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GEOMETRIC MORPHISMS BETWEEN TOPOSES OF MONOID ACTIONS: FACTORIZATION SYSTEMS

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Abstract. Let M, N be monoids, and $\mathbf{PSh}(M)$, $\mathbf{PSh}(N)$ their respective categories of right actions on sets. In this paper, we systematically investigate correspondences between properties of geometric morphisms $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ and properties of the semigroup homomorphisms $M \to N$ or flat-left-N-right-M-sets inducing them. More specifically, we consider properties of geometric morphisms featuring in factorization systems, namely: surjections, inclusions, localic morphisms, hyperconnected morphisms, terminal-connected morphisms, étale morphisms, pure morphisms and complete spreads. We end with an application of topos-theoretic Galois theory to the special case of toposes of the form $\mathbf{PSh}(M)$.

Contents

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1. Introduction

This article is part of an ongoing project in which we study toposes of presheaves $\mathbf{PSh}(M)$ with M a monoid, and the geometric morphisms between these toposes. A topos of this form appeared in the construction of the Arithmetic Site of Connes and Consani [CC14], in the special case where M is the monoid of nonzero natural numbers under multiplication. Variations on the Arithmetic Site, with different choices of monoid M, were considered in [Sag20], [Hem19] and [LB]. Further, if we think of a commutative monoid M as dual to an "affine $\mathbb{F}_1$-scheme", as in [Man95], then $\mathbf{PSh}(M)$ can be seen as a category of quasi-coherent modules on such an $\mathbb{F}_1$-scheme, see [Pir19].

In semigroup theory, studying the topos $\mathbf{PSh}(M)$ can give a helpful alternative point of view: in [HR21b] it was demonstrated that various known facts from semigroup theory have natural topos-theoretic interpretations. In [HR21c] a problem in semigroup theory was solved by the present authors with the help of topos-theoretic language; conversely, in [HR21a] a geometric morphism between toposes of this form provided a counterexample to an open question in topos theory.

In [HR21b], we restricted our attention to the study of the global section geometric morphism $\mathbf{PSh}(M) \to \mathbf{Set}$. This time, we will look at more general geometric morphisms $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$, with M and N monoids. In [Roga] the second named author presented a 2-categorical equivalence between a 2-category of discrete monoids and a 2-category whose objects are their (presheaf) toposes of right actions, whereby essential geometric morphisms between the toposes correspond to semigroup homomorphisms between the monoids; the global sections morphism of $\mathbf{PSh}(M)$ corresponds to the unique semigroup homomorphism $M \to 1$, for example. As explained in [HR21b], general geometric morphisms $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ correspond to sets equipped with a flat left N-action and a compatible right M-action. In this paper, we refer to these as $[N, M)$-sets; see Definition 2.2.2. A natural next step in studying toposes of discrete monoid actions is an investigation of how properties of geometric morphisms descend to properties of the corresponding semigroup homomorphisms or $[N, M)$-sets. Since properties of geometric morphisms are far too varied to examine exhaustively in a single article, we focus here on *factorization systems*.

The first factorization systems that we will consider are the (surjection, inclusion) factorization and the (hyperconnected, localic) factorization. These are the two most well-known factorization systems for geometric morphisms. For essential geometric morphisms between presheaf toposes, an explicit construction for these two factorizations is given in [Joh02, §A4.2 and §A4.6]. If we apply this to the special case of an essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup homomorphism $\phi : M \to N$, then we get a factorization

$$M \xrightarrow{\pi} M/{\sim} \xrightarrow{\psi} eNe \xrightarrow{\iota} N$$

$$\mathbf{PSh}(M) \xrightarrow[\text{hyperconnected}]{} \mathbf{PSh}(M/{\sim}) \xrightarrow[\text{localic surj.}]{} \mathbf{PSh}(eNe) \xrightarrow[\text{inclusion}]{} \mathbf{PSh}(N)$$

where the hyperconnected part is induced by the projection of M onto its image $M/{\sim} = \phi(M)$, the localic surjection part is induced by the inclusion of $M/{\sim}$ in eNe (with $e = \phi(1)$), and the inclusion part is induced by the semigroup inclusion $eNe \subseteq N$. The localic part is the composition of the localic surjection part and the inclusion part, while the surjection part is the composition of the hyperconnected part and the localic surjection part. For a general geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ given by a $[N, M)$-set A, we can also consider the (surjection, inclusion) factorization $\mathbf{PSh}(M) \to \mathcal{E} \to \mathbf{PSh}(N)$, but in this case the intermediate topos is not necessarily a topos of monoid actions, or even a presheaf topos. However, we can still give concrete characterizations of when f is surjective, localic or hyperconnected, in terms of the $[N, M)$-set A.

Another factorization system that we will discuss is the (terminal-connected, étale) factorization, which exists for all essential geometric morphisms, see [Car, §4.7]. For an essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$, induced by a semigroup homomorphism $\phi : M \to N$, it follows from the definition that the intermediate topos is again a presheaf topos. We describe the factorization as explicitly as possible, which leads to a characterization of when f is terminal-connected (resp. étale) in terms of the semigroup homomorphism ϕ. For a more general geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$, induced by a $[N, M)$-set A, we again give a characterization of terminal-connectedness (in the sense of Osmond [Osm21, Definition 5.3.3]). Because étale geometric morphisms are always essential, they do not have to be considered separately here. However, note that the (terminal-connected, étale) factorization does not always exist for general geometric morphisms.

A last factorization that we will consider is the (pure, complete spread) factorization, as studied extensively by Bunge and Funk, see [BF96], [BF98] and [BF06]. This factorization exists whenever the domain topos is locally connected, and it is conceptually dual to the (terminal-connected, étale) factorization mentioned above. For an essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup homomorphism $\phi : M \to N$, the factorization is dual in a literal sense: the geometric morphism induced by ϕ is pure (resp. a complete spread) if and only if the geometric morphism induced by $\phi^{\mathrm{op}} : M^{\mathrm{op}} \to N^{\mathrm{op}}$ is terminal-connected (resp. étale). For a general geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ given by a $[N, M)$-set A, we give a characterization of when f is pure. In our setting, f can only be a complete spread if it is essential, so only a study of general pure geometric morphisms is needed here.

It follows from the work of Bunge and Funk [BF98, Corollary 7.9] that the intersection of étale geometric morphisms and complete spreads is (over connected presheaf toposes, including our case of interest) given by the *locally constant étale* morphisms. These are employed in a topos-theoretic version of Galois theory. As an application of our investigation, we recover the result that the Galois groupoid for a topos of the form $\mathbf{PSh}(M)$ is a group, and is exactly the *groupification* of M.

Overview. In Section 2, we recall how semigroup homomorphisms and biactions of monoids induce geometric morphisms, as well as some basic categorical constructions which we shall need later. We tackle the (surjection, inclusion) and (hyperconnected, localic) factorization systems in Section 3, the (terminal-connected, étale) factorization system in Section 4 and finally the (pure, complete spread) factorization in Section 5. Each of these sections begin with some background on the types of morphism involved, followed by an investigation of the factorization system for essential geometric morphisms coming from semigroup homomorphisms. The latter part of each section contains an attempt to characterize the biactions producing geometric morphisms in the various classes.

In Section 6, we investigate the relationship between the latter two factorization systems, in particular giving examples illustrating the various possible relationships between étale morphisms and complete spreads. We apply this in Section 7 to streamline the application in the Galois theory of our toposes of discrete monoid actions.

Throughout, the reader may assume that M and N denote monoids.

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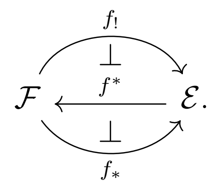
2. Background

2.1. Essential Geometric Morphisms. Let $\mathcal{E}$ and $\mathcal{F}$ be Grothendieck toposes. Recall that a *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ is by definition an adjunction

$$\mathcal{F} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{E}, \quad (f^* \dashv f_*)$$

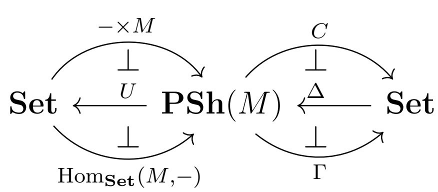
with f_*, the *direct image functor*, right adjoint to f^*, the *inverse image functor*, where the latter is required to preserve finite limits. We follow the convention that a 2-morphism or *geometric transformation* $f \Rightarrow g$ between geometric morphisms $f, g : \mathcal{F} \to \mathcal{E}$ is a natural transformation $f^* \Rightarrow g^*$.

A geometric morphism is said to be **essential** if f^* has a left adjoint, denoted $f_!$:



By the Special Adjoint Functor Theorem, a geometric morphism $(f^* \dashv f_*)$ is essential precisely if f^* preserves not just finite limits but all small limits. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories $\mathcal{C}$ and $\mathcal{D}$ induces an essential geometric morphism $f : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ whose inverse image functor is precomposition with F^{op}, so that f_* and $f_!$ are given by right and left Kan extensions along F^{op}, respectively. Conversely, any essential geometric morphism between presheaf toposes is (up to natural isomorphism) induced by some functor F in this way, which is recovered by restricting $f_!$ to the representable presheaves.

From Theorem 6.5 of [Roga], we have an equivalence between the 2-category of monoids, semigroup homomorphisms and 'conjugations' (monoid elements which commute appropriately with homomorphisms), and the 2-category of the corresponding presheaf toposes, essential geometric morphisms and natural transformations, up to reversing the direction of the conjugations. These presheaf toposes have a great deal more structure than the monoids from which they are constructed, and as such this equivalence gives us access to a variety of approaches for examining the subtler properties of monoids and their right actions. We recall from [HR21b] that, given a monoid M, its topos $\mathbf{PSh}(M)$ of actions is equipped with a canonical point and a global sections morphism:



(1)

where the functors not explicitly specified are:

- the forgetful functor U sending a right M-set to its underlying set;
- the global sections functor Γ sending an M-set A to its set

$$\mathrm{Fix}_M(A) = \mathrm{Hom}_{\mathbf{PSh}(M)}(1, A)$$

of fixed points under the action of M;

- the constant sheaf functor Δ sending a set B to the same set with trivial M-action;
- the connected components functor C sending an M-set A to its set of components under the action of M (that is, to its quotient under the equivalence relation generated by $a \sim a \cdot m$ for $a \in A$, $m \in M$).

It should also be noted that for a set X, the M-action on $\mathrm{Hom}_{\mathbf{Set}}(M,X)$ by $m \in M$ sends f to $(n \mapsto f(mn))$, while the M action on $X \times M$ is by right multiplication on the M-component.

These geometric morphisms correspond under the equivalence to the canonical monoid homomorphisms $1 \to M$ and $M \to 1$. More generally, an arbitrary monoid homomorphism ϕ gets sent to the essential geometric morphism whose inverse image is restriction of the action along ϕ. The geometric morphism corresponding to an arbitrary semigroup homomorphism is a little more complicated, and can be most concisely described in terms of a $[N, M)$ set and a (M, N)-set; see Lemma 2.2.4 below.

2.2. General Geometric Morphisms. In previous work [HR21b, Propositions 1.5 and 1.8], we discussed how more generally a geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ can be understood as a tensor–hom adjunction. We recall those results here.

2.2.1. Definition. *If X is a set equipped with a left N-action and a right M-action, then we say that the left N-action and right M-action are* ***compatible*** *if $(n \cdot x) \cdot m = n \cdot (x \cdot m)$ for all $n \in N$, $x \in X$ and $m \in M$. Sets with a compatible left N-action and right M-action will be called* (N, M)***-sets***.[1] *As homomorphisms between these, we of course consider functions commuting with both actions.*

For a right N-set X and a left N-set A, recall that we define the **tensor product** $X \otimes_N A$ to be the quotient of $X \times A$ by the equivalence relation $\sim$, generated by $(x \cdot n, a) \sim (x, n \cdot a)$ for $x \in X$, $a \in A$ and $n \in N$. A left N-set A is then said to be **flat** if the functor

$$- \otimes_N A : [N^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$$
$$X \mapsto X \otimes_N A$$

preserves finite limits, which is equivalent (see e.g. [MLM94, VII.6, Theorem 3]) to the conditions that

1. A is non-empty;
2. for elements $b, b' \in A$ there exists $a \in A$ and $n, n' \in N$ with $n \cdot a = b$ and $n' \cdot a = b'$; and
3. whenever $c \in A$ and $n, n' \in N$ with $n \cdot c = n' \cdot c$, there exists $d \in A$, $p \in N$ with $p \cdot d = c$ and $np = n'p$.

[1] We read this as 'left-N-right-M-set'.

2.2.2. Definition. *We say a (N, M)-set A is **flat**, or a $[N, M)$**-set**, if it is flat as a left N-set. The category of $[N, M)$-sets forms a full subcategory of the category of (N, M)-sets.*

As shown below, the category of geometric morphisms $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is equivalent to the category of $[N, M)$-sets. More generally, a (Lawvere) **distribution** $f : \mathcal{F} \to \mathcal{E}$ between toposes is any adjoint pair $f^* \dashv f_*$, where f^* does not necessarily preserve finite limits (see [Law] or [BF96]). A morphism $f \Rightarrow g$ between distributions $f, g : \mathcal{F} \to \mathcal{E}$ is still a natural transformation $f^* \Rightarrow g^*$. We mention the following special case of Diaconescu's Theorem:

2.2.3. Theorem. *There is an equivalence between the category of geometric morphisms $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ and the category of $[N, M)$-sets. More generally, there is an equivalence between the category of distributions $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ and the category of (N, M)-sets.*

Proof. At the level of objects, the equivalences send an adjunction $f^* \dashv f_*$ to the (N, M)-set $f^*(N)$, which has a right M-action by virtue of being an object of $\mathbf{PSh}(M)$, and a left N action coming from the images of the endomorphisms of N as an object of $\mathbf{PSh}(N)$, which consist of left multiplication by elements of N. Conversely, a (N, M)-set A is sent to the tensor-hom adjunction $(- \otimes_N A) \dashv \mathrm{Hom}_M(A, -)$; see [HR21b, Proposition 1.5].

Given two adjunctions $f^* \dashv f_*$ and $g^* \dashv g_*$, a natural transformation $f^* \Rightarrow g^*$ is determined by its component $f^*(N) \to g^*(N)$, which is automatically a right-M-set homomorphism; it is also a left-N-set homomorphism by naturality with respect to the endomorphisms of N. Conversely, a (N, M)-set homomorphism $A \to B$ induces a natural transformation $- \otimes_N A \to - \otimes_N B$ by composition on the second component; commutation with the respective actions ensures that this is well-defined and an M-set homomorphism at each object X.

Finally, the geometric morphisms f are precisely the distributions such that f^* preserves finite limits, so correspond under this equivalence to the full subcategory of $[N, M)$-sets, as required. ∎

Thus we have an algebraic characterization of arbitrary geometric morphisms between toposes of discrete monoid actions, as well as an alternative perspective on the extra adjunction $(f_! \dashv f^*)$ in an essential geometric morphism f. Explicitly, by direct calculation:

2.2.4. Lemma. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be an essential geometric morphism induced by a semigroup homomorphism $\phi : M \to N$. Then the $[N, M)$-set corresponding to $(f^* \dashv f_*)$ is the left ideal $N\phi(1)$ equipped with left N-action by multiplication and right M-action by multiplication after applying ϕ. In particular, when ϕ is a monoid homomorphism, the $[N, M)$-set is simply N equipped with the respective actions.*

Meanwhile, the (M, N)-set corresponding to the extra adjunction $(f_! \dashv f^*)$ is the right ideal $\phi(1)N$ of N, similarly equipped with respective multiplication actions but with the handedness reversed.

The correspondence from Theorem 2.2.3 is well-behaved with respect to composition of geometric morphisms.

2.2.5. Lemma. *Suppose* $g : \mathbf{PSh}(M) \to \mathbf{PSh}(L)$ *and* $f : \mathbf{PSh}(L) \to \mathbf{PSh}(N)$ *are induced by the* $[L, M)$*-set* B *and the* $[N, L)$*-set* A *respectively. Then* $f \circ g$ *is induced by* $A \otimes_L B$ *(up to isomorphism). This result extends to all distributions.*

Proof. This is immediate from the fact that $g^* f^*(X) \simeq (X \otimes_N A) \otimes_L B$ and the tensor product is associative up to isomorphism. ■

2.3. Categories of Elements and Slice Toposes. Let $\mathcal{C}$ be a small category and let $X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$. Recall that the **category of elements** of X is the category $\int_{\mathcal{C}} X$ having

- as objects, pairs (C, a), where C is an object of $\mathcal{C}$ and $a \in X(C)$;
- as morphisms $(C, a) \to (D, b)$ the morphisms $f : C \to D$ such that $b \cdot f = a$;
- composition given by composition in $\mathcal{C}$.

We shall also need the dual construction: viewing a functor $Y : \mathcal{C} \to \mathbf{Set}$ as a contravariant functor defined on $\mathcal{C}^{\mathrm{op}}$, we define $\int^{\mathcal{C}} Y := \left(\int_{\mathcal{C}^{\mathrm{op}}} Y\right)^{\mathrm{op}}$.

When $\mathcal{C} = M$ is a monoid, a presheaf on M is precisely a right M-set. Since there is only one object, we simplify the description of categories of elements by dropping the indexing over the objects. If M is commutative, then $\int_M M$ agrees with the category $C(M)$ appearing in [CC21, §4.1].

Categories of elements are useful for studying slice toposes. Recall that for any category $\mathcal{E}$ and object X in $\mathcal{E}$, the **slice category** $\mathcal{E}/X$ is the category with,

- as objects the morphisms $f : E \to X$ in $\mathcal{E}$ with codomain X;
- as morphisms from $f : E \to X$ to $f' : E' \to X$ the morphisms $g : E \to E'$ such that $f' \circ g = f$;
- composition given by composition of morphisms in $\mathcal{E}$.

For a topos $\mathcal{E}$ and an object X in $\mathcal{E}$, the slice category $\mathcal{E}/X$ is again a topos, inheriting all of the required properties from $\mathcal{E}$ (this fact is sometimes called the *fundamental theorem of topos theory*). We shall refer to a topos of the form $\mathcal{E}/X$ for a generic object X as a **slice of** $\mathcal{E}$. The relation between categories of elements above and slice toposes can be described as follows:

2.3.1. Proposition. *Consider a small category* $\mathcal{C}$ *and a presheaf* X *on* $\mathcal{C}$*. Then there is an equivalence of categories*

$$\mathbf{PSh}(\mathcal{C})/X \;\simeq\; \mathbf{PSh}\left(\int_{\mathcal{C}} X\right).$$

Proof. In one direction, an object $g : Y \to X$ on the left hand side is sent to the presheaf $\hat{g}$ on $\int_{\mathcal{C}} X$ sending (C, x) to $g(C)^{-1}(\{x\}) \subseteq Y(C)$. In the opposite direction, a presheaf G on $\int_{\mathcal{C}} X$ is sent to the object $\tilde{G} : Y \to X$, where $Y(C) = \coprod_{x \in X(C)} G(C, x)$ and $\tilde{G}$ sends the elements in each $G(C, x)$ to x. We leave the remaining details to the reader; this features as Exercise III.8 in [MLM94]. ■

2.4. Idempotent Completion.

2.4.1. Definition. *Recall that a category $\mathcal{C}$ is **idempotent complete** (also known as Cauchy complete or Karoubi complete) if every idempotent splits, in the sense that given any idempotent $e : C \to C$ in $\mathcal{C}$, there exist morphisms $r : C \to D$ and $s : D \to C$ with $r \circ s = \mathrm{id}_D$ and $s \circ r = e$, which are automatically unique up to unique isomorphism with D.*

We can construct the idempotent completion of any (small) category. The category of presheaves on the resulting category is equivalent to the category of presheaves on the original category; conversely, an idempotent complete category can be recovered up to equivalence from its category of presheaves as the subcategory of indecomposable projective objects [sga72, Exercice 7.6(c,e)]. Hence there is a unique idempotent complete category up to equivalence representing any presheaf category.

We recall the following from [Roga, Section 2]:

2.4.2. Lemma. *The idempotent completion $\check{M}$ of a monoid M is given by the category with,*

- *as objects the idempotents of M (the object corresponding to an idempotent $e \in M$ is denoted by $\underline{e}$);*
- *as morphisms $\underline{e} \to \underline{d}$ the elements $m \in M$ such that $me = m = dm$;*
- *identity morphism on an object $\underline{e}$ given by the corresponding element e;*
- *composition given by multiplication in M.*

As the name suggests, this category is idempotent complete, and is the unique idempotent complete category up to equivalence such that $\mathbf{PSh}(M) \simeq \mathbf{PSh}(\check{M})$.

2.4.3. Remark. *Any semigroup homomorphism $\phi : M \to N$ induces a functor $\check{\phi} : \check{M} \to \check{N}$ mapping $\underline{e}$ to $\underline{\phi(e)}$ and $m : \underline{e} \to \underline{d}$ to $\phi(m) : \underline{\phi(e)} \to \underline{\phi(d)}$, and this in turn induces the essential geometric morphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ corresponding to ϕ under the 2-equivalence mentioned at the start of Section 2.1.*

Extending Proposition 2.3.1, we have the following result.

2.4.4. Corollary. *Suppose that $\mathcal{C}$ is a small idempotent-complete category and X is a presheaf on it. Then $\int_{\mathcal{C}} X$ is idempotent complete. It follows that this is (up to equivalence) the unique idempotent complete category with*

$$\mathbf{PSh}(\mathcal{C})/X \;\simeq\; \mathbf{PSh}\left(\int_{\mathcal{C}} X\right).$$

Proof. Let (C, a) be an object in $\int_{\mathcal{C}} X$ and suppose that there is an idempotent morphism $e : (C, a) \to (C, a)$ indexed by a morphism $e \in \mathcal{C}$. Then e must itself be an idempotent in $\mathcal{C}$. Consider the splitting $e = sr$ of e in $\mathcal{C}$; let D be the domain of s. Since $a \cdot e = a$, we have $r : (C, a) \to (D, a \cdot r)$ and $s : (D, a \cdot r) \to (C, a)$ in $\int_{\mathcal{C}} X$. This defines the desired splitting of the original idempotent. ∎

We can also use the idempotent completion to characterize toposes of discrete monoid actions amongst presheaf toposes.

2.4.5. Lemma. *Let $\mathcal{C}$ be a small category. Then there is a monoid M such that $\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(M)$ if and only if there is an object C in $\mathcal{C}$ such that every other object in $\mathcal{C}$ is a retract of C. In this case, we can take $M = \mathrm{End}_{\mathcal{C}}(C)$.*

Proof. If C is an object such that every other object is a retract of C, then consider the full subcategory of $\mathcal{C}$ on the single object C, which we can identify with the monoid $\mathrm{End}_{\mathcal{C}}(C)$. The idempotent completions of $\mathrm{End}_{\mathcal{C}}(C)$ and $\mathcal{C}$ agree, so $\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(\mathrm{End}_{\mathcal{C}}(C))$.

Conversely, suppose that $\mathbf{PSh}(\mathcal{C}) \simeq \mathbf{PSh}(M)$ for some monoid M. Then there is an object C' in the idempotent completion $\check{\mathcal{C}}$ of $\mathcal{C}$ such that every other object is a retract of C'. Because C' lies in the idempotent completion, it is itself a retract of an object C in $\mathcal{C}$. The statement of the lemma then follows from transitivity of retracts. ∎

3. The (surjection, inclusion) and (hyperconnected, localic) factorizations

There are a number of standard factorization systems for geometric morphisms, some applicable to all morphisms, others only to particular classes. While we describe a variety of them here, we focus on the cases which do not take us outside the realm of presheaf toposes, and especially on refinements of these which keep us in the realm of toposes of discrete monoid actions. We begin with factorization systems for essential geometric morphisms, and use these to factorize more general geometric morphisms later.

3.0.1. Definition. *Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is:*

- *a **surjection** if f^* is faithful;*
- *an **inclusion** if f_* is full and faithful;*
- ***hyperconnected** if f^* is full and faithful and has image closed under subquotients (quotients of subobjects);*
- ***localic** if every object in $\mathcal{F}$ is a subquotient of an object of the form $f^*(X)$ for some X in $\mathcal{E}$.*

There are equivalent characterizations of these classes of geometric morphism which we shall employ at various points.

The two factorization systems for geometric morphisms that are most well-known are the (surjection, inclusion) factorization, and the (hyperconnected, localic) factorization. That is, every geometric morphism canonically factors as a surjection followed by an inclusion, or as a hyperconnected morphism followed by a localic morphism, uniquely up to compatible equivalence of the intermediate topos. Moreover, these factorizations are compatible in the sense illustrated in (2) below.

3.1. The essential case. Conveniently, both of these factorizations restrict in a canonical way to the class of essential geometric morphisms between presheaf toposes.

3.1.1. Lemma. *Suppose $f : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ is an essential geometric morphism induced by a functor $F : \mathcal{C} \to \mathcal{D}$. Then,*

- *f is surjective $\Leftrightarrow$ F is essentially surjective up to retracts;*
- *f is an inclusion $\Leftrightarrow$ F is full and faithful;*
- *f is hyperconnected $\Leftrightarrow$ F is full and essentially surjective up to retracts;*
- *f is localic $\Leftrightarrow$ F is faithful.*

Here "essentially surjective up to retracts" means that for every D in $\mathcal{D}$ there is some C in $\mathcal{C}$ such that D is a retract of $F(C)$.

In particular, since we can factor any functor F between idempotent-complete small categories as a functor which is essentially surjective up to retracts followed by one which is full and faithful, we obtain a canonical representation of the (surjection, inclusion) factorization of f, and the intermediate topos is a presheaf topos. The analogue is true for the (hyperconnected, localic) factorization of f.

Proof. See Johnstone, [Joh02, Examples A4.2.7(b), A4.2.12(b), A4.6.2(c) and A4.6.9]; the case of monoid homomorphisms is even explicitly discussed after Example A4.6.9 there. To sketch a short proof, one can verify directly that the given conditions are sufficient; conversely, since the stated factorizations of F must give factorizations of f which we know to be unique up to equivalence of the intermediate topos (or equivalently up to equivalence of the intermediate idempotent-complete category) and since the given conditions are invariant under equivalence, they must also be necessary. ∎

Let $\phi : M \to N$ be a semigroup homomorphism between monoids M and N (so the identity need not be preserved). Applying Lemma 3.1.1 to the functor $\check{\phi} : \check{M} \to \check{N}$ from Remark 2.4.3 and the corresponding essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$, we deduce the following corollaries for essential geometric morphisms between toposes of discrete monoid actions.

3.1.2. Corollary. *The **(surjection, inclusion) factorization** of f is canonically represented by the factorization of $\phi : M \to N$ as a monoid homomorphism followed by an inclusion of semigroups of the form $\iota : eNe \hookrightarrow N$, where $e = \phi(1)$ is the idempotent of N which is the image of the identity element of M.*

$$M \xrightarrow{\ \psi\ } eNe \xhookrightarrow{\ \iota\ } N$$

$$\mathbf{PSh}(M) \xrightarrow[\text{surjection}]{} \mathbf{PSh}(eNe) \xrightarrow[\text{inclusion}]{} \mathbf{PSh}(N)$$

In particular, essential geometric morphisms induced by monoid homomorphisms are always surjective. Conversely, given an essential surjection, the inclusion part of its (surjection, inclusion) factorization must be an equivalence. That is, the inclusion $\iota : eNe \to N$

of the image of the corresponding semigroup homomorphism induces an equivalence of toposes. We may therefore assume, up to replacing the monoid presenting the codomain topos with a Morita-equivalent one, that an essential surjection is induced by a monoid homomorphism, rather than a mere semigroup homomorphism.

3.1.3. Corollary. *The* ***(hyperconnected, localic) factorization*** *of f corresponds to the (quotient, injection) factorization of ϕ, which factors $\phi : M \to N$ through the quotient monoid homomorphism $\pi : M \to M/{\sim}$, where $m \sim n$ if and only if $\phi(m) = \phi(n)$. Diagrammatically:*

$$M \xrightarrow{\ \pi\ } M/{\sim} \xhookrightarrow{\ \psi\ } N$$

$$\mathbf{PSh}(M) \xrightarrow[\text{hyperconnected}]{} \mathbf{PSh}(M/{\sim}) \xrightarrow[\text{localic}]{} \mathbf{PSh}(N)$$

These two factorization systems are compatible: we can factorize any semigroup homomorphism ϕ and the corresponding essential geometric morphism f into three parts. We factorize ϕ into a quotient map $\pi : M \to M/{\sim}$, followed by an injective monoid homomorphism $\psi : M/{\sim} \to eNe$, followed by an inclusion $\iota : eNe \to N$, where $e = \phi(1)$. The induced geometric morphisms give a factorization as follows:

$$\begin{array}{c} M \xrightarrow{\ \pi\ } M/{\sim} \xhookrightarrow{\ \psi\ } eNe \xhookrightarrow{\ \iota\ } N \\ \mathbf{PSh}(M) \xrightarrow[\text{hyperconnected}]{} \mathbf{PSh}(M/{\sim}) \xrightarrow[\text{localic surj.}]{} \mathbf{PSh}(eNe) \xrightarrow[\text{inclusion}]{} \mathbf{PSh}(N) \end{array} \tag{2}$$

In this situation, the surjective part is the composition of the hyperconnected and localic surjection parts, and the localic part is the composition of the localic surjection and inclusion parts. It will often be helpful to consider the three parts of this (hyperconnected, localic surjection, inclusion) factorization separately.

3.2. The general case. Unfortunately, the latter part of Lemma 3.1.1 is not true for a geometric morphism that is not essential: the intermediate topos in the (surjection, inclusion) or (hyperconnected, localic) factorization of a typical geometric morphism $g : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is not a presheaf topos, let alone a topos of discrete monoid actions[2]. Nonetheless, we can identify conditions on $[N, M]$-sets which produce morphisms in these classes.

3.2.1. Proposition. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be the geometric morphism corresponding to the $[N, M]$-set A. Then f is localic if and only if M is a retract of some subobject of A, as a right M-set.*

[2]On the other hand, it was recently shown by the second author that the (hyperconnected, localic) factorization does have a presentation in terms of actions of *topological* monoids, [Rogb].

PROOF. By definition, f being localic requires that every object X of $\mathbf{PSh}(M)$ be a subquotient of one of the form $f^*(Y)$. By pulling back along (the image under f^* of) the cover of Y by a disjoint union of copies of N, we conclude that f is localic if and only if for every object X in $\mathbf{PSh}(M)$ there is a subobject $C \subseteq \bigsqcup_{i \in I} A$ and an epimorphism $C \to X$. In the special case where $X = M$, we get a surjection $p : C \twoheadrightarrow M$, which splits since M is projective. Letting A' be the connected component of C containing the section, we conclude that A' must be a subobject of just one of the copies of A, and hence by restricting p to A', we conclude that M is a subquotient of A, as required. Conversely, if A has a subobject A' of which M is a retract, then each object X in $\mathbf{PSh}(M)$ admits a surjection $\bigsqcup_{i \in I} A' \twoheadrightarrow \bigsqcup_{i \in I} M \twoheadrightarrow X$. Since $\bigsqcup_{i \in I} A'$ is a subobject of $\bigsqcup_{i \in I} A$, this shows that f is localic. ∎

We shall extend this proposition to a necessary and sufficient condition for the direct image f_* to be faithful in Scholium 5.2.5, but fullness of f_* is challenging in general. We shall at least see a sufficient condition for f to be an inclusion in Corollary 5.2.6.

We can characterize surjections in terms of an algebraic condition, albeit a not very enlightening one.

3.2.2. LEMMA. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be the geometric morphism corresponding to the $[N, M]$-set A. Then f is a surjection if and only if for all N-sets X and elements $x, y \in X$, if we have $x \otimes a = y \otimes a$ in $X \otimes_N A$ for all $a \in A$, then $x = y$.*

PROOF. Composing with the canonical essential surjective point of $\mathbf{PSh}(M)$, we see that f is a surjection if and only if the composite point $\mathbf{Set} \to \mathbf{PSh}(N)$ is. The stated condition is a translation of the requirement that the unit of this point is a monomorphism. The statement then follows from the classical result that the unit of an adjunction is a monomorphism if and only if the left adjoint is faithful. ∎

Finding necessary and sufficient conditions for f^* to be full is difficult, but fortunately we have other ways to characterize hyperconnected morphisms.

3.2.3. PROPOSITION. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be the geometric morphism corresponding to the $[N, M]$-set A. Then f is hyperconnected if and only if the condition of Lemma 3.2.2 is satisfied and every sub-M-set of A is of the form $I \otimes_N A$ for some right ideal $I \subseteq N$.*

PROOF. If f is hyperconnected, then f is certainly a surjection. Moreover, since f^* is full and faithful and closed under subobjects, every monomorphism $A' \hookrightarrow A$ must be of the form $f^*(g)$ for some right N-set homomorphism $g : X \to N$. But f^* preserves epimorphisms and monomorphisms, which means that if we take the epi-mono factorization of g, the epimorphic part must be sent to an isomorphism by f^*, so the monomorphic part induces the same subobject. The conclusion follows, since sub-right-N-sets of N are precisely right ideals.

Conversely, given the conditions on A, we know from Lemma 3.2.2 that f is a surjection; we shall show that f^* is closed under subobjects. Indeed, given a right N-set X, consider the image under f^* of a cover of X by copies of N, which simplifies to

$\coprod_{k \in K} A \twoheadrightarrow f^*(X)$. Given a subobject Z of $f^*(X)$ in $\mathbf{PSh}(M)$, we can pull back the cover to obtain a cover of Z of the form $\coprod_{k \in K} I_k \otimes_N A \twoheadrightarrow Z$ (taking advantage of the fact that subobjects of coproducts are coproducts of subobjects). This lifts to a morphism $\coprod_{k \in K} I_k \hookrightarrow \coprod_{k \in K} N \twoheadrightarrow X$; applying f^* to the epi-mono factorization of this composite produces the desired presentation of Z. Being faithful and closed under subobjects means that, for each object A of $\mathbf{PSh}(N)$, f^* induces an equivalence of subobject lattices $\mathrm{Sub}(A) \cong \mathrm{Sub}(f^*(A))$, which is one of the equivalent characterizations of hyperconnected morphisms (see [Joh02, Proposition A4.6.6(vi)]). ∎

4. The (terminal-connected, étale) factorization

Recall that a geometric morphism is **locally connected** if its inverse image functor is locally cartesian closed (preserves dependent products); any locally connected morphism is essential. There is a well-known (connected and locally connected, étale) factorization system for locally connected morphisms, constructed for a given morphism f by slicing the codomain topos over the object $f_!(1)$; see [Joh02, Lemma C3.3.5]. This factorization system extends with an identical construction to essential geometric morphisms, as observed by Caramello in [Car, §4.7]. Recent work of Osmond [Osm21, Theorem 5.4.10] demonstrates how this can be extended to a factorization system for arbitrary geometric morphisms, after replacing étale geometric morphisms by more general pro-étale geometric morphisms.

We begin from the following definitions, which appear as [Osm21, Definitions 5.2.3 and 5.3.3], respectively.

4.0.1. Definition. *A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be* ***étale*** *if $\mathcal{F}$ is equivalent to $\mathcal{E}/X$ for some object X, and f factors as the equivalence followed by the canonical geometric morphism $\mathcal{E}/X \to \mathcal{E}$; we refer to the latter as the étale geometric morphism corresponding to X.*

On the other hand, a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be ***terminal-connected*** *if there is a bijection $\mathrm{Hom}_{\mathcal{F}}(1, f^*(X)) \cong \mathrm{Hom}_{\mathcal{E}}(1, X)$, natural in X.*

A geometric morphism into **Set** is terminal-connected if and only if it is **connected**, meaning that its inverse image functor is full and faithful. Indeed, it is clear that connected morphisms are always terminal-connected morphisms (since inverse image functors preserve the terminal object); conversely, for $p : \mathcal{E} \to \mathbf{Set}$ the global sections morphism, we have

$$\mathrm{Hom}_{\mathcal{E}}(p^*(X), p^*(Y)) \cong \prod_{x \in X} \mathrm{Hom}_{\mathcal{E}}(1, p^*(Y)) \cong \prod_{x \in X} \mathrm{Hom}_{\mathbf{Set}}(1, Y) \cong \mathrm{Hom}_{\mathbf{Set}}(X, Y),$$

naturally in X and Y, whence p^* is full and faithful. Terminal-connectedness was previously defined only for the class of essential geometric morphisms. Indeed, for such morphisms we can recover the definition stated in [Car, §4.7] by adjointness, as Osmond observes in [Osm21].

4.0.2. Lemma. [Osm21, Proposition 5.3.4] *An essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is terminal-connected if and only if $f_!$ preserves the terminal object.*

Meanwhile, the following result from [Joh02] suggests that we should think of étale geometric morphisms over a topos $\mathcal{E}$ as corresponding to discrete internal locales (rather than merely as objects of the topos).

4.0.3. Lemma. [Joh02, Lemma C3.5.4] *A geometric morphism is étale if and only if it is localic and* ***atomic****, meaning that its inverse image functor is logical (preserves exponential objects and the subobject classifier); note that Johnstone calls étale geometric morphisms local homeomorphisms.*

4.1. The essential case. Given an essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$, there is a factorization

$$\mathcal{F} \xrightarrow{g} \mathcal{E}/f_!(1) \xrightarrow{h} \mathcal{E},$$

where both factors are essential, g is terminal-connected and h is the local homeomorphism corresponding to $f_!(1)$. This **(terminal-connected, étale) factorization**, is again unique up to compatible equivalence of the intermediate topos, see [Car, Proposition 4.62], so we may refer to g as the *terminal-connected part of f* and to h as the *étale part of f*.

4.1.1. Proposition. *Let $f : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ be an essential geometric morphism induced by a functor $F : \mathcal{C} \to \mathcal{D}$. Then F has a factorization $\mathcal{C} \to \mathcal{B} \to \mathcal{D}$ into a final functor followed by a discrete fibration (unique up to equivalence). Further, the induced factorization*

$$\mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{B}) \to \mathbf{PSh}(\mathcal{D})$$

coincides with the (terminal-connected, étale) factorization of f.

Proof. In [SW73] it is shown that each functor can be factorized as an initial functor followed by a discrete opfibration. By dualizing we get a factorization of a functor into a final functor followed by a discrete fibration. To show that the induced factorization coincides with the (terminal-connected, étale) factorization, we write out the factorization explicitly below.

Using the notations from Subsection 2.3, we may consider the factorization:

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \int_{\mathcal{D}} f_!(1) & \longrightarrow & \mathcal{D}, \\ & & & & \\ C & \longmapsto & (F(C), x \cdot Ft) & \longmapsto & F(C) \\ \downarrow t & & \downarrow Ft & & \downarrow Ft \\ C' & \longmapsto & (F(C'), x) & \longmapsto & F(C') \end{array} \tag{3}$$

where $\int_{\mathcal{D}} f_!(1)$ is the category of elements of the object $f_!(1)$ in $\mathbf{PSh}(\mathcal{D})$, the right-hand functor is the forgetful functor and $x \in f_!(1)(F(C'))$ corresponds to the morphism

$\mathbf{y}(F(C')) \cong f_!(C') \xrightarrow{f_!(!)} f_!(1)$. We omit the proof of the respective properties and uniqueness, although we refer the reader to the original reference for the dual factorization of a functor into an initial functor followed by a discrete opfibration in [SW73]; we shall use this dual factorization in the next section.

To see that this produces a (terminal-connected, étale) factorization of f, we employ Proposition 2.3.1 to observe that the right-hand factor is étale, since the extra left adjoint of the geometric morphism induced by the projection functor is identified under the equivalence with the forgetful functor from the slice topos. Meanwhile, the geometric morphism induced by the left-hand factor has left adjoint sending a presheaf X on $\mathcal{C}$ to the object $f_!(X \to 1)$ of $\mathbf{PSh}(\mathcal{D})/f_!(1)$, whence it is terminal-connected by Lemma 4.0.2. ∎

Now suppose $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is an essential geometric morphism induced by a semigroup homomorphism $\phi : M \to N$, and let $e := \phi(1)$. Equivalently, f is induced by the functor $\phi' : M \to \check{N}$ sending the unique object of M to the object $\underline{e}$ of $\check{N}$ and sending $m \in M$ to the endomorphism of $\underline{e}$ indexed by $\phi(m)$ (we use the notation of Lemma 2.4.2). We now apply Proposition 4.1.1 to ϕ'. We see that $\mathbf{PSh}(N)/f_!(1)$ is the category of presheaves on the category of elements $\int_{\check{N}} f_!(1)$, and moreover the terminal-connected part of f is the geometric morphism induced by the functor $M \to \int_{\check{N}} f_!(1)$ such that:

- the unique object of M is sent to the object $(\underline{e}, * \otimes e)$ of $\int_{\check{N}} f_!(1)$, where $* \otimes e \in 1 \otimes_M eN \cong f_!(1)$, and
- the morphism corresponding to $m \in M$ is sent to the endomorphism of $(\underline{e}, * \otimes e)$ indexed by $\phi(m)$.

Denote the monoid of endomorphisms of $(\underline{e}, * \otimes e)$ in $\int_{\check{N}} f_!(1)$ by D. More explicitly, since morphisms in this category are indexed by morphisms in $\check{N}$, we can identify D with the following subsemigroup of N:

$$D = \{n \in N : ene = n \;\text{ and }\; * \otimes en = * \otimes e \;\text{ in }\; 1 \otimes_M eN\} \subseteq eNe. \tag{4}$$

It will also be useful for us to consider the object $(\underline{1}, 1 \otimes e)$ of $\int_{\check{N}} f_!(1)$; letting E be the monoid of endomorphisms of $(\underline{1}, 1 \otimes e)$, we can identify E with a submonoid of N:

$$E = \{n \in N : * \otimes en = * \otimes e \text{ in } 1 \otimes_M eN\}. \tag{5}$$

In particular, $D = eEe$. Further, we have a diagram of semigroup homomorphisms,

$$\begin{array}{ccccc} M & \xrightarrow{\psi} & D & \longrightarrow & eNe \\ & & {\scriptstyle\iota}\downarrow & & \downarrow \\ & & E & \xrightarrow[\tau]{} & N \end{array} , \tag{6}$$

where the horizontal maps are monoid homomorphisms, the vertical maps are inclusions of subsemigroups which reduce to identities when ϕ is a monoid homomorphism, and both paths $M \to N$ compose to give ϕ.

We will see in Proposition 4.1.6 and Proposition 4.1.7 that the terminal-connected part of f factors through both $\mathbf{PSh}(D)$ and $\mathbf{PSh}(E)$, in such a way that each of the factors is again terminal-connected, see also diagram (10). First, we give a different interpretation of D and E in terms of right-factorability.

4.1.2. DEFINITION. *Recall from [HR21b, Definition 2.9] that a non-empty subset S of a monoid M is called* ***right-factorable*** *if whenever $x \in S$ and $y \in M$ with $xy \in S$, then $y \in S$; such a subset automatically contains the identity. Further, for an arbitrary subset T, we defined $\langle T \rangle\rangle_M \subseteq M$ to be the smallest right-factorable submonoid of M containing T. We say that $\langle T \rangle\rangle_M$ is the submonoid of M* ***right-factorably generated*** *by T.*

4.1.3. LEMMA. *In $1 \otimes_M eN$, we have $* \otimes en = * \otimes e$ if and only if $n \in \langle \phi(M) \rangle\rangle_N$. With the notation established above, we find that $E = \langle \phi(M) \rangle\rangle_N$, and similarly that $D = \langle \phi(M) \rangle\rangle_{eNe}$.*

PROOF. By definition of equality in $1 \otimes_M eN$, for $n \in N$ we have $* \otimes e = * \otimes en$ if and only if $1 \sim_M n$, where $\sim_M$ is the right congruence generated by the basic relations $1 \sim \phi(m)$ for all $m \in M$. But by [HR21b, Lemma 2.13], we have $\langle \phi(M) \rangle\rangle_N = \{n \in N : 1 \sim_M n\}$. In other words, $E = \langle \phi(M) \rangle\rangle_N$. To show the analogous result for D, note that eNe is a retract of eN (as left M-sets). The functor $1 \otimes_M -$ preserves retracts, in particular $1 \otimes_M eNe \subseteq 1 \otimes_M eN$. So for $n \in eNe$ the equation $* \otimes en = * \otimes e$ holds in $1 \otimes_M eN$ if and only if it holds in $1 \otimes_M eNe$. The proof that $D = \langle \phi(M) \rangle\rangle_{eNe}$ is now analogous to the above proof that $E = \langle \phi(M) \rangle\rangle_N$. ∎

So the diagram (6) can be written more explicitly as

$$\begin{array}{ccccc} M & \xrightarrow{\psi} & \langle \phi(M) \rangle\rangle_{eNe} & \longrightarrow & eNe \\ & & {\scriptstyle \iota}\downarrow & & \downarrow \\ & & \langle \phi(M) \rangle\rangle_N & \xrightarrow[\tau]{} & N \end{array} . \tag{7}$$

It turns out that f is terminal-connected if and only if the inclusion τ in (7) is the identity.

4.1.4. COROLLARY. *An essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup homomorphism $\phi : M \to N$ is terminal-connected if and only if $\langle \phi(M) \rangle\rangle_N = N$.*

PROOF. Let $e = \phi(1)$. Then $f_!(1) = 1 \otimes_M eN$, whence f is terminal-connected if and only if $1 \otimes_M eN \simeq 1$, which is to say that $* \otimes e = * \otimes en$ for all $n \in N$. By Lemma 4.1.3, this is equivalent to $\langle \phi(M) \rangle\rangle_N = N$. ∎

4.1.5. Example.

1. Let $\mathbb{Z}$ be the group of integers under addition and let $\mathbb{N} \subseteq \mathbb{Z}$ be the submonoid of natural numbers. For each $n \in \mathbb{N}$, we have that $n + (-n) = 0$, and as a result $-n$ is contained in the right-factorable submonoid generated by $\mathbb{N}$, and hence $\langle\mathbb{N}\rangle\rangle_{\mathbb{Z}} = \mathbb{Z}$. It follows that the induced essential localic surjection $\mathbf{PSh}(\mathbb{N}) \to \mathbf{PSh}(\mathbb{Z})$ is terminal-connected.

2. More generally, let G be a group and let $N \subseteq G$ be a submonoid such that $G = \{a^{-1}b \ : \ a, b \in N\}$; for example, if R is a valuation ring in a field K then we may take $N = R - \{0\}$ and $G = K^*$ to be the respective multiplicative monoids of non-zero elements. Then for every $a, b \in N$ we have $a(a^{-1}b) = b$, and this shows that $\langle N\rangle\rangle_G = G$. So the induced geometric morphism $\mathbf{PSh}(N) \to \mathbf{PSh}(G)$ is terminal-connected.

3. Consider any ring R as a monoid with its multiplication operation, and consider the subsemigroup $\{0\} \subseteq R$. Because $0 \cdot r = 0$ for all $r \in R$, we have that the right-factorable submonoid generated by $\{0\}$ is equal to R itself. It follows that the induced essential geometric morphism $\mathbf{Set} \simeq \mathbf{PSh}(\{0\}) \to \mathbf{PSh}(R)$ is terminal-connected.

4. Let N be a commutative idempotent monoid. We denote the multiplication in N by $\wedge$, and in this way we can view N as a meet-semilattice. Let $M \subseteq N$ be a subsemigroup, i.e. a subset closed under $\wedge$. We can compute that $\langle M\rangle\rangle_N$ is then the upwards closure of M. So $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is terminal-connected if and only if the upwards closure of $M \subseteq N$ is all of N.

4.1.6. Proposition. *Given a semigroup homomorphism $\phi : M \to N$, let*

$$f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$$

be the induced essential geometric morphism. Then the terminal-connected part of f has (surjection, inclusion) factorization given by

$$\mathbf{PSh}(M) \xrightarrow{k} \mathbf{PSh}(\langle\phi(M)\rangle\rangle_{eNe}) \xrightarrow{j} \mathbf{PSh}(N)/f_!(1), \tag{8}$$

where k is the essential surjection induced by the factor

$$\psi : M \to \langle\phi(M)\rangle\rangle_{eNe}$$

of ϕ from (7), and j is the essential inclusion induced by the inclusion of the monoid $\langle\phi(M)\rangle\rangle_{eNe}$ as a full subcategory of $\int_{\check{N}} f_!(1)$ on the single object $(\underline{e}, * \otimes e)$. Moreover, both k and j are terminal-connected.

Proof. That this is a canonical representation of the (surjection, inclusion) factorization follows from Lemma 3.1.1, so we only need to verify the last claim. We can deduce from Lemma 4.1.3 that D can be identified with $\langle\psi(M)\rangle\rangle_{eNe}$, whence k is terminal-connected by Corollary 4.1.4. To see that j is terminal-connected, observe that $j_!(1) \cong j_!k_!(1) \cong 1$ since both k and $j \circ k$ are terminal-connected. ∎

4.1.7. PROPOSITION. *With the same set-up as Proposition 4.1.6, the geometric inclusion j in (8) further factors as,*

$$\mathbf{PSh}(\langle\phi(M)\rangle\rangle_{eNe}) \xrightarrow{j_1} \mathbf{PSh}(\langle\phi(M)\rangle\rangle_N) \xrightarrow{j_2} \mathbf{PSh}(N)/f_!(1), \tag{9}$$

where j_1 is the essential inclusion induced by the inclusion of semigroups

$$\iota : \langle\phi(M)\rangle\rangle_{eNe} \to \langle\phi(M)\rangle\rangle_N$$

from (7), and j_2 is the essential inclusion induced by the inclusion of $\langle\phi(M)\rangle\rangle_N$ as a full subcategory of $\int_{\check{N}} f_!(1)$ on the object $(\underline{1}, 1 \otimes e)$. Again, both j_1 and j_2 are terminal-connected.

PROOF. Replacing all of the monoids with their idempotent completions and extending semigroup homomorphisms to functors accordingly, the fact that these geometric morphisms are inclusions is another application of Lemma 3.1.1. Because, $\langle\phi(M)\rangle\rangle_{eNe}$ contains $\phi(M)$, we have that $\langle\phi(M)\rangle\rangle_{eNe}$ right-factorably generates $\langle\phi(M)\rangle\rangle_N$, so using Corollary 4.1.4 we see that j_1 is terminal-connected. That j_2 is terminal-connected follows just as for j in the proof of Proposition 4.1.6. ■

In summary, the geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup morphism $\phi : M \to N$, with $\phi(1) = e$, factors as

$$\begin{array}{ccccc}
\mathbf{PSh}(M) & \xrightarrow{\text{tc surj.}} & \mathbf{PSh}(\langle\phi(M)\rangle\rangle_{eNe}) & & \\
 & & \downarrow{\scriptstyle \text{tc incl.}} & & \\
 & & \mathbf{PSh}(\langle\phi(M)\rangle\rangle_N) & \xrightarrow{\text{tc incl.}} & \mathbf{PSh}(N)/f_!(1) \\
 & & & & \downarrow{\scriptstyle \text{étale}} \\
 & & & & \mathbf{PSh}(N),
\end{array} \tag{10}$$

where 'tc' is short-hand for terminal-connected. If $\phi(1) = 1$, then this reduces to

$$\mathbf{PSh}(M) \xrightarrow{\text{tc surj.}} \mathbf{PSh}(\langle\phi(M)\rangle\rangle_N) \xrightarrow{\text{tc incl.}} \mathbf{PSh}(N)/f_!(1) \xrightarrow{\text{étale}} \mathbf{PSh}(N). \tag{11}$$

4.1.8. REMARK. *Note that since f is induced by a semigroup homomorphism, $f_!(1) = 1 \otimes_M \phi(1)N$ is an inhabited set, so the map $f_!(1) \to 1$ is an epimorphism, which implies that the induced geometric morphism $\mathbf{PSh}(N)/f_!(1) \to \mathbf{PSh}(N)$ is always a surjection. Indeed, since $\mathbf{PSh}(N)$ is hyperconnected, all of its non-initial objects are well-supported, so an étale morphism from any non-degenerate topos will be a surjection.*

We now consider conditions under which the morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ we have been considering is étale. By orthogonality, if f is étale, then the terminal-connected factor $\mathbf{PSh}(M) \to \mathbf{PSh}(N)/f_!(1)$ must be an equivalence, and by the above remark, f must be a surjection. Since the (surjection, inclusion) factorization of f is given by $\mathbf{PSh}(M) \to \mathbf{PSh}(eNe) \to \mathbf{PSh}(N)$, with $e = \phi(1)$, it follows that f is étale if and only if the inclusion part $\mathbf{PSh}(eNe) \to \mathbf{PSh}(N)$ is an equivalence and the surjection part $\mathbf{PSh}(M) \to \mathbf{PSh}(eNe)$ is étale. So we may without loss of generality restrict our attention to the case where ϕ is a monoid homomorphism.

4.1.9. Proposition. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be a geometric morphism induced by a monoid homomorphism $\phi : M \to N$. Then the terminal-connected surjection part in the factorization* (11) *is an equivalence if and only if ϕ is injective and $\phi(M)$ is a right-factorable submonoid of N.*

Proof. The terminal-connected surjection part in (11) is induced by the monoid homomorphism $M \to \langle\phi(M)\rangle\rangle_N$. So we get an equivalence if and only if this monoid homomorphism is a bijection. ∎

To understand when the terminal-connected inclusion part of f is an equivalence, we need the following definition.

4.1.10. Definition. *Given a monoid N, we write $N^{\ltimes}$ for the submonoid of **right-invertible elements**. That is,*

$$N^{\ltimes} := \{u \in N : \exists v \in N, \, uv = 1\}.$$

*Dually, we write $N^{\rtimes}$ for the submonoid of **left-invertible elements**, so*

$$N^{\rtimes} := \{v \in N \mid \exists u \in N, \, uv = 1\}.$$

4.1.11. Proposition. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be a geometric morphism induced by a monoid homomorphism $\phi : M \to N$. Then the terminal-connected inclusion part in the factorization* (11) *is an equivalence if and only if for all $n \in N$ there is some $u \in N^{\ltimes}$ such that $nu \in \langle\phi(M)\rangle\rangle_N$.*

Proof. The terminal-connected inclusion part $\mathbf{PSh}(\langle\phi(M)\rangle\rangle_N) \to \mathbf{PSh}(N)/f_!(1)$ is an essential inclusion induced by the functor $\langle\phi(M)\rangle\rangle_N \to \int_N f_!(1)$ (note that we don't need to take the category of elements over the idempotent completion because ϕ is a monoid homomorphism). It is enough to show that this geometric morphism is surjective as well, which by Lemma 3.1.1 is the case if and only if the functor $\langle\phi(M)\rangle\rangle_N \to \int_N f_!(1)$ is essentially surjective up to retracts. In other words, we need that every element $* \otimes n \in f_!(1)$ is a retract of $*\otimes 1$ in the category of elements $\int_N f_!(1)$. Equivalently, for each $n \in N$ there are $u, v \in N$ such that $uv = 1$ and $* \otimes nu = * \otimes 1$. By the proof of Lemma 4.1.3, $* \otimes nu = * \otimes 1$ if and only if $nu \in \langle\phi(M)\rangle\rangle_N$. ∎

Combining the two propositions above, we obtain a characterization of étale geometric morphisms induced by monoid homomorphisms.

4.1.12. Theorem. *Let f be an essential geometric morphism induced by a monoid homomorphism $\phi : M \to N$. Then the following are equivalent:*

1. *f is étale;*

2. *ϕ is injective, $\phi(M) \subseteq N$ is right-factorable and for any $n \in N$ there is some $u \in N^{\ltimes}$ such that $nu \in \phi(M)$.*

More generally, if ϕ is merely a semigroup homomorphism, then f is étale if and only if the monoid homomorphism part of ϕ satisfies the conditions above, and the inclusion $eNe \subseteq N$ induces an equivalence, where $e = \phi(1)$.

Further, we remark that étale geometric morphisms are locally connected, so in particular they are always essential [Joh02, C3.3]. As a result, every étale geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is induced by some semigroup homomorphism $\phi : M \to N$. We already mentioned in Remark 4.1.8 that f is necessarily surjective. So by Corollary 3.1.2, we can even assume that ϕ is a monoid homomorphism, after replacing N by a Morita-equivalent monoid.

4.1.13. Corollary. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be an étale geometric morphism. If $N^{\ltimes} = \{1\}$, then f is an equivalence.*

Proof. Since f is étale, it is essential. So up to equivalence, it is induced by some semigroup homomorphism $\phi : M \to N$. If $\phi(1) = e$, then the inclusion $eNe \subseteq N$ induces an equivalence. Because $N^{\ltimes} = \{1\}$, this implies $e = 1$, see [Kna72, Corollary 6.2(3)]. So ϕ is a monoid homomorphism. Applying Theorem 4.1.12, for all $n \in N$, there is $u \in N^{\ltimes}$ such that $nu \in \phi(M)$. Because $N^{\ltimes} = \{1\}$, this means that ϕ is bijective, so f is an equivalence. ∎

4.1.14. Example.

1. For $H \subseteq G$ an inclusion of groups, we have that the induced geometric morphism $\mathbf{PSh}(H) \to \mathbf{PSh}(G)$ is étale.

2. Consider the monoid $\mathbb{Z}_p^{\mathrm{ns}}$ of nonzero p-adic integers under multiplication. Each p-adic integer can be written as up^k for $k \in \{0, 1, 2, \dots\}$ and $u \in \mathbb{Z}_p$ an invertible element. Further, if x is a nonzero p-adic integer, then $xp^k = p^l$ implies $x = p^{l-k}$. From this it follows that the inclusion $\mathbb{N} \to \mathbb{Z}_p^{\mathrm{ns}}$, $k \mapsto p^k$ induces an étale geometric morphism $\mathbf{PSh}(\mathbb{N}) \to \mathbf{PSh}(\mathbb{Z}_p^{\mathrm{ns}})$, where $\mathbb{N}$ is the monoid of natural numbers under addition.

4.2. The general case. Since all étale morphisms are essential, there is limited benefit to considering étale geometric morphisms induced by $[N, M)$-sets, any such being necessarily isomorphic to one induced by a semigroup homomorphism ϕ via Lemma 2.2.4. We leave it to the reader to translate the conditions of Theorem 4.1.12 into properties of that $[M, N)$-set. It remains to consider more general (i.e. non-essential) terminal-connected morphisms.

By definition, a geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by an $[N, M)$-set A is terminal-connected if and only if $\mathrm{Hom}_N(1, X) \cong \mathrm{Hom}_M(1, X \otimes_N A)$, naturally in X. Translating this into algebra, this is equivalent to requiring that for every right N-set X, the M-fixed points of $X \otimes_N A$ are all of the form $x \otimes a$, for x an N-fixed point of X. Indeed, consider the mapping

$$\mathrm{Hom}_N(1, X) \to \mathrm{Hom}_M(1, X \otimes_N A)$$
$$x \mapsto x \otimes a.$$

This is independent of the choice of $a \in A$, since $x \otimes a = x \otimes (n \cdot a)$ for all $n \in N$ and A is connected as a left N-set since it is filtered; $x \otimes a$ is an M-fixed point by a similar argument. Also, if x, y are distinct N-fixed points of X, then we have a monomorphism $(x, y) : 1 + 1 \hookrightarrow X$ which is preserved by $- \otimes_N A$ due to flatness, whence we have a monomomorphism $(x \otimes a, y \otimes a) \hookrightarrow X \otimes_N A$ and the mapping above is injective. So terminal-connectedness reduces to the requirement of surjectivity of this mapping.

Recall that a right M-set is called **principal** (or cyclic) if it is generated by a single element, or equivalently if it can be presented as a quotient of M (as a right M-set). Observe that, while $\mathrm{Hom}_N(1, -)$ does not preserve arbitrary colimits, it does preserve the expression of an N-set as a colimit of the principal sub-N-sets generated by its elements. As such, we can reduce the condition to the special case of principal N-sets, to conclude:

4.2.1. LEMMA. *A geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by an $[N, M)$-set A is terminal-connected if and only if for every principal N-set X, the M-fixed points of $X \otimes_N A$ are all of the form $x \otimes a$, for x an N-fixed point of X.*

4.2.2. EXAMPLE. Consider the geometric morphism $f : \mathbf{PSh}(\mathbb{Z}) \to \mathbf{PSh}(\mathbb{N})$ given by the $[\mathbb{N}, \mathbb{Z})$-set $\mathbb{Z}$, with $\mathbb{N}$ and $\mathbb{Z}$ both seen as monoids under addition. For integers $a \geq 0$ and $b \geq 1$, we write $N_{a,b}$ for the quotient of $\mathbb{N}$ (as a right $\mathbb{N}$-set) by the congruence generated by $a \sim a+b$. The elements of $N_{a,b}$ can be written as $\{0, 1, \ldots, a+b-1\}$ and the generator $1 \in \mathbb{N}$ acts by sending each x to $x + 1$ for $x \leq a + b - 2$ and by sending $a + b - 1$ to a. Every principal $\mathbb{N}$-set is either isomorphic to $\mathbb{N}$ or to some $N_{a,b}$. One can compute $\mathbb{N} \otimes_{\mathbb{N}} \mathbb{Z} \cong \mathbb{Z}$ and $N_{a,b} \otimes_{\mathbb{N}} \mathbb{Z} \cong \mathbb{Z}/b\mathbb{Z}$. So for a principal right $\mathbb{N}$-set X, either both X and $X \otimes_{\mathbb{N}} \mathbb{Z}$ have no fixed points, or they both have precisely one fixed point. It follows that f is terminal-connected.

4.2.3. REMARK. *Recently, Osmond in [Osm21] demonstrated that the (terminal-connected, étale) factorization can be generalized to arbitrary geometric morphisms (between Grothendieck toposes) as the (terminal-connected, pro-étale) factorization. From [Osm21, Definition 5.4.6], a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is **pro-étale** if it can be expressed as a cofiltered bilimit of étale morphisms over $\mathcal{E}$. It might therefore be interesting to investigate this factorization system and its application to the special case of geometric morphisms between toposes of discrete monoid actions, but we leave this to future work.*

5. The (pure, complete spread) factorization

In this section, we discuss the (pure, complete spread) factorization, sometimes called the comprehensive factorization. We shall see that this factorization system is dual in a concrete sense to the (terminal-connected, étale) factorization system of the last section; see Proposition 5.1.1, for example.

We first recall the definition of pure geometric morphisms.

5.0.1. Definition. *A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be* ***dominant*** *if the canonical map $0 \to f_*(0)$ is an isomorphism.*

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is ***pure*** *if the natural map $1 \sqcup 1 \to f_*(1 \sqcup 1)$ is an isomorphism. Equivalently, f is pure if and only if f_* preserves finite coproducts [Joh02, C3.4.12(i)], so in particular any pure geometric morphism is dominant.*

5.0.2. Remark. *Bunge and Funk's definition of pure geometric morphism in [BF96] and [BF98] is different (aside from the fact that we take as fixed base topos $\mathcal{S} = \mathbf{Set}$): there a geometric morphism is called pure if the natural map $1 \sqcup 1 \to f_*(1 \sqcup 1)$ is an epimorphism, and pure dense if it is an isomorphism. In Definition 5.0.1, we follow the convention originally used by Johnstone [Joh82], which was later also followed by Bunge and Funk in [BF06]. We recommend all of these references to a reader interested in a deeper treatment of these properties.*

5.0.3. Lemma. *All geometric morphisms $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ are dominant.*

Proof. Suppose f is induced by the $[N, M]$-set A. Then $f_*(0) = \mathrm{Hom}_M(A, 0) = 0$ since A (being flat as a left-N-set) is non-empty. ∎

5.0.4. Definition. *An object X of a topos is called* ***connected*** *if it has exactly two (distinct) complemented subobjects, the initial subobject 0 and X itself, or equivalently (since coproducts are stable under pullback in a topos) if $\mathrm{Hom}(X, -)$ preserves finite coproducts.*

In a locally connected Grothendieck topos, every object is a coproduct of connected objects, and the latter can be identified as the objects whose image under the extra left adjoint of the global sections geometric morphism is terminal (see [Joh02, Lemma C3.3.6]). The following lemma is a special case of [BF96, Proposition 2.7], in the setting of locally connected Grothendieck toposes over the base topos $\mathcal{S} = \mathbf{Set}$. We give a simplified proof in this special case.

5.0.5. Lemma. *Suppose $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism (between Grothendieck toposes) and both $\mathcal{F}$ and $\mathcal{E}$ are locally connected. Then the following are equivalent:*

1. *f is a pure geometric morphism;*
2. *f^* preserves connected objects;*
3. *The unit of f is an isomorphism at objects of the form $p^*(S)$, where p is the global sections geometric morphism of $\mathcal{E}$.*
4. *f_* preserves small coproducts.*

PROOF. Let q, p be the global sections morphisms of $\mathcal{F}, \mathcal{E}$ respectively.

(1) ⇒ (2) Suppose f is pure (so f_* preserves finite coproducts) and let X be a connected object of $\mathcal{E}$. Given a decomposition $f^*(X) = A \sqcup B$, we have $\mathrm{Hom}_{\mathcal{F}}(f^*(X), f^*(X)) \cong \mathrm{Hom}_{\mathcal{E}}(X, f_*(A \sqcup B)) \cong \mathrm{Hom}_{\mathcal{E}}(X, f_*(A)) \sqcup \mathrm{Hom}_{\mathcal{E}}(X, f_*(B)) \cong \mathrm{Hom}_{\mathcal{F}}(f^*(X), A) \sqcup \mathrm{Hom}_{\mathcal{E}}(f^*(X), B)$. Chasing the isomorphisms involved, this means that the identity on $f^*(X)$ must factor through the inclusion of A or B into $f^*(X)$, meaning that one of the components must be all of $f^*(X)$. We can similarly check that f_* preserving 0 implies that $f^*(X) \not\cong 0$, so $f^*(X)$ is connected, as required.

(2) ⇒ (3) Preservation of connected objects means that $q_! f^* \cong p_!$, since all of these functors preserve the expression of objects of $\mathcal{E}$ as coproducts of connected objects. In particular, their adjoints are isomorphic, which is to say that $p^* \cong f_* q^* = f_* f^* p^*$ (and more specifically this forces the desired unit morphisms to be isomorphisms).

(3) ⇒ (4) Given a set S indexing a family of objects $\{X_s \mid s \in S\}$ of objects of $\mathcal{F}$, observe that their coproduct is determined by the collection of pullbacks,

$$\begin{array}{ccc} X_s & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \coprod_{s \in S} X_s & \longrightarrow & q^*(S); \end{array}$$

because coproducts are stable under pullback in a topos. Applying f_*, which preserves these pullbacks, we see that we have:

$$\begin{array}{ccc} f_*(X_s) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ f_*(\coprod_{s \in S} X_s) & \longrightarrow & f_* q^*(S), \end{array}$$

and so given that $f_* q^*(S) \cong p^*(S)$ (compatibly with the inclusions of the elements $1 \to p^*(S)$) this expresses $f_*(\coprod_{s \in S} X_s)$ as the coproduct of the objects $f_*(X_s)$, as required.

(4) ⇒ (1) Immediate from the definition of pure morphisms. ■

It follows from the proof above that to check that f is pure, it is enough to check that $f^*(X)$ is connected, for each X in a generating family for $\mathcal{E}$.

Connected geometric morphisms are pure, cf. [Joh02, Lemma C3.4.14]. Moreover, a locally connected geometric morphism to **Set** is pure if and only if it is connected (if and only if the terminal object is connected).

5.0.6. EXAMPLE. If X is a topological space, then the unique geometric morphism $\mathbf{Sh}(X) \to \mathbf{Set}$ is pure if and only if it is connected, which is the case if and only if X is connected as a topological space. More generally, let X and Y be locally connected topological spaces and let $\phi : Y \to X$ be a continuous map. Then by Lemma 5.0.5 the induced geometric morphism $f : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ is pure if and only if the inverse image of any connected open set is connected, since ϕ^{-1} can be identified with the restriction of f^* to the subterminal objects, and any connected object of $\mathbf{Sh}(X)$ can be expressed as a (connected) colimit of connected subterminal objects. In particular, consider an inclusion $\{x\} \subseteq X$ and let $p : \mathbf{Set} \to \mathbf{Sh}(X)$ be the induced geometric morphism. If X is locally connected, then p is pure if and only if x is contained in any connected open subset $U \subseteq X$. This is the case if and only if x is contained in any open subset, i.e. if and only if x is a dense point.

For example, consider the spectrum $\mathrm{Spec}(\mathbb{Z})$. Each nonempty open subset contains the generic point $x = (0)$, so it cannot be written as a disjoint union of two smaller nonempty open subsets. This shows that every nonempty open subset of $\mathrm{Spec}(\mathbb{Z})$ is connected, in particular $\mathrm{Spec}(\mathbb{Z})$ is locally connected. For the same reason, $U \cap \{x\}$ is a singleton for each connected open subset, so the geometric morphism induced by $\{x\} \subseteq \mathrm{Spec}(\mathbb{Z})$ is pure. Note that the geometric morphism induced by $\{x\} \subseteq \mathrm{Spec}(\mathbb{Z})$ is not surjective, since its inverse image functor identifies all of the non-initial subterminal objects; in particular it is not connected.

We now recall the definition of spreads, based on [BF96, Definition 1.1] and the equivalent conditions of their Proposition 1.5. Note that in [BF96, Proposition 1.5] the authors work over a general base topos and mention *definable subobjects*, which are those which can be presented as pullbacks of subobjects coming from the base topos; when the base topos is Boolean (such as **Set** for Grothendieck toposes in this paper), definable subobjects coincide with complemented subobjects thanks to stability of coproduct decompositions under pullback, which simplifies the definition.

5.0.7. Definition. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism, and let S be a generating family of $\mathcal{E}$. Then f is a **spread** if for every object F in $\mathcal{F}$, there is a complemented subobject $C \subseteq \coprod_{i \in I} f^*(X_i)$, with each $X_i \in S$, such that there is an epimorphism $C \to F$. This definition does not depend on the choice of generating family S.*

To see that the definition of spread does not depend on the choice of generating family, suppose we have an inclusion $S' \subseteq S$ of generating families for $\mathcal{E}$. Let $C \subseteq \coprod_{i \in I} f^*(X_i)$ be a complemented subobject, with each $X_i \in S$. Then we can find for each $i \in I$ an epimorphism $\coprod_{j \in I(i)} X_{ij} \to X_i$ for some set $I(i)$ and $X_{ij} \in S'$ for all $j \in I(i)$. Pulling back C along the epimorphism

$$\coprod_{\substack{i \in I \\ j \in I(i)}} f^*(X_{ij}) \to \coprod_{i \in I} f^*(X_i)$$

gives a complemented subobject $C' \subseteq \coprod_{i \in I,\ j \in I(i)} f^*(X_{ij})$, together with an epimorphism $C' \to C$. Any epimorphism $C \to F$ will then extend to an epimorphism $C' \to F$. Thus, given any pair of generating families, we can take their union and conclude via two applications of this argument that the definitions involving the respective families agree.

5.0.8. Proposition. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then f is a spread if and only if for every object F in $\mathcal{F}$ there is a complemented subobject $C \subseteq f^*(X)$, for some X in $\mathcal{E}$, and an epimorphism $C \to F$. In particular, every spread is localic.*

PROOF. We do not assume that the generating family is small in the discussion above. In particular, taking S to be the class of all objects in $\mathcal{E}$, we obtain the hypothesised characterization of spreads. This result appears in [BF06, Corollary 3.1.8]. ■

5.0.9. EXAMPLE. Let X be a topological space. Then the unique geometric morphism $\mathbf{Sh}(X) \to \mathbf{Set}$ is a spread if and only if X has a basis of clopen subsets (i.e. if and only if X is zero-dimensional). Indeed, since the terminal object is a generator in $\mathbf{Set}$, we require (following Definition 5.0.7) that every object of $\mathbf{Sh}(X)$ admits an epimorphism from a disjoint union of complemented subterminal objects, which in particular means that X must have a basis of clopens, and conversely every object in $\mathbf{Sh}(X)$ is a quotient of a coproduct of (subterminals corresponding to) opens in a base.

As a special case of spreads, we also include the following definition:

5.0.10. DEFINITION. *We say a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is an* ***injection*** *if its direct image f_* is faithful. Equivalently, f is an injection if and only if every object of $\mathcal{F}$ is a quotient of one in the image of f^*, whence by Proposition 5.0.8 every injection is a spread.*

We can prove the equivalence of the two definitions as follows. A general fact from category theory is that f_* is faithful if and only if the counit of the adjunction $f^*f_*X \to X$ is an epimorphism. So if f_* is faithful, then X is a quotient of f^*f_*X. Conversely, if X is a quotient of f^*Y for some Y in $\mathcal{E}$, then the quotient map $f^*Y \to X$ factorizes through the counit $f^*f_*X \to X$, so the counit is an epimorphism.

In particular, inclusions are injections, so *a fortiori* inclusions are spreads.

We now discuss the notion of completeness for geometric morphisms with locally connected domain. Again, the definition we give below is simplified by the fact that we are working with Grothendieck toposes over $\mathbf{Set}$. For the more general definition, we refer to [BF06] [BF96].

We first follow [BF96, p. 19], keeping to their notation as much as possible. Consider a Grothendieck topos $\mathcal{F}$ with site of definition $(\mathcal{D}, J)$, and write $i : \mathcal{F} \to \mathbf{PSh}(\mathcal{D})$ for its inclusion into the topos of presheaves. Let $\chi : \mathbf{Set} \to \mathcal{F}$ be a distribution (defined after Definition 2.2.2, above), let

$$\mathcal{C} = \int^{\mathcal{D}} \chi^* \circ i^* \circ \mathbf{y}$$

with $\mathbf{y}$ the Yoneda embedding, and let $U : \mathcal{C} \to \mathcal{D}$ be the induced discrete opfibration. The functor U induces an essential geometric morphism $u : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$.

Still following [BF96, p. 19], we define $\mathbf{D}_\chi$ via the following pullback diagram,

$$\begin{array}{ccc} \mathbf{D}_\chi & \xrightarrow{\pi} & \mathbf{PSh}(\mathcal{C}) \\ \chi' \downarrow & & \downarrow u \\ \mathcal{F} & \xrightarrow{i} & \mathbf{PSh}(\mathcal{D}) \end{array} ;$$

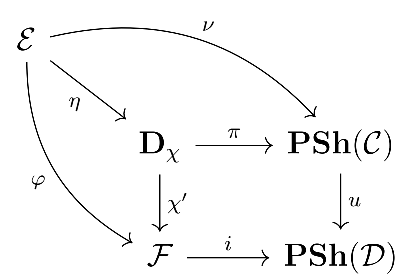
here π is an inclusion because i is. We may present $\mathbf{D}_\chi$ as the topos of sheaves on the site $(\mathcal{C}, J_\chi)$, with J_χ the pullback topology of J along χ. We call this topology the **amalgamation topology**, following [BF06, §2.3]; its covering sieves will be called **amalgamation covering sieves**. Let (D, c) be an object of $\mathcal{C}$, so D is an object in $\mathcal{D}$ and $c \in \chi^*(i^*\mathbf{y}(D))$. Then a sieve S on (D, c) is an amalgamation covering sieve if and only if there is a J-covering sieve R on D such that for every $a' : D' \to D$ in R and each element $c' \in \chi^*(i^*\mathbf{y}(D'))$ with $a' \cdot c = c'$, we have that the corresponding morphism $a' : (D', c') \to (D, c)$ is contained in S, using the abbreviation $a' \cdot c = \chi^*(i^*\mathbf{y}(a'))(c)$.

Now let $\varphi : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, with $\mathcal{E}$ locally connected. Then we can define the distribution Λ_φ whose left adjoint functor is,

$$\Lambda_\varphi^* = e_! \circ \varphi^* : \mathcal{F} \longrightarrow \mathbf{Set}$$

with $e : \mathcal{E} \to \mathbf{Set}$ the global sections geometric morphism; see [BF96, p. 20].

Bunge and Funk showed that for each topos $\mathcal{F}$, the constructions $\Lambda_{(-)}$ and $\mathbf{D}_{(-)}$ are functorial, and form an adjunction $\Lambda_{(-)} \dashv \mathbf{D}_{(-)}$ between geometric morphisms with locally connected domain and fixed codomain $\mathcal{F}$, and distributions on $\mathcal{F}$ [BF96, Proposition 2.11]. In particular, in our situation above, there is a natural geometric morphism $\eta : \mathcal{E} \to \mathbf{D}_{\Lambda_\varphi}$ over $\mathcal{F}$, the unit of the adjunction. So we arrive at a commutative diagram of the form



(12)

with $\chi = \Lambda_\varphi$. Bunge and Funk then prove that η is pure [BF96, Theorem 2.15] and that χ' is a spread [BF96, Proposition 2.10]. The factorization of $\varphi = \chi' \circ \eta$ is called the *comprehensive factorization* in [BF06].

5.0.11. Definition. *For a geometric morphism $\varphi : \mathcal{E} \to \mathcal{F}$ with $\mathcal{E}$ locally connected, we say that φ is **complete** if the geometric morphism η in the diagram (12) is a surjection. A spread that is complete will be called a **complete spread**.*

We can now recall the main result regarding the comprehensive factorization:

5.0.12. Theorem. [BF96, Theorem 2.15], [BF06, Theorem 2.4.8] *For any geometric morphism $\varphi : \mathcal{E} \to \mathcal{F}$ with locally connected domain, the comprehensive factorization $\varphi = \chi' \circ \eta$ in (12) is a factorization into a pure geometric morphism η followed by a complete spread χ'.*

This factorization is unique, in the sense that for any other factorization of φ in a pure part $p : \mathcal{E} \to \mathcal{G}$ followed a complete spread $g : \mathcal{G} \to \mathcal{F}$ there is an equivalence $\xi : \mathcal{G} \to \mathbf{D}_\chi$ such that $p \cong \xi^{-1} \circ \eta$ and $g \cong \chi' \circ \xi$.

We shall refer to the comprehensive factorization as the **(pure, complete spread) factorization**. The uniqueness part of Theorem 5.0.12 implies in particular that the geometric morphisms η and χ' in (12) are independent of the choice of site for $\mathcal{F}$.

5.0.13. Remark. *The concept of completeness for geometric morphisms is due to Bunge and Funk. It was first introduced only for spreads with locally connected domain by Bunge and Funk in [BF96]; they later generalized the notion to arbitrary geometric morphisms with locally connected domain in [BF06]. Their original definition of completeness is different from the one we gave in Definition 5.0.11, but they show in [BF06, Theorem 3.5.3] that the property is equivalent. The notion of complete spreads in topos theory was inspired by complete spreads in topology, as introduced by Fox [Fox57].*

In this paper, complete spreads will always have locally connected domain. This implies that the identity geometric morphism $\mathcal{F} \to \mathcal{F}$ does not qualify as a complete spread if $\mathcal{F}$ is not locally connected. A perhaps counterintuitive consequence of this is that it is possible for a geometric morphism $\varphi : \mathcal{E} \to \mathcal{F}$ to be both pure and a complete spread, without being an equivalence; Bunge and Funk give an example, see [BF96, Example 2.8].

A wider definition of complete spreads (including all identity morphisms) appears in [BF07]; by Proposition 2.2 there, that definition coincides with the one here for geometric morphisms with locally connected domain. For this wider definition of complete spreads, the uniqueness part of Theorem 5.0.12 is no longer valid; however, in this setting there is a unique factorization into a hyperpure part followed by a complete spread, see [BF07].

It is also worth noting that the quoted results from [BF96], [BF06] and [BF07] are formulated for Grothendieck toposes over an arbitrary elementary base topos $\mathcal{S}$; we have restricted here to Grothendieck toposes over **Set**.

It will be helpful to have a closer look at the diagram (12). To simplify the notation, we will identify objects D in $\mathcal{D}$ with their image in $\mathcal{F}$ via $i^* \circ \mathbf{y}$. The category

$$\mathcal{C} = \int^{\mathcal{D}} e_! \circ \varphi^* \circ i^* \circ \mathbf{y} \tag{13}$$

then has as objects the pairs (D, c) with D in $\mathcal{D}$ and $c \hookrightarrow \varphi^*(D)$ a connected component, and as morphisms $(D, c) \to (D', c')$ the maps $a : D \to D'$ such that the image of c along $\varphi^*(a)$ is contained in c'.

There is a concrete description of the geometric morphism $\nu : \mathcal{E} \to \mathbf{PSh}(\mathcal{C})$ in the proof of [BF96, Proposition 2.11]. It is completely determined by the associated flat functor

$$\nu^* \circ \mathbf{y} : \mathcal{C} \to \mathcal{E}, \tag{14}$$

and in this case $\nu^* \circ \mathbf{y}$ sends an object (D, c) in $\mathcal{C}$ to the connected component $c \subseteq \varphi^*(D)$ in $\mathcal{E}$. A morphism $(D, c) \to (D', c')$ determined by a map $a : D \to D'$ is sent to the morphism $\varphi^*(a)|_c : c \to c'$ in $\mathcal{E}$.

We are now ready to discuss an equivalent, explicit formulation of the notion of completeness. Consider again the geometric morphisms (12). We defined φ to be complete if η is surjective. Since π is always an inclusion, φ is complete if and only if $\nu = \pi \circ \eta$

is a (surjection, inclusion) factorization. By the construction of the (surjection, inclusion) factorization [MLM94, Chapter VII, §4], this is in turn equivalent to the statement that the amalgamated topology J_χ has as covering sieves on (D, c) precisely the sieves that become jointly epimorphic after applying $\nu^* \circ \mathbf{y}$, or in other words, the sieves $S = \{a_i : (D_i, c_i) \to (D, c)\}_{i \in I}$ such that the morphisms $\varphi^*(a_i)|_{c_i} : c_i \to c$ are jointly epimorphic.

5.0.14. Definition. *We keep the notation as above. Let $S = \{a_i : (D_i, c_i) \to (D, c)\}_{i \in I}$ be a sieve on (D, c) in the category $\mathcal{C}$. Then we say that S is a **φ-covering sieve** if the associated family $\{\varphi^*(a_i)|_{c_i} : c_i \to c\}_{i \in I}$ is jointly epimorphic in $\mathcal{E}$.*

As discussed above, the φ-covering sieves are precisely the covering sieves determining the image of the geometric morphism $\nu : \mathcal{E} \to \mathbf{PSh}(\mathcal{C})$. Because this image is contained in $\mathbf{D}_\chi$, the topology determining $\mathbf{D}_\chi$, the amalgamated topology, must be weaker. In other words, every amalgamated covering sieve is a φ-covering sieve. On the other hand, by the discussion above:

5.0.15. Proposition. *The geometric morphism φ is complete if and only if every φ-covering sieve is an amalgamation covering sieve.*

To provide some intuition regarding complete geometric morphisms, we include an example and a result for the special case of maps between topological spaces.

5.0.16. Example. Consider the inclusion $W \subseteq \mathbb{R}^2$, with W the unit circle minus the point $(1, 0)$ (these spaces are locally connected), and let $\varphi : \mathcal{E} \to \mathcal{F}$ be the induced geometric morphism, with $\mathcal{E} = \mathbf{Sh}(W)$ and $\mathcal{F} = \mathbf{Sh}(\mathbb{R}^2)$. We look at the comprehensive factorization as discussed above, using the same notation. Note that φ is an inclusion, in particular a spread. We will show that it is not complete, using Proposition 5.0.15. We take a base for the topology on $\mathbb{R}^2$ consisting of the open balls with radius at most $1/2$. These open balls and the inclusions between them form the site $(\mathcal{D}, J)$. The category $\mathcal{C}$ as in (12) then has as objects the pairs (U, c), where U is an open ball of radius at most $1/2$ and $c \subseteq W \cap U$ is a connected component and a morphism $(V, c') \to (U, c)$ is an inclusion $V \subseteq U$ such that $c' \subseteq c$. Take a pair (U, c) with U containing $(1, 0)$ and an open set $V \subseteq U$ such that $V \cap W = c$ and $(1, 0) \notin V$. Then the sieve generated by $(V, c) \to (U, c)$ is an φ-covering sieve. However, it is not an amalgamation covering sieve, because any amalgamation covering sieve must contain a pair (U', c') with $(1, 0) \in U' \subseteq U$ and $c' \subseteq c$.

5.0.17. Proposition. *Let Y be a topological space and $X \subseteq Y$ a subspace, with X locally connected. If each connected component X' of X is closed in Y, then the geometric morphism induced by $X \subseteq Y$ is complete. The converse holds if we assume that all points of Y are closed.*

Proof. We keep using the same notations as in our discussion above of the comprehensive factorization. Let $\varphi : \mathcal{E} \to \mathcal{F}$ the geometric inclusion induced by the inclusion $X \subseteq Y$, with $\mathcal{E} = \mathbf{Sh}(X)$ and $\mathcal{F} = \mathbf{Sh}(Y)$. As site $(\mathcal{D}, J)$ for $\mathcal{F}$, we take the canonical site of open subsets of Y. The category $\mathcal{C}$ as in (12) then has as objects the pairs (U, c) with $U \subseteq Y$ open and $c \subseteq U \cap X$ a connected component.

Suppose that every connected component of X is closed in Y. We show that φ is complete using Proposition 5.0.15. Let $S = \{(U_i, c_i) \to (U, c)\}_{i \in I}$ be an φ-covering sieve. We claim that it is an amalgamation covering sieve as well. For each $x \in c$, take a pair (U_i, c_i) in S with $x \in c_i$. Since c_i is open in $X \cap U_i$, we can take an open subset $V_x \subseteq U_i$ such that $V_x \cap X = c_i$. Note that (V_x, c_i) is still contained in S. Now consider the covering sieve R on U generated by the inclusions $V_x \to U$ for $x \in c$ and the inclusion $U - c \to U$ (to show that c is closed in U, use that c is clopen in $X' \cap U$ for some connected component X' of X, and that in turn $X' \cap U$ is closed in U). The pullback of R to (U, c) is the sieve generated by the inclusions $(V_x, V_x \cap X) \to (U, c)$, and this pullback sieve is contained in S. So S is indeed an amalgamation covering sieve.

Conversely, suppose that the induced geometric morphism φ is complete. Take a connected component X' of X and an element $y \in \overline{X'} - X'$, with $\overline{X'}$ the closure of X' in Y. Because y is closed in Y and $y \notin X'$, we can consider the φ-covering sieve S generated by $(Y - \{y\}, X') \to (Y, X')$. We claim that this is not an amalgamation covering sieve, which gives a contradiction. To see this, take an arbitrary covering sieve R on Y. Then R contains an inclusion $V \to Y$ with $y \in V$. Because $y \in \overline{X'}$, we see that $V \cap X' \neq \varnothing$. So if S contains the pullback of R, then S must contain a morphism $(V, c') \to (Y, X')$ for some connected component $c' \subseteq V \cap X'$, which leads to a contradiction. As a result, S is not an amalgamation covering sieve. ∎

5.0.18. Remark. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, with $\mathcal{E}$ locally connected. The comprehensive factorization of φ can be used to construct a (pure surjection, spread) factorization as follows. Let $\varphi = \chi' \circ \eta$ be the comprehensive factorization, and take the (surjection, inclusion) factorization $\eta = j \circ p$ of η. Then both j and p are again pure [BF06, Proposition 2.2.8] and since j is an inclusion, it is in particular a spread. In this way, we get a (pure surjection, spread) factorization*

$$\mathcal{E} \xrightarrow{p} \mathcal{G} \xrightarrow{\chi' \circ j} \mathcal{F}.$$

It turns out that the middle topos $\mathcal{G}$ in this factorization is locally connected, and that any (pure surjection, spread) factorization, with the middle topos locally connected, is equivalent to this one, see [BF06, Theorem 5.12].

In particular, if a spread with locally connected domain is pure, then its pure surjection part is trivial, so it is an inclusion. Conversely, inclusions are spreads, so a geometric morphism with locally connected domain is a pure spread if and only if it is a pure inclusion.

5.0.19. Remark. *We would be remiss not to also mention the (pure, entire) factorization described by Johnstone in [Joh02, C3.4]. A geometric morphism is entire if it is localic and the corresponding internal locale is compact and zero-dimensional. For comparison, under the wider definition of complete spreads referenced in Remark 5.0.13, any entire geometric morphism is a complete spread, but not conversely.*

The (pure, entire) factorization of a morphism $f : \mathcal{F} \to \mathcal{E}$ is obtained by taking the intermediate topos to be the topos of internal sheaves on (the zero-dimensional locale *dual to) the subframe of $f_*(\Omega_{\mathcal{F}})$ generated by $f_*(2_{\mathcal{F}})$. This is typically different from the factorizations we consider here. For example, for an infinite set X viewed as a discrete space, the geometric morphism $\mathbf{Sh}(X) \to \mathbf{Set}$ is a complete spread, but its (pure, entire) factorization has sheaves on the Stone-Čech compactification of X as the intermediate topos. The reason we do not extensively consider this factorization system in this paper is exactly the reason illustrated by that example: the intermediate topos in this factorization is rarely a presheaf topos, even for an essential geometric morphism between presheaf toposes. We leave deeper consideration of entire morphisms to future work.*

5.1. The essential case. The following proposition is the dual of Proposition 4.1.1.

5.1.1. Proposition. *Let $f : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ be an essential geometric morphism induced by a functor $F : \mathcal{C} \to \mathcal{D}$. Then F has a factorization as an initial functor followed by a discrete opfibration, namely*

$$\mathcal{C} \to \int^{\mathcal{D}} g_!(1) \to \mathcal{D},$$

where $g : \mathbf{PSh}(\mathcal{C}^{\mathrm{op}}) \to \mathbf{PSh}(\mathcal{D}^{\mathrm{op}})$ is the essential geometric morphism induced by F^{op}. This is the unique such factorization up to equivalence of the intermediate category. Further, the induced factorization of f coincides with the (pure, complete spread) factorization of f.

Proof. The unique factorization of a functor into an initial functor followed by a discrete opfibration is due to Street and Walters [SW73]. Explicitly, we can obtain this factorization by applying the factorization from Proposition 4.1.1 to

$$F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$$

and then dualizing; recall that a functor is final if and only if its opposite functor is initial, and similarly a functor is a discrete fibration if and only if its opposite is a discrete opfibration. Recall also that we also defined the dual category of elements appearing in this factorization in Section 2.3. That this induces the (pure, complete spread) factorization at the level of geometric morphisms is given as Example 2.16(2) in [BF96]. ∎

Given this construction, we can immediately dualize the results of Section 4 to get the corresponding results for the (pure, complete spread) factorization of a geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup homomorphism $\phi : M \to N$. We first introduce the dual of Definition 4.1.2.

5.1.2. Definition. *A non-empty subset S of a monoid M is called **left-factorable** if whenever $x \in M$ and $y \in S$ with $xy \in S$, then $x \in S$. For an arbitrary subset T, we define $\langle\langle T \rangle_M \subseteq M$ to be the smallest left-factorable submonoid of M containing T, and call this the submonoid of M **left-factorably generated** by T.*

For a geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup homomorphism $\phi : M \to N$, we constructed in Section 4 a factorization

$$\begin{array}{ccccc}
\mathbf{PSh}(M) & \xrightarrow{\text{tc surj.}} & \mathbf{PSh}(\langle\phi(M)\rangle\rangle_{eNe}) & & \\
 & & \big\downarrow{\scriptstyle \text{tc incl.}} & & \\
 & & \mathbf{PSh}(\langle\phi(M)\rangle\rangle_{N}) & \xrightarrow{\text{tc incl.}} & \mathbf{PSh}(\int_N 1 \otimes_M eN) \\
 & & & & \big\downarrow{\scriptstyle \text{étale}} \\
 & & & & \mathbf{PSh}(N).
\end{array}$$

If we apply this factorization to $\phi^{\mathrm{op}} : M^{\mathrm{op}} \to N^{\mathrm{op}}$ and then take opposites, then we get the factorization

$$\begin{array}{ccccc}
\mathbf{PSh}(M) & \xrightarrow{\text{pure surj.}} & \mathbf{PSh}(\langle\langle\phi(M)\rangle_{eNe}) & & \\
 & & \big\downarrow{\scriptstyle \text{pure incl.}} & & \\
 & & \mathbf{PSh}(\langle\langle\phi(M)\rangle_{N}) & \xrightarrow{\text{pure incl.}} & \mathbf{PSh}(\int^N eN \otimes_M 1) \\
 & & & & \big\downarrow{\scriptstyle \text{complete spread}} \\
 & & & & \mathbf{PSh}(N).
\end{array} \tag{15}$$

Here we have made use of the following equalities:

$$(\langle\phi(M^{\mathrm{op}})\rangle\rangle_{N^{\mathrm{op}}})^{\mathrm{op}} = \langle\langle\phi(M)\rangle_N, \qquad \left(\langle\phi(M^{\mathrm{op}})\rangle\rangle_{(eNe)^{\mathrm{op}}}\right)^{\mathrm{op}} = \langle\langle\phi(M)\rangle_{eNe},$$
$$\left(\int_{N^{\mathrm{op}}} Y\right)^{\mathrm{op}} = \int^N Y, \qquad 1 \otimes_{M^{\mathrm{op}}} X = X \otimes_M 1,$$

with Y a left N-set and X a right M-set. We can deduce from this the dual to Corollary 4.1.4.

5.1.3. Corollary. *An essential geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ induced by a semigroup homomorphism $\phi : M \to N$ is pure if and only if $\langle\langle\phi(M)\rangle_N = N$.*

Dualizing the argument preceding Proposition 4.1.9, we deduce that essential complete spreads induced by semigroup homomorphisms are surjective. This produces the following dual to Theorem 4.1.12.

5.1.4. Theorem. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be an essential geometric morphism induced by a monoid homomorphism $\phi : M \to N$. Then the following are equivalent:*

1. *f is a complete spread;*
2. *ϕ is injective, $\phi(M) \subseteq N$ is left-factorable and for any $n \in N$ there is some $v \in N^{\rtimes}$ such that $vn \in \phi(M)$.*

More generally, if ϕ is merely a semigroup homomorphism, then f is a complete spread if and only if the monoid homomorphism part of ϕ satisfies the conditions above, and the inclusion $eNe \subseteq N$ induces an equivalence, where $e = \phi(1)$.

The condition that for any $n \in N$ there is some $v \in N^{\rtimes}$ such that $vn \in \phi(M)$, corresponds to the essential inclusion $\mathbf{PSh}(\langle\langle \phi(M) \rangle_N) \to \mathbf{PSh}(\int^N eN \otimes_M 1)$ being an equivalence. Further, the condition that ϕ is injective with $\phi(M) \subseteq N$ left-factorable, corresponds to the condition that the pure surjection part $\mathbf{PSh}(M) \to \mathbf{PSh}(\langle\langle \phi(M) \rangle_N)$ is an equivalence. The geometric morphisms such that the pure surjection part is an equivalence are precisely the spreads, so:

5.1.5. Proposition. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be an essential geometric morphism induced by a monoid homomorphism $\phi : M \to N$. Then f is a spread if and only if ϕ is injective and $\phi(M) \subseteq N$ is left-factorable.*

More generally, if f is induced by a semigroup homomorphism $\phi : M \to N$, then the pure surjection part is given by $\mathbf{PSh}(M) \to \mathbf{PSh}(\langle\langle \phi(M) \rangle_{eNe})$, as shown in (15). Again, f is a spread if and only if the pure surjection part is an equivalence, so if and only if ϕ is injective and $\phi(M) \subseteq eNe$ is left-factorable.

Finally, we give an updated version of Example 4.1.14.

5.1.6. Example.

1. For $H \subseteq G$ an inclusion of groups, we have that the induced geometric morphism $\mathbf{PSh}(H) \to \mathbf{PSh}(G)$ is both étale and a complete spread.

2. Consider the monoid $\mathbb{Z}_p^{\mathrm{ns}}$ of nonzero p-adic integers under multiplication. Then the inclusion $\mathbb{N} \to \mathbb{Z}_p^{\mathrm{ns}}$, $k \mapsto p^k$ induces an essential geometric morphism $\mathbf{PSh}(\mathbb{N}) \to \mathbf{PSh}(\mathbb{Z}_p^{\mathrm{ns}})$ that is both étale and complete spread.

In general, an étale geometric morphism $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is not necessarily a complete spread (and vice versa). In Subsection 6.4, we give an extreme example of a geometric morphism which is both terminal-connected and a complete spread (so its étale part is an equivalence), and dually, an example of a morphism which is both pure and étale (so its complete spread part is an equivalence).

5.1.7. Remark. *When we apply the dualization procedure employed in this section to the (surjection, inclusion) and (hyperconnected, localic) factorization systems of Section 3, we find (by inspection of Corollaries 3.1.2 and 3.1.3) that we obtain the same result as factorizing directly. In this sense, those factorization systems are self-dual.*

5.2. The general case. As a consequence of Theorem 5.0.12 due to Bunge–Funk, the (pure, complete spread) factorization works for general geometric morphisms with locally connected domain. In the following, we describe the construction of this factorization in detail, following [BF96] (with the same or similar notation), in the special case of presheaf toposes.

In this case, the comprehensive factorization discussed in the beginning of this section simplifies a lot. Going back to diagram (12), if $\mathcal{F}$ is a presheaf topos, then we can take $\mathcal{F} \simeq \mathbf{PSh}(\mathcal{D})$, with the geometric morphism i in (12) the identity geometric morphism. The pullback morphism $\pi : \mathbf{D}_\chi \to \mathbf{PSh}(\mathcal{C})$ is then an equivalence as well. So the comprehensive factorization is of the form

$$\mathcal{E} \xrightarrow{\ \nu\ } \mathbf{PSh}\left(\int^{\mathcal{D}} e_! \circ \varphi^* \circ \mathbf{y}\right) \longrightarrow \mathbf{PSh}(\mathcal{D}), \tag{16}$$

see (13), with the right-hand factor induced by the projection functor $\int^{\mathcal{D}} e_! \circ \varphi^* \circ \mathbf{y} \longrightarrow \mathcal{D}$.

We now restrict our attention to geometric morphisms

$$f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$$

between toposes of presheaves on monoids M and N. The geometric morphism f is given by a $[N, M)$-set A, in the sense that $f^*(X) \simeq X \otimes_N A$, see Theorem 2.2.3.

In this case, the geometric morphism ϕ in (16) is given by f, and e in (16) corresponds to the global sections geometric morphism $\gamma_M : \mathbf{PSh}(M) \to \mathbf{Set}$. We have $\gamma_{M,!}(X) \simeq X \otimes_M 1$ for any right M-set X, so we find that $\gamma_{M,!} \circ f^*$ is given by tensoring with the left N-set $A \otimes_M 1$. So the comprehensive factorization (16) is of the form

$$\mathbf{PSh}(M) \xrightarrow{\ \nu\ } \mathbf{PSh}\left(\int^{N} A \otimes_M 1\right) \xrightarrow{\ \pi\ } \mathbf{PSh}(N), \tag{17}$$

with π induced by the projection functor $\int^N A \otimes_M 1 \longrightarrow N$.

We can further describe ν via its associated flat functor $\nu^* \circ \mathbf{y}$, see the discussion around (14). In this special case, we find:

5.2.1. Proposition. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be a geometric morphism, determined by a $[N, M)$-set A via Theorem 2.2.3. The pure part $\nu : \mathbf{PSh}(M) \to \mathbf{PSh}(\int^N A \otimes_M 1)$ is determined by the flat functor*

$$V : \int^N A \otimes_M 1 \longrightarrow \mathbf{PSh}(M)$$

with $V(c)$ the connected component of A, as right M-set, corresponding to the element $c \in A \otimes_M 1$. For $n : c \to n \cdot c$ a morphism in $\int^N A \otimes_M 1$, $V(n)$ is the morphism $V(c) \to V(n \cdot c)$, $x \mapsto nx$.

From the above discussion, we can also deduce:

5.2.2. Proposition. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be a complete spread, for monoids M and N. Then f is essential.*

So complete spreads $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ are completely characterized by Theorem 5.1.4. Note that from the work of Bunge and Funk it follows more generally that any complete spread between presheaf toposes is essential, see the factorization (16). We shall see that this is not the case for pure geometric morphisms.

In the setting of this paper, it is natural to ask when the intermediate topos $\mathbf{PSh}(\int^N A \otimes_M 1)$ in (17) is equivalent to $\mathbf{PSh}(B)$ for some monoid B.

5.2.3. PROPOSITION. *Let* $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ *be the geometric morphism determined by a* $[N, M)$*-set* A. *Then* $\mathbf{PSh}(\int^N A \otimes_M 1) \simeq \mathbf{PSh}(B)$ *for some monoid* B *if and only if there is some* $c \in A \otimes_M 1$ *such that for every* $c' \in A \otimes_M 1$ *there is some* $v \in N^{\rtimes}$ *such that* $vc' = c$. *In this case, the (pure, complete spread) factorization has the following more concrete description. We can take* $B = \{n \in N : nc = c\}$. *Let* $A' \subseteq A$ *be the component of* A *corresponding to* c. *Then the left* N*-action on* A *restricts to a left* B*-action on* A', *and* A' *is flat as left* B*-set. The geometric morphism* $\nu : \mathbf{PSh}(M) \to \mathbf{PSh}(B)$ *from* (17) *is determined by the* $[B, M)$*-set* A', *and the (essential) geometric morphism* $\pi : \mathbf{PSh}(B) \to \mathbf{PSh}(N)$ *is induced by the inclusion of monoids* $B \subseteq N$.

PROOF. By Lemma 2.4.5, there is an equivalence $\mathbf{PSh}(\int^N A \otimes_M 1) \simeq \mathbf{PSh}(B)$ for some monoid B if and only if there is an object c of $\int^N A \otimes_M 1$ of which every object in $\int^N A \otimes_M 1$ is a retract. Now c' is a retract of c if and only if there are $u, v \in N$ with $uv = 1$ and $vc' = c$ and $uc = c'$ (the last equation follows from the first two). The stated conditions follow, with B the endomorphism monoid of c in $\int^N A \otimes_M 1$.

The equivalence $\mathbf{PSh}(B) \to \mathbf{PSh}(\int^N A \otimes_M 1)$ is induced by the inclusion of B as a full subcategory of $\int^N A \otimes_M 1$ (as the endomorphism monoid of c). If we compose this with the projection to N, then we get the monoid inclusion $B \subseteq N$. By the construction of the (pure, complete spread) factorization, the complete spread part $\pi : \mathbf{PSh}(B) \to \mathbf{PSh}(N)$ is the geometric morphism induced by this monoid inclusion $B \subseteq N$. We now consider the pure part $\nu : \mathbf{PSh}(M) \to \mathbf{PSh}(B)$. It is determined by the $[B, M)$-set $\nu^*(B)$ via Theorem 2.2.3. Through the equivalence $\mathbf{PSh}(B) \simeq \mathbf{PSh}(\int^N A \otimes_M 1)$, B corresponds to the object $\mathbf{y}(c)$. By Proposition 5.2.1, we find $\nu^*(B) = A'$, with the left B-action on A' being the restriction of the left N-action on A; this is well-defined since $B = \{n \in N : nc = c\}$. ■

The form of the pure part should come as no surprise after the following characterization.

5.2.4. PROPOSITION. *Let* $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ *be the geometric morphism corresponding to the* $[N, M)$*-set* A. *Then* f *is pure if and only if* A *is connected as a right* M*-set.*

PROOF. Since $\mathbf{PSh}(M)$ and $\mathbf{PSh}(N)$ are locally connected, f is pure if and only if f_* preserves small coproducts, see Lemma 5.0.5. We have $f_* \simeq \mathrm{Hom}_M(A, -)$, which preserves small coproducts if and only if A is connected as right M-set. ■

We can apply this in particular to a geometric morphism f induced by a semigroup morphism $\phi : M \to N$, where $A = Ne$ with $e = \phi(1)$ as in Lemma 2.2.4. We then find that f is pure if and only if Ne is connected as right M-set, i.e. if and only if $Ne \otimes_M 1 \simeq 1$. This provides an alternative route to Corollary 5.1.3. Indeed, by dualizing Lemma 4.1.3 we see that $Ne \otimes_M 1 \simeq 1$ if and only if $\langle\langle \phi(M) \rangle\rangle_N = N$.

For the sake of completeness, we also characterize spreads and injections by extending the argument we saw in Proposition 3.2.1.

5.2.5. SCHOLIUM. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be the geometric morphism corresponding to the $[N, M)$-set A. Then f is a spread if and only if M is a retract of some connected component of A, as a right M-set. Moreover, f is an injection if and only if M is a retract of A.*

PROOF. We simply replace 'subobject' with 'complemented subobject' and 'object', respectively, in the proof of Proposition 3.2.1. ∎

Note that M is a retract of a connected component of A if and only if this connected component generates $\mathbf{PSh}(M)$, see [KKM00, II, Theorem 3.16].

We can further combine Proposition 5.2.4 and Scholium 5.2.5 to give a characterization of pure spreads, or equivalently by Remark 5.0.18, pure inclusions.

5.2.6. COROLLARY. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be the geometric morphism corresponding to the $[N, M)$-set A. Then f is a pure spread (or equivalently, a pure inclusion) if and only if A is connected as a right M-set and has M as a retract.*

5.2.7. EXAMPLE. Consider the $[\mathbb{N}, \mathbb{Z})$-set $\mathbb{Z}$, with the left and right action given by addition. Here $\mathbb{Z}$ is connected as a right $\mathbb{Z}$-set and there is an epimorphism of right $\mathbb{Z}$-sets $\mathbb{Z} \to \mathbb{Z}$ (the identity map). So the geometric morphism $\mathbf{PSh}(\mathbb{Z}) \to \mathbf{PSh}(\mathbb{N})$ described by the $[\mathbb{N}, \mathbb{Z})$-set $\mathbb{Z}$ is a pure inclusion.

More generally, let $\phi : N \to Z$ be a monoid map such that Z is flat as a left N-set. Equivalently, ϕ is flat as a functor, see [Bén96, 4.7]. Then the geometric morphism $\mathbf{PSh}(Z) \to \mathbf{PSh}(N)$ described by the $[N, Z)$-set Z is a pure inclusion.

5.2.8. REMARK. *Since we have established that the (terminal-connected, étale) factorization is dual to the (pure, complete spread) factorization, one might wonder why we did not simply dualize the construction of the latter factorization in this section in order to obtain the former factorization for arbitrary geometric morphisms. The reason is that 'reversing' an $[N, M)$-set produces a $(M^{\mathrm{op}}, N^{\mathrm{op}}]$-set, and hence a distribution going in the opposite direction. This distribution can be factorized via the dual of the construction above, but the result cannot in general be dualized back to a factorization of the original geometric morphism. Alternatively, one can directly observe that the conditions of Lemma 4.2.1 and Proposition 5.2.4 are not dual to one another.*

6. Comparing étale and complete spread geometric morphisms

In Example 5.1.6, we saw examples of geometric morphisms which are both étale and complete spreads. In this section, we examine the relationship between these classes of morphism in more detail, first in general and then applied to our case of interest.

6.1. LOCALLY CONSTANT ÉTALE MORPHISMS. By definition, objects of a topos $\mathcal{E}$ correspond (up to equivalence of domain toposes) to étale geometric morphisms with codomain $\mathcal{E}$. The most basic kind of étale maps are the constant étale maps. These correspond to the objects A with $A = \coprod_{i \in I} 1$ a disjoint union of copies of the terminal object; these are called the **constant objects**, and when $\mathcal{E}$ is a Grothendieck topos they can equivalently be expressed as being of the form $p^*(I)$, where p is the global sections morphism of $\mathcal{E}$. The corresponding étale geometric morphism is equivalent to the codiagonal morphism $\coprod_{i\in I}\mathcal{E} \to \mathcal{E}$, where $\coprod_{i\in I}\mathcal{E}$ denotes the coproduct of I copies of $\mathcal{E}$ in the category of toposes (beware that a *colimit* in the bicategory of Grothendieck toposes is constructed as the *limit* of the corresponding diagram of inverse image functors in the bicategory of cocomplete categories).

For objects A and U we say that A is **trivialized** by U if there is a commutative diagram

$$\begin{array}{ccc} A\times U & \xrightarrow[\cong]{\psi} & \coprod_{i\in I} U \\ & \searrow \qquad \swarrow & \\ & U & \end{array}$$

with ψ an isomorphism, where the diagonal maps are the evident projection and codiagonal map.

6.1.1. Definition. *An object A of a topos $\mathcal{E}$ is said to be* ***locally constant*** *if there is a family of objects $\{U_k\}_{k\in K}$ whose morphisms to the terminal object are jointly epimorphic such that A is trivialized by each of the U_k; we call $\{U_k\}_{k\in K}$ a* ***trivializing family*** *for A. An étale geometric morphism with codomain $\mathcal{E}$ is called* ***locally constant étale*** *if it is (up to equivalence of the domain) of the form $\mathcal{E}/A \to \mathcal{E}$ with A locally constant.*

Let S be a generating family for $\mathcal{E}$. If $\{U_k\}_{k\in K}$ is a trivializing family for an object A, then any refinement of $\{U_k\}_{k\in K}$ is again a trivializing family for A. So if we can find a trivializing family, then we can also find a trivializing family where each U_k is contained in S.

In particular, a locally constant object in the topos $\mathbf{Sh}(X)$, for X a topological space, admits a trivializing family of the form $\{U_k\}_{k\in K}$ with each U_k given by an open set of X. For an object A in $\mathbf{Sh}(X)$, the associated étale geometric morphism $\mathbf{Sh}(X)/A \to \mathbf{Sh}(X)$ is of the form $\mathbf{Sh}(Y) \to \mathbf{Sh}(X)$, induced by a local homeomorphism $\phi : Y \to X$, see [sga72, Exposé IV, §5.7]. The condition that A is locally constant then translates to the condition that ϕ is a covering map. So the locally constant étale geometric morphisms with codomain $\mathbf{Sh}(X)$ are precisely those of the form $\mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ induced by a covering map $Y \to X$.

We have seen that all étale morphisms over presheaf toposes are induced by discrete fibrations (Proposition 4.1.1), and dually that complete spreads are induced by discrete opfibrations (Proposition 5.1.1).

6.1.2. Definition. *A functor $F : \mathcal{C} \to \mathcal{D}$ is a* ***discrete bifibration*** *if it is both a discrete fibration and a discrete opfibration.*

Locally constant étale geometric morphisms to presheaf toposes are characterized as in the following proposition. The equivalence (1) $\Leftrightarrow$ (4) is proven by Bunge and Funk [BF98, Corollary 7.9] for connected presheaf toposes, but note that we do not require connectedness in our proof.

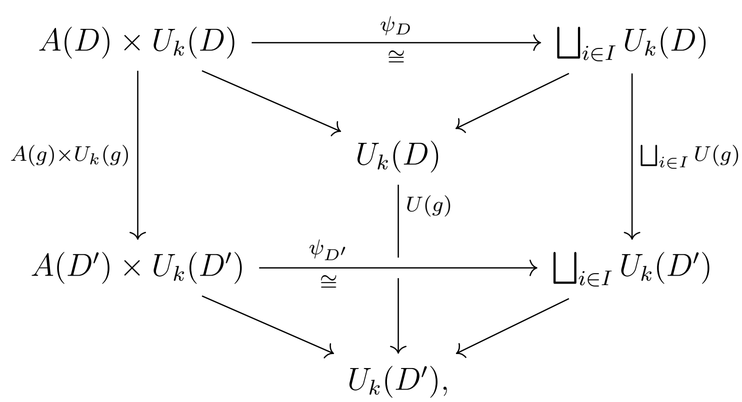
6.1.3. Proposition. *For a presheaf A on a small category $\mathcal{D}$, the following are equivalent:*

1. $\mathbf{PSh}(\mathcal{D})/A \to \mathbf{PSh}(\mathcal{D})$ *is locally constant étale,*
2. *A is locally constant as an object of* $\mathbf{PSh}(\mathcal{D})$*,*
3. *Given any morphism $g : D' \to D$ of $\mathcal{D}$, $A(g)$ is an isomorphism,*
4. *The discrete fibration $\int_{\mathcal{D}} A \to \mathcal{D}$ is a discrete bifibration,*
5. *The étale geometric morphism* $\mathbf{PSh}(\mathcal{D})/A \to \mathbf{PSh}(\mathcal{D})$ *is a complete spread.*

Proof. The equivalence (2) ⇔ (3) follows from a more general result by Leroy in [Ler79, Proposition 2.2.1]; we give a simplified argument in this special case.

(1) ⇔ (2) By definition.

(2) ⇒ (3) Given a trivializing family $\{U_k\}_{k\in K}$ for A which is jointly epimorphic over 1, there must in particular be some $k \in K$ such that $U_k(D) \neq \emptyset$, whence also $U_k(D') \neq \emptyset$. Consider the naturality diagram,



where I is fixed. Let $u \in U_k(D)$ and $u' := U_k(g)(u)$. Given $x, y \in A(D)$ with $A(g)(x) = A(g)(y) = z$, say, we write $\psi_D(x, u) = (i, u)$ and $\psi_D(y, u) = (j, u)$ for certain indices $i, j \in I$. We have $\psi_{D'}(z, u') = (i, u')$, and similarly $\psi_{D'}(z, u') = (j, u')$, so $i = j$ and as a result $x = y$. Conversely, given $z \in A(D')$, we write $\psi_{D'}(z, u') = (i, u')$ for some index $i \in I$. Now take $x \in A(D)$ with $\psi_D(x, u) = (i, u)$. It follows that $A(g)(x) = z$.

(3) ⇒ (2) For A satisfying the given condition, let us take $\{\mathbf{y}(D)\}_{D\in\mathrm{ob}(\mathcal{D})}$ as our set of trivializing objects. We have an isomorphism $\mathbf{y}(D) \times A \to \coprod_{a\in A(D)} \mathbf{y}(D)$ which at an object D' is defined by

$$\mathrm{Hom}(D', D) \times A(D') \to \coprod_{a\in A(D)} \mathrm{Hom}(D', D)$$
$$(g, a) \mapsto (A(g)(a), g),$$

which is a bijection since $A(g)$ is.

(3) ⇔ (4) We already know that $\int_{\mathcal{D}} A \to \mathcal{D}$ is a discrete fibration, so it suffices to check whether it is a discrete opfibration. Given an object (D, a) of $\int_{\mathcal{D}} A$ and a morphism $g : D \to D'$ in $\mathcal{D}$, we know that $A(g)$ is a bijection, so we have a unique lifting of g to the morphism $g : (D, a) \to (D', A(g)^{-1}(a))$, as required. Conversely, if the projection is a discrete opfibration, then each $a' \in A(D')$ has a unique pre-image along $A(g)$ for any g, so $A(g)$ is a bijection.

(4) ⇔ (5) Recall from Proposition 2.3.1 that the projection $F : \int_{\mathcal{D}} A \to A$ induces the étale morphism $\mathbf{PSh}(\mathcal{D})/A \to \mathbf{PSh}(\mathcal{D})$ up to equivalence of the domain. The equivalence then follows from the fact that a functor induces a complete spread if and only if it is a discrete opfibration up to equivalence of the domain category (by Proposition 5.1.1); F being a discrete fibration forces this equivalence to be an isomorphism. ■

This result applies in particular to the case where $\mathcal{D}$ is a monoid. We can combine it with the characterizations of Theorems 4.1.12 and 5.1.4 to deduce the following.

6.1.4. Corollary. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be an essential geometric morphism induced by a monoid homomorphism $\phi : M \to N$. Then the following are equivalent:*

1. *f is locally constant étale;*
2. *ϕ is injective, $\phi(M) \subseteq N$ is both left-factorable and right-factorable, and for any $n \in N$ there are elements $u \in N^{\ltimes}$, $v \in N^{\rtimes}$ such that $nu \in \phi(M)$ and $vn \in \phi(M)$.*

More generally, if ϕ is merely a semigroup homomorphism, then f is locally constant étale if and only if the monoid homomorphism part of ϕ satisfies the conditions above, and the inclusion $eNe \subseteq N$ induces an equivalence, where $e = \phi(1)$.

It follows from this corollary that if N is commutative, then f is étale if and only if it is a complete spread, if and only if it is locally constant étale. Independently, if N is a group, then any inclusion of a subgroup into N induces a locally constant étale morphism.

6.2. Étale geometric morphisms with fixed codomain. While the abstract characterizations of subsemigroups inducing étale geometric morphisms and complete spreads from the previous sections are useful for recognizing these properties, classifying such morphisms of the form $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ for a fixed monoid N can still be challenging. Here we explore a different approach in terms of N-sets.

We can use Lemma 2.4.5 and Proposition 2.3.1 to identify objects X of $\mathbf{PSh}(N)$ such that

$$\mathbf{PSh}(N)/X \simeq \mathbf{PSh}(M)$$

for some monoid M; namely, this happens if there is some object $x \in \int_N X$ of which every object is a retract. Given an element $y \in X$, y is a retract of x as an object of $\int_N X$ if and only if $\exists u \in N^{\ltimes}$ with $x = yu$. Letting v be the right inverse of u, this can be expressed in the following diagram:

$$vu \circlearrowright x \underset{u}{\overset{v}{\rightleftarrows}} y \circlearrowleft \mathrm{id}\,.$$

6.2.1. DEFINITION. *Let N be a monoid and let X be a right N-set. An element $x \in X$ will be called a* ***strong generator*** *if for all $y \in X$ there is an element $u \in N^{\ltimes}$ such that $yu = x$.*

Note that a strong generator is in particular a generator, so a right N-set that admits a strong generator is cyclic. If x is a strong generator and $u \in N^{\ltimes}$, then xu is again a strong generator.

6.2.2. THEOREM. *Let N be a monoid. Fix a right N-set X and a strong generator $x \in X$, and set $N_x = \{n \in N : xn = x\}$. Then the inclusion $N_x \subseteq N$ induces an étale geometric morphism*

$$\mathbf{PSh}(N_x) \longrightarrow \mathbf{PSh}(N).$$

Conversely, every étale geometric morphism $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is of this form (up to precomposition with an equivalence).

PROOF. The étale geometric morphisms with codomain $\mathbf{PSh}(N)$ are precisely the geometric morphisms $\mathbf{PSh}(\int_N X) \to \mathbf{PSh}(N)$ for some right N-set X. Further, from the above we see that $\mathbf{PSh}(\int_N X) \simeq \mathbf{PSh}(M)$ for some monoid M if and only if X has a strong generator x, and in this case we can take M to be N_x, which is the endomorphism monoid of x in $\int_N X$. ∎

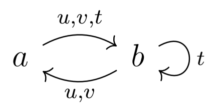
Alternatively, one direction of the statement can be deduced from Theorem 4.1.12.

6.2.3. REMARK. *Suppose $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is an étale geometric morphism induced by a semigroup homomorphism $\phi : M \to N$. We have already seen that f must be surjective: by the argument before Proposition 4.1.11, if $e := \phi(1)$ then the inclusion $eNe \subseteq N$ must induce an equivalence. In other words, we can replace N with the Morita equivalent monoid eNe to obtain a monoid homomorphism (factoring ϕ) inducing f. On the other hand, using Theorem 6.2.2 we see that we may instead replace the domain monoid M by a Morita equivalent monoid M' such that there is a monoid morphism $\phi' : M' \to N$ inducing f.*

This does not work for arbitrary surjective geometric morphisms. For example, take N a monoid with a nontrivial idempotent $e \in N$, and consider the semigroup map $\phi : 1 \to N$ with $\phi(1) = e$. This induces a geometric morphism $f : \mathbf{Set} \to \mathbf{PSh}(N)$ which is surjective whenever the inclusion $eNe \subseteq N$ induces an equivalence, by Corollary 3.1.2. However, there are no monoids Morita equivalent to 1 (other than 1 itself), so it is impossible for f to be induced by a monoid map $M' \to N$ for some monoid M'.

6.2.4. EXAMPLE. If we have a monoid N, a right N-set X, and strong generators $x, x' \in X$, then N_x and $N_{x'}$ are Morita equivalent, since both $\mathbf{PSh}(N_x)$ and $\mathbf{PSh}(N_{x'})$ are equivalent to $\mathbf{PSh}(\int_N X)$. We now show with an example that N_x and $N_{x'}$ are not necessarily isomorphic.

Let $N = \langle u, v, t : uv = 1, t^2 = t \rangle$ and $X = \{a, b\}$. Consider the right N-action on X defined on generators as $a \cdot u = a \cdot v = a \cdot t = b \cdot t = b$ and $b \cdot u = b \cdot v = a$.



Clearly, both a and b are strong generators, so the endomorphism monoids N_a and N_b in $\int_N X$ are Morita equivalent, since both present the topos $\mathbf{PSh}(N)/X$.

We shall show that $N_a \not\cong N_b$ by examining the idempotents of the involved monoids. First, observe that every element of N can be reduced to the canonical form $v^{l_0}u^{k_0}tv^{l_1}u^{k_1}t\cdots tv^{l_n}u^{k_n}$, where $k_i, l_i, n \geq 0$ and $k_i + l_i \geq 1$ for each $1 \leq i \leq n-1$; call n the *breadth* of the element. Squaring the canonical form expression, the result will have breadth $2n$ unless $k_n = l_0$ and $k_0 = l_n = 0$, in which case the breadth will be $2n - 1$. From these possibilities, we conclude that any idempotent element of N can be written as either $v^k u^k$ or $v^k t u^k$ for some $k \in \mathbb{N}$. The idempotents lying in N_a are all those of the form $v^k u^k$, plus those of the form $v^{2i+1}tu^{2i+1}$ for some $i \in \mathbb{N}$. On the other hand, N_b contains the idempotents of the form $v^k u^k$ and $v^{2j}tu^{2j}$ for $j \in \mathbb{N}$. In N_a there is an idempotent $e = vu \neq 1$ such that $ef = f$ for all other idempotent $f \neq 1$ in N_a. In N_b there is no idempotent with this property, simply because $t \in N_b$. Indeed, from $et = t$ and $e \neq 1$ it would follow that $e = t$, but this idempotent does not qualify because $tvu \neq vu$ (indeed, tvu is not even idempotent).

6.3. Complete spreads with fixed codomain. We can dualize the results from the previous subsection to complete spreads.

6.3.1. Definition. *Let N be a monoid and let Y be a left N-set. An element $y \in Y$ will be called a* ***strong generator*** *if for all $x \in Y$ there is an element $v \in N^{\rtimes}$ such that $vx = y$.*

6.3.2. Theorem. *Let N be a monoid. Fix a left N-set Y and a strong generator $y \in Y$, and set $N^y = \{n \in N : ny = y\}$. Then the inclusion $N^y \subseteq N$ induces a complete spread*

$$\mathbf{PSh}(N^y) \longrightarrow \mathbf{PSh}(N).$$

Conversely, every complete spread $\mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is of this form (up to precomposition with an equivalence).

6.3.3. Remark. *The dual of Remark 6.2.3 holds here. If $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is a complete spread, then we can assume that it is induced by a monoid map $M \to N$, either by replacing N with a Morita equivalent monoid (using that f is surjective), or by replacing M with a Morita equivalent monoid (using Theorem 6.3.2).*

Observe that while we were able to characterize locally constant étale geometric morphisms in terms of subsemigroups in Corollary 6.1.4, we cannot combine Theorems 6.2.2 and 6.3.2 so easily since they refer to fundamentally different objects (right and left N-sets, respectively). We can instead employ the characterization from Proposition 6.1.3 to deduce the following.

6.3.4. COROLLARY. *Let X be a right N-set with a strong generator x. Then the induced étale geometric morphism $\mathbf{PSh}(N_x) \to \mathbf{PSh}(N)$ is locally constant étale if and only if N acts on X by automorphisms, meaning that for each element $n \in N$, the map $(- \cdot n) : X \to X$ is a bijection. Dually, if Y is a left N-set with a strong generator y, then the induced complete spread $\mathbf{PSh}(N^y) \to \mathbf{PSh}(N)$ is locally constant étale if and only if N acts on Y by automorphisms.*

6.4. A MATRIX MONOID EXAMPLE. In this subsection, we deliver on our promise at the end of Subsection 5.1: we give an example of a monoid homomorphism such that the induced geometric morphism is both terminal-connected and a complete spread. After dualizing, this additionally gives an example where the induced geometric morphism is both pure and étale.

The example is inspired by some of the literature on the Arithmetic Site of Connes and Consani [CC14], [CC19], [Hem19], [LB].

For a prime number p, consider the monoid

$$Q_p = \left\{ \begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix} : n \in \mathbb{N},\ k \in \mathbb{Z} \right\}$$

under matrix multiplication, and the submonoid

$$F_p = \left\{ \begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix} : n, k \in \mathbb{N},\ 0 \leq k < p^n \right\} \ \subseteq \ Q_p.$$

Here we think of $\mathbf{PSh}(F_p)$ as corresponding to the prime p part of Conway's site as introduced in [LB]. Further, Q_p is the prime p part of (the opposite of) the $(ax + b)$-monoid, which is related to the study of parabolic $\mathbb{Q}$-lattices, see [CC19]. The topos $\mathbf{PSh}(Q_p)$ is the prime p part of the topos associated to the $(ax + b)$-monoid, as studied in [Hem19, §2.5].

6.4.1. PROPOSITION. *The monoid F_p is free, with as generators the matrices*

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} p & 0 \\ 1 & 1 \end{pmatrix}, \ \dots \begin{pmatrix} p & 0 \\ p-1 & 1 \end{pmatrix}.$$

PROOF. For $n \in \mathbb{N}$, take natural numbers $a_0, \dots, a_{n-1}$ with $0 \leq a_i < p$ for each $i \in \{0, \dots, n-1\}$. We then calculate

$$\begin{pmatrix} p & 0 \\ a_{n-1} & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ a_{n-2} & 1 \end{pmatrix} \cdots \begin{pmatrix} p & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ a_0 & 1 \end{pmatrix} = \begin{pmatrix} p^n & 0 \\ \sum_{i=0}^{n-1} a_i p^i & 1 \end{pmatrix}.$$

The submonoid generated by the given matrices is then free, by the uniqueness of p-adic expansions. ∎

6.4.2. Proposition. *Let $f : \mathbf{PSh}(F_p) \to \mathbf{PSh}(Q_p)$ be the geometric morphism induced by the inclusion $F_p \subseteq Q_p$. Then f is terminal-connected and a complete spread.*

Proof. We first prove that f is terminal-connected. By Corollary 4.1.4, f is terminal-connected if and only if $\langle F_p \rangle\rangle_{Q_p} = Q_p$. Take arbitrary n and k, with $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Choose a natural number r such that $p^{n+r} + k \geq 0$ and then choose $m > r$ large enough such that $p^{n+r} + k < p^{m+n}$. We compute

$$\begin{pmatrix} p^m & 0 \\ p^r & 1 \end{pmatrix} \begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix} = \begin{pmatrix} p^{m+n} & 0 \\ p^{n+r} + k & 1 \end{pmatrix}.$$

The matrices $\begin{pmatrix} p^m & 0 \\ p^r & 1 \end{pmatrix}$ and $\begin{pmatrix} p^{m+n} & 0 \\ p^{n+r} + k & 1 \end{pmatrix}$ are both contained in F_p. It follows that $\begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix}$ is contained in the right-factorable closure $\langle F_p \rangle\rangle_{Q_p}$. Because n and k were arbitrary, we conclude that $\langle F_p \rangle\rangle_{Q_p} = Q_p$. So f is terminal-connected.

We now prove that f is a complete spread. By Theorem 5.1.4, it is enough to show that $F_p \subseteq Q_p$ is left-factorable, and that for any $x \in Q_p$ there is some $v \in Q_p^{\times}$ such that $vx \in F_p$. To show that $F_p \subseteq Q_p$ is left-factorable, we compute

$$\begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} p^m & 0 \\ l & 1 \end{pmatrix} = \begin{pmatrix} p^{n+m} & 0 \\ kp^m + l & 1 \end{pmatrix}.$$

We now have to show that if $0 \leq kp^m + l < p^{n+m}$ and $0 \leq l < p^m$, then $0 \leq k < p^n$. We leave it to the reader to verify this. Now take

$$x = \begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix} \in Q_p.$$

We have to find $v \in Q_p^{\times}$ such that $vx \in F_p$. Note that

$$Q_p^{\times} = Q_p^{\times} = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{Z} \right\}$$

and

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} p^n & 0 \\ k & 1 \end{pmatrix} = \begin{pmatrix} p^n & 0 \\ zp^n + k & 1 \end{pmatrix}.$$

So we need to find an integer z such that $0 \leq zp^n + k < p^n$. There is a unique such z, namely the smallest z with $0 \leq zp^n + k$. ∎

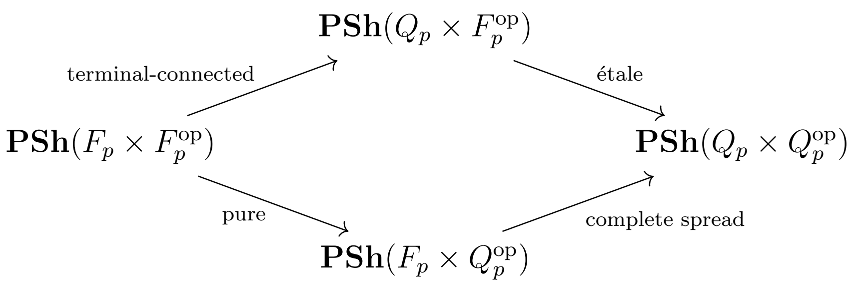
Because terminal-connected is dual to pure, and étale is dual to being a complete spread, we can dualize to get an example of a pure and étale geometric morphism.

6.4.3. Corollary. *Let $g : \mathbf{PSh}(F_p^{\mathrm{op}}) \to \mathbf{PSh}(Q_p^{\mathrm{op}})$ be the geometric morphism induced by the inclusion $F_p^{\mathrm{op}} \subseteq Q_p^{\mathrm{op}}$. Then g is pure and étale.*

Now consider the inclusion $F_p \times F_p^{\mathrm{op}} \subseteq Q_p \times Q_p^{\mathrm{op}}$, and let

$$h : \mathbf{PSh}(F_p \times F_p^{\mathrm{op}}) \to \mathbf{PSh}(Q_p \times Q_p^{\mathrm{op}})$$

be the induced essential geometric morphism. Then the (terminal-connected, étale) factorization and (pure, complete spread) factorization are given by



with each geometric morphism induced by the inclusion of submonoids. This gives an example of an essential geometric morphism where the (terminal-connected, étale) factorization and (pure, complete spread) factorization are both nontrivial and distinct from each other.

To verify that the diagram above gives the correct (terminal-connected, étale) and (pure, complete spread) factorizations, we can either use the characterizations of Corollary 4.1.4, Theorem 4.1.12, Proposition 5.2.4 and Theorem 5.1.4, or use the following shortcut:

6.4.4. Lemma. *Let $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ be the essential geometric morphism induced by a monoid map $\phi : M \to N$. For a monoid P, consider the monoid map $\phi_P : M \times P \to N \times P$ with $\phi_P(m,p) = (\phi(m), p)$. Let $f_P : \mathbf{PSh}(M \times P) \to \mathbf{PSh}(N \times P)$ be the geometric morphism induced by ϕ_P. If f is terminal-connected (resp. étale, pure, a complete spread), then f_P is terminal-connected (resp. étale, pure, a complete spread) as well.*

Proof. It is enough to prove the statement for terminal-connected or étale geometric morphisms; the statement for pure geometric morphisms and complete spreads then follows by dualization.

We can write $\mathbf{PSh}(N \times P) \simeq \mathbf{PSh}(N) \times \mathbf{PSh}(P)$, i.e. $\mathbf{PSh}(N \times P)$ is the product of $\mathbf{PSh}(N)$ and $\mathbf{PSh}(P)$ in the category of toposes, see [Joh77, Corollary 4.36]. If $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is étale, then so is $f_P : \mathbf{PSh}(M \times P) \to \mathbf{PSh}(N \times P)$, because étale geometric morphisms are stable under base change.

Now suppose that $f : \mathbf{PSh}(M) \to \mathbf{PSh}(N)$ is terminal-connected. By two applications of Lemma 2.2.4 and Proposition 5.2.4, we see that N is connected as left M-set, from which it follows that $N \times P$ is connected as left $(M \times P)$-set, so $f_P : \mathbf{PSh}(M \times P) \to \mathbf{PSh}(N \times P)$ is terminal-connected. ∎

7. Application: Galois theory

7.1. Background on Galois theory for toposes. For a locally connected topos $\mathcal{E}$, there is a well-known notion of Galois theory founded on the notion of locally constant object we gave in Definition 6.1.1. We recall how this works below, following [Zoo02], [Ler79], [BF98].

First, observe that if $\{A_i\}_{i\in I}$ is a family of locally constant objects, then in general the coproduct $\coprod_{i\in I} A_i$ need not be locally constant. For example, in the topos of continuous actions of the profinite integers, each action of the form $\mathbb{Z}/n\mathbb{Z}$ is locally constant and trivialized by itself, but the disjoint union $\coprod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is not locally constant, since there is no object which can trivialize all of these cycles at once. If we consider the full subcategory $\mathbf{SLC}(\mathcal{E}) \subseteq \mathcal{E}$ consisting of the objects that are disjoint unions of locally constant objects, however, it turns out that $\mathbf{SLC}(\mathcal{E})$ is again a topos and the functor including $\mathbf{SLC}(\mathcal{E})$ into $\mathcal{E}$ is the inverse image functor of a connected geometric morphism $g : \mathcal{E} \to \mathbf{SLC}(\mathcal{E})$; see [Ler79, Théorème 2.4.(i)]. Moreover, $\mathbf{SLC}(\mathcal{E})$ is a **Galois topos**, i.e. a 2-categorical cofiltered limit of toposes of the form $\mathbf{PSh}(G)$, with G a (discrete) groupoid; see [Zoo02, Théorème 1.1]. More precisely, the cofiltered limit is taken over the different sieves $\{\phi_k : U_k \to 1\}_{k\in K}$ in $\mathcal{E}$ that are covering sieves in the sense that the morphisms $\{\phi_k\}_k$ are jointly epimorphic. For each such covering sieve $\mathcal{U}$, the category of locally constant objects in $\mathcal{E}$ that are trivialized by $\mathcal{U}$, in the sense described before Definition 6.1.1, is equivalent to $\mathbf{PSh}(G_{\mathcal{U}})$ for a certain discrete groupoid $G_{\mathcal{U}}$ [Zoo02, Théorème 1.1]. Finally, the topos $\mathbf{SLC}(\mathcal{E})$ is then the inverse limit of the toposes $\mathbf{PSh}(G_{\mathcal{U}})$.

We say that a locally connected topos $\mathcal{E}$ is **locally simply connected** [BD81] if there exists a *single* covering sieve $\mathcal{U}$ which trivializes each locally constant object in $\mathcal{E}$. In this case, $\mathbf{SLC}(\mathcal{E}) \simeq \mathbf{PSh}(G_{\mathcal{U}})$ is itself a topos of presheaves on a (discrete) groupoid. For example, let X be a path-connected, locally path-connected, semilocally simply connected space. Then there exists an open covering $\bigcup_{k\in K} U_k = X$ by path-connected open subsets U_k such that each $\pi_1(U_k) \to \pi_1(X)$ is the zero map (this property is independent of the choice of basepoints). Now if Y is a covering space over X, then for each $k \in K$ the monodromy action of $\pi_1(U_k)$ on Y is trivial, so the restriction of Y to U_k is trivial. Recalling that the covering maps over X correspond to the locally constant objects in $\mathbf{Sh}(X)$, we see that the sieve generated by the subterminal objects $\{\phi_k : U_k \to 1\}_{k\in K}$ is a covering sieve that trivializes all locally constant objects, so the topos $\mathbf{Sh}(X)$ is locally simply connected as one would hope. In this case, we can identify $\mathbf{SLC}(\mathbf{Sh}(X))$ with the category of right actions of the fundamental group $\pi_1(X)$ (with the discrete topology), or with the category of covering spaces $Y \to X$.

We can also consider the small étale topos $\mathcal{E} = \mathrm{Spec}(K)_{\text{ét}}$ associated to a field K, which is equivalent to the topos $\mathbf{Cont}(\mathrm{Gal}(K^s/K))$ of continuous right $\mathrm{Gal}(K^s/K)$-sets, with $\mathrm{Gal}(K^s/K)$ the absolute Galois group of K [Sta22, Theorem 03QT].

We claim that $\mathcal{E} \simeq \mathbf{SLC}(\mathcal{E})$ in this case. Indeed, $\mathcal{E}$ is equivalent to the category of étale morphisms $X \to \mathrm{Spec}(K)$ with X a scheme [Sta22, Lemma 03QR]. The connected objects in the topos then correspond to morphisms of the form $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ with $K \subseteq L$ a separable field extension. If L' is the normal closure of L in K^s, then $L \otimes_K L' \cong$

$\prod_{i=1}^{n} L'$ with $n = [L : K]$. So on the geometric side, we see that $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is locally constant, trivialized by $\mathrm{Spec}(L')$. Any object is the sum of connected objects, corresponding to the decomposition into orbits in $\mathbf{Cont}(\mathrm{Gal}(K^s/K))$, and we just showed that any connected object is locally constant. We conclude that $\mathcal{E} \simeq \mathbf{SLC}(\mathcal{E})$.

7.2. Galois theory for toposes of monoid actions. We now apply the concepts above to the topos $\mathbf{PSh}(N)$ for a monoid N.

For every small category $\mathcal{C}$ there is a functor $\eta : \mathcal{C} \to \Pi(\mathcal{C})$ to a groupoid $\Pi(\mathcal{C})$, unique up to equivalence, such that every functor from $\mathcal{C}$ to a groupoid factors uniquely through η. Concretely, $\Pi(\mathcal{C})$ can be constructed as the groupoid with the same objects as $\mathcal{C}$ in which morphisms are equivalence classes of composites of morphisms and formal inverses of morphisms in $\mathcal{C}$. In the case that $\mathcal{C}$ is a monoid N, this construction produces a group, that we will call the **groupification** and denote by $\pi_1(N)$; for N commutative, $\pi_1(N)$ is known as the *Grothendieck group* of N.

We can deduce from Proposition 6.1.3 that in the special case of presheaf toposes, coproducts of locally constant objects are locally constant, so $\mathbf{SLC}(\mathbf{PSh}(\mathcal{C}))$ consists precisely of the locally constant objects, and moreover we can recover the result that $\mathbf{SLC}(\mathbf{PSh}(\mathcal{C})) \simeq \mathbf{PSh}(\Pi(\mathcal{C}))$ ([Ler79, Remark after Corollary 4.6.5]), by observing that the locally constant presheaves on $\mathcal{C}$ are precisely those which extend along $\eta : \mathcal{C} \to \Pi(\mathcal{C})$. The connected geometric morphism $\mathbf{PSh}(\mathcal{C}) \to \mathbf{SLC}(\mathbf{PSh}(\mathcal{C}))$ then agrees with the essential geometric morphism induced by the functor $\eta : \mathcal{C} \to \Pi(\mathcal{C})$. In particular, if N is a monoid, then we will in the remainder denote by $g : \mathbf{PSh}(N) \to \mathbf{PSh}(\pi_1(N))$ the essential geometric morphism induced by the homomorphism $N \to \pi_1(N)$. The locally constant objects in $\mathbf{PSh}(N)$ are precisely the objects of the form $g^*(X)$ for X in $\mathbf{PSh}(\pi_1(N))$, and hence a geometric morphism with codomain $\mathbf{PSh}(N)$ is locally constant étale if and only if it is of the form

$$\mathbf{PSh}(N)/g^*(X) \longrightarrow \mathbf{PSh}(N)$$

for X in $\mathbf{PSh}(\pi_1(N))$. In light of the discussion above, the following result should not be unexpected.

7.2.1. Corollary. *For any monoid N, $\mathbf{PSh}(N)$ is a locally simply connected topos.*

Proof. We show that N, as a right N-set, trivializes every locally constant object. Indeed, if A is locally constant then by Proposition 6.1.3 N acts by automorphisms on A, so the mapping

$$\begin{aligned} \coprod_{a \in A} N &\to A \times N \\ (a, n) &\mapsto (a \cdot n, n) \end{aligned}$$

is easily verified to be an isomorphism which commutes with the required maps. ∎

More generally, any connected presheaf topos is locally simply connected, see [BF98, Corollary 7.9]. The proof there works for general presheaf toposes as well.

We can rephrase Corollary 6.3.4 in terms of $\pi_1(N)$.

7.2.2. THEOREM. *Let N be a monoid and let X be an object of $\mathbf{PSh}(\pi_1(N))$. Let $g : \mathbf{PSh}(N) \to \mathbf{PSh}(\pi_1(N))$ be the geometric morphism induced by the groupification map $\eta : N \to \pi_1(N)$. Then the following are equivalent:*

1. *there is an equivalence $\mathbf{PSh}(N)/g^*(X) \simeq \mathbf{PSh}(M)$ for some monoid M;*
2. *there is a subgroup $H \subseteq \pi_1(N)$ such that $X \cong H\backslash\pi_1(N)$ (the latter being the set of right cosets of H) and for all $y \in \pi_1(N)$ there is some $u \in \eta(N^\ltimes)$ such that $yu \in H$.*

In this case, $M = \eta^{-1}(H)$, and

$$\mathbf{PSh}(N)/g^*(X) \simeq \mathbf{PSh}(M) \longrightarrow \mathbf{PSh}(N)$$

agrees with the essential geometric morphism induced by the inclusion $M \subseteq N$.

PROOF. We know from Theorem 6.2.2 that $\mathbf{PSh}(N)/g^*(X)$, being étale over $\mathbf{PSh}(N)$, is equivalent to $\mathbf{PSh}(M)$ for some monoid M if and only if $g^*(X)$ contains a strong generator, i.e. an element $x \in g^*(X)$ such that for all $y \in g^*(X)$ there is some $u \in N^\ltimes$ such that $yu = x$. This requires $g^*(X)$ to be connected as a right N-set, so *a fortiori* it must be connected (and hence transitive) as a right $\pi_1(N)$-set. Because $\pi_1(N)$ is a group, we may apply a version of the orbit-stabilizer theorem to deduce that X can be written as a quotient $X \cong H\backslash\pi_1(N)$, where $H \subseteq \pi_1(N)$ is the stabilizer of x. The condition that x is a strong generator can be reformulated by saying that for any $y \in \pi_1(N)$ there is some $u \in \eta(N^\ltimes)$ such that $yu \in H$, as required. Conversely, given a presentation of X as $H\backslash\pi_1(N)$ satisfying the given conditions, it follows that the coset $H1$ is a strong generator of X, and applying the formula from Theorem 6.2.2, a representing monoid M is then given by

$$N_x = \{n \in N : Hn = H\} = \eta^{-1}(H),$$

as required. ■

7.2.3. EXAMPLE. Consider the monoid $N = \mathbb{Z}_p^{\mathrm{ns}}$ of nonzero p-adic integers under multiplication. The groupification of $\mathbb{Z}_p^{\mathrm{ns}}$ is the group $\mathbb{Q}_p^*$ of nonzero p-adic rational numbers. Consider the subgroup $H = \{p^k : k \in \mathbb{Z}\} \subseteq \mathbb{Q}_p^*$. For all $g \in \mathbb{Q}_p^*$ there is an $u \in \mathbb{Z}_p^*$ such that $gu = p^k$ for some $k \in \mathbb{Z}$. So we are in the setting of Theorem 7.2.2. We find $M = N \cap H = \{p^k : k \in \mathbb{N}\} \cong \mathbb{N}$. So we see that the geometric morphism $\mathbf{PSh}(\mathbb{N}) \to \mathbf{PSh}(\mathbb{Z}_p^{\mathrm{ns}})$ induced by the inclusion $\mathbb{N} \to \mathbb{Z}_p^{\mathrm{ns}},\ k \mapsto p^k$ is not only étale (as we already saw in Example 4.1.14), but even locally constant étale. We can think of it as the covering map of $\mathbf{PSh}(\mathbb{Z}_p^{\mathrm{ns}})$ corresponding to the subgroup $H \subseteq \mathbb{Q}_p^*$.

Observe that if we are given a presentation of a monoid, we can easily compute the groupification by interpreting the same presentation as a presentation of a group.

7.2.4. Example. Consider the *bicyclic semigroup* B with presentation

$$B = \langle u, v : uv = 1 \rangle.$$

Every element in B can be written in a unique way as $v^i u^j$ for some $i, j \in \mathbb{N}$. The right-invertible elements are $B^{\ltimes} = \{u^k : k \in \mathbb{N}\}$ and the left-invertible elements are $B^{\rtimes} = \{v^k : k \in \mathbb{N}\}$. We find $\pi_1(B)$ is the group with the same presentation as B, which can be identified with $\mathbb{Z}$, taking u as the generator; the groupification map is $\eta : B \to \mathbb{Z}$, $v^i u^j \mapsto j - i$. The subgroups of $\mathbb{Z}$ are of the form $d\mathbb{Z} \subseteq \mathbb{Z}$ for $d \in \mathbb{N}$. The equivalent conditions of Theorem 7.2.2 are satisfied, for X the right $\mathbb{Z}$-set $d\mathbb{Z}\backslash\mathbb{Z}$, if and only if $d \neq 0$, so for each $d \in \{1, 2, 3, \dots\}$ we get a locally constant étale geometric morphism

$$f_d : \mathbf{PSh}(B_d) \to \mathbf{PSh}(B)$$

with $B_d = \eta^{-1}(d\mathbb{Z}) = \{v^i u^j \in B : i \equiv j \mod d\}$. We borrowed the notation B_d from [Mun68, (1.4)], where it is proved that B_d is a regular, simple semigroup whose idempotents form a submonoid isomorphic to $(\mathbb{N}, \max)$.

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