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HIGHER COVERINGS OF RACKS AND QUANDLES – PART I

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ABSTRACT. This article is the first part of a series of three articles, in which we develop a higher covering theory of racks and quandles. This project is rooted in M. Eisermann's work on quandle coverings, and the categorical perspective brought to the subject by V. Even, who characterizes coverings as those surjections which are central, relatively to trivial quandles. We extend this work by applying the techniques from higher categorical Galois theory, in the sense of G. Janelidze, and in particular we identify meaningful higher-dimensional centrality conditions defining our higher coverings of racks and quandles.

In this first article (Part I), we revisit the foundations of the covering theory of interest, we extend it to the more general context of racks and mathematically describe how to navigate between racks and quandles. We explain the algebraic ingredients at play, and reinforce the homotopical and topological interpretations of these ingredients. In particular we study and insist on the crucial role of the left adjoint of the conjugation functor Conj between groups and racks (or quandles). We rename this functor Pth, and explain in which sense it sends a rack to its group of homotopy classes of paths. We characterize coverings and relative centrality using Pth, but also develop a more visual "geometrical" understanding of these conditions. We use alternative generalizable and visual proofs for the characterization of central extensions of racks and quandles. We complete the recovery of M. Eisermann's suitable constructions of weakly universal covers, and fundamental groupoids from a Galois-theoretic perspective. We sketch how to deduce M. Eisermann's detailed classification results from the fundamental theorem of categorical Galois theory. As we develop this complementary understanding of the subject, we lay down all the ideas and results which will articulate the higher-dimensional theory developed in Part II and III.

1. Introduction

Over the last decades, racks (see [41] for a detailed introduction to the subject and its history) and quandles [61, 69] have been applied to knot theory, physics and computer sciences in various works – see for instance [62, 9, 41, 26, 42, 23, 22, 63, 29] and references there. In geometry, the earlier notion of *symmetric space*, as studied by O. Loos in [65] (see also [77]), gives yet another context for applications (see [2, 47] for up-to-date

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introductions to the field). Racks and quandles have also received a lot of attention from experts in categorical algebra [30, 32, 33, 34, 6, 8, 7], for the development of the notion of Σ-local properties as well as in relation to the concept quandle covering.

This concept is due to M. Eisermann who developed a covering theory for quandles, published in [29], where he studies quandle coverings in analogy with topological coverings. In particular, he derives several classification results for coverings, in the form of Galois correspondences as in topology (or Galois theory). In order to do so, he works with some suitable constructions such as a (weakly) universal covering or a fundamental group(oid) of a quandle.

In his Ph.D. thesis [30], V. Even applies categorical Galois theory, in the sense of G. Janelidze [49], to the context of quandles. By doing so, he establishes that M. Eisermann's coverings arise from the admissible adjunction between *trivial quandles* (i.e. sets) and quandles, in the same way that topological coverings arise from the admissible adjunction between discrete topological spaces (i.e. sets) and locally connected topological spaces (see Section 6.3 in [4]). He also derives that M. Eisermann's notion of fundamental group of a connected, pointed quandle coincides with the corresponding notion from categorical Galois theory. This, in turn, makes the bridge with the fundamental group of a pointed, connected topological space. By doing so, V. Even strengthens the analogy with topology and opens the door for the application of many tools from categorical Galois theory, and categorical algebra.

In this article, we investigate the lower dimensional covering theory of quandles with the perspective of developing a higher-dimensional covering theory in this context. In order to do so, we explicitly extend M. Eisermann and V. Even's work to the more general context of racks, as it was already suggested in their articles. We investigate visual representations and a suitable categorical understanding of the different algebraic and topological ingredients of these covering theories in order to prepare the generalization of these tools in higher dimensions. We use categorical Galois theory in order to access the concepts and tools which are required for the development of such a higher-dimensional covering theory, which is further developed in *Higher coverings of racks and quandles – Part II* [73] and the forthcoming third part of this project [74].

In Section 1.1, we describe enough of categorical Galois theory to motivate the overall project and explain the results we seek. We describe those key properties of the adjunction between the category of racks (or quandles) and the category of trivial racks (i.e. sets) that we need in order to achieve our higher-dimensional goals. Finally we comment on the use of projective presentations, and a global strategy to characterize the central extensions (coverings) arising in the context of such a suitable adjunction.

In Section 2, we recall and study the basics of the theory of racks and quandles that we need for the investigation of the covering theories of interest. We start (Section 2.1) with a short study of the axioms, our first comments relating groups, racks and quandles, and the basic concepts of symmetry, inner automorphisms, and their actions. Next (Section 2.2), we develop some intuition about the geometrical features of a rack. We illustrate our comments on the construction of the free rack, and recall the construction of the canonical projective presentation of a rack, which presents the elements in a rack with the geometrical features of those in the appropriate free rack. We then introduce the connected component adjunction (Section 2.3), from which the covering theory of interest arises. The concepts of trivializing relation, connectedness, primitive path, orbit congruence, etc. are recalled. We propose to derive the trivializing relation from the geometrical understanding of free objects via projective presentations. We recall the admissibility results for the connected component adjunction and comment on the non-local character of connectedness. We illustrate our visual approach to coverings on the characterization of trivial extensions.

Section 2.8 follows with a description of the links between the construction of Pth (the group of paths functor), left adjoint of the conjugation functor, and the equivalence classes of *tails* of formal terms in the language of racks. Again, we propose to look at the simple description of Pth on free objects, and extend this description to all objects, via the canonical projective presentations. We describe the action of the group of paths and how it relates to inner automorphisms and equivalence classes of primitive paths in general. The free action of this group on free objects is recalled. We emphasize the functoriality of Pth on all morphisms by contrast with the non-functorial construction of inner automorphisms. We describe the kernels of the induced maps between groups of paths, in preparation for the characterizations of centrality. We insist on the fact that the role of Pth (as left adjoint of the conjugation functor) is the same in racks and in quandles, although it appears as more intimately related to racks in design. We conclude this survey with a study of the adjunction between racks and quandles (Section 2.22). We derive its admissibility and deduce that all extensions are central with respect to this adjunction. We build the free quandle $\mathrm{F_q}(A)$ on a set A in a way that illustrates best the journey from one context to the other. By doing so, we rephrase the interest for pairs of generators with opposite exponents (the transvection group, understood via the functor Pth°). We show that the normal subgroup $\mathrm{Pth}^{\circ}(\mathrm{F_q}(A)) \leq \mathrm{Pth}(\mathrm{F_q}(A))$ of the group of paths acts freely on $\mathrm{F_q}(A)$ as expected.

In Section 3, we give a comprehensive Galois-theoretic account of the low-dimensional covering theory of quandles, which we extend to the suitable context of racks. Coverings are described, as well as their different characterizations, using the kernels of induced maps $\mathrm{Pth}(f)$ between groups of paths, but also via the concept of *closing horns*. We recall that primitive extensions are coverings, and coverings are preserved and reflected by pullbacks along surjections (i.e. central extensions are coverings). We find counterexamples for Theorem 4.2 in [16], and finally illustrate our "geometrical" approach to centrality on the characterization of normal extensions. In Section 3.15, we give a new proof – generalizable to higher dimensions – for the characterization of central extensions of racks and quandles. We investigate how the concepts of centrality in racks and in quandles relate (Section 3.19), using the factorization of the connected component adjunction through the adjunction between racks and quandles. Amongst other results, we derive that the centralizing relations, if they exist, should be the same in both contexts. We then prove (Section 3.25) several characterizations of these centralizing relations, and extend the results from

[28] on the reflectivity of coverings in extensions. In preparation for the admissibility in dimension 2, we show that coverings are closed under quotients along double extensions (towards "Birkhoff") and we show the commutativity property of the kernel pair of the centralization unit (towards "strongly Birkhoff"). We then move to Section 3.32, and the construction of weakly universal covers from the centralization of canonical projective presentations. From there, we build the fundamental Galois groupoid of a rack and of a quandle, establishing the homotopical interpretations of Pth and Pth°. In Section 3.40 we illustrate the use of the fundamental theorem of categorical Galois theory in this context.

1.1. The point of view of categorical Galois theory. Categorical Galois theory (in the sense of G. Janelidze [49], see also [55]) is a very general theory with rich and various interpretations depending on the numerous contexts of application. On a theoretical level, Galois theory exhibits strong links with, for example, factorization systems, commutator theory, homology and homotopy theory (see for instance [57, 18, 54]). Looking at applications, it unifies, in particular, the theory of field extensions from classical Galois theory (as well as both of its generalizations by A. Grothendieck and A. R. Magid.), the theory of coverings of locally connected topological spaces, and the theory of central extensions of groups. The covering theory of racks and quandles [29] is yet another example [30], which combines intuitive interpretations inspired by the topological example with features of the group theoretic case. A detailed historical account of the developments of Galois theory is given in [4] and [54] gives an overview of the developments of categorical Galois theory (from the perspective of universal algebra). In this introduction we avoid the technical details of the general theory, but hint at the very essentials needed by us. Another suitable and more comprehensive introduction can be found in [72].

Categorical Galois theory always arises from an *adjunction* (say "relationship") between two categories (think "contexts"). For our purposes, there shall be a supposedly better understood "primitive context", say $\mathcal{X}$, which sits inside a supposedely more difficult "sophisticated context", say $\mathcal{C}$; such that moreover $\mathcal{C}$ *reflects* back on $\mathcal{X}$ – e.g. sets, considered as discrete topological spaces, sit inside locally connected topological spaces which reflect back on sets via the *connected component functor* π_0 [4, Section 6.3]. Under certain hypotheses on these contexts and their relationship, categorical Galois theory studies a (specific) *sphere of influence* of the context $\mathcal{X}$ in the context $\mathcal{C}$, with respect to this relationship – the idea of a relative notion of centrality [43, 66]. This influence (centrality) is discussed in terms of a chosen class of morphisms in these categories which we call *extensions* (e.g. the class of *surjective étale maps*). The data of such an adjunction $F \dashv \mathrm{I}$ (with unit η and counit ϵ) and a chosen class of extensions $\mathcal{E}$ is called a *Galois structure* $\Gamma \coloneqq (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ and provides the axiomatic framework for categorical Galois theory to be applicable (provided that $(\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ satisfies some conditions, see [50, 55]).

Given such a Galois structure Γ, the idea is that extensions "which live in $\mathcal{X}$", which we call *primitive extensions*, induce, in two steps, two other notions of extensions in $\mathcal{C}$, which are naturally related to primitive extensions "in a tractable way". The first step influence: *trivial extensions*, are those extensions t of $\mathcal{C}$ that are directly constructed from a primitive extension p in $\mathcal{X}$, by pullback along a component of the *unit* η (see Figure 1 – this gives topological *trivial coverings* in our example). Then the second step influence: *central extensions* (which are topological *coverings* in our example), are those extensions which “are locally trivial extensions”, i.e. extensions which can be *split* by another extension, where an extension e *splits* an extension c when the pullback t of c along e is a trivial extension – see Figure 1.

Under certain conditions on the Galois structure Γ (see *admissibility* in [55] and Section 1.1.1) the central extensions above a given object can be classified using data which is internal to $\mathcal{X}$ – in a form which is often called a *Galois correspondence*, as in the theory of topological coverings.

More precisely, we need the Galois-theoretic concept of the *fundamental groupoid* of an object. In topology, the classical concept of fundamental group(oid) of a (pointed) space may be viewed as the Galois-theoretic concept of fundamental groupoid defined for a Galois structure Γ. Given an extension $n\colon A \to B$, the kernel pair $p_1, p_2\colon \operatorname{Ker}(n) \rightrightarrows A$ of n always determines the structure of an internal groupoid in $\mathcal{C}$. If n is moreover a central extension, which is split by itself, i.e. the projections of the *kernel pair* $p_1, p_2\colon \operatorname{Ker}(n) \rightrightarrows A$ of n are trivial extensions, then n is called a *normal extensions* and the image $\mathcal{G} \coloneqq F(p_1, p_2\colon \operatorname{Ker}(n) \rightrightarrows A)$ by the reflector F of this *groupoid* induced by $\ker(n)$ is still a groupoid in $\mathcal{X}$. The *fundamental theorem of categorical Galois theory* then says that *internal presheaves* over that groupoid $\mathcal{G}$ (think “groupoid actions in $\mathcal{X}$”) yield a category which is equivalent to the category of those extensions above B which are split by n. If n splits all central extensions above B, for instance, in the contexts of interest, when it is a *weakly universal central extension* above B (see Section 1.1.2), then $\mathcal{G}$ is the *fundamental groupoid* of B, which thus classifies all central extensions above B. A *weakly universal central extension above* B is a central extension with codomain B, which factors through any other central extension above B. Note that the conditions – connectedness, local path-connectedness and semi-local simply-connectedness – on the space X in [46, Theorem 1.38] are there to guarantee the existence of a weakly universal covering above X (see

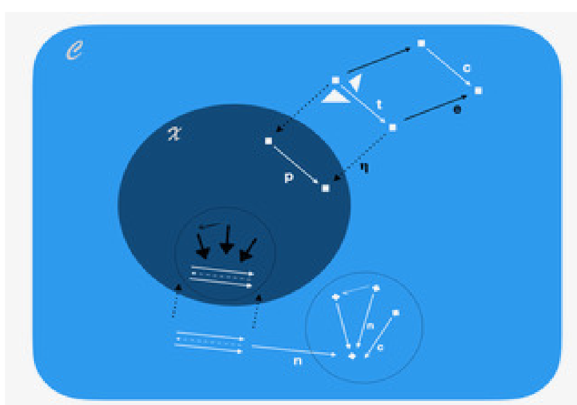


Figure 1: A kid’s drawing of categorical Galois theory

[4, Section 6.6-8]). Internal groupoids and internal actions are well explained in [60], a standard reference for the use of groupoids is [11].

In the case of groups, the adjunction of interest is $\mathrm{ab} \dashv \mathrm{I}$, the *abelianization adjunction*, where the *left adjoint* $\mathrm{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ sends a group G to the abelian group $G/[G,G]$, constructed by quotienting out the *commutator subgroup* $[G,G]$ of G. In this context, the *extensions* are chosen to be the *regular epimorphisms*, which are merely the surjective group homomorphisms. Given this Galois structure, the Galois-theoretic concept of central extension coincides with the concept of a *central extension* from group theory. The fundamental theorem can for instance be used to show that given a *perfect group* G, the *second integral homology group* of G can be *presented* as a "Galois group" (see [49, Remark 5.4], [4, Section 5.2.(10-17)] and [53]).

Note that in Part I, the adjunction which gives rise to the covering theory of racks and quandles is related to $\mathrm{ab} \dashv \mathrm{I}$ and is also such that $\mathcal{X}$ is a *subvariety of algebras* in $\mathcal{C} \coloneqq \mathsf{Rck}/\mathsf{Qnd}$. Such data always gives a Galois structure, by defining extensions to be the surjective maps [55]. Moreover, $\mathcal{X} = \mathsf{Set}$ is here equivalent to the category of sets, such that the left adjoint $F \coloneqq \pi_0\colon \mathsf{Rck}/\mathsf{Qnd} = \mathcal{C} \to \mathcal{X}$ can be interpreted as a connected component functor like in topology.

Now from the example of groups, and the aforementioned links with homology, the development of Galois theory led for instance to a generalization of the Hopf formulae for the (integral) homology of groups [12] to other non-abelian settings, leading to a whole new approach to non-abelian homology and cohomology, by means of higher central extensions [51, 52, 38, 53, 45, 35, 36, 39, 37, 76, 27]. In order to access the relevant higher-dimensional information, one actually "iterates" categorical Galois theory. The increase in dimension consists in shifting from the context of $\mathcal{C}$ to the *category of extensions* of $\mathcal{C}$: $\mathsf{Ext}\mathcal{C}$ defined as the full subcategory of the arrow category $\mathsf{Arr}\mathcal{C}$ with objects being extensions. A morphism $\alpha\colon f_A \to f_B$ in such a category of morphisms is given by a pair of morphisms in $\mathcal{C}$, which we denote $\alpha = (\alpha_\top, \alpha_\bot)$ (the *top* and *bottom components* of α), such that these form an (oriented) commutative square (on the left).

$$\begin{tikzcd} A_\top \arrow[r, "\alpha_\top"] \arrow[d, "f_A"'] \arrow[dr, phantom, "(\to)"] & B_\top \arrow[d, "f_B"] \\ A_\bot \arrow[r, "\alpha_\bot"'] & B_\bot \end{tikzcd} \qquad \begin{tikzcd} A_\top \arrow[rr, "\alpha_\top"] \arrow[dr, dashed, "p"] \arrow[dd, "f_A"'] & & B_\top \arrow[dd, "f_B"] \\ & P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\ A_\bot \arrow[rr, "\alpha_\bot"'] & & B_\bot \end{tikzcd}$$

We call the *comparison map* of such a morphism (or commutative square) the unique map $p\colon A_\top \to P$ induced by the universal property of $P \coloneqq A_\bot \times_{B_\bot} B_\top$, the pullback of $\alpha_\bot$ and f_B. Now from the study of the *admissible adjunction* $F \dashv \mathrm{I}$ (within the Galois structure Γ), Galois theory produces the concept of a central extension, and thus we may look at the *full subcategory* $\mathsf{CExt}\mathcal{C}$ of $\mathsf{Ext}\mathcal{C}$ whose objects are central extensions. The category of central extensions $\mathsf{CExt}\mathcal{C}$ is not reflective (even less so *admissible*) in the category of extensions $\mathsf{Ext}\mathcal{C}$ in general (see [56]). In groups one can universally *centralize* an extension, along a quotient of its domain, and there $\mathsf{CExt}\mathcal{C}$ is actually a full replete *(regular epi)-reflective* subcategory of $\mathsf{Ext}\mathcal{C}$. When such a reflection exists, one may further wonder whether there is a Galois structure behind it, and whether it is admissible. What is the sphere of influence of central extensions in extensions, and with respect to which class of *extensions of extensions*, i.e. can we re-instantiate Galois theory in this induced (two-dimensional) context?

An appropriate class of morphisms to work with, in order to obtain an admissible Galois structure in such a two-dimensional setting, is the class of *double extensions* (see for instance [51, 44, 40, 35]). A *double extension* is a morphsim $\alpha = (\alpha_{\top}, \alpha_{\bot})$ in $\mathsf{Ext}\mathcal{C}$ such that both $\alpha_{\top}$ and $\alpha_{\bot}$ are extensions and the comparison map of α is also an extension. *Double central extensions of groups* were described in [51], and higher-dimensional Galois theory developed further [52, 38], leading to the aforementioned results in homology and cohomology.

Similarly in topology, higher homotopical information of spaces can be studied via the higher fundamental groupoids in the higher-dimensional Galois theory of locally connected topological spaces. A detailed survey about the study of higher-dimensional homotopy group(oid)s can be found in [10], see also [13]. Some insights are given in [15] where higher Galois theory is used to build a homotopy double groupoid for maps of spaces (see also [14]).

In this article we consolidate the understanding of the one-dimensional covering theory of racks and quandles, and introduce all the necessary ideas to start a higher-dimensional Galois theory in this context. Note that the generalization of the covering theory to higher dimensions is far from trivial and the existing literature on the lower-dimensional theory (see for instance [29, 30, 31, 33, 28]) was not aimed at facilitating such a development. In this article, we thus enable the expansion of the theory's scope to higher dimensions with the interesting homological and homotopical perspectives that have led to many new results in previous applications.

1.1.1. Admissibility via the strong Birkhoff condition, in two steps. Note that in the literature, most instantiations of higher categorical Galois theory are such that the "base" category $\mathcal{C}$ is a *Mal'tsev category* (see [20, 21, 19]), and such that moreover all the induced higher-dimensional categories of extensions ($\mathsf{Ext}\mathcal{C}$, $\mathsf{ExtExt}\mathcal{C}$, and so on) are also Mal'tsev categories. Admissibility conditions (for Galois theory to be applicable) as well as computations with higher extensions are easier to handle in such a context. The categories we are interested in are not Mal'tsev categories. Showing how higher categorical Galois theory can apply in this more general setting thus requires some refinements on the arguments which are used in the existing examples (see for instance [31]).

The difficulty is in the induction for higher dimensions which will be detailed in Part II and Part III [73, 75, 74]. As a necessary foundation for these higher-dimensional goals, we sketch, without technical details, which properties of the adjunctions (or Galois structures) of interest to focus on in lower-dimensions.

Our starting context is that of [55] which we refer to for more details. The Galois structures $\Gamma = (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ of interest are then such that $\mathcal{X}$ is a *Birkhoff subcategory* of $\mathcal{C}$ [17, 55]. In particular, $\mathcal{X}$ is closed in $\mathcal{C}$ under quotients along extensions, which in this context is equivalent to the fact that the *reflection squares* of extensions are *pushouts*.

Given $f\colon A \to B$ in $\mathcal{C}$, *the reflection square at* f (with respect to Γ) is the morphism (η_A, η_B) with domain f and codomain $\mathrm{I}\,F(f)$ in $\mathsf{Arr}(\mathcal{C})$. The subcategory $\mathcal{X}$ is then said to be *strongly Birkhoff* in $\mathcal{C}$ if moreover these reflection squares of extensions are themselves double extensions.

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr, "p"] \arrow[dd, "f"'] & & \mathrm{I}\,F(A) \arrow[dd, "\mathrm{I}\,F(f)"] \\
& P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
B \arrow[rr, "\eta_B"'] & & \mathrm{I}\,F(B)
\end{tikzcd} \tag{1}$$

Proposition 2.6 in [38] implies that if Γ is *strongly Birkhoff*, then it is in particular admissible (and thus categorical Galois theory is applicable). This strongly Birkhoff condition is the condition of interest to us. For the Galois structures Γ we consider (such as in [55]) Proposition 5.4 in [19] implies that if Γ is Birkhoff, it is strongly Birkhoff if and only if, for any object A in $\mathcal{C}$, the kernel pair of η_A *commutes* (in the sense of the composition of relations) with any other equivalence relation on A (see [68, 19]). For instance, in the category of groups, any two equivalence relations commute with each other (see *Mal'tsev categories* [19]). Hence since Ab is a Birkhoff subcategory of Grp, it is actually strongly Birkhoff in Grp, which implies the *admissibility* of $\mathrm{ab} \dashv \mathrm{I}$ (see [55, Theorem 3.4]). However, working in a Mal'tsev category is not necessary, as it was already known (see for instance [55]), and observed again by V. Even in [30] and [31], where he uses the permutability property of the kernel pairs of unit morphisms to conclude the admissibility of his Galois structure. In Part I, we briefly re-discuss these results and illustrate the argument on a new adjunction. In higher dimensions, we shall also aim to obtain strongly Birkhoff Galois structures by splitting the work in two steps: (1) closure by quotients along higher extensions and (2) the permutability condition on the kernel pairs of the unit morphisms.

1.1.2. Splitting along projective presentations and weakly universal covers. Remember that in any category, an object E is *projective* – with respect to a given class of morphisms, which we always take to be our extensions – if for any extension $f\colon A \twoheadrightarrow B$ and any morphism $p\colon E \to B$, there exists a factorization of p through f i.e. $g\colon E \to A$ such that $f \circ g = p$. A projective presentation of an object B is then given by an extension $p\colon E \twoheadrightarrow B$ such that E is projective (with respect to extensions). For instance, in *varieties of algebras* (in the sense of universal algebra), there are *enough projectives*, in particular each object has a *canonical* projective presentation given by the *counit* of the "free-forgetful" *monadic* adjunction with sets [67].

We may assume that in the Galois structures Γ (as it is the case in groups or in Part II-III) the "sophisticated context" $\mathcal{C}$ has *enough projectives.* Then any central extension

f is in particular split by any projective presentation p of its codomain. We have

$$\begin{tikzcd}
E \times_B A \arrow[rrr, "p_A"] \arrow[dr, dotted, "t"] \arrow[dd, "p_E"'] & & & A \arrow[dd, "f"] \\
& T \times_B A \arrow[urr] \arrow[dd, "p_T"] & & \\
E \arrow[rrr, "p"] \arrow[dr, dashed, "p'"'] & & & B \\
& T \arrow[urr] & &
\end{tikzcd} \tag{2}$$

where p' is induced by E being projective, t is induced by the universal property of $T \times_B A$ and p_T is a trivial extension by assumption. Then with no assumptions on $\mathcal{C}$, the left hand face is a pullback since the back face and the right hand face are. Assume that the Galois structure we consider is admissible; trivial, central and normal extensions are then pullback stable (see for instance [55]), and thus p_E is a trivial extension, since it is the pullback of a trivial extension. Hence if $\mathcal{C}$ has enough projectives, then for any object B in $\mathcal{C}$ the category of central extensions $\mathsf{CExt}(B)$ above B is the same as the category of those extensions which are split by one given morphism such as the foregoing projective presentation p of B.

Now when central extensions are reflective in extensions, a weakly universal central extension can always be obtained from the centralization of a projective presentation. One can for example recover this idea from [70]. Consider an extension $f \colon A \to B$, and the centralization of a projective presentation of B:

$$\begin{tikzcd}
& E \arrow[dl, "\text{centralization}"'] \arrow[dr, dashed, "a"] \arrow[dd, "p"] & \\
E' \arrow[rr, dotted, "b"] \arrow[dr, "p'"'] & & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

We get a since E is projective and b by the universal property of p'. In the contexts of interest (see for instance Proposition 3.34), a central extension is split by each weakly universal central extension of its codomain. Such weakly universal central extensions above an object B are then split by themselves which makes them normal extensions. The reflection of the kernel pair of such is then the *fundamental Galois groupoid of* B, which classifies central extensions above B.

1.1.3. General strategy for characterizing central extensions. Finally we describe our general strategy, suggested by G. Janelidze, when it comes to identifying a property which characterizes central extensions. It is easy to show that if a central extension f is split by a split epimorphism p, then it is a trivial extension. As a consequence, those central extensions that have projective codomains are trivial extensions. Now suppose one has identified a special class of extensions, called *candidate-coverings*, such that candidate-coverings are preserved and reflected by pullbacks along extensions. Provided primitive extensions are candidate-coverings, then all trivial extensions are candidate-coverings and also central extensions are. Moreover, given a candidate-covering $f \colon A \to B$, pulling back f along a projective presentation p of B yields a candidate-covering with projective codomain. Since f is central if and only if it is split by such a p, we see that candidate-coverings are central extensions if and only if all candidate-coverings with projective codomains are actually trivial extensions, which is usually easier to check.

2. An introduction to racks and quandles

We introduce all the ingredients of the theory of racks and quandles needed for this work, which we describe and develop from the perspective inspired by the covering theory of interest.

2.1. Axioms and basic concepts.

2.1.1. Racks and quandles as a system of symmetries. Symmetry is classically modeled/studied using groups. Informally speaking: given a space X, one studies the group of automorphisms $\mathrm{Aut}(X)$ of X. In his Ph.D. thesis [61], D.E. Joyce describes quandles as another algebraic approach to symmetry such that, locally, each point x in a space X would be equipped with a global symmetry S_x of the space X. Groups always come with such a system of symmetries given by conjugation and the definition of inner automorphisms. Quandles, and more primitively racks, can be seen as an algebraic generalisation of such.

2.1.2. Describing the algebraic axioms. Consider a set X that comes equipped with two functions

$$X \underset{\mathrm{S}^{-1}}{\overset{\mathrm{S}}{\rightrightarrows}} X^X,$$

which assign functions S_x and S_x^{-1} in X^X (the set of functions from X to X) to each element x in X. Each element x then acts on any other y in X via those functions S_x and S_x^{-1}. By convention we shall always write actions on the right:

$$y \cdot \mathrm{S}_x \coloneqq \mathrm{S}_x(y) \qquad\qquad y \cdot \mathrm{S}_x^{-1} \coloneqq \mathrm{S}_x^{-1}(y)$$

The functions S_x and S_x^{-1} at a given point $x \in X$ are required to be inverses of one another, in particular for all y in X we have

$$(y \cdot \mathrm{S}_x^{-1}) \cdot \mathrm{S}_x = y = (y \cdot \mathrm{S}_x) \cdot \mathrm{S}_x^{-1}.$$

Note that, under this assumption, S^{-1} and S determine each other. Now we want to call such bijections S_x *symmetries* (or *inner automorphisms*) of X. But observe that the set X is now equipped with two binary operations

$$X \times X \underset{\triangleleft^{-1}}{\overset{\triangleleft}{\rightrightarrows}} X,$$

defined by $x \triangleleft y \coloneqq x \cdot \mathrm{S}_y$ and $x \triangleleft^{-1} y \coloneqq x \cdot \mathrm{S}_y^{-1}$ for each x and y in X. Read "y acts on x (positively or negatively)". Automorphisms of X should then preserve these operations. In particular we thus require that for each x, y and z in X:

$$(x \triangleleft y) \triangleleft z = (x \triangleleft y) \cdot \mathrm{S}_z = (x \cdot \mathrm{S}_z) \triangleleft (y \cdot \mathrm{S}_z) = (x \triangleleft z) \triangleleft (y \triangleleft z).$$

2.1.3. Defining a rack. Any set X equipped with such structure, i.e. two binary operations $\triangleleft$ and $\triangleleft^{-1}$ on X such that for all x, y and z in X:

(R1) $(x \triangleleft y) \triangleleft^{-1} y = x = (x \triangleleft^{-1} y) \triangleleft y$;

(R2) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$;

is called a *rack*. We write Rck for the category of racks with rack homomorphisms defined as usual (functions preserving the operations).

We refer to the axiom (R2) as *self-distributivity*. For each x in X, the *positive* (resp. *negative*) *symmetry at* x is the automorphism S_x (resp. S_x^{-1}). A *symmetry*, also called *right-translation*, of X is S_x or S_x^{-1} for some x in X. The *symmetries* of X refers to the set of those.

2.1.4. Racks from group conjugation. A crucial class of examples is given by group conjugation. D.E. Joyce describes quandles as "the algebraic theory of conjugation" [61]. We have the functor $\mathrm{Conj}\colon \mathsf{Grp} \to \mathsf{Rck}$ which sends a group G to the rack $\mathrm{Conj}(G)$ with same underlying set, and whose rack operations are defined by conjugation: $x \triangleleft a \coloneqq a^{-1}xa$ and $x \triangleleft^{-1} a \coloneqq axa^{-1}$, for a and x in G. Group homomorphisms are sent to rack homomorphisms by just keeping the same underlying function. The forgetful functor $\mathrm{U}\colon \mathsf{Grp} \to \mathsf{Set}$ thus factors through $\mathrm{U}\colon \mathsf{Rck} \to \mathsf{Set}$ via Conj. However the functor Conj is not *full*, since given groups G and H, there are more rack homomorphisms between $\mathrm{Conj}(G)$ and $\mathrm{Conj}(H)$ than there are group homomorphisms between G and H.

An important ingredient for understanding the relationship between groups, racks and quandles is the left adjoint of Conj (Subsection 2.8). The study of this left adjoint (first defined by D.E. Joyce as *Adconj*, referred to as Adj in [29]) is central to this piece of work. In what follows, we often consider groups as racks without necessarily mentioning the functor Conj.

2.1.5. Other identities and self-distributivity. Note that for the symmetries S_x to define automorphisms of racks, one needs distributivity of $\triangleleft$ on $\triangleleft^{-1}$, distributivity of $\triangleleft^{-1}$ on $\triangleleft$, and self-distributivity of $\triangleleft^{-1}$. All these identities are induced by the chosen axioms. Besides, it suffices for a function f to preserve one of the operations in order for it to preserve the other. Actually the roles of $\triangleleft$ and $\triangleleft^{-1}$ are interchangeable. Swapping them in a given equation, gives again a valid equation. Finally we recall that under the axiom (R1), the axiom (R2) is equivalent to

(R2') $x \triangleleft (y \triangleleft z) = ((x \triangleleft^{-1} z) \triangleleft y) \triangleleft z$.

or similarly to $x \triangleleft (y \triangleleft^{-1} z) = ((x \triangleleft z) \triangleleft y) \triangleleft^{-1} z$. From the preceding discussion we also have

$$x \triangleleft^{-1} (y \triangleleft^{-1} z) = ((x \triangleleft z) \triangleleft^{-1} y) \triangleleft^{-1} z, \quad \text{and finally} \quad x \triangleleft^{-1} (y \triangleleft z) = ((x \triangleleft z) \triangleleft y) \triangleleft^{-1} z.$$

Considering these as identities between *formal terms in the language of racks* (see for instance Chapter II, Section 10 in [17]), we say that the term on the right-hand side is *unfolded*, whereas the term on the left hand side isn't.

2.1.6. Composing symmetries – inner automorphisms. By construction, given a rack X, the images of S and S^{-1} are in the group of automorphisms of X. The *group of inner automorphisms* is defined as the subgroup $\mathrm{Inn}(X)$ of $\mathrm{Aut}(X)$ generated by the image of S. For each rack X, we may then restrict S to the morphism $\mathrm{S}\colon X \to \mathrm{Inn}(X)$. Note that the construction of the group of inner automorphisms Inn does not define a functor from Rck to Grp. It does so when restricted to surjective morphisms (see for instance [16]). Observe that if $z = x \triangleleft y$ in X, then $\mathrm{S}_z = \mathrm{S}_y^{-1} \circ \mathrm{S}_x \circ \mathrm{S}_y$ by self-distributivity (R2'). The function S is actually a rack homomorphism from X to $\mathrm{Conj}(\mathrm{Inn}(X))$.

Of course inner automorphisms of a group coincide with the inner automorphisms of the associated conjugation rack. However, observe that for a group G, a composite of symmetries is always a symmetry, whereas in a general rack, the composite of a sequence of symmetries does not always reduce to a one-step symmetry.

2.1.7. Acting with inner automorphisms – representing sequences of symmetries. Given a rack X, we have of course an action of $\mathrm{Inn}(X)$ on X given by evaluation. Remember that we write actions on the right, hence we use the notation $z \cdot (\mathrm{S}_x \circ \mathrm{S}_y) \coloneqq \mathrm{S}_y(\mathrm{S}_x(z))$ for x, y, and z in X. Now any $g \in \mathrm{Inn}(X)$ decomposes as a product $g = \mathrm{S}_{x_1}^{\delta_n} \circ \cdots \circ \mathrm{S}_{x_n}^{\delta_1}$ for some elements $x_1, \ \ldots, \ x_n$ in X and exponents $\delta_1, \ \ldots, \ \delta_n$ in $\{-1, \ 1\}$. Such a decomposition is not necessarily unique, but for any x in X the action of g on x is well defined by

$$x \cdot g \coloneqq x \cdot (\mathrm{S}_{x_1}^{\delta_n} \circ \cdots \circ \mathrm{S}_{x_n}^{\delta_1}) = x \triangleleft^{\delta_1} x_1 \triangleleft^{\delta_2} x_2 \cdots \triangleleft^{\delta_n} x_n,$$

where we omit parentheses using the convention that one should always compute the left-most operation first.

2.1.7.1. As we shall see, we need these successive applications of symmetries in order to study *connectedness* in racks. For our purposes, using the group of inner automorphisms for their study is not satisfactory. Note that given $x \neq y$ in a rack X, two symmetries S_x and S_y are identified in $\mathrm{Inn}(X)$ if they define the same automorphism. Motivated by the covering theories of interest, we study different ways to organize the *set of symmetries* $\{\mathrm{S}_x, \ \mathrm{S}_x^{-1}\}_{x \in X}$ into a group acting on X. Note that we may understand the definition of *augmented quandles* (or racks) [61], see Paragraph 2.10.1, as a tool to abstract away from "representing" sequences of symmetries via composites of such (in the sense of the group of inner automorphisms).

2.1.8. Quandles, the idempotency axiom. As explained by D.E. Joyce, it is reasonable (in reference to applications) to require that a symmetry at a given point fixes that point. If for each x in a rack X we have moreover that

(Q1) $x \triangleleft x = x$;

then X is called a *quandle*. We have the category of quandles Qnd defined as before. Again, (Q1) is equivalent to (Q1'): $x \triangleleft^{-1} x = x$, under the axiom (R1).

For the purpose of this article, we shall mainly be working in the more general context of racks since these exhibit all the necessary features for the covering theory of interest.

Actually all concepts of centrality and coverings remain the same whether one works with the category of racks or of quandles. The addition of the idempotency axiom still has certain consequences on ingredients of the theory such as the fundamental groupoid or the *homotopy classes of paths*. We shall always make explicit these differences and similarities, also using the enlightening study of the "free-forgetful" adjunction between racks and quandles.

2.1.9. Idempotency in racks. Note that even though (Q1) doesn't hold in each rack, a weaker version of the idempotency axiom still holds in all racks as a consequence of self-distributivity. Observe that in a rack X, given any y and $x \in X$, we have

$$x \triangleleft (y \triangleleft y) = x \triangleleft^{-1} y \triangleleft y \triangleleft y = x \triangleleft y.$$

The symmetries S_y and $\mathrm{S}_{(y \triangleleft y)}$, at y and $y \triangleleft y$ are always identified in $\mathrm{Inn}(X)$, even when $y \neq (y \triangleleft y)$ in X. Similarly, for x and y in X any chain $y \triangleleft^k y$ (for $k \in \mathbb{Z}$, the action of y on y, repeated $|k|$ times – use $\triangleleft^{-1}$ when $k < 0$) is such that $x \triangleleft (y \triangleleft^k y) = x \triangleleft y$ (see also Sections 2.22.1 and 3.19).

2.2. From axioms to geometrical features.

We informally highlight two additional elementary features of the axioms which play an important role in what follows. We then illustrate them in the characterization of the free rack on a set A

2.2.1. Heads and tails – detachable tails. Observe that on either side of the identities defining racks, the *head* x of each term is the same and does not play any role in the described identifications.

(R1) $x \triangleleft y \triangleleft^{-1} y = x = x \triangleleft^{-1} y \triangleleft y$ (R2') $x \triangleleft (y \triangleleft z) = x \triangleleft^{-1} z \triangleleft y \triangleleft z$

Now consider any *formal term* in the language of racks (built inductively from atomic variables and the rack operations – see Chapter II Section 10 in [17]), such as for instance

$$(x \triangleleft y) \triangleleft^{-1} (\cdots ((a \triangleleft b) \triangleleft^{-1} c) \triangleleft d) \cdots \triangleleft z. \tag{3}$$

Remember that roughly speaking, the elements of the *free rack* on a set A can be constructed as equivalence classes of such formal terms, built inductively from the atomic variables in A, where two terms are identified if one can be obtained from the other by replacing subterms according to the axioms, or according to any provable equations derived from the axioms.

Given any term such as above, we shall distinguish the *head* x of the term from the rest of it which is called the *tail* of the term. The informal idea is that the "behaviour" of the tail is independent from the head it is attached to. It thus makes sense to consider the tails (or equivalence classes of such) separately from the heads these tails might act upon.

Observe that the idempotency axiom plays a slightly different role in that respect since, although the heads of terms are left unchanged under the use of (Q1), the identifications in the tails of terms might depend on the heads these are attached to. We shall however see that the discussion about racks still lays a clear foundation for understanding the case of quandles which we discuss in Section 2.22.

2.2.2. Tails as sequences of symmetries. By Paragraph 2.1.5, acting with a symmetry of the form $\mathrm{S}_{(x \triangleleft y)}$ translates into successive applications of $\mathrm{S}_y^{-1}, \mathrm{S}_x, \mathrm{S}_y$ from left to right.

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{S}_y^{-1}"] \arrow[d, "\mathrm{S}_{x \triangleleft y}"'] & \bullet \arrow[d, "\mathrm{S}_x"] \\
\bullet & \bullet \arrow[l, "\mathrm{S}_y"]
\end{tikzcd}$$

Now consider any formal term such as in Equation (3) for instance. Using (R2') repeatedly, we may *unfold* the tail of a term into a string of successive actions of the form

$$x \triangleleft y \triangleleft^{-1} c \triangleleft c \triangleleft^{-1} b \triangleleft^{-1} a \triangleleft b \triangleleft^{-1} c \triangleleft c \triangleleft d \cdots \triangleleft z.$$

We can then interpret the tail as a *path* of successive actions of the symmetries which are applied to the head x. Using (R1) repeatedly again, we may also discard all possible occurrences of the successive application of a symmetry and its inverse

$$x \triangleleft y \triangleleft^{-1} b \triangleleft^{-1} a \triangleleft b \triangleleft d \cdots \triangleleft z.$$

Such *unfolded* and *reduced* terms provide normal forms (unique representatives) for elements in the free rack. The elements of a free rack on a set A are thus described with this architectural feature of having a head in A and an independent tail, such that the tail is a sequence of "representatives" of the symmetries which organize themselves as the elements of the free group on A.

2.2.3. The free rack. The following construction can be found in [41]. It was also studied in [64].

Given a set A, the free rack on A is given by

$$\mathrm{F_r}(A) \coloneqq A \rtimes \mathrm{F_g}(A) \coloneqq \{(a, g) \mid g \in \mathrm{F_g}(A);\ a \in A\},$$

where $\mathrm{F_g}(A)$ is the free group on A and the operations on $\mathrm{F_r}(A)$ are defined for (a, g) and (b, h) in $A \rtimes \mathrm{F_g}(A)$ by

$$(a, g) \triangleleft (b, h) \coloneqq (a, gh^{-1}\underline{b}h) \qquad \text{and} \qquad (a, g) \triangleleft^{-1} (b, h) \coloneqq (a, gh^{-1}\underline{b}^{-1}h).$$

In order to distinguish elements x in A from their images under the injection $\eta_A^g \colon A \to \mathrm{F_g}(A)$, we shall use the convention to write

$$\underline{a} \coloneqq \eta_A^g(a).$$

Looking for the unit of the adjunction, we then have the injective function which sends an element in A to the trivial path starting at that element, i.e. $\eta_A^r \colon A \to \mathrm{F_r}(A) \colon a \mapsto (a, e)$, where e is the empty word (neutral element) in $\mathrm{F_g}(A)$.

Note that since any element $g \in \mathrm{F_g}(A)$ decomposes as a product $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{F_g}(A)$ for some $g_i \in A$ and exponents $\delta_i = 1$ or -1, with $1 \leq i \leq n$, we have, for any $(a, g) \in \mathrm{F_r}(A)$, a decomposition as

$$(a, g) = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = (a, e) \triangleleft^{\delta_1} (g_1, e) \triangleleft^{\delta_2} (g_2, e) \cdots \triangleleft^{\delta_n} (g_n, e).$$

Using such decompositions, any group cancellation in g can be expressed as an instance of the first axiom of racks, and conversely, any instance of the first axiom of racks translates as a group cancellation in the path component. The universal property of the unit η^r and the definition of $\mathrm{F_r}\colon \mathsf{Set} \to \mathsf{Rck}$ on morphisms then follows easily, yielding the left adjoint of the forgetful functor $\mathrm{U}\colon \mathsf{Rck} \to \mathsf{Set}$.

2.2.3.1. Terminology and visual representation In order to emphasize its visual representation, we call an element $(a, g) \in \mathrm{F_r}(A)$ a *trail*. We call g the *path* (or *tail*) component and a the *head* component of the trail (a, g). It is understood that the path g formally acts on a to produce an *endpoint* of the trail (see Paragraph 2.2.3). Formally (a, g) stands for both the trail and its endpoint:

$$a \xrightarrow{g} (a, g).$$

The action of a trail (b, h) on another trail (a, g) consists in adding, at the end of the path g, the contribution of the *symmetry associated to the endpoint* of (b, h) (see Subsection 2.2.4 and further). We say that a trail acts on another *by endpoint*, as in the diagram below, where composition of arrows is computed by multiplication in the path component:

$$\begin{tikzcd} a \arrow[dd, "g"'] & & b \arrow[dd, "h"'] & & a \arrow[d, "g"] \\ & \triangleleft & & = & (a, g) \arrow[d, "h^{-1}\underline{b}h"] \\ (a, g) & & (b, h) & & (a, gh^{-1}\underline{b}h) \end{tikzcd} \qquad (4)$$

2.2.4. Canonical projective presentations. Since Rck is a variety of algebras, any object X can be canonically presented as the quotient

$$\begin{tikzcd} \mathrm{F_r}\,\mathrm{F_r}\,X \arrow[r, shift left=2, "\mathrm{F_r}\,\epsilon^r_X"] \arrow[r, shift right=2, "\epsilon^r_{\mathrm{F_r}\,X}"'] & \mathrm{F_r}\,X \arrow[l, "\mathrm{F_r}\,\eta^r_X" description] \arrow[r, "\epsilon^r_X"] & X \end{tikzcd}$$

where we have omitted the forgetful functor $\mathrm{U}\colon \mathsf{Rck} \to \mathsf{Set}$ (understand X alternatively as a rack or a set), and ϵ^r_X is the counit of the "free-forgetful" adjunction $\mathrm{F_r} \dashv \mathrm{U}$. This counit ϵ^r_X is the coequalizer of the reflexive graph on the left. This canonical presentation of racks allows us to capture a sense in which the geometrical features of free objects are carried through to any general rack. We shall illustrate this on the important functorial constructions of the Galois theory of interest. Let us make explicit these objects and morphisms to exhibit some of the mechanics at play. Think of what this *right-exact fork* represents for groups, where the operation is associative.

First of all we may exhibit heads and tails and rewrite this *right-exact fork* as

$$\begin{tikzcd}[column sep=large] (X \rtimes \mathrm{F_g}(X)) \rtimes \mathrm{F_g}(X \rtimes \mathrm{F_g}(X)) \arrow[r, shift left=2, "\epsilon^r_X \times \mathrm{F_g}[\epsilon^r_X]"] \arrow[r, shift right=2, "\epsilon^r_{\mathrm{F_r}\, X}"'] & X \rtimes \mathrm{F_g}\, X \arrow[l, "\mathrm{F_r}\,\eta^r_X" description] \arrow[r, "\epsilon^r_X"] & X \end{tikzcd}$$

By Paragraph 2.2.3, the counit ϵ^r_X should send a pair $(x, g) = (x, g_1{}^{\delta_1} \cdots g_n{}^{\delta_n})$ for $g_i \in X$ to the element in the rack X given by $\epsilon^r_X(x, g) = x \cdot g \coloneqq x \triangleleft^{\delta_1} g_1 \cdots \triangleleft^{\delta_n} g_n$.

Hence the canonical projective presentation ϵ^r_X of a rack X covers each element $x \in X$ by all possible formal decompositions (x_0, g) of that element x, such that x *is the endpoint of the trail* (x_0, g), i.e. the result of the action of a *path* on a *head*: $x = x_0 \cdot g$. Now this head x_0 and each "representative of a symmetry" $g_i{}^{\delta_i}$ in the path component $g = g_1{}^{\delta_1} \cdots g_n{}^{\delta_n}$ may itself be expressed as the endpoint of some trail (i.e. $x_0 = x_{00} \cdot h$, and $g_i = y_i \cdot k_i$ for h and k_i in $\mathrm{F_g}\, X$). This is what is captured by the object $\mathrm{F_r}\,\mathrm{F_r}(X)$ on the left of the fork.

Then from the definition of the counit, we may derive the two projections. These may be understood as expressing two things:

First observe that an element $t = [(a, g); e]$ in $\mathrm{F_r}\,\mathrm{F_r}(X)$ (i.e. an element which has a trivial path component, but an interesting head) is sent to $((a \cdot g), e)$ by the first projection and to (a, g) by the second projection. The two projections thus allow us to move part of the tail of a trail towards the head of that trail and part of the head towards the tail.

Then an element $[(a, e); (b, h)]$ – i.e. an element with a trivial head component and a non trivial (but simple) tail – is sent by the first projection to $(a, (b \cdot h))$, and by the second projection to $(a, h^{-1}bh)$. Coequalizing these two projections expresses self-distributivity (see Paragraphs 2.1.5 and 2.2.2). In other words it illustrates how to compute the representative of the symmetry associated to the endpoint of a trail. This is already part of the definition of the rack operation in the free rack. We have the rack homomorphism on the left

$$\begin{tikzcd} X \rtimes \mathrm{F_g}(X) \arrow[r, "i_X"] & \mathrm{F_g}(X) \\ (x, g) \arrow[r, mapsto, "i_X"] & g^{-1}xg \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\eta^g_X"] \arrow[dr, "\eta^r_X"'] & & \mathrm{Conj}(\mathrm{F_g}(X)) \\ & \mathrm{F_r}(X) \arrow[ur, dashed, "i_X"'] & \end{tikzcd}$$

which sends a path to the symmetry associated to its endpoint. It is actually induced by the universal property of free racks as displayed in the diagram on the right.

2.3. The connected component adjunction.

2.3.1. Trivial racks and trivializing congruence. Another important theoretical example of racks is given by the so-called *trivial racks* (or trivial quandles) for which each symmetry at a given point is chosen to be the identity. Each point acts trivially on the rest of the rack. This may be expressed as an additional axiom:

(Triv) $x \triangleleft y = x$.

Since each set comes with a unique structure of trivial rack and each function between trivial racks is a homomorphism, we get an isomorphism between the category of sets (Set)

and the category of trivial racks. The category of sets is thus a subvariety of algebras within racks.

The inclusion functor $\mathrm{I}\colon \mathsf{Set} \to \mathsf{Rck}$ sends a set to the trivial rack on that set. Now this inclusion functor has a left adjoint, which sends a rack to the *freely trivialized* rack. The trivialization $A/(\mathrm{C}_0\, A)$ of a rack A can be easily obtained by quotienting out the congruence $\mathrm{C}_0\, A$ generated by the pairs $(x, x \triangleleft y)$ for x and y in A. However, the congruence $\mathrm{C}_0\, A$ can be conveniently characterised by the fact that x and y in X are in relation by $\mathrm{C}_0\, A$ if and only if they are *connected* in the following sense (see [61]).

2.3.2. Connectedness and primitive paths. Two elements x and y in a rack A are said to be *connected* ($[x] = [y]$) if there exists $n \in \mathbb{N}$ and elements a_1, a_2, ..., a_n in A such that
$$y = x \triangleleft^{\delta_1} a_1 \triangleleft^{\delta_2} a_2 \cdots \triangleleft^{\delta_n} a_n,$$
for some coefficients $\delta_i \in \{-1, 1\}$ for $1 \leq i \leq n$.

Such a sequence of elements together with the choice of coefficients is viewed as a formal sequence of symmetries (see Paragraph 2.1.7.1). Bearing in mind Paragraphs 2.2.1 and 2.2.2, we call such a formal sequence of symmetries $(a_i, \delta_i)_{1 \leq i \leq n}$ a *primitive path* of the rack A. In particular this specific primitive path *connects* x to y but may be applied to different elements in the rack. We call the data of such a pair $T = (x, (a_i, \delta_i)_{1 \leq i \leq n})$ a *primitive trail* in X, where x is the head of T and y the *endpoint* of T. As we mentioned before, (x, y) is in $\mathrm{C}_0\, A$ if and only if there exists a primitive path which connects x to y.

Observe that for any rack homomorphism $f\colon A \to X$ for some trivial rack X we have $\mathrm{C}_0\, A \leq \mathrm{Eq}(f)$. The functor $\pi_0\colon \mathsf{Rck} \to \mathsf{Set}$, such that $\pi_0(A) \coloneqq A/(\mathrm{C}_0\, A)$ is the set of connected components of A (i.e. the set of $\mathrm{C}_0\, A$-equivalence classes), is left adjoint to $\mathrm{I}\colon \mathsf{Set} \to \mathsf{Rck}$ with unit $\eta_A\colon\ A \ \to\ \pi_0(A)$, sending an element $a \in A$ to its connected component $\eta_A(a)$ (also denoted $[a]$) in $\pi_0(A)$.

2.3.3. From free objects to all – definition as a colimit. Observe that the composite
$$\mathsf{Set} \xrightarrow{\ \mathrm{I}\ } \mathsf{Rck} \xrightarrow{\ \mathrm{U}\ } \mathsf{Set}$$
gives the identity functor. As a consequence, the composite of left adjoints $\pi_0\, \mathrm{F_r}$ also gives the identity functor. More precisely we may deduce from the composite of adjunctions that, given a set X, the unit $\eta_{\mathrm{F_r}(X)}\colon X \rtimes \mathrm{F_g}(X) \to X$ is "projection on X", i.e. the connected component of a trail $(x, g) \in \mathrm{F_r}(X)$ is given by projection on its head x.

Since π_0 is a left adjoint, it preserves colimits, hence $\pi_0(X)$ should be the coequalizer, in Set, of the pair:
$$\pi_0((X \rtimes \mathrm{F_g}(X)) \rtimes \mathrm{F_g}(X \rtimes \mathrm{F_g}(X))) \underset{\pi_0(\epsilon^r_{\mathrm{F_r}\,\mathrm{U}\,X})}{\overset{\pi_0(\epsilon^r_X \times \mathrm{F_g}[\epsilon^r_X])}{\rightrightarrows}} \pi_0(X \rtimes \mathrm{F_g}\, X),$$
which indeed reduces to being the coequalizer of the pair $p_1, p_2\colon X \times \mathrm{F_g}(X) \rightrightarrows X$, where $p_1(x, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = x \triangleleft^{\delta_1} g_1 \cdots \triangleleft^{\delta_n} g_n$ and $p_2(x, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = x$.

2.3.4. Equivalence classes of primitive paths. The term *primitive path* is used to express the idea that it is the most unrefined way we shall use to acknowledge that two elements are connected. Literally it is just a formal sequence of symmetries.

As explained in Paragraph 2.1.7, inner automorphisms also "represent" sequences of symmetries. Again, each primitive path naturally reduces to an inner automorphism simply by composing all the symmetries in the sequence. We also have that (x, y) is in $\mathrm{C}_0\, A$ if and only if there exists $g \in \mathrm{Inn}(A)$ such that $x \cdot g = y$. In other words, $\mathrm{C}_0\, A$ is the congruence generated by the action of $\mathrm{Inn}(A)$. It is called the *orbit congruence* of $\mathrm{Inn}(A)$ (see Paragraph 2.3.7). In what follows, we like to view inner automorphisms as equivalence classes of primitive paths. As mentioned earlier we shall consider other such equivalence classes of primitive paths which lie in between formal sequences of symmetries and composites of such. Each of these represent different witnesses of how to connect elements in a rack A. All of these generate the same trivializing congruence $\mathrm{C}_0\, A$.

2.3.5. Conjugacy classes. Observe that for a group G, the set of connected components of $\mathrm{Conj}(G)$ is given by the set of conjugacy classes in G. In this case the congruence $\mathrm{C}_0(\mathrm{Conj}(G))$ is characterised as follows: $(a, b) \in \mathrm{C}_0(\mathrm{Conj}(G))$ if and only if there exists $c \in G$ such that $b = c^{-1}ac$. Again, any primitive path, or sequence of symmetries, can be described via a single symmetry obtained as the symmetry of the product of the elements in the sequence. Note that if H is an abelian group, then $\mathrm{Conj}(H)$ is the trivial rack on the underlying set of H. More precisely the restriction to Ab of the functor Conj yields the forgetful functor $\mathrm{U}\colon \mathsf{Ab} \to \mathsf{Set}$.

2.3.6. Racks and quandles have the same connected components. The functor π_0 may be restricted to the domain Qnd and is then left adjoint to the inclusion functor $\mathrm{I}\colon \mathsf{Set} \to \mathsf{Qnd}$ by the same arguments as above. More precisely we have for any rack X that $\pi_0\ {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X) = \pi_0(X)$, where ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X)$ is the free quandle on the rack X.

2.3.7. Orbit congruences permute. In order to obtain the admissibility of Set in Qnd, V. Even shows that certain classes of congruences commute with all congruences. As for quandles, we define *orbit congruences* [16] as the congruences induced by the action of a normal subgroup of the group of inner automorphisms. More precisely, if X is a rack, and N a normal subgroup of $\mathrm{Inn}(X)$ we shall write $\sim_N$ for the N*-orbit congruence* defined for elements x and y in X by: $x \sim_N y$ if and only if there exists $g \in N$ such that $x \cdot g = y$. As it is explained in [31] (see Proposition 2.3.9), this is well defined and yields a congruence (also in Rck). We then have the following – see [32] and [31, Lemma 3.1.2] for the proof, which also holds in Rck.

2.4. Lemma. *Let X be a rack, R a reflexive (internal) relation on X and N a normal subgroup of* $\mathrm{Inn}(X)$*, then the relations $\sim_N$ and R permute: $\sim_N \circ R = R \circ \sim_N$.*

2.4.1. Admissibility for Galois theory. Of course the kernel pair of any unit morphism $\eta_X\colon X \to \pi_0(X)$ is an orbit congruence, since by Paragraph 2.3.4, two elements are in the same connected component if and only if they are in the same orbit under the action of $\mathrm{Inn}(X)$.

As it was recalled in Section 1.1.1 (see also [55]), this yields Theorem 1 of [30]:

2.5. PROPOSITION. *The subvariety* Set *is strongly Birkhoff and thus admissible in* Rck. *Similarly for* Set *in* Qnd.

The Galois structure $\Gamma \coloneqq (\mathsf{Rck}, \mathsf{Set}, \pi_0, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ (respectively for quandles $\Gamma^q \coloneqq (\mathsf{Qnd}, \mathsf{Set}, \pi_0, \mathrm{I}, \eta, \epsilon, \mathcal{E})$) (see [55]) where $\mathcal{E}$ is the class of surjective morphisms of racks (respectively quandles), is thus admissible, i.e. the study of Galois theory is relevant in this context and gives rise, in principle, to a meaningful notion of relative centrality.

2.5.1. CONNECTED COMPONENTS ARE NOT CONNECTED. Given an element a in a rack A, we may consider its *connected component* C_a, i.e. the elements of A which are connected to a. The set C_a is actually a *subrack* of A as it is closed under the operations in A. We may construct the rack C_a as a pullback in Rck:

$$\begin{tikzcd} \mathrm{C}_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "{[a]}"] \\ A \arrow[r, "\eta_A"] & \pi_0(A), \end{tikzcd} \tag{5}$$

where $1 = \{*\}$ is the one element set, which is the terminal object in Rck and also the free quandle on the one element set. Note that if A is connected, then by definition $\pi_0(A) = \{*\}$ and thus $\mathrm{C}_a = A$. However if $\mathrm{C}_a \subset A$, then C_a might have more than one connected component itself (i.e. $\pi_0(\mathrm{C}_a)$ has cardinality $|\pi_0(\mathrm{C}_a)| > 1$), since the existence of a primitive path between some c and b in C_a, might depend on elements which are not connected to a.

2.6. EXAMPLE. A rack A is called *involutive* if the two operations $\triangleleft$ and $\triangleleft^{-1}$ coincide. The subvariety of *involutive racks* is thus obtained by adding the axiom

(Inv) $x \triangleleft y \triangleleft y = x$.

We define the involutive quandle $Q_{ab\star}$ with three elements a, b and $\star$ such that the operation $\triangleleft$ is defined by the following table (see $Q_{(2,1)}$ from [29, Example 1.3]).

$\triangleleft$	a	b	$\star$
a	a	a	b
b	b	b	a
$\star$	$\star$	$\star$	$\star$

The connected component of a is the trivial rack $\mathrm{C}_a = \{a, b\}$ which has itself two connected components $\{a\}$ and $\{b\}$.

We like to say that, for racks (and quandles) the notion of connectedness is not local. In categorical terms, we may say that the functor π_0 is not *semi-left-exact* [24, 18]. This property is indeed characterised, in this context, by the preservation of pullbacks such as in Equation (5) above, i.e. π_0 is semi-left-exact if and only if any such connected component (C_a) is connected $(\pi_0(\mathrm{C}_a) = \{*\})$ (see for instance [4] and [78, Theorem 2.1]). This is an important difference with the case of topological spaces for instance, where the connected components are connected and thus the corresponding π_0 functor is semi-left-exact. See also [32] for further insights on connectedness. Looking at [25, Corollary 2.5], we further compute that $\pi_0(\mathrm{F_r}(1) \times \mathrm{F_r}(1)) = \mathbb{Z}$ and thus that $\pi_0 \colon \mathsf{Rck} \to \mathsf{Set}$ does not preserve finite products; wheareas $\pi_0 \colon \mathsf{Qnd} \to \mathsf{Set}$ does, as was shown in [30, Lemma 3.6.5].

2.6.1. Towards covering theory. Knowing that Γ is admissible, we may now wonder what is the "sphere of influence" of Set in Rck, with respect to surjective maps, and start to develop the corresponding covering theory. Since Set is strongly Birkhoff in Rck, trivial extensions (first step influence) are easy to characterize as those surjections which are "injective on connected components":

2.7. Proposition. *(See also [30, 31]) Given a surjective morphism of racks $t \colon X \to Y$, the following conditions are equivalent:*

(i) t *is a trivial extension;*

(ii) $\mathrm{Eq}(t) \cap \mathrm{C}_0\, X = \Delta_X$*;*

(iii) if a *and* b *in* X *are connected, then* $t(a) = t(b)$ *implies* $a = b$.

Recall that the construction of inner automorphisms (Inn) induces a functor on surjective morphisms: given a surjective morphism $t \colon X \to Y$, we write $\mathrm{Inn}(t) \colon \mathrm{Inn}(X) \to \mathrm{Inn}(Y)$ or $\hat{t}$ for the induced homomorphism between the inner automorphism groups (see first two sections of [16]).

We may then also describe a trivial extension as an extension which *reflects loops*: trivial extensions are those extensions such that for any a in A, if g in $\mathrm{Inn}(A)$ is such that $t(a) \cdot \hat{t}(g) = t(a)$, then $a \cdot g = a$.

$$(a \xrightarrow{g} a \cdot g) \quad \overset{t}{\longmapsto} \quad t(a) \overset{\hat{t}(g)}{=} t(a \cdot g) \qquad \Rightarrow \qquad a \overset{g}{=} a \cdot g$$

In what follows, we shall use such geometrical interpretations to make sense of the algebraic conditions of interest for the covering theory of racks and quandles. However, the non-functoriality of Inn on general morphisms appears as a serious weakness (see for instance the need for Remark 2.12 in the proof of Proposition 3.16). It will become clear from what follows that a more suitable way to represent sequences of symmetries is needed. This is achieved by the *group of paths* which we motivate and describe in the next section. It is not a new concept, but our name for the left adjoint of the conjugation functor, which was described by D.E. Joyce and then used by M. Eisermann to construct weakly universal covers and a suitable fundamental groupoid for quandles. We provide an alternative description of the construction and the role of this functor.

2.8. The group of paths.

2.8.1. Definition. Consider a rack X and two elements x and y in X which are connected by a primitive path $\mathrm{S}_{x_1}^{\delta_1}, \ldots, \mathrm{S}_{x_n}^{\delta_n}$:

$$x \cdot (\mathrm{S}_{x_1}^{\delta_1}, \ldots, \mathrm{S}_{x_n}^{\delta_n}) \coloneqq x \triangleleft^{\delta_1} x_1 \cdots \triangleleft^{\delta_n} x_n = y.$$

Because of (R1), we discussed that it makes sense to identify such formal sequences so as to obtain elements of the free group on X. Now in the same way that we used Paragraph 2.1.5 to unfold formal terms, we still have that whenever $x_i = b \triangleleft c$ for $1 \leq i \leq n$ and b, c in X, acting with S_{x_i} amounts to successively acting with S_c^{-1}, S_b and S_c. From a rack X we may thus build the quotient:

$$\mathrm{F_g}(X) \xrightarrow{q_X} \mathrm{Pth}(X) \coloneqq \mathrm{F_g}(X)/\langle \underline{c}^{-1}\underline{a}^{-1}\underline{x}\,\underline{a} \mid a, x, c \in X \text{ and } c = x \triangleleft a \rangle,$$

which is understood as a group of equivalence classes of primitive paths. Two primitive paths are identified in the group of paths if and only if one can be formally obtained from the other, using the identities induced by the graph of the rack operations (such as $c = x \triangleleft a$), as well as the axioms of racks (or more precisely the axiom-induced identities between tails of formal terms).

2.8.2. Unit and universal property. The function $\eta^g \colon X \to \mathrm{F_g}(X)$ composed with this quotient $q_X \colon \mathrm{F_g}(X) \to \mathrm{Pth}(X)$ yields a morphism of racks $\mathrm{pth}_X \colon X \to \mathrm{Conj}(\mathrm{Pth}(X))$, which sends each element x of X to $\mathrm{pth}_X(x)$ in $\mathrm{Pth}(X)$, such that $\mathrm{pth}_X(x)$ "represents" the positive symmetry at x in the same way S_x does in $\mathrm{Inn}(X)$ (see Paragraph 2.10.1). As for the inclusion in the free group, we shall use the convention

$$\underline{x} \coloneqq \mathrm{pth}_X(x).$$

Now given a rack homomorphism $f \colon X \to \mathrm{Conj}(G)$ for some group G, there is a unique group homomorphism f' induced by the universal property of the free group, which, moreover, factors uniquely through $q_X \colon \mathrm{F_g}(X) \to \mathrm{F_g}(X)/\langle (\underline{x \triangleleft a})^{-1}\underline{a}^{-1}\underline{x}\,\underline{a} \mid a, x \in X \rangle$, since for any a and x in X, $f(x \triangleleft a) = f(a)^{-1} f(x) f(a)$ in G:

$$\begin{tikzcd} X \arrow[r, "\eta^g_X"] \arrow[dr, "f"'] & \mathrm{F_g}(X) \arrow[r, "q_X"] \arrow[d, dotted, "\exists! f'"] & \mathrm{Pth}(X) \arrow[dl, dashed, "\exists! \bar{f}"] \\ & G & \end{tikzcd}$$

Hence, the construction Pth uniquely defines a functor which is the left adjoint of Conj with unit $\mathrm{pth} \colon 1_{\mathsf{Rck}} \to \mathrm{Conj}\,\mathrm{Pth}$. As usual, given $f \colon X \to Y$ in Rck, there is a unique morphism $\mathrm{Pth}(f)$, such that

$$\begin{tikzcd} X \arrow[r, "\mathrm{pth}_X"] \arrow[d, "f"'] & \mathrm{Conj}(\mathrm{Pth}(X)) \arrow[d, dotted, "\exists!\, \mathrm{Conj}(\mathrm{Pth}(f))"] \\ Y \arrow[r, "\mathrm{pth}_Y"] & \mathrm{Conj}(\mathrm{Pth}(Y)), \end{tikzcd}$$

which defines the functor Pth on morphisms.

2.9. Notation. In what follows, we write $\vec{f}$ for the image $\mathrm{Pth}(f)$ of a morphism f from Rck.

2.9.1. From free objects to all – construction as a colimit. Again, observe that the composite $\mathrm{Pth}\,\mathrm{F_r}$ is left adjoint to the forgetful functor $\mathrm{U}\colon \mathsf{Grp} \to \mathsf{Set}$, i.e. $\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X)$. More precisely, we may interpret pth as the extension to all objects of the functorial construction on free objects

$$i_X\colon X \rtimes \mathrm{F_g}(X) \to \mathrm{F_g}(X) \;:\; (x, g) \mapsto g^{-1}\underline{x}g$$

which sends a trail to the "representative of the symmetry" associated to its endpoint (Subsection 2.2.4). Indeed, by the composition of adjunctions, as before, this i is easily seen to define the restriction to free objects of the unit pth of the $\mathrm{Pth} \dashv \mathrm{Conj}$ adjunction:

$$\begin{tikzcd}
X \arrow[rr, "\eta^g_X"] \arrow[dr, "\eta^r_X"'] & & \mathrm{Conj}(\mathrm{F_g}(X)) \arrow[dd, dashed, "\exists!\,\mathrm{Conj}(f'')"] \\
& \mathrm{F_r}(X) \arrow[ur, "i_X = \mathrm{pth}_{\mathrm{F_r}(X)}"] \arrow[dr, "\forall f"'] & \\
& & \mathrm{Conj}(G)
\end{tikzcd}
\quad \text{where} \quad i_X(x, e) = i_X\eta^r_X(x) = \eta^g_X(x) = \underline{x}. \qquad (6)$$

Then since Pth is a left adjoint, $q_X\colon \mathrm{F_g}(X) \to \mathrm{Pth}(X)$ should be the coequalizer of the pair

$$\mathrm{Pth}((X \rtimes \mathrm{F_g}(X)) \rtimes \mathrm{F_g}(X \rtimes \mathrm{F_g}(X))) \underset{\mathrm{Pth}(\epsilon^r_{\mathrm{F_r}\,\mathrm{U}\,X})}{\overset{\mathrm{Pth}(\epsilon^r_X \times \mathrm{F_g}[\epsilon^r_X])}{\rightrightarrows}} \mathrm{Pth}(X \rtimes \mathrm{F_g}\,X)$$

which, using i above, we compute to be the pair $p_1, p_2\colon\ \mathrm{F_g}(X \times \mathrm{F_g}(X)) \rightrightarrows \mathrm{F_g}(X)$, where p_1 and p_2 are defined by

$$p_1(x, g) = i_X(x \cdot g, e) = \eta^g_X(x \cdot g) = \underline{x \cdot g} \quad \text{and} \quad p_2(x, g) = i_X(x, g) = g^{-1}\underline{x}g.$$

The universal property of the unit and definition on morphisms then follows easily as before. We use this detailed construction of Pth as a colimit, in the proof of Proposition 2.19.

2.10. Remark. Note that the relationship between $\pi_0 \dashv \mathrm{I}$ in Rck (or Qnd) and the abelianization in groups using the adjunction $\mathrm{Pth} \dashv \mathrm{Conj}$ has played an important role in the study of the present paper. It is conveniently pictured in the square of adjunctions of Diagram (8), where all squares of functors (I, Pth, $\mathrm{F_{ab}}$, I), (π_0, $\mathrm{F_{ab}}$, Pth, ab) and (U, I, I, Conj) commute while the square (π_0, Conj, U, ab) does not. Given a group G, the image $\pi_0(\mathrm{Conj}(G))$ is given by the set of conjugacy classes. The corresponding congruence in Qnd is given by

$$a \sim b \Leftrightarrow (\exists c \in G)(c^{-1}ac = b). \qquad (7)$$

Then the abelianization $\mathrm{ab}(G)$ is the quotient of G by the congruence generated in Grp by the identities $\{c^{-1}ac = a \mid a, c \in G\}$. In general the equivalence relation defined in (7) does not define a group congruence. A counter-example is given by the group of permutations S_3. It has three conjugacy classes given by cycles, two permutations and the unit. The derived subgroup is the alternating group A_3 which is of order 2. This shows that there are less elements in the abelianization of S_3 than conjugacy classes in S_3.

$$\begin{tikzcd} \mathsf{Rck} \arrow[r, bend right, "\pi_0"'] \arrow[d, bend right, "\mathrm{Pth}"'] \arrow[d, phantom, "\dashv"] & \mathsf{Set} \arrow[l, bend right, "\mathrm{I}"'] \arrow[l, phantom, "\top"] \arrow[d, bend right, "\mathrm{F_{ab}}"'] \arrow[d, phantom, "\dashv"] \\ \mathsf{Grp} \arrow[u, bend right, "\mathrm{Conj}"'] \arrow[r, bend right, "\mathrm{ab}"'] & \mathsf{Ab} \arrow[u, bend right, "\mathrm{U}"'] \arrow[l, bend right, "\mathrm{I}"'] \arrow[l, phantom, "\top"] \end{tikzcd} \tag{8}$$

As we mentioned before, the restriction of Conj to abelian groups gives the forgetful functor to Set. By uniqueness of left adjoints we must also have $\mathrm{F_{ab}}\,\pi_0 = \mathrm{ab}\,\mathrm{Pth}$. Finally, starting with a set X in Set we may consider it as a trivial quandle by application of I. Then we compute

$$\mathrm{Pth}(\mathrm{I}(X)) \coloneqq \mathrm{F_g}(X)/\langle (x \triangleleft a)^{-1}a^{-1}xa | a, x \in X\rangle = \mathrm{F_g}(X)/\langle x^{-1}a^{-1}xa | a, x \in X\rangle,$$

which shows that for each set X we have $\mathrm{Pth}(\mathrm{I}(X)) = \mathrm{I}\,\mathrm{F_{ab}}(X)$, which then easily gives $\mathrm{Pth}\,\mathrm{I} = \mathrm{I}\,\mathrm{F_{ab}}$, i.e. the restriction of Pth to trivial racks gives the free abelian group functor.

2.10.1. Action by inner automorphisms. It is already clear from the construction of Pth that the group of paths $\mathrm{Pth}(X)$ acts on the rack X "via representatives of the symmetries". For any x and y in X we have

$$x \cdot (\underline{y}) = x \triangleleft y,$$

which uniquely defines the action of any element in $\mathrm{Pth}(X)$.

Compare this action with the action by inner automorphisms: for each rack X, the universal property of pth_X on $\mathrm{S}\colon X \to \mathrm{Inn}(X)$ (defined in Subsection 2.1.6) gives

$$\begin{tikzcd} X \arrow[r, "\mathrm{pth}_X"] \arrow[dr, "\mathrm{S}"'] & \mathrm{Pth}(X) \arrow[d, dashed, "s"] \\ & \mathrm{Inn}(X), \end{tikzcd} \tag{9}$$

where we have omitted Conj, and s is the group homomorphism which relates the representatives of symmetries in $\mathrm{Pth}(X)$ to those in $\mathrm{Inn}(X)$. The morphism s is called the *excess* of X in [41]. It is shown to be a central extension of groups in [29, Proposition 2.26]. Note that if $N \triangleleft \mathrm{Pth}(X)$ is a normal subgroup of $\mathrm{Pth}(X)$, then $s(N)$ is a normal subgroup of $\mathrm{Inn}(X)$. Hence the congruence $\sim_N$ induced by the action of N on X always defines an *orbit congruence* ($\sim_N = \sim_{s(N)}$) in the sense of Paragraph 2.3.7.

We extend the concept of a *trail* from Paragraph 2.2.3.1.

2.11. Definition. *Given a rack X, a* trail *(in X) is the data of a pair (x, g) given by a* head $x \in X$ *and a* path $g \in \mathrm{Pth}(X)$. *The* endpoint *of such a trail is then the element obtained by the action $x \cdot g$, of g on x.*

Let us recall (see for instance [41, Section 2]) that, using the notion of an *augmented rack*, $\mathrm{Pth}(X)$ is the initial group containing representatives of the symmetries of X and acting via those symmetries on X – whereas $\mathrm{Inn}(X)$ is the terminal such. Augmented racks are given by a group G and a rack homomorphism $\iota\colon X \to \mathrm{Conj}(G)$ together with a right action of G on X such that for g, h in G and x, y in X,

1. if e is the neutral element in G, then $x \cdot e = x$;
2. $x \cdot (gh) = (x \cdot g) \cdot h$;
3. $(x \triangleleft y) \cdot g = (x \cdot g) \triangleleft (y \cdot g)$;
4. $\iota(x \cdot g) = g^{-1}\iota(x)g$.

Looking at augmented racks on a fixed rack X, a morphism between augmented racks $\iota\colon X \to G$ and $\iota'\colon X \to G'$ is given by a group homomorphism $f\colon G \to G'$ such that $f\iota = \iota'$. An example of such is given by $s\colon \mathrm{Pth}(X) \to \mathrm{Inn}(X)$ from Diagram (9). It is then easy to derive that $\mathrm{pth}_X\colon X \to \mathrm{Pth}(X)$ is initial amongst augmented racks (on X) whereas $\mathrm{S}\colon X \to \mathrm{Inn}(X)$ is terminal. This describes why Inn can be used as the reference to define such *actions by representatives of the symmetries*, described as *actions by inner automorphisms*. On the other hand, it exhibits $\mathrm{Pth}(A)$ as the *freest way* to produce an augmented rack.

2.12. Remark. As mentioned before, Pth has the crucial advantage of functoriality, i.e. for any morphism of racks $f\colon X \to Y$ (including non-surjective ones), and for any $x \in Y$, $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{Pth}(X)$, we have that

$$x \cdot (\vec{f}(g)) = x \cdot (\vec{f}(\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n})) = x \cdot (\underline{f(g_1)}^{\delta_1} \cdots \underline{f(g_n)}^{\delta_n}) = x \triangleleft^{\delta_1} f(g_1) \cdots \triangleleft^{\delta_n} f(g_n).$$

In the next paragraph, we observe that in the case of free objects $\mathrm{F_r}(X)$, these two constructions coincide ($\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{Inn}(\mathrm{F_r}(X))$ is $\mathrm{F_g}(X)$) and, most importantly for what follows, they act freely on $\mathrm{F_r}(X)$ (results first discussed in [41, 64]).

2.12.1. Free actions on free objects. By Paragraph 2.9.1, and for any set X, the group of paths $\mathrm{Pth}(\mathrm{F_r}(X)) \cong \mathrm{F_g}(X)$ is freely generated by the elements

$$\mathrm{pth}_{\mathrm{F_r}(X)}[\eta^r_X(x)] = \mathrm{pth}_{\mathrm{F_r}(X)}[(x, e)] = \underline{(x, e)}$$

for $x \in X$. Using the identification $\underline{(x, e)} \leftrightarrow \underline{x}$, for any element (x, g) of $\mathrm{F_r}(X)$ and any word $h = \underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n}$ in $\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X)$, with $h_i \in X$ and $\delta_i \in \{-1, 1\}$ for each $1 \leq i \leq n$, we have that

$$(x, g) \cdot h = (x, g) \cdot (\underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n}) = (x, g) \triangleleft^{\delta_1} (h_1, e) \cdots \triangleleft^{\delta_n} (h_n, e) = (x, gh).$$

2.13. Proposition. *The action of* $\mathrm{F_g}(X) = \mathrm{Pth}(\mathrm{F_r}(X))$ *on* $\mathrm{F_r}(X) = X \rtimes \mathrm{F_g}(X)$ *corresponds to the usual* $\mathrm{F_g}(X)$ *right action in* Set

$$(X \times \mathrm{F_g}(X)) \times \mathrm{F_g}(X) \to X \times \mathrm{F_g}(X) \colon ((a,g),h) \mapsto (a,g)\cdot h = (a, gh),$$

given by multiplication in $\mathrm{F_g}(X)$*. Such an action is free, since if* $(a,hg) = (a,g)$*, then* $hg = g$ *and thus* $h = e$*.*

Observe that $\mathrm{Inn}(\mathrm{F_r}(X))$ is generated as a group by the elements in the image of $\mathrm{S}\,\eta^r_X$. Indeed for each $(a,g) = (a, g_1^{\delta_1}\cdots g_n^{\delta_n}) = (a,e) \triangleleft^{\delta_1} (g_1,e) \cdots \triangleleft^{\delta_n} (g_n,e) = (a,e)\cdot g$, in $\mathrm{F_r}(A)$, as before, we have

$$\mathrm{S}_{(a,g)} = \mathrm{S}^{-\delta_n}_{(g_n,e)} \cdots \mathrm{S}^{-\delta_1}_{(g_1,e)}\, \mathrm{S}_{(a,e)}\, \mathrm{S}^{\delta_1}_{(g_1,e)} \cdots \mathrm{S}^{\delta_n}_{(g_n,e)};$$

see identity (4) from page 502: $\mathrm{S}_{(a,e)\cdot g} = g^{-1}\,\mathrm{S}_{(a,e)}\,g$.

We conclude that $\mathrm{Inn}(\mathrm{F_r}(X))$ is actually freely generated. Indeed, the group homomorphism $s\colon \mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X) \to \mathrm{Inn}(\mathrm{F_r}(X))$ defined in Subsection 2.10.1, is such that:

- it is surjective, since the generating set $s(X) = \{\mathrm{S}_{(x,e)} \mid x \in X\} \subset \mathrm{Inn}(\mathrm{F_r}(X))$ is the image of $X \subset \mathrm{F_g}(X)$ by s;
- it is injective, since $s(\underline{h_1}^{\delta_1}\cdots \underline{h_n}^{\delta_n}) = e$ for some $h_i \in X$ and $\delta_i \in \{-1, 1\}$ for $1 \leq i \leq n$, if and only if

$$(x,g) = (x,g)\cdot(\mathrm{S}^{\delta_1}_{(h_1,e)} \cdots \mathrm{S}^{\delta_n}_{(h_n,e)}) = (x,g)\cdot(\underline{h_1}^{\delta_1}\cdots\underline{h_n}^{\delta_n}),$$

  for all $(x,g) \in \mathrm{F_r}(X)$, which implies that $\underline{h_1}^{\delta_1}\cdots\underline{h_n}^{\delta_n} = e$ since the action of $\mathrm{F_g}(X)$ is free.

2.14. Proposition. *We may always identify* $\mathrm{Inn}(\mathrm{F_r}(X))$, $\mathrm{Pth}(\mathrm{F_r}(X))$ *and* $\mathrm{F_g}(X)$ *as well as their action on* $\mathrm{F_r}(X)$*, which is free. We refer to them as the* group of paths of $\mathrm{F_r}(X)$.

2.14.1. The kernels of induced morphisms $\vec{f}$. In this section we introduce the results which we use to describe the relationship between the group of paths Pth and the central extensions (coverings) and centralizing relations of racks and quandles.

Our Lemma 2.16 is only a slight generalization of a Lemma in [3]. We further generalize to *higher dimensions* in Part II.

2.15. Definition. *Given a group homomorphism* $f\colon G \to H$*, and a generating set* $A \subseteq G$ *(i.e. such that* $G = \langle a \mid a \in A\rangle_G$*), we define (implicitly with respect to* A*)*

(i) two elements g_a *and* g_b *in* G *are* f-symmetric (to each other) *if there exists* $n \in \mathbb{N}$ *and a sequence of pairs* (a_1,b_1), …, (a_n,b_n) *in the set* $(A\times A)\cap \mathrm{Eq}(f)$*, such that*

$$g_a = a_1^{\delta_1}\cdots a_n^{\delta_n}, \qquad \text{and} \qquad g_b = b_1^{\delta_1}\cdots b_n^{\delta_n},$$

for some $\delta_i \in \{-1, 1\}$*, where* $1 \leq i \leq n$*. Alternatively say that* g_a *and* g_b *are an* f-symmetric pair*.*

(ii) K_f *is the set of* f-symmetric paths *defined as the elements* $g \in G$ *such that* $g = g_a g_b^{-1}$ *for some* g_a *and* $g_b \in G$ *which are* f*-symmetric to each other.*

2.16. Lemma. *Given the hypotheses of Definition 2.15, the set of f-symmetric paths $\mathrm{K}_f \subseteq G$ defines a normal subgroup in G. More precisely it is the normal subgroup generated by the elements of the form ab^{-1} such that a, $b \in A$, and $(a,b) \in \mathrm{Eq}(f)$:*

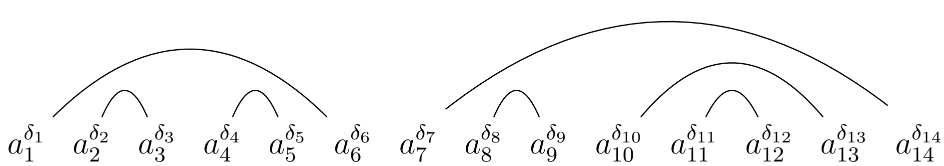
$$\mathrm{K}_f = G_f \coloneqq \langle\langle ab^{-1} \mid (a,b) \in (A \times A) \cap \mathrm{Eq}(f) \rangle\rangle_G.$$

Proof. First we show that K_f is a normal subgroup of G. Let g_a and g_b be f-symmetric (to each other). Observe that g_b^{-1} and g_a^{-1} are also f-symmetric, and thus K_f is closed under inverses. Moreover, if h_a and h_b are f-symmetric, and $g = g_a g_b^{-1}$, $h = h_a h_b^{-1}$, then $gh = k_a k_b^{-1}$, with $k_a = h_a h_a^{-1} g_a$ and $k_b = h_b h_a^{-1} g_b$ which are f-symmetric. Finally since A generates G, for any $k \in G$, kg_a and kg_b are f-symmetric to each other, and thus $kgk^{-1} \in \mathrm{K}_f$ is an f-symmetric path.

Since the generators of G_f are in the normal subgroup K_f, it suffices to show that $\mathrm{K}_f \leq G_f$. Given an f-symmetric pair g_a and g_b, we show that $g = g_a g_b^{-1} \in G_f$ by induction, on the minimum length n_g of the sequences $(a_i, b_i)_{1 \leq i \leq n}$ in the set $(A \times A) \cap \mathrm{Eq}(f)$ such that $g_a = a_1^{\delta_1} \cdots a_n^{\delta_n}$ and $g_b = b_1^{\delta_1} \cdots b_n^{\delta_n}$ for some $\delta_i \in \{-1, 1\}$. If $n_g = 1$, then g is a generator of G_f. Suppose that $g = g_a g_b^{-1} \in G_f$ for all such f-symmetric pair with $n_g < n$ for some fixed $n \in \mathbb{N}$. Then given a pair $g_a = a_1^{\delta_1} \cdots a_n^{\delta_n}$ and $g_b = b_1^{\delta_1} \cdots b_n^{\delta_n}$ for some (a_1, b_1), ..., (a_n, b_n) in the set $(A \times A) \cap \mathrm{Eq}(f)$, and $\delta_i \in \{-1, 1\}$, we have that $h_a \coloneqq a_1^{-\delta_1} g_a$ and $h_b \coloneqq b_1^{-\delta_1} g_b$ are such that $h = h_a h_b^{-1} \in G_f$ by assumption. Moreover, $g = a_1^{\delta_1} h a_1^{-\delta_1} a_1^{\delta_1} b_1^{-\delta_1}$ which is a product of elements in G_f. ∎

2.17. Observation. *Consider a function $f\colon A \to B$, and a word $\nu = a_1^{\delta_1} \cdots a_n^{\delta_n}$ with $a_i \in A$ and $\delta_i \in \{-1, 1\}$, for $1 \leq i \leq n$. This word represents an element g in the free group $\mathrm{F_g}(A)$. As usual, a* reduction *of ν consists in eliminating, in the word ν, an adjacent pair $a_i^{\delta_i} a_{i+1}^{\delta_{i+1}}$ such that $\delta_i = -\delta_{i+1}$ and $a_i = a_{i+1}$. Every element $g \in \mathrm{F_g}(A)$ represented by a word ν admits a unique* normal form *i.e. a word ν' obtained from ν after a sequence of reductions, such that there is no possible reduction in ν', but ν' still represents the same element g in $\mathrm{F_g}(A)$.*

Suppose that ν represents an element g which is in the kernel $\mathrm{Ker}(\mathrm{F_g}(f))$. The normal form of the word $f[\nu] \coloneqq f(a_1)^{\delta_1} \cdots f(a_n)^{\delta_n}$ (which represents $\mathrm{F_g}(f)(g) = e \in \mathrm{F_g}(B)$) is the empty word $\emptyset$, and thus there is a sequence of reductions of $f[\nu]$ such that the end result is $\emptyset$. From this sequence of reductions, we may deduce that $n = 2m$ for some $m \in \mathbb{N}$ and the letters in the word (or sequence) ν organize themselves in m pairs $(a_i^{\delta_i}, a_j^{\delta_j})$ (the pre-images of those pairs that are reduced at some point in the aforementioned sequence of reductions) such that $i < j$, $f(a_i) = f(a_j)$, $\delta_i = -\delta_j$, each letter of the word g appears in only one such pair and finally given any two such pairs $(a_i^{\delta_i}, a_j^{\delta_j})$ and $(a_l^{\delta_l}, a_m^{\delta_m})$, then $l < i$ (respectively $l > i$) if and only $m > j$ (respectively $m < j$), i.e. drawing lines which link those letters of the word ν that are identified by the pairing, none of these lines can cross.



Given such a pairing of the letters of ν, for each $k \in \{1,\ \ldots,\ n\}$ we write $(a_{i_k}^{\delta_{i_k}}, a_{j_k}^{\delta_{j_k}})$ for the unique pair such that either $i_k = k$ or $j_k = k$. Note that, conversely, any element g in $\mathrm{F_g}(A)$ which is represented by a word ν which admits such a pairing of its letters, is necessarily in $\mathrm{Ker}(\mathrm{F_g}(f))$.

Using this observation, we characterize the kernels of maps between free groups.

2.18. PROPOSITION. *Given a function $f\colon A \to B$, the kernel $\mathrm{Ker}(\mathrm{F_g}(f))$ of the induced group homomorphism $\mathrm{F_g}(f)\colon \mathrm{F_g}(A) \to \mathrm{F_g}(B)$ is given by the normal subgroup $\mathrm{K}_{\mathrm{F_g}(f)}$ of $\mathrm{F_g}(f)$-symmetric paths (as in Definition 2.15): $\mathrm{Ker}(\mathrm{F_g}(f)) = \mathrm{K}_{\mathrm{F_g}(f)}$.*

PROOF. The inclusion $\mathrm{Ker}(\mathrm{F_g}(f)) \supseteq \mathrm{K}_{\mathrm{F_g}(f)}$ is obvious. Consider a reduced word $\nu = a_1^{\delta_1} \cdots a_n^{\delta_n}$ of length $n \in \mathbb{N}$ which represents an element g in $\mathrm{F_g}(A)$ with $\delta_i \in \{-1, 1\}$, for $1 \leq i \leq n$ and suppose that $g \in \mathrm{Ker}(\mathrm{F_g}(f))$. Then the letters $a_k^{\delta_k}$ of the sequence (or word) $\nu \coloneqq (a_k^{\delta_k})_{1 \leq k \leq n}$ organize themselves in pairs $(a_{i_k}^{\delta_{i_k}}, a_{j_k}^{\delta_{j_k}})$ as in Observation 2.17. Define the word $\nu' = b_1^{\delta_1} \cdots b_n^{\delta_n}$ such that for each $1 \leq k \leq n$, $b_k \coloneqq a_{i_k}$. Then by construction ν' represents an element h which reduces to the empty word in $\mathrm{F_g}(A)$, so that $g = gh^{-1}$. Moreover, g and h form an f-symmetric pair, which shows that $g \in \mathrm{K}_{\mathrm{F_g}(f)}$.

■

Finally we obtain the following result.

2.19. PROPOSITION. *Given a surjective morphism of racks $f\colon X \twoheadrightarrow Y$, the kernel $\mathrm{Ker}(\vec{f})$ of the group homomorphism $\vec{f} \coloneqq \mathrm{Pth}(f)\colon \mathrm{Pth}(X) \twoheadrightarrow \mathrm{Pth}(Y)$ is given by the normal subgroup $\mathrm{K}_{\vec{f}}$ of $\vec{f}$-symmetric paths (as in Definition 2.15):*

$$\mathrm{Ker}(\vec{f}) = \mathrm{K}_{\vec{f}} = \langle\langle ab^{-1} \mid (a,b) \in \mathrm{Eq}(f) \rangle\rangle_{\mathrm{Pth}(X)}.$$

PROOF. From Subsection 2.9.1, we reconstruct the image $\vec{f}$ as in Diagram (10), where we also draw the kernels of $\mathrm{F_g}(f)$ and $\vec{f}$. Since q_X and q_Y are the coequalizers of the pairs above (see Subsection 2.9.1 for more details), and the map $\mathrm{F_g}(f \times \mathrm{F_g}(f))$ is surjective, by Lemma 1.2 in [5], the square $(*)$ is a double extension (regular pushout), and thus the comparison map k_1 is surjective. Then $\mathrm{Ker}(\vec{f})$ coincides with the image $\ker \mathrm{F_g}(f)$ along q_X, by uniqueness of (regular epi)-mono factorizations in Grp. We may compute this image to be $\mathrm{K}_{\vec{f}}$. Indeed, in elementary terms, any $g \in \mathrm{Pth}(X)$ such that $\vec{f}(g) = e$ can be "covered" by an element $h \in \mathrm{F_g}(X)$ such that $q_X(h) = g$ and $\mathrm{F_g}(f)[h] = e$ as well.

$$\begin{tikzcd}
 & \mathrm{F_g}(X \times \mathrm{F_g}(X)) \arrow[r, two heads, "\mathrm{F_g}(f \times \mathrm{F_g}(f))"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{F_g}(Y \times \mathrm{F_g}(Y)) \arrow[d, shift left] \arrow[d, shift right] \\
\mathrm{Ker}(\mathrm{F_g}(f)) \arrow[r, tail, "\ker(\mathrm{F_g}(f)"] \arrow[d, dashed, "k_1"'] & \mathrm{F_g}(X) \arrow[r, two heads, "\mathrm{F_g}(f)"] \arrow[d, two heads, "q_X"] \arrow[dr, phantom, "(*)"] & \mathrm{F_g}(Y) \arrow[d, two heads, "q_Y"] \\
\mathrm{Ker}(\vec{f}) \arrow[r, tail, "\ker(\mathrm{F_g}(f)"] & \mathrm{Pth}(X) \arrow[r, dashed, two heads, "\vec{f}"] & \mathrm{Pth}(Y)
\end{tikzcd} \tag{10}$$

By Proposition 2.18, we have that $h = h_a h_b^{-1}$ for some h_a and h_b in $\mathrm{F_g}(X)$ which are $\mathrm{F_g}(f)$-symmetric to each other. The images $q_X(h_a)$ and $q_X(h_b)$ are thus $\vec{f}$-symmetric by commutativity of $(*)$, and the quotient $g = q_X(h) = q_X(h_a)q_X(h_b)^{-1} \in \mathrm{K}_{\vec{f}}$ is an $\vec{f}$-symmetric path. ■

2.20. Notation. For a morphism of racks f, we often write f*-symmetric* (pair or path) instead of $\vec{f}$-symmetric (pair or path). An f*-symmetric trail* (x, g) is a trail with an f-symmetric path g.

2.20.1. The left adjoint Pth is not faithful. Observe that given a set A, the morphism

$$\mathrm{F_r}(A) \xrightarrow{i_A = \mathrm{Pth}_{\mathrm{F_r}(A)}} \mathrm{F_g}(A) ,$$

is not injective. Indeed the elements $(a, \underline{a}g)$ and (a, g) have the same image. We shall see that the kernel pair of i_A yields the quotient producing the free quandle from the free rack. Then the free quandle $\mathrm{F_q}(A)$ on the set A embeds in the group $\mathrm{Conj}(\mathrm{F_g}(A))$, which is why D.E. Joyce calls quandles the *algebraic theory of conjugation*. Observe, though, that not all quandles embed in a group.

2.21. Example. In the involutive quandle $Q_{ab\star}$ defined in Example 2.6, the elements $\underline{a}$ and $\underline{b}$ are identified in $\mathrm{Pth}(Q_{ab\star})$. Indeed, $\underline{a}$ and $\underline{b}$ act trivially on $Q_{ab\star}$, hence they are in the center of the group $\mathrm{Pth}(Q_{ab\star})$. Moreover, a and b are in the same connected component, and thus they are also sent to conjugates in $\mathrm{Pth}(Q_{ab\star})$, which yields $\underline{a} = \underline{b}$. Note that from there we have $\mathrm{Pth}(Q_{ab\star}) = \mathrm{F_g}(\{a, \star\})/\langle\langle a^{-1} \star^{-1} a\star \rangle\rangle_{\mathrm{F_g}(\{a,\star\})} = \mathrm{F_{ab}}(\{a, \star\}) = \mathbb{Z} \times \mathbb{Z}$, where $\mathrm{F_{ab}}$ is the free abelian group functor, and in $\mathbb{Z} \times \mathbb{Z}$, we have $\underline{a} = \underline{b} = (1, 0)$ and $\underline{\star} = (0, 1)$ (also see [29, Proposition 2.27]).

In particular, the unit of the adjuntion $\mathrm{Pth} \dashv \mathrm{Conj}$ is not injective and Pth is not faithful (note that the right adjoint Conj is faithful, but not full). As a consequence $Q_{ab\star}$ is not a subquandle of a quandle in $\mathrm{Conj}(\mathsf{Grp})$ since this would imply that $\mathrm{pth}_{Q_{ab\star}}$ is injective. We may also observe that a subquandle of a conjugation quandle is such that $(x \triangleleft y = x) \Leftrightarrow (y \triangleleft x = y)$.

2.21.1. Racks and quandles have the same group of paths. Observe that we may restrict Pth to the domain Qnd. By the same argument $\mathrm{Pth}\,\mathrm{I}\colon \mathsf{Qnd} \to \mathsf{Grp}$ (which we denote Pth) is then left adjoint to $\mathrm{Conj}\colon \mathsf{Grp} \to \mathsf{Qnd}$. We may conclude by uniqueness of left adjoints that if ${}_{\mathrm{r}}\mathrm{F_q}$ is the left adjoint to the inclusion $\mathrm{I}\colon \mathsf{Qnd} \to \mathsf{Rck}$, then $\mathrm{Pth}\ {}_{\mathrm{r}}\mathrm{F_q} \cong \mathrm{Pth}\colon \mathsf{Rck} \to \mathsf{Grp}$. The adjunction between racks and groups factorizes into Diagram (11) (where $\mathrm{Pth} \cdot \mathrm{I} = \mathrm{Pth}$, $\mathrm{I} \cdot \mathrm{Conj} = \mathrm{Conj}$, $\mathrm{Pth} \cdot {}_{\mathrm{r}}\mathrm{F_q} = \mathrm{Pth}$, ${}_{\mathrm{r}}\mathrm{F_q} \cdot \mathrm{Conj} = \mathrm{Conj}$). Considering the comment of Paragraph 2.1.9 about the idempotency axiom, we may want to rephrase this as follows: for each rack X, the quotient defining $\mathrm{Pth}(X)$ always identifies generators that would be identified in the free quandle on X.

$$
\begin{tikzcd}
\mathsf{Rck} \arrow[rr, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right, "\mathrm{Pth}"'] \arrow[dr, phantom, "\dashv"] & & \mathsf{Qnd} \arrow[ll, bend left, "\mathrm{I}"] \arrow[dl, bend right, "\mathrm{Pth}"'] \arrow[dl, phantom, "\dashv"] \\
& \mathsf{Grp} \arrow[ul, bend right, "\mathrm{Conj}"'] \arrow[ur, bend right, "\mathrm{Conj}"'] &
\end{tikzcd}
\tag{11}
$$

2.22. Working with quandles. We introduce the necessary material to make the transition from the context of racks to the context of quandles. See also the *associated quandle* in [41].

2.22.1. The free quandle on a rack. Remember from Paragraph 2.1.9 that the idempotency axiom is a consequence of the axioms of racks "for elements in the tail of a term". In order to turn a rack into a quandle the identifications that matter are thus of the form

$$x \triangleleft^{\delta_x} x \triangleleft^{\delta_x} \cdots x \triangleleft^{\delta_x} x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n,$$

where a use of the idempotency axiom cannot be avoided. Now by self-distributivity of the operations, we may write $y \coloneqq (x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n)$, and then rewrite these identities as

$$y \triangleleft^{\delta_x} y \triangleleft^{\delta_x} \cdots y \triangleleft^{\delta_x} y = y.$$

2.23. Definition. *Given a rack X, define Q_X as the relation (in Set) defined for $(x, y) \in X \times X$ by $(x, y) \in Q_X$ if and only if $x = y \triangleleft^k y$ for some integer k (see Paragraph 2.1.9), where $y \triangleleft^0 y \coloneqq y$.*

2.24. Lemma. *Given a rack X, the relation Q_X defines a congruence on X.*

Proof.

1. The relation Q_X is reflexive by definition.

2. As aforementioned, for x and a in some rack, any chain $a \triangleleft^k a$ for some $k \in \mathbb{Z}$ is such that $x \triangleleft (a \triangleleft^k a) = x \triangleleft a$. Hence Q_X is symmetric since $b = a \triangleleft^k a$ implies that $b \triangleleft^{-k} b = b \triangleleft^{-k} a = a$.

3. Now Q_X is transitive by self-distributivity.

4. And finally it is internal since if $a = b \triangleleft^k b$ and $c = d \triangleleft^l d$ then $a \triangleleft c = (b \triangleleft^k b) \triangleleft (d \triangleleft^l d) = (b \triangleleft^k b) \triangleleft d = (b \triangleleft d) \triangleleft^k (b \triangleleft d)$.

∎

2.25. Lemma. *Given a rack X, then a pair of elements $(x,y) \in X \times X$ is in the kernel pair $\mathrm{Eq}({}^r\eta^q_X)$ of ${}^r\eta^q_X \colon X \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X)$ if and only if $y = x \triangleleft^n x$ for some integer n, i.e. $Q_X = \mathrm{Eq}({}^r\eta^q_X)$.*

Proof. Since Rck is a Barr-exact category [1], it suffices to show that the quotient of X by the equivalence relation Q_X (on the left) is the same as the quotient of X by $\mathrm{Eq}({}^r\eta^q_X)$ (on the right):

$$X \xrightarrow{q} X/Q_X \qquad\qquad X \xrightarrow{{}^r\eta^q_X} {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X).$$

For this we show that X/Q_X is a quandle and that q has the same universal property as ${}^r\eta^q_X$. Indeed we have that $q(a) \triangleleft q(a) = q(a \triangleleft a) = q(a)$ since $(a, a \triangleleft a) \in Q_X$ for each a. Finally observe that if $f \colon X \to Q$ is a rack homomorphism such that Q is a quandle, then we necessarily have that f coequalizes the projections $\pi_1,\ \pi_2 \colon Q_X \rightrightarrows X$ of the congruence Q_X. We then conclude by the universal property of the coequalizer. ∎

2.25.1. Galois theory of quandles in racks. We may now study the Galois structure ${}_r\Gamma_q :=(\mathsf{Rck},\ \mathsf{Qnd},\ {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}},\ \mathrm{I},\ {}^r\eta^q,\ {}^r\epsilon^q,\ \mathcal{E})$ where $\mathcal{E}$ is the class of surjective morphisms (see Section 1.1 and [55]).

Since Qnd is a Birkhoff subcategory of Rck, for ${}_r\Gamma_q$ to be admissible, it suffices to show that for each rack X the kernel pair $\mathrm{Eq}({}^r\eta^q_X)$ of the unit permutes with other congruences on X (see Section 1.1.1). Observe that this is not a consequence of Lemma 2.4.

2.26. Lemma. *Given a rack X, then the congruence $Q_X = \mathrm{Eq}({}^r\eta^q_X)$ permutes with any other internal relation R on X.*

Proof. We prove that a pair $(a,b) \in X \times X$ is in $\mathrm{Eq}({}^r\eta^q_X)\, R$ if and only if it is in $R\ \mathrm{Eq}({}^r\eta^q_X)$. As in Lemma 2.4, we show that if there is $c \in X$ such that (a,c) is in one of these relations (say for instance $\mathrm{Eq}({}^r\eta^q_X)$) and (c,b) in the other one (R), then there is a $c' \in X$ such that (a,c') is in the latter (R) and (c',b) in the former ($\mathrm{Eq}({}^r\eta^q_X)$). Now observe that if $(x,y) \in R$, then $(x,y) \triangleleft^k (x,y) = (x \triangleleft^k x, y \triangleleft^k y)$ is in R for any integer k. The result then follows from reading the following diagram for any $k \in \mathbb{Z}$, where horizontal arrows represent membership in $\mathrm{Eq}({}^r\eta^q_X)$ and vertical arrows represent membership in R. Indeed from the top right corner below we construct the bottom left corner and the other way around:

$$\begin{tikzcd}
c_1 \triangleleft^{-k} c_1 = a \arrow[r, shift left, "\mathrm{S}_a^k"] \arrow[d, dash] & c_1 = a \triangleleft^k a \arrow[l, shift left, "\mathrm{S}_{c_1}^{-k}"] \arrow[d, dash] \\
b \triangleleft^{-k} b = c_2 \arrow[r, shift left, "\mathrm{S}_{c_2}^k"] & b = c_2 \triangleleft^k c_2 \arrow[l, shift left, "\mathrm{S}_b^{-k}"]
\end{tikzcd}$$

where we use the fact that if $x = y \triangleleft^k y$ then $\mathrm{S}_x = \mathrm{S}_y$. Algebraically we read $(a, c_1) \in Q_X$ implies $c_1 \triangleleft^{-k} c_1 = a$ for some $k \in \mathbb{Z}$ and $(c_1, b) \in R$ implies $(c_1 \triangleleft^{-k} c_1, b \triangleleft^{-k} b) \in R$, thus choosing $c_2 = b \triangleleft^{-k} b$ yields one of the implications. The other direction translates similarly. ∎

2.27. Remark. Given a rack X, the congruence Q_X is not an orbit congruence in general. For instance, observe that $Q_{\mathrm{F_r}(\{a,b\})}$ contains the pairs $(a, a \triangleleft a)$ and $(b, b \triangleleft b)$. Suppose by contradiction that there is a normal subgroup $N \leq \mathrm{Inn}(\mathrm{F_r}(\{a,b\})) = \mathrm{F_g}(\{a,b\})$ for which $\sim_N = Q_{\mathrm{F_r}(\{a,b\})}$. Then since $\mathrm{F_g}(\{a,b\})$ acts freely on $\mathrm{F_r}(X)$, both inner automorphisms S_a and S_b need to be in N. This leads to a contradiction since $a \sim_N (a \triangleleft b)$ but $(a, a \triangleleft b) \notin Q_{\mathrm{F_r}(\{a,b\})}$. By contrast $Q_{\mathrm{F_r}(\{*\})}$ is of course an orbit congruence.

2.28. Corollary. *Quandles form a strongly Birkhoff (and thus admissible) subcategory of* Rck.

Proof. By Proposition 5.4 in [19], the reflection squares of surjective morphisms are double extensions (see Section 1.1.1). This implies the admissibility of the Galois structure ${}_r\Gamma_q$, for instance by [38, Proposition 2.6]. ∎

Note that the left adjoint ${}_{\mathrm{r}}\mathrm{F_q}$ is actually semi-left-exact as we may deduce from the fact that "connected components are connected" (see Paragraph 2.5.1).

2.29. Proposition. *Any pullback of the form*

$$\begin{tikzcd} C_a \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "{[a]}"] \\ X \arrow[r, "{}^r\eta^q_X"'] & {}_{\mathrm{r}}\mathrm{F_q}(X), \end{tikzcd}$$

in Rck*, is preserved by the reflector* ${}_{\mathrm{r}}\mathrm{F_q}$*, i.e.* ${}_{\mathrm{r}}\mathrm{F_q}(C_a) = 1$*; and thus by [78, Theorem 2.1], we conclude that* ${}_{\mathrm{r}}\mathrm{F_q}$ *is* semi-left-exact *in the sense of [24, 18].*

Proof. Observe that $X \times 1 \cong X$ and thus elements of the pullback C_a are merely elements $x \in X$ such that that ${}^r\eta^q(x) = [a] \in {}_{\mathrm{r}}\mathrm{F_q}(X)$ i.e. all elements x and y in C_a are such that there is $k \in \mathbb{Z}$ such that $x = y \triangleleft^k y$. Hence by Lemma 2.25 the image of this pullback by ${}_{\mathrm{r}}\mathrm{F_q}$ gives indeed 1, which concludes the proof. ∎

Observe that there is a limit to the exactness properties satisfied by ${}_{\mathrm{r}}\mathrm{F_q}$: we already saw in Paragraph 2.5.1 that ${}_{\mathrm{r}}\mathrm{F_q}$ cannot preserve finite products, since $\pi_0 \colon \mathsf{Qnd} \to \mathsf{Set}$ does but $\pi_0\ {}_{\mathrm{r}}\mathrm{F_q} \colon \mathsf{Rck} \to \mathsf{Set}$ does not. Moreover, since Qnd is an idempotent subvariety of Rck, Proposition 2.6 of [25] induces that ${}_{\mathrm{r}}\mathrm{F_q}$ does not have *stable units* (in the sense of [24]).

To conclude, we show that, besides semi-left-exactness, the ${}_{\mathrm{r}}\mathrm{F_q}$-covering theory is "trivial" in the sense that all surjections are ${}_{\mathrm{r}}\mathrm{F_q}$-central. We use the general strategy which was stated in Section 1.1.3. Since the Galois structure is strongly Birkhoff, the "first step influence" is as usual:

2.30. Lemma. *A surjective morphism* $f \colon X \to Y$*, in the category of racks, is* ${}_{\mathrm{r}}\mathrm{F_q}$*-trivial if and only if* $Q_X \cap \mathrm{Eq}(f) = \Delta_X$*.*

PROOF. The morphism f is trivial if and only if the reflection square at f is a pullback (see Section 1.1.1, Diagram (1)). Since this reflection square is a double extension, it suffices for the comparison map to be injective. Since the square is a pushout, the kernel pair of the comparison map is given by the intersection $Q_X \cap \mathrm{Eq}(f)$ of the kernel pairs of q_X and f respectively. ∎

2.31. PROPOSITION. *All surjections $f\colon X \to Y$ in the category of racks are ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$-central.*

PROOF. Consider the canonical projective presentation $\epsilon^r_Y\colon \mathrm{F_r}(UY) \to Y$, and take the pullback of f along ϵ^r_Y. This yields a morphism

$$\bar{f}\colon X \times_Y \mathrm{F_r}(UY) \to \mathrm{F_r}(UY).$$

Now any morphism $g\colon X \to \mathrm{F_r}(Y)$ with free codomain is ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$-trivial since if $x = x \triangleleft^k x$ in X for some integer k and if, moreover, $f(x) = f(x) \triangleleft^k f(x)$ in $\mathrm{F_r}(Y)$, then $f(x)^k = e$ by the free action of $\mathrm{Pth}(\mathrm{F_r}(Y))$ on $\mathrm{F_r}(Y)$. However this can only be if $k = 0$, which implies that $Q_X \cap \mathrm{Eq}(f) = \Delta_X$. ∎

2.31.1. TOWARDS THE FREE QUANDLE. Given a set A, in order to develop a good candidate description for the free quandle on A (see also [61]), we may now consider $\mathrm{F_q}(A)$ as the free quandle on the rack $\mathrm{F_r}(A)$. As aforementioned and roughly speaking, the following identifications between terms:

$$x \triangleleft^{\delta_x} x \triangleleft^{\delta_x} \cdots x \triangleleft^{\delta_x} x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k, \tag{12}$$

define the relation $Q_{\mathrm{F_r}(A)}$ such that $\mathrm{F_q}(A) = \mathrm{F_r}(A)/Q_{\mathrm{F_r}(A)}$.

We want to select one representative $(a, g) \in A \rtimes \mathrm{F_g}(A)$ for each equivalence class determined by these identifications. Thinking in terms of trails, we observe that if (a, g) and (b, h) are identified, then they must have the same head $a = b$. We thus focus on the paths and use a clever semi-direct product decomposition of $\mathrm{F_g}(A)$.

2.31.1.1. Characteristic of a path We have the following commutative diagram in Set,

$$\begin{tikzcd} A \arrow[r, "\eta^g_A"] \arrow[d, "\mathrm{Cst}"'] & \mathrm{F_g}(A) \arrow[d, dotted, "\chi := \mathrm{F_g}(\mathrm{Cst})"] \\ 1 \arrow[r, "\eta^g_1"'] & \mathbb{Z} = \mathrm{F_g}(1), \end{tikzcd}$$

where $\mathbb{Z}$ is the underlying set of the additive group of integers, and the composite $\eta^g_1\,\mathrm{Cst}$ is the constant function with image $1 \in \mathbb{Z}$. Given an element $g \in \mathrm{F_g}(A)$, there exists a decomposition $g = g_1^{\delta_1} \cdots g_n^{\delta_n}$ for some $g_i \in A$ and exponents $\delta_i = \{-1, 1\}$, with $1 \leq i \leq n$. The characteristic function sums up the exponents $\chi(g) = \sum_{i=1}^n \delta_i$ (of course the result doesn't depend on the chosen decomposition of g). We may then classify paths in $\mathrm{F_g}(A)$ in terms of their characteristic (i.e. their image by χ). Looking at Equation (12), two terms with same head, and same characteristic, that are moreover identified by $Q_{\mathrm{F_r}(A)}$, must actually be equal. In other words, given a fixed head a each equivalence class $[(a, g)]$ in $\mathrm{F_q}(A)$ has only one representative (a, g') such that the path g' is of a given characteristic.

2.31.1.2. Characteristic zero and semi-direct product decomposition The kernel of χ defines a normal subgroup $\mathrm{F}^{\circ}_{\mathrm{g}}(A) \leq \mathrm{F}_{\mathrm{g}}(A)$ which is characterized (see [61] and Proposition 2.18) by

$$\mathrm{F}^{\circ}_{\mathrm{g}}(A) = \langle ab^{-1} \mid a, b \in A \rangle_{\mathrm{F}_{\mathrm{g}}(A)}.$$

Then for each $a \in A$, we may identify $\mathbb{Z}$ with the subgroup $\langle a^n \mid n \in \mathbb{Z} \rangle \leq \mathrm{F}_{\mathrm{g}}(A)$ which may be seen as the subgroup of $\mathrm{F}_{\mathrm{g}}(A)$ which fixes $[(a, e)] \in \mathrm{F}_{\mathrm{q}}(A) \coloneqq \mathrm{F}_{\mathrm{r}}(A)/Q_{\mathrm{F}_{\mathrm{r}}(A)}$. This then gives a splitting for χ, on the left, yielding the split short exact sequence on the right:

$$\iota_a \colon \mathbb{Z} \to \mathrm{F}_{\mathrm{g}}(A) \;:\; k \mapsto a^k \qquad\qquad \begin{tikzcd} \mathrm{F}^{\circ}_{\mathrm{g}}(A) \arrow[r, tail, "\nu_A"] & \mathrm{F}_{\mathrm{g}}(A) \arrow[r, two heads, "\chi"] & \mathbb{Z} \arrow[l, bend left, "\iota_a"] \end{tikzcd}$$

2.31.1.3. Characteristic zero representatives Given an element $a \in A$, any $g \in \mathrm{F}_{\mathrm{g}}(A)$ decomposes uniquely as $\underline{a}^{\chi(g)} g_0$, where $g_0 = \underline{a}^{-\chi(g)} g$. This defines a function sending equivalence classes $[(a, g)] \in \mathrm{F}_{\mathrm{q}}(A)$, to their representatives of characteristic zero (a, g_0). Note that, for two different a and b in A, the construction of g_0 will vary, however elements of $\mathrm{F}_{\mathrm{r}}(A)$ with different heads are always sent to different equivalence classes in $\mathrm{F}_{\mathrm{q}}(A)$.

2.31.1.4. Transporting structure This function is indeed bijective, and thus we may transport the quandle structure from the quotient $\mathrm{F}_{\mathrm{r}}(A)/Q_{\mathrm{F}_{\mathrm{r}}(A)}$ to the set of representatives $A \times \mathrm{F}^{\circ}_{\mathrm{g}}(A)$. More explicitly we compute for (b, h) and (a, g) in $\mathrm{F}_{\mathrm{r}}(A)$ that

$$(a, g_0) \triangleleft (b, h_0) = (a, g_0 h_0^{-1} \underline{b} h_0),$$

where $w \coloneqq g_0 h_0^{-1} \underline{b} h_0$ is not of characteristic zero. We then want to take $w_0 = \underline{a}^{-1} g_0 h_0^{-1} \underline{b} h_0$ and define in $\mathrm{F}_{\mathrm{q}}(A)$:

$$(a, g_0) \triangleleft (b, h_0) \coloneqq (a, w_0).$$

2.31.2. The free quandle. After this analysis, we may confidently build the free quandle (first described in [61]) as follows.

Given a set A the free quandle on A is given by

$$\mathrm{F}_{\mathrm{q}}(A) \coloneqq A \rtimes \mathrm{F}^{\circ}_{\mathrm{g}}(A) \coloneqq \{(a, g) \mid g \in \mathrm{F}^{\circ}_{\mathrm{g}}(A);\; a \in A\},$$

where the operations on $\mathrm{F}_{\mathrm{q}}(A)$ are defined for (a, g) and (b, h) in $A \rtimes \mathrm{F}^{\circ}_{\mathrm{g}}(A)$ by

$$(a, g) \triangleleft (b, h) \coloneqq (a, \underline{a}^{-1} g h^{-1} \underline{b} h) \quad \text{and} \quad (a, g) \triangleleft^{-1} (b, h) \coloneqq (a, \underline{a} g h^{-1} \underline{b}^{-1} h).$$

As before, g is the *path* component and a is the *head* component of the so-called *trail* $(a, g) \in \mathrm{F}_{\mathrm{q}}(A)$ and we say that an element (b, h) *acts on an element* (a, g) *by endpoint.* These operations indeed define a quandle structure.

From there, we translate all main results from the construction of free racks. Looking for the unit of the adjunction, we have the injective function $\eta^q_A \colon A \to \mathrm{F}_{\mathrm{q}}(A) : a \mapsto (a, e)$.

Moreover, since any element $g \in \mathrm{F}^{\circ}_{\mathrm{g}}(A)$ decomposes as a product $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{F}_{\mathrm{g}}(A)$ for some $g_i \in A$ and exponents $\delta_i \in \{-1, 1\}$, with $1 \leq i \leq n$, and $\sum_i \delta_i = 0$, we have, for any $(a, hg) \in \mathrm{F_q}(A)$ with g and $h \in \mathrm{F}^{\circ}_{\mathrm{g}}(A)$, a decomposition as

$$\begin{aligned}(a, hg) &= (a, h\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = (a, \underline{a}^{\sum_i -\delta_i} h\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = (a, \underline{a}^{-\delta_n} \cdots \underline{a}^{-\delta_1} h\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \\ &= (a, h) \lhd^{\delta_1} (g_1, e) \cdots \lhd^{\delta_n} (g_n, e).\end{aligned}$$

Observing that if $\underline{g_i}^{-\delta_i} = \underline{g_{i+1}}^{\delta_{i+1}}$ for some $(a, g) = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \in \mathrm{F_q}(A)$ as above, then

$$\begin{aligned}&(a, e) \lhd^{\delta_1} (g_1, e) \cdots \lhd^{\delta_{i-1}} (g_{i-1}, e) \lhd^{\delta_{i+2}} (g_{i+2}, e) \cdots \lhd^{\delta_n} (g_n, e) = \\ &\quad = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_{i-1}}^{\delta_{i-1}} \underline{g_{i+2}}^{\delta_{i+2}} \cdots \underline{g_n}^{\delta_n}) = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_{i-1}}^{\delta_{i-1}} \underline{g_i}^{\delta_i} \underline{g_{i+1}}^{\delta_{i+1}} \underline{g_{i+2}}^{\delta_{i+2}} \cdots \underline{g_n}^{\delta_n}) \\ &\quad = (a, e) \lhd^{\delta_1} (g_1, e) \cdots \lhd^{\delta_n} (g_n, e),\end{aligned}$$

which expresses the first axiom of racks, using group cancellation, as before.

From there we derive the universal property of the unit: given a function $f\colon A \to Q$ for some quandle Q, we show that f factors uniquely through η^q_A. Given an element $(a, g) \in \mathrm{F_q}(A)$, we have that for any decomposition $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}$ as above, we must have

$$f(a, g) = f(a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = f((a, e) \lhd^{\delta_1} (g_1, e) \cdots \lhd^{\delta_n} (g_n, e)) = f(a) \lhd^{\delta_1} f(g_1) \cdots \lhd^{\delta_n} f(g_n)$$

which uniquely defines the morphism of quandles $f\colon \mathrm{F_q}(A) \to Q$ as extension of f along η^q_A. This extension is well defined since equal such decompositions in $\mathrm{F_q}(A)$ are equal after f by the first axiom of racks.

Finally the left adjoint $\mathrm{F_q}\colon \mathsf{Set} \to \mathsf{Qnd}$ of the forgetful functor $\mathrm{U}\colon \mathsf{Qnd} \to \mathsf{Set}$ with unit η^q is then defined on functions $f\colon A \to B$ by

$$\mathrm{F_q}(f) \coloneqq f \times \mathrm{F}^{\circ}_{\mathrm{g}}(f)\colon A \rtimes \mathrm{F}^{\circ}_{\mathrm{g}}(A) \to B \rtimes \mathrm{F}^{\circ}_{\mathrm{g}}(B),$$

where $\mathrm{F}^{\circ}_{\mathrm{g}}(f)$ is the restriction of $\mathrm{F_g}(f)$ to the normal subgroup $\mathrm{F}^{\circ}_{\mathrm{g}}(A) \leq \mathrm{F_g}(A)$, whose image is in $\mathrm{F}^{\circ}_{\mathrm{g}}(B)$. This defines quandle homomorphisms. Also functoriality of $\mathrm{F_q}$ and naturality of η^q are immediate.

2.31.2.1. Free action of $\mathrm{F}^{\circ}_{\mathrm{g}}(A)$ Now remember the action by inner automorphisms of $\mathrm{F_g}(A) = \mathrm{Pth}(\mathrm{F_q}(A))$ defined by the commutative diagram in Set:

$$\begin{tikzcd} A \arrow[rr, "\eta^g_A"] \arrow[dr, "\eta^q_A"'] & & \mathrm{F_g}(A) \arrow[dd, dotted, "s"] \\ & \mathrm{F_q}(A) \arrow[ur, "\mathrm{pth}_{\mathrm{F_q}(A)}"'] \arrow[dr, "\mathrm{S}"'] & \\ & & \mathrm{Inn}(\mathrm{F_q}(A)), \end{tikzcd}$$

where s is the group homomorphism induced by the universal property of η^g_A or equivalently that of $\mathrm{pth}_{\mathrm{F_q}(A)}$.

This action is *not* in general given by left multiplication in $\mathrm{F}^{\circ}_{\mathrm{g}}(A)$, since in particular an h in $\mathrm{F_g}(A)$ is of course not always of characteristic zero. However, from Paragraph 2.31.2 we deduce that whenever $h \in \mathrm{F}^{\circ}_{\mathrm{g}}(A)$, the action of h on an element $(a, g) \in \mathrm{F_q}(A)$ gives (a, gh) as before.

2.32. Proposition. *The action of* $\mathrm{F}^{\circ}_{\mathrm{g}}(A)$ *on* $\mathrm{F}_{\mathrm{q}}(A)$ *given via the restriction*

$$\mathrm{F}^{\circ}_{\mathrm{g}}(A) \xrightarrow{s^{\circ}} \mathrm{Inn}^{\circ}(\mathrm{F}_{\mathrm{q}}(A)),$$

of s *thus corresponds to the usual left-action of* $\mathrm{F}^{\circ}_{\mathrm{g}}(A)$ *in* Set*:* $(A \times \mathrm{F}^{\circ}_{\mathrm{g}}(A)) \times \mathrm{F}^{\circ}_{\mathrm{g}}(A)) \to A \times \mathrm{F}^{\circ}_{\mathrm{g}}(A)$*, given by multiplication in* $\mathrm{F}^{\circ}_{\mathrm{g}}(A)$*. Such an action is free since if* $(a, gh) = (a, g)$*, then* $gh = g$ *and thus* $h = e$*.*

2.32.1. The group of paths of a quandle. Observe that the construction of χ for the free group $\mathrm{F}_{\mathrm{g}}(A) = \mathrm{Pth}(\mathrm{F}_{\mathrm{r}}(A))$ generalizes to any rack X. The function $\mathrm{Cst}\colon X \to 1$ is actually a rack homomorphism to the trivial rack 1. It thus induces a group homomorphism $\chi = \mathrm{Pth}(\mathrm{Cst})$:

$$\begin{tikzcd} X \arrow[r, "\mathrm{pth}_X"] \arrow[d, "\mathrm{Cst}"'] & \mathrm{Pth}(X) \arrow[d, dashed, "\chi = \mathrm{Pth}(\mathrm{Cst})"] \\ 1 \arrow[r, "\mathrm{pth}_1"'] & \mathbb{Z} = \mathrm{Pth}(1). \end{tikzcd}$$

As in the case of the free rack, we have the short exact sequence of groups:

$$\mathrm{Pth}^{\circ}(X) \overset{\nu_X}{\rightarrowtail} \mathrm{Pth}(X) \overset{\chi}{\twoheadrightarrow} \mathbb{Z} = \mathrm{Pth}(1),$$

where $\nu_X\colon \mathrm{Pth}^{\circ}(X) \to \mathrm{Pth}(X)$ is the kernel of χ. This construction defines a functor $\mathrm{Pth}^{\circ}\colon \mathsf{Rck} \to \mathsf{Grp}$. Most importantly it defines a functor $\mathrm{Pth}^{\circ}\colon \mathsf{Qnd} \to \mathsf{Grp}$ which can be interpreted as sending a quandle to its group of equivalence classes of primitive paths, such that two primitive paths are identified if one can be obtained from the other with respect to the axioms defining quandles. In the same way that Pth describes homotopy classes of paths in racks, Pth° describes homotopy classes of paths in quandles, as it was already explained in [29] and we shall rediscover in the covering theory described below.

2.32.1.1. The transvection group As in the case of free groups, given a rack X, Proposition 2.19 implies that the kernel $\mathrm{Pth}^{\circ}(X)$ of χ is characterized as the subgroup:

$$\mathrm{Pth}^{\circ}(X) = \langle \underline{a}\, \underline{b}^{-1} \mid a,\, b \in X \rangle_{\mathrm{Pth}(X)}, \tag{13}$$

which is the definition that was used by D.E. Joyce in [61]. Then the restriction of the quotient $s\colon \mathrm{Pth}(X) \to \mathrm{Inn}(X)$ (defined in Subsection 2.1.7) yields the normal subgroup

$$\mathrm{Inn}^{\circ}(X) \coloneqq \langle \underline{a}\, \underline{b}^{-1} \mid a,\, b \in X \rangle_{\mathrm{Inn}(X)},$$

which was called the *transvection group* of X by D.E. Joyce.

This transvection group plays an important role in the literature. In the context of this work, we understand that the construction Pth° has better properties such as functoriality, and is of more significance to the theory of coverings than its image Inn° within inner automorphisms.

2.32.1.2. The case of free quandles Observe that for a set X, $\mathrm{Pth}^{\circ}(\mathrm{F}_{\mathrm{q}}(X)) = \mathrm{F}^{\circ}_{\mathrm{g}}(X)$ (for instance by Equation (13)). As in the case of free racks we get that:

2.33. Proposition. *Given a set A, we may identify the three groups* $\mathrm{Inn}^\circ(\mathrm{F_q}(A)) = \mathrm{Pth}^\circ(\mathrm{F_q}(A)) = \mathrm{F}^\circ_{\mathrm{g}}(A)$*, and their actions on* $\mathrm{F_q}(A)$*. We refer to them as the* group of paths of $\mathrm{F_q}(A)$*. This group acts freely on* $\mathrm{F_q}(A)$ *by Corollary 2.32.*

Proof. Given a set A, the morphism $s^\circ \colon \mathrm{F}^\circ_{\mathrm{g}}(A) \to \mathrm{Inn}^\circ(\mathrm{F_q}(A))$ is a group isomorphism:

- it is surjective, since $\mathrm{Inn}^\circ(\mathrm{F_q}(A))$ is generated by the image of $AA^{-1} \subset \mathrm{F}^\circ_{\mathrm{g}}(A)$ by s which is the set $s(A)s(A)^{-1} = \{\mathrm{S}_{(a,e)}(\mathrm{S}_{(b,e)})^{-1} \mid a, b \in A\} \subset \mathrm{Inn}^\circ(\mathrm{F_q}(A))$.
- it is injective, as before because of the free action of $\mathrm{F}^\circ_{\mathrm{g}}(A)$ via s°.

■

2.33.0.1. Inner automorphism groups In the case of quandles, the group of inner automorphisms $\mathrm{Inn}(\mathrm{F_q}(A))$ is not isomorphic to $\mathrm{F_g}(A)$ in general. However, the only counter-example is actually the case $A = \{1\}$: $\mathrm{F_q}(\{1\}) = \{1\}$ is the trivial quandle on one element and $\mathrm{Inn}(\{1\}) = \{e\}$ is the trivial group, whereas $\mathrm{F_g}(\{1\})$ is $\mathbb{Z}$. Of course we do have $\mathrm{F}^\circ_{\mathrm{g}}(\{1\}) = \{e\}$. Now in all the other cases $\mathrm{Inn}(\mathrm{F_q}(A)) \cong \mathrm{F_g}(A)$. The case $A = \emptyset$ is trivial. Then whenever

$$x \lhd^{\delta_x} x \lhd^{\delta_x} \cdots x \lhd^{\delta_x} x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_k} a_k = x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_k} a_k,$$

it suffices to pick $y \neq x \in A$ and then $y \lhd^{\delta_x} x \lhd^{\delta_x} x \lhd^{\delta_x} \cdots x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_k} a_k \neq y \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_k} a_k$, showing that in $\mathrm{Inn}(\mathrm{F_q}(A))$: $\underline{x}^{\delta_x}\underline{x}^{\delta_x} \cdots \underline{x}^{\delta_x}\underline{a_1}^{\delta_1} \cdots \underline{a_k}^{\delta_k} \neq \underline{a_1}^{\delta_1} \cdots \underline{a_k}^{\delta_k}$, just as in $\mathrm{Inn}(\mathrm{F_r}(A))$.

3. Covering theory of racks and quandles

In this section we study the relative notion of centrality induced by the sphere of influence of Set in Rck, with respect to extensions (surjective homomorphisms). Remember that pullbacks of primitive extensions (surjections in Set) along the unit η induce the concept of trivial extensions, which we saw are those extensions which reflect *loops*. Central extensions in Rck are those from which a trivial extension can be reconstructed by pullback along another extension. Equivalently, central extensions are those extensions whose pullback, along a projective presentation of their codomain, is trivial. In Section 3.1 we thus look for a condition (C) such that, if a surjective rack homomorphism $f \colon A \to B$ satisfies (C), then the pullback t of f along $\epsilon^r_B \colon \mathrm{F_r}(B) \to B$ reflects loops (see Section 1.1 and references there).

3.1. One-dimensional coverings. Quandle coverings were defined in [29], and shown to characterize Γ_q-central extensions of quandles in [30]. We give the same definition for rack coverings (already suggested in M. Eisermann's work), which we then characterize in several ways. In Section 3.15 we further show that these are exactly the central extensions of racks.

Remember that in dimension zero, a rack A is actually a set, if *zero-dimensional data*, i.e. an element $a \in A$, acts trivially on any element $x \in A$: $x \lhd a = x$. We saw that this may be expressed by the fact that $\mathrm{Pth}(A)$ acts trivially on A or alternatively by the fact that any two elements which are connected by a primitive path are actually equal.

Now in *dimension one*, an extension $f\colon A \twoheadrightarrow B$ is a covering if *one-dimensional data*, i.e. a pair (a, b) in the kernel pair of f, acts trivially on any element in A:

3.2. Definition. *A morphism of racks $f\colon A \to B$ is said to be a* covering *if it is surjective and for each pair $(a, b) \in \mathrm{Eq}(f)$, and any $x \in A$ we have $x \triangleleft a \triangleleft^{-1} b = x$.*

Of course a trivial example is given by surjective functions between sets (the primitive extensions). The following implies that central extensions are coverings:

3.3. Lemma. *Coverings are preserved and reflected by pullbacks along surjections in* Rck.

Proof. Same proof as in [31] see also [30]. ∎

3.3.1. Coverings and the group of paths. Observe that given data f, x, a and b, such as in Definition 3.2, we have in particular that $x \triangleleft^{-1} a = x \triangleleft^{-1} a \triangleleft a \triangleleft^{-1} b = x \triangleleft^{-1} b$. In fact we can easily deduce that f is a covering if and only if for all such x, a and b as before

$$x \triangleleft^{-1} a \triangleleft b = x.$$

This is to say that f is a covering if and only if any path of the form $\underline{a}\,\underline{b}^{-1}$ or $\underline{a}^{-1}\underline{b} \in \mathrm{Pth}(A)$, for a and b in A, such that $f(a) = f(b)$, acts trivially on elements in A. But then f is a covering if and only if the subgroup of $\mathrm{Pth}(A)$ generated by those elements acts trivially on elements of A. Now, given $g \in \mathrm{Pth}(A)$, if $z \cdot g = z$ for all z in A, then also $x \cdot a^{-1} \cdot g \cdot a = (x \triangleleft^{-1} a) \cdot g \cdot a = (x \triangleleft^{-1} a) \cdot a = x$ for all $a \in A$. Hence we conclude that f is a covering if and only if the normal subgroup $\langle\langle ab^{-1} \mid (a, b) \in \mathrm{Eq}(f) \rangle\rangle_{\mathrm{Pth}(A)}$ acts trivially on elements of A. Finally by Proposition 2.19 we get the following result which illustrates the importance of Pth in the covering theory of racks and quandles.

3.4. Theorem. *Given a surjective morphism $f\colon A \to B$ in* Rck *(or in* Qnd*), the following conditions are equivalent:*

1. *f is a covering;*
2. *the group of $\vec{f}$-symmetric paths $\mathrm{K}_{\vec{f}}$ acts trivially on A (as a subgroup of $\mathrm{Pth}(A)$) – i.e. any f-symmetric trail loops in A;*
3. *$\mathrm{Ker}(\vec{f})$ acts trivially on A (as a subgroup of $\mathrm{Pth}(A)$);*
4. *$\mathrm{Ker}(\vec{f})$ is a subobject of the kernel $\mathrm{Ker}(s)$, where $s\colon \mathrm{Pth}(A) \to \mathrm{Inn}(A)$ is the canonical quotient described in Paragraph 2.10.1.*

Proof. The statements (1), (2) and (3) are equivalent by the previous paragraph (and thus by Proposition 2.19). Statement (4) is merely a way to rephrase (3) using the fact that elements of the inner automorphism groups are defined by their action. ∎

3.5. Remark. Note that a morphism of groups is central if and only if it gives a covering in racks [29, Example 2.34], see also [29, Example 1.2] and comments below. Conversely any covering in racks induces a central extension between the groups of paths [29, Proposition 2.39]. However, certain morphisms, such as $f\colon Q_{ab\star} \to \{*\}$, which are not central in Rck (or Qnd) are sent by Pth to central extensions of groups, e.g.

$$\vec{f}: \ \mathrm{Pth}(Q_{ab\star}) = \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} = \mathrm{Pth}(\{*\}) \ : \ (k,l) \mapsto k+l.$$

As it was observed by M. Eisermann in Qnd, we have:

3.6. Corollary. *A rack covering $f\colon A \to B$ induces a surjective morphism of groups $\bar{f}\colon \mathrm{Pth}(B) \to \mathrm{Inn}(A)$ such that $\vec{f}\bar{f} = s$ and thus induces an action of $\mathrm{Pth}(B)$ on A given for $g_B \in \mathrm{Pth}(B)$ and $x \in A$ by $x \cdot g_B \coloneqq x \cdot g_A$, where g_A is any element in the pre-image $\vec{f}^{-1}(g_b)$.*

Observe that an easy way to obtain a rack covering is by constructing a quotient $f\colon A \twoheadrightarrow B$ such that $\vec{f}$ is an isomorphism.

3.7. Example. The components of the unit ${}^{r}\eta^{q}$ of the ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ adjunction are rack coverings. Indeed, we discussed in Paragraph 2.21.1 that $\mathrm{Pth}\,{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} = \mathrm{Pth}$, see also Paragraph 2.1.9. In particular, we look at the one element set 1 and consider the map $f \coloneqq {}^{r}\eta^{q}_{\mathrm{F_r}(1)}\colon \mathrm{F_r}(1) \to \mathrm{F_q}(1) = 1$. We then compute that $\vec{f} = \mathrm{Pth}({}^{r}\eta^{q}_{\mathrm{F_r}(1)})$ and $\mathrm{Inn}(f) = \mathrm{Inn}({}^{r}\eta^{q}_{\mathrm{F_r}(1)})$ are respectively the morphisms

$$\mathrm{Pth}(\mathrm{F_r}(1)) = \mathbb{Z} \xrightarrow{\mathrm{id}_{\mathbb{Z}}} \mathrm{Pth}(\mathrm{F_q}(1)) = \mathbb{Z} \ \text{ and } \ \mathrm{Inn}(\mathrm{F_r}(1)) = \mathbb{Z}_3 \longrightarrow \mathrm{Inn}(\mathrm{F_q}(1)) = \{e\}\ ,$$

where $\mathbb{Z}$ is the infinite cyclic group, $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ is the cyclic group with 3 elements and $\{e\}$ the trivial group. In this case $\vec{f}$ is an isomorphism, but $\mathrm{Inn}(f)$ is not.

3.8. Remark. In the article [16], Theorem 4.2 says that quandle coverings (such as in (3) of Proposition 3.4 above) should coincide with *rigid quotients* of quandles, i.e. surjective morphisms $f\colon A \to B$ which induce an isomorphism $\mathrm{Inn}(f)\colon \mathrm{Inn}(A) \to \mathrm{Inn}(B)$. Looking at the proof on page 1150, the authors assume "by construction" that the map η (between the *excess* of Q and R [41]) is surjective, which is equivalent to asking for the bottom right-hand square $c_R\ Adconj(h) = \mathrm{Inn}(h)\ c_Q$ to be a pushout. This doesn't seem to hold in the generality asked for in [16]. Note that these results are presented in such a way that they should also hold in Rck, since the idempotency axiom is never used. Then the example above provides a counter-example to [16, Theorem 4.2] in Rck. We further give a counter-example in Qnd, which shows that [16, Theorem 4.2] must be incorrect.

3.9. Example. Consider the quandle $Q_{ab\star}$ from Example 2.6, which by Example 2.21 is such that $\mathrm{Pth}(Q_{ab\star}) = \mathbb{Z}\times\mathbb{Z}$ with $\underline{a} = \underline{b} = (1,0)$ and $\underline{\star} = (0,1)$. Moreover, observe that the trivial quandle with two elements $\pi_0(Q_{ab\star})$ is also such that $\mathrm{Pth}(\pi_0(Q_{ab\star})) = \mathrm{F_{ab}}(\{[a],[\star]\}) = \mathbb{Z}\times\mathbb{Z}$ where $\underline{[a]} = (1,0)$ and $\underline{[\star]} = (0,1)$. Hence the morphism of quandles $f \coloneqq \eta_{Q_{ab\star}}\colon Q_{ab\star} \to \pi_0(Q_{ab\star})$ is such that $\vec{f} = \mathrm{id}_{\mathbb{Z}\times\mathbb{Z}}$. In particular $\mathrm{Ker}(\vec{f}) = \{e\}$

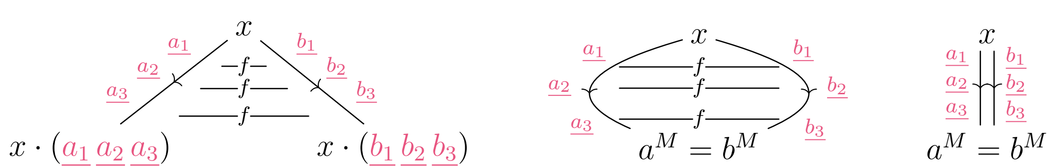
is the trivial group, but $\mathrm{Inn}(f)\colon \mathbb{Z}/2\mathbb{Z} \to \{e\}$ is not an isomorphism. Other such examples can be built using morphisms between quandles from Example 1.3, as well as Proposition 2.27 and Remark 2.28 in [29].

3.9.1. VISUALIZING COVERINGS. Coverings are characterized by the trivial action of f-symmetric paths, which are the elements $g = g_a g_b^{-1} \in \mathrm{Pth}(A)$ such that g_a and g_b are f-symmetric to each other. Notice that an f-symmetric pair g_a, g_b is obtained from the projections of a primitive path in $\mathrm{Eq}(f)$. We emphasize the geometrical aspect of these 2-dimensional primitive paths by defining *membranes* and *horns*. An f-symmetric trail is a compact 1-dimensional concept which remains so when generalized to higher dimensions. The concept of f-horn allows for a more visual, geometrical and elementary description of these ingredients as well as their higher-dimensional generalizations.

3.10. DEFINITION. *Given a morphism $f\colon A \to B$ in* Rck *(or* Qnd*), we define an f-membrane $M = ((a_0, b_0), ((a_i, b_i), \delta_i)_{1\leq i\leq n})$ to be the data of a primitive trail in* $\mathrm{Eq}(f)$ *(see Paragraph 2.3.2). We call such an f-membrane M a f-horn if $a_0 = b_0 =: x$ which we denote $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$. The* associated f-symmetric pair *of the membrane or horn M is given by the paths $g_a^M := \underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n}$ and $g_b^M := \underline{b_1}^{\delta_1} \cdots \underline{b_n}^{\delta_n}$ in* $\mathrm{Pth}(A)$*. The* top trail *is $t_a = (a_0, g_a^M)$ and the* bottom trail *is $t_b = (b_0, g_b^M)$. The* endpoints *of the membrane or horn are given by $a_M = a_0 \cdot g_a^M$ and $b_M = b_0 \cdot g_b^M$.*

Given an f-symmetric trail (x, g) for $g = g_a g_b^{-1} \in \mathrm{Ker}(\vec{f})$ as before, there is an f-horn such that its associated f-symmetric pair is given by g_a and g_b (in particular the associated f-symmetric trail is then (x, g)). Given a horn $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$, we represent it (with $n = 3$ and $\delta_i = 1$ for $1 \leq i \leq 3$) as in the left-hand diagram below.

3.11. DEFINITION. *A horn $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$ is said to* close *(into a* disk*) if its endpoints are equal: $a^M = x \cdot g_a^M = x \cdot g_b^M = b^M$. The horn M is said to* retract *if for each $1 \leq k \leq n$, the truncated horn $M_{\leq k} := (x, (a_i, b_i, \delta_i)_{1\leq i\leq k})$ closes.*



3.12. COROLLARY. *A surjective morphism $f\colon A \twoheadrightarrow B$ in* Rck *(or* Qnd*) is a covering if and only if every f-horn retracts (or equivalently, if every f-horn closes into a disk).*

3.12.1. VISUALIZING NORMAL EXTENSIONS. Normal extensions of quandles are described by V. Even in [30]. The same description works in racks. We reinterpret it using our own terminology.

3.13. DEFINITION. *Given a surjective morphism $f\colon A \to B$ in* Rck*, together with an f-membrane $M = (a_i, b_i, \delta_i)_{0\leq i\leq n}$, we say that the membrane M* forms a cylinder *if both the top and the bottom trails of M are loops.*

3.14. Proposition. *A surjective morphism $f\colon A \to B$ in* Rck *(or* Qnd*) is a normal extension if and only if f-membranes are* rigid*, i.e. if and only if given any f-membrane $M = (a_i, b_i, \delta_i)_{0\leq i\leq n}$, M forms a cylinder as soon as either the top or the bottom trail of M is a loop.*

Proof. The surjection f is normal if and only if the projections $\pi_1, \pi_2\colon \mathrm{Eq}(f) \rightrightarrows A$ of the kernel pair of f are trivial. Such projections are trivial if and only if they reflect loops. The π_1 (resp. π_2) projection of a trail $t = ((a_0, b_0), h)$ in $\mathrm{Eq}(f)$ loops if and only if there is an f-membrane $M = ((a_0, b_0), ((a_i, b_i), \delta_i)_{1\leq i\leq n})$ such that $\vec{\pi}_1(h) = g_a^M$, $\vec{\pi}_2(h) = g_b^M$ and the top (resp. bottom) trail of M loops (see also [30, Proposition 3.2.3]). ∎

3.15. Characterizing central extensions. V. Even's strategy to prove the characterization is to split coverings along the weakly universal covers constructed by M. Eisermann. These weakly universal covers can be understood as the centralization of the canonical projective presentations (using free objects – see Section 3.32). Their structure and properties used to show V. Even's result derive from the structure and properties of the free objects we described before. Thus even though V. Even's proof can be translated to the context of racks, we prefer to work directly with free objects in the alternative proof below. This approach then easily generalizes to higher dimensions without us having to build the weakly universal higher-dimensional coverings from scratch.

3.16. Proposition. *Any rack-covering with free codomain $f\colon A \to \mathrm{F_r}(B)$ is a trivial extension.*

Proof. In order to test whether f is a trivial extension, consider $x \in A$ and $g = \underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n} \in \mathrm{Pth}(A)$ for $n \in \mathbb{N}$, $a_1, \ldots, a_n$ in A and $\delta_1, \ldots, \delta_n$ in $\{-1, 1\}$. Assume that f sends the trail (x, g) to the loop $(f(x), \vec{f}(g))$:

$$f(a) \cdot (\underline{f(a_1)}^{\delta_1} \cdots \underline{f(a_n)}^{\delta_n}) = f(x) \triangleleft^{\delta_1} f(a_1) \cdots \triangleleft^{\delta_n} f(a_n) = f(x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n) = f(x),$$

where we write $\underline{f(a_i)} \coloneqq \mathrm{pth}_{\mathrm{F_r}(B)}(f(a_i))$ (which does not mean that $f(a_i)$ is in B). We have to show that (x, g) was a loop in the first place:

$$x \cdot g = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n = x.$$

Now since the action of $\mathrm{Pth}(\mathrm{F_r}(B))$ on $\mathrm{F_r}(B)$ is free, any loop in $\mathrm{Pth}(\mathrm{F_r}(B))$ must be trivial, and in particular $\underline{f(a_1)}^{\delta_1} \cdots \underline{f(a_n)}^{\delta_n} = e$. Hence $g \in \mathrm{Ker}(\vec{f})$, and thus by Theorem 3.4, $x \cdot g = x$, which concludes the proof. ∎

Note finally that the exact same proof works for quandle coverings, using the fact that if A is a quandle, we may then always choose a_i's and δ_i's such that $\sum_i \delta_i = 0$. Then $\underline{f(a_1)}^{\delta_1} \cdots \underline{f(a_n)}^{\delta_n}$ is in $\mathrm{Pth}^{\circ}(\mathrm{F_q}(B))$ which acts freely on $\mathrm{F_q}(B)$. The rest of the proof remains identical.

3.17. Proposition. *If a quandle-covering $f\colon A \to \mathrm{F_q}(B)$ has a free codomain, then it is a trivial extension.*

By Lemma 3.3, and the previous propositions, the strategy of Section 1.1.3 yields Theorem 2 from [30], as well as:

3.18. Theorem. *Rack coverings are the same as central extensions of racks.*

3.19. Comparing admissible adjunctions by factorization. The notions of trivial object and connectedness, or trivialising relation C_0, coincide in racks and quandles. These are understood as the zero-dimensional central extensions and centralizing relations. In dimension 1, the notions of central extensions in racks and quandles also coincide. Further we also have coincidence of the centralizing relations and the corresponding notions in dimension 2. Before we move on, we show how these results are no coincidence and can be studied systematically as a consequence of the tight relationship between the π_0-admissible adjunctions of interest.

Expanding on Paragraph 2.3.6 we get a factorization (Diagram (14)) as in 2.21.1 (where $\pi_0 \cdot \mathrm{I} = \pi_0$, $\mathrm{I} \cdot \mathrm{I} = \mathrm{I}$, $\pi_0 \cdot {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} = \pi_0$, ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} \cdot \mathrm{I} = \mathrm{I}$) and all the adjunctions are admissible. Since we are dealing here with several different Galois structures: Γ from Rck to Set, ${}_r\Gamma_q$ from Rck to Qnd and say $\Gamma_q \coloneqq (\mathsf{Qnd}, \mathsf{Set}, \pi_0, \mathrm{I}, \eta, \epsilon, \mathcal{E})$, we specify the Galois structure with respect to which the concepts of interest are discussed.

$$
\begin{tikzcd}
\mathsf{Rck} \arrow[rr, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right, "\pi_0"'] \arrow[dr, phantom, "\dashv"] & & \mathsf{Qnd} \arrow[ll, bend left, "\mathrm{I}"] \arrow[dl, bend right, "\pi_0"'] \arrow[dl, phantom, "\dashv"] \\
& \mathsf{Set.} \arrow[ul, bend right, "\mathrm{I}"'] \arrow[ur, bend right, "\mathrm{I}"'] &
\end{tikzcd}
\tag{14}
$$

3.20. Lemma. *If $f\colon A \to B$ is a Γ-trivial extension, then f is also ${}_r\Gamma_q$-trivial, and the image ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ of f is a Γ_q-trivial extension in* Qnd.

Proof. The Γ-canonical square of f in Rck is given on the left, and factorizes into the composite of double extensions on the right in Diagram (15). Hence if f is a trivial extension,

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \pi_0(A) \arrow[d, "\pi_0(f)"] \\
B \arrow[r, "\eta_A"'] & \pi_0(B),
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "{}^r\eta^q_A"] \arrow[d, "f"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A) \arrow[r, "\eta_{{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A)}"] \arrow[d, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)"'] & \pi_0(A) = \pi_0({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A)) \arrow[d, "\pi_0(f)"] \\
B \arrow[r, "{}^r\eta^q_B"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B) \arrow[r, "\eta_{{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)}"'] & \pi_0(B) = \pi_0({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)).
\end{tikzcd}
\tag{15}
$$

then this composite is a pullback square. The composite of two double extensions is a pullback if and only if both double extensions are pullbacks (see for instance [73, Lemma 2.1.4]). ∎

3.21. Lemma. *An extension $f\colon A \to B$ in* Qnd *is*

(i) Γ_q*-trivial in* Qnd *if and only if* $\mathrm{I}(f)$ *is* Γ*-trivial in* Rck;

(ii) Γ_q*-central in* Qnd *if and only if* $\mathrm{I}(f)$ *is* Γ*-central in* Rck.

Proof. The first point (i) is immediate by the previous lemma, and the fact that the π_0-canonical squares of $\mathrm{I}(f)$ in Rck is the same as the image by I of the Γ_q-canonical square of f in Qnd. Note also that I preserves and reflects pullbacks.

For the second statement (ii), if f is Γ_q-central, then there is an extension $p\colon E \to B$ such that the pullback of f along p is Γ_q-trivial. We may conclude by taking the image by I of this pullback square. Now if $\mathrm{I}(f)$ is Γ-central in Rck, there exists $p\colon E \to B$ in Rck such that the pullback t of $\mathrm{I}(f)$ along p is Γ-trivial in Rck. Taking the quotient along ${}^r\eta^q$ of this pullback square (1) yields a factorization of (1):

$$\begin{tikzcd}
E \times_B A \arrow[r, "{}^r\eta^q_P"] \arrow[d, "t"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(E \times_B A) \arrow[r] \arrow[d, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(t)"'] & A \arrow[d, "f"] \\
E \arrow[r, "{}^r\eta^q_E"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(E) \arrow[r, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(p)"'] & B.
\end{tikzcd}$$

Again, since the left hand square is a double extension, and the composite is a pullback, both squares are actually pullbacks and thus f is Γ_q-central. ■

Now since the π_0-adjunction is strongly Birkhoff (both in Rck and Qnd), central extensions are closed by quotients along double extensions in ExtRck (or ExtQnd – see also Proposition 3.30).

3.22. Corollary. *The image by* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ *of a* Γ*-central extension* $f\colon A \to B$ *in* Rck *is a* Γ_q*-central extension in* Qnd*.*

Proof. The image ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ is Γ_q-central extension if and only if $\mathrm{I}({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f))$ is Γ-central. Since Set is strongly Birkhoff in Rck, $\mathrm{I}({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f))$ is the quotient of a Γ-central extension in Rck along a double extension and thus is still Γ-central in Rck. ■

3.23. Proposition. *If the image by* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ *of an* ${}_r\Gamma_q$*-trivial extension* $f\colon A \to B$ *in* Rck *is a* Γ_q*-central extension in* Qnd*, then* f *is* Γ*-central in* Rck*.*

Proof. Consider the following commutative cube in Rck where we omit the inclusion $\mathrm{I}\colon \mathsf{Qnd} \to \mathsf{Rck}$. The back face is a pullback by construction. The right hand face is a pullback by assumption, and the left hand face is a pullback by Proposition 2.31.

$$\begin{tikzcd}
 & P_1 \arrow[rr] \arrow[dl, "{}^r\eta^q_{P_1}"'] \arrow[dd, "t"] & & A \arrow[dl, "{}^r\eta^q_A"'] \arrow[dd, "f"] \\
{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(P_1) \arrow[rr] \arrow[dd, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(t)"'] & & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A) \arrow[dd, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)"] & \\
 & \mathrm{F}_{\mathrm{r}}(B) \arrow[rr, "\epsilon^r_B"] \arrow[dl, "{}^r\eta^q_{\mathrm{F}_{\mathrm{r}}(B)}"] & & B \arrow[dl, "{}^r\eta^q_B"] \\
\mathrm{F}_{\mathrm{q}}(B) \arrow[rr, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\epsilon^r_B)"'] & & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B) &
\end{tikzcd}$$

We deduce that the front face is a pullback as well. Since ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ is Γ_q-central by assumption, and since ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\epsilon^r_B)\colon \mathrm{F}_{\mathrm{q}}(B) = {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\mathrm{F}_{\mathrm{r}}(B)) \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)$ factorizes as

$$\mathrm{F}_{\mathrm{q}}(B) \xrightarrow{\mathrm{F}_{\mathrm{q}}({}^r\eta^q_B)} \mathrm{F}_{\mathrm{q}}({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)) \xrightarrow{\mathrm{F}_{\mathrm{q}}(\epsilon^q_{{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)})} {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B),$$

both ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(t)$ and t are Γ-trivial as the pullback of a trivial extension. ■

Note that some extensions of racks which are not central, still have central images under ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ (see example in [72]). Of course some morphisms of racks which are not ${}_r\Gamma_q$-trivial are still Γ-central: we already mentioned the important example of ${}^r\eta^q_A$ for any rack A.

Before even studying the next steps of the covering theory, we can predict that what happens in Qnd directly follows from what happens in Rck.

3.24. Corollary. *If the full subcategory* $\mathsf{CExtRck}$ *of central extensions of racks is reflective within the category of extensions* ExtRck *(see Theorem 3.26 for details), then also* $\mathsf{CExtQnd}$ *is reflective in* ExtQnd *and the reflection is computed as in* ExtRck*, via the inclusion* $\mathrm{I}\colon \mathsf{Qnd} \to \mathsf{Rck}$*.*

Proof. Since Qnd is closed under quotients in Rck, the centralization of an extension in $\mathsf{Qnd} \preceq \mathsf{Rck}$ yields an extension in Qnd which is moreover central by Lemma 3.21. The universality in $\mathsf{CExtQnd}$ directly derives from the universality in $\mathsf{CExtRck}$ by the same arguments. ■

3.25. Centralizing extensions. We adapt the result from [28], showing the reflectivity of quandle coverings in the category of extensions, to the context of racks. We put the emphasis on our new characterizations of the centralizing relation which works the same for racks and for quandles. We also prepare the ingredients to show the admissibility of coverings within extensions, and the forthcoming covering theory in dimension 2.

Let us define $\mathcal{E}_1$ to be the class of double extensions in ExtRck.

3.26. Theorem. *The category* $\mathsf{CExtRck}$ *is an* $(\mathcal{E}_1)$*-reflective subcategory of the category* ExtRck *with left adjoint* F_1 *and unit* η^1 *defined for an object* $f\colon A \to B$ *in* ExtRck *by* $\eta^1_f := (\eta^1_A, \mathrm{id}_B)$*, where* $\eta^1_A\colon A \to A/\,\mathrm{C}_1(f)$ *is the quotient of* A *by the* centralizing *congruence* $\mathrm{C}_1(f)$*, which can be defined in the following equivalent ways:*

(i) $\mathrm{C}_1(f)$ *is the equivalence relation on* A *generated by the pairs* $(x \triangleleft a \triangleleft^{-1} b, x)$ *for* x*,* a*, and* b *in* A *such that* $f(a) = f(b)$*,*

(ii) a pair (a, b) *of elements from* A *is in the equivalence relation* $\mathrm{C}_1(f)$ *if and only if* a *and* b *are the endpoints of a horn, i.e. there exists a horn* $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$ *such that* $x \cdot g^M_a = a$ *and* $x \cdot g^M_b = b$*,*

(iii) $\mathrm{C}_1(f)$ *is the orbit relation* $\sim_{\mathrm{Ker}(\vec{f})}$ *(or equivalently* $\sim_{\mathrm{K}_{\vec{f}}}$*) induced by the action of the kernel of* $\vec{f}$ *(i.e. the group of* f*-symmetric paths).*

Observing that $\mathrm{C}_1(f) \leq \mathrm{Eq}(f)$*, the image of* f *by* F_1 *is defined as the unique factorization of* f *through this quotient:*

$$\begin{tikzcd} A \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\eta^1_A"'] & & B \\ & A/\,\mathrm{C}_1(f) \arrow[ur, two heads, "\mathrm{F}_1(f)"'] & \end{tikzcd}$$

The definition of F_1 *on morphisms* $\alpha = (\alpha_{\top}, \alpha_{\bot})\colon f_A \to f_B$ *decomposes into the top component* $\mathrm{F}_1^{\top}(\alpha)\colon A_{\top}/\,\mathrm{C}_1(f_A) \to B_{\top}/\,\mathrm{C}_1(f_B)$ *defined by the universal property of the quotients* $\eta^1_{A_{\top}}$ *for* $f_A\colon A_{\top} \to A_{\bot}$*; and the bottom component* $\mathrm{F}_1^{\bot}(\alpha) = \alpha_{\bot}$ *which simply returns the bottom component of* α*.*

Proof. Using definition (i) for the centralizing relation, the proof of Theorem 5.5 in [28] easily translates to the context of racks. Then given an extension $f\colon A \to B$, the unit $\eta^1_f = (\eta^1_A, \mathrm{id}_B)$ is a double extension since its bottom component is an isomorphism. It remains to show that the definitions (ii) and (i) are equivalent, since (iii) is equivalent to (ii) by Proposition 2.19.

First we show by induction on $n \in \mathbb{N}$ that $\mathrm{C}_1(f)$, defined as in (i), contains all pairs that are endpoints of a horn. Then we show that the collection of such pairs defines a congruence containing the generators of $\mathrm{C}_1(f)$. This then concludes the proof.

Step 0 is satisfied by reflexivity of $\mathrm{C}_1(f)$. Now assume that if (a, b) is a pair of elements in A, which are endpoints of a horn $M = (x, (a_i, b_i, \delta_i)_{1 \leq i \leq n})$ of length $n \leq k$, for some fixed natural number k, then $(a, b) \in \mathrm{C}_1(f)$. We show that the endpoints $a \coloneqq x \cdot g^M_a$ and $b \coloneqq x \cdot g^M_b$ of any given horn $M = (x, (a_i, b_i, \delta_i)_{1 \leq i \leq k+1})$ of length $k+1$ are in relation by $\mathrm{C}_1(f)$. Indeed, define $a' = a \triangleleft^{-\delta_{k+1}} a_{k+1}$ and $b' = b \triangleleft^{-\delta_{k+1}} b_{k+1}$. Then we have that $(a', b') \in \mathrm{C}_1(f)$ by assumption and, moreover,

$$\big(a = a' \triangleleft^{\delta_{k+1}} a_{k+1}\big) \quad \mathrm{C}_1(f) \quad \big(b' \triangleleft^{\delta_{k+1}} a_{k+1}\big) \quad \mathrm{C}_1(f) \quad \big(b' \triangleleft^{\delta_{k+1}} b_{k+1} = b\big)\,,$$

by compatibility of $\mathrm{C}_1(f)$ with the rack operation, together with reflexivity, and further by definition (i) of $\mathrm{C}_1(f)$. We may conclude by transitivity of $\mathrm{C}_1(f)$.

Now define the symmetric set relation S as the subset of $A \times A$, given by pairs of endpoints of f-horns. Looking at horns of length 0 and 1, S defines a reflexive relation containing the generators of $\mathrm{C}_1(f)$. It is also easy to observe that it is compatible with the rack operation. Thus it remains to show transitivity. In order to do so, for k and n in $\mathbb{N}$, consider a horn $M = (x, (a_i, b_i, \delta_i)_{1 \leq i \leq k})$, and its endpoints a and b as before, as well as a horn $N = (z, (c_i, d_i, \gamma_i)_{1 \leq i \leq n})$ with endpoints $c = z \cdot g^N_a$ and $d = z \cdot g^N_b$. If $b = c$ then also (a, d) is in S since:

$$\begin{aligned} a &= x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k \triangleleft^{-\gamma_n} c_n \cdots \triangleleft^{-\gamma_1} c_1 \triangleleft^{\gamma_1} c_1 \cdots \triangleleft^{\gamma_n} c_n, \\ d &= x \triangleleft^{\delta_1} b_1 \cdots \triangleleft^{\delta_k} b_k \triangleleft^{-\gamma_n} c_n \cdots \triangleleft^{-\gamma_1} c_1 \triangleleft^{\gamma_1} d_1 \cdots \triangleleft^{\gamma_n} d_n. \end{aligned}$$

■

By Corollary 3.24, what we deduced about the functor F_1 restricts to the domain $\mathsf{CExtQnd}$, and so also describes the left adjoint to the inclusion in ExtQnd from Theorem 5.5. in [28]. In addition to Corollary 3.24, we further describe how centralization behaves with respect to ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$.

3.26.1. Navigating between racks and quandles. Observe that the adjunction ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} \colon \mathsf{Rck} \rightleftarrows \mathsf{Qnd} \colon \mathrm{I}$ induces (in the obvious way) an adjunction ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1} \colon \mathsf{ExtRck} \rightleftarrows \mathsf{ExtQnd} \colon \mathrm{I}$ with unit given by ${}^{r}_{1}\eta^{q} = ({}^{r}\eta^{q}, {}^{r}\eta^{q})$. Then by Corollary 3.22 this adjunction restricts to the full subcategories ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1} \colon \mathsf{CExtRck} \rightleftarrows \mathsf{CExtQnd} \colon \mathrm{I}$.

3.27. Proposition. *We have the following square of adjunctions, in which the* 4 *obvious squares of functors (one for each oriented diagonal) commute (up to isomorphism), i.e. :*

$$\begin{tikzcd}
\mathsf{ExtRck} \arrow[r, bend right, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\mathrm{F}_1"'] \arrow[d, phantom, "\dashv"] & \mathsf{ExtQnd} \arrow[l, bend right, "\mathrm{I}"'] \arrow[d, bend right, "\mathrm{F}_1"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{CExtRck} \arrow[u, bend right, "\mathrm{I}"'] \arrow[r, bend right, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}"'] \arrow[r, phantom, "\top"] & \mathsf{CExtQnd}. \arrow[u, bend right, "\mathrm{I}"'] \arrow[l, bend right, "\mathrm{I}"']
\end{tikzcd}$$

Proof. Corollary 3.28 gives commutativity of the square $\mathrm{F}_1\,\mathrm{I} = \mathrm{I}\,\mathrm{F}_1$ from the top right to the bottom left. In the opposite direction, $\mathrm{I}\;{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1} = {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}\,\mathrm{I}$ by Corollary 3.22 again. Finally bottom-right to top-left $\mathrm{I}\,\mathrm{I} = \mathrm{I}\,\mathrm{I}$ commutes trivially, from which we can deduce, by uniqueness of left adjoints, that ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}\,\mathrm{F}_1 = \mathrm{F}_1\;{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}$. ■

3.28. Corollary. *In particular, if $f \colon A \to B$ is a morphism of racks, then the centralization*

$$\mathrm{F}_1({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)) \colon {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A)/\,\mathrm{C}_1({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)) \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)$$

of ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ is equal (up to isomorphism) to ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\mathrm{F}_1(f)) \colon {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A/\,\mathrm{C}_1(f)) \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)$ which is the reflection of the centralization $\mathrm{F}_1(f)$ of f.

3.28.1. Towards admissibility in dimension 2. A reflector such as F_1, of a subcategory of morphisms containing the identities into a larger class of morphisms can always be chosen such that the bottom component of the unit of the adjunction is the identity [48, Corollary 5.2]. This is important in order to obtain higher order reflections and admissibility, for we relate certain problems back to the first level context. For dimension 2, we need this reflection to be strongly Birkhoff. Below we have the results we need for the permutability condition on the kernel pair of the unit ("strongly") and for the closure by quotients of central extensions ("Birkhoff").

3.29. Proposition. *Given a rack extension $f \colon A \to B$ (or in particular an extension in* Qnd*) as before, the kernel pair $\mathrm{C}_1(f)$ of the domain-component η^1_A of the unit $\eta^1_f := (\eta^1_A, \mathrm{id}_B)$, commutes with all congruences on A, in* Rck *(and so also in particular in* Qnd*).*

Proof. By Theorem 3.26, the centralizing relation $\mathrm{C}_1(f)$ is an orbit congruence which thus commutes with any other congruence on A. ■

As we shall see in Part II and III, the following property is a consequence of the fact that the Galois structure Γ, in dimension 0, is strongly Birkhoff. For now we show by hand:

3.30. Proposition. *If $\alpha = (\alpha_\top, \alpha_\bot)$ is a double extension of racks (or in particular quandles)*

$$\begin{tikzcd}
A_\top \arrow[rr, two heads, "\alpha_\top"] \arrow[dr, two heads, "p"] \arrow[dd, "f_A"'] & & B_\top \arrow[dd, "f_B"] \\
& A_\bot \times_{B_\bot} B_\top \arrow[ur, two heads, "\pi_2"'] \arrow[dl, two heads, "\pi_1"] & \\
A_\bot \arrow[rr, "\alpha_\bot"'] & & B_\bot
\end{tikzcd}$$

then the morphism $\bar{\alpha}$ induced between the centralizing relations $\mathrm{C}_1(f_A)$ *and* $\mathrm{C}_1(f_B)$ *is a regular epimorphism. Moreover, if f_A is a central extension then f_B is a central extension.*

Proof. Certainly if we show that $\bar{\alpha}$ is a regular epimorphism, then assuming that f_A is central, then its centralizing relation is trivial, hence the centralizing relation of f_B is trivial, showing that f_B is central (note that in this context, it is enough to have preservation of centrality by quotients along double extensions in order to have surjectivity of $\bar{\alpha}$, see Part II and III).

We pick a pair $(x \triangleleft y, x \triangleleft z)$ amongst the generators of $\mathrm{C}_1(f_B)$ (i.e. with $f_B(y) = f_B(z)$). Since $\alpha_\bot$ is surjective we get $a \in A_\bot$ such that $\alpha_\bot(a) = f_B(y)$. Now both pairs (a, y) and (a, z) are in the pullback $A_\bot \times_{B_\bot} B_\top$ hence there exist t and s in $A_\top$ such that $\alpha_\top(t) = y$, $\alpha_\top(s) = z$ and $f_A(t) = f_A(s) = a$, by surjectivity of p. Now there is also $u \in A_\top$ such that $t(u) = x$ and the pair $(u \triangleleft t, u \triangleleft s)$ is a generator of $\mathrm{C}_1(f_A)$ by definition. It is also sent to $(x \triangleleft y, x \triangleleft z) \in \mathrm{C}_1(f_B)$ by $\bar{\alpha}$ by construction. All generators of $\mathrm{C}_1(f_B)$ are thus in the image of $\bar{\alpha}$, and this concludes the proof. ∎

3.31. Corollary. *Given a morphism $\alpha = (\alpha_\top, \alpha_\bot)\colon f_A \to f_B$ in* ExtRck *such that $\alpha_\top$ and $\alpha_\bot$ are surjections, then the square below (where $P \coloneqq (A_\top / \mathrm{C}_1(f_A)) \times_{(B_\top / \mathrm{C}_1(f_B))} B_\top$) is a double extension of racks. Similarly in* ExtQnd.

$$\begin{tikzcd}
A_\top \arrow[rr, two heads, "\alpha_\top"] \arrow[dr, two heads, "p"] \arrow[dd, "\eta^1_{A_\top}"'] & & B_\top \arrow[dd, "\eta^1_{B_\top}"] \\
& P \arrow[ur, two heads, "\pi_2"'] \arrow[dl, two heads, "\pi_1"] & \\
(A_\top / \mathrm{C}_1(f_A)) \arrow[rr, "\mathrm{F}_1^\top(\alpha)"'] & & (B_\top / \mathrm{C}_1(f_B))
\end{tikzcd}$$

Proof. By Lemma 1.2 in [5], this square is a pushout as a consequence of Proposition 3.30. Then by Proposition 5.4 in [19], p is a surjection as well, making α into a double extension. ∎

In Part II we complete the proof that $\Gamma_1 = (\mathsf{ExtRck}, \mathsf{CExtRck}, \mathrm{F}_1, \mathrm{I}, \eta^1, \epsilon^1, \mathcal{E}^1)$ forms an admissible Galois structure such that morphisms in $\mathcal{E}^1$ are of *effective $\mathcal{E}^1$-descent* [59, 58].

3.32. Weakly universal covers and the fundamental groupoid.

3.32.1. CENTRALIZING THE CANONICAL PRESENTATIONS. Weakly universal covers for quandles were described by M. Eisermann. He also indicated how to adapt his theory to the case of racks. In this section, we recover his constructions from the centralization of the canonical projective presentations as explained in the introduction. Note that the difference between the weakly universal covers (w.u.c.) in racks and in quandles is then due to the difference between the canonical projective presentations rather than the centralizations which are the same.

Given the canonical projective presentation of a rack $\epsilon^r_X\colon \mathrm{F_r}(X) \to X$, we saw in Paragraph 2.9.1 that the induced morphism $\vec{\epsilon}^{\,r}_X$ is actually the quotient map $\vec{\epsilon}^{\,r}_X = q_X\colon \mathrm{F_g}(X) \to \mathrm{Pth}(X)$ from Subsection 2.8. Hence the kernel of $\vec{\epsilon}^{\,r}_X$ is given by

$$\mathrm{Ker}(\vec{\epsilon}^{\,r}_X) = \langle\langle \underline{c}^{-1}\underline{a}^{-1}\underline{x}\,\underline{a} \mid a, x, c \in X \text{ and } c = x \triangleleft a\rangle\rangle_{\mathrm{F_g}(X)}.$$

Since the action of $\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X)$ is by right multiplication, two elements (a, g) and (b, h) in $\mathrm{F_r}(X)$ are identified by the centralizing relation $\mathrm{C}_1(\epsilon^r_X)$ if and only if $a = b$ and there is $k \in \mathrm{Ker}(\vec{\epsilon}^{\,r}_X)$ such that $g = hk$. In other words, the domain component $\eta^1_{\mathrm{F_r}(X)}$ of the centralization unit is given by the product $\mathrm{id}_X \times q_X\colon X \rtimes \mathrm{F_g}(X) \to X \rtimes \mathrm{Pth}(X)$, where the operation in $\tilde{X} \coloneqq X \rtimes \mathrm{Pth}(X)$ is defined as in Paragraph 2.2.3.1, Equation (4).

3.33. DEFINITION. *Given a rack X, we define the associated weakly universal cover of X to be the centralised map $\omega_X \coloneqq \mathrm{F}_1(\epsilon^r_X)$*

$$\tilde{X} \coloneqq X \rtimes \mathrm{Pth}(X) \xrightarrow{\omega_X} X,$$

where ω_X sends a trail $(a, g) \in \tilde{X}$ to its endpoint $a \cdot g$, and trails in $\tilde{X}$ "act by endpoint" as in $\mathrm{F_r}(X)$. Note that this construction is functorial in X, yielding a functor $\tilde{-}\colon \mathsf{Rck} \to \mathsf{Rck}$ which sends a morphism of racks $f\colon A \to B$ to the morphism $\tilde{f} \coloneqq f \times \vec{f}\colon \tilde{A} \to \tilde{B}$; and a natural transformation $\omega\colon \tilde{-} \to \mathrm{id}_{\mathsf{Rck}}$, whose component at X is ω_X.

Then the action of $\mathrm{Pth}(X)$ induced by the covering ω_X on $\tilde{X} = X \rtimes \mathrm{Pth}(X)$ is by right multiplication, and is thus free. Given any other covering $f\colon B \to X$, together with a splitting function $s\colon X \to B$ in Set such that $fs = \mathrm{id}_X$, a factorization $\omega_f\colon \tilde{X} \to B$ of ω_X through f is given by $\omega_f(a, e) \coloneqq s(a)$ and compatibility with the action of $\mathrm{Pth}(X)$ on $\tilde{X}$ and B (see Corollary 3.6).

Considering the canonical projective presentation of a quandle $\epsilon^q_X\colon X \rtimes \mathrm{Pth}^\circ(X) \to X$, the same reasoning yields a w.u.c. with the same properties $\omega^q_X\colon \tilde{X}^\circ \coloneqq X \rtimes \mathrm{Pth}^\circ(X) \to X$, such that the quandle structure on $X \rtimes \mathrm{Pth}^\circ(X)$ is as for $\mathrm{F_q}(X)$ (Paragraph 2.31.2). As in the case of racks, this describes a functor as well as a natural transformation whose component at any quandle A is ω^q_A. Observe that Corollary 3.28 implies that $\tilde{X}^\circ \coloneqq X \rtimes \mathrm{Pth}^\circ(X)$ is actually the free quandle on the rack $\tilde{X} \coloneqq X \rtimes \mathrm{Pth}(X)$ and thus if X is a quandle, then ω^q_X is merely the image of ω_X by ${}_{\mathrm{r}}\mathrm{F_q}$.

As it was proved by V. Even [31], every covering of X is split by ω^q_X in Qnd and a similar argument shows that every covering of X is split by ω_X in Rck. This derives more generally from Corollary 3.31:

3.34. PROPOSITION. *If the extension $c\colon A \to B$ is split by an extension $e\colon E \to B$, then it is also split by the centralization $\mathrm{F}_1(e)\colon E \to B$ of this extension e. As a consequence, c must be split by any weakly universal cover above B.*

PROOF. Consider the reflection by F_1 of the pullback P of e and c as on the right-hand side of Diagram (16). Since the composite of two double extensions is a pullback if and only if both double extensions are pullbacks themselves, Corollary 3.31 implies that the commutative squares $t'\eta^1_P = \eta^1_E t$ and $c\,\mathrm{F}_1(f) = \mathrm{F}_1(e)t'$ are pullback squares, where $t' := \mathrm{F}_1^{\top}[(t,c)]$.

$$\begin{tikzcd}
& \pi_0(P) \arrow[dl, equal] \arrow[dd, "\pi_0(t)"] & & P \arrow[ll, "\eta_P"'] \arrow[rr, "f"] \arrow[dl, "\eta^1_P"'] \arrow[dd, "t"] \arrow[drr, phantom, "\lrcorner", very near start] & & A \arrow[dl, equal] \arrow[dd, "c"] \\
\pi_0(P) \arrow[dd, "\pi_0(t)=\pi_0(t')"'] & & P/\mathrm{C}_1(f) \arrow[ll, "\eta_{(P/\mathrm{C}_1(f))}"'] \arrow[rr, "\mathrm{F}_1(f)"'] \arrow[dd, "t'"] & & A \arrow[dd, "c"] & \\
& \pi_0(E) \arrow[dl, equal] & & E \arrow[ll, "\eta_E"'] \arrow[rr, "e"] \arrow[dl, "\eta^1_E"'] & & B \arrow[dl, equal] \\
\pi_0(E) & & E/\mathrm{C}_1(e) \arrow[ll, "\eta_{(E/\mathrm{C}_1(e))}"] \arrow[rr, "\mathrm{F}_1(e)"'] & & B &
\end{tikzcd} \tag{16}$$

Hence, since $\eta_E = \eta_{(E/\mathrm{C}_1(e))}\eta^1_E$, and similarly $\eta_P = \eta_{(P/\mathrm{C}_1(f))}\eta^1_P$, the F-reflection square $\eta_E t = \pi_0(t)\eta_P$ at t (which is a pullback by assumption) factors through the F-reflection square $\pi_0(t')\eta_{(P/\mathrm{C}_1(f))} = \eta_{(E/\mathrm{C}_1(e))}t'$ at t' via the pullback square $t'\eta^1_P = \eta^1_E t$. Since the square $\pi_0(t')\eta_{(P/\mathrm{C}_1(f))} = \eta_{(E/\mathrm{C}_1(e))}t'$ is a double extension, it is actually a pullback, which shows that t' is a trivial extension. We conclude by observing that a weakly universal cover above B factors through $\mathrm{F}_1(e)$ and trivial extensions are stable by pullbacks (see also Diagram 2). ∎

Given any X in Rck (respectively Qnd), the covering ω_X (respectively ω^q_X) is split by itself and thus it is a normal covering. Hence its kernel pair is sent to a groupoid by the reflection π_0 (see [4, Lemma 5.1.22]) and thus we can construct the *fundamental groupoid* (see *Galois groupoid* of a *weakly universal central extension* as in [4]) yielding functors $\pi^r_1\colon \mathsf{Rck} \to \mathsf{Grpd}$ and $\pi^q_1\colon \mathsf{Qnd} \to \mathsf{Grpd}$, with codomain the category of ordinary groupoids Grpd (i.e. the category of internal groupoids in Set).

3.35. DEFINITION. *The functor $\pi_1\colon \mathsf{Rck} \to \mathsf{Grpd}$ is defined on objects by sending a rack X to $\pi^r_1(X)$, the image by π_0 of the groupoid induced by taking the kernel pair of ω_X. Functoriality is induced by functoriality of ω.*

Similarly the functor $\pi^q_1\colon \mathsf{Qnd} \to \mathsf{Grpd}$ is defined by sending a quandle X to $\pi^q_1(X)$, the image by π_0 of the groupoid induced by taking the kernel pair of ω^q_X.

From there, the Galois theorem yields an equivalence of categories between the category of coverings of X and the category of internal covariant presheaves over $\pi_1(X)$ (and similarly for Qnd, see Section 1.1 and references).

3.35.1. The fundamental groupoid. We show that the fundamental groupoid $\pi_1(X)$ (respectively $\pi_1^q(X)$) for an object X in the category Rck (respectively Qnd) is indeed the groupoid induced by the action of $\mathrm{Pth}(X)$ (respectively $\mathrm{Pth}^\circ(X)$) on X, as suggested in M. Eisermann's work (see [29, Section 8]). As was mentioned in the introduction, these results, and categorical Galois theory, give a positive answer to M. Eisermann's questions about the relevance of his analogies with topology. Results about the fundamental group of a connected pointed quandle were given by V. Even in [30]. We generalize these results to the non-connected, non-pointed context in both categories Rck and Qnd. Exploiting the analogy with the covering theory of locally connected topological spaces, this result confirms the intuition that the elements of the group $\mathrm{Pth}(X)$ (respectively $\mathrm{Pth}^\circ(X)$) are representatives of the classes of *homotopically equivalent paths* which connect elements in the rack (respectively quandle) X.

3.36. Definition. *Given a set X and a group G together with an action of G on X, we build the ordinary groupoid (of elements) $\mathcal{G}_{(X,G)}$ (in Set)*

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[loop above, "(\ )^{-1}"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X \arrow[l, "i" description] \end{tikzcd}$$

where $X_0 := X$, $X_1 := X \times G$ and for $a \in X_0$, $(a,g) \in X_1$,

$$d(a,g) := a; \quad c(a,g) := a \cdot g; \quad i(a) := (a,e); \quad (a,g)^{-1} := (a \cdot g, g^{-1});$$

$p_1, p_2\colon X_2 \rightrightarrows X_1$ form the pullback of c and d; and m is the composition function defined for $\langle (a,g),(b,h)\rangle$ in X_2 by $m\langle (a,g),(b,h)\rangle := (a,g)\cdot(b,h) := (a,gh)$. Note that this construction actually defines a functor from the category of group actions to the category of ordinary groupoids.

3.37. Theorem. *Given an object X in Rck (respectively Qnd), the fundamental groupoid $\pi_1(X)$ (resp. $\pi_1^q(X)$) is given by the set groupoid $\mathcal{G}_{(X,\mathrm{Pth}(X))}$ (resp. $\mathcal{G}_{(X,\mathrm{Pth}^\circ(X))}$). Moreover, the groupoid morphisms induced by $f\colon X \to Y$ via Pth (resp. Pth°) and $\mathcal{G}$ correspond to $\pi_1(f)$ (resp. $\pi_1^q(f)$).*

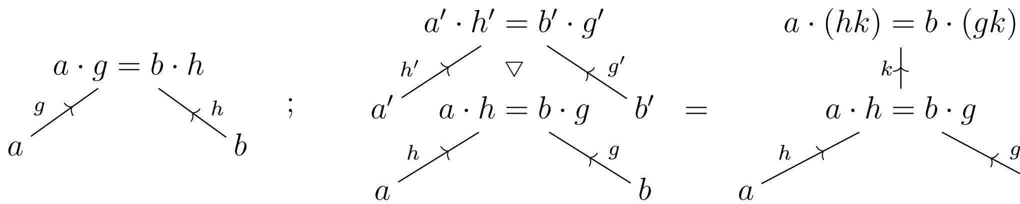
Proof. Given the kernel pair $d_1, d_2\colon X_1' \rightrightarrows X'$ of the weakly universal cover $\omega_X\colon \tilde{X} \to X$ (resp. $\omega_X^q\colon \tilde{X}^\circ \to X$), we define the groupoid $\mathcal{G}$ as:

$$\begin{tikzcd} X_2' \arrow[r, shift left=2, "p_1'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "p_2'"'] & X_1' \arrow[loop above, "(\ )^{-1}"] \arrow[r, shift left=2, "d_2"] \arrow[r, shift right=2, "d_1"'] & X' \arrow[l, "u" description] \end{tikzcd}$$

where X_2' is the pullback of d_2 and d_1, and m' is the composition function defined by the unique factorization of $d_2 \circ p_2', d_1 \circ p_1'\colon X_2' \rightrightarrows X'$ through $d_2, d_1\colon X_1' \rightrightarrows X'$.

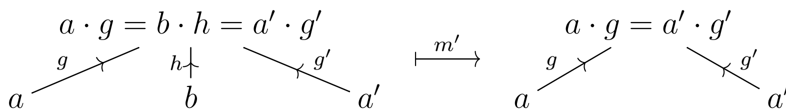
Remember that a trail $(a,g) \in X'$ is represented as an arrow $g\colon\ a \to a \cdot g$; and the action of a trail on another is as in Paragraph 2.2.3.1, Equation (4), where the composition of arrows is understood by multiplication in $\mathrm{Pth}(X)$ (resp. $\mathrm{Pth}^\circ(X)$).

By definition, the elements in X'_1 are then pairs of trails with same endpoint (diagram on the left), and the rack (resp. quandle) operation is defined component-wise such that we have the equality on the right:



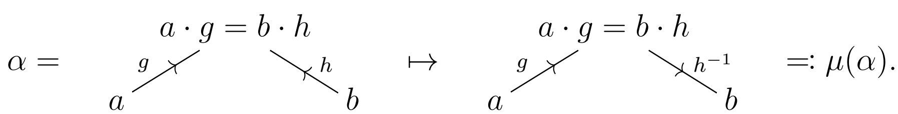
(17)

where $k \coloneqq (h')^{-1}\underline{a'}h'$ (resp. $k \coloneqq \underline{(a \cdot h)}^{-1}(h')^{-1}\underline{a'}h'$). Finally observe that X'_2 is composed of pairs of elements in X'_1 with one matching leg (such as represented on the left), which images by m' are given as in the right-hand diagram:

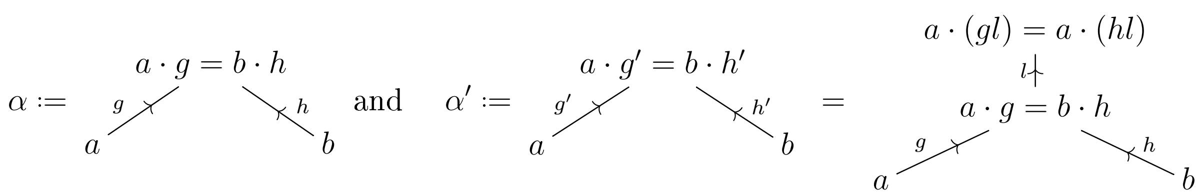


Again the operation in X'_2 is defined component-wise and behaves as in X'_1.

We compute the image $\pi_0(\mathcal{G})$ which is $\pi_1(X)$ (resp. $\pi_1^q(X)$) by definition. Working on each object separately, first observe that as for $\mathrm{F_r}(X)$ (resp. $\mathrm{F_q}(X)$), the unit $\eta_{X'}\colon X' \to \pi_0(X') = X$ sends a trail $(a, g) \in X \rtimes \mathrm{Pth}(X)$ (resp. in $X \rtimes \mathrm{Pth}^\circ(X)$) to its head $a \in X$, i.e. $\eta_{X'}$ is given by the product projection on X. Now for each pair of trails $\alpha = \langle (a, g), (b, h)\rangle$ in X'_1, we define the trail $\mu(\alpha) \coloneqq (a, gh^{-1})$ in X':



Observe that this trail $\mu(\alpha)$ is invariant under the action on α, of other pairs $\beta = \langle (a', g'), (b', h')\rangle$ in X'_1, since $\mu(\alpha \triangleleft \beta) = (a, hkk^{-1}g^{-1}) = \mu(\alpha)$, where $k = (h')^{-1}\underline{a'}h'$ (resp. $k = \underline{(a \cdot h)}^{-1}\ (h')^{-1}\underline{a'}h'$) is the common part of both left and right legs as in Equation (17). Conversely suppose that α, α' in X'_1 have the same image by μ, we show that α and α' are connected in X'_1. Indeed, α and α' must then be of the form $\alpha = \langle (a, g), (b, h)\rangle$ and $\alpha' = \langle (a, g'), (b, h')\rangle$, such that moreover $gh^{-1} = g'h'^{-1}$. Then the path $l \coloneqq h^{-1}h' = g^{-1}g' \in \mathrm{Pth}(X)$ (resp. in $\mathrm{Pth}^\circ(X)$) decomposes as a product $l = \underline{x_0}^{\delta_0} \cdots \underline{x_n}^{\delta_n}$, such that all the pairs $\langle (x_i, e), (x_i, e)\rangle$ are in X'_1 (and we have moreover $\sum_{i=0}^{n} \delta_i = 0$ in the context of Qnd). By acting with these pairs "$- \triangleleft^{\delta_i} \langle (x_i, e), (x_i, e)\rangle$" on α, we may obtain α' as in the diagram on the right:



Hence we have the unit morphism $\eta_{X_1'} = \mu\colon X_1' \to \pi_0(X_1')$ where $\pi_0(X_1')$ is $\pi_0(\mathrm{Eq}(\omega_X)) = X \times \mathrm{Pth}(X)$ (resp. $\pi_0(\mathrm{Eq}(\omega_X^q)) = X \times \mathrm{Pth}^\circ(X)$). We may then compute $\pi_0(d_2) = c$, $\pi_0(d_1) = d$, $\pi_0(i) = u$ and $\pi_0(-1) = -1$, as displayed in the commutative Diagram (18) of plain arrows, where the bottom groupoid is the inclusion in Rck (resp. Qnd) of the groupoid $\mathcal{G}_{(X,\mathrm{Pth}(X))}$ (resp. $\mathcal{G}_{(X,\mathrm{Pth}^\circ(X))}$) from Set. Hence $X_1 = X \times \mathrm{Pth}(X)$ (resp. $X_1 = X \times \mathrm{Pth}^\circ(X)$) has the same underlying set as X', and the underlying functions of $\eta_{X'}$ and d are both given by "projection on X".

$$\begin{tikzcd}
X_2' \arrow[r, shift left=2, "p_1'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "p_2'"'] \arrow[d, dotted, "\eta_{X_2'}=\mu\times\mu"'] & X_1' \arrow[loop above, "-1"] \arrow[r, shift left=2, "d_2"] \arrow[r, shift right=2, "d_1"'] \arrow[d, dashed, "\eta_{X_1'}=\mu"'] & X' \arrow[l, "u" description] \arrow[r, "\omega_X\ (\text{resp. } \omega_X^q)"] \arrow[d, "\eta_{X'}=d"] & X \\
X_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[loop below, "-1"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X \arrow[l, "i" description] &
\end{tikzcd} \tag{18}$$

Then since ω_X (resp. ω_X^q) is a normal covering, d_1 and d_2 are trivial extensions, so that the commutative squares $dd_1 = d\mu$ and $dd_2 = c\mu$ are actually pullback squares. Hence the pullback $p_1', p_2'\colon X_2' \rightrightarrows X_1'$ of d_2 and d_1 and the pullback $p_1, p_2\colon X_2 \rightrightarrows X_1$ of c and d, induce a morphism $f\colon X_2' \to X_2$ which is thus the pullback of $\eta_{X_1'} = \mu$ and computed componentwise as $f = \mu \times \mu$. By admissibility of the Galois structure Γ (see Paragraph 2.4.1 and [55]), this morphism is also the unit component $f = \eta_{X_2'}$. Finally the commutativity of the square $\mu m' = m\eta_{X_2'}$ is given by construction (and easy to check by hand), which concludes the proof that $\pi_1(X) = \pi_0(\mathcal{G}) = \mathcal{G}_{(X,\mathrm{Pth}(X))}$ (resp. $\pi_1^q(X) = \mathcal{G}_{(X,\mathrm{Pth}^\circ(X))}$ in Qnd). ∎

3.37.0.1. Remarks One of D.E. Joyce's main results is to show that the *knot quandle* is a complete invariant for oriented knots. Now the *knot group* [71] of an oriented knot, which is the fundamental group of the ambient space of the knot, is also computed as the group of paths of the knot quandle. In other words, the knot group is the *fundamental group* of the knot quandle, in the sense of the covering theory of racks (not in the sense of the covering theory of quandles).

Finally observe that $\pi_1(X)$ (resp. $\pi_1^q(X)$) can be equipped with a non-trivial ad-hoc structure of rack (resp. quandle) making it into an internal groupoid in Rck (resp. Qnd) with internal object of objects the rack (resp. quandle) X. Given two trails (a, g) and (b, h) in X_1, define $(a, g) \triangleleft (b, h) := (a \triangleleft b, \underline{b}^{-1} g h^{-1} \underline{b} h)$ (note that if g, $h \in \mathrm{Pth}^\circ(X)$, then $\underline{b}^{-1} g h^{-1} \underline{b} h \in \mathrm{Pth}^\circ(X)$). Unlike in $\hat{X}$ (resp. $\hat{X}^\circ$), trails act on each other with both their heads and end-points, which means that both projections to X are morphisms in Rck (resp. Qnd). The rest of the structure is easy to derive.

3.37.0.2. Working with skeletons As we shall see in the next section, we are interested in the fundamental groupoid, up to equivalence. Given a rack A, we thus also describe a *skeleton* S of $\pi_1(A)$ (in the sense of [67, Section IV.4]). The resulting groupoid S is not regular like $\pi_1(A)$, it is totally disconnected and its vertices are the connected components of A. With the objective of interpreting the fundamental theorem of Galois theory, the homotopical information contained in $\pi_1(A)$ can be made more explicit using its skeleton.

3.38. Definition. *Given an object A in Rck (respectively in Qnd), we call a* pointing *of A any choice of representatives $I \coloneqq \{a_i\}_{i\in\pi_0(A)} \subseteq A$ such that $\eta_A(a_i) = [a_i] = i$ for each equivalence class $i \in \pi_0(A)$. Then for any element $a \in A$, define Loop_a as the group of loops $l \in \mathrm{Pth}(A)$ (resp. $l \in \mathrm{Pth}^\circ(A)$) such that $a \cdot l = a$. Observe that if $[a] = [b]$, for some a and b in A, then there is $g \in \mathrm{Pth}(A)$ (resp. $g \in \mathrm{Pth}^\circ(A)$) such that $a = b\cdot g$ and thus the subgroups Loop_a and Loop_b are isomorphic, via the automorphism of $\mathrm{Pth}(A)$ (resp. Pth°) given by conjugation with g.*

Let us fix a pointing $I \coloneqq \{a_i\}_{i\in\pi_0(A)} \subseteq A$ of A, then we define the groupoid $\pi_1(A, I)$ (resp. $\pi_1^q(A, I)$) as

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & A_1 \arrow[loop above, "-1"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & \pi_0(A), \arrow[l, "i" description] \end{tikzcd}$$

where $A_1 \coloneqq \coprod_{i\in\pi_0(A)} \mathrm{Loop}_{a_i}$ is defined as the disjoint union, of the underlying sets of Loop_{a_i}'s indexed by $i \in \pi_0(A)$. The domain and codomain maps send a loop $l \in \mathrm{Loop}_{a_i}$ to the index $i \in \pi_0(A)$. The set A_2 is then the disjoint union of products $A_2 \coloneqq \coprod_{i\in\pi_0(A)}(\mathrm{Loop}_{a_i} \times \mathrm{Loop}_{a_i})$ and m is defined by multiplication in $\mathrm{Loop}_{a_i} \leq \mathrm{Pth}(A)$ (resp. $\mathrm{Loop}_{a_i} \leq \mathrm{Pth}^\circ(A)$).

From the description of the skeleton of a groupoid obtained as in Definition 3.36, we deduce:

3.39. Lemma. *For each I pointing of A object of Rck (respectively of Qnd), $\pi_1(A, I)$ (respectively $\pi_1^q(A, I)$) is a skeleton of the fundamental groupoid $\pi_1(A)$ (respectively $\pi_1^q(A)$).*

3.40. The fundamental theorem of categorical Galois theory. In sections 5, 6 and 7 of [29], M. Eisermann studies in detail different classification results for quandle coverings. We will not go into so much depth ourselves, however we show how to recover and extend the main theorems from these sections using categorical Galois theory.

Given an object A in Rck (respectively Qnd), the category of *internal covariant presheaves* over $\pi \coloneqq \pi_1(A)$ (resp. $\pi \coloneqq \pi_1^q(A)$) are externally described as the category of functors from π to Set and thus as the category of π-groupoid actions on sets Set^π. Given a pointing I of A, define $\pi(I) \coloneqq \pi_1(A, I)$ (resp. $\pi(I) \coloneqq \pi_1^q(A, I)$ and deduce from $\pi(I) \cong \pi$ that $\mathsf{Set}^\pi \cong \mathsf{Set}^{\pi(I)}$. Now $\pi(I)$ is totally disconnected, thus the category of $\pi(I)$-actions is equivalent to the category $\coprod_{i\in\pi_0(A)} \mathsf{Set}^{\mathrm{Loop}_{a_i}}$ whose objects are sequences of Loop_{a_i}-group actions (see Definition 3.38), indexed by $i \in \pi_0(A)$, and morphisms between these are π_0-indexed sums of group-action morphisms. From the fundamental theorem of categorical Galois theory (see for instance [55, Theorem 6.2]), classifying central extensions above an object we deduce in particular:

3.41. Theorem. *Given an object A in Rck and a pointing $I \coloneqq \{a_i\}_{i\in\pi_0(A)} \subseteq A$ of A, there is a natural equivalence of categories between the category $\mathsf{CExt}(A)$ of central extensions above A and the category $\mathsf{Set}^{\pi_1(A)}$. The latter category is then also equivalent (but not naturally) to $\mathsf{Set}^{\pi_1(A,I)} \cong \coprod_{i\in\pi_0(A)} \mathsf{Set}^{\mathrm{Loop}_{a_i}}$. The same theorem holds in Qnd,* *using the appropriate definition of* Loop_{a_i} *and using* $\pi_1^q(A)$ *and* $\pi_1^q(A,I)$ *instead of* $\pi_1(A)$ *and* $\pi_1(A,I)$.

3.42. Corollary. *The category of central extensions above a connected rack A is equivalent to the category of* Loop_a*-actions (from Definition 3.38), for any given element $a \in A$. The same is true in* Qnd.

3.43. Example. We illustrate this result on a trivial example, to show the difference between the context of Rck and that of Qnd. Consider the one element set 1. The coverings above 1 in Qnd should all be surjective maps to 1 in Set, whereas the coverings above 1 in Rck include for instance the unit morphism ${}^r\eta^q_{\mathrm{F_r}(1)} = \eta_{\mathrm{F_r}1} \colon \mathrm{F_r}(1) \to \mathrm{F_q}(1) = 1$, whose domain is not a set. Then observe that $\mathrm{Pth}(1) = \mathbb{Z}$ and thus $\mathrm{Pth}^{\circ}(1) = \{e\}$ and since there is only one element $* \in 1$, Loop_* is the former in Rck and the latter in Qnd. Hence the category of coverings above 1 in Qnd is $\mathsf{Set}^{\{e\}}$ which is indeed equivalent to Set. The category of coverings above 1 in Rck is given by $\mathsf{Set}^{\mathbb{Z}}$, the category of $\mathbb{Z}$-actions on sets, where $\mathbb{Z}$ is the additive group of integers.

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