$, subject to the conditions (2.10.1)–(2.10.4). By Lemma 2.11, it remains to prove that the partial maps $\vartheta_{Y'}(\mathcal{U}) \circ \tilde{\varphi}$ and $\tilde{\varphi} \circ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ agree for any slice $\mathcal{U} \in \mathcal{S}(F)$. We pick $\mathcal{U}$. Then $\mathcal{U}$ is a slice in $\mathcal{X}_g$ for some $x, x' \in \mathcal{C}^0$ and $g \in \mathcal{C}(x, x')$.

First, we check that our two partial maps have the same domain. Since $\tilde{\varphi}$ is a globally defined map, the domain of $\tilde{\varphi} \circ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ is the domain of $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ and the domain of $\vartheta_{Y'}(\mathcal{U}) \circ \tilde{\varphi}$ is the $\tilde{\varphi}$-preimage of the domain of $\vartheta_{Y'}(\mathcal{U})$. The domains of $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ and $\vartheta_{Y'}(\mathcal{U})$ are $r^{-1}_{\beta_{\mathcal{G}^0}Y_x}(s(\mathcal{U}))$ and $r^{-1}_{Y'_x}(s(\mathcal{U}))$, respectively. Since $\tilde{\varphi}$ is a $\mathcal{G}^0$-map, the domain of $\vartheta_{Y'}(\mathcal{U}) \circ \tilde{\varphi}$ is also equal to the $r^{-1}_{\beta_{\mathcal{G}^0}Y_x}(s(\mathcal{U}))$. This proves the claim that both partial maps have the same domain.

Since i_Y and φ are F-equivariant, we know that $\vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U}) \circ i_Y = i_Y \circ \vartheta_Y(\mathcal{U})$ and $\vartheta_{Y'}(\mathcal{U}) \circ \varphi = \varphi \circ \vartheta_Y(\mathcal{U})$. Together with $\tilde{\varphi} \circ i_Y = \varphi$, this implies

$$(\vartheta_{Y'}(\mathcal{U}) \circ \tilde{\varphi}) \circ i_Y = \vartheta_{Y'}(\mathcal{U}) \circ \varphi = \varphi \circ \vartheta_Y(\mathcal{U}) = \tilde{\varphi} \circ i_Y \circ \vartheta_Y(\mathcal{U}) = (\tilde{\varphi} \circ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})) \circ i_Y.$$

These partial maps have domain $r^{-1}_{Y'_x}(s(\mathcal{U}))$. The i_Y-image of this is dense in $r^{-1}_{\beta_{\mathcal{G}^0}Y_x}(s(\mathcal{U}))$ because of Lemma 4.18 and Lemma 4.7. Since the target Y' of $\vartheta_{Y'}(\mathcal{U}) \circ \tilde{\varphi}$ and $\tilde{\varphi} \circ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ is Hausdorff and these maps agree on a dense subset, we get $\vartheta_{Y'}(\mathcal{U}) \circ \tilde{\varphi} = \tilde{\varphi} \circ \vartheta_{\beta_{\mathcal{G}^0}Y}(\mathcal{U})$ as needed. ∎

6.2. PROPOSITION. [7, Corollary 5.6.6] *The inclusion of a reflective full subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ creates all limits that $\mathcal{C}$ admits. As a consequence, if a diagram in $\mathcal{D}$ has a limit in $\mathcal{C}$, then it also has a limit in $\mathcal{D}$, which is isomorphic to the limit in $\mathcal{C}$.*

6.3. THEOREM. *Let $F \colon \mathcal{C} \to \mathfrak{Gr}_{\mathrm{lc,proper}}$ be a diagram of proper, locally compact groupoid correspondences. Then the universal F-action takes place on a space Ω that is Hausdorff, locally compact and proper over $\mathcal{G}^0 := \bigsqcup_{x \in \mathcal{C}^0} \mathcal{G}^0_x$. The groupoid model of F is a locally compact groupoid.*

PROOF. We give two proofs. First, a universal F-action is the same as a terminal object in the category of F-actions, and this is an example of a limit, namely, of the empty diagram. Theorem 3.2 says that a terminal object exists in the category of all F-actions. Proposition 6.1 and Proposition 6.2 imply that this terminal object is isomorphic to an object in the subcategory of Hausdorff proper $\mathcal{G}$-spaces. Actually, our subcategory is closed under isomorphism, and so the terminal object belongs to it. Then its underlying space is locally compact by Remark 4.3.

The second proof is more explicit. Let Ω be the universal F-action. The relative Stone–Čech compactification comes with a canonical F-equivariant map $\iota \colon \Omega \hookrightarrow \beta_{\mathcal{G}^0}\Omega$; here we use the canonical F-action on $\beta_{\mathcal{G}^0}\Omega$. Since Ω is universal, there is a canonical map $\beta_{\mathcal{G}^0}\Omega \to \Omega$ as well. The composite map $\Omega \to \beta_{\mathcal{G}^0}\Omega \to \Omega$ is the identity map because Ω is terminal. The composite map $\beta_{\mathcal{G}^0}\Omega \to \Omega \to \beta_{\mathcal{G}^0}\Omega$ and the identity map have the same composite with ι. Since the range of ι is dense by Lemma 4.7 and $\beta_{\mathcal{G}^0}\Omega$ is Hausdorff, it

follows that the composite map $\beta_{\mathcal{G}^0}\Omega \to \Omega \to \beta_{\mathcal{G}^0}\Omega$ is equal to the identity map as well. So $\Omega \cong \beta_{\mathcal{G}^0}\Omega$, and this means that Ω is Hausdorff and proper over $\mathcal{G}^0$. ∎

6.4. **COROLLARY.** *Let $F\colon \mathcal{C} \to \mathfrak{Gr}_{\mathrm{lc,proper}}$ be a diagram of proper, locally compact groupoid correspondences. Assume that $\mathcal{C}^0$ is finite and that each object space $\mathcal{G}^0_x$ in the diagram is compact. Then the universal F-action takes place on a compact Hausdorff space. The groupoid model of F is a locally compact groupoid with compact object space.*

PROOF. Our extra assumptions compared to Theorem 6.3 say that $\mathcal{G}^0$ is compact. Then Hausdorff spaces that are proper over $\mathcal{G}^0$ are compact. ∎

6.5. **EXAMPLE.** The (m,n)-dynamical systems of Ara, Exel and Katsura [2] are described in [5, Section 4.4] as actions of a certain diagram of proper groupoid correspondences. The diagram is an equaliser diagram of the form $\mathcal{G} \rightrightarrows \mathcal{G}$, where $\mathcal{G}$ is the one-arrow one-object groupoid. A proper groupoid correspondence $\mathcal{G} \to \mathcal{G}$ is just a finite set, and it is determined up to isomorphism by its cardinality. We get (m,n)-dynamical systems when we pick the two sets to have cardinality m and n, respectively. Corollary 6.4 applies to this diagram and shows that its universal action takes place on a compact Hausdorff space. Ara, Exel and Katsura describe in [2, Theorem 3.8] an (m,n)-dynamical system that is universal among (m,n)-dynamical systems on compact Hausdorff spaces. Corollary 6.4 shows that it remains universal if we allow (m,n)-dynamical systems on arbitrary topological spaces.

6.6. **EXAMPLE.** Let $\mathcal{C} = (\mathbb{N},+)$ be the category with a single object and morphisms the nonnegative integers. A diagram $F\colon \mathcal{C} \to \mathfrak{Gr}$ is determined by a single groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ for an étale groupoid $\mathcal{G}$ (see [5, Section 3.4]). Let $\mathcal{G}$ be the trivial groupoid with one arrow and one object. Then $\mathcal{X}$ is just a discrete set because the source map $\mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism. The groupoid model of the resulting diagram is a special case of the self-similar groups treated in [5, Section 9.2], in the case when the group is trivial. It is shown there that the universal action takes place on the space $\Omega := \prod_{n\in\mathbb{N}} \mathcal{X}$. If $\mathcal{X}$ is finite, then Ω is compact by Tychonoff's Theorem. In contrast, if $\mathcal{X}$ is infinite, then Ω is not even locally compact. This example shows that we need a diagram of proper correspondences for the groupoid model to be locally compact.

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