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ON THE METRICAL AND QUANTALIC VERSIONS OF THE $*$-AUTONOMOUS CATEGORY OF SUP-LATTICES

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In memory of Marta Bunge

ABSTRACT. In 1984, Joyal and Tierney presented the category Sup of complete lattices and their suprema-preserving maps as a $*$-autonomous category in the sense of Barr. Work on this paper was motivated by the question whether the Joyal-Tierney proof may be extended to a metrical context, so that the order of the lattice gets replaced by a generalized metric in the sense of Lawvere. The affirmative answer we give relies crucially on working with not necessarily symmetric metrics. It applies not only to small separated and cocomplete categories enriched in the Boolean quantale $\mathsf{2}$ (reproducing Sup), or in the Lawvere quantale $[0,\infty]$ (producing the category we were looking for), but in any commutative and unital quantale $\mathcal{V}$. Benefitting from previous work by Stubbe, Hofmann, and others, with rather explicit constructions of its tensor product and the internal hom we give an alternative proof that the resulting category $\mathcal{V}$-Sup is $*$-autonomous, a result first established by Eklund, Gutiérrez García, Höhle, and Kortelainen in 2018 from a predominantly order-theoretic perspective.

1. Introduction

Ever since the pioneering work [Joyal, Tierney 1984] appeared, many authors utilized and extended the remarkable properties of the category Sup of complete lattices and their suprema-preserving maps, often in generalized contexts—already Joyal and Tierney worked in an elementary topos. We mention here in particular the article [Stubbe 2007] which showed that many aspects of the Joyal-Tierney work hold in a quantaloidic setting. In this paper we work in the more restrictive context of a commutative and unital quantale, with the primary goal of establishing a metrical counterpart of the category Sup as $*$-autonomous in the sense of [Barr 1979], being guided by enriched category theory but staying accessible to readers with knowledge of only basic category theory.

As sup-lattices are precisely the small, separated and (co)complete categories X enriched in the Boolean quantale $\mathsf{2}$, it is not surprising that we will simply replace $\mathsf{2}$ by the

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quantale $[0,\infty]$ of [Lawvere 1973] (ordered by the natural $\geq$), so that the order of the lattice X is induced by a (necessarily) non-symmetric metric on X (unless X is trivial). Cocompleteness, *i.e.*, the existence of all weighted colimits in X, may then be described by a tensorial action of $[0,\infty]$ on X. This gives us a hands-on description of the desired category. It enables us to mimic the Joyal-Tierney proof that Sup is $*$-autonomous and show that $[0,\infty]$, ordered by the natural $\leq$ (!), serves as a dualizing object in that category; see Theorem 2.4.

Following to a large extent the Joyal-Tierney template, the proof of Theorem 2.4 is actually carried out in the generality of a commutative and unital quantale $\mathcal{V}$, rather than just $[0,\infty]$, eventually leading us to a proof of Theorem 7.4. In there we confirm that the category $\mathcal{V}$-Sup of small, separated and cocomplete categories enriched in $\mathcal{V}$ and their cocontinuous $\mathcal{V}$-functors, after being presented equivalently as the category $\mathsf{Sup}^{\mathcal{V}}$ of cocontinuous $\mathcal{V}$-acts in Sup, is $*$-autonomous. This result was first established in [Eklund, Gutierrez Garcia, Hoehle, Kortelainen 2018] who, in their Theorem 3.1.30, first proved that $\mathsf{Sup}^{\mathcal{V}}$ (denoted by $\mathsf{Mod}(\mathcal{V})$ by them, a notation we avoid in order not to run into a terminological or notational clash with [Lawvere 1973, Hofmann, Seal, Tholen 2014]) is $*$-autonomous, before identifying this category as isomorphic to $\mathcal{V}$-Sup (in Section 3.3 of their book).

But the main difference between the two proofs certainly does not lie in the order of presentation, and not even in our emphasis of the categorical, rather than the order-theoretic, perspective which leads us naturally to the metric application. Rather, it lies in the fact that the approach in [Eklund, Gutierrez Garcia, Hoehle, Kortelainen 2018] *uses* the symmetric monoidal-closed structure of Sup, rather than *generalizing* it directly to the $\mathcal{V}$-context as we do in this paper. A key ingredient of our argumentation is the observation that the elegant Joyal-Tierney presentation of quotients in Sup "works" also in $\mathcal{V}$-Sup and thereby allows for a quite direct construction of the tensor product via a universal bi-morphism, rather than via its internal hom and its dualizing object as done in [Eklund, Gutierrez Garcia, Hoehle, Kortelainen 2018]. Indeed, the relevant Proposition 6.2 on quotients in this paper is a straight generalization of Proposition 3 in [Joyal, Tierney 1984].

Not to rely on standard texts on enriched category theory, such as [Kelly 1982, Borceux 1994, Stubbe 2005, Stubbe 2006], whose generality is not needed here, for the convenience of some readers we recall in Sections 3 and 4 the elements of the much slimmer quantale-enriched category theory as used in this paper; some of these elements may also be found in [Akhvlediani, Clementino, Tholen 2010, Hofmann, Seal, Tholen 2014, Hofmann, Nora 2018], and the literature mentioned in there. Other readers are referred to these two sections just for clarification of notation or terminology.

The equivalent presentation of the objects and morphisms of $\mathcal{V}$-Sup in terms of an equationally defined action of the quantale $\mathcal{V}$ is given in Section 5, where we follow and extend the presentation of $\mathcal{V}$-categories with finitary weighted colimits as in [Hofmann, Nora 2018], along with a brief discussion of the monadicity of $\mathcal{V}$-Sup (Theorem 5.5). In Section 6 we prove that $\mathcal{V}$-Sup becomes symmetric monoidal with a box-tensor product classifying its bi-morphisms. Its closedness is shown in Section 7, followed by the proof that $\mathcal{V}$-Sup is $*$-autonomous.

The approach presented in this paper was first sketched in the conference talk [Tholen 2022]. We leave it for future work to determine to which extent the results may hold for categories of relational algebras [Barr 1970], of T-categories [Burroni 1971], or of monad-quantale-enriched categories as presented in [Hofmann 2007, Hofmann, Seal, Tholen 2014]. The paper [Bunge 1974] offers an early and very deep analysis of the abstract structures underlying the categories studied by Barr and Burroni. We dedicate this paper to the memory of Marta Bunge, for her long-term leadership in the development of category theory.

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2. Inf-complete metric acts

In this paper, a *metric* on a set X is a function $d : X\times X \to [0,\infty]$ satisfying the conditions $(d(x,y) = 0 = d(y,x) \iff x = y)$ and $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$. Hence, our metric spaces differ from the ordinary notion of [Fréchet 1906] insofar as d may assume the value ∞ and, more importantly, as d may fail to be symmetric. Whenever convenient, we adopt the convention of writing $X(x,y)$ instead of $d(x,y)$. A *non-expansive* (or 1-*Lipschitz*) map $f : X \to Y$ of metric spaces satisfies $Y(fx, fy) \le X(x,y)$ for all $x, y \in X$. These are the objects and morphisms of the category

$$\mathsf{Met}_{\mathrm{sep}}.$$

It is a full reflective subcategory of Lawvere's category Met whose objects are not necessarily separated, *i.e.,* they may fail the "$\Longrightarrow$"-part of our first condition for a metric.

2.1. Definition. *An* inf-complete metric act *is a complete lattice* $(X,\le)$ *that comes with an action* $+ : [0,\infty] \times X \longrightarrow X$ *of the ordered monoid* $([0,\infty],\le,+,0)$ *on* X *which preserves infima in each variable separately; hence, the action satisfies the following conditions for all* $x, x_i\,(i \in I)$ *in* X *and* $u, v, u_i\,(i \in I)$ *in* $[0,\infty]$*:*

(Act1) $0 + x = x$*;*

(Act2) $u + (v + x) = (u + v) + x$*;*

(Inf1) $(\inf_i u_i) + x = \inf_i(u_i + x)$*;*

(Inf2) $u + \inf_i x_i = \inf_i(u + x_i)$.

A morphism $f : X \to Y$ *of inf-complete metric acts must preserve all infima and the action (i.e., be* $+$*-equivariant). This defines the category*

$$\mathsf{ICMetA}\,.$$

A bi-morphism $g : X \times Y \longrightarrow Z$ of inf-complete metric acts is a mere set map which is a morphism in each variable separately.

Every inf-metric act X becomes a metric space with

$$X(x, y) = \inf\{u \in [0, \infty] \mid x \le u + y\}.$$

This describes the object assignment of a forgetful functor $\mathsf{ICMetA} \to \mathsf{Met}_{\mathrm{sep}}$. It leaves us with the question of what additional structure, or property, may be needed to make a given metric space an inf-complete metric act, such that its induced metric is the given one and, in fact, such that the resulting codomain modification of the forgetful functor becomes an isomorphism of categories. In the more general quantalic context, this question will be answered in the following sections.

2.2. Examples.

1. The complete lattice $[0, \infty]$ with its natural order becomes an inf-complete metric act when we take the monoid operation $+$ as the action of $[0, \infty]$ on itself. The induced (non-symmetric) metric d on $[0, \infty]$ is given by $d(v, w) = \max\{v - w, 0\}$.

2. For every metric space X we make the set

$$\mathsf{P}X = \{\phi : X \to [0, \infty] \mid \forall x, y \in X : \phi y \le \phi x + X(x, y)\}$$

an inf-complete act by defining the order and action pointwise in $[0, \infty]$:

$$\phi \le \psi \iff \forall x \in X : \phi x \le \psi x \qquad \text{and} \qquad (u + \phi)x = u + \phi x.$$

The induced metric on $\mathsf{P}X$ is the usual (non-symmetrized) sup-metric inherited from $[0, \infty]$, that is: $\mathsf{P}X(\phi, \psi) = \sup_{x \in X} d(\phi x, \psi x)$. The functions in $\mathsf{P}X$ are precisely the non-expansive maps $X \to [0, \infty]$. Such function ϕ may also be thought of as an isometric extension of the metric of X to the set $X \cup \{*\}$ with a new point $*$, whereby ϕ gives the distances from $*$ to the points in X, with the understanding that the reverse distances are always ∞.

$$\begin{tikzcd} & x \arrow[dr, "X(x,y)"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & \\ * \arrow[ur, "\phi x"] \arrow[rr, "\phi y"'] & {} & y \end{tikzcd}$$

3. For every inf-complete metric act X, the action $+ : [0, \infty] \times X \to X$ is a bi-morphism of inf-complete metric acts.

2.3. Definition. *For every inf-complete metric act X we call the subobject*

$$X^\star := \mathsf{ICMetA}(X, [0,\infty])$$

of $\mathsf{P}X$*, given by all* ICMetA*-morphisms* $X \to [0,\infty]$*, the* dual act *of* X*.*

The justification for this terminology rests with the following theorem, the proof of which will be carried out in the more general quantalic context afterwards, eventually leading us to the overarching Theorem 7.4.

2.4. Theorem. *The object assignment $X \longmapsto X^\star$ extends to a dual equivalence of categories and, thus, makes* ICMetA *a self-dual category. In fact, with the tensor product that classifies its bi-morphisms, the category* ICMetA *becomes symmetric monoidal-closed and even $*$-autonomous, such that $[0,\infty]$ serves as a dualizing object.*

3. Quantale-enriched categories, functors, and distributors

In this section we summarize our notations and some basic properties and known results for quantale-enriched categories as used and needed in the following sections.

3.1. Quantales. A (commutative and unital) *quantale* $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$ is simultaneously a complete lattice $(\mathcal{V}, \leq)$ and a commutative monoid $(\mathcal{V}, \otimes, \mathrm{k})$, such that $\otimes$ distributes over arbitrary suprema in $\mathcal{V}$, always denoted by $\bigvee$. In particular, $\mathcal{V}$ has a bottom element, $\bot = \bigvee \emptyset$, and a top element $\top = \bigwedge \emptyset$. For the $\otimes$-neutral element k we allow $\mathrm{k} < \top$ (as well as $\mathrm{k} = \bot$, in which case, however, $\mathcal{V}$ is trivial: $\bot = \top$).

For every $v \in \mathcal{V}$, the (by distributivity) $\bigvee$-preserving map $- \otimes v : \mathcal{V} \to \mathcal{V}$ has a right adjoint $[v, -]$ in the lattice $\mathcal{V}$, whose values $[v, w]$ are characterized by

$$u \leq [v, w] \iff u \otimes v \leq w$$

for all $u \in \mathcal{V}$. In other words, considered as a small, thin (*i.e.,* arrows are given by order), and symmetric monoidal category, $\mathcal{V}$ is closed. Alternatively, $\mathcal{V}$ is an internal commutative monoid in the symmetric monoidal-closed category Sup.

Our guiding examples of quantales are the *Boolean quantale* $\mathsf{2} = (\{\bot, \top\}, \Rightarrow, \&, \top)$ and the *Lawvere quantale* $[0,\infty]_+ = ([0,\infty], \geq, +, 0)$, where $\geq$ is the inversion of the natural order $\leq$ of the reals; to avoid confusion with the quantalic notation $\bigvee, \bigwedge$, we continue to use respectively $\inf, \sup$ (as in Section 2) whenever referring to the natural order of the reals. The internal hom in $[0,\infty]_+$ is computed as $[v, w] = \max\{w - v, 0\}$; hence, with d as in Examples 2.2, $[v, w] = d(w, v)$.

3.2. $\mathcal{V}$-relations. A *$\mathcal{V}$-relation* $r : X \mathrel{-\!\!\!\!\mapstochar\!\!\!\to} Y$ of sets X, Y is a map $r : X \times Y \to \mathcal{V}$. Its composite $s \cdot r$ with $s : Y \mathrel{-\!\!\!\!\mapstochar\!\!\!\to} Z$ is given by $(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$. With the pointwise order of its hom-sets inherited from $\mathcal{V}$, we have the 2-category $\mathcal{V}\text{-}\mathsf{Rel}$, equipped with the involution $r \mapsto r^\circ$ with $r^\circ(y, x) = r(x, y)$. Every set map $f : X \to Y$

gives the $\mathcal{V}$-relation $f_\circ : X \nrightarrow Y$, with $f_\circ(x,y) = \mathrm{k}$ if $fx = y$ and $\perp$ otherwise, as well as $f^\circ := (f_\circ)^\circ : Y \nrightarrow X$. One has the adjunction $f_\circ \dashv f^\circ$, and $1_X^\circ = (1_X)_\circ$ serves as the identity morphism on X in $\mathcal{V}$-Rel.

The order on the hom-sets of $\mathcal{V}$-Rel is complete, and the composition of $\mathcal{V}$-relations (as an extension of $\otimes$ in $\mathcal{V}$) distributes over arbitrary joins in $\mathcal{V}$-Rel, these being formed pointwise. Consequently, $\mathcal{V}$-Rel is Sup-enriched, *i.e.*, a *quantaloid*. Since, of course, the composition is generally *not* commutative, for $\mathcal{V}$-relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ we must distinguish between the right adjoints $[r,-]$ and $]s,-[$ of the monotone maps

$$- \cdot r \colon \mathcal{V}\text{-}\mathsf{Rel}(Y,Z) \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}(X,Z) \quad \text{and} \quad s \cdot - \colon \mathcal{V}\text{-}\mathsf{Rel}(X,Y) \longrightarrow \mathcal{V}\text{-}\mathsf{Rel}(X,Z)$$

respectively. One calls $[r,t] : Y \nrightarrow Z$ the *extension of* $t : X \nrightarrow Z$ *along* r, and $]s,t[: X \nrightarrow Y$ the *lifting of* t *along* s:

$$\begin{tikzcd} & Z \\ X \arrow[ur, "t", "|" marking] \arrow[r, "r"', "|" marking] \arrow[ur, phantom, "\geq"', shift right=3] & Y \arrow[u, "{[r,t]}"', "|" marking] \end{tikzcd} \qquad\qquad \begin{tikzcd} Y \arrow[r, "s", "|" marking] & Z \\ X \arrow[u, "{]s,t[}", "|" marking] \arrow[ur, "t"', "|" marking] \arrow[ur, phantom, "\leq", shift left=3] & \end{tikzcd}$$

By definition, these $\mathcal{V}$-relations are respectively determined by the equivalences

$$s \leq [r,t] \iff s \cdot r \leq t \quad \text{and} \quad s \cdot r \leq t \iff r \leq]s,t[.$$

For their pointwise computation in terms of the internal hom of $\mathcal{V}$ one has the formulae

$$[r,t](y,z) = \bigwedge_{x\in X} [r(x,y), t(x,z)], \qquad]s,t[(x,y) = \bigwedge_{z\in Z} [s(y,z), t(x,z)].$$

3.3. $\mathcal{V}$-CATEGORIES AND $\mathcal{V}$-FUNCTORS. From now on, $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$ will always be a quantale. A (small) $\mathcal{V}$-*category* is a set X with a $\mathcal{V}$-relation $d : X \nrightarrow X$ that is reflexive ($1_X^\circ \leq d$) and transitive ($d \cdot d \leq d$). Writing $X(x,y)$ for $d(x,y)$, this means

$$\mathrm{k} \leq X(x,x) \quad \text{and} \quad X(x,y) \otimes X(y,z) \leq X(x,z),$$

for all $x,y,z \in X$. A $\mathcal{V}$-*functor* $f : X \to Y$ is a map satisfying $d \leq f^\circ \cdot e \cdot f_\circ$ (where e is the structure of Y); that is, for all $x, y \in X$ one has

$$X(x,y) \leq Y(fx,fy).$$

This defines the category $\mathcal{V}$-Cat. It contains the important object $\mathcal{V}$, with $\mathcal{V}(v,w) = [v,w]$.

Every $\mathcal{V}$-category X has an induced (pre)order, defined by ($x \leq y \iff \mathrm{k} \leq X(x,y)$). The induced order of the $\mathcal{V}$-category $\mathcal{V}$ coincides with its given lattice order. $\mathcal{V}$-functors become monotone maps w.r.t. the induced orders. The hom-set $\mathcal{V}$-$\mathsf{Cat}(X,Y)$ inherits

its pointwise order from Y (so that $f \leq g \iff \forall x \in X : fx \leq gx$), making $\mathcal{V}$-Cat a 2-category and producing the induced-order 2-functors

$$\mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Ord} \qquad \text{and} \qquad \mathcal{V}\text{-}\mathsf{Cat}_{\rm sep} \to \mathsf{Pos}\,;$$

here $\mathcal{V}$-$\mathsf{Cat}_{\rm sep}$ is the full subcategory of $\mathcal{V}$-Cat, containing the *separated* $\mathcal{V}$-categories whose induced order makes them posets. Ord=2-Cat is the category of (pre)ordered sets and monotone maps, and Pos = 2-$\mathsf{Cat}_{\rm sep}$ is its full subcategory of posets. Note also: Met = $[0,\infty]_+$-Cat.

The dual $X^{\rm op}$ of a $\mathcal{V}$-category X, given by $X^{\rm op}(x,y) = X(y,x)$, defines the isomorphism $(-)^{\rm op} : \mathcal{V}\text{-}\mathsf{Cat}^{\rm co} \to \mathcal{V}\text{-}\mathsf{Cat}$ of 2-categories; it is covariant on the identically mapped morphisms, but contravariant on 2-cells, since $(f \leq f' \iff f^{\rm op} \geq (f')^{\rm op})$.

An adjunction of $\mathcal{V}$-functors $f \dashv g : Y \to X$ in $\mathcal{V}$-Cat is already determined by the adjunction of the induced monotone maps, that is, by the conditions $fg \leq 1_Y$ and $1_X \leq gf$, in which case one has $Y(fx,y) = X(x,gy)$ for all $x \in X, y \in Y$. This latter condition is actually strong enough to force the $\mathcal{V}$-functoriality of the maps f and g, without any further assumption on them. But note that one may have adjoint monotone maps between $\mathcal{V}$-categories which fail to be $\mathcal{V}$-functors, *i.e.*, their adjunction in Ord does not guarantee the adjunction in $\mathcal{V}$-Cat.

The *tensor product* $X \otimes Y$ of $\mathcal{V}$-categories X, Y rests on the set $X \times Y$, structured by

$$(X \otimes Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y')$$

for all $x, x' \in X, y, y' \in Y$. Together with the tensor-neutral object $\mathrm{E} = \{*\}$ with $\mathrm{E}(*,*) = \mathrm{k}$, it makes $\mathcal{V}$-Cat symmetric monoidal, even closed: the *internal hom* $[X,Y]$ is given by the set $\mathcal{V}$-$\mathsf{Cat}(X,Y)$, structured by

$$[X,Y](f,g) = \bigwedge_{x \in X} Y(fx,gx)$$

for all $\mathcal{V}$-functors $f, g : X \to Y$. Therefore, $[X,[Y,Z]] \cong [X \otimes Y, Z] \cong [Y,[X,Z]]$, naturally in all variables. We note that, whilst $[X,Y]$ is separated when Y is so, $X \otimes Y$ generally fails to be separated even when both X and Y are separated.

3.4. $\mathcal{V}$-DISTRIBUTORS. For $\mathcal{V}$-categories X, Y, a $\mathcal{V}$*-distributor* $\rho : X \to\!\!\!\!\!\circ\, Y$ is a $\mathcal{V}$-relation $\rho : X \to\!\!\!\!\!\!\shortmid\;\, Y$ satisfying $e \cdot \rho \cdot d \leq \rho$, where d, e denote the structures of X, Y, respectively; equivalently,

$$X(x',x) \otimes \rho(x,y) \otimes Y(y,y') \leq \rho(x',y')$$

for all $x, x' \in X$, $y, y' \in Y$, or: $\rho : X^{\rm op} \otimes Y \to \mathcal{V}$ is a $\mathcal{V}$-functor.

Every $\mathcal{V}$-functor $f : X \to Y$ gives the $\mathcal{V}$-distributors $f_* := e \cdot f_\circ : X \to\!\!\!\!\!\circ\, Y$ and $f^* := f^\circ \cdot e : Y \to\!\!\!\!\!\circ\, X$; in particular, one has $1_X^* : X \to\!\!\!\!\!\circ\, X$, coinciding with the structure d of X. Identifying elements of X with $\mathcal{V}$-functors $\mathrm{E} \to X$ and elements of $\mathcal{V}$ with $\mathcal{V}$-distributors $\mathrm{E} \to\!\!\!\!\!\circ\, \mathrm{E}$, one has the useful formula $y^* \cdot x_* = X(x,y)$ for all $x, y \in X$.

$\mathcal{V}$-distributors are closed under $\mathcal{V}$-relational composition. Inheriting also the order from $\mathcal{V}$-Rel, the $\mathcal{V}$-categories and their $\mathcal{V}$-distributors form the 2-category $\mathcal{V}$-Dist, in which 1_X^* serves as the identity morphism of X. One has the identity-on-objects 2-functors

$$\mathcal{V}\text{-}\mathsf{Cat}^{\mathrm{co}} \xrightarrow{(-)_*} \mathcal{V}\text{-}\mathsf{Dist} \xleftarrow{(-)^*} \mathcal{V}\text{-}\mathsf{Cat}^{\mathrm{op}}\,.$$

For every $\mathcal{V}$-functor $f : X \to Y$ there is the adjunction $f_* \dashv f^*$ in $\mathcal{V}$-Dist, and for a $\mathcal{V}$-functor $g : Y \to X$ one has $f \dashv g$ in $\mathcal{V}$-Cat if, and only if, $f_* = g^*$. The $\mathcal{V}$-functor f is called *fully faithful* if $f^* \cdot f_* = 1_X^*$.

Since every hom-set $\mathcal{V}\text{-}\mathsf{Dist}(X, Y) = [X^{\mathrm{op}} \otimes Y, \mathcal{V}]$ is a $\mathcal{V}$-category, $\mathcal{V}$-Dist becomes enriched in $\mathcal{V}$-Cat. We note that the tensor product of $\mathcal{V}$-Cat may be used to make $\mathcal{V}$-Dist symmetric monoidal, such that the functors $(-)_*$ and $(-)^*$ become homomorphisms of monoidal categories.

Most importantly, $\mathcal{V}$-Dist is closed under the formation of suprema and infima in the hom-sets of $\mathcal{V}$-Rel, as well as under the formation of extensions and liftings. Hence, for $\mathcal{V}$-distributors ρ, σ, τ as shown in the diagram below, the $\mathcal{V}$-relational extension $[\rho, \tau]$ and the lifting $]\sigma, \tau[$ are $\mathcal{V}$-distributors and get respectively characterized by the equivalences

$$\sigma \leq [\rho, \tau] \iff \sigma \cdot \rho \leq \tau \quad \text{and} \quad \sigma \cdot \rho \leq t \iff \rho \leq\,]\sigma, \tau[\ ,$$

in the case of $[\rho, \tau]$ to be universally quantified over σ, and for $]\sigma, \tau[$ over ρ.

$$\begin{tikzcd} & Z \\ X \arrow[ur, "\tau", "\circ" marking] \arrow[r, "\rho"', "\circ" marking] & Y \arrow[u, "{[\rho,\tau]}"', "\circ" marking] \arrow[ul, phantom, "\geq"] \end{tikzcd} \qquad\qquad \begin{tikzcd} Y \arrow[r, "\sigma", "\circ" marking] & Z \\ X \arrow[u, "{]\sigma,\tau[}", "\circ" marking] \arrow[ur, "\tau"', "\circ" marking] \arrow[ur, phantom, "\leq", near start, shift left=2] & \end{tikzcd}$$

4. Presheaves, weighted colimits, and cocompletion

We continue our rapid tour through quantale-enriched category theory to the extent needed in the following section. For detailed proofs, the reader may consult, for example, [Stubbe 2005, Hofmann, Seal, Tholen 2014, Hofmann, Nora 2018], and the references given in there.

4.1. THE $\mathcal{V}$-PRESHEAF ADJUNCTION. In what follows, we identify X with $\mathrm{E} \otimes X$ and $X \otimes \mathrm{E}$, for every $\mathcal{V}$-category X. One then has the $\mathcal{V}$-categories

$$\mathcal{P}_{\mathcal{V}} X := \mathcal{V}\text{-}\mathsf{Dist}(X, \mathrm{E}) = [X^{\mathrm{op}}, \mathcal{V}] \qquad \text{and} \qquad \mathcal{P}^{\sharp}_{\mathcal{V}} X := \mathcal{V}\text{-}\mathsf{Dist}^{\mathrm{co}}(\mathrm{E}, X) = [X, \mathcal{V}]^{\mathrm{op}}.$$

of $\mathcal{V}$*-presheaves* and $\mathcal{V}$*-copresheaves* on X, respectively. Of course, $\mathcal{P}_{\mathcal{V}}(X^{\mathrm{op}}) = (\mathcal{P}^{\sharp}_{\mathcal{V}} X)^{\mathrm{op}}$. Every $x \in X$ gives the representable $\mathcal{V}$-(co)presheaves

$$\mathbf{y}_X x := x^* = X(-, x) = X^{\mathrm{op}} \to \mathcal{V} \qquad \text{and} \qquad \mathbf{y}^{\sharp}_X x := x_* = X(x, -) : X \to \mathcal{V}.$$

This defines the fully faithful *Yoneda* $\mathcal{V}$*-functors*

$$\mathbf{y}_X : X \longrightarrow \mathcal{P}_{\mathcal{V}} X \qquad \text{and} \qquad \mathbf{y}^{\sharp}_X : X \longrightarrow \mathcal{P}^{\sharp}_{\mathcal{V}} X\,.$$

These are the respective units at X of the $\mathcal{V}$-Cat-enriched (co)presheaf adjunctions

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Dist}^{\mathrm{op}} \arrow[r, bend right=10, "\mathcal{P}_{\mathcal{V}} = \mathcal{V}\text{-}\mathsf{Dist}(-,\mathrm{E})"'] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathsf{Cat} \arrow[l, bend right=10, "(-)^*"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{V}\text{-}\mathsf{Dist}^{\mathrm{co}} \arrow[r, bend right=10, "\mathcal{P}^{\sharp}_{\mathcal{V}} = \mathcal{V}\text{-}\mathsf{Dist}^{\mathrm{co}}(\mathrm{E},-)"'] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathsf{Cat} \arrow[l, bend right=10, "(-)_*"'] \end{tikzcd} .$$

the first of which is based on the natural isomorphism

$$\mathcal{V}\text{-}\mathsf{Dist}(X,Y) = [X^{\mathrm{op}} \otimes Y, \mathcal{V}] \cong [Y, [X^{\mathrm{op}}, \mathcal{V}]] = \mathcal{V}\text{-}\mathsf{Cat}(Y, \mathcal{P}_{\mathcal{V}}X).$$

4.2. The $\mathcal{V}$-presheaf monad. The $\mathcal{V}$-(co)presheaf adjunctions give us 2-monads on $\mathcal{V}$-Cat, briefly denoted by $\mathcal{P}$ and $\mathcal{P}^{\sharp}$. Their crucial property is their *lax idempotency* which, for $\mathcal{P}$, means $\mathcal{P}\mathbf{y} \leq \mathbf{y}\mathcal{P}$, *i.e.*, $\mathcal{P}_{\mathcal{V}}(\mathbf{y}_X^*) \leq \mathbf{y}_{\mathcal{P}_{\mathcal{V}}X} : \mathcal{P}_{\mathcal{V}}X \longrightarrow \mathcal{P}_{\mathcal{V}}\mathcal{P}_{\mathcal{V}}X$, for every $\mathcal{V}$-category X. Consequently, for X to carry a $\mathcal{P}$-pseudo-algebra structure $c : \mathcal{P}X \to X$, it suffices that c be a pseudo-retraction of $\mathbf{y}_X$ or, equivalently, to satisfy $c \dashv \mathbf{y}_X$ in $\mathcal{V}$-Cat. Such c is unique up to order-equivalence, and its existence is therefore a property of X, which we call *$\mathcal{P}$-cocompleteness*. In the dual case we say that X is *$\mathcal{P}^{\sharp}$-complete*.

The hands-on characterization of these properties reads as follows: X is $\mathcal{P}$-cocomplete if, and only if, for every $\omega \in \mathcal{P}_{\mathcal{V}}X$ there is some chosen element $c(\omega) \in X$ with $c(\omega)_* \simeq [\omega, 1_X^*]$. Then c becomes automatically a $\mathcal{V}$-functor $\mathcal{P}X \to X$. Dually, X is $\mathcal{P}^{\sharp}$-complete if, and only if, for every $\upsilon \in \mathcal{P}^{\sharp}_{\mathcal{V}}X$ there is some chosen element $d(\upsilon) \in X$ with $d(\upsilon)^* \simeq]\upsilon, 1_X^*[$. In pointwise terms, this means that, respectively, one has for all $x \in X$ the formulae

$$X(c(\omega), x) = \bigwedge_{z \in X} [\omega z, X(z,x)] \qquad \text{and} \qquad X(x, d(\upsilon)) = \bigwedge_{z \in X} [\upsilon z, X(x,z)]$$

$$\begin{tikzcd} & X \\ X \arrow[ur, "1_X^*"] \arrow[r, "\omega"'] \arrow[r, phantom, shift left=3, "\geq"] & \mathrm{E} \arrow[u, "c(\omega)_* = {[\omega, 1_X^*]}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{E} \arrow[r, "\upsilon"] \arrow[r, phantom, shift right=3, "\leq"] & X \\ X \arrow[u, "d(\upsilon)^* = {]\upsilon, 1_X^*[}"] \arrow[ur, "1_X^*"'] & \end{tikzcd}$$

4.3. Weighted colimits and cocompletion. For $\mathcal{V}$-categories X, Z, W, let $h : Z \to X$ be a $\mathcal{V}$-functor and $\omega : Z \mathrel{-\!\!\circ\!\!\rightarrow} W$, $\upsilon : W \mathrel{-\!\!\circ\!\!\rightarrow} Z$ be $\mathcal{V}$-distributors. A *colimit of h weighted by ω* is a $\mathcal{V}$-functor $q : W \to X$ such that $q_* = [\omega, h_*]$. Dually, a *limit of h weighted by υ* is a $\mathcal{V}$-functor $p : W \to X$ such that $p^* =]\upsilon, h^*[$. The respective formulae are ($x \in X$, $t \in W$):

$$X(qt, x) = \bigwedge_{z \in Z} [\omega(z,t), X(hz, x)] \quad \text{and} \quad X(x, pt) = \bigwedge_{z \in Z} [\upsilon(t,z), X(x, hz)];$$

one then writes

$$q \simeq \mathrm{colim}^{\omega} h \quad \text{and} \quad p \simeq \lim{}^{\upsilon} h.$$

$$\begin{tikzcd} & X \\ Z \arrow[ur, "h_*"] \arrow[r, "\omega"'] \arrow[r, phantom, shift left=3, "\geq"] & W \arrow[u, "q_* = {[\omega, h_*]}"'] \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "\upsilon"] \arrow[r, phantom, shift right=3, "\leq"] & Z \\ X \arrow[u, "p^* = {]\upsilon, h^*[}"] \arrow[ur, "h^*"'] & \end{tikzcd}$$

The duality of the notions of colimit and limit formally reads as $(\mathrm{colim}^{\omega}h)^{\mathrm{op}} \simeq \lim^{\omega^{\circ}}(h^{\mathrm{op}})$.

A $\mathcal{V}$-functor $f : X \to Y$ *preserves* the colimit $q \simeq \mathrm{colim}^{\omega}h$ if $fq \simeq \mathrm{colim}^{\omega}(fh)$; dually for limits. The $\mathcal{V}$-functor f is *(co)continuous* if it preserves all existing weighted (co)limits. The $\mathcal{V}$-category X is *cocomplete* if, for all h, ω as above, there is a chosen colimit of h weighted by ω. Dually, X is *complete* if, for all h, υ as above, there is a chosen limit of h weighted by υ. With the formulae

$$\mathrm{colim}^{\omega}h \simeq \mathrm{colim}^{\omega\cdot h^*}1_X \qquad \text{and} \qquad \lim^{\upsilon}h \simeq \lim^{h_*\cdot\upsilon}1_X,$$

one shows that cocompleteness equivalently means $\mathcal{P}$-cocompleteness, and that completeness means $\mathcal{P}^{\sharp}$-completeness. Moreover, cocontinuity of a $\mathcal{V}$-functor $f : X \to Y$ of cocomplete $\mathcal{V}$-categories translates to f being a pseudo-homomorphism of the respective $\mathcal{P}$-pseudo-algebras; likewise for continuity. Briefly: the category

$$\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{colim}}$$

of cocomplete $\mathcal{V}$-categories and their cocontinuous $\mathcal{V}$-functors is equivalent to the category of $\mathcal{P}$-pseudo-algebras and their pseudo-homomorphisms. Identified as the free $\mathcal{P}$-pseudo-algebras, all $\mathcal{V}$-presheaf-categories (in particular $\mathcal{V} = \mathcal{P}\mathrm{E}$) are cocomplete, and one has:

Cocompletion Theorem: Up to order-equivalence, every $\mathcal{V}$-functor $f : X \to Y$ into a cocomplete $\mathcal{V}$-category Y factors through $\mathbf{y}_X$, by a cocontinuous $\mathcal{V}$-functor $\tilde{f} : \mathcal{P}_{\mathcal{V}}X \to Y$ that is unique up to order-equivalence.

$$\begin{tikzcd} X \arrow[r, "\mathbf{y}_X"] \arrow[dr, "f"'] \arrow[dr, phantom, bend left=15, "\simeq"] & \mathcal{P}_{\mathcal{V}}X \arrow[d, dashed, "\tilde{f}"] \\ & Y \end{tikzcd}$$

As one readily shows, for every $\omega \in \mathcal{P}_{\mathcal{V}}X$ one has $\omega \simeq \mathrm{colim}^{\omega}\mathbf{y}_X$. Therefore, the cocontinuous $\mathcal{V}$-functor $\tilde{f}$ evaluates at ω according to

$$\tilde{f}(\omega) \simeq \mathrm{colim}^{\omega}f.$$

Dually, working with $\mathcal{P}^{\sharp}$ rather than $\mathcal{P}$, one obtains a completion theorem for $\mathcal{V}$-categories. The qualifier "up to order equivalence" is not needed when Y is separated.

4.4. Tensors and conical suprema. For a $\mathcal{V}$-category X and all $x \in X$, $u \in \mathcal{V}$, one defines the *tensor* $u \odot x$ and the *cotensor* $u \pitchfork x$ to exist in X if the respective colimit and limit

$$u \odot x \simeq \mathrm{colim}^{u}x \quad \text{and} \quad u \pitchfork x \simeq \lim{}^{u}x,$$

exists in X; this means: if, for all $y \in X$, one respectively has

$$X(u \odot x, y) = [u, X(x,y)] \qquad \text{and} \qquad X(y, u \pitchfork x) = [u, X(y,x)].$$

$$\begin{tikzcd} & X \\ \mathrm{E} \arrow[ur, "x_*"] \arrow[r, "u"'] \arrow[ur, phantom, "\geq", shift right=3] & \mathrm{E} \arrow[u, "(u\odot x)_*={[u,x_*]}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{E} \arrow[r, "u"] & \mathrm{E} \\ X \arrow[u, "(u\pitchfork x)^*={]u,x^*[}"] \arrow[ur, "x^*"'] \arrow[ur, phantom, "\leq", shift left=3] & \end{tikzcd}$$

X is *(co)tensored* if all (co)tensors exist in X. Equivalently, X is tensored if the $\mathcal{V}$ functor $X(x,-) : X \to \mathcal{V}$ has a left adjoint for all $x \in X$, and X is cotensored if X^{op} is tensored.

For every $\mathcal{V}$-category X, the presheaf category $\mathcal{P}_{\mathcal{V}}X$ is not only tensored, but also cotensored, with the tensor $u \odot \phi$ and the cotensor $u \pitchfork \phi$ for $u \in \mathcal{V},\ \phi \in \mathcal{P}_{\mathcal{V}}X$ given for all $x \in X$ by

$$(u \odot \phi)x = u \otimes (\phi x) \qquad \text{and} \qquad (u \pitchfork \phi)x = [u, \phi x],$$

By duality, also $\mathcal{P}^\sharp X$ is tensored and cotensored.

For $x, y,\ x_i (i \in I)$ in a $\mathcal{V}$-category X, the element x is a *conical supremum* of $(x_i)_{i\in I}$, and y is a *conical infimum* of $(x_i)_{i\in I}$ if, respectively,

$$X(x,z) = \bigwedge_{i\in I} X(x_i, z) \qquad \text{and} \qquad X(z,y) = \bigwedge_{i\in I} X(z, x_i)$$

for all $z \in X$; equivalently: if x is a supremum of $(x_i)_{i\in I}$ in X w.r.t. its induced order and the Yoneda $\mathcal{V}$-functor $\mathbf{y}^\sharp_X : X \longrightarrow \mathcal{P}^\sharp_{\mathcal{V}}X = [X, \mathcal{V}]^{\mathrm{op}}$ preserves this supremum, and dually, if y is an infimum of $(x_i)_{i\in I}$ in the induced order of X and the Yoneda $\mathcal{V}$-functor $\mathbf{y}_X : X \to \mathcal{P}_{\mathcal{V}}X = [X^{\mathrm{op}}, \mathcal{V}]$ preserves the infimum.

These are particular types of weighted (co)limits: let $\omega := \bigvee_i x_i^*$ in $\mathcal{P}_{\mathcal{V}}X$ and $\upsilon := \bigwedge_i (x_i)_*$ in $\mathcal{P}^\sharp_{\mathcal{V}}X = [X, \mathcal{V}]^{\mathrm{op}}$. Then, using $\triangledown$ to signal that a supremum or infimum in the induced order is conical, we have

$$\bigvee_{i\in I}^{\triangledown} x_i \simeq \mathrm{colim}^{\omega} 1_X^* \qquad \text{and} \qquad \bigwedge_{i\in I}^{\triangledown} x_i \simeq \lim{}^{\upsilon} 1_X^*,$$

with the term on either side of the $\simeq$-sign existing when the term on the other side exists.

$$\begin{tikzcd} & X \\ X \arrow[ur, "1_X^*"] \arrow[r, "\omega"'] \arrow[ur, phantom, "\geq", shift right=3] & \mathrm{E} \arrow[u, "(\bigvee_i^{\triangledown} x_i)_*={[\omega,1_X^*]}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{E} \arrow[r, "\upsilon"] & X \\ X \arrow[u, "(\bigwedge_i^{\triangledown} x_i)^*={]\upsilon,1_X^*[}"] \arrow[ur, "1_X^*"'] \arrow[ur, phantom, "\leq", shift left=3] & \end{tikzcd}$$

X is *order-complete* if every family in X has a chosen supremum or, equivalently, an infimum, with respect to its induced order. X is *conically cocomplete* (*conically complete*) if every family in X has a chosen conical supremum (conical infimum, respectively) in X.

Tensors and conical suprema are the building blocks to obtain arbitrary weighted colimits; dually for weighted limits: for a $\mathcal{V}$-functor $h : Z \to X$ and $\mathcal{V}$-distributors $\omega : Z \nrightarrow W, \upsilon : W \nrightarrow Z$, one respectively has, when X is tensored or cotensored,

$$(\mathrm{colim}^{\omega} h)t \simeq \bigvee_{z\in Z}^{\triangledown} \omega(z,t) \odot hz \qquad \text{and} \qquad (\lim{}^{\upsilon} h)t \simeq \bigwedge_{z\in Z}^{\triangledown} \upsilon(t,z) \pitchfork hz,$$

with the colimit (limit) existing precisely when the conical supremum (infimum, respectively) exist for all $t \in W$. Consequently, a $\mathcal{V}$-functor $f : X \to Y$ of a tensored category X is cocontinuous if, and only if, f preserves tensors and conical suprema; likewise in the dual situation.

4.5. Completeness Theorem. Let us first record the easily established Adjoint Functor Theorem: For a complete $\mathcal{V}$-category Y, a $\mathcal{V}$-functor $g : Y \to X$ admits a left adjoint $\mathcal{V}$-functor $f \dashv g$ if, and only if, g is continuous. When X is cocomplete and $x \in X$, one may apply the theorem to $g := X(-, x) : X^{\mathrm{op}} \to \mathcal{V}$ to show the implication (ii)$\Longrightarrow$(iv) of the following equivalent statements characterizing (co)completeness of a $\mathcal{V}$-category X:

(i) X is $\mathcal{P}$-cocomplete;

(ii) X is cocomplete;

(iii) X is tensored and conically cocomplete;

(iv) X is tensored, cotensored, and order-complete;

(v) X is cotensored and conically complete;

(vi) X is complete;

(vii) X is $\mathcal{P}^{\sharp}$-complete.

We note that, whilst each, completeness and order-completeness, are self-dual properties of a $\mathcal{V}$-category, in general neither conical completeness nor being tensored are self-dual properties.

5. Equational presentation of separated cocomplete $\mathcal{V}$-categories

The rules of the following proposition for tensorial action and suprema are well-known to hold (and to be easily proved) in every separated cocomplete $\mathcal{V}$-category. Actually, separation is not essential if one is willing to trade the strict equality for order-equivalence. But for simplicity of the presentation, we restrict ourselves to considering only separated $\mathcal{V}$-categories in the rest of the paper.

$\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$ continues to be a (commutative and unital) quantale.

5.1. Proposition. *In a separated cocomplete $\mathcal{V}$-category X equipped with its induced partial order, the following identities hold for all $u, v, u_i \in \mathcal{V}$ $(i \in I)$ and $x, x_i \in X, (i \in I)$:*

(Act1) $\mathrm{k} \odot x = x$;

(Act2) $u \odot (v \odot x) = (u \otimes v) \odot x$;

(Sup1) $(\bigvee_{i\in I} u_i) \odot x = \bigvee_{i\in I}(u_i \odot x)$*;*

(Sup2) $u \odot (\bigvee_{i\in I} x_i) = \bigvee_{i\in I}(u \odot x_i)$.

PROOF. For example, to validate (Sup2), since suprema in a cocomplete $\mathcal{V}$-category are conical, we just need to note that, for all $y \in X$, one has

$$X(\bigvee_{i\in I} u \odot x_i, y) = \bigwedge_{i\in I} X(u \odot x_i, y) = \bigwedge_{i\in I} [u, X(x_i, y)] = [u, \bigwedge_{i\in I} X(x_i, y)] = [u, X(\bigvee_{i\in I} x_i, y)].$$

∎

The point of the above proposition is that there is a converse proposition, as follows:

5.2. PROPOSITION. *Let the complete lattice X be endowed with an action $* : \mathcal{V} \times X \to X$ satisfying* (Act1,2) *and* (Sup1,2)*, with $\odot$ replaced by $*$. Then there is a unique $\mathcal{V}$-category structure on X which makes X separated and cocomplete as such, with tensors given by $*$, and with its induced partial order coinciding with the originally given lattice order.*

PROOF. Since we want the $\mathcal{V}$-category structure on X to make X tensored, with the tensor action given by $*$, one necessarily needs to have $X(u * x, y) = [u, X(x, y)]$ for all $x, y \in X$ and $u \in \mathcal{V}$. Hence, $X(x, -)$ must necessarily be the uniquely determined right adjoint of the (by (Sup1)) monotone map $- * x : \mathcal{V} \to X$; explicitly, $X(x, y) = \bigvee\{u \in \mathcal{V} \mid u * x \leq y\}$, for all $x, y \in X$.

For the existence of the structure, we note that (Sup1) actually makes $- * x$ a sup-map, so we have its right adjoint $X(x, -)$, for all $x \in X$, and in particular the inequality $X(x, y) * x \leq y$ (representing the counit of the adjunction $(- * x) \dashv X(x, -)$ at y). We show that X(-,-) makes X indeed a $\mathcal{V}$-category. Clearly, $\mathrm{k} \leq X(x, x)$ holds by (Act1), and with (Act2) and (Sup2) we see

$$(X(y, z) \otimes X(x, y)) * x = X(y, z) * (X(x, y) * x) \leq X(y, z) * y \leq z.$$

This implies the desired inequality $X(x, y) \otimes X(y, z) \leq X(x, z)$, for all $x, y, z \in X$. Since trivially ($\mathrm{k} \leq X(x, y) \iff x = \mathrm{k} * x \leq y$), the induced order of the $\mathcal{V}$-category X just defined is precisely the given order of X, so that the $\mathcal{V}$-category X is in particular order-complete. Also, trivially, X is tensored via $*$, by the definition of its $\mathcal{V}$-category structure. Finally, (Sup2) guarantees that the given suprema of the lattice X are conical in its $\mathcal{V}$-category structure, making the latter (co)complete, by the Completeness Theorem. ∎

The two propositions compel us to consider the 2-category

$$\mathsf{Sup}^{\mathcal{V}}$$

whose objects are complete lattices X, provided with an action $* : \mathcal{V} \times X \to X$ satisfying (Act1,2) and (Sup1,2), with $\odot$ replaced by $*$. Its morphisms are suprema-preserving and $*$-equivariant maps $f : X \to Y$; that is, f satisfies the conditions $f(u * x) = u * fx$ and $f(\bigvee_i x_i) = \bigvee_i fx_i$. The 2-cells are given by the pointwise order of sup-maps.

The propositions compare the objects of $\mathsf{Sup}^{\mathcal{V}}$ with those of the 2-category

$$\mathcal{V}\text{-}\mathsf{Sup} := \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{colim,sep}}\,,$$

which are separated, cocomplete $\mathcal{V}$-categories. Its morphisms are cocontinuous $\mathcal{V}$-functors, with 2-cells given by their pointwise order. The two propositions above validate the object-part of the following (no longer surprising) claim.

5.3. Theorem. *The object assignment $\mathcal{O} : \mathcal{V}\text{-}\mathsf{Sup} \to \mathsf{Sup}^{\mathcal{V}}$, defined by taking the induced order and the tensorial action, extends to morphisms and becomes an isomorphism of 2-categories.*

Proof. We just need to make sure that morphisms stay morphisms when mapped identically in either direction. That is certainly the case for the direction of $\mathcal{O}$. Conversely, for $f : X \to Y$ in $\mathsf{Sup}^{\mathcal{V}}$, using consecutively the unit of the adjunction $(- * fx) \dashv Y(fx, -)$ at $X(x, x')$, the $*$-equivariance of f, and the counit of the adjunction $(- * x) \dashv X(x, -)$ at x' in conjunction with the monotonicity of $Y(fx, f-)$, we obtain for all $x, x' \in X$ the desired inequality

$$\begin{aligned} X(x,x') &\leq Y(fx, X(x,x') * fx) \\ &= Y(fx, f(X(x,x') * x)) \\ &\leq Y(fx, fx')\,. \end{aligned}$$

This makes f a $\mathcal{V}$-functor for the $\mathcal{V}$-category structures on X and Y as constructed. As a $*$-equivariant sup-map, it preserves tensors and conical suprema and is thus cocontinuous. ∎

5.4. Examples. (1) For $\mathcal{V} = \mathsf{2}$ the Boolean quantale, $\mathcal{V}\text{-}\mathsf{Sup} \cong \mathsf{Sup}^{\mathcal{V}}$ is just the category Sup. There is, of course, another isomorphic description of Sup, in the guise of the category Inf of complete lattices and infima-preserving maps, thanks to the isomorphism $\mathsf{Sup} \longrightarrow \mathsf{Inf},\ X \longmapsto X^{\mathrm{op}}$. For any quantale $\mathcal{V}$, this isomorphism gives us the isomorphism

$$\mathsf{Sup}^{\mathcal{V}} \cong \mathsf{Inf}^{\mathcal{V}^{\mathrm{op}}},$$

where the objects of $\mathsf{Inf}^{\mathcal{V}^{\mathrm{op}}}$ are described like those of $\mathsf{Sup}^{\mathcal{V}}$, except that $\bigvee$ in (Sup1,2) needs to be replaced by $\bigwedge$; the morphisms in $\mathsf{Inf}^{\mathcal{V}^{\mathrm{op}}}$ are, of course, infima-preserving equivariant maps.

(2) For $\mathcal{V} = [0, \infty]_+$ the Lawvere quantale, $\mathcal{V}^{\mathrm{op}}$ takes us back to $[0, \infty]$ with its natural order. Hence, the category $\mathsf{Inf}^{\mathcal{V}^{\mathrm{op}}}$ is precisely the category ICMetA of Section 2. It is thus isomorphic to $[0, \infty]_+\text{-}\mathsf{Sup}$.

With our general quantale $\mathcal{V}$, we have the following commutative diagram where all unnamed functors are forgetful:

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Sup} \arrow[rr, "\mathcal{O}", "\cong"'] \arrow[d] & & \mathsf{Sup}^{\mathcal{V}} \arrow[d] \\ \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}} \arrow[dr] & & \mathsf{Sup} \arrow[dl] \\ & \mathsf{Pos} \arrow[d] & \\ & \mathsf{Set} & \end{tikzcd}$$

5.5. Theorem. *In the diagram above, the upper two isomorphic categories are monadic over any other category of the diagram.*

Proof. The presheaf monad $\mathcal{P}$ of $\mathcal{V}$-Cat can be restricted to the full subcategory $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$. It follows from 4.3 that the category of Eilenberg-Moore algebras of the restricted monad is isomorphic to $\mathcal{V}$-Sup, which is therefore monadic over $\mathcal{V}$-$\mathsf{Cat}_{\mathrm{sep}}$. With the [Duskin 1969] version of the Beck monadicity criterion, one can show that $\mathcal{V}$-Sup is even monadic over both, Pos and Set; a proof is contained in the thesis [Martinelli 2021]. For $\mathcal{V} = \mathsf{2}$, this gives the well known fact that Sup is monadic over both, Pos and Set, *qua* the down-set and power-set monads, respectively. Consequently, since we already know that $\mathsf{Sup}^{\mathcal{V}} \cong \mathcal{V}$-$\mathsf{Sup}$ is monadic over Pos and Set, with a cancellation theorem for right adjointness (see [Dubuc 1968]) and the Beck criterion, one may conclude that $\mathsf{Sup}^{\mathcal{V}}$ is monadic over Sup – which is hardly surprising since we defined the category in terms of Sup-based operations and equations. ■

6. The box-tensor product classifying the bi-morphisms of $\mathcal{V}$-Sup

The goal of this section is to classify the *bi-morphisms* of the category $\mathcal{V}$-Sup. For objects $X, Y, Z \in \mathcal{V}$-Sup, these are the set maps $f : X \times Y \to Z$ such that, for all $x \in X$ and $y \in Y$, the maps $f(-, y) : X \to Z$ and $f(x, -) : Y \to Z$ are morphisms in $\mathcal{V}$-Sup. By the Lemma below, such maps are necessarily $\mathcal{V}$-functors $X \otimes Y \to Z$. Hence, denoting the set of all these bi-morphisms by $\mathsf{BiMor}(\mathsf{X}, \mathsf{Y}; \mathsf{Z})$, we search for an object $X \boxtimes Y$ such that one has a natural bijective correspondence

$$\mathsf{BiMor}(\mathsf{X}, \mathsf{Y}; \mathsf{Z}) \cong \mathcal{V}\text{-}\mathsf{Sup}(X \boxtimes Y, Z) \ .$$

This entails finding a universal bi-morphism $X \otimes Y \to X \boxtimes Y$, and the roadmap to that end is clear (see [Banaschewski, Nelson 1976]): form the free cocomplete $\mathcal{V}$-category $\mathcal{P}_{\mathcal{V}}(X \otimes Y)$ and find the least congruence on it which forces the universal map, *i.e.*, the Yoneda $\mathcal{V}$-functor $\mathbf{y} = \mathbf{y}_{X \otimes Y}$, once composed with the canonical projection r onto the quotient object, to become a bi-morphism

$$X \otimes Y \xrightarrow{\ \mathbf{y}\ } \mathcal{P}_{\mathcal{V}}(X \otimes Y) \xrightarrow{\ r\ } X \boxtimes Y \ .$$

The challenge is to describe the quotient $\mathcal{V}$-category in such a way that proving its cocompleteness becomes manageable. Fortunately, this challenge can be overcome by a straight generalization of the method used by [Joyal, Tierney 1984] in case $\mathcal{V} = \mathsf{2}$, as we show in the Proposition below.

6.1. Lemma. *For $\mathcal{V}$-categories X, Y, Z, let the set map $f : X \times Y \to Z$ be such that the maps $f(x, -)$ and $f(-, y)$ are $\mathcal{V}$-functors, for all $x \in X, y \in Y$. Then $f : X \otimes Y \to Z$ is a $\mathcal{V}$-functor.*

Proof. By hypothesis, for $x, x' \in X,\ y, y' \in Y$, one has $X(x, x') \le Z(f(x, y), f(x', y))$ and $Y(y, y') \le Z(f(x', y), f(x', y'))$. This gives

$$\begin{aligned}(X \otimes Y)((x, y), (x', y')) &= X(x, x') \otimes Y(y, y')\\ &\le Z(f(x, y), f(x', y)) \otimes Z(f(x', y), f(x', y'))\\ &\le Z(f(x, y), f(x', y')).\end{aligned}$$

■

6.2. Proposition. *Let $R \subseteq P \times P$ be any relation on a separated (co)complete $\mathcal{V}$-category P, and let the set*

$$Q = \{y \in P \mid \forall (z_1, z_2) \in R \ \ (P(z_1, y) = P(z_2, y))\}$$

be considered as a full $\mathcal{V}$-subcategory of P. Then:

(1) *Q is (co)complete;*

(2) *there is a surjective morphism $r : P \to Q$ in $\mathcal{V}$-Sup whose kernel relation contains R; moreover,*

(3) *any morphism $r' : P \to Q'$ in $\mathcal{V}$-Sup whose kernel relation contains R factors (uniquely) through r, by a morphism $h : Q \to Q'$ in $\mathcal{V}$-Sup.*

Proof. (1) For all $y \in Q$, $(z_1, z_2) \in Q$ and $u \in \mathcal{V}$ one has

$$P(z_1, u \pitchfork y) = [u, P(z_1, y)] = [u, P(z_2, y)] = P(z_2, u \pitchfork y).$$

This shows that Q is closed under cotensors in P; likewise for conical infima. Consequently, Q is closed under all weighted limits in P and, hence, (co)complete.

(2) Being continuous, the fully faithful inclusion $\mathcal{V}$-functor $i : Q \hookrightarrow P$ has a left adjoint, $r : P \to Q$, exhibiting Q as a full reflective $\mathcal{V}$-subcategory of P, satisfying the strict identity $ri = 1_Q$, by separation in P. By adjunction, we have $Q(rx, y) = P(x, y)$ for all $x \in P$, $y \in Q$, and, in the induced order of P, we have the pointwise description of r by $rx = \bigwedge^{\triangledown}\{y \in Q \mid x \le y\}$. Consequently, for all $(z_1, z_2) \in R$ one obtains

$$rz_1 = \bigwedge\{y \in Q \mid \mathrm{k} \le P(z_1, y)\} = \bigwedge\{y \in Q \mid \mathrm{k} \le P(z_2, y)\} = rz_2.$$

(3) For $r' : P \to Q'$ in $\mathcal{V}$-Sup with $r'z_1 = r'z_2$ for all $(z_1, z_2) \in R$ we have the right adjoint $j : Q' \to P$ of r', satisfying $P(x, jy') = Q'(r'x, y')$ for all $x \in P$ and $y' \in Q'$. Consequently, with

$$P(z_1, jy') = Q'(r'z_1, y') = Q'(r'z_2, y') = P(z_2, jy')$$

holding for all $(z_1, z_2) \in R$, we see that the continuous $\mathcal{V}$-functor j takes values in Q, and we obtain the left adjoint $h : Q \to Q'$ of the restriction of j. Moreover, the left adjoints commute as $hr = r'$, since their respective right adjoints commute trivially. ■

Following the general strategy sketched above, given $X, Y \in \mathcal{V}\text{-}\mathsf{Sup}$, we consider $P := \mathcal{P}_{\mathcal{V}}(X \otimes Y)$ and, writing $z_1 \, R \, z_2$ for $(z_1, z_2) \in R$, we let $R \subseteq P \times P$ be the relation defined by

$$\mathbf{y}(u \odot x, y) \;\; R \;\; u \odot \mathbf{y}(x, y) \;\; R \;\; \mathbf{y}(x, u \odot y),$$

$$\mathbf{y}(\bigvee_{i \in I} x_i, y) \;\; R \;\; \bigvee_{i \in I} \mathbf{y}(x_i, y), \qquad \mathbf{y}(x, \bigvee_{i \in I} y_i) \;\; R \;\; \bigvee_{i \in I} \mathbf{y}(x, y_i),$$

for all $u \in \mathcal{V}$, $x, x_i \in X$, $y, y_i \in Y$, $i \in I$, with any set I. Then Proposition 6.2 gives us the $\mathcal{V}\text{-}\mathsf{Sup}$-object $X \boxtimes Y := Q$ and its morphism $r : P \to Q$ satisfying the properties (1-3). With $t_{X,Y} := r \, \mathbf{y}$, these properties let us conclude instantaneously:

6.3. THEOREM. *Given $X, Y \in \mathcal{V}\text{-}\mathsf{Sup}$, the map $t_{X,Y} : X \otimes Y \to X \boxtimes Y$ is a universal bi-morphism into a $\mathcal{V}\text{-}\mathsf{Sup}$-object: every bi-morphism $f : X \otimes Y \to Z$ with $Z \in \mathcal{V}\text{-}\mathsf{Sup}$ has the form $f = \overline{f} \, t_{X,Y}$, with a unique $\mathcal{V}\text{-}\mathsf{Sup}$-morphism $\overline{f} : X \boxtimes Y \to Z$.*

PROOF. Since Z is separated and (co)complete, the bi-morphism f factors uniquely through the free $\mathcal{P}$-algebra $\mathcal{P}_{\mathcal{V}}(X \otimes Y)$, as $f = \tilde{f} \mathbf{y}$ with a $\mathcal{V}\text{-}\mathsf{Sup}$-morphism $\tilde{f}$ (see 4.3) which, in turn, factors uniquely as $\tilde{f} = \overline{f} r$.

$$\begin{tikzcd} X \otimes Y \arrow[r, "\mathbf{y}"] \arrow[drr, "f"'] & \mathcal{P}_{\mathcal{V}}(X \otimes Y) \arrow[r, "r"] \arrow[dr, "\tilde{f}"] & X \boxtimes Y \arrow[d, "\overline{f}"] \\ & & Z \end{tikzcd}$$

■

6.4. COROLLARY. *The box-tensor product as defined above makes $\mathcal{V}\text{-}\mathsf{Sup}$ a symmetric monoidal category.*

PROOF. It is obvious that the box-tensor product in $\mathcal{V}\text{-}\mathsf{Sup}$ inherits its natural associativity and commutativity isomorphisms, as well as the coherence conditions, from the corresponding morphisms and conditions of the tensor product of $\mathcal{V}\text{-}\mathsf{Cat}$, *qua* universality of its definition, as indicated by the following diagram:

$$\begin{tikzcd} X \otimes Y \arrow[r, "\mathbf{y}_{X \otimes Y}"] \arrow[d, "c_{X,Y}"', "\cong"] & \mathcal{P}_{\mathcal{V}}(X \otimes Y) \arrow[r, "r_{X,Y}"] \arrow[d, "\cong"', "\mathcal{P}_{\mathcal{V}}(c^*_{X,Y})"] & X \boxtimes Y \arrow[d, "\cong"', "\overline{c}_{X,Y}"] \\ Y \otimes X \arrow[r, "\mathbf{y}_{Y \otimes X}"] & \mathcal{P}_{\mathcal{V}}(Y \otimes X) \arrow[r, "r_{Y,X}"] & Y \boxtimes X \end{tikzcd}$$

The $\boxtimes$-neutrality of the $\mathcal{V}$-category $\mathcal{V} = \mathcal{P}_{\mathcal{V}}\mathrm{E}$ is witnessed by the natural isomorphism $\ell_X : \mathcal{V} \boxtimes X \to X$ that is the mate of the bi-morphism $\mathcal{V} \otimes X \to X$, $(u, x) \mapsto u \odot x$. Indeed, showing that this $\mathcal{V}$-functor is universal amongst all bi-morphisms $\mathcal{V} \otimes X \to Z$ into $\mathcal{V}\text{-}\mathsf{Sup}$-objects Z is a routine exercise, by Proposition 5.1. ■

7. Finishing the proof that the category $\mathcal{V}$-Sup is $*$-autonomous

Our first task is to show that the symmetric monoidal structure of $\mathcal{V}$-Sup (Corollary 6.4) is closed. For that, we note that Theorem 6.3 gives us, for any $Z \in \mathcal{V}$-Sup, a natural bijection

$$\mathcal{V}\text{-}\mathsf{Cat}(\mathrm{E}, Z) \cong \mathcal{V}\text{-}\mathsf{Sup}(\mathcal{P}_{\mathcal{V}}\mathrm{E}, Z)\,.$$

Hence, if Z is supposed to serve as the internal hom of the given objects $X, Y \in \mathcal{V}$-Sup, the $\boxtimes$-neutrality of $\mathcal{P}_{\mathcal{V}}\mathrm{E}$ in $\mathcal{V}$-Sup forces the underlying set of Z to be (within isomorphism) $\mathcal{V}$-$\mathsf{Sup}(X, Y)$. Consequently, we set

$$[\![X, Y]\!] := \mathcal{V}\text{-}\mathsf{Sup}(X, Y)$$

and consider it as a $\mathcal{V}$-subcategory of the $\mathcal{V}$-category $[X, Y]$ of all $\mathcal{V}$-functors $X \to Y$.

7.1. Proposition. *For all $X, Y \in \mathcal{V}$-Sup, the $\mathcal{V}$-category $[\![X, Y]\!]$ is separated and cocomplete and, in fact, serves as the internal hom of X and Y in the symmetric monoidal-closed category $(\mathcal{V}\text{-}\mathsf{Sup}, \boxtimes, \mathcal{V})$.*

Proof. The $\mathcal{V}$-category $[\![X, Y]\!]$ inherits separation from $[X, Y]$. For its cocompleteness, defining tensors and conical suprema pointwise by

$$(u \odot f)x = u \odot fx \qquad \text{and} \qquad (\bigvee_{i\in I}^{\triangledown} f_i)x = \bigvee_{i\in I}^{\triangledown} f_i x$$

for $u \in \mathcal{V}$, $f, f_i \in [\![X, Y]\!]$ $(i \in I)$ and all $x \in X$, one routinely shows that $u \odot f$ and $\bigvee_{i\in I} f_i$ are well-defined (*i.e.*, preserve tensors and conical suprema) and actually do the job indicated by their names. For example, the computation

$$\begin{aligned}[\![X, Y]\!](u \odot f, g) &= \bigwedge_{x\in X} Y(u \odot fx, gx) = \bigwedge_{x\in X} [u, Y(fx, gx)]\\ &= [u, \bigwedge_{x\in X} Y(fx, gx)] = [u, [\![X, Y]\!](f, g)]\end{aligned}$$

for all $g \in [\![X, Y]\!]$ confirms that $u \odot f$ is indeed the tensor of u and f in $[\![X, Y]\!]$.

In order to show that $[\![X, Y]\!]$ serves as the internal hom in $\mathcal{V}$-Sup, for all $Z \in \mathcal{V}$-Sup, we now sketch how to establish a natural $\mathcal{V}$-Cat-isomorphism

$$[\![Z \boxtimes X, Y]\!] \cong [\![Z, [\![X, Y]\!]]\!] \qquad (1)$$

Considering $\mathrm{BiMor}(Z, X; Y)$ as a $\mathcal{V}$-subcategory of $[Z \otimes X, Y]$, by Theorem 6.3 one has the natural isomorphism $[\![Z \boxtimes X, Y]\!] \to \mathrm{BiMor}(Z, X; Y)$ which actually lives in $\mathcal{V}$-Cat. It then remains to be shown that the natural isomorphism $[Z \otimes X, Y] \to [Z, [X, Y]]$ in $\mathcal{V}$-Cat can be restricted as indicated by the diagram

$$\begin{tikzcd}
 & {[Z\otimes X,Y]} \arrow[r, "\cong"] & {[Z,[X,Y]]} \\
{[\![Z\boxtimes X,Y]\!]} \arrow[r, "\cong"] & \mathrm{BiMor}(Z,X;Y) \arrow[u] \arrow[r, "\cong"] & {[\![Z,[\![X,Y]\!]]\!]} \arrow[u]
\end{tikzcd}$$

This, however, is now a routine exercise, as one may use tensors and conical suprema of cocontinuous $\mathcal{V}$-functors, all defined pointwise, to guarantee well-definedness of the restricted isomorphism. ■

7.2. Corollary. *For all* $X \in \mathcal{V}\text{-}\mathsf{Sup}$ *and* $Z \in \mathcal{V}\text{-}\mathsf{Cat}$, *one has a natural isomorphism* $[Z,X] \cong [\![\mathcal{P}_{\mathcal{V}}Z, X]\!]$ *in* $\mathcal{V}\text{-}\mathsf{Cat}$. *For* $Z = \mathrm{E}$, *it gives the natural isomorphism*

$$X \cong [\![\mathcal{V}, X]\!]. \tag{2}$$

Explicitly, it follows from 4.3 that the canonical isomorphism $X \to [\![\mathcal{V}, X]\!]$ assigns to $x \in X$ the $\mathcal{V}$-functor $- \odot x : \mathcal{V} \to X$; its inverse map evaluates $\phi \in [\![\mathcal{V}, X]\!]$ at k.

The next essential ingredient to establishing the category $\mathcal{V}\text{-}\mathsf{Sup}$ as $*$-autonomous is its easily seen self-duality, as follows.

7.3. Proposition. *There is an isomorphism of categories* $\mathcal{V}\text{-}\mathsf{Sup}^{\mathrm{op}} \longrightarrow \mathcal{V}\text{-}\mathsf{Sup}$ *which assigns to a cocontinuous* $\mathcal{V}$*-functor* $f : X \to Y$ *the* $\mathcal{V}$*-functor* $(f^\bullet)^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$, *where* $f \dashv f^\bullet$ *in* $\mathcal{V}\text{-}\mathsf{Cat}$.

Proof. By the Adjoint Functor Theorem, the cocontinuous $\mathcal{V}$-functor f has a right adjoint $f^\bullet$. Hence, as a left adjoint of $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$, the $\mathcal{V}$-functor $(f^\bullet)^{\mathrm{op}}$ is a morphism of $\mathcal{V}\text{-}\mathsf{Sup}$. ■

The Proposition gives us the natural isomorphism

$$[\![X, Y]\!] \cong [\![Y^{\mathrm{op}}, X^{\mathrm{op}}]\!]. \tag{3}$$

Replacing X in (2) by X^{op} and then applying (3), one obtains the natural isomorphism

$$X^{\mathrm{op}} \cong [\![X, \mathcal{V}^{\mathrm{op}}]\!], \tag{4}$$

which turns out to send $x \in X$ to $X(-, x) : X \to \mathcal{V}^{\mathrm{op}}$.

As we confirm next, since $(X^{\mathrm{op}})^{\mathrm{op}} = X$, and likewise for morphisms, the isomorphism (4) exhibits the $\mathcal{V}$-category $\mathcal{V}^{\mathrm{op}}$ as a *global dualizing object* in $\mathcal{V}\text{-}\mathsf{Sup}$. This fact, in conjunction with Proposition 7.1, makes the category $\mathcal{V}\text{-}\mathsf{Sup}$ a $*$*-autonomous category* in the sense of [Barr 1979], a result first proved by [Eklund, Gutierrez Garcia, Hoehle, Kortelainen 2018], with a different proof (see the Introduction).

7.4. Theorem. [Duality Theorem] *The functor* $[\![-, \mathcal{V}^{\mathrm{op}}]\!] : \mathcal{V}\text{-}\mathsf{Sup}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathsf{Sup}$ *is an equivalence of categories, with pseudo-inverse* $[\![-, \mathcal{V}^{\mathrm{op}}]\!]^{\mathrm{op}}$, *so that*

$$X \cong [\![[\![X, \mathcal{V}^{\mathrm{op}}]\!], \mathcal{V}^{\mathrm{op}}]\!],$$

naturally for all $X \in \mathcal{V}\text{-}\mathsf{Sup}$. *The category* $\mathcal{V}\text{-}\mathsf{Sup}$ *is* $*$*-autonomous.*

PROOF. We must confirm that, for $f : X \to Y$ in $\mathcal{V}$-Sup, under the natural isomorphism (4) the map $(f^\bullet)^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ corresponds to precomposition by f, *i.e.*, that the following diagram commutes:

$$\begin{tikzcd} X^{\mathrm{op}} \arrow[r, "\cong"] & \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket \\ Y^{\mathrm{op}} \arrow[u, "(f^\bullet)^{\mathrm{op}}"] \arrow[r, "\cong"] & \llbracket Y, \mathcal{V}^{\mathrm{op}} \rrbracket \arrow[u, "-f"'] \end{tikzcd}$$

But the commutativity is just a consequence of the adjunction $X(x, f^\bullet y) = Y(fx, y)$. ■

7.5. REMARKS. (1) For the Lawvere quantale $\mathcal{V} = [0, \infty]_+$, the Duality Theorem takes the form of Theorem 2.4 when we take into account Examples 5.4.

(2) The dualizing natural isomorphism $\delta_X : X \to \llbracket \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket, \mathcal{V}^{\mathrm{op}} \rrbracket$ is, up to $\boxtimes$-symmetry, simply the transpose of the adjunction counit ε_X, determined by the commutative diagram

$$\begin{tikzcd} \llbracket \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket, \mathcal{V}^{\mathrm{op}} \rrbracket \boxtimes \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket \arrow[rr, "\varepsilon_{\llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket}"] & & \mathcal{V}^{\mathrm{op}} \\ X \boxtimes \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket \arrow[u, "\delta_X \boxtimes 1_{\llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket}"] \arrow[rr, "\cong"] & & \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket \boxtimes X \arrow[u, "\varepsilon_X"'] \end{tikzcd}$$

As consequences of (1-4), we mention three further natural isomorphisms, the first of which was used to *define* the (box-)tensor product *qua* its internal hom by [Eklund, Gutierrez Garcia, Hoehle, Kortelainen 2018]. The second formula says that, in principle, one could also introduce the internal hom *qua* the box tensor product, and the third one is often used in characterizing a symmetric monoidal-closed category with a fully faithful dualization functor as $*$-autonomous.

7.6. COROLLARY. *For all $X, Y, Z \in \mathcal{V}$-Sup, one has natural isomorphisms*

$$(X \boxtimes Y)^{\mathrm{op}} \cong \llbracket X, Y^{\mathrm{op}} \rrbracket \qquad \textit{and} \qquad \llbracket X, Y \rrbracket^{\mathrm{op}} \cong X \boxtimes Y^{\mathrm{op}},$$

$$\llbracket X \boxtimes Y, Z^{\mathrm{op}} \rrbracket \cong \llbracket X, (Y \boxtimes Z)^{\mathrm{op}} \rrbracket.$$

PROOF. In light of $(Y^{\mathrm{op}})^{\mathrm{op}} = Y$, the first isomorphism is a replica of the second isomorphism, which follows from

$$\begin{aligned} \llbracket X, Y \rrbracket &\cong \llbracket Y^{\mathrm{op}}, X^{\mathrm{op}} \rrbracket && (3) \\ &\cong \llbracket Y^{\mathrm{op}}, \llbracket X, \mathcal{V}^{\mathrm{op}} \rrbracket \rrbracket && (4) \\ &\cong \llbracket Y^{\mathrm{op}} \boxtimes X, \mathcal{V}^{\mathrm{op}} \rrbracket && (1) \\ &\cong (Y^{\mathrm{op}} \boxtimes X)^{\mathrm{op}} && (4) \\ &\cong (X \boxtimes Y^{\mathrm{op}})^{\mathrm{op}}. \end{aligned}$$

The third isomorphism follows from the first in conjunction with the isomorphism (1). ■

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References

A. Akhvlediani, M.M. Clementino and W. Tholen: On the categorical meaning of Hausdorff and Gromov distances, I. *Topology and its Applications* 157:1275–1295, 2010.

B. Banaschewski, E. Nelson: Tensor products and bimorphisms. *Canadian Mathematical Bulletin* 19(4):385–402, 1976.

M. Barr: Relational Algebras. In: *Lecture Notes in Mathematics* 137, pp 39–55, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

M. Barr: **-Autonomous Categories.* Lecture Notes in Mathematics 752, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

F. Borceux: *Handbook of Categorical Algebra 2: Categories and Structures.* Cambridge University Press, Cambridge, 1994.

M. Bunge: Coherent extensions and relational algebras. *Transactions of the Americal Mathematical Society* 197:355–390, 1974.

A. Burroni: T-catégories (catégories dans un triple). *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 51(2):215–321, 1971.

E. Dubuc: Adjoint triangles. In: *Lecture Notes in Mathematics* 61, pp 69–91, Springer-Verlag, Berlin-Heidelberg-NewYork, 1968.

J. Duskin: Variations on Beck's tripleability criterion. In: *Reports of the Midwest Category Seminar III*, Lecture Notes in Mathematics 102, pp 74–129 , Springer-Verlag, Berlin-Heidelberg-New York, 1969.

P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen: *Semigroups in Complete Lattices. Quantales, Modules and Related Topics.* Springer, 2018.

M. Fréchet: Sur quelqes points du calcul fonctionel. *Rendiconti del Circolo Matematico di Palermo* 22(1):1–72, 1906.

D. Hofmann: Topological theories and closed objects. *Advances in Mathematics* 215:789–824, 2018.

D. Hofmann and P. Nora: Enriched Stone-type dualities. *Advances in Mathematics* 330:307–360, 2018.

D. Hofmann, G.J. Seal and W. Tholen (editors): *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology.* Cambridge University Press, Cambridge, 2014.

A. Joyal and M. Tierney: *An extension of the Galois theory of Grothendieck*. Memoirs of the American Mathematical Society 309, Providence RI, 1984.

G.M. Kelly: *Basic Concepts of Enriched Category Theory*. Cambridge Universiy Press, Cambridge, 1982. Republished in *Reprints in Theory and Applications of Categories* 10, 2005.

F.W. Lawvere: Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano* 43:135–166, 1973. Republished in *Reprints in Theory and Applications of Categories* 1, 2002.

S. Mac Lane: *Categories for the Working Mathematician*. Springer-Verlag, Berlin-Heidelberg-New York, 1971; 2nd ed. 1994.

E. Martinelli: *Actions, Injectives and Injective Hulls in Quantale-Enriched Multicategories*. Thesis, University of Aveiro, Aveiro, 2021.

I. Stubbe: Categorical structures enriched in a quantaloid: categories distributors, functors. *Theory and Applications of Categories* 14(1):1–45, 2005.

I. Stubbe: Categorical structures enriched in a quantaloid: tensored and cotensored categories. *Theory and Applications of Categories* 16(14):283–306, 2006.

I. Stubbe: $\mathcal{Q}$-modules and $\mathcal{Q}$-suplattices. *Theory and Applications of Categories* 19(4):50–60, 2007.

W. Tholen: Another categorical look at monoids, quantales, metrics, *etc.* Talk given at the *5th Workshop on Categorical Algebra: Recent Developments and Future Perspectives*, Gargnano del Garda (Italy), 26-30 April 2022. Available at https://tholen.mathstats.yorku.ca .

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