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SURJECTION-LIKE CLASSES OF MORPHISMS

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ABSTRACT. We characterize 'good' classes of epimorphisms in a finitely complete category, i.e., those which 'interact with finite limits as surjections do in the category Set of sets and functions'. More precisely, we prove that given a class E of morphisms in a small finitely complete category $\mathcal{C}$, there exists a faithful conservative (respectively fully faithful) embedding $\mathcal{C} \hookrightarrow \mathsf{Set}^{\mathcal{D}}$ into a presheaf category which preserves and reflects finite limits and which sends morphisms in E, and only those, to componentwise surjections if and only if E contains the identities, is closed under composition, has the strong right cancellation property, is stable under pullbacks and does not contain any proper monomorphisms (respectively any morphism in it is a regular epimorphism). The classes of split epimorphisms and descent morphisms are such examples and the corresponding full embedding theorems are given by Yoneda and Barr's embeddings. As new examples, we get a conservative embedding theorem for the class of pullback-stable strong epimorphisms and a full embedding theorem for the class of effective descent morphisms. The proof presented here is not based on transfinite inductions and is therefore rather explicit, in contrast with similar embedding theorems.

Introduction

The most natural examples of categories one may think of are given by mathematical structures of some kind together with the appropriate functions between them as morphisms. Among many others, we can cite the categories Set of sets and functions, Gp of groups and group homomorphisms, Mon of monoids and monoid homomorphisms, Top of topological spaces and continuous functions or Pos of partially ordered sets and order preserving maps. In all these examples, as in most of the classical examples, monomorphisms are characterized as exactly the injective morphisms. For this reason, monomorphisms are commonly thought of as the 'right' categorical generalization of injective morphisms.

The analogous question of generalizing surjective morphisms is much more subtle and no definite answer has been given so far, each author often using his/her own preferred classes of morphisms. A first class of morphisms one can think of to generalize surjective morphisms is the class of epimorphisms. However, it is commonly accepted not to be a good solution. Although epimorphisms in Set are exactly surjective functions, it is far from being the case in many classical categories. For instance, in Mon, the embedding

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$\mathbb{N} \hookrightarrow \mathbb{Z}$ of natural numbers into integers is an epimorphism. This is therefore an example of a morphism which is both a monomorphism and an epimorphism but which is not an isomorphism. As this phenomenon cannot occur in our leading example of surjective functions in Set, this brings us to our first axiom for a 'good class E of surjections' in a category $\mathcal{C}$:

(NoPMono) Every monomorphism in E is an isomorphism.

Another candidate to generalize surjective morphisms to an arbitrary category is the class of regular epimorphisms, i.e., coequalizers. This class satisfies our first axiom (NoPMono). In addition, in every algebraic category (i.e. variety of universal algebras), regular epimorphisms coincide with surjective homomorphisms. However, in general, they fail to be closed under composition. Since this is an important property shared by surjective morphisms in classical categories, we will also impose this as an axiom on E:

(ClComp) E is closed under composition.

Let us now make our question more precise. As finite limits are overwhelming in many subfields of category theory, we will look for classes E of morphisms in a finitely complete category $\mathcal{C}$ which 'interact with finite limits in the same way surjections interact with finite limits in Set'. Setting aside concerns about size, we can be even more precise: we are looking for classes E of morphisms for which there exists an 'embedding' $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ into a presheaf category which preserves and reflects finite limits and such that, for each morphism $e \in \mathcal{C}$, one has $e \in E$ if and only if $\varphi(e)$ is a componentwise surjective natural transformation. The word 'embedding' will have two different meanings: in the 'conservative case', it will be a faithful conservative functor; while in the 'full case', we mean a fully faithful functor. As we have seen so far, in both cases, the axioms (NoPMono) and (ClComp) have to be satisfied, which makes the classes of epimorphisms and of regular epimorphisms counter-examples in general. However, Barr's embedding theorem for regular categories [3] exactly means that the class of regular epimorphisms in a *regular category* [4] satisfies the full embedding theorem.

Going back to the general case of a finitely complete category $\mathcal{C}$, we can consider the class of strong epimorphisms, or equivalently the class of extremal epimorphisms, i.e., epimorphisms which do not factorize through any proper subobject of their codomain. This class satisfies both axioms (NoPMono) and (ClComp) and coincide with surjective homomorphisms in any algebraic category. However, in general, it fails to be stable under pullbacks. Since surjections in Set are pullback-stable, this means that strong epimorphisms do not interact with finite limits in a general finitely complete category in the same way surjections do in Set. We therefore need to add an additional axiom for our 'good classes of surjections', namely:

(StPb) E is stable under pullbacks.

To avoid the degenerate empty class of morphisms and that of isomorphisms (which in general do not satisfy any of the two embedding theorems mentioned above), we need to add the following two axioms:

(Id) E contains the identities;

(SRightCancP) E has the strong right cancellation property, i.e., $gf \in E \Rightarrow g \in E$.

We have therefore the five axioms (NoPMono), (ClComp), (StPb), (Id) and (SRightCancP) which we prove to be together equivalent to the conservative embedding theorem (Theorem 2.1). For this reason, we call a class of morphisms satisfying these five axioms a *surjection-like class of morphisms*, as referenced in the title. In any finitely complete category, the largest such class of morphisms is given by the class $E_{\mathsf{pb\,strong\,epi}}$ of pullback-stable strong epimorphisms, i.e., morphisms whose pullback along any morphism is a strong epimorphism. We also show that in general, this class fails to satisfy the full embedding theorem mentioned above. The reason is that morphisms in $E_{\mathsf{pb\,strong\,epi}}$ are in general not regular epimorphisms, which is a necessary condition for the full embedding theorem. We thus add a last axiom:

(Reg) Morphisms in E are regular epimorphisms.

A surjection-like class of morphisms satisfying (Reg) is said to be *regular* and we show (Theorem 2.2) that this is exactly what is needed to have a full embedding theorem. The biggest such class of morphisms is the class E_{descent} of descent morphisms, also described as pullback-stable regular epimorphisms. The fact that this class satisfies the axioms to be a regular surjection-like class of morphisms has been shown in [9]. The full embedding theorem obtained from this class E_{descent} has been obtained in [2] from the embedding theorem for regular categories [3].

The smallest example of a (regular) surjection-like class of morphisms is the class $E_{\mathsf{split\,epi}}$ of split epimorphisms (i.e., epimorphisms admitting a section). In that case, the full embedding can be taken to be just the classical Yoneda embedding. Between these two examples, one also has the class $E_{\mathsf{eff\,descent}}$ of effective descent morphisms, which, in view of the results from [9, 11, 12], is a regular surjection-like class of morphisms (assuming the axiom of universes [1] to avoid some size issues). The full embedding theorem coming from $E_{\mathsf{eff\,descent}}$, as well as the conservative embedding theorem coming from $E_{\mathsf{pb\,strong\,epi}}$, are to our knowledge new from this paper. It is also worth mentioning that from the results in [5, 9], we know that in the category Cat of small categories and functors, the four classes $E_{\mathsf{split\,epi}} \subsetneqq E_{\mathsf{eff\,descent}} \subsetneqq E_{\mathsf{descent}} \subsetneqq E_{\mathsf{pb\,strong\,epi}}$ are distinct and there exist infinitely many other regular surjection-like classes of morphisms in Cat.

Let us also mention that, given a surjection-like class of morphisms E in a finitely complete category $\mathcal{C}$, one cannot expect in general to have an (E, M)-factorization system where M is the class of monomorphisms in $\mathcal{C}$. Indeed, all morphisms f in $\mathcal{C}$ factorize as $f = me$ with m a monomorphism and $e \in E$ if and only if $\mathcal{C}$ is a regular category and E is the class of regular epimorphisms in it.

Our Embedding Theorems 2.1 and 2.2 have two main applications. On the practical side, one can prove many statements about finite limits and morphisms in a surjection-like class of morphisms E in any finitely complete category $\mathcal{C}$ just by producing a proof for the particular case $\mathcal{C} = \mathsf{Set}$ and $E = \{\text{surjections}\}$. Therefore, for these statements, a proof using elements will be enough to prove the result in full generality. To avoid size issues, one has to assume here the axiom of universes. For such proof reductions, the Conservative Embedding Theorem 2.1 (and so surjection-like classes of morphisms) is almost as useful as the Full Embedding Theorem 2.2 (and so regular surjection-like classes of morphisms) since, even in the full case, one has to take extra caution when dealing with the existence of some morphism. This technique is well-known and have been briefly discussed e.g. in [2] and more precisely described in [7]. We will give as such an application here a generalization of the so-called Barr–Kock theorem where one requires that a morphism is (only) a pullback-stable strong epimorphism (see Corollary 2.3).

More importantly, on the theoretical side, these embedding theorems give potential criteria to decide which classes of morphisms form 'good' categorical generalizations of surjections.

The major part of the paper is devoted to the proof of our Embedding Theorems 2.1 and 2.2, so let us say a word on it now. A possible way to tackle this problem is to consider the Grothendieck topology $\mathcal{T}_E$ on $\mathcal{C}$ induced by E and the composite functor

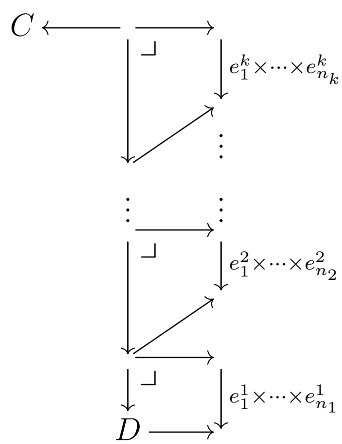
$$\mathcal{C} \xrightarrow{\mathsf{Y}} \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{\mathsf{a}} \mathsf{Sh}(\mathcal{C}, \mathcal{T}_E) \xrightarrow{\mathsf{B}} \mathsf{Set}^{\mathcal{D}}$$

where Y is the classical Yoneda embedding, a is the sheafification functor for the topology $\mathcal{T}_E$ and B is Barr's full embedding [3] for the regular category $\mathsf{Sh}(\mathcal{C}, \mathcal{T}_E)$ of sheaves on the site $(\mathcal{C}, \mathcal{T}_E)$. If E is a surjection-like class of morphisms, it can be proved that aY is conservative and faithful and it sends a morphism e to a (regular) epimorphism in $\mathsf{Sh}(\mathcal{C}, \mathcal{T}_E)$ if and only if $e \in E$. Moreover, if E is a regular surjection-like class of morphisms, $\mathcal{T}_E$ is a subcanonical topology and aY is a fully faithful embedding. Although this proof might be short and elegant, it is has several drawbacks. Firstly, even if the category $\mathcal{C}$ is assumed to be small, the sheaf category $\mathsf{Sh}(\mathcal{C}, \mathcal{T}_E)$ is in general not small. Therefore, Barr's embedding theorem cannot be applied as such to this regular category $\mathsf{Sh}(\mathcal{C}, \mathcal{T}_E)$. One thus needs to change universe even in the case where $\mathcal{C}$ is small which is very unpleasant. In particular, in the codomain of this embedding, the category $\mathcal{D}$ need not be small and the category Set is not any more the usual category of (small) sets and functions. Secondly, in the construction of B from [3], some transfinite inductions are used, based on the axiom of choice to well-order the sets on which these transfinite inductions run. As a result, the way the category $\mathcal{D}$ and the embedding B are constructed is quite obscure and this has raised a certain scepticism in the community. Finally, this proof technique seems not to be easily generalizable and this will probably prevent us to prove similar results in the future (see the section on future work below).

Another strategy could be to generalize Barr's proof of his embedding theorem for regular categories by replacing small regular categories by small finitely complete categories and the class of regular epimorphisms by a (regular) surjection-like class of morphisms. This would have the advantage of avoiding the size issue raised by the above proof technique, but this would still be based on some not explicit transfinite inductions. Moreover, it is not clear how one could achieve such a generalization directly. Indeed, the first step in Barr's proof is to show that, if $\mathcal{C}$ is a small regular category, the opposite category

$\widetilde{\mathcal{C}} = \mathsf{Lex}(\mathcal{C}, \mathsf{Set})^{\mathsf{op}}$ of the category of finite limit preserving functors $\mathcal{C} \to \mathsf{Set}$ is also regular and the fully faithful embedding $\mathcal{C} \to \widetilde{\mathcal{C}}$ preserves and reflects finite limits and regular epimorphisms. Barr then uses this bigger category $\widetilde{\mathcal{C}}$ to construct (using transfinite inductions) a $\mathcal{C}$-projective covering of any object of $\mathcal{C}$ (and even of $\widetilde{\mathcal{C}}$); this argument being based on an argument by Grothendieck. The problem in our generalized setting is that, if we replace regular epimorphisms in $\mathcal{C}$ by morphisms in E, by what should we replace regular epimorphisms in $\widetilde{\mathcal{C}}$? Although we could close the class E in $\widetilde{\mathcal{C}}$ under certain properties, it is not clear to us how to adapt the proof in that context.

Our idea will be instead to 'deconstruct' Barr's proof in order to make it explicit in terms of $\mathcal{C}$ (and not working in $\widetilde{\mathcal{C}}$). The hardest step in that process was to make explicit how to construct pullbacks and cofiltered limits in $\widetilde{\mathcal{C}}$. That way, we achieved to turn the $\mathcal{C}$-projective covering of a finite limit preserving functor $F \colon \mathcal{C} \to \mathsf{Set}$ constructed by Barr into an explicit (transfinite induction free) construction of a componentwise injective natural transformation $\iota_F \colon F \rightarrowtail \overline{F}$ where $\overline{F} \colon \mathcal{C} \to \mathsf{Set}$ preserves finite limits and sends elements of E to surjections. As a result, we get a much more explicit construction of the small category $\mathcal{D}$ and the embedding $\varphi \colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ in our Embedding Theorems 2.1 and 2.2. In particular, in the conservative case, the objects of the category $\mathcal{D}$ can be chosen to be the objects of $\mathcal{C}$, and given objects C, D, the set $\varphi(C)(D)$ can be defined as some quotient of the set of diagrams of the form



where the k quadrilaterals are pullbacks of finite products of specified elements of E.

Except from this Introduction, the paper contains only two sections. In the first one, the definitions of (regular) surjection-like classes of morphisms are given, their basic properties are studied and some examples are discussed. The second section is devoted to our embedding theorems. They are proved simultaneously in a common proof which is divided into 42 steps, but the proof in the conservative case only requires the first 30 steps.

Future work. If possible, we would like to study the case of jointly epimorphic cospans in a future work. Actually, we want to generalize our work in order to answer the question: which classes of cospans (or which Grothendieck topologies) interact with finite limits in the same way as jointly surjective cospans interact with finite limits in Set?

We are also interested in the analogous question where finite limits are not considered, i.e., which classes of morphisms behave like surjections do in Set? More precisely, we would like to characterize those classes E of morphisms in a small category $\mathcal{C}$ for which there exists a (fully) faithful conservative functor $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ to a presheaf category such that a morphism $e \in \mathcal{C}$ is in E if and only if $\varphi(e)$ is a componentwise surjection. Of course, the axioms (Id), (ClComp) and (SRightCancP) still have to hold for E, but the axioms (StPb) and (NoPMono) do not seem to be needed any more. Indeed, these two axioms are 'finite limit statements' (for (NoPMono), we recall that the property of being a monomorphism is equivalent to the condition that the two projections of the kernel pair are equal). In view of that, the class of epimorphisms might be an example of such a class and therefore be considered as a 'good class of surjections', although we did not consider it to be so since the very beginning of this introduction as it has 'poor interactions with finite limits'.

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1. Axioms and examples

Let us fix in this whole section a finitely complete category $\mathcal{C}$ (i.e., a category admitting finite limits).

Axioms for a conservative embedding. Let us consider a class of morphisms E in $\mathcal{C}$. We say that E is a *surjection-like class of morphisms* (in $\mathcal{C}$) if it satisfies the following five axioms:

(Id) E *contains identities*, i.e., for each object $A \in \mathcal{C}$, one has $1_A \in E$.

(ClComp) E is *closed under composition*, i.e., given any pair of composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{C}$, if f and g are in E, then so is the composite gf.

(SRightCancP) E has the *strong right cancellation property*, i.e., given any pair of composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, if $gf \in E$, then $g \in E$.

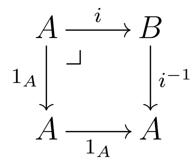
(StPb) E is *stable under pullbacks*, i.e., given any pullback square

$$\begin{array}{ccc} P & \xrightarrow{\overline{f}} & B \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

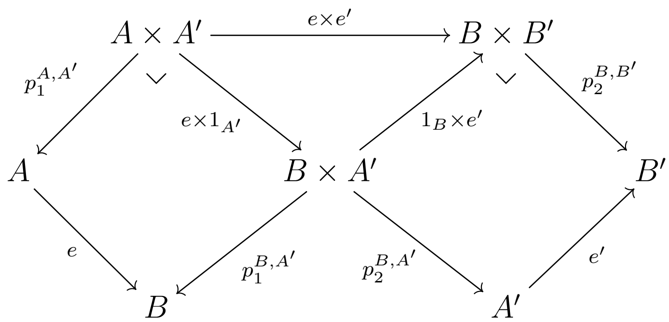
in $\mathcal{C}$, if $f \in E$, then $\overline{f} \in E$.

(NoPMono) *E does not contain any proper monomorphisms*, i.e., each monomorphism in E is an isomorphism.

In that case, since E satisfies (Id) and (StPb), we know it *contains isomorphisms*. This can be seen via the following pullback square



for any isomorphism i. In addition, since E satisfies (ClComp) and (StPb), it is well-known that E is *closed under binary products*, i.e., if $e, e' \in E$, then $e \times e' \in E$. If $e\colon A \to B$ and $e'\colon A' \to B'$ are two morphisms, then $e \times e'\colon A \times A' \to B \times B'$ is canonically induced by e and e' between the product of A and A' and that of B and B'. The following commutative diagram



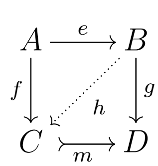
where the squares are pullbacks and where the morphisms $p_1^{A,A'}$, $p_1^{B,A'}$, $p_2^{B,A'}$ and $p_2^{B,B'}$ are product projections indicates how to prove this property. Therefore, each surjection-like class of morphisms E is *closed under finite products*, i.e., given any finite family $(e_i \in E)_{i \in I}$ of elements of E, their product $\prod_{i \in I} e_i$ also belongs to E.

Let us now denote by $E_{\mathsf{split\,epi}}$ the class of *split epimorphisms* in $\mathcal{C}$, i.e., the class of morphisms $e\colon A \to B$ such that there exists $s\colon B \to A$ with $es = 1_B$. It is well-known and routine to prove that $E_{\mathsf{split\,epi}}$ is a surjection-like class of morphisms. Moreover, in view of (Id) and (SRightCancP), it is the smallest one, i.e., the inclusion

$$E_{\mathsf{split\,epi}} \subseteq E$$

holds for any surjection-like class of morphisms E in $\mathcal{C}$.

We recall that a *strong epimorphism* in $\mathcal{C}$ is a morphism $e\colon A \to B$ such that for any commutative square of plain morphisms



where m is a monomorphism, there exists a (necessarily unique) dotted morphism h making the two triangles commutative. Since $\mathcal{C}$ is finitely complete, this property implies that e is an epimorphism, and it is equivalent to the property of being an *extremal epimorphism*, i.e., an (epi)morphism such that given any commutative triangle

$$\begin{array}{ccc} & & C \\ & \overset{f}{\nearrow} & \big\downarrow m \\ A & \xrightarrow[e]{} & B \end{array}$$

where m is a monomorphism, then m is an isomorphism. The class of strong epimorphisms is in general not stable under pullbacks. We thus need the notion of *pullback-stable strong epimorphism*, which is a morphism $e\colon A \to B$ such that, for any pullback square

$$\begin{array}{ccc} P & \xrightarrow{\overline{e}} & C \\ {\scriptstyle \overline{f}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow[e]{} & B, \end{array}$$

the morphism $\overline{e}$ is a strong epimorphism. In particular, this implies that e is itself a strong epimorphism. We denote by $E_{\mathsf{pb\,strong\,epi}}$ the class of pullback-stable strong epimorphisms in $\mathcal{C}$. Again, it is well-known and routine to prove that $E_{\mathsf{pb\,strong\,epi}}$ is a surjection-like class of morphisms. Moreover, it is the largest one. Indeed, given a surjection-like class of morphisms E, by (SRightCancP) and (NoPMono), each element of E is an extremal epimorphism. Hence, by (StPb), each element of E is a pullback-stable strong epimorphism. Therefore, for each surjection-like class of morphisms E in $\mathcal{C}$, the inclusions

$$E_{\mathsf{split\,epi}} \subseteq E \subseteq E_{\mathsf{pb\,strong\,epi}}$$

hold.

Axioms for a full embedding. We recall that a *regular epimorphism* in $\mathcal{C}$ is a morphism e which is the coequalizer of two parallel morphisms. Since $\mathcal{C}$ is finitely complete, this is equivalent to require that e is the coequalizer of its kernel pair.

A surjection-like class of morphisms E in the finitely complete category $\mathcal{C}$ is said to be *regular* when it satisfies the additional axiom:

(Reg) Every morphism in E is a regular epimorphism.

Since (Reg) is stronger than (NoPMono), a regular surjection-like class of morphisms in $\mathcal{C}$ is a class of morphisms satisfying the axioms (Id), (ClComp), (SRightCancP), (StPb) and (Reg).

It is classical that each split epimorphism is a regular epimorphism. Therefore, the class $E_{\mathsf{split\,epi}}$ is a regular surjection-like class of morphisms and it is the smallest such. Besides, not all pullback-stable strong epimorphisms are regular epimorphisms (e.g. in the category Cat of small categories, see below). Therefore, the surjection-like class of morphisms $E_{\mathsf{pb\,strong\,epi}}$ is in general not regular.

Since $\mathcal{C}$ is finitely complete, a *descent morphism* can be characterized [10] as a *pullback-stable regular epimorphism*, i.e., a morphism $e\colon A \to B$ such that, for any pullback square

$$\begin{array}{ccc} P & \xrightarrow{\overline{e}} & C \\ {\scriptstyle \overline{f}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow[e]{} & B, \end{array}$$

the morphism $\overline{e}$ is a regular epimorphism. We denote by E_{descent} the class of descent morphisms in $\mathcal{C}$. It has been proved in [9] that E_{descent} is a regular surjection-like class of morphisms. Moreover, in view of (StPb) and (Reg), it is the largest one. Therefore, for each regular surjection-like class of morphisms E in $\mathcal{C}$, the inclusions

$$E_{\mathsf{split\,epi}} \subseteq E \subseteq E_{\mathsf{descent}}$$

hold.

The finitely complete category $\mathcal{C}$ is said to be *regular* in the sense of [4] if it has coequalizers of kernel pairs and if regular epimorphisms are stable under pullbacks. In that case, the classes $E_{\mathsf{pb\,strong\,epi}}$ and E_{descent} both coincide with the class of regular epimorphisms, which is thus a regular surjection-like class of morphisms. Given a surjection-like class of morphisms E in a general finitely complete category $\mathcal{C}$, the following conditions are equivalent:

- the class E has the additional property that each morphism f in $\mathcal{C}$ factorizes as $f = me$ where m is a monomorphism and $e \in E$;
- $\mathcal{C}$ is a regular category and E is the class of regular epimorphisms in it.

Indeed, the factorization property is well-known in regular categories, while the other direction can be proved as follows. Given a strong epimorphism f, we can factorize it as $f = me$ with $e \in E$ and m a monomorphism. Since f is a strong epimorphism, m is an isomorphism and $f \in E$. This proves that E is the class of strong epimorphisms. Using Proposition 2.2.2 in [6], this proves that $\mathcal{C}$ is regular and E is the class of regular epimorphisms.

Examples. We have already seen the examples $E_{\mathsf{split\,epi}}$ and E_{descent} of regular surjection-like classes of morphisms and $E_{\mathsf{pb\,strong\,epi}}$ of (in general not regular) surjection-like class of morphisms.

The class $E_{\mathsf{eff\,descent}}$ of *effective descent morphisms* in $\mathcal{C}$ is easily seen to satisfy (Id) and (Reg). Moreover, it is proved in [12] that $E_{\mathsf{eff\,descent}}$ satisfies (StPb) and in [11] that it satisfies (ClComp). Finally, it is shown in [9], under some smallness assumption on $\mathcal{C}$, that $E_{\mathsf{eff\,descent}}$ satisfies (SRightCancP). This smallness assumption can be removed assuming for instance the *axiom of universes* [1]. The class $E_{\mathsf{eff\,descent}}$ is thus a regular surjection-like class of morphisms, which sits, in general, strictly between $E_{\mathsf{split\,epi}}$ and E_{descent} (see e.g. the case $\mathcal{C} = \mathsf{Cat}$ below).

In the category Set of sets, assuming the axiom of choice, the four classes $E_{\mathsf{pb\,strong\,epi}}$, E_{descent}, $E_{\mathsf{eff\,descent}}$ and $E_{\mathsf{split\,epi}}$ all coincide with the class of surjective functions. However, this situation is very specific to Set.

Given an object X in the finitely complete category $\mathcal{C}$, we denote by E_X the class of morphisms $e\colon A \to B$ such that, for each morphism $f\colon X \to B$, there exists a morphism $g\colon X \to A$ such that $eg = f$.

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle e} \\ & & B \end{array}$$

Clearly, E_X satisfies the axioms (Id), (ClComp), (SRightCancP) and (StPb), but in general it does not satisfies (NoPMono). Let us consider the particular case where $\mathcal{C} = \mathsf{Cat}$ is the category of small categories. For a non-negative integer n, let $\mathcal{J}_n$ be the totally ordered set on $n+1$ elements seen as a category, i.e., the category generated by the graph

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n.$$

We have the strict inclusions

$$E_{\mathcal{J}_0} \supsetneqq E_{\mathcal{J}_1} \supsetneqq E_{\mathcal{J}_2} \supsetneqq E_{\mathcal{J}_3} \supsetneqq E_{\mathcal{J}_4} \supsetneqq \cdots$$

and $\bigcap_{n\geqslant 0} E_{\mathcal{J}_n} \supsetneqq E_{\mathsf{split\,epi}}$. The class $E_{\mathcal{J}_0}$ is the class of functors which are surjective on objects. It does not satisfies (NoPMono). It is shown in [5] that $E_{\mathcal{J}_1} = E_{\mathsf{pb\,strong\,epi}}$ and $E_{\mathcal{J}_2} = E_{\mathsf{descent}}$; and it is shown in [9] that $E_{\mathcal{J}_3} = E_{\mathsf{eff\,descent}}$. In view of the above strict inclusions, we therefore have that $E_{\mathcal{J}_1}$ is a surjection-like class of morphisms which is not regular; and for $n \geqslant 2$, the class $E_{\mathcal{J}_n}$ is a regular surjection-like class of morphisms. Hence, $\bigcap_{n\geqslant 0} E_{\mathcal{J}_n}$ is also a regular surjection-like class of morphisms. We therefore have infinitely many such classes in Cat.

Let us conclude this section by proving that none of the axioms (Id), (ClComp), (SRightCancP), (StPb) and (NoPMono) (respectively (Id), (ClComp), (SRightCancP), (StPb) and (Reg)) can be removed from the definition of surjection-like classes of morphisms (respectively of regular surjection-like classes of morphisms). If the finitely complete category $\mathcal{C}$ is not a preorder, the empty class $E = \varnothing$ satisfies all axioms except (Id); the class of isomorphisms in $\mathcal{C}$ satisfies all axioms except (SRightCancP) and the class of all morphisms in $\mathcal{C}$ satisfies all axioms except (NoPMono) and (Reg). Let now E and E' be two distinct regular surjection-like classes of morphisms in $\mathcal{C}$ (which exist, e.g., if $\mathcal{C} = \mathsf{Cat}$). The class $(E \times E') \cup (E' \times E)$ in the product category $\mathcal{C} \times \mathcal{C}$ satisfies all axioms except (ClComp). Finally, if $\mathcal{C}$ is the category of preordered sets, the class of regular epimorphisms, which coincides with the class of strong epimorphisms, has been described in [8]. It satisfies (Id), (ClComp), (SRightCancP), (NoPMono) and (Reg), but not (StPb).

2. The embedding theorems

We are now ready to state and prove our embedding theorems. We recall that a functor is said to be *conservative* if it reflects isomorphisms. For a small category $\mathcal{D}$, we denote by $\mathsf{Set}^{\mathcal{D}}$ the category of functors $\mathcal{D} \to \mathsf{Set}$ and their natural transformations. In this category $\mathsf{Set}^{\mathcal{D}}$, the classes $E_{\mathsf{pb\,strong\,epi}}$, E_{descent} and $E_{\mathsf{eff\,descent}}$ all coincide with the class of *componentwise surjections*, i.e., the class of natural transformations $\alpha\colon F \Rightarrow G\colon \mathcal{D} \to \mathsf{Set}$ such that, for each object $D \in \mathcal{D}$, α_D is a surjective function.

For set-theoretical reasons, the theorems below are stated under the assumption that the category $\mathcal{C}$ is *small*. However, using the *axiom of universes* [1], this condition can easily be overcome.

2.1. THEOREM. *Let $\mathcal{C}$ be a small finitely complete category. The following conditions on a class E of morphisms in $\mathcal{C}$ are equivalent:*

- *There exists a small category $\mathcal{D}$ and a faithful conservative functor $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ which preserves and reflects finite limits and such that, for each morphism e in $\mathcal{C}$, $\varphi(e)$ is a componentwise surjection if and only if $e \in E$.*
- *E is a surjection-like class of morphisms, i.e., it satisfies the axioms*

  (Id) *E contains all identities;*

  (ClComp) *E is closed under composition;*

  (SRightCancP) *E has the strong right cancellation property;*

  (StPb) *E is stable under pullbacks;*

  (NoPMono) *every monomorphism in E is an isomorphism.*

2.2. THEOREM. *Let $\mathcal{C}$ be a small finitely complete category. The following conditions on a class E of morphisms in $\mathcal{C}$ are equivalent:*

- *There exists a small category $\mathcal{D}$ and a fully faithful functor $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ which preserves and reflects finite limits and such that, for each morphism e in $\mathcal{C}$, $\varphi(e)$ is a componentwise surjection if and only if $e \in E$.*
- *E is a regular surjection-like class of morphisms, i.e., it satisfies the axioms*

  (Id) *E contains all identities;*

  (ClComp) *E is closed under composition;*

  (SRightCancP) *E has the strong right cancellation property;*

  (StPb) *E is stable under pullbacks;*

  (Reg) *every morphism in E is a regular epimorphism.*

Before proving these theorems, let us make a few comments about them. Considering $\mathcal{C}$ to be a small regular category and E the class of regular epimorphisms in it, one recovers from Theorem 2.2 Barr's full embedding theorem [3] for regular categories. More generally, if in Theorem 2.2 one chooses E to be the class E_{descent} of descent morphisms in a small finitely complete category $\mathcal{C}$, one recovers the embedding theorem established in [2].

The particular instance of Theorem 2.2 for $E = E_{\mathsf{split\,epi}}$ is also known. Indeed, in that case, it is sufficient to consider the Yoneda embedding $\mathsf{Y}\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ since a morphism e is a split epimorphism if and only if $\mathsf{Y}(e)$ is a componentwise surjection.

To our knowledge, the conservative embedding theorem obtained from Theorem 2.1 for $E = E_{\mathsf{pb\,strong\,epi}}$ and the full embedding theorem obtained from Theorem 2.2 for $E = E_{\mathsf{eff\,descent}}$ are new.

In our opinion, the main theoretical application of Theorems 2.1 and 2.2 is to justify why, and in which sense, (regular) surjection-like classes of morphisms are exactly the ones interacting with finite limits in the same way as surjections do in Set.

On the practical side, as it is explained in [2, 7], these embedding theorems (and the axiom of universes) enable one to reduce the proof of some results about the interaction between finite limits and a surjection-like class of morphisms E in a finitely complete category $\mathcal{C}$ to the case where $\mathcal{C} = \mathsf{Set}$ and E is the class of surjective functions. Arguments using elements can therefore be applied in those contexts. As an example of this technique, we prove a generalization of the so-called Barr–Kock theorem where the morphism e below is only required to be a pullback-stable strong epimorphism (in this particular case, taking $E = E_{\mathsf{pb\,strong\,epi}}$ gives the most general version of the result).

2.3. Corollary. *(Barr–Kock) Given a finitely complete category $\mathcal{C}$, we consider the following diagram*

$$\begin{array}{ccccc} R[e] & \overset{r_1^e}{\underset{r_2^e}{\rightrightarrows}} & A & \xrightarrow{e} & B \\ {\scriptstyle k}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ R[f] & \overset{r_1^f}{\underset{r_2^f}{\rightrightarrows}} & X & \xrightarrow[f]{} & Y \end{array} \tag{1}$$

where (r_1^e, r_2^e) is the kernel pair of e, (r_1^f, r_2^f) is the kernel pair of f, the right hand square is commutative and the morphism k is the induced morphism making the upper left hand and down left hand squares commute. If e is a pullback-stable strong epimorphism and the upper left hand square is a pullback, then the right hand square is also a pullback.

Proof. Let $E = E_{\mathsf{pb\,strong\,epi}}$ so that e is supposed to be in E. By the axiom of universes, we can suppose without loss of generality that $\mathcal{C}$ is a small category. We can thus consider the embedding $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ given by Theorem 2.1. Since φ preserves finite limits and commutative diagrams, the diagram in $\mathsf{Set}^{\mathcal{D}}$

$$\begin{array}{ccccc} \varphi(R[e]) & \underset{\varphi(r_2^e)}{\overset{\varphi(r_1^e)}{\rightrightarrows}} & \varphi(A) & \xrightarrow{\varphi(e)} & \varphi(B) \\ \downarrow{\scriptstyle \varphi(k)} & & \downarrow{\scriptstyle \varphi(g)} & & \downarrow{\scriptstyle \varphi(h)} \\ \varphi(R[f]) & \underset{\varphi(r_2^f)}{\overset{\varphi(r_1^f)}{\rightrightarrows}} & \varphi(X) & \xrightarrow[\varphi(f)]{} & \varphi(Y) \end{array} \tag{2}$$

satisfies similar assumptions as (1), with $e \in E$ being replaced with $\varphi(e)$ is a componentwise surjection. Since finite limits are computed componentwise in $\mathsf{Set}^{\mathcal{D}}$, for any object $D \in \mathcal{D}$, we know that the diagram in Set

$$\begin{array}{ccccc} \varphi(R[e])(D) & \underset{\varphi(r_2^e)_D}{\overset{\varphi(r_1^e)_D}{\rightrightarrows}} & \varphi(A)(D) & \xrightarrow{\varphi(e)_D} & \varphi(B)(D) \\ \downarrow{\scriptstyle \varphi(k)_D} & & \downarrow{\scriptstyle \varphi(g)_D} & & \downarrow{\scriptstyle \varphi(h)_D} \\ \varphi(R[f])(D) & \underset{\varphi(r_2^f)_D}{\overset{\varphi(r_1^f)_D}{\rightrightarrows}} & \varphi(X)(D) & \xrightarrow[\varphi(f)_D]{} & \varphi(Y)(D) \end{array} \tag{3}$$

satisfies similar assumptions as (1), with $\varphi(e)_D$ a surjective function. Since Barr–Kock theorem can easily be proved in Set using elements (see e.g. [7]), we deduce that for each object $D \in \mathcal{D}$, the right hand square of diagram (3) is a pullback. Since finite limits are computed componentwise in $\mathsf{Set}^{\mathcal{D}}$, this means that the right hand square in diagram (2) is a pullback. Since φ reflects finite limits, this finally implies that the right hand square of diagram (1) is also a pullback as desired. ■

PROOF OF THEOREMS 2.1 AND 2.2. We are now going to prove both Theorem 2.1 and Theorem 2.2 simultaneously.

PROOF. The proof is divided in 42 steps, but the proof of Theorem 2.1 already ends after 30 steps.

STEP 1. THE EASY DIRECTION. Let us first suppose that such a category $\mathcal{D}$ and such a functor $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ exist. Since the axioms (Id), (ClComp), (SRightCancP), (StPb) and (NoPMono) are (or can be) stated in terms of finite limits and the class E, and in view of the properties of φ, they hold for E in $\mathcal{C}$ just because they hold for the class of componentwise surjections in $\mathsf{Set}^{\mathcal{D}}$. If φ is full, let us prove that E satisfies (Reg). Let $e\colon A \to B$ be a morphism in E and let $r_1, r_2\colon R[e] \rightrightarrows A$ be its kernel pair. We must show that e is the coequalizer of r_1 and r_2. Let $x\colon A \to X$ be a morphism such that $xr_1 = xr_2$. Using the properties of φ, we know that $\varphi(e)$ is a componentwise surjection and $\varphi(r_1), \varphi(r_2)\colon \varphi(R[e]) \rightrightarrows \varphi(A)$ is its kernel pair. Therefore, since componentwise surjections in $\mathsf{Set}^{\mathcal{D}}$ are regular epimorphisms, $\varphi(e)$ is the coequalizer of $\varphi(r_1)$ and $\varphi(r_2)$. Since $\varphi(x)\varphi(r_1) = \varphi(x)\varphi(r_2)$, there exists a unique natural transformation $\alpha\colon \varphi(B) \to \varphi(X)$ such that $\alpha\varphi(e) = \varphi(x)$. One then concludes that there is a unique morphism $y\colon B \to X$ in $\mathcal{C}$ such that $ye = x$ using the fact that φ is full and faithful.

Step 2. Desired properties of the category $\mathcal{D}$. Conversely, let us suppose that E is a surjection-like class of morphisms (respectively a regular surjection-like class of morphisms). For each morphism $f\colon C \to C'$ in $\mathcal{C}$, we denote by $\mathcal{C}(f,-)\colon \mathcal{C}(C',-) \to \mathcal{C}(C,-)$ the corresponding natural transformation between the representable functors. Let us suppose we have constructed a small full subcategory $\mathcal{D}$ of the functor category $\mathsf{Set}^{\mathcal{C}}$ satisfying the following properties:

(i) Each functor $F\colon \mathcal{C} \to \mathsf{Set}$ in $\mathcal{D}$ preserves finite limits.

(ii) For each $e \in E$ and each $F \in \mathcal{D}$, the function $F(e)$ is surjective.

(iii) For each object $C \in \mathcal{C}$, there exists a natural transformation $\iota_C\colon \mathcal{C}(C,-) \to F_C$ with $F_C \in \mathcal{D}$ such that:

  (a) ι_C is a monomorphism in $\mathsf{Set}^{\mathcal{C}}$;

  (b) for each morphism $e\colon C' \to C$ in $\mathcal{C}$, if there exists a natural transformation $\alpha\colon \mathcal{C}(C',-) \to F_C$ making the triangle

$$\begin{array}{ccc} \mathcal{C}(C,-) & \xrightarrow{\mathcal{C}(e,-)} & \mathcal{C}(C',-) \\ {\scriptstyle \iota_C}\downarrow & \swarrow{\scriptstyle \alpha} & \\ F_C & & \end{array}$$

  commute, then $e \in E$;

and moreover, in the case the proof of the full embedding of Theorem 2.2 is concerned,

  (c) there exists two parallel morphisms $\rho^1, \rho^2\colon F_C \rightrightarrows G_C$ in $\mathcal{D}$ of which ι_C is an equalizer of in $\mathsf{Set}^{\mathcal{C}}$;

  (d) for each natural transformation $\alpha\colon \mathcal{C}(C,-) \to H$ with $H \in \mathcal{D}$, there exists a natural transformation $\beta\colon F_C \to H$ making the triangle

$$\begin{array}{ccc} \mathcal{C}(C,-) & \xrightarrow{\iota_C} & F_C \\ {\scriptstyle \alpha}\downarrow & \swarrow{\scriptstyle \beta} & \\ H & & \end{array}$$

  commute.

Step 3. The functor φ. In the case where such a category $\mathcal{D}$ exists, we are able to construct the desired functor $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$ as follows. Given an object $C \in \mathcal{C}$, the functor $\varphi(C)\colon \mathcal{D} \to \mathsf{Set}$ is defined on an object $F \in \mathcal{D}$ by $\varphi(C)(F) = F(C)$ and on a natural transformation $\alpha\colon F \to F'$ in $\mathcal{D}$ by $\varphi(C)(\alpha) = \alpha_C\colon F(C) \to F'(C)$. Given a morphism $f\colon C \to C'$ in $\mathcal{C}$, the natural transformation $\varphi(f)\colon \varphi(C) \to \varphi(C')$ is defined on an object F of $\mathcal{D}$ as $\varphi(f)_F = F(f)\colon F(C) \to F(C')$. It is routine to show that this indeed gives a functor $\varphi\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{D}}$.

STEP 4. φ PRESERVES FINITE LIMITS. Since finite limits are computed in $\mathsf{Set}^{\mathcal{D}}$ componentwise and since each functor $F \in \mathcal{D}$ preserves finite limits (i), we can immediately see that φ also preserves finite limits.

STEP 5. φ IS FAITHFUL. To see that φ is faithful, let $f, g \colon C \rightrightarrows C'$ be two morphisms in $\mathcal{C}$ such that $\varphi(f) = \varphi(g)$. In particular, we know that $F_C(f) = \varphi(f)_{F_C} = \varphi(g)_{F_C} = F_C(g)$ for the functor F_C given by (iii). We can thus compute:

$$\begin{aligned}\iota_{C,C'}(f) &= \iota_{C,C'}(\mathcal{C}(C,-)(f)(1_C))\\ &= F_C(f)(\iota_{C,C}(1_C))\\ &= F_C(g)(\iota_{C,C}(1_C))\\ &= \iota_{C,C'}(\mathcal{C}(C,-)(g)(1_C))\\ &= \iota_{C,C'}(g).\end{aligned}$$

Since ι_C is a monomorphism in $\mathsf{Set}^{\mathcal{C}}$ (iii)(a), its C'-component $\iota_{C,C'}$ is an injection and so $f = g$ which proves that φ is faithful.

STEP 6. $e \in E$ IF AND ONLY IF $\varphi(e)$ IS A COMPONENTWISE SURJECTION. Given a morphism $e \colon C \to C'$ in $\mathcal{C}$, if $e \in E$, then $\varphi(e)$ is a componentwise surjection in view of (ii). Conversely, if $\varphi(e)$ is a componentwise surjection, in particular, $F_{C'}(e) \colon F_{C'}(C) \to F_{C'}(C')$ is a surjection. There exists thus an element $a \in F_{C'}(C)$ such that $F_{C'}(e)(a) = \iota_{C',C'}(1_{C'})$. By the Yoneda Lemma, this element a corresponds to a natural transformation $\alpha \colon \mathcal{C}(C,-) \to F_{C'}$ such that $\alpha \circ \mathcal{C}(e,-) = \iota_{C'}$. Using (iii)(b), we deduce that $e \in E$.

STEP 7. φ IS CONSERVATIVE. It is now easy to prove that φ is conservative. Let f be a morphism in $\mathcal{C}$ for which $\varphi(f)$ is an isomorphism in $\mathsf{Set}^{\mathcal{D}}$. Since $\varphi(f)$ is in particular a componentwise surjection, $f \in E$ by Step 6. But since $\varphi(f)$ is a monomorphism and φ is faithful, f is also a monomorphism. Since f is a monomorphism in E, using (NoPMono), we know that f is an isomorphism.

STEP 8. φ REFLECTS FINITE LIMITS. Since φ is a conservative functor which preserves finite limits from a finitely complete category, we know that φ reflects finite limits.

STEP 9. IN THE REGULAR CASE, φ IS FULL. In the case the proof of the full embedding of Theorem 2.2 is concerned, let us also show that φ is full. Let C, C' be two objects of $\mathcal{C}$ and $\gamma \colon \varphi(C) \to \varphi(C')$ a natural transformation. We thus have a function $\gamma_{F_C} \colon F_C(C) \to F_C(C')$. Let us consider the natural transformation $\alpha \colon \mathcal{C}(C',-) \to F_C$ corresponding via the Yoneda Lemma to $\gamma_{F_C}(\iota_{C,C}(1_C)) \in F_C(C')$. Considering also the two natural transformations $\rho^1, \rho^2 \colon F_C \rightrightarrows G_C$ given by (iii)(c), we can compute

$$\begin{aligned}\rho^1_{C'}(\alpha_{C'}(1_{C'})) &= \rho^1_{C'}(\gamma_{F_C}(\iota_{C,C}(1_C)))\\ &= \varphi(C')(\rho^1)(\gamma_{F_C}(\iota_{C,C}(1_C)))\\ &= \gamma_{G_C}(\varphi(C)(\rho^1)(\iota_{C,C}(1_C)))\\ &= \gamma_{G_C}(\rho^1_C(\iota_{C,C}(1_C)))\end{aligned}$$

$$
\begin{aligned}
&= \gamma_{G_C}(\rho^2_C(\iota_{C,C}(1_C)))\\
&= \gamma_{G_C}(\varphi(C)(\rho^2)(\iota_{C,C}(1_C)))\\
&= \varphi(C')(\rho^2)(\gamma_{F_C}(\iota_{C,C}(1_C)))\\
&= \rho^2_{C'}(\gamma_{F_C}(\iota_{C,C}(1_C)))\\
&= \rho^2_{C'}(\alpha_{C'}(1_{C'}))
\end{aligned}
$$

which, by the Yoneda Lemma, proves that $\rho^1\alpha = \rho^2\alpha$. Since ι_C is the equalizer of ρ^1 and ρ^2 in Set^C (iii)(c), we know that there exists a unique natural transformation $\beta\colon \mathcal{C}(C',-) \to \mathcal{C}(C,-)$ such that $\iota_C\beta = \alpha$. Using the Yoneda Lemma again, there exists a unique morphism $f\colon C \to C'$ in $\mathcal{C}$ such that $\beta = \mathcal{C}(f,-)$. We shall prove that $\gamma = \varphi(f)$. Given $F \in \mathcal{D}$ and $d \in F(C)$, it suffices to prove that $\gamma_F(d) = F(f)(d)$. By the Yoneda Lemma, this element d corresponds to a natural transformation $\delta\colon \mathcal{C}(C,-) \to F$. Using (iii)(d), there exists a natural transformation $\varepsilon\colon F_C \to F$ such that $\varepsilon\iota_C = \delta$. It thus suffices to compute:

$$
\begin{aligned}
\gamma_F(d) &= \gamma_F(\delta_C(1_C))\\
&= \gamma_F(\varepsilon_C(\iota_{C,C}(1_C)))\\
&= \varepsilon_{C'}(\gamma_{F_C}(\iota_{C,C}(1_C)))\\
&= \varepsilon_{C'}(\alpha_{C'}(1_{C'}))\\
&= \varepsilon_{C'}(\iota_{C,C'}(\beta_{C'}(1_{C'})))\\
&= \varepsilon_{C'}(\iota_{C,C'}(f))\\
&= \delta_{C'}(f)\\
&= F(f)(\delta_C(1_C))\\
&= F(f)(d).
\end{aligned}
$$

STEP 10. THE SET $\widehat{F}(C)$. It remains now to construct a full subcategory $\mathcal{D}$ of $\mathsf{Set}^{\mathcal{C}}$ satisfying the properties described in Step 2. In order to do so, for a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$, let us construct a finite limit preserving functor $\widehat{F}\colon \mathcal{C} \to \mathsf{Set}$ as follows. Given an object $C \in \mathcal{C}$, we consider the set of 7-tuples $(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)$ where

- $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ is non-negative integer;
- $(e_i)_i$ is a family $(e_i\colon B_i \to D_i)_{1\leqslant i\leqslant n}$ of n morphisms in E;
- $(f_i)_i$ is a family $(f_i\colon A \to D_i)_{1\leqslant i\leqslant n}$ of n morphisms in $\mathcal{C}$;
- $\overline{f}$ and $\overline{e}$ are the projections of a pullback square

$$
\begin{array}{ccc}
P & \xrightarrow{\ \overline{f}\ } & B_1 \times \cdots \times B_n\\
{\scriptstyle \overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1\times\cdots\times e_n}\\
A & \xrightarrow[(f_1,\dots,f_n)]{} & D_1 \times \cdots \times D_n
\end{array}
$$

where $e_1 \times \cdots \times e_n$ is the product of $e_1, \ldots, e_n$ and $(f_1, \ldots, f_n)$ is the morphism induced by $f_1, \ldots, f_n$;

- $g \colon P \to C$ is a morphism in $\mathcal{C}$;
- $a \in F(A)$ is an element.

Notice that since E is closed under finite products (since E satisfies (Id), (ClComp) and (StPb)), the morphism $e_1 \times \cdots \times e_n$ is in E. Since E is stable under pullbacks (StPb), this implies that $\overline{e} \in E$. We define $\widehat{F}(C)$

$$\widehat{F}(C) = \left\{ \big(n \in \mathbb{N}, (e_i \in E)_i, (f_i)_i, \overline{f}, \overline{e}, g, a \in F(A)\big) \;\middle|\; \begin{array}{ccccc} & P & \xrightarrow{\overline{f}} & \prod_{i=1}^{n} B_i \\ {\scriptstyle g}\swarrow & {\scriptstyle \overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle \prod_{i=1}^{n} e_i} \\ C & A & \xrightarrow[(f_1,\ldots,f_n)]{} & \prod_{i=1}^{n} D_i \end{array} \right\} / \cong_{F,C}$$

to be the quotient of this set of such 7-tuples by the equivalence relation $\cong_{F,C}$ defined as follows. For two such 7-tuples $(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)$ and $(n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a')$, represented by

$$\begin{array}{cccc} & P & \xrightarrow{\overline{f}} & B_1 \times \cdots \times B_n \\ {\scriptstyle g}\swarrow & {\scriptstyle \overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1 \times \cdots \times e_n} \\ C & A & \xrightarrow[(f_1,\ldots,f_n)]{} & D_1 \times \cdots \times D_n \end{array} \quad \text{and} \quad \begin{array}{cccc} & P' & \xrightarrow{\overline{f}'} & B'_1 \times \cdots \times B'_{n'} \\ {\scriptstyle g'}\swarrow & {\scriptstyle \overline{e}'}\downarrow & \lrcorner & \downarrow{\scriptstyle e'_1 \times \cdots \times e'_{n'}} \\ C & A' & \xrightarrow[(f'_1,\ldots,f'_{n'})]{} & D'_1 \times \cdots \times D'_{n'} \end{array} ,$$

we have

$$(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a) \cong_{F,C} (n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a')$$

if and only if there exists a span

$$\begin{array}{ccc} & A'' & \\ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle h'} \\ A & & A' \end{array}$$

in $\mathcal{C}$ together with an element $a'' \in F(A'')$ such that

- $F(h)(a'') = a$;
- $F(h')(a'') = a'$;

- considering the pullback diagram

$$\begin{array}{ccc} P'' & \xrightarrow{\left(k_1,\dots,k_n,k'_1,\dots,k'_{n'}\right)} & B_1\times\cdots\times B_n\times B'_1\times\cdots\times B'_{n'} \\ {\scriptstyle \overline{e}''}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle e_1\times\cdots\times e_n\times e'_1\times\cdots\times e'_{n'}} \\ A'' & \xrightarrow[\left(f_1h,\dots,f_nh,f'_1h',\dots,f'_{n'}h'\right)]{} & D_1\times\cdots\times D_n\times D'_1\times\cdots\times D'_{n'} \end{array} ,$$

the sets

$$\mathcal{X}=\{(i,i')\in\mathbb{N}^2\,|\,1\leqslant i\leqslant n,\ 1\leqslant i'\leqslant n',\ e_i=e'_{i'}\text{ and }F(f_i)(a)=F(f'_{i'})(a')\},$$

$$\mathcal{Y}=\{(i_1,i_2)\in\mathbb{N}^2\,|\,1\leqslant i_1\leqslant n,\ 1\leqslant i_2\leqslant n,\ e_{i_1}=e_{i_2}\text{ and }F(f_{i_1})(a)=F(f_{i_2})(a)\}$$

and

$$\mathcal{Y}'=\{(i'_1,i'_2)\in\mathbb{N}^2\,|\,1\leqslant i'_1\leqslant n',\ 1\leqslant i'_2\leqslant n',\ e'_{i'_1}=e'_{i'_2}\text{ and }F(f'_{i'_1})(a')=F(f'_{i'_2})(a')\},$$

and the equalizer

$$M \overset{m}{\rightarrowtail} P'' \underset{\left((k'_{i'})_{(i,i')\in\mathcal{X}},(k_{i_2})_{(i_1,i_2)\in\mathcal{Y}},(k'_{i'_2})_{(i'_1,i'_2)\in\mathcal{Y}'}\right)}{\overset{\left((k_i)_{(i,i')\in\mathcal{X}},(k_{i_1})_{(i_1,i_2)\in\mathcal{Y}},(k'_{i'_1})_{(i'_1,i'_2)\in\mathcal{Y}'}\right)}{\rightrightarrows}} \prod_{(i,i')\in\mathcal{X}} B_i\times\prod_{(i_1,i_2)\in\mathcal{Y}} B_{i_1}\times\prod_{(i'_1,i'_2)\in\mathcal{Y}'} B'_{i'_1}\ ,$$

the diagram

$$\begin{array}{ccccc} M & \overset{m}{\rightarrowtail} & P'' & \xrightarrow{\left(h\overline{e}'',(k_i)_{1\leqslant i\leqslant n}\right)} & P \\ {\scriptstyle m}\big\downarrow & & & & \big\downarrow{\scriptstyle g} \\ P'' & \xrightarrow[\left(h'\overline{e}'',(k'_{i'})_{1\leqslant i'\leqslant n'}\right)]{} & P' & \xrightarrow[g']{} & C \end{array}$$

commutes where $(h\overline{e}'',(k_i)_{1\leqslant i\leqslant n})$ is the unique morphism $P''\to P$ such that

$$\overline{e}\,(h\overline{e}'',(k_i)_{1\leqslant i\leqslant n})=h\overline{e}''$$

and

$$\overline{f}\,(h\overline{e}'',(k_i)_{1\leqslant i\leqslant n})=(k_1,\dots,k_n)$$

and where $(h'\overline{e}'',(k'_{i'})_{1\leqslant i'\leqslant n'})$ is the unique morphism $P''\to P'$ such that

$$\overline{e}'\,(h'\overline{e}'',(k'_{i'})_{1\leqslant i'\leqslant n'})=h'\overline{e}''$$

and

$$\overline{f}'\,(h'\overline{e}'',(k'_{i'})_{1\leqslant i'\leqslant n'})=(k'_1,\dots,k'_{n'})\,.$$

Remark: Before proceeding with the rest of the proof, let us give a little intuition of this construction. The dream would be that $F(A) \subseteq \widehat{F}(A)$ for every object A (see Step 22) and that for every $\overline{e}\colon P \to A \in E$, the function $\widehat{F}(\overline{e})$ would be surjective. Although we will not reach this dream directly (see Step 25), we want to add, for any $a \in F(A)$, an element $p \in \widehat{F}(P)$ such that $\widehat{F}(\overline{e})(p) = a$. Moreover, for every $g\colon P \to C$, we need to add an element $\widehat{F}(g)(p) \in \widehat{F}(C)$. When we tried to define $\widehat{F}(C)$ as a quotient of the mere set of triples $(\overline{e}, g, a)$, we were not able to prove the required properties of $\widehat{F}$. Instead, for each $a \in F(A)$, we needed to consider many liftings to elements of $\widehat{F}(P)$. A careful analysis of the required properties resulted in the above definition.

Step 11. An equivalent definition of $\cong_{F,C}$. Before proving that $\cong_{F,C}$ is indeed an equivalence relation, let us prove that, in the above definition of $\cong_{F,C}$, we can equivalently require the additional following conditions (together with those from Step 10):

- $f_i h = f'_{i'} h'$ for all $(i, i') \in \mathcal{X}$;
- $f_{i_1} h = f_{i_2} h$ for all $(i_1, i_2) \in \mathcal{Y}$;
- $f'_{i'_1} h' = f'_{i'_2} h'$ for all $(i'_1, i'_2) \in \mathcal{Y}'$.

In order to do so, let us consider the following equalizer diagram.

$$A''' \overset{h''}{\rightarrowtail} A'' \underset{\left((f'_{i'}h')_{(i,i')\in\mathcal{X}},(f_{i_2}h)_{(i_1,i_2)\in\mathcal{Y}},(f'_{i'_2}h')_{(i'_1,i'_2)\in\mathcal{Y}'}\right)}{\overset{\left((f_ih)_{(i,i')\in\mathcal{X}},(f_{i_1}h)_{(i_1,i_2)\in\mathcal{Y}},(f'_{i'_1}h')_{(i'_1,i'_2)\in\mathcal{Y}'}\right)}{\rightrightarrows}} \prod_{(i,i')\in\mathcal{X}} D_i \times \prod_{(i_1,i_2)\in\mathcal{Y}} D_{i_1} \times \prod_{(i'_1,i'_2)\in\mathcal{Y}'} D'_{i'_1}$$

Since F preserves finite limits and in view of the definitions of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Y}'$, there exists a unique $a''' \in F(A''')$ such that $F(h'')(a''') = a''$. Let us show that the span

$$A \xleftarrow{\;hh''\;} A''' \xrightarrow{\;h'h''\;} A'$$

together with the element $a''' \in F(A''')$ is also a witness of

$$(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a) \cong_{F,C} (n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a').$$

One obviously has $F(hh'')(a''') = F(h)(a'') = a$ and $F(h'h'')(a''') = F(h')(a'') = a'$. As far as the last condition is concerned, we consider the diagram

$$\begin{array}{ccccc}
R & \xrightarrow{\;r\;} & P'' & \xrightarrow{\;(k_1,\dots,k_n,k'_1,\dots,k'_{n'})\;} & B_1 \times \cdots \times B_n \times B'_1 \times \cdots \times B'_{n'} \\
{\scriptstyle \overline{e}'''}\downarrow & \lrcorner & {\scriptstyle \overline{e}''}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1\times\cdots\times e_n\times e'_1\times\cdots\times e'_{n'}} \\
A''' & \xrightarrow[\;h''\;]{} & A'' & \xrightarrow[\;(f_1h,\dots,f_nh,f'_1h',\dots,f'_{n'}h')\;]{} & D_1 \times \cdots \times D_n \times D'_1 \times \cdots \times D'_{n'}
\end{array}$$

where both rectangles are pullbacks and the equalizer diagram

$$R' \overset{r'}{\rightarrowtail} R \underset{\left((k'_{i'}r)_{(i,i')\in\mathcal{X}},(k_{i_2}r)_{(i_1,i_2)\in\mathcal{Y}},(k'_{i'_2}r)_{(i'_1,i'_2)\in\mathcal{Y}'}\right)}{\overset{\left((k_ir)_{(i,i')\in\mathcal{X}},(k_{i_1}r)_{(i_1,i_2)\in\mathcal{Y}},(k'_{i'_1}r)_{(i'_1,i'_2)\in\mathcal{Y}'}\right)}{\rightrightarrows}} \prod_{(i,i')\in\mathcal{X}} B_i \times \prod_{(i_1,i_2)\in\mathcal{Y}} B_{i_1} \times \prod_{(i'_1,i'_2)\in\mathcal{Y}'} B'_{i'_1}\,.$$

Using the universal property of the equalizer m, there exists a unique morphism $r''\colon R' \to M$ such that $mr'' = rr'$. The last condition then follows from

$$\begin{aligned} g\left(hh''\overline{e}''', (k_ir)_{1\leqslant i\leqslant n}\right) r' &= g\left(h\overline{e}'', (k_i)_{1\leqslant i\leqslant n}\right) rr' \\ &= g\left(h\overline{e}'', (k_i)_{1\leqslant i\leqslant n}\right) mr'' \\ &= g'\left(h'\overline{e}'', (k'_{i'})_{1\leqslant i'\leqslant n'}\right) mr'' \\ &= g'\left(h'\overline{e}'', (k'_{i'})_{1\leqslant i'\leqslant n'}\right) rr' \\ &= g'\left(h'h''\overline{e}''', (k'_{i'}r)_{1\leqslant i'\leqslant n'}\right) r'. \end{aligned}$$

In addition, from the definition of h'', we immediately have $f_ihh'' = f'_{i'}h'h''$ for all $(i,i') \in \mathcal{X}$, $f_{i_1}hh'' = f_{i_2}hh''$ for all $(i_1,i_2) \in \mathcal{Y}$ and $f'_{i'_1}h'h'' = f'_{i'_2}h'h''$ for all $(i'_1,i'_2) \in \mathcal{Y}'$.

STEP 12. $\cong_{F,C}$ IS SYMMETRIC. Let us now prove that this relation $\cong_{F,C}$ is indeed an equivalence relation. The symmetry of this relation is easily obtained by exchanging the roles of h and h'.

STEP 13. $\cong_{F,C}$ IS REFLEXIVE. In order to prove the reflexivity of $\cong_{F,C}$, let us consider a 7-tuple $(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)$ and the set

$$\mathcal{X} = \{(i,i') \in \mathbb{N}^2 \mid 1 \leqslant i \leqslant n,\, 1 \leqslant i' \leqslant n,\, e_i = e_{i'} \text{ and } F(f_i)(a) = F(f_{i'})(a)\}.$$

The relation

$$(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a) \cong_{F,C} (n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)$$

is attested by the span

$$A \xleftarrow{\;1_A\;} A \xrightarrow{\;1_A\;} A$$

together with the element a. Indeed, while the first two conditions are obviously satisfied, the equality

$$g\left(\overline{e}'', (k_i)_{1\leqslant i\leqslant n}\right) m = g\left(\overline{e}'', (k'_i)_{1\leqslant i\leqslant n}\right) m$$

can be deduced in this case from the fact that $k_im = k'_im$ for each $1 \leqslant i \leqslant n$ since $(i,i) \in \mathcal{X}$.

STEP 14. $\cong_{F,C}$ IS TRANSITIVE. As far as the transitivity of $\cong_{F,C}$ is concerned, we consider three 7-tuples, denoted

$$(n_j, (e_i^j)_i, (f_i^j)_i, \overline{f}_j, \overline{e}_j, g_j, a_j)$$

and represented by

$$\begin{array}{cccc} & P_j & \xrightarrow{\;\overline{f}_j\;} & B_1^j \times \cdots \times B_{n_j}^j \\ {}^{g_j}\swarrow & \downarrow{\scriptstyle \overline{e}_j} \quad \lrcorner & & \downarrow{\scriptstyle e_1^j \times \cdots \times e_{n_j}^j} \\ C & A_j & \xrightarrow[(f_1^j, \dots, f_{n_j}^j)]{} & D_1^j \times \cdots \times D_{n_j}^j \end{array}$$

for each $j \in \{1, 2, 3\}$. We suppose

$$(n_1, (e_i^1)_i, (f_i^1)_i, \overline{f}_1, \overline{e}_1, g_1, a_1) \cong_{F,C} (n_2, (e_i^2)_i, (f_i^2)_i, \overline{f}_2, \overline{e}_2, g_2, a_2)$$

and

$$(n_2, (e_i^2)_i, (f_i^2)_i, \overline{f}_2, \overline{e}_2, g_2, a_2) \cong_{F,C} (n_3, (e_i^3)_i, (f_i^3)_i, \overline{f}_3, \overline{e}_3, g_3, a_3)$$

and we shall prove that

$$(n_1, (e_i^1)_i, (f_i^1)_i, \overline{f}_1, \overline{e}_1, g_1, a_1) \cong_{F,C} (n_3, (e_i^3)_i, (f_i^3)_i, \overline{f}_3, \overline{e}_3, g_3, a_3).$$

We consider the sets

$$\mathcal{X}_{12} = \{(i_1, i_2) \in \mathbb{N}^2 \mid 1 \leqslant i_1 \leqslant n_1,\ 1 \leqslant i_2 \leqslant n_2,\ e_{i_1}^1 = e_{i_2}^2 \text{ and } F(f_{i_1}^1)(a_1) = F(f_{i_2}^2)(a_2)\},$$

$$\mathcal{X}_{13} = \{(i_1, i_3) \in \mathbb{N}^2 \mid 1 \leqslant i_1 \leqslant n_1,\ 1 \leqslant i_3 \leqslant n_3,\ e_{i_1}^1 = e_{i_3}^3 \text{ and } F(f_{i_1}^1)(a_1) = F(f_{i_3}^3)(a_3)\},$$

$$\mathcal{X}_{23} = \{(i_2, i_3) \in \mathbb{N}^2 \mid 1 \leqslant i_2 \leqslant n_2,\ 1 \leqslant i_3 \leqslant n_3,\ e_{i_2}^2 = e_{i_3}^3 \text{ and } F(f_{i_2}^2)(a_2) = F(f_{i_3}^3)(a_3)\},$$

$$\mathcal{Y}_1 = \{(i_1, i_1') \in \mathbb{N}^2 \mid 1 \leqslant i_1 \leqslant n_1,\ 1 \leqslant i_1' \leqslant n_1,\ e_{i_1}^1 = e_{i_1'}^1 \text{ and } F(f_{i_1}^1)(a_1) = F(f_{i_1'}^1)(a_1)\},$$

$$\mathcal{Y}_2 = \{(i_2, i_2') \in \mathbb{N}^2 \mid 1 \leqslant i_2 \leqslant n_2,\ 1 \leqslant i_2' \leqslant n_2,\ e_{i_2}^2 = e_{i_2'}^2 \text{ and } F(f_{i_2}^2)(a_2) = F(f_{i_2'}^2)(a_2)\}$$

and

$$\mathcal{Y}_3 = \{(i_3, i_3') \in \mathbb{N}^2 \mid 1 \leqslant i_3 \leqslant n_3,\ 1 \leqslant i_3' \leqslant n_3,\ e_{i_3}^3 = e_{i_3'}^3 \text{ and } F(f_{i_3}^3)(a_3) = F(f_{i_3'}^3)(a_3)\}.$$

By assumption, we know there exist two spans

$$A_1 \xleftarrow{\;h_1\;} A \xrightarrow{\;h_2\;} A_2 \qquad \text{and} \qquad A_2 \xleftarrow{\;h_2'\;} A' \xrightarrow{\;h_3'\;} A_3$$

together with elements $a \in F(A)$ and $a' \in F(A')$ satisfying

- $F(h_1)(a) = a_1$, $F(h_2)(a) = a_2 = F(h_2')(a')$ and $F(h_3')(a') = a_3$;

- $$g_1\left(h_1\overline{e},(k^1_{i_1})_{1\leqslant i_1\leqslant n_1}\right)m = g_2\left(h_2\overline{e},(k^2_{i_2})_{1\leqslant i_2\leqslant n_2}\right)m \tag{4}$$
  and
  $$g_2\left(h'_2\overline{e}',(k'^2_{i_2})_{1\leqslant i_2\leqslant n_2}\right)m' = g_3\left(h'_3\overline{e}',(k'^3_{i_3})_{1\leqslant i_3\leqslant n_3}\right)m' \tag{5}$$
  where
  $$\begin{array}{ccc} P & \xrightarrow{\left(k^1_1,\dots,k^1_{n_1},k^2_1,\dots,k^2_{n_2}\right)} & B^1_1\times\cdots\times B^1_{n_1}\times B^2_1\times\cdots\times B^2_{n_2} \\ {\scriptstyle\overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle e^1_1\times\cdots\times e^1_{n_1}\times e^2_1\times\cdots\times e^2_{n_2}} \\ A & \xrightarrow[\left(f^1_1h_1,\dots,f^1_{n_1}h_1,f^2_1h_2,\dots,f^2_{n_2}h_2\right)]{} & D^1_1\times\cdots\times D^1_{n_1}\times D^2_1\times\cdots\times D^2_{n_2} \end{array}$$
  and
  $$\begin{array}{ccc} P' & \xrightarrow{\left(k'^2_1,\dots,k'^2_{n_2},k'^3_1,\dots,k'^3_{n_3}\right)} & B^2_1\times\cdots\times B^2_{n_2}\times B^3_1\times\cdots\times B^3_{n_3} \\ {\scriptstyle\overline{e}'}\downarrow & \lrcorner & \downarrow{\scriptstyle e^2_1\times\cdots\times e^2_{n_2}\times e^3_1\times\cdots\times e^3_{n_3}} \\ A' & \xrightarrow[\left(f^2_1h'_2,\dots,f^2_{n_2}h'_2,f^3_1h'_3,\dots,f^3_{n_3}h'_3\right)]{} & D^2_1\times\cdots\times D^2_{n_2}\times D^3_1\times\cdots\times D^3_{n_3} \end{array}$$
  are pullback diagrams and
  $$M \overset{m}{\rightarrowtail} P \underset{\left((k^2_{i_2})_{(i_1,i_2)\in\mathcal{X}_{12}},(k^1_{i'_1})_{(i_1,i'_1)\in\mathcal{Y}_1},(k^2_{i'_2})_{(i_2,i'_2)\in\mathcal{Y}_2}\right)}{\overset{\left((k^1_{i_1})_{(i_1,i_2)\in\mathcal{X}_{12}},(k^1_{i_1})_{(i_1,i'_1)\in\mathcal{Y}_1},(k^2_{i_2})_{(i_2,i'_2)\in\mathcal{Y}_2}\right)}{\rightrightarrows}} \prod_{(i_1,i_2)\in\mathcal{X}_{12}} B^1_{i_1}\times\prod_{(i_1,i'_1)\in\mathcal{Y}_1} B^1_{i_1}\times\prod_{(i_2,i'_2)\in\mathcal{Y}_2} B^2_{i_2}$$
  and
  $$M' \overset{m'}{\rightarrowtail} P' \underset{\left((k'^3_{i_3})_{(i_2,i_3)\in\mathcal{X}_{23}},(k'^2_{i'_2})_{(i_2,i'_2)\in\mathcal{Y}_2},(k'^3_{i'_3})_{(i_3,i'_3)\in\mathcal{Y}_3}\right)}{\overset{\left((k'^2_{i_2})_{(i_2,i_3)\in\mathcal{X}_{23}},(k'^2_{i_2})_{(i_2,i'_2)\in\mathcal{Y}_2},(k'^3_{i_3})_{(i_3,i'_3)\in\mathcal{Y}_3}\right)}{\rightrightarrows}} \prod_{(i_2,i_3)\in\mathcal{X}_{23}} B^2_{i_2}\times\prod_{(i_2,i'_2)\in\mathcal{Y}_2} B^2_{i_2}\times\prod_{(i_3,i'_3)\in\mathcal{Y}_3} B^3_{i_3}$$
  are equalizer diagrams.

Moreover, using Step 11, we can assume without loss of generality that

- $f^1_{i_1}h_1 = f^2_{i_2}h_2$ for all $(i_1,i_2)\in\mathcal{X}_{12}$;
- $f^2_{i_2}h_2 = f^2_{i'_2}h_2$ for all $(i_2,i'_2)\in\mathcal{Y}_2$;
- $f^2_{i_2}h'_2 = f^3_{i_3}h'_3$ for all $(i_2,i_3)\in\mathcal{X}_{23}$.

We consider the pullback

$$\begin{array}{ccc} A'' & \xrightarrow{l'} & A' \\ {\scriptstyle l}\downarrow & \lrcorner & \downarrow{\scriptstyle h'_2} \\ A & \xrightarrow[h_2]{} & A_2 \end{array} \tag{6}$$

and since $F(h_2)(a) = a_2 = F(h'_2)(a')$ and F preserves finite limits, we know that there exists a unique element $a'' \in F(A'')$ such that $F(l)(a'') = a$ and $F(l')(a'') = a'$. We are going to show that the span

$$A_1 \xleftarrow{h_1 l} A'' \xrightarrow{h'_3 l'} A_3$$

together with the element $a'' \in F(A'')$ is a witness of

$$(n_1, (e^1_i)_i, (f^1_i)_i, \overline{f}_1, \overline{e}_1, g_1, a_1) \cong_{F,C} (n_3, (e^3_i)_i, (f^3_i)_i, \overline{f}_3, \overline{e}_3, g_3, a_3).$$

We immediately notice that $F(h_1 l)(a'') = F(h_1)(a) = a_1$ and $F(h'_3 l')(a'') = F(h'_3)(a') = a_3$. In order to check the last condition, we consider the pullback

$$\begin{array}{ccc} P'' & \xrightarrow{\left(k''^1_1,\dots,k''^1_{n_1},k''^3_1,\dots,k''^3_{n_3}\right)} & B^1_1 \times \cdots \times B^1_{n_1} \times B^3_1 \times \cdots \times B^3_{n_3} \\ {\scriptstyle \overline{e}''}\downarrow & \lrcorner & \downarrow{\scriptstyle e^1_1\times\cdots\times e^1_{n_1}\times e^3_1\times\cdots\times e^3_{n_3}} \\ A'' & \xrightarrow[\left(f^1_1 h_1 l,\dots,f^1_{n_1} h_1 l, f^3_1 h'_3 l',\dots,f^3_{n_3} h'_3 l'\right)]{} & D^1_1 \times \cdots \times D^1_{n_1} \times D^3_1 \times \cdots \times D^3_{n_3} \end{array} \tag{7}$$

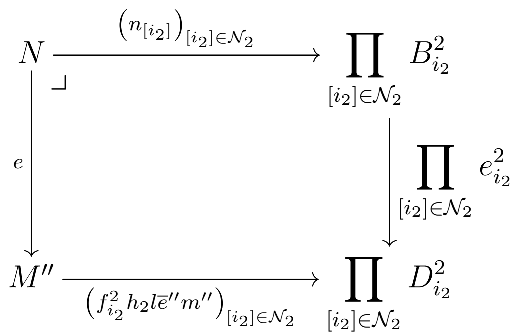
and the equalizer

$$M'' \overset{m''}{\rightarrowtail} P'' \underset{\left((k''^3_{i_3})_{(i_1,i_3)\in\mathcal{X}_{13}},(k''^1_{i'_1})_{(i_1,i'_1)\in\mathcal{Y}_1},(k''^3_{i'_3})_{(i_3,i'_3)\in\mathcal{Y}_3}\right)}{\overset{\left((k''^1_{i_1})_{(i_1,i_3)\in\mathcal{X}_{13}},(k''^1_{i_1})_{(i_1,i'_1)\in\mathcal{Y}_1},(k''^3_{i_3})_{(i_3,i'_3)\in\mathcal{Y}_3}\right)}{\rightrightarrows}} \prod_{(i_1,i_3)\in\mathcal{X}_{13}} B^1_{i_1} \times \prod_{(i_1,i'_1)\in\mathcal{Y}_1} B^1_{i_1} \times \prod_{(i_3,i'_3)\in\mathcal{Y}_3} B^3_{i_3}$$

and we must show that

$$g_1\left(h_1 l \overline{e}'', (k''^1_{i_1})_{1\leqslant i_1\leqslant n_1}\right) m'' = g_3\left(h'_3 l' \overline{e}'', (k''^3_{i_3})_{1\leqslant i_3\leqslant n_3}\right) m''.$$

We now consider the set $\mathcal{Y}_2$ as an equivalence relation on the set $\{1,\dots,n_2\}$ and denote by $\mathcal{N}_2$ the quotient of $\{1,\dots,n_2\}$ by $\mathcal{Y}_2$. We denote by $[i_2] \in \mathcal{N}_2$ the element represented by $i_2 \in \{1,\dots,n_2\}$. From the definition of $\mathcal{Y}_2$ and using our assumptions, we know that if $(i_2, i'_2) \in \mathcal{Y}_2$, then $e^2_{i_2} = e^2_{i'_2}$ and $f^2_{i_2} h_2 l \overline{e}'' m'' = f^2_{i'_2} h_2 l \overline{e}'' m''$. This shows that the pullback square



(8)

is well-defined. Since E is closed under finite products (by (Id), (ClComp) and (StPb)) and since E is stable under pullbacks (StPb), we know that $e \in E$. Since morphisms in E are (strong) epimorphisms (by (SRightCancP) and (NoPMono)), we only need to show that the identity

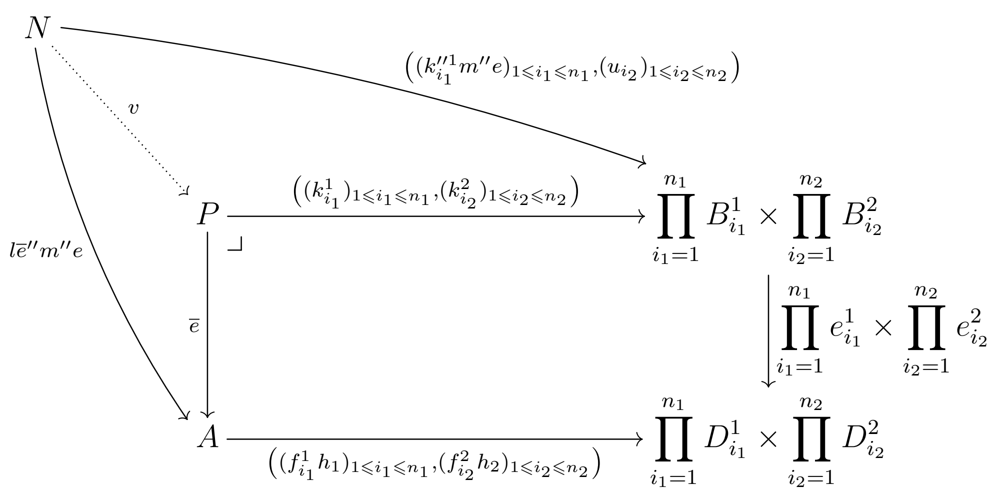
$$g_1\left(h_1 l\overline{e}'', (k''^1_{i_1})_{1\leqslant i_1\leqslant n_1}\right)m''e = g_3\left(h'_3 l'\overline{e}'', (k''^3_{i_3})_{1\leqslant i_3\leqslant n_3}\right)m''e$$

holds. In order to do so, for each $i_2 \in \{1,\dots,n_2\}$, we define a morphism $u_{i_2}\colon N \to B^2_{i_2}$ as follows:

$$u_{i_2} = \begin{cases} k''^1_{i_1}m''e & \text{for some } i_1 \in \{1,\dots,n_1\} \text{ such that } (i_1,i_2)\in\mathcal{X}_{12} \text{ if it exists,}\\ k''^3_{i_3}m''e & \text{for some } i_3 \in \{1,\dots,n_3\} \text{ such that } (i_2,i_3)\in\mathcal{X}_{23} \text{ if it exists,}\\ n_{[i_2]} & \text{if no such } i_1 \text{ or } i_3 \text{ exists.}\end{cases}$$

Let us check that this definition makes sense. If $i_1, i'_1 \in \{1,\dots,n_1\}$ are such that (i_1,i_2) and (i'_1,i_2) are in $\mathcal{X}_{12}$, then $(i_1,i'_1)\in\mathcal{Y}_1$ and $k''^1_{i_1}m'' = k''^1_{i'_1}m''$ by definition of m''. The other cases are treated similarly.

The outer part of the diagram



is commutative. Indeed, for $i_1 \in \{1,\dots,n_1\}$, $f^1_{i_1}h_1 l\overline{e}''m''e = e^1_{i_1}k''^1_{i_1}m''e$ follows from the commutativity of (7); for $i_2 \in \{1,\dots,n_2\}$, $f^2_{i_2}h_2 l\overline{e}''m''e = e^2_{i_2}u_{i_2}$ is obtained from

$$f^2_{i_2}h_2 l\overline{e}''m''e = f^1_{i_1}h_1 l\overline{e}''m''e = e^1_{i_1}k''^1_{i_1}m''e = e^2_{i_2}k''^1_{i_1}m''e$$

if $(i_1, i_2) \in \mathcal{X}_{12}$ for some $1 \leqslant i_1 \leqslant n_1$, from

$$f^2_{i_2} h_2 l \overline{e}'' m'' e = f^2_{i_2} h'_2 l' \overline{e}'' m'' e = f^3_{i_3} h'_3 l' \overline{e}'' m'' e = e^3_{i_3} k''^3_{i_3} m'' e = e^2_{i_2} k''^3_{i_3} m'' e$$

if $(i_2, i_3) \in \mathcal{X}_{23}$ for some $1 \leqslant i_3 \leqslant n_3$ and from the commutativity of (8) if $u_{i_2} = n_{[i_2]}$. Hence, there exists a unique morphism $v\colon N \to P$ retaining commutativity of the diagram. For each $(i_1, i_2) \in \mathcal{X}_{12}$, one has $k^1_{i_1} v = k''^1_{i_1} m'' e = u_{i_2} = k^2_{i_2} v$ and for each $(i_1, i'_1) \in \mathcal{Y}_1$, one has $k^1_{i_1} v = k''^1_{i_1} m'' e = k''^1_{i'_1} m'' e = k^1_{i'_1} v$. Moreover, for each $(i_2, i'_2) \in \mathcal{Y}_2$, one has

$$k^2_{i_2} v = u_{i_2} = u_{i'_2} = k^2_{i'_2} v$$

where the second equality is proved case-by-case:

- if there exists $i_1 \in \{1, \ldots, n_1\}$ such that $(i_1, i_2) \in \mathcal{X}_{12}$, then $(i_1, i'_2) \in \mathcal{X}_{12}$ and $u_{i_2} = k''^1_{i_1} m'' e = u_{i'_2}$;
- if there exists $i_3 \in \{1, \ldots, n_3\}$ such that $(i_2, i_3) \in \mathcal{X}_{23}$, then $(i'_2, i_3) \in \mathcal{X}_{23}$ and $u_{i_2} = k''^3_{i_3} m'' e = u_{i'_2}$;
- if none of the above occurs, then $u_{i_2} = n_{[i_2]} = n_{[i'_2]} = u_{i'_2}$.

By definition of the equalizer m, it follows from the above identities that there exists a unique morphism $w\colon N \to M$ such that $mw = v$.

Analogously, one can show that there exists a unique morphism $v'\colon N \to P'$ satisfying $\overline{e}' v' = l' \overline{e}'' m'' e$ and

$$\big((k'^2_{i_2})_{1 \leqslant i_2 \leqslant n_2}, (k'^3_{i_3})_{1 \leqslant i_3 \leqslant n_3}\big)\, v' = \big((u_{i_2})_{1 \leqslant i_2 \leqslant n_2}, (k''^3_{i_3} m'' e)_{1 \leqslant i_3 \leqslant n_3}\big)$$

and a unique morphism $w'\colon N \to M'$ such that $m' w' = v'$.

It follows immediately from the definitions of v and v' and the commutativity of (6) that the diagrams

$$\begin{array}{ccc} N & \xrightarrow{\quad v \quad} & P \\ {\scriptstyle m''e}\downarrow & & \downarrow{\scriptstyle \left(h_1 \overline{e}, (k^1_{i_1})_{1 \leqslant i_1 \leqslant n_1}\right)} \\ P'' & \xrightarrow[\left(h_1 l \overline{e}'', (k''^1_{i_1})_{1 \leqslant i_1 \leqslant n_1}\right)]{} & P_1 \end{array} \tag{9}$$

$$\begin{array}{ccc} N & \xrightarrow{\quad v \quad} & P \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle \left(h_2 \overline{e}, (k^2_{i_2})_{1 \leqslant i_2 \leqslant n_2}\right)} \\ P' & \xrightarrow[\left(h'_2 \overline{e}', (k'^2_{i_2})_{1 \leqslant i_2 \leqslant n_2}\right)]{} & P_2 \end{array} \tag{10}$$

and

$$\begin{array}{ccc} N & \xrightarrow{\quad v' \quad} & P' \\ {\scriptstyle m''e}\downarrow & & \downarrow{\scriptstyle \left(h_3'\overline{e}',(k_{i_3}'^3)_{1\leqslant i_3\leqslant n_3}\right)} \\ P'' & \xrightarrow[\left(h_3'l'\overline{e}'',(k_{i_3}''^3)_{1\leqslant i_3\leqslant n_3}\right)]{} & P_3 \end{array} \tag{11}$$

commute. We then have the required equality as follows:

$$\begin{aligned} g_1\left(h_1 l\overline{e}'', (k_{i_1}''^1)_{1\leqslant i_1\leqslant n_1}\right) m''e &= g_1\left(h_1\overline{e}, (k_{i_1}^1)_{1\leqslant i_1\leqslant n_1}\right) v && \text{by (9)} \\ &= g_1\left(h_1\overline{e}, (k_{i_1}^1)_{1\leqslant i_1\leqslant n_1}\right) mw \\ &= g_2\left(h_2\overline{e}, (k_{i_2}^2)_{1\leqslant i_2\leqslant n_2}\right) mw && \text{by (4)} \\ &= g_2\left(h_2\overline{e}, (k_{i_2}^2)_{1\leqslant i_2\leqslant n_2}\right) v \\ &= g_2\left(h_2'\overline{e}', (k_{i_2}'^2)_{1\leqslant i_2\leqslant n_2}\right) v' && \text{by (10)} \\ &= g_2\left(h_2'\overline{e}', (k_{i_2}'^2)_{1\leqslant i_2\leqslant n_2}\right) m'w' \\ &= g_3\left(h_3'\overline{e}', (k_{i_3}'^3)_{1\leqslant i_3\leqslant n_3}\right) m'w' && \text{by (5)} \\ &= g_3\left(h_3'\overline{e}', (k_{i_3}'^3)_{1\leqslant i_3\leqslant n_3}\right) v' \\ &= g_3\left(h_3'l'\overline{e}'', (k_{i_3}''^3)_{1\leqslant i_3\leqslant n_3}\right) m''e && \text{by (11)} \end{aligned}$$

proving the transitivity of $\cong_{F,C}$.

Step 15. A good representative. We will denote by $[(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)]$ the element of $\widehat{F}(C)$ represented by the 7-tuple $(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)$. Let us show now that each element $[(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)]$ of $\widehat{F}(C)$ can be represented by a 7-tuple $(n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',g',a')$ for which the equivalence relation

$$\mathcal{Y}' = \{(i_1',i_2')\in\mathbb{N}^2 \,|\, 1\leqslant i_1'\leqslant n',\ 1\leqslant i_2'\leqslant n',\ e'_{i_1'}=e'_{i_2'} \text{ and } F(f'_{i_1'})(a')=F(f'_{i_2'})(a')\}$$

on $\{1,\ldots,n'\}$ is simply the diagonal

$$\Delta_{n'} = \{(i',i')\,|\, i'\in\{1,\ldots,n'\}\}.$$

To fix notation, we display the morphisms forming the 7-tuple $(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)$ in the following diagram

$$\begin{array}{cccc} & P & \xrightarrow{\quad \overline{f} \quad} & B_1\times\cdots\times B_n \\ {\scriptstyle g}\swarrow & {\scriptstyle \overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1\times\cdots\times e_n} \\ C & A & \xrightarrow[(f_1,\ldots,f_n)]{} & D_1\times\cdots\times D_n \end{array}$$

and denote as before by $\mathcal{Y}$ the set

$$\mathcal{Y} = \{(i_1,i_2)\in\mathbb{N}^2 \,|\, 1\leqslant i_1\leqslant n,\ 1\leqslant i_2\leqslant n,\ e_{i_1}=e_{i_2} \text{ and } F(f_{i_1})(a)=F(f_{i_2})(a)\}.$$

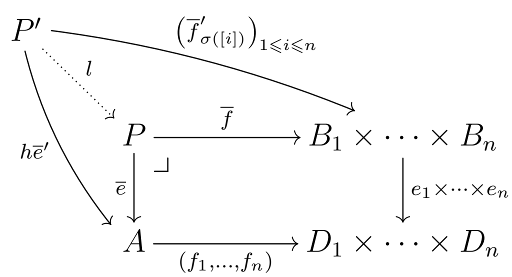
We denote by $\mathcal{N}$ the quotient of the set $\{1,\dots,n\}$ by the equivalence relation $\mathcal{Y}$ and consider a bijection $\sigma\colon \mathcal{N} \to \{1,\dots,n'\}$ where n' is the cardinality of the finite set $\mathcal{N}$. We also consider the equalizer diagram

$$A' \overset{h}{\rightarrowtail} A \underset{(f_{i_2})_{(i_1,i_2)\in\mathcal{Y}}}{\overset{(f_{i_1})_{(i_1,i_2)\in\mathcal{Y}}}{\rightrightarrows}} \prod_{(i_1,i_2)\in\mathcal{Y}} D_{i_1}\ .$$

Since F preserves finite limits and by definition of $\mathcal{Y}$, there exists a unique element $a' \in F(A')$ such that $F(h)(a') = a$. For each $i' \in \{1,\dots,n'\}$, we choose a representative $i \in \{1,\dots,n\}$ such that $i' = \sigma([i])$ and denote by $e'_{i'}\colon B'_{i'} \to D'_{i'}$ the morphism $e_i\colon B_i \to D_i$ and by $f'_{i'}\colon A' \to D'_{i'}$ the morphism $f_i h\colon A' \to D_i$. Note that the definitions of $e'_{i'}$ and $f'_{i'}$ are independent of the chosen i. We then form the pullback

$$\begin{array}{ccc} P' & \xrightarrow{\overline{f}'=(\overline{f}'_1,\dots,\overline{f}'_{n'})} & B'_1 \times \cdots \times B'_{n'} \\ {\scriptstyle \overline{e}'}\downarrow & \lrcorner & \downarrow{\scriptstyle e'_1\times\cdots\times e'_{n'}} \\ A' & \xrightarrow[(f'_1,\dots,f'_{n'})]{} & D'_1 \times \cdots \times D'_{n'} \end{array}$$

and consider the diagram



whose outer part is commutative, inducing the existence of a unique morphism l making this diagram commute. We denote by $g'\colon P' \to C$ the morphism gl. We thus have defined a 7-tuple $(n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a')$. This 7-tuple satisfies the implication

$$\Big(e'_{i'_1} = e'_{i'_2} \wedge F(f'_{i'_1})(a') = F(f'_{i'_2})(a')\Big) \Longrightarrow i'_1 = i'_2$$

for each $i'_1, i'_2 \in \{1,\dots,n'\}$. Indeed, for i'_1 and i'_2 satisfying the premise in the above implication, if $i_1, i_2 \in \{1,\dots,n\}$ are such that $i'_1 = \sigma([i_1])$ and $i'_2 = \sigma([i_2])$, one has

$$e_{i_1} = e'_{i'_1} = e'_{i'_2} = e_{i_2}$$

and

$$F(f_{i_1})(a) = F(f_{i_1}h)(a') = F(f'_{i'_1})(a') = F(f'_{i'_2})(a') = F(f_{i_2}h)(a') = F(f_{i_2})(a).$$

This implies that $[i_1] = [i_2]$ and thus $i'_1 = i'_2$. To conclude this step, it remains to show that the span

$$A \xleftarrow{\;h\;} A' \xrightarrow{\;1_{A'}\;} A'$$

together with the element $a' \in F(A')$ is a witness of

$$(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a) \cong_{F,C} (n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a').$$

We already know that $F(h)(a') = a$ and $F(1_{A'})(a') = a'$. The equalities

$$\begin{aligned}\mathcal{X} &= \{(i_1, i'_2) \in \mathbb{N}^2 \,|\, 1 \leqslant i_1 \leqslant n,\ 1 \leqslant i'_2 \leqslant n',\ e_{i_1} = e'_{i'_2} \text{ and } F(f_{i_1})(a) = F(f'_{i'_2})(a')\} \\ &= \{(i_1, \sigma([i_2])) \in \mathbb{N}^2 \,|\, (i_1, i_2) \in \mathcal{Y}\}\end{aligned}$$

follow easily from the definitions introduced above. In order to check the last condition of the definition of $\cong_{F,C}$, we need the pullback

$$\begin{array}{ccc} P'' & \xrightarrow{\left(k_1,\dots,k_n,k'_1,\dots,k'_{n'}\right)} & B_1 \times \cdots \times B_n \times B'_1 \times \cdots \times B'_{n'} \\ {\scriptstyle \overline{e}''}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1 \times \cdots \times e_n \times e'_1 \times \cdots \times e'_{n'}} \\ A' & \xrightarrow[\left(f_1 h,\dots,f_n h,f'_1,\dots,f'_{n'}\right)]{} & D_1 \times \cdots \times D_n \times D'_1 \times \cdots \times D'_{n'} \end{array}$$

and, in view of the description of $\mathcal{X}$ and $\mathcal{Y}'$, the equalizer

$$M \overset{m}{\rightarrowtail} P'' \underset{\left((k'_{\sigma([i_2])})_{(i_1,i_2)\in\mathcal{Y}},(k_{i_2})_{(i_1,i_2)\in\mathcal{Y}}\right)}{\overset{\left((k_{i_1})_{(i_1,i_2)\in\mathcal{Y}},(k_{i_1})_{(i_1,i_2)\in\mathcal{Y}}\right)}{\rightrightarrows}} \prod_{(i_1,i_2)\in\mathcal{Y}} B_{i_1} \times \prod_{(i_1,i_2)\in\mathcal{Y}} B_{i_1}\ .$$

In particular, we know that for each $i \in \{1, \dots, n\}$, the identity $k_i m = k'_{\sigma([i])} m$ holds. Using this and the definition of l, it is straightforward to prove that the diagram

$$\begin{array}{ccccc} M & & \overset{m}{\rightarrowtail} & & P'' \\ {\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle \left(h\overline{e}'',(k_i)_{1\leqslant i\leqslant n}\right)} \\ P'' & \xrightarrow[\left(\overline{e}'',(k'_{i'})_{1\leqslant i'\leqslant n'}\right)]{} & P' & \xrightarrow[l]{} & P \end{array}$$

commutes. Since $g' = gl$, this proves that $g\,(h\overline{e}'', (k_i)_{1\leqslant i\leqslant n})\,m = g'\,(\overline{e}'', (k'_{i'})_{1\leqslant i'\leqslant n'})\,m$ as required.

Step 16. The functor $\widehat{F}$. Given a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$, we have constructed, for each object $C \in \mathcal{C}$, a set $\widehat{F}(C)$. We now turn this construction into a functor $\widehat{F}\colon \mathcal{C} \to \mathsf{Set}$. Given a morphism $u\colon C \to C'$ in $\mathcal{C}$, we define the function $\widehat{F}(u)\colon \widehat{F}(C) \to \widehat{F}(C')$ as

$$\widehat{F}(u)([(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)]) = [(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},ug,a)]$$

for each element $[(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)]$ of $\widehat{F}(C)$. It is obvious that this function is well-defined, i.e., that the implication

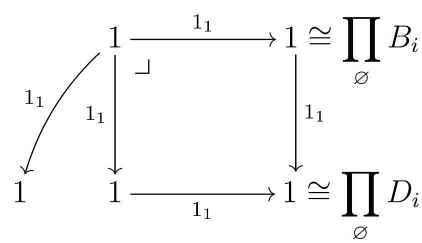
$$\begin{aligned} &(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a) \cong_{F,C} (n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',g',a') \\ \Longrightarrow\ &(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},ug,a) \cong_{F,C'} (n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',ug',a') \end{aligned}$$

holds and that this gives rise to a functor $\widehat{F}\colon \mathcal{C} \to \mathsf{Set}$.

Step 17. $\widehat{F}$ preserves the terminal object. We now want to prove that, given a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$, the functor $\widehat{F}\colon \mathcal{C} \to \mathsf{Set}$ also preserves finite limits. To do this, it is sufficient to prove that $\widehat{F}$ preserves the terminal object and pullbacks. Let us start with the terminal object. So let 1 be the terminal object of $\mathcal{C}$ and let us prove that $\widehat{F}(1)$ is a singleton set. Since F preserves finite limits, $F(1)$ is a singleton set, say $\{*\}$. We can immediately notice that the 7-tuple

$$(0,\varnothing,\varnothing,1_1,1_1,1_1,*),$$

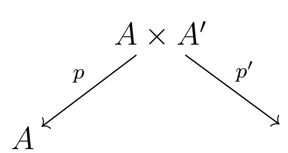
displayed as in



represents an element of $\widehat{F}(1)$. In order to prove uniqueness, let $(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)$ and $(n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',g',a')$ be two 7-tuples representing elements of $\widehat{F}(1)$, with $a \in F(A)$ and $a' \in F(A')$, and let us prove that

$$(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a) \cong_{F,1} (n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',g',a'). \tag{12}$$

We consider the product diagram



(13)

and, since F preserves finite limits, the unique element $a'' \in F(A \times A')$ such that $F(p)(a'') = a$ and $F(p')(a'') = a'$. The span (13) together with the element a'' is a witness of the relation (12). Indeed, the first two conditions hold by definition of a'' and the last condition trivially holds since it requires that two parallel morphisms with codomain 1 are identical.

Step 18. $\widehat{F}$ preserves jointly monomorphic pairs of morphisms. As a preliminary step to prove that $\widehat{F}$ preserves pullbacks, let us show it preserves jointly monomorphic pairs of morphisms. This means we must show that given two morphisms

$$C_1 \xleftarrow{p_1} C \xrightarrow{p_2} C_2$$

in $\mathcal{C}$ satisfying the implication

$$(p_1 u = p_1 v \wedge p_2 u = p_2 v) \Longrightarrow u = v$$

for any pair of morphisms $u, v \colon X \rightrightarrows C$, then the induced function

$$\widehat{F}(C) \xrightarrow{\left(\widehat{F}(p_1), \widehat{F}(p_2)\right)} \widehat{F}(C_1) \times \widehat{F}(C_2)$$

is injective. Let $[(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)]$ and $[(n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a')]$ be two elements of $\widehat{F}(C)$ such that

$$(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, p_1 g, a) \cong_{F,C_1} (n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', p_1 g', a') \tag{14}$$

and

$$(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, p_2 g, a) \cong_{F,C_2} (n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', p_2 g', a'). \tag{15}$$

Using Step 15, we can assume without loss of generality that the two implications

$$(e_{i_1} = e_{i_2} \wedge F(f_{i_1})(a) = F(f_{i_2})(a)) \Longrightarrow i_1 = i_2$$

and

$$\left(e'_{i'_1} = e'_{i'_2} \wedge F(f'_{i'_1})(a') = F(f'_{i'_2})(a')\right) \Longrightarrow i'_1 = i'_2$$

hold for indices $i_1, i_2 \in \{1, \ldots, n\}$ and $i'_1, i'_2 \in \{1, \ldots, n'\}$. To fix notation, we say that the morphisms involved in those elements of $\widehat{F}(C)$ can be displayed as in

$$\begin{array}{ccccc} & P & \xrightarrow{\overline{f}} & B_1 \times \cdots \times B_n \\ \swarrow^{g} & \downarrow{\scriptstyle \overline{e}} \;\lrcorner & & \downarrow{\scriptstyle e_1 \times \cdots \times e_n} \\ C & A & \xrightarrow[(f_1, \ldots, f_n)]{} & D_1 \times \cdots \times D_n \end{array} \quad \text{and} \quad \begin{array}{ccccc} & P' & \xrightarrow{\overline{f}'} & B'_1 \times \cdots \times B'_{n'} \\ \swarrow^{g'} & \downarrow{\scriptstyle \overline{e}'} \;\lrcorner & & \downarrow{\scriptstyle e'_1 \times \cdots \times e'_{n'}} \\ C & A' & \xrightarrow[(f'_1, \ldots, f'_{n'})]{} & D'_1 \times \cdots \times D'_{n'} \end{array}$$

and, for $j \in \{1,2\}$, the relation mentioned in (14) if $j=1$, respectively in (15) if $j=2$, is witnessed by the span

$$A \xleftarrow{\;h_j\;} A''_j \xrightarrow{\;h'_j\;} A'$$

together with the element $a''_j \in F(A''_j)$. Therefore, for each $j \in \{1,2\}$, we have $F(h_j)(a''_j) = a$, $F(h'_j)(a''_j) = a'$ and, considering the pullback diagram

$$\begin{array}{ccc} P''_j & \xrightarrow{\left(k_{j1},\dots,k_{jn},k'_{j1},\dots,k'_{jn'}\right)} & B_1 \times \cdots \times B_n \times B'_1 \times \cdots \times B'_{n'} \\ {\scriptstyle \overline{e}''_j}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle e_1\times\cdots\times e_n\times e'_1\times\cdots\times e'_{n'}} \\ A''_j & \xrightarrow[\left(f_1h_j,\dots,f_nh_j,f'_1h'_j,\dots,f'_{n'}h'_j\right)]{} & D_1 \times \cdots \times D_n \times D'_1 \times \cdots \times D'_{n'} \end{array} \quad , \tag{16}$$

the set

$$\mathcal{X} = \{(i,i') \in \mathbb{N}^2 \,|\, 1 \leqslant i \leqslant n,\ 1 \leqslant i' \leqslant n',\ e_i = e'_{i'} \text{ and } F(f_i)(a) = F(f'_{i'})(a')\}$$

and the equalizer

$$M_j \overset{m_j}{\rightarrowtail} P''_j \underset{(k'_{ji'})_{(i,i')\in\mathcal{X}}}{\overset{(k_{ji})_{(i,i')\in\mathcal{X}}}{\rightrightarrows}} \prod_{(i,i')\in\mathcal{X}} B_i \ , \tag{17}$$

the diagram

$$\begin{array}{ccccc} M_j & \overset{m_j}{\rightarrowtail} & P''_j & \xrightarrow{\left(h_j\overline{e}''_j,(k_{ji})_{1\leqslant i\leqslant n}\right)} & P \\ {\scriptstyle m_j}\big\downarrow & & & & \big\downarrow{\scriptstyle p_jg} \\ P''_j & \xrightarrow[\left(h'_j\overline{e}''_j,(k'_{ji'})_{1\leqslant i'\leqslant n'}\right)]{} & P' & \xrightarrow[p_jg']{} & C_j \end{array} \tag{18}$$

commutes. We consider the pullback

$$\begin{array}{ccc} A'' & \xrightarrow{l_2} & A''_2 \\ {\scriptstyle l_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (h_2,h'_2)} \\ A''_1 & \xrightarrow[(h_1,h'_1)]{} & A \times A' \end{array} \tag{19}$$

and, since F preserves finite limits, we know there exists a unique element $a'' \in F(A'')$ such that $F(l_1)(a'') = a''_1$ and $F(l_2)(a'') = a''_2$. Let us show that the span

$$A \xleftarrow{\;h_1l_1\;} A'' \xrightarrow{\;h'_1l_1\;} A'$$

together with the element $a'' \in F(A'')$ is a witness of the relation

$$(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a) \cong_{F,C} (n', (e'_{i'})_{i'}, (f'_{i'})_{i'}, \overline{f}', \overline{e}', g', a').$$

We already know that $F(h_1 l_1)(a'') = F(h_1)(a''_1) = a$ and $F(h'_1 l_1)(a'') = F(h'_1)(a''_1) = a'$. In order to check the last condition, we need to consider the diagram

$$\begin{array}{ccccc} M & \xrightarrow{r'_1} & M_1 & & \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle m_1} & & \\ P'' & \xrightarrow{r_1} & P''_1 & \xrightarrow{(k_{11},\dots,k_{1n},k'_{11},\dots,k'_{1n'})} & B_1 \times \cdots \times B_n \times B'_1 \times \cdots \times B'_{n'} \\ {\scriptstyle \overline{e}''}\downarrow & \lrcorner & \downarrow{\scriptstyle \overline{e}''_1} \;\lrcorner & & \downarrow{\scriptstyle e_1 \times \cdots \times e_n \times e'_1 \times \cdots \times e'_{n'}} \\ A'' & \xrightarrow[l_1]{} & A''_1 & \xrightarrow[(f_1h_1,\dots,f_nh_1,f'_1h'_1,\dots,f'_{n'}h'_1)]{} & D_1 \times \cdots \times D_n \times D'_1 \times \cdots \times D'_{n'} \end{array} \qquad (20)$$

where all rectangles are pullbacks. Using the equalizer (17) for $j = 1$, it is routine to show that the diagram

$$M \overset{m}{\rightarrowtail} P'' \underset{(k'_{1i'}r_1)_{(i,i')\in\mathcal{X}}}{\overset{(k_{1i}r_1)_{(i,i')\in\mathcal{X}}}{\rightrightarrows}} \prod_{(i,i')\in\mathcal{X}} B_i$$

is also an equalizer diagram. We must therefore prove that the diagram

$$\begin{array}{ccccc} M & \overset{m}{\rightarrowtail} & P'' & \xrightarrow{(h_1 l_1 \overline{e}'', (k_{1i}r_1)_{1\leqslant i\leqslant n})} & P \\ {\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle g} \\ P'' & \xrightarrow[(h'_1 l_1 \overline{e}'', (k'_{1i'}r_1)_{1\leqslant i'\leqslant n'})]{} & P' & \xrightarrow[g']{} & C \end{array}$$

commutes. Using the commutativity of (20), this means we must show that the identity

$$g\,(h_1\overline{e}''_1, (k_{1i})_{1\leqslant i\leqslant n})\, m_1 r'_1 = g'\,(h'_1\overline{e}''_1, (k'_{1i'})_{1\leqslant i'\leqslant n'})\, m_1 r'_1$$

holds. Since p_1 and p_2 are jointly monomorphic, and since this identity composed with p_1 holds by commutativity of (18) for $j = 1$, it remains to show that the identity

$$p_2 g\,(h_1\overline{e}''_1, (k_{1i})_{1\leqslant i\leqslant n})\, m_1 r'_1 = p_2 g'\,(h'_1\overline{e}''_1, (k'_{1i'})_{1\leqslant i'\leqslant n'})\, m_1 r'_1$$

holds. In view of the pullback (16) for $j = 2$, and since

$$\begin{aligned} &(f_1h_2, \dots, f_nh_2, f'_1h'_2, \dots, f'_{n'}h'_2)\, l_2\overline{e}'' \\ &= (f_1h_1, \dots, f_nh_1, f'_1h'_1, \dots, f'_{n'}h'_1)\, l_1\overline{e}'' \\ &= (e_1 \times \cdots \times e_n \times e'_1 \times \cdots \times e'_{n'})\, (k_{11}, \dots, k_{1n}, k'_{11}, \dots, k'_{1n'})\, r_1, \end{aligned}$$

there exists a unique morphism $r_2 \colon P'' \to P''_2$ such that

$$\overline{e}''_2 r_2 = l_2 \overline{e}'' \tag{21}$$

and

$$(k_{21}, \ldots, k_{2n}, k'_{21}, \ldots, k'_{2n'})\, r_2 = (k_{11}, \ldots, k_{1n}, k'_{11}, \ldots, k'_{1n'})\, r_1. \tag{22}$$

For each $(i, i') \in \mathcal{X}$, we know that

$$k_{2i} r_2 m = k_{1i} r_1 m = k'_{1i'} r_1 m = k'_{2i'} r_2 m.$$

Hence, using the equalizer (17) for $j = 2$, we know that there exists a unique morphism $r'_2 \colon M \to M_2$ such that $m_2 r'_2 = r_2 m$. To conclude this step, it remains to compute

$$\begin{aligned}
p_2 g\, (h_1 \overline{e}''_1, (k_{1i})_{1 \leqslant i \leqslant n})\, m_1 r'_1 &= p_2 g\, (h_1 \overline{e}''_1 r_1, (k_{1i} r_1)_{1 \leqslant i \leqslant n})\, m && \text{by (20)} \\
&= p_2 g\, (h_1 l_1 \overline{e}'', (k_{2i} r_2)_{1 \leqslant i \leqslant n})\, m && \text{by (20) and (22)} \\
&= p_2 g\, (h_2 l_2 \overline{e}'', (k_{2i} r_2)_{1 \leqslant i \leqslant n})\, m && \text{by (19)} \\
&= p_2 g\, (h_2 \overline{e}''_2, (k_{2i})_{1 \leqslant i \leqslant n})\, r_2 m && \text{by (21)} \\
&= p_2 g\, (h_2 \overline{e}''_2, (k_{2i})_{1 \leqslant i \leqslant n})\, m_2 r'_2 && \\
&= p_2 g'\, (h'_2 \overline{e}''_2, (k'_{2i'})_{1 \leqslant i' \leqslant n'})\, m_2 r'_2 && \text{by (18) for } j = 2 \\
&= p_2 g'\, (h'_2 \overline{e}''_2 r_2, (k'_{2i'} r_2)_{1 \leqslant i' \leqslant n'})\, m && \\
&= p_2 g'\, (h'_2 l_2 \overline{e}'', (k'_{1i'} r_1)_{1 \leqslant i' \leqslant n'})\, m && \text{by (21) and (22)} \\
&= p_2 g'\, (h'_1 l_1 \overline{e}'', (k'_{1i'} r_1)_{1 \leqslant i' \leqslant n'})\, m && \text{by (19)} \\
&= p_2 g'\, (h'_1 \overline{e}''_1, (k'_{1i'})_{1 \leqslant i' \leqslant n'})\, r_1 m && \text{by (20)} \\
&= p_2 g'\, (h'_1 \overline{e}''_1, (k'_{1i'})_{1 \leqslant i' \leqslant n'})\, m_1 r'_1 && \text{by (20).}
\end{aligned}$$

STEP 19. $\widehat{F}$ PRESERVES PULLBACKS. Let us now show that $\widehat{F}$ preserves pullbacks. Let

$$\begin{array}{ccc}
C & \xrightarrow{p_2} & C_2 \\
{\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle u_2} \\
C_1 & \xrightarrow[u_1]{} & C_3
\end{array} \tag{23}$$

be a pullback diagram in $\mathcal{C}$. Let $[(n_1, (e^1_{i_1})_{i_1}, (f^1_{i_1})_{i_1}, \overline{f}_1, \overline{e}_1, g_1, a_1)]$ be an element of $\widehat{F}(C_1)$ and $[(n_2, (e^2_{i_2})_{i_2}, (f^2_{i_2})_{i_2}, \overline{f}_2, \overline{e}_2, g_2, a_2)]$ be an element of $\widehat{F}(C_2)$ such that

$$(n_1, (e^1_{i_1})_{i_1}, (f^1_{i_1})_{i_1}, \overline{f}_1, \overline{e}_1, u_1 g_1, a_1) \cong_{F, C_3} (n_2, (e^2_{i_2})_{i_2}, (f^2_{i_2})_{i_2}, \overline{f}_2, \overline{e}_2, u_2 g_2, a_2). \tag{24}$$

Using Step 15, we can assume without loss of generality that the two implications

$$\Big(e^1_{i_1} = e^1_{i'_1} \wedge F(f^1_{i_1})(a_1) = F(f^1_{i'_1})(a_1) \Big) \Longrightarrow i_1 = i'_1 \tag{25}$$

and

$$\Big(e^2_{i_2} = e^2_{i'_2} \wedge F(f^2_{i_2})(a_2) = F(f^2_{i'_2})(a_2) \Big) \Longrightarrow i_2 = i'_2 \tag{26}$$

hold for indices $i_1, i'_1 \in \{1, \dots, n_1\}$ and $i_2, i'_2 \in \{1, \dots, n_2\}$. To fix notation, we say that the morphisms involved in the chosen elements of $\widehat{F}(C_1)$ and of $\widehat{F}(C_2)$ can be displayed as in

$$\begin{array}{ccccc} & P_1 & \xrightarrow{\overline{f}_1} & B^1_1 \times \cdots \times B^1_{n_1} \\ {\scriptstyle g_1}\swarrow & \downarrow{\scriptstyle \overline{e}_1} & \lrcorner & \downarrow{\scriptstyle e^1_1 \times \cdots \times e^1_{n_1}} \\ C_1 & A_1 & \xrightarrow[(f^1_1, \dots, f^1_{n_1})]{} & D^1_1 \times \cdots \times D^1_{n_1} \end{array} \quad \text{and} \quad \begin{array}{ccccc} & P_2 & \xrightarrow{\overline{f}_2} & B^2_1 \times \cdots \times B^2_{n_2} \\ {\scriptstyle g_2}\swarrow & \downarrow{\scriptstyle \overline{e}_2} & \lrcorner & \downarrow{\scriptstyle e^2_1 \times \cdots \times e^2_{n_2}} \\ C_2 & A_2 & \xrightarrow[(f^2_1, \dots, f^2_{n_2})]{} & D^2_1 \times \cdots \times D^2_{n_2} \end{array}$$

and the relation mentioned in (24) is witnessed by the span

$$\begin{array}{ccc} & A & \\ {\scriptstyle h_1}\swarrow & & \searrow{\scriptstyle h_2} \\ A_1 & & A_2 \end{array}$$

together with the element $a \in F(A)$. We thus know that $F(h_1)(a) = a_1$, $F(h_2)(a) = a_2$ and, considering the pullback

$$\begin{array}{ccc} P & \xrightarrow{\left(k^1_1, \dots, k^1_{n_1}, k^2_1, \dots, k^2_{n_2}\right)} & B^1_1 \times \cdots \times B^1_{n_1} \times B^2_1 \times \cdots \times B^2_{n_2} \\ \downarrow{\scriptstyle \overline{e}} & \lrcorner & \downarrow{\scriptstyle e^1_1 \times \cdots \times e^1_{n_1} \times e^2_1 \times \cdots \times e^2_{n_2}} \\ A & \xrightarrow[\left(f^1_1 h_1, \dots, f^1_{n_1} h_1, f^2_1 h_2, \dots, f^2_{n_2} h_2\right)]{} & D^1_1 \times \cdots \times D^1_{n_1} \times D^2_1 \times \cdots \times D^2_{n_2} \end{array} \quad , \tag{27}$$

the set

$$\mathcal{X} = \{(i_1, i_2) \in \mathbb{N}^2 \,|\, 1 \leqslant i_1 \leqslant n_1,\, 1 \leqslant i_2 \leqslant n_2,\, e^1_{i_1} = e^2_{i_2} \text{ and } F(f^1_{i_1})(a_1) = F(f^2_{i_2})(a_2)\}$$

and the equalizer

$$M \overset{m}{\rightarrowtail} P \underset{(k^2_{i_2})_{(i_1,i_2) \in \mathcal{X}}}{\overset{(k^1_{i_1})_{(i_1,i_2) \in \mathcal{X}}}{\rightrightarrows}} \prod_{(i_1,i_2) \in \mathcal{X}} B^1_{i_1} \ , \tag{28}$$

the diagram

$$\begin{array}{ccccc} M & \overset{m}{\rightarrowtail} & P & \xrightarrow{\left(h_1 \overline{e}, (k^1_{i_1})_{1 \leqslant i_1 \leqslant n_1}\right)} & P_1 \\ {\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle u_1 g_1} \\ P & \xrightarrow[\left(h_2 \overline{e}, (k^2_{i_2})_{1 \leqslant i_2 \leqslant n_2}\right)]{} & P_2 & \xrightarrow[u_2 g_2]{} & C_3 \end{array} \tag{29}$$

commutes. Moreover, using Step 11, we can suppose without loss of generality that $f^1_{i_1} h_1 = f^2_{i_2} h_2$ for all $(i_1, i_2) \in \mathcal{X}$. Using the commutativity of (29) and the pullback (23), we know that there is a unique morphism $g\colon M \to C$ such that

$$p_1 g = g_1 \left(h_1 \overline{e}, (k^1_{i_1})_{1 \leqslant i_1 \leqslant n_1}\right) m \tag{30}$$

and

$$p_2 g = g_2\left(h_2\overline{e}, (k^2_{i_2})_{1\leqslant i_2\leqslant n_2}\right) m. \tag{31}$$

We consider the set

$$\{1,\dots,n_1\} \amalg \{1,\dots,n_2\} = \{(i_1,1)\,|\,1\leqslant i_1\leqslant n_1\}\cup\{(i_2,2)\,|\,1\leqslant i_2\leqslant n_2\}$$

and its quotient $\mathcal{N}$ by the equivalence relation $R_{\mathcal{X}}$ on it generated by the relations

$$(i_1,1)R_{\mathcal{X}}(i_2,2)$$

for all $(i_1,i_2)\in\mathcal{X}$. We consider a bijection $\sigma\colon\mathcal{N}\to\{1,\dots,n\}$ where n is the cardinality of the finite set $\mathcal{N}$. For each $i\in\{1,\dots,n\}$, we choose a representative $(i_j,j)\in\{1,\dots,n_1\}\amalg\{1,\dots,n_2\}$ such that $\sigma([(i_j,j)])=i$ and denote by $e_i\colon B_i\to D_i$ the morphism $e^j_{i_j}\colon B^j_{i_j}\to D^j_{i_j}$, by $f_i\colon A\to D_i$ the morphism $f^j_{i_j}h_j\colon A\to D^j_{i_j}$ and by $\overline{f}'_i\colon M\to B_i$ the morphism $k^j_{i_j}m\colon M\to B^j_{i_j}$. Note that the morphisms e_i, f_i and $\overline{f}'_i$ are independent of the chosen representative (i_j,j). Let us show that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\overline{f}'=(\overline{f}'_1,\dots,\overline{f}'_n)} & B_1\times\cdots\times B_n \\ {\scriptstyle \overline{e}m}\big\downarrow & & \big\downarrow{\scriptstyle e_1\times\cdots\times e_n} \\ A & \xrightarrow[(f_1,\dots,f_n)]{} & D_1\times\cdots\times D_n \end{array} \tag{32}$$

is a pullback. The diagram commutes since, for each $i\in\{1,\dots,n\}$, $f_i\overline{e}m = f^j_{i_j}h_j\overline{e}m = e^j_{i_j}k^j_{i_j}m = e_i\overline{f}'_i$ where (i_j,j) is the representative of i. Moreover, given morphisms $x\colon X\to A$ and $(y_1,\dots,y_n)\colon X\to B_1\times\cdots\times B_n$ such that $f_ix=e_iy_i$ for all $i\in\{1,\dots,n\}$, the outer part of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\left(y_{\sigma([(1,1)])},\dots,y_{\sigma([(n_1,1)])},y_{\sigma([(1,2)])},\dots,y_{\sigma([(n_2,2)])}\right)} & B^1_1\times\cdots\times B^1_{n_1}\times B^2_1\times\cdots\times B^2_{n_2} \\ {\scriptstyle x'}\searrow\ \ {\scriptstyle x}\downarrow & & \\ P & \xrightarrow{\left(k^1_1,\dots,k^1_{n_1},k^2_1,\dots,k^2_{n_2}\right)} & B^1_1\times\cdots\times B^1_{n_1}\times B^2_1\times\cdots\times B^2_{n_2} \\ {\scriptstyle \overline{e}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle e^1_1\times\cdots\times e^1_{n_1}\times e^2_1\times\cdots\times e^2_{n_2}} \\ A & \xrightarrow[\left(f^1_1h_1,\dots,f^1_{n_1}h_1,f^2_1h_2,\dots,f^2_{n_2}h_2\right)]{} & D^1_1\times\cdots\times D^1_{n_1}\times D^2_1\times\cdots\times D^2_{n_2} \end{array}$$

commutes. Therefore, using the pullback (27), there exists a unique dotted morphism $x'\colon X\to P$ making the diagram commutative. In addition, given $(i_1,i_2)\in\mathcal{X}$, $k^1_{i_1}x' = y_{\sigma([(i_1,1)])} = y_{\sigma([(i_2,2)])} = k^2_{i_2}x'$. Using the equalizer (28), this means that there exists a unique $x''\colon X\to M$ such that $mx''=x'$. In particular, $\overline{e}mx''=\overline{e}x'=x$ and, for each

$i \in \{1, \ldots, n\}$, $\overline{f}'_i x'' = k^j_{i_j} m x'' = k^j_{i_j} x' = y_i$ where (i_j, j) is the representative of i. A morphism satisfying these properties is unique. Indeed, if $x''' \colon X \to M$ is such that $\overline{e} m x''' = x$ and, for each $i \in \{1, \ldots, n\}$, $\overline{f}'_i x''' = y_i$, then $k^j_{i_j} m x''' = \overline{f}'_{\sigma([(i_j,j)])} x''' = y_{\sigma([(i_j,j)])}$ for all $(i_j, j) \in \{1, \ldots, n_1\} \amalg \{1, \ldots, n_2\}$. Since (27) is a pullback, this implies that $m x''' = x'$ and so $x''' = x''$. This proves that the diagram (32) is indeed a pullback. We thus have an element $[(n, (e_i)_i, (f_i)_i, \overline{f}', \overline{e} m, g, a)]$ in $\widehat{F}(C)$ which can be displayed as in

$$\begin{array}{ccccc} & & M & \xrightarrow{\overline{f}' = (\overline{f}'_1, \ldots, \overline{f}'_n)} & B_1 \times \cdots \times B_n \\ & \swarrow^{g} & \Big\downarrow{\scriptstyle \overline{e} m} & \lrcorner & \Big\downarrow{\scriptstyle e_1 \times \cdots \times e_n} \\ C & & A & \xrightarrow[(f_1, \ldots, f_n)]{} & D_1 \times \cdots \times D_n \end{array} .$$

We can notice that the implication

$$(e_i = e_{i'} \wedge F(f_i)(a) = F(f_{i'})(a)) \Longrightarrow i = i'$$

holds for indices $i, i' \in \{1, \ldots, n\}$. Indeed, if $(i_j, j), (i'_{j'}, j') \in \{1, \ldots, n_1\} \amalg \{1, \ldots, n_2\}$ are such that $i = \sigma([(i_j, j)])$ and $i' = \sigma([(i'_{j'}, j')])$, then

$$\begin{aligned} (e_i = e_{i'} \wedge F(f_i)(a) = F(f_{i'})(a)) &\Longleftrightarrow \left(e^j_{i_j} = e^{j'}_{i'_{j'}} \wedge F(f^j_{i_j})(a_j) = F(f^{j'}_{i'_{j'}})(a_{j'}) \right) \\ &\Longrightarrow [(i_j, j)] = [(i'_{j'}, j')] \end{aligned}$$

where the last implication holds by definition of $R_{\mathcal{X}}$ if $j \neq j'$ and by our assumptions (25) and (26) if $j = j'$.

For each $j \in \{1, 2\}$, we shall prove that the span

$$\begin{array}{ccc} & A & \\ {}^{1_A}\swarrow & & \searrow^{h_j} \\ A & & A_j \end{array}$$

together with the element $a \in F(A)$ is a witness of the relation

$$(n, (e_i)_i, (f_i)_i, \overline{f}', \overline{e} m, p_j g, a) \cong_{F, C_j} (n_j, (e^j_{i_j})_{i_j}, (f^j_{i_j})_{i_j}, \overline{f}_j, \overline{e}_j, g_j, a_j).$$

We already know that $F(1_A)(a) = a$, $F(h_j)(a) = a_j$. We then consider the pullback diagram

$$\begin{array}{ccc} P'_j & \xrightarrow{\left(k_{j1}, \ldots, k_{jn}, k'_{j1}, \ldots, k'_{jn_j}\right)} & B_1 \times \cdots \times B_n \times B^j_1 \times \cdots \times B^j_{n_j} \\ \Big\downarrow{\scriptstyle \overline{e}'_j} & \lrcorner & \Big\downarrow{\scriptstyle e_1 \times \cdots \times e_n \times e^j_1 \times \cdots \times e^j_{n_j}} \\ A & \xrightarrow[\left(f_1, \ldots, f_n, f^j_1 h_j, \ldots, f^j_{n_j} h_j\right)]{} & D_1 \times \cdots \times D_n \times D^j_1 \times \cdots \times D^j_{n_j} \end{array}$$

and the equalizer diagram

$$M'_j \overset{m'_j}{\rightarrowtail} P'_j \underset{\left((k'_{ji_j})_{(i,i_j)\in\mathcal{X}'_j}\right)}{\overset{\left((k_{ji})_{(i,i_j)\in\mathcal{X}'_j}\right)}{\rightrightarrows}} \prod_{(i,i_j)\in\mathcal{X}'_j} B_i$$

where the set $\mathcal{X}'_j$ is defined by

$$\begin{aligned}\mathcal{X}'_j &= \{(i,i_j)\in\mathbb{N}^2 \,|\, 1\leqslant i\leqslant n,\ 1\leqslant i_j\leqslant n_j,\ e_i = e^j_{i_j} \text{ and } F(f_i)(a) = F(f^j_{i_j})(a_j)\}\\ &= \{(\sigma([(i_j,j)]),i_j) \,|\, 1\leqslant i_j\leqslant n_j\}.\end{aligned}$$

We need to prove that the diagram

$$\begin{array}{ccccc} M'_j & \overset{m'_j}{\rightarrowtail} & P'_j & \xrightarrow{\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right)} & M \\ {\scriptstyle m'_j}\downarrow & & & & \downarrow{\scriptstyle p_j g} \\ P'_j & \xrightarrow[\left(h_j\overline{e}'_j,(k'_{ji_j})_{1\leqslant i_j\leqslant n_j}\right)]{} & P_j & \xrightarrow[g_j]{} & C_j \end{array}$$

commutes where $\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right)$ is the unique morphism $P'_j\to M$ such that

$$\overline{e}m\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right) = \overline{e}'_j \tag{33}$$

and

$$\overline{f}'_i\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right) = k_{ji}$$

for each $i\in\{1,\dots,n\}$. In view of (30) and (31), it is sufficient to prove

$$\left(h_j\overline{e},(k^j_{i_j})_{1\leqslant i_j\leqslant n_j}\right)m\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right)m'_j = \left(h_j\overline{e}'_j,(k'_{ji_j})_{1\leqslant i_j\leqslant n_j}\right)m'_j.$$

Composing with the pullback projection $\overline{e}_j$, this amounts to

$$h_j\overline{e}m\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right)m'_j = h_j\overline{e}'_jm'_j$$

which follows from (33); and composing with the other pullback projection $\overline{f}_j$, this amounts to

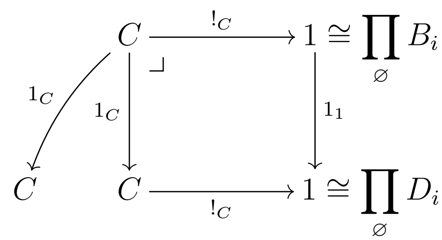
$$k^j_{i_j}m\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right)m'_j = \overline{f}'_{\sigma([(i_j,j)])}\left(\overline{e}'_j,(k_{ji})_{1\leqslant i\leqslant n}\right)m'_j = k_{j\sigma([(i_j,j)])}m'_j = k'_{ji_j}m'_j$$

for each $i_j\in\{1,\dots,n_j\}$, where the last identity holds in view of the definition of m'_j since $(\sigma([(i_j,j)]),i_j)\in\mathcal{X}'_j$. We have therefore proved the existence of an element in $\widehat{F}(C)$ whose image under $\widehat{F}(p_1)$ is $[(n_1,(e^1_{i_1})_{i_1},(f^1_{i_1})_{i_1},\overline{f}_1,\overline{e}_1,g_1,a_1)]$ and whose image under $\widehat{F}(p_2)$ is $[(n_2,(e^2_{i_2})_{i_2},(f^2_{i_2})_{i_2},\overline{f}_2,\overline{e}_2,g_2,a_2)]$. In view of Step 18, it is the unique such, proving that $\widehat{F}$ preserves pullbacks. In view of Step 17, this means that $\widehat{F}$ preserves finite limits.

STEP 20. THE NATURAL TRANSFORMATION $\lambda_F\colon F \to \widehat{F}$. Given a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$, we have so far constructed a finite limit preserving functor $\widehat{F}\colon \mathcal{C} \to \mathsf{Set}$. Let us now construct a natural transformation $\lambda_F\colon F \to \widehat{F}$. As before, we denote by 1 the terminal object of $\mathcal{C}$. For an object $C \in \mathcal{C}$, we define the function $\lambda_{F,C}\colon F(C) \longrightarrow \widehat{F}(C)$ by

$$\lambda_{F,C}(c) = [(0, \varnothing, \varnothing, !_C, 1_C, 1_C, c)]$$

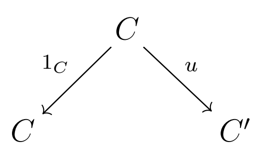
for each element $c \in F(C)$, where $!_C$ is the unique morphism $C \to 1$. The morphisms involved here can be displayed as in



and we define the C-component of λ_F to be $\lambda_{F,C}$. As far as the naturality of λ_F is concerned, one is required to show that for a morphism $u\colon C \to C'$ in $\mathcal{C}$ and an element $c \in F(C)$, the relation

$$(0, \varnothing, \varnothing, !_C, 1_C, u, c) \cong_{F,C'} (0, \varnothing, \varnothing, !_{C'}, 1_{C'}, 1_{C'}, F(u)(c))$$

holds, which is easily seen to be attested by the span

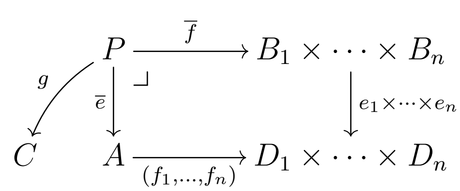


together with the element $c \in F(C)$.

STEP 21. THE RELATION $\cong_{F,C}$ IN A PARTICULAR CASE. We now give an easier description of when, given an object $C \in \mathcal{C}$, an element of $\widehat{F}(C)$ (represented by a 7-tuple in the form as in Step 15) is equal to $\lambda_{F,C}(c)$ for an element c of $F(C)$. More precisely, given a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$, an object $C \in \mathcal{C}$, an element $c \in F(C)$ and an element $[(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)] \in \widehat{F}(C)$ satisfying the implication

$$(e_i = e_{i'} \wedge F(f_i)(a) = F(f_{i'})(a)) \Longrightarrow i = i'$$

for indices $i, i' \in \{1, \dots, n\}$ and whose morphisms can be displayed as in



,

the equality

$$[(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)] = \lambda_{F,C}(c) \tag{34}$$

holds if and only if there exists a span

$$\begin{array}{ccc} & A' & \\ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle h'} \\ A & & C \end{array} \tag{35}$$

together with an element $a' \in F(A')$ such that $F(h)(a') = a$, $F(h')(a') = c$ and, considering the pullback

$$\begin{array}{ccc} P' & \xrightarrow{h''} & P \\ {\scriptstyle \overline{e}'}\downarrow & \lrcorner & \downarrow{\scriptstyle \overline{e}} \\ A' & \xrightarrow[h]{} & A \end{array},$$

the identity $h'\overline{e}' = gh''$ holds. Indeed, the identity (34) holds if and only if the relation

$$(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a) \cong_{F,C} (0,\varnothing,\varnothing,!_C,1_C,1_C,c)$$

holds, i.e., if and only if there exists a span as in (35) and an element $a' \in F(A')$ satisfying $F(h)(a') = a$, $F(h')(a') = c$ and a third condition. For this last condition, we need to consider the pullback made of the composite of the two pullback rectangles in the diagram below.

$$\begin{array}{ccccc} P' & \xrightarrow{h''} & P & \xrightarrow{\overline{f}} & B_1 \times \cdots \times B_n \\ {\scriptstyle \overline{e}'}\downarrow & \lrcorner & {\scriptstyle \overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1\times\cdots\times e_n} \\ A' & \xrightarrow[h]{} & A & \xrightarrow[(f_1,\dots,f_n)]{} & D_1 \times \cdots \times D_n \end{array}$$

In addition, using the notation of Step 10, the sets $\mathcal{X}$ and $\mathcal{Y}'$ are empty and the set $\mathcal{Y}$ is just $\mathcal{Y} = \{(i,i)\} \mid i \in \{1,\dots,n\}\}$ which shows that the equalizer m is the identity $1_{P'}$. In view of this, the last condition turns out to be $h'\overline{e}' = gh''$ as required.

STEP 22. $\lambda_F\colon F \rightarrowtail \widehat{F}$ IS A MONOMORPHISM. Let us now show that the natural transformation $\lambda_F\colon F \to \widehat{F}$ is a monomorphism in $\mathsf{Set}^{\mathcal{C}}$, i.e., for each object $C \in \mathcal{C}$, the function $\lambda_{F,C}$ is injective. Let c, c' be two elements of $F(C)$ such that $\lambda_{F,C}(c) = \lambda_{F,C}(c')$. In view of Step 21, there exists a span

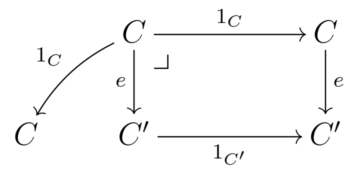
$$\begin{array}{ccc} & A & \\ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle h'} \\ C & & C \end{array}$$

together with an element $a \in F(A)$ such that $F(h)(a) = c$, $F(h')(a) = c'$ and, considering the pullback

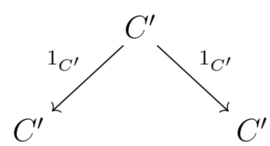
$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle 1_A}\downarrow & \lrcorner & \downarrow{\scriptstyle 1_C} \\ A & \xrightarrow[h]{} & C \end{array},$$

the identity $h' = h$ holds. Therefore, $c = F(h)(a) = F(h')(a) = c'$ and $\lambda_{F,C}$ is injective.

STEP 23. IF $e\colon C \to C'$ IS IN E, THE IMAGE OF $\widehat{F}(e)$ CONTAINS THE IMAGE OF $\lambda_{F,C'}$. Given a morphism $e\colon C \to C'$ in E and an element $c' \in F(C')$, we shall show that there exists an element $x \in \widehat{F}(C)$ such that $\widehat{F}(e)(x) = \lambda_{F,C'}(c')$. This element x may be given by $[(1,(e),(1_{C'}),1_C,e,1_C,c')]$ whose morphisms can be displayed as in the following diagram.

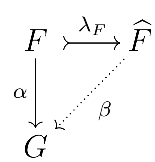


We thus need to show that the identity $[(1,(e),(1_{C'}),1_C,e,e,c')] = \lambda_{F,C'}(c')$ holds. Using Step 21, we can immediately see that this is the case as attested by the span

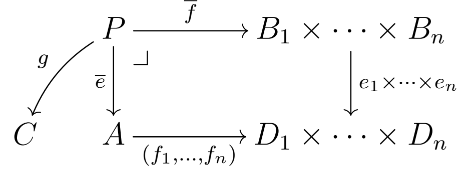


together with the element $c' \in F(C')$.

STEP 24. EXTENSION PROPERTY ALONG $\lambda_F\colon F \rightarrowtail \widehat{F}$. This step is a preparation step for the regular case (i.e., for the proof of Theorem 2.2), but holds in the general case. Let $F, G\colon \mathcal{C} \rightrightarrows \mathsf{Set}$ be two finite limit preserving functors such that, for each $e \in E$, $G(e)$ is a surjective function. Then, for any natural transformation $\alpha\colon F \to G$, there exists a natural transformation $\beta\colon \widehat{F} \to G$ such that the triangle



commutes. For each morphism $e\colon C \to C'$ in E, using the axiom of choice, we choose a section for $G(e)$, i.e., a function $s_e\colon G(C') \to G(C)$ such that $G(e)s_e = 1_{G(C')}$. Given an object $C \in \mathcal{C}$, we define the function $\beta_C\colon \widehat{F}(C) \to G(C)$ as follows. Given a 7-tuple $(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)$ representing an element of $\widehat{F}(C)$, whose morphisms can be displayed as in the diagram

 ,

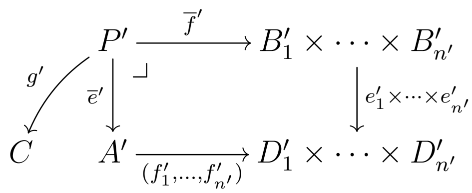
we know that, since G preserves finite limits, the n-tuple

$$(s_{e_i}(G(f_i)(\alpha_A(a))))_{1\leqslant i\leqslant n}$$

represents an element of $G(B_1 \times \cdots \times B_n)$. This element is sent by $G(e_1 \times \cdots \times e_n)$ to the n-tuple $(G(f_i)(\alpha_A(a)))_{1\leqslant i\leqslant n}$ in $G(D_1 \times \cdots \times D_n)$. Therefore, since G preserves finite limits, there exists a unique element $p \in G(P)$ such that $G(\overline{e})(p) = \alpha_A(a)$ and $G(\overline{f})(p) = (s_{e_i}(G(f_i)(\alpha_A(a))))_{1\leqslant i\leqslant n}$. We define

$$\beta_C([(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a)]) = G(g)(p).$$

Let us show that this function is well-defined. If $(n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',g',a')$ is another 7-tuple, whose morphisms can be displayed as in the diagram

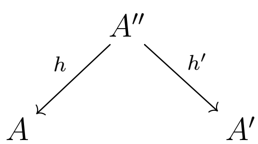


,

such that

$$(n,(e_i)_i,(f_i)_i,\overline{f},\overline{e},g,a) \cong_{F,C} (n',(e'_{i'})_{i'},(f'_{i'})_{i'},\overline{f}',\overline{e}',g',a'),$$

let us prove that $G(g)(p) = G(g')(p')$ where $p' \in G(P')$ is defined analogously to p. There exists a span



together with an element $a'' \in F(A'')$ such that $F(h)(a'') = a$, $F(h')(a'') = a'$ and, using the notation of Step 10, $g\,(h\overline{e}'',(k_i)_{1\leqslant i\leqslant n})\,m = g'\,(h'\overline{e}'',(k'_{i'})_{1\leqslant i'\leqslant n'})\,m$. For each $i \in \{1,\ldots,n\}$, we have

$$G(f_ih)(\alpha_{A''}(a'')) = G(f_i)(\alpha_A(F(h)(a'')) = G(f_i)(\alpha_A(a)) = G(e_i)(s_{e_i}(G(f_i)(\alpha_A(a))))$$

and similarly, for each $i' \in \{1,\ldots,n'\}$, we have

$$G(f'_{i'}h')(\alpha_{A''}(a'')) = G(e'_{i'})(s_{e'_{i'}}(G(f'_{i'})(\alpha_{A'}(a')))).$$

Again since G preserves finite limits, this implies that there exists a unique element $p'' \in G(P'')$ such that

- $G(\overline{e}'')(p'') = \alpha_{A''}(a'')$;
- $G(k_i)(p'') = s_{e_i}(G(f_i)(\alpha_A(a)))$ for each $i \in \{1,\ldots,n\}$;

- $G(k'_{i'})(p'') = s_{e'_{i'}}(G(f'_{i'})(\alpha_{A'}(a')))$ for each $i' \in \{1, \dots, n'\}$.

For each $(i, i') \in \mathcal{X}$, we know that

$$\begin{aligned} G(k_i)(p'') &= s_{e_i}(G(f_i)(\alpha_A(a))) \\ &= s_{e_i}(\alpha_{D_i}(F(f_i)(a))) \\ &= s_{e'_{i'}}(\alpha_{D'_{i'}}(F(f'_{i'})(a'))) \\ &= s_{e'_{i'}}(G(f'_{i'})(\alpha_{A'}(a'))) \\ &= G(k'_{i'})(p''). \end{aligned}$$

Similarly, $G(k_{i_1})(p'') = G(k_{i_2})(p'')$ holds for each $(i_1, i_2) \in \mathcal{Y}$ and $G(k'_{i'_1})(p'') = G(k'_{i'_2})(p'')$ holds for each $(i'_1, i'_2) \in \mathcal{Y}'$. Therefore, since G preserves finite limits, there exists a unique element $p''' \in G(M)$ such that $G(m)(p''') = p''$. We can now compute

$$\begin{aligned} G(g)(p) &= G\left(g\left(h\overline{e}'', (k_i)_{1\leqslant i\leqslant n}\right)\right)(p'') \\ &= G\left(g\left(h\overline{e}'', (k_i)_{1\leqslant i\leqslant n}\right) m\right)(p''') \\ &= G\left(g'\left(h'\overline{e}'', (k'_{i'})_{1\leqslant i'\leqslant n'}\right) m\right)(p''') \\ &= G\left(g'\left(h'\overline{e}'', (k'_{i'})_{1\leqslant i'\leqslant n'}\right)\right)(p'') \\ &= G(g')(p') \end{aligned}$$

proving that the function β_C is well-defined.

Given a morphism $u\colon C \to C'$ and using the above notation, we have, for each element $[(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)]$ of $\widehat{F}(C)$,

$$\begin{aligned} \beta_{C'}(\widehat{F}(u)([(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)])) &= \beta_{C'}([(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, ug, a)]) \\ &= G(ug)(p) \\ &= G(u)(\beta_C([(n, (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)])). \end{aligned}$$

This shows that the functions β_C's form a natural transformation $\beta\colon \widehat{F} \to G$. We conclude this step by remarking that, for each object $C \in \mathcal{C}$ and each element $c \in F(C)$, the identities

$$\beta_C(\lambda_{F,C}(c)) = \beta_C([(0, \varnothing, \varnothing, !_C, 1_C, 1_C, c)]) = G(1_C)(\alpha_C(c)) = \alpha_C(c)$$

hold.

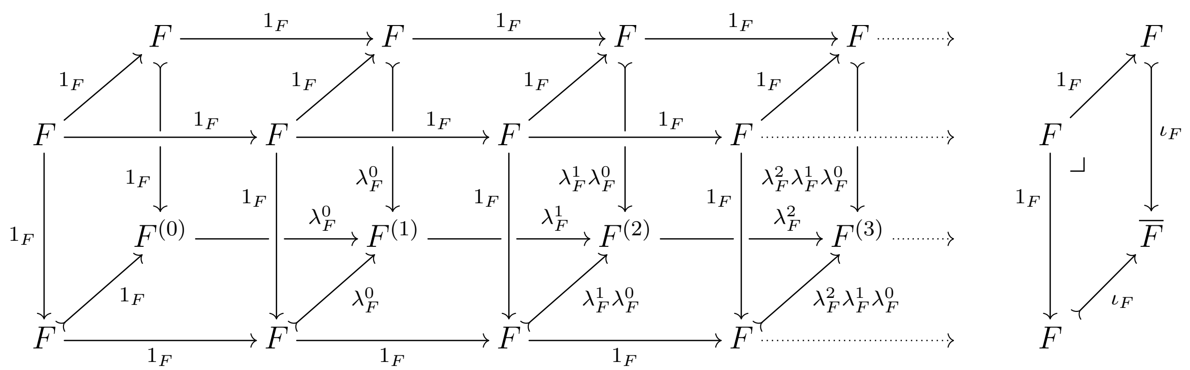
Step 25. The monomorphism $\iota_F\colon F \rightarrowtail \overline{F}$. Since Step 23 does not give surjectivity of $\widehat{F}(e)$ for $e \in E$, but only surjectivity 'relative to the elements in the image of $\lambda_{F,C'}$', we need to use this construction $\lambda_F\colon F \rightarrowtail \widehat{F}$ infinitely many times. Given a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$, we thus construct by (ordinary) induction a sequence

$$F^{(0)} \overset{\lambda_F^0}{\rightarrowtail} F^{(1)} \overset{\lambda_F^1}{\rightarrowtail} F^{(2)} \overset{\lambda_F^2}{\rightarrowtail} F^{(3)} \overset{\lambda_F^3}{\rightarrowtail} \cdots$$

of finite limit preserving functors $F^{(n)}\colon \mathcal{C} \to \mathsf{Set}$ and monomorphisms $\lambda_F^n\colon F^{(n)} \rightarrowtail F^{(n+1)}$ in $\mathsf{Set}^{\mathcal{C}}$ as follows:

- $F^{(0)} = F$;
- if $F^{(n)}$ is constructed, $\lambda^n_F \colon F^{(n)} \rightarrowtail F^{(n+1)}$ is defined as $\lambda_{F^{(n)}} \colon F^{(n)} \rightarrowtail \widehat{F^{(n)}}$.

We now consider the colimit $\overline{F}$ in $\mathsf{Set}^{\mathcal{C}}$ of this filtered diagram. Since filtered colimits commute with finite limits in Set, and since the colimit $\overline{F}$ is constructed componentwise, the functor $\overline{F} \colon \mathcal{C} \to \mathsf{Set}$ also preserves finite limits. Moreover, the canonical natural transformation $F^{(0)} \to \overline{F}$, denoted by $\iota_F \colon F \rightarrowtail \overline{F}$, is also a monomorphism in $\mathsf{Set}^{\mathcal{C}}$ since the filtered colimit of the pullbacks in the left part of the diagram



is the pullback of the respective filtered colimits, represented in the right part of the diagram.

To fix notation, let us recall that, given an object $C \in \mathcal{C}$, the set $\overline{F}(C)$ is constructed as the quotient

$$\overline{F}(C) = \left(\coprod_{n \in \mathbb{N}} F^{(n)}(C) \right) / \cong^{\infty}_{F,C}$$

of the disjoint union of the sets $F^{(n)}(C)$ by the equivalence relation $\cong^{\infty}_{F,C}$ defined as follows. We denote an element in this disjoint union by (n, c) where $n \in \mathbb{N}$ and $c \in F^{(n)}(C)$. Two such elements (n, c) and (n', c') satisfy the relation $(n, c) \cong^{\infty}_{F,C} (n', c')$ if and only if

$$\begin{cases} \lambda^{n'-1}_{F,C}(\cdots(\lambda^{n}_{F,C}(c))\cdots) = c' & \text{if } n < n' \\ c = c' & \text{if } n = n' \\ \lambda^{n-1}_{F,C}(\cdots(\lambda^{n'}_{F,C}(c'))\cdots) = c & \text{if } n > n'. \end{cases}$$

An element (n, c) in the above disjoint union represents the element denoted by $[(n, c)]$ in $\overline{F}(C)$. For a morphism $u \colon C \to C'$ in $\mathcal{C}$, this element is mapped by $\overline{F}(u)$ to

$$\overline{F}(u)([(n, c)]) = [(n, F^{(n)}(u)(c))].$$

The C-component of the monomorphism ι_F is simply the injection $\iota_{F,C} \colon F(C) \rightarrowtail \overline{F}(C)$ defined by $\iota_{F,C}(c) = [(0, c)]$ for each $c \in F(C)$.

Step 26. $\overline{F}(e)$ is surjective for all $e \in E$. We shall now prove that, given a finite limit preserving functor $F\colon \mathcal{C} \to \mathsf{Set}$ and a morphism $e\colon C \to C'$ in E, the function $\overline{F}(e)\colon \overline{F}(C) \to \overline{F}(C')$ is surjective. Let $[(n, c')]$ be an element of $\overline{F}(C')$ with $n \in \mathbb{N}$ and $c' \in F^{(n)}(C')$. Using Step 23, there exists an element $x \in F^{(n+1)}(C)$ such that $F^{(n+1)}(e)(x) = \lambda^n_{F,C'}(c')$. We therefore have

$$\overline{F}(e)([(n+1, x)]) = [(n+1, F^{(n+1)}(e)(x))] = [(n+1, \lambda^n_{F,C'}(c'))] = [(n, c')],$$

proving the surjectivity of $\overline{F}(e)$.

Step 27. Extension property along $\iota_F\colon F \rightarrowtail \overline{F}$. This step is a preparation step for the regular case (i.e., for the proof of Theorem 2.2), but holds in the general case. Let $F, G\colon \mathcal{C} \rightrightarrows \mathsf{Set}$ be two finite limit preserving functors such that, for each $e \in E$, $G(e)$ is a surjective function. Then, for any natural transformation $\alpha\colon F \to G$, there exists a natural transformation $\beta\colon \overline{F} \to G$ such that the triangle

$$\begin{array}{ccc} F & \overset{\iota_F}{\rightarrowtail} & \overline{F} \\ {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\beta} & \\ G & & \end{array}$$

commutes. This fact is an immediate consequence of Step 24 and the colimit definition of $\overline{F}$ in Step 25.

Step 28. The natural transformation $\iota_{\mathcal{C}(C,-)}\colon \mathcal{C}(C,-) \rightarrowtail \overline{\mathcal{C}(C,-)}$. Given an object $C \in \mathcal{C}$, it is well-known and routine to prove that the representable functor $\mathcal{C}(C,-)\colon \mathcal{C} \to \mathsf{Set}$ preserves finite limits. We can thus apply the previous constructions and results to the case $F = \mathcal{C}(C,-)$. In particular, we get a monomorphism $\iota_{\mathcal{C}(C,-)}\colon \mathcal{C}(C,-) \rightarrowtail \overline{\mathcal{C}(C,-)}$ in $\mathsf{Set}^{\mathcal{C}}$.

Step 29. For $e\colon C' \to C$, if $\overline{\mathcal{C}(C,-)}(e)(x) = \iota_{\mathcal{C}(C,-),C}(1_C)$, then $e \in E$. Given a morphism $e\colon C' \to C$ in $\mathcal{C}$, $\iota_{\mathcal{C}(C,-),C}(1_C)$ is an element of $\overline{\mathcal{C}(C,-)}(C)$. If this element is in the image of $\overline{\mathcal{C}(C,-)}(e)$, i.e., if $\overline{\mathcal{C}(C,-)}(e)(x) = \iota_{\mathcal{C}(C,-),C}(1_C)$ for some $x \in \overline{\mathcal{C}(C,-)}(C')$, we now show that $e \in E$. We know that x is written in the form $x = [(n, c')]$ for some $n \in \mathbb{N}$ and some $c' \in \mathcal{C}(C,-)^{(n)}(C')$, hence

$$\iota_{\mathcal{C}(C,-),C}(1_C) = \overline{\mathcal{C}(C,-)}(e)(x) = \overline{\mathcal{C}(C,-)}(e)([(n,c')]) = [(n, \mathcal{C}(C,-)^{(n)}(e)(c'))].$$

We thus need to prove by induction on n that

$$\iota_{\mathcal{C}(C,-),C}(1_C) = [(n, \mathcal{C}(C,-)^{(n)}(e)(c'))] \tag{36}$$

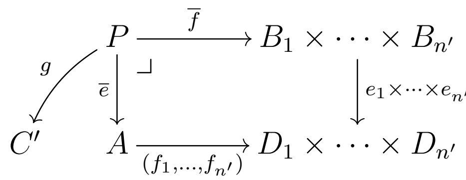
implies that $e \in E$, for any $n \in \mathbb{N}$, any morphism $e\colon C' \to C$ and any element $c' \in \mathcal{C}(C,-)^{(n)}(C')$.

If $n = 0$, we have $\mathcal{C}(C,-)^{(0)} = \mathcal{C}(C,-)$ and thus c' is a morphism $C \to C'$. The identity (36) becomes $[(0, 1_C)] = [(0, ec')]$, which means $1_C = ec'$ and therefore e is a split epimorphism. Since E contains split epimorphisms (by (Id) and (SRightCancP)), we know that $e \in E$.

Let us suppose now that our thesis holds for $n \geqslant 0$ and let us prove it also holds for $n+1$. Our hypothesis (36) becomes in this case $[(0, 1_C)] = [(n+1, \mathcal{C}(C,-)^{(n+1)}(e)(c'))]$. The element $c' \in \mathcal{C}(C,-)^{(n+1)}(C') = \mathcal{C}\widehat{(C,-)^{(n)}}(C')$ can be represented by a 7-tuple as

$$c' = [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)]$$

where the morphisms involved can be displayed as in the diagram



and where a is an element of $\mathcal{C}(C,-)^{(n)}(A)$. Using Step 15, we can suppose without loss of generality that the implication

$$(e_i = e_{i'} \wedge F(f_i)(a) = F(f_{i'})(a)) \Longrightarrow i = i'$$

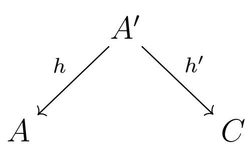
holds for indices $i, i' \in \{1, \ldots, n'\}$. Our hypothesis can now be rewritten as

$$\begin{aligned}[(0, 1_C)] &= [(n+1, \mathcal{C}(C,-)^{(n+1)}(e)(c'))] \\ &= [(n+1, \mathcal{C}\widehat{(C,-)^{(n)}}(e)([(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a)]))] \\ &= [(n+1, [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, eg, a)])]\end{aligned}$$

which means

$$\lambda^n_{\mathcal{C}(C,-),C}(\cdots(\lambda^0_{\mathcal{C}(C,-),C}(1_C))\cdots) = [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, eg, a)].$$

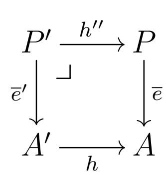
Since $\lambda^n_{\mathcal{C}(C,-),C} = \lambda_{\mathcal{C}(C,-)^{(n)},C}$ holds by definition, we know from Step 21 that there exists a span



together with an element $a' \in \mathcal{C}(C,-)^{(n)}(A')$ such that $\mathcal{C}(C,-)^{(n)}(h)(a') = a$,

$$\mathcal{C}(C,-)^{(n)}(h')(a') = \lambda^{n-1}_{\mathcal{C}(C,-),C}(\cdots(\lambda^0_{\mathcal{C}(C,-),C}(1_C))\cdots) \quad (37)$$

and, considering the pullback



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the identity $h'\overline{e}' = egh''$ holds. Since E is closed under finite products (by (Id), (ClComp) and (StPb)), we know that $e_1 \times \cdots \times e_{n'} \in E$. Since E is stable under pullbacks (StPb), this implies that $\overline{e} \in E$ and so $\overline{e}' \in E$. Moreover, $[(n, a')]$ is an element of $\overline{\mathcal{C}(C, -)}(A')$ and we know, by (37), that

$$\iota_{\mathcal{C}(C,-),C}(1_C) = [(0, 1_C)] = [(n, \mathcal{C}(C, -)^{(n)}(h')(a'))].$$

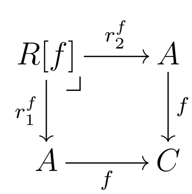
Using our induction hypothesis, this implies that $h' \in E$. Since $h', \overline{e}' \in E$ and E is closed under composition (ClComp), we deduce from this that $h'\overline{e}' = egh'' \in E$. Since E has the strong right cancellation property (SRightCancP), this implies that $e \in E$, concluding this step.

Step 30. Conclusion of the proof of Theorem 2.1. We are now able to construct a small full subcategory $\mathcal{D}$ of $\mathsf{Set}^{\mathcal{C}}$ satisfying properties (i), (ii) and (iii) described in Step 2 in the case the proof of Theorem 2.1 is concerned. We define in this case $\mathcal{D}$ to be the full subcategory of $\mathsf{Set}^{\mathcal{C}}$ made of the functors $\overline{\mathcal{C}(C, -)}$ for each object $C \in \mathcal{C}$. Since $\mathcal{C}$ is small, so is $\mathcal{D}$. Property (i), requiring that each functor $F \in \mathcal{D}$ preserves finite limits, has been proved in Step 25. Property (ii), requiring that for each $e \in E$ and each $F \in \mathcal{D}$ the function $F(e)$ is surjective, is the content of Step 26. For each object $C \in \mathcal{C}$, the natural transformation $\iota_C \colon \mathcal{C}(C, -) \to F_C$ mentioned in property (iii) is given by $\iota_{\mathcal{C}(C,-)} \colon \mathcal{C}(C, -) \rightarrowtail \overline{\mathcal{C}(C, -)}$. Step 28 shows that property (iii)(a) holds, i.e., that $\iota_{\mathcal{C}(C,-)}$ is a monomorphism in $\mathsf{Set}^{\mathcal{C}}$. By the Yoneda Lemma, property (iii)(b) can be reformulated exactly as Step 29. Notice that, in view of Step 27, property (iii)(d) is also satisfied, although it is not require here. This concludes the proof of Theorem 2.1. It remains to conclude also the proof of Theorem 2.2.

Step 31. The set $F_c^C(D)$. In order to construct the morphisms ρ^1, ρ^2 of (iii)(c) from Step 2, we need to consider an explicit description of the cokernel pair of $\iota_{\mathcal{C}(C,-)}$ in the category of finite limit preserving functors $\mathcal{C} \to \mathsf{Set}$ and their natural transformations. Since it is not harder to do this construction more generally, we are going to construct this cokernel pair for any natural transformation $\mathcal{C}(C, -) \to F$ for a finite limit preserving functor F and an object C (see the remark at the end of Step 40). By the Yoneda lemma, such a natural transformation corresponds to an element of $F(C)$. This construction will take several steps.

Therefore, given a finite limit preserving functor $F \colon \mathcal{C} \to \mathsf{Set}$, an object $C \in \mathcal{C}$ and an element $c \in F(C)$, we now construct a functor $F_c^C \colon \mathcal{C} \to \mathsf{Set}$ as follows. Given an object $D \in \mathcal{C}$, we consider the set of 5-tuples (f, r_1^f, r_2^f, g, a) where

- $f \colon A \to C$ is a morphism;
- r_1^f and r_2^f are the projections of the kernel pair of f, i.e.,



is a pullback diagram;

- $g\colon R[f] \to D$ is a morphism;
- $a \in F(A)$ is an element such that $F(f)(a) = c$.

We define $F_c^C(D)$

$$F_c^C(D) = \left\{ (f, r_1^f, r_2^f, g, a \in F(A)) \;\middle|\; \begin{array}{ccccc} & & R[f] & \xrightarrow{r_2^f} & A \\ & \swarrow^{g} & {\scriptstyle r_1^f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ D & & A & \xrightarrow[f]{} & C \end{array} \text{ and } F(f)(a) = c \right\} / \cong_{F,C,c,D}$$

to be the quotient of this set of such 5-tuples by the equivalence relation $\cong_{F,C,c,D}$ defined as follows. Given two such 5-tuples (f, r_1^f, r_2^f, g, a) and $(f', r_1^{f'}, r_2^{f'}, g', a')$, whose morphisms can be displayed as in the diagrams

$$D \xleftarrow{g} R[f] \overset{r_1^f}{\underset{r_2^f}{\rightrightarrows}} A \xrightarrow{f} C \quad \text{and} \quad D \xleftarrow{g'} R[f'] \overset{r_1^{f'}}{\underset{r_2^{f'}}{\rightrightarrows}} A' \xrightarrow{f'} C \quad ,$$

one has

$$(f, r_1^f, r_2^f, g, a) \cong_{F,C,c,D} (f', r_1^{f'}, r_2^{f'}, g', a')$$

if and only if there exists a span

$$\begin{array}{ccc} & A'' & \\ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle h'} \\ A & & A' \end{array}$$

in $\mathcal{C}$ together with an element $a'' \in F(A'')$ such that

- $F(h)(a'') = a$;
- $F(h')(a'') = a'$;
- $fh = f'h'$;
- considering the kernel pair $r_1^{fh}, r_2^{fh}\colon R[fh] \rightrightarrows A''$ of $fh = f'h'$ and the unique morphisms $\overline{h}\colon R[fh] \to R[f]$ and $\overline{h}'\colon R[fh] \to R[f']$ making the diagrams

$$\begin{array}{ccc} R[fh] & \overset{r_1^{fh}}{\underset{r_2^{fh}}{\rightrightarrows}} & A'' \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle h} \\ R[f] & \overset{r_1^f}{\underset{r_2^f}{\rightrightarrows}} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} R[fh] & \overset{r_1^{fh}}{\underset{r_2^{fh}}{\rightrightarrows}} & A'' \\ {\scriptstyle \overline{h}'}\downarrow & & \downarrow{\scriptstyle h'} \\ R[f'] & \overset{r_1^{f'}}{\underset{r_2^{f'}}{\rightrightarrows}} & A' \end{array}$$

(reasonably) commutative (in the sense $hr_1^{fh} = r_1^f \overline{h}$, $hr_2^{fh} = r_2^f \overline{h}$, $h'r_1^{fh} = r_1^{f'} \overline{h}'$ and $h'r_2^{fh} = r_2^{f'} \overline{h}'$), the diagram

$$\begin{array}{ccc} R[fh] & \xrightarrow{\overline{h}'} & R[f'] \\ {\scriptstyle \overline{h}} \downarrow & & \downarrow {\scriptstyle g'} \\ R[f] & \xrightarrow[g]{} & D \end{array}$$

also commutes.

STEP 32. $\cong_{F,C,c,D}$ IS AN EQUIVALENCE RELATION. To see that the relation $\cong_{F,C,c,D}$ is symmetric, it suffices to exchange the roles of h and h'. To see that $\cong_{F,C,c,D}$ is reflexive, one can consider, for a 5-tuple (f, r_1^f, r_2^f, g, a), the span

$$A \xleftarrow{1_A} A \xrightarrow{1_A} A$$

together with the element $a \in F(A)$ (where A is the domain of f) to prove

$$(f, r_1^f, r_2^f, g, a) \cong_{F,C,c,D} (f, r_1^f, r_2^f, g, a).$$

It remains to prove that $\cong_{F,C,c,D}$ is transitive. We consider, for each $j \in \{1, 2, 3\}$, a 5-tuple $(f_j, r_1^{f_j}, r_2^{f_j}, g_j, a_j)$ whose morphisms can be displayed as in the diagram

$$D \xleftarrow{g_j} R[f_j] \underset{r_2^{f_j}}{\overset{r_1^{f_j}}{\rightrightarrows}} A_j \xrightarrow{f_j} C \quad .$$

We suppose that

$$(f_1, r_1^{f_1}, r_2^{f_1}, g_1, a_1) \cong_{F,C,c,D} (f_2, r_1^{f_2}, r_2^{f_2}, g_2, a_2)$$

and

$$(f_2, r_1^{f_2}, r_2^{f_2}, g_2, a_2) \cong_{F,C,c,D} (f_3, r_1^{f_3}, r_2^{f_3}, g_3, a_3)$$

and we shall show that

$$(f_1, r_1^{f_1}, r_2^{f_1}, g_1, a_1) \cong_{F,C,c,D} (f_3, r_1^{f_3}, r_2^{f_3}, g_3, a_3). \tag{38}$$

We thus know there exist two spans

$$A_1 \xleftarrow{h_1} A \xrightarrow{h_2} A_2 \quad \text{and} \quad A_2 \xleftarrow{h_2'} A' \xrightarrow{h_3'} A_3$$

together with two elements $a \in F(A)$ and $a' \in F(A')$ satisfying

- $F(h_1)(a) = a_1$, $F(h_2)(a) = a_2 = F(h_2')(a')$ and $F(h_3')(a') = a_3$;
- $f_1h_1 = f_2h_2$ and $f_2h_2' = f_3h_3'$;
- considering the kernel pairs $r_1^{f_1h_1}, r_2^{f_1h_1} \colon R[f_1h_1] \rightrightarrows A$ and $r_1^{f_2h_2'}, r_2^{f_2h_2'} \colon R[f_2h_2'] \rightrightarrows A'$ of f_1h_1 and f_2h_2' respectively, and the unique morphisms $\overline{h}_1$, $\overline{h}_2$, $\overline{h}_2'$ and $\overline{h}_3'$ making the four diagrams

$$\begin{array}{ccc} R[f_1h_1] & \underset{r_2^{f_1h_1}}{\overset{r_1^{f_1h_1}}{\rightrightarrows}} & A \\ {\scriptstyle \overline{h}_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ R[f_1] & \underset{r_2^{f_1}}{\overset{r_1^{f_1}}{\rightrightarrows}} & A_1 \end{array} \qquad \begin{array}{ccc} R[f_1h_1] & \underset{r_2^{f_1h_1}}{\overset{r_1^{f_1h_1}}{\rightrightarrows}} & A \\ {\scriptstyle \overline{h}_2}\downarrow & & \downarrow{\scriptstyle h_2} \\ R[f_2] & \underset{r_2^{f_2}}{\overset{r_1^{f_2}}{\rightrightarrows}} & A_2 \end{array}$$

$$\begin{array}{ccc} R[f_2h_2'] & \underset{r_2^{f_2h_2'}}{\overset{r_1^{f_2h_2'}}{\rightrightarrows}} & A' \\ {\scriptstyle \overline{h}_2'}\downarrow & & \downarrow{\scriptstyle h_2'} \\ R[f_2] & \underset{r_2^{f_2}}{\overset{r_1^{f_2}}{\rightrightarrows}} & A_2 \end{array} \qquad \begin{array}{ccc} R[f_2h_2'] & \underset{r_2^{f_2h_2'}}{\overset{r_1^{f_2h_2'}}{\rightrightarrows}} & A' \\ {\scriptstyle \overline{h}_3'}\downarrow & & \downarrow{\scriptstyle h_3'} \\ R[f_3] & \underset{r_2^{f_3}}{\overset{r_1^{f_3}}{\rightrightarrows}} & A_3 \end{array}$$

(reasonably) commutative, the diagrams

$$\begin{array}{ccc} R[f_1h_1] & \xrightarrow{\overline{h}_2} & R[f_2] \\ {\scriptstyle \overline{h}_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ R[f_1] & \xrightarrow[g_1]{} & D \end{array} \quad \text{and} \quad \begin{array}{ccc} R[f_2h_2'] & \xrightarrow{\overline{h}_3'} & R[f_3] \\ {\scriptstyle \overline{h}_2'}\downarrow & & \downarrow{\scriptstyle g_3} \\ R[f_2] & \xrightarrow[g_2]{} & D \end{array}$$

also commute.

We consider the pullback square

$$\begin{array}{ccc} P & \xrightarrow{k'} & A' \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle h_2'} \\ A & \xrightarrow[h_2]{} & A_2 \end{array}$$

and, since F preserves finite limits and $F(h_2)(a) = F(h_2')(a')$, we know that there exists a unique element $p \in F(P)$ such that $F(k)(p) = a$ and $F(k')(p) = a'$. Let us show that the span

$$\begin{array}{ccc} & P & \\ {\scriptstyle h_1k}\swarrow & & \searrow{\scriptstyle h_3'k'} \\ A_1 & & A_3 \end{array}$$

together with the element p is a witness of the relation (38). We already know that the identities

$$F(h_1k)(p) = F(h_1)(a) = a_1,$$
$$F(h_3'k')(p) = F(h_3')(a') = a_3$$

and

$$f_1h_1k = f_2h_2k = f_2h_2'k' = f_3h_3'k'$$

hold. For the last condition to check, we consider the kernel pair $r_1^{f_1h_1k}, r_2^{f_1h_1k} \colon R[f_1h_1k] \rightrightarrows P$ of f_1h_1k and the unique morphisms l and l' making the diagrams

$$\begin{array}{ccc} R[f_1h_1k] & \overset{r_1^{f_1h_1k}}{\underset{r_2^{f_1h_1k}}{\rightrightarrows}} & P \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle k} \\ R[f_1h_1] & \overset{r_1^{f_1h_1}}{\underset{r_2^{f_1h_1}}{\rightrightarrows}} & A \\ {\scriptstyle \overline{h}_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ R[f_1] & \overset{r_1^{f_1}}{\underset{r_2^{f_1}}{\rightrightarrows}} & A_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R[f_1h_1k] & \overset{r_1^{f_1h_1k}}{\underset{r_2^{f_1h_1k}}{\rightrightarrows}} & P \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle k'} \\ R[f_2h_2'] & \overset{r_1^{f_2h_2'}}{\underset{r_2^{f_2h_2'}}{\rightrightarrows}} & A' \\ {\scriptstyle \overline{h}_3'}\downarrow & & \downarrow{\scriptstyle h_3'} \\ R[f_3] & \overset{r_1^{f_3}}{\underset{r_2^{f_3}}{\rightrightarrows}} & A_3 \end{array}$$

(reasonably) commute, and we must show that $g_1\overline{h}_1l = g_3\overline{h}_3'l'$. For each $i \in \{1, 2\}$, we can compute

$$r_i^{f_2}\overline{h}_2l = h_2r_i^{f_1h_1}l = h_2kr_i^{f_1h_1k} = h_2'k'r_i^{f_1h_1k} = h_2'r_i^{f_2h_2'}l' = r_i^{f_2}\overline{h}_2'l',$$

showing that $\overline{h}_2l = \overline{h}_2'l'$. We can then conclude this step by the computation

$$g_1\overline{h}_1l = g_2\overline{h}_2l = g_2\overline{h}_2'l' = g_3\overline{h}_3'l'.$$

Step 33. The functor F_c^C. Given a finite limit preserving functor $F \colon \mathcal{C} \to \mathsf{Set}$, an object $C \in \mathcal{C}$ and an element $c \in F(C)$, we have constructed, for each object $D \in \mathcal{C}$, a set $F_c^C(D)$. We now turn this construction into a functor $F_c^C \colon \mathcal{C} \to \mathsf{Set}$. Given a morphism $u \colon D \to D'$ in $\mathcal{C}$, we define the function $F_c^C(u) \colon F_c^C(D) \to F_c^C(D')$ as

$$F_c^C(u)([(f, r_1^f, r_2^f, g, a)]) = [(f, r_1^f, r_2^f, ug, a)]$$

for each element $[(f, r_1^f, r_2^f, g, a)]$ of $F_c^C(D)$ (represented by the 5-tuple (f, r_1^f, r_2^f, g, a)). It is obvious that this function is well-defined, i.e., that the implication

$$\begin{aligned} &(f, r_1^f, r_2^f, g, a) \cong_{F,C,c,D} (f', r_1^{f'}, r_2^{f'}, g', a') \\ \implies\ &(f, r_1^f, r_2^f, ug, a) \cong_{F,C,c,D'} (f', r_1^{f'}, r_2^{f'}, ug', a') \end{aligned}$$

holds and that this gives rise to a functor $F_c^C \colon \mathcal{C} \to \mathsf{Set}$.

STEP 34. F^C_c PRESERVES THE TERMINAL OBJECT. In order to apply the construction $\iota_{F^C_c} \colon F^C_c \rightarrowtail \overline{F^C_c}$, we shall now prove that this functor F^C_c also preserves finite limits. As for $\widehat{F}$, it is enough to show that F^C_c preserves the terminal object 1 of $\mathcal{C}$ and pullbacks. Let us start with the terminal object. We must thus show that $F^C_c(1)$ is a singleton set. The element $[(1_C, 1_C, 1_C, !_C, c)] \in F^C_c(1)$, whose morphisms can be displayed as in the diagram

$$1 \xleftarrow{!_C} C \underset{1_C}{\overset{1_C}{\rightrightarrows}} C \xrightarrow{1_C} C \quad ,$$

shows that $F^C_c(1)$ is not empty. Let $[(f, r^f_1, r^f_2, g, a)]$ and $[(f', r^{f'}_1, r^{f'}_2, g', a')]$ be two elements of $F^C_c(1)$ whose morphisms can be displayed as in the diagrams

$$1 \xleftarrow{g} R[f] \underset{r^f_2}{\overset{r^f_1}{\rightrightarrows}} A \xrightarrow{f} C \quad \text{and} \quad 1 \xleftarrow{g'} R[f'] \underset{r^{f'}_2}{\overset{r^{f'}_1}{\rightrightarrows}} A' \xrightarrow{f'} C \quad .$$

We consider the pullback square

$$\begin{array}{ccc} A'' & \xrightarrow{h'} & A' \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle f'} \\ A & \xrightarrow[f]{} & C \end{array}$$

and, since F preserves finite limits and $F(f)(a) = c = F(f')(a')$, we know there exists a unique element $a'' \in F(A'')$ such that $F(h)(a'') = a$ and $F(h')(a'') = a'$. The span

$$A \xleftarrow{h} A'' \xrightarrow{h'} A'$$

together with the element a'' is a witness of the relation

$$(f, r^f_1, r^f_2, g, a) \cong_{F,C,c,1} (f', r^{f'}_1, r^{f'}_2, g', a').$$

Indeed, the first three conditions follow immediately from the definitions and the last condition is trivially satisfied since it requires that two parallel morphisms with codomain 1 are equal. This proves that $[(f, r^f_1, r^f_2, g, a)] = [(f', r^{f'}_1, r^{f'}_2, g', a')]$ and so $F^C_c(1)$ is indeed a singleton set.

STEP 35. F^C_c PRESERVES JOINTLY MONOMORPHIC PAIRS OF MORPHISMS. As a preliminary step to prove that F^C_c preserves pullbacks, we show that it preserves jointly monomorphic pairs of morphisms. So let

$$D_1 \xleftarrow{p_1} D \xrightarrow{p_2} D_2$$

be a pair of morphisms satisfying the implication

$$(p_1x = p_1y \wedge p_2x = p_2y) \Longrightarrow x = y$$

for each pair of morphisms $x, y\colon X \rightrightarrows D$ and let us prove that the function

$$F_c^C(D) \xrightarrow{\left(F_c^C(p_1), F_c^C(p_2)\right)} F_c^C(D_1) \times F_c^C(D_2)$$

is injective. We consider two elements $[(f, r_1^f, r_2^f, g, a)]$ and $[(f', r_1^{f'}, r_2^{f'}, g', a')]$ of $F_c^C(D)$ whose morphisms can be displayed as in the diagrams

$$D \xleftarrow{\ g\ } R[f] \underset{r_2^f}{\overset{r_1^f}{\rightrightarrows}} A \xrightarrow{\ f\ } C \quad \text{and} \quad D \xleftarrow{\ g'\ } R[f'] \underset{r_2^{f'}}{\overset{r_1^{f'}}{\rightrightarrows}} A' \xrightarrow{\ f'\ } C$$

and satisfying

$$(f, r_1^f, r_2^f, p_jg, a) \cong_{F,C,c,D_j} (f', r_1^{f'}, r_2^{f'}, p_jg', a') \tag{39}$$

for each $j \in \{1, 2\}$. We shall prove that

$$(f, r_1^f, r_2^f, g, a) \cong_{F,C,c,D} (f', r_1^{f'}, r_2^{f'}, g', a'). \tag{40}$$

For each $j \in \{1, 2\}$, the relation (39) gives us a span

$$A \xleftarrow{\ h_j\ } A''_j \xrightarrow{\ h'_j\ } A'$$

together with an element $a''_j \in F(A''_j)$ such that $F(h_j)(a''_j) = a$, $F(h'_j)(a''_j) = a'$, $fh_j = f'h'_j$ and, considering the kernel pair $r_1^{fh_j}, r_2^{fh_j}\colon R[fh_j] \rightrightarrows A''_j$ of fh_j and the unique morphisms $\overline{h}_j$ and $\overline{h}'_j$ making the diagrams

$$\begin{array}{ccc} R[fh_j] & \underset{r_2^{fh_j}}{\overset{r_1^{fh_j}}{\rightrightarrows}} & A''_j \\ {\scriptstyle \overline{h}_j}\downarrow & & \downarrow{\scriptstyle h_j} \\ R[f] & \underset{r_2^{f}}{\overset{r_1^{f}}{\rightrightarrows}} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} R[fh_j] & \underset{r_2^{fh_j}}{\overset{r_1^{fh_j}}{\rightrightarrows}} & A''_j \\ {\scriptstyle \overline{h}'_j}\downarrow & & \downarrow{\scriptstyle h'_j} \\ R[f'] & \underset{r_2^{f'}}{\overset{r_1^{f'}}{\rightrightarrows}} & A' \end{array}$$

(reasonably) commute, the identity

$$p_jg\overline{h}_j = p_jg'\overline{h}'_j \tag{41}$$

holds. We consider the pullback diagram

$$\begin{array}{ccc} A'' & \xrightarrow{k_2} & A''_2 \\ {\scriptstyle k_1}\downarrow & \lrcorner & \downarrow{\scriptstyle (h_2,h'_2)} \\ A''_1 & \xrightarrow[(h_1,h'_1)]{} & A \times A' \end{array}$$

and, since F preserves finite limits, $F(h_1)(a''_1) = F(h_2)(a''_2)$ and $F(h'_1)(a''_1) = F(h'_2)(a''_2)$, there exists a unique element $a'' \in F(A'')$ such that $F(k_1)(a'') = a''_1$ and $F(k_2)(a'') = a''_2$. Let us show that the span

$$A \xleftarrow{h_1k_1} A'' \xrightarrow{h'_1k_1} A'$$

together with the element a'' is a witness of the relation (40). We already know that $F(h_1k_1)(a'') = F(h_1)(a''_1) = a$, $F(h'_1k_1)(a'') = F(h'_1)(a''_1) = a'$ and $fh_1k_1 = f'h'_1k_1$. For the last condition to check, we consider the kernel pair $r_1^{fh_1k_1}, r_2^{fh_1k_1} \colon R[fh_1k_1] \rightrightarrows A''$ of fh_1k_1 and the unique morphisms $\overline{k}_1$ and $\overline{k}_2$ making the diagrams

$$\begin{array}{ccc} R[fh_1k_1] & \overset{r_1^{fh_1k_1}}{\underset{r_2^{fh_1k_1}}{\rightrightarrows}} & A'' \\ {\scriptstyle \overline{k}_1}\downarrow & & \downarrow{\scriptstyle k_1} \\ R[fh_1] & \overset{r_1^{fh_1}}{\underset{r_2^{fh_1}}{\rightrightarrows}} & A''_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} R[fh_1k_1] & \overset{r_1^{fh_1k_1}}{\underset{r_2^{fh_1k_1}}{\rightrightarrows}} & A'' \\ {\scriptstyle \overline{k}_2}\downarrow & & \downarrow{\scriptstyle k_2} \\ R[fh_2] & \overset{r_1^{fh_2}}{\underset{r_2^{fh_2}}{\rightrightarrows}} & A''_2 \end{array}$$

(reasonably) commute. In particular, $\overline{k}_1$ makes the diagrams

$$\begin{array}{ccc} R[fh_1k_1] & \overset{r_1^{fh_1k_1}}{\underset{r_2^{fh_1k_1}}{\rightrightarrows}} & A'' \\ {\scriptstyle \overline{h}_1\overline{k}_1}\downarrow & & \downarrow{\scriptstyle h_1k_1} \\ R[f] & \overset{r_1^{f}}{\underset{r_2^{f}}{\rightrightarrows}} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} R[fh_1k_1] & \overset{r_1^{fh_1k_1}}{\underset{r_2^{fh_1k_1}}{\rightrightarrows}} & A'' \\ {\scriptstyle \overline{h}'_1\overline{k}_1}\downarrow & & \downarrow{\scriptstyle h'_1k_1} \\ R[f'] & \overset{r_1^{f'}}{\underset{r_2^{f'}}{\rightrightarrows}} & A' \end{array}$$

(reasonably) commute. To conclude this step, it remains to prove that $g\overline{h}_1\overline{k}_1 = g'\overline{h}'_1\overline{k}_1$. Since p_1 and p_2 are jointly monomorphic, we actually need only to prove that $p_1g\overline{h}_1\overline{k}_1 = p_1g'\overline{h}'_1\overline{k}_1$ and $p_2g\overline{h}_1\overline{k}_1 = p_2g'\overline{h}'_1\overline{k}_1$. The first equality follows immediately from the identity (41) for $j = 1$. Since $h_1k_1 = h_2k_2$ and $h'_1k_1 = h'_2k_2$, we know that $\overline{h}_1\overline{k}_1 = \overline{h}_2\overline{k}_2$ and $\overline{h}'_1\overline{k}_1 = \overline{h}'_2\overline{k}_2$. Using (41) for $j = 2$, we can then conclude this step with the computation

$$p_2g\overline{h}_1\overline{k}_1 = p_2g\overline{h}_2\overline{k}_2 = p_2g'\overline{h}'_2\overline{k}_2 = p_2g'\overline{h}'_1\overline{k}_1.$$

STEP 36. F_c^C PRESERVES PULLBACKS. Let us now show that F_c^C preserves pullbacks. We consider a pullback square

$$\begin{array}{ccc} D & \xrightarrow{p_2} & D_2 \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle u_2} \\ D_1 & \xrightarrow[u_1]{} & D_3 \end{array} \tag{42}$$

in $\mathcal{C}$ together with elements $[(f_1, r_1^{f_1}, r_2^{f_1}, g_1, a_1)] \in F_c^C(D_1)$ and $[(f_2, r_1^{f_2}, r_2^{f_2}, g_2, a_2)] \in F_c^C(D_2)$ satisfying

$$F_c^C(u_1)([(f_1, r_1^{f_1}, r_2^{f_1}, g_1, a_1)]) = F_c^C(u_2)([(f_2, r_1^{f_2}, r_2^{f_2}, g_2, a_2)])$$

i.e.,

$$(f_1, r_1^{f_1}, r_2^{f_1}, u_1g_1, a_1) \cong_{F,C,c,D_3} (f_2, r_1^{f_2}, r_2^{f_2}, u_2g_2, a_2). \tag{43}$$

The morphisms involved in these elements can be displayed as in the diagrams

$$D_1 \xleftarrow{g_1} R[f_1] \underset{r_2^{f_1}}{\overset{r_1^{f_1}}{\rightrightarrows}} A_1 \xrightarrow{f_1} C \quad \text{and} \quad D_2 \xleftarrow{g_2} R[f_2] \underset{r_2^{f_2}}{\overset{r_1^{f_2}}{\rightrightarrows}} A_2 \xrightarrow{f_2} C$$

and the relation (43) gives us a span

$$\begin{array}{ccc} & A & \\ {\scriptstyle h_1}\swarrow & & \searrow{\scriptstyle h_2} \\ A_1 & & A_2 \end{array}$$

together with an element $a \in F(A)$ such that $F(h_1)(a) = a_1$, $F(h_2)(a) = a_2$, $f_1h_1 = f_2h_2$ and, considering the kernel pair $r_1^{f_1h_1}, r_2^{f_1h_1} \colon R[f_1h_1] \rightrightarrows A$ of f_1h_1 and the unique morphisms $\overline{h}_1$ and $\overline{h}_2$ making the diagrams

$$\begin{array}{ccc} R[f_1h_1] & \underset{r_2^{f_1h_1}}{\overset{r_1^{f_1h_1}}{\rightrightarrows}} & A \\ {\scriptstyle \overline{h}_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ R[f_1] & \underset{r_2^{f_1}}{\overset{r_1^{f_1}}{\rightrightarrows}} & A_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} R[f_1h_1] & \underset{r_2^{f_1h_1}}{\overset{r_1^{f_1h_1}}{\rightrightarrows}} & A \\ {\scriptstyle \overline{h}_2}\downarrow & & \downarrow{\scriptstyle h_2} \\ R[f_2] & \underset{r_2^{f_2}}{\overset{r_1^{f_2}}{\rightrightarrows}} & A_2 \end{array}$$

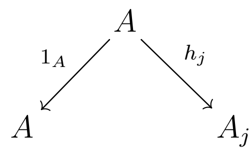
(reasonably) commute, the identity $u_1g_1\overline{h}_1 = u_2g_2\overline{h}_2$ holds. In view of this identity and of the pullback in (42), there exists a unique morphism $g \colon R[f_1h_1] \to D$ such that $p_1g = g_1\overline{h}_1$ and $p_2g = g_2\overline{h}_2$. Since $F(f_1h_1)(a) = F(f_1)(a_1) = c$, we have defined in this way an element $[(f_1h_1, r_1^{f_1h_1}, r_2^{f_1h_1}, g, a)]$ of $F_c^C(D)$ whose morphisms can be displayed as in the diagram

$$D \xleftarrow{g} R[f_1h_1] \underset{r_2^{f_1h_1}}{\overset{r_1^{f_1h_1}}{\rightrightarrows}} A \xrightarrow{f_1h_1} C \quad .$$

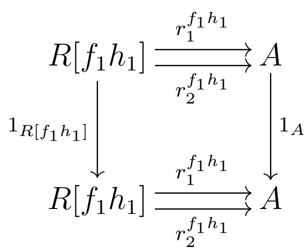
For each $j \in \{1,2\}$, let us prove that the relation

$$(f_1h_1, r_1^{f_1h_1}, r_2^{f_1h_1}, p_jg, a) \cong_{F,C,c,D_j} (f_j, r_1^{f_j}, r_2^{f_j}, g_j, a_j)$$

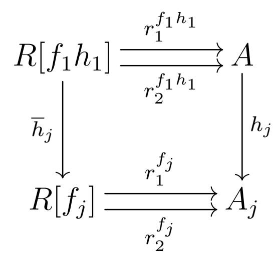
is attested by the span



together with the element $a \in F(A)$. We already know that $F(1_A)(a) = a$, $F(h_j)(a) = a_j$ and $f_1h_1 = f_jh_j$. Since the diagrams

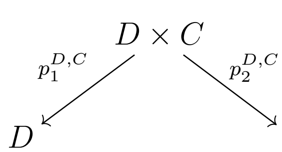


and

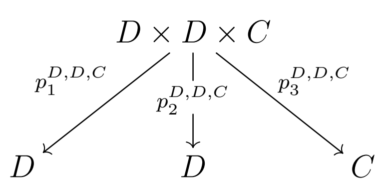


(reasonably) commute, the last condition we need to check reduces to $p_jg = g_j\overline{h}_j$ which holds by definition of g. We have thus constructed an element of $F_c^C(D)$ which is sent by $F_c^C(p_j)$ to $[(f_j, r_1^{f_j}, r_2^{f_j}, g_j, a_j)]$ for each $j \in \{1,2\}$. By Step 35, this element is unique, proving that F_c^C preserves pullbacks. Combining this with Step 34, we have shown that F_c^C preserves finite limits.

STEP 37. THE NATURAL TRANSFORMATIONS $\xi_1^{F,C,c}, \xi_2^{F,C,c} \colon F \rightrightarrows F_c^C$. Given a finite limit preserving functor $F \colon \mathcal{C} \to \mathsf{Set}$, an object $C \in \mathcal{C}$ and an element $c \in F(C)$, we have constructed a finite limit preserving functor $F_c^C \colon \mathcal{C} \to \mathsf{Set}$. Let us now construct two natural transformations $\xi_1^{F,C,c}, \xi_2^{F,C,c} \colon F \rightrightarrows F_c^C$. Given $i \in \{1,2\}$ and an object $D \in \mathcal{C}$, we define the function $\xi_{i,D}^{F,C,c} \colon F(D) \to F_c^C(D)$ as follows. We consider the binary and ternary products



and



and remark that the kernel pair of $p_2^{D,C}$ is given by

$$(p_1^{D,D,C}, p_3^{D,D,C}), (p_2^{D,D,C}, p_3^{D,D,C}) \colon D \times D \times C \rightrightarrows D \times C.$$

For each element $d \in F(D)$, since F preserves finite limits, there is a unique element of $F(D \times C)$, denoted by (d,c), such that $F(p_1^{D,C})((d,c)) = d$ and $F(p_2^{D,C})((d,c)) = c$. For each such element $d \in F(D)$, we then define

$$\xi_{i,D}^{F,C,c}(d) = [(p_2^{D,C}, (p_1^{D,D,C}, p_3^{D,D,C}), (p_2^{D,D,C}, p_3^{D,D,C}), p_i^{D,D,C}, (d,c))]$$

where morphisms involved in this element of $F_c^C(D)$ can be displayed as in the diagram

$$D \xleftarrow{p_i^{D,D,C}} D \times D \times C \underset{(p_2^{D,D,C},\,p_3^{D,D,C})}{\overset{(p_1^{D,D,C},\,p_3^{D,D,C})}{\rightrightarrows}} D \times C \xrightarrow{p_2^{D,C}} C \quad .$$

To show that $\xi_i^{F,C,c}$ is indeed a natural transformation, we consider a morphism $u\colon D \to D'$ in $\mathcal{C}$ and an element $d \in F(D)$. We must show that

$$F_c^C(u)(\xi_{i,D}^{F,C,c}(d)) = \xi_{i,D'}^{F,C,c}(F(u)(d))$$

or in other words that the relation

$$\begin{aligned} &(p_2^{D,C}, (p_1^{D,D,C}, p_3^{D,D,C}), (p_2^{D,D,C}, p_3^{D,D,C}), up_i^{D,D,C}, (d,c)) \\ \cong_{F,C,c,D'}\ &(p_2^{D',C}, (p_1^{D',D',C}, p_3^{D',D',C}), (p_2^{D',D',C}, p_3^{D',D',C}), p_i^{D',D',C}, (F(u)(d),c)) \end{aligned}$$

holds. Let us show that this relation is witnessed by the span

$$\begin{array}{ccc} & D \times C & \\ {}^{1_{D\times C}}\swarrow & & \searrow^{u \times 1_C} \\ D \times C & & D' \times C \end{array}$$

together with the element $(d,c) \in F(D \times C)$. We obviously have $F(1_{D\times C})((d,c)) = (d,c)$, $F(u \times 1_C)((d,c)) = (F(u)(d),c)$ and $p_2^{D,C} = p_2^{D',C}(u \times 1_C)$. Since the diagrams

$$\begin{array}{ccc} D \times D \times C & \underset{(p_2^{D,D,C},\,p_3^{D,D,C})}{\overset{(p_1^{D,D,C},\,p_3^{D,D,C})}{\rightrightarrows}} & D \times C \\ {\scriptstyle 1_{D\times D\times C}}\downarrow & & \downarrow{\scriptstyle 1_{D\times C}} \\ D \times D \times C & \underset{(p_2^{D,D,C},\,p_3^{D,D,C})}{\overset{(p_1^{D,D,C},\,p_3^{D,D,C})}{\rightrightarrows}} & D \times C \end{array} \quad \text{and} \quad \begin{array}{ccc} D \times D \times C & \underset{(p_2^{D,D,C},\,p_3^{D,D,C})}{\overset{(p_1^{D,D,C},\,p_3^{D,D,C})}{\rightrightarrows}} & D \times C \\ {\scriptstyle u\times u\times 1_C}\downarrow & & \downarrow{\scriptstyle u\times 1_C} \\ D' \times D' \times C & \underset{(p_2^{D',D',C},\,p_3^{D',D',C})}{\overset{(p_1^{D',D',C},\,p_3^{D',D',C})}{\rightrightarrows}} & D' \times C \end{array}$$

(reasonably) commute, the last condition reduces to $up_i^{D,D,C} = p_i^{D',D',C}(u \times u \times 1_C)$ which obviously holds.

STEP 38. CHARACTERIZATION OF $\xi_{1,D}^{F,C,c}(d) = \xi_{2,D}^{F,C,c}(d)$. For an object $D \in \mathcal{C}$ and an element $d \in F(D)$, we now show that the equality $\xi_{1,D}^{F,C,c}(d) = \xi_{2,D}^{F,C,c}(d)$ holds if and only if there exists a span

$$\begin{array}{ccc} & D' & \\ {}^{h}\swarrow & & \searrow^{h'} \\ D & & C \end{array} \tag{44}$$

together with an element $d' \in F(D')$ such that $F(h)(d') = d$, $F(h')(d') = c$ and $hr_1^{h'} = hr_2^{h'}$ where $r_1^{h'}, r_2^{h'} \colon R[h'] \rightrightarrows D'$ is the kernel pair of h'. The equality $\xi_{1,D}^{F,C,c}(d) = \xi_{2,D}^{F,C,c}(d)$ means

$$
\begin{aligned}
&(p_2^{D,C}, (p_1^{D,D,C}, p_3^{D,D,C}), (p_2^{D,D,C}, p_3^{D,D,C}), p_1^{D,D,C}, (d,c)) \\
\cong_{F,C,c,D}\ &(p_2^{D,C}, (p_1^{D,D,C}, p_3^{D,D,C}), (p_2^{D,D,C}, p_3^{D,D,C}), p_2^{D,D,C}, (d,c))
\end{aligned}
$$

or, in other words, that there exists a span

$$
\begin{array}{ccc}
 & D'' & \\
{}^{(h_1,h'')}\swarrow & & \searrow^{(h_2,h'')} \\
D \times C & & D \times C
\end{array}
\tag{45}
$$

together with an element $d'' \in F(D'')$ such that $F(h_1)(d'') = d = F(h_2)(d'')$, $F(h'')(d'') = c$ and, since the diagrams

$$
\begin{array}{ccc}
R[h''] & \overset{r_1^{h''}}{\underset{r_2^{h''}}{\rightrightarrows}} & D'' \\
{\scriptstyle (h_1 r_1^{h''}, h_1 r_2^{h''}, h'' r_1^{h''})}\downarrow & & \downarrow{\scriptstyle (h_1,h'')} \\
D \times D \times C & \overset{(p_1^{D,D,C}, p_3^{D,D,C})}{\underset{(p_2^{D,D,C}, p_3^{D,D,C})}{\rightrightarrows}} & D \times C
\end{array}
$$

and

$$
\begin{array}{ccc}
R[h''] & \overset{r_1^{h''}}{\underset{r_2^{h''}}{\rightrightarrows}} & D'' \\
{\scriptstyle (h_2 r_1^{h''}, h_2 r_2^{h''}, h'' r_1^{h''})}\downarrow & & \downarrow{\scriptstyle (h_2,h'')} \\
D \times D \times C & \overset{(p_1^{D,D,C}, p_3^{D,D,C})}{\underset{(p_2^{D,D,C}, p_3^{D,D,C})}{\rightrightarrows}} & D \times C
\end{array}
$$

(reasonably) commute (where $r_1^{h''}, r_2^{h''} \colon R[h''] \rightrightarrows D''$ is the kernel pair of h''), the identity $h_1 r_1^{h''} = h_2 r_2^{h''}$ holds. If a span (h, h') and an element d' as in (44) exist, one can get a span as in (45) simply as

$$
\begin{array}{ccc}
 & D' & \\
{}^{(h,h')}\swarrow & & \searrow^{(h,h')} \\
D \times C & & D \times C
\end{array}
$$

together with the same element $d' \in F(D')$. Conversely, if a span $((h_1, h''), (h_2, h''))$ and an element d'' as in (45) exist, we consider the equalizer

$$
D' \overset{m}{\rightarrowtail} D'' \overset{h_1}{\underset{h_2}{\rightrightarrows}} D
$$

and, since F preserves finite limits and $F(h_1)(d'') = F(h_2)(d'')$, there exists a unique element $d' \in F(D')$ such that $F(m)(d') = d''$. Then, the span

$$\begin{array}{ccc} & D' & \\ {\scriptstyle h_1 m}\swarrow & & \searrow{\scriptstyle h''m} \\ D & & C \end{array}$$

together with the element d' satisfies the required properties. Indeed, we know that $F(h_1 m)(d') = F(h_1)(d'') = d$, $F(h''m)(d') = F(h'')(d'') = c$ and, considering the kernel pair $r_1^{h''m}, r_2^{h''m} \colon R[h''m] \rightrightarrows D'$ of $h''m$ and the unique morphism $\overline{m}$ making the diagram

$$\begin{array}{ccc} R[h''m] & \overset{r_1^{h''m}}{\underset{r_2^{h''m}}{\rightrightarrows}} & D' \\ {\scriptstyle \overline{m}}\downarrow & & \downarrow{\scriptstyle m} \\ R[h''] & \overset{r_1^{h''}}{\underset{r_2^{h''}}{\rightrightarrows}} & D'' \end{array}$$

(reasonably) commute, the identities

$$h_1 m r_1^{h''m} = h_1 r_1^{h''} \overline{m} = h_2 r_2^{h''} \overline{m} = h_2 m r_2^{h''m} = h_1 m r_2^{h''m}$$

hold.

Step 39. $\xi_{1,C}^{F,C,c}(c) = \xi_{2,C}^{F,C,c}(c)$. Particularizing the characterization of Step 38 to the case where $D = C$ and $d = c$, we get that $\xi_{1,C}^{F,C,c}(c) = \xi_{2,C}^{F,C,c}(c)$. Indeed, it suffices to consider the span

$$\begin{array}{ccc} & C & \\ {\scriptstyle 1_C}\swarrow & & \searrow{\scriptstyle 1_C} \\ C & & C \end{array}$$

together with the element $c \in F(C)$.

Step 40. Characterization of when $\theta_c \colon \mathcal{C}(C, -) \to F$ is the equalizer of $\xi_1^{F,C,c}$ and $\xi_2^{F,C,c}$. Given a finite limit preserving functor $F \colon \mathcal{C} \to \mathsf{Set}$, an object $C \in \mathcal{C}$ and an element $c \in F(C)$, we have constructed a finite limit preserving functor F_c^C together with two natural transformations $\xi_1^{F,C,c}, \xi_2^{F,C,c} \colon F \rightrightarrows F_c^C$. By the Yoneda Lemma, the element $c \in F(C)$ corresponds to a natural transformation $\theta_c \colon \mathcal{C}(C, -) \to F$, defined for each object $D \in \mathcal{C}$ and each morphism $f \colon C \to D$ by $\theta_{c,D}(f) = F(f)(c)$. Using the Yoneda Lemma, Step 39 means exactly that $\xi_1^{F,C,c}\theta_c = \xi_2^{F,C,c}\theta_c$. We shall now prove that

$$\mathcal{C}(C, -) \xrightarrow{\theta_c} F \overset{\xi_1^{F,C,c}}{\underset{\xi_2^{F,C,c}}{\rightrightarrows}} F_c^C \qquad (46)$$

is an equalizer diagram in $\mathsf{Set}^{\mathcal{C}}$ if and only if for each span

$$\begin{array}{ccc} & A & \\ {}^{h}\swarrow & & \searrow^{k} \\ D & & C \end{array} \tag{47}$$

such that $hr_1^k = hr_2^k$ where $r_1^k, r_2^k \colon R[k] \rightrightarrows A$ is the kernel pair of k and for each element $a \in F(A)$ such that $F(k)(a) = c$, there exists a unique morphism $f \colon C \to D$ in $\mathcal{C}$ such that $F(f)(c) = F(h)(a)$.

The diagram (46) is an equalizer diagram in $\mathsf{Set}^{\mathcal{C}}$ if and only if for each object $D \in \mathcal{C}$ and each element $d \in F(D)$ such that $\xi_{1,D}^{F,C,c}(d) = \xi_{2,D}^{F,C,c}(d)$, there exists a unique morphism $f \in \mathcal{C}(C, D)$ such that $\theta_{c,D}(f) = d$, i.e., $F(f)(c) = d$. If this property holds, given a span (h, k) and an element $a \in F(A)$ as in (47), the element $d = F(h)(a) \in F(D)$ satisfies $\xi_{1,D}^{F,C,c}(d) = \xi_{2,D}^{F,C,c}(d)$ by Step 38, and therefore there exists a unique morphism $f \colon C \to D$ such that $F(f)(c) = F(h)(a)$.

Conversely, suppose the property about spans (47) holds and let D be an object of $\mathcal{C}$ and d be an element of $F(D)$ such that $\xi_{1,D}^{F,C,c}(d) = \xi_{2,D}^{F,C,c}(d)$. By Step 38, there exists a span (h, k) and an element $a \in F(A)$ as in (47) such that, in addition of $hr_1^k = hr_2^k$ and $F(k)(a) = c$, the identity $F(h)(a) = d$ also holds. Therefore, there exists a unique morphism $f \colon C \to D$ such that $F(f)(c) = d$.

Remark: It can be proved that $\xi_1^{F,C,c}, \xi_2^{F,C,c} \colon F \rightrightarrows F_c^C$ is the cokernel pair of θ_c in the category of finite limit preserving functors $\mathcal{C} \to \mathsf{Set}$ and their natural transformations. Therefore, θ_c is a regular monomorphism in that category exactly when (46) is an equalizer diagram (in the category of finite limit preserving functors $\mathcal{C} \to \mathsf{Set}$, or equivalently, in $\mathsf{Set}^{\mathcal{C}}$). Since we will not need that in this proof, we omit details here.

Step 41. In the regular case, $\iota_{\mathcal{C}(C,-)} \colon \mathcal{C}(C, -) \rightarrowtail \overline{\mathcal{C}(C, -)}$ is the equalizer of ξ_1^C and ξ_2^C. Given an object $C \in \mathcal{C}$, we have defined in Step 28 a monomorphism $\iota_{\mathcal{C}(C,-)} \colon \mathcal{C}(C, -) \rightarrowtail \overline{\mathcal{C}(C, -)}$ in $\mathsf{Set}^{\mathcal{C}}$. By the Yoneda Lemma, this natural transformation corresponds to an element $c = \iota_{\mathcal{C}(C,-),C}(1_C) \in \overline{\mathcal{C}(C, -)}(C)$. To shorten notation, for each $i \in \{1, 2\}$, we denote by ξ_i^C the natural transformation $\xi_i^{\overline{\mathcal{C}(C,-)},C,c} \colon \overline{\mathcal{C}(C, -)} \to \overline{\mathcal{C}(C, -)}_c^C$. Applying the characterization of Step 40, we shall now prove that, if E is a regular surjection-like class of morphisms,

$$\mathcal{C}(C, -) \overset{\iota_{\mathcal{C}(C,-)}}{\rightarrowtail} \overline{\mathcal{C}(C, -)} \underset{\xi_2^C}{\overset{\xi_1^C}{\rightrightarrows}} \overline{\mathcal{C}(C, -)}_c^C$$

is an equalizer diagram in $\mathsf{Set}^{\mathcal{C}}$. That is, for each span

$$\begin{array}{ccc} & A & \\ {}^{h}\swarrow & & \searrow^{k} \\ D & & C \end{array}$$

such that $hr_1^k = hr_2^k$, where $r_1^k, r_2^k \colon R[k] \rightrightarrows A$ is the kernel pair of k, and for each element $a \in \overline{\mathcal{C}(C,-)}(A)$ such that

$$\overline{\mathcal{C}(C,-)}(k)(a) = \iota_{\mathcal{C}(C,-),C}(1_C),$$

we must show that there exists a unique morphism $f \colon C \to D$ such that

$$\overline{\mathcal{C}(C,-)}(f)(\iota_{\mathcal{C}(C,-),C}(1_C)) = \overline{\mathcal{C}(C,-)}(h)(a).$$

In view of the construction of $\overline{\mathcal{C}(C,-)}$ given in Step 25, an element $a \in \overline{\mathcal{C}(C,-)}(A)$ can be written in the form $a = [(n,x)]$ for some $n \in \mathbb{N}$ and some $x \in \mathcal{C}(C,-)^{(n)}(A)$. In that case, $\overline{\mathcal{C}(C,-)}(k)(a) = [(n, \mathcal{C}(C,-)^{(n)}(k)(x))]$ and $\overline{\mathcal{C}(C,-)}(h)(a) = [(n, \mathcal{C}(C,-)^{(n)}(h)(x))]$. Moreover, $\iota_{\mathcal{C}(C,-),C}(1_C) = [(0, 1_C)]$. Therefore, we must prove that, for each $n \in \mathbb{N}$, for each span

$$\begin{array}{ccc} & A & \\ {}^{h}\swarrow & & \searrow^{k} \\ D & & C \end{array} \tag{48}$$

and for each $x \in \mathcal{C}(C,-)^{(n)}(A)$, if $hr_1^k = hr_2^k$ and

$$[(n, \mathcal{C}(C,-)^{(n)}(k)(x))] = [(0, 1_C)], \tag{49}$$

there exists a unique morphism $f \colon C \to D$ such that

$$[(n, \mathcal{C}(C,-)^{(n)}(h)(x))] = [(0, f)]. \tag{50}$$

The uniqueness of f follows immediately from the fact that $\iota_{\mathcal{C}(C,-)} \colon \mathcal{C}(C,-) \rightarrowtail \overline{\mathcal{C}(C,-)}$ is a monomorphism (Step 28). We shall prove the existence of f by induction on n.

If $n = 0$, since $\mathcal{C}(C,-)^{(0)} = \mathcal{C}(C,-)$, x is a morphism $C \to A$ and the equality (50) becomes $hx = f$. We thus have to prove that there exists a morphism $f \colon C \to D$ such that $hx = f$, which is trivial.

We now assume that the above property holds for n and we shall prove it also holds for $n+1$. We are thus given a span (h,k) as in (48) and an element $x \in \mathcal{C}(C,-)^{(n+1)}(A)$. Since $\mathcal{C}(C,-)^{(n+1)} = \mathcal{C}\widehat{(C,-)^{(n)}}$, the element x can be written as $x = [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, g, a')]$, where the morphisms involved can be displayed as in the diagram

$$\begin{array}{ccccc} & P & \xrightarrow{\ \overline{f}\ } & B_1 \times \cdots \times B_{n'} \\ {}^{g}\swarrow & {\scriptstyle \overline{e}}\downarrow & \lrcorner & \downarrow{\scriptstyle e_1 \times \cdots \times e_{n'}} \\ A & A' & \xrightarrow[(f_1,\ldots,f_{n'})]{} & D_1 \times \cdots \times D_{n'} \end{array}$$

and where $a' \in \mathcal{C}(C,-)^{(n)}(A')$. Using Step 15, we can assume without loss of generality that the implication

$$\big(e_i = e_{i'} \wedge \mathcal{C}(C,-)^{(n)}(f_i)(a') = \mathcal{C}(C,-)^{(n)}(f_{i'})(a')\big) \Longrightarrow i = i'$$

holds for indices $i, i' \in \{1, \ldots, n'\}$. Moreover, we assume that $hr_1^k = hr_2^k$ as well as the hypothesis (49) which now becomes

$$[(n+1, [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, kg, a')])] = [(0, 1_C)],$$

i.e.,

$$[(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, kg, a')] = \lambda^n_{\mathcal{C}(C,-),C}(\cdots(\lambda^0_{\mathcal{C}(C,-),C}(1_C))\cdots).$$

Since $\lambda^n_{\mathcal{C}(C,-),C} = \lambda_{\mathcal{C}(C,-)^{(n)},C}$ holds by definition, we know from Step 21 that there exists a span

$$\begin{array}{ccc} & A'' & \\ {\scriptstyle l_1}\swarrow & & \searrow{\scriptstyle l_2} \\ A' & & C \end{array}$$

together with an element $a'' \in \mathcal{C}(C,-)^{(n)}(A'')$ such that

$$\mathcal{C}(C,-)^{(n)}(l_1)(a'') = a', \tag{51}$$

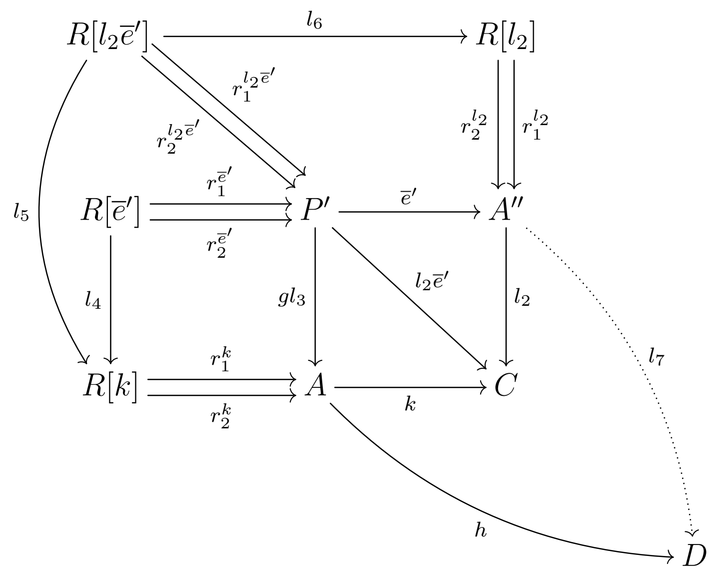
$$\mathcal{C}(C,-)^{(n)}(l_2)(a'') = \lambda^{n-1}_{\mathcal{C}(C,-),C}(\cdots(\lambda^0_{\mathcal{C}(C,-),C}(1_C))\cdots) \tag{52}$$

and, considering the pullback

$$\begin{array}{ccc} P' & \xrightarrow{l_3} & P \\ {\scriptstyle \overline{e}'}\downarrow & \lrcorner & \downarrow{\scriptstyle \overline{e}} \\ A'' & \xrightarrow[l_1]{} & A' \end{array} \quad , \tag{53}$$

the identity $l_2\overline{e}' = kgl_3$ holds. Since E is closed under finite products (by (Id), (ClComp) and (StPb)), we know that $e_1 \times \cdots \times e_{n'} \in E$. Since E is stable under pullbacks (StPb), this implies that $\overline{e} \in E$ and so $\overline{e}' \in E$. Since morphisms in E are regular epimorphisms (Reg), $\overline{e}'$ is a regular epimorphism, i.e., the coequalizer of its kernel pair $r_1^{\overline{e}'}, r_2^{\overline{e}'} \colon R[\overline{e}'] \rightrightarrows P'$. Note that this is the only place where we use the axiom (Reg) in the entire proof. We now consider the kernel pairs $r_1^{l_2}, r_2^{l_2} \colon R[l_2] \rightrightarrows A''$ of l_2 and $r_1^{l_2\overline{e}'}, r_2^{l_2\overline{e}'} \colon R[l_2\overline{e}'] \rightrightarrows P'$ of $l_2\overline{e}'$

and the unique morphisms l_4, l_5 and l_6 making the diagram of plain arrows



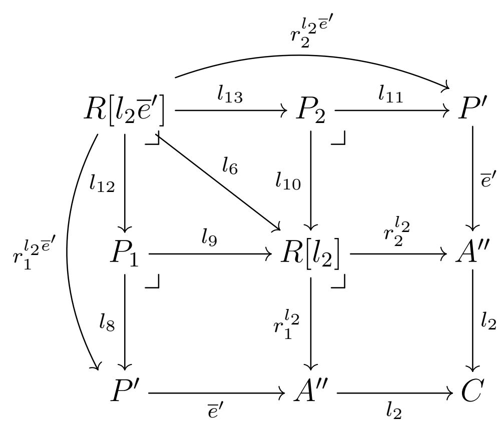
(reasonably) commute. Since

$$hgl_3r_1^{\overline{e}'} = hr_1^k l_4 = hr_2^k l_4 = hgl_3r_2^{\overline{e}'},$$

we know there is a unique morphism $l_7 \colon A'' \to D$ such that

$$l_7\overline{e}' = hgl_3. \tag{54}$$

In the diagram below,



P_1 and P_2 are defined as pullbacks and the morphisms l_{12} and l_{13} as the unique ones making the diagram commutative. Since the outer square is a pullback, the top left square is a also pullback by the usual properties of composition and cancellation of pullbacks. Since $\overline{e}' \in E$ and since E is stable under pullbacks (StPb), we know that $l_9, l_{10}, l_{12}, l_{13} \in E$.

Since E is closed under composition (ClComp), this proves that $l_6 = l_9 l_{12} \in E$. Moreover, we can compute

$$l_7 r_1^{l_2} l_6 = l_7 \overline{e}' r_1^{l_2 \overline{e}'} = h g l_3 r_1^{l_2 \overline{e}'} = h r_1^{k} l_5 = h r_2^{k} l_5 = h g l_3 r_2^{l_2 \overline{e}'} = l_7 \overline{e}' r_2^{l_2 \overline{e}'} = l_7 r_2^{l_2} l_6.$$

Since morphisms in E are (strong) epimorphisms (by (SRightCancP) and (NoPMono)) and since $l_6 \in E$, this shows that $l_7 r_1^{l_2} = l_7 r_2^{l_2}$. We thus have a span

$$D \xleftarrow{\;l_7\;} A'' \xrightarrow{\;l_2\;} C$$

together with an element $a'' \in \mathcal{C}(C,-)^{(n)}(A'')$ such that $l_7 r_1^{l_2} = l_7 r_2^{l_2}$ and

$$[(n, \mathcal{C}(C,-)^{(n)}(l_2)(a''))] = [(0, 1_C)]$$

by (52). Using our induction hypothesis, there exists a morphism $f \colon C \to D$ such that

$$[(n, \mathcal{C}(C,-)^{(n)}(l_7)(a''))] = [(0, f)].$$

It remains to prove that

$$[(n, \mathcal{C}(C,-)^{(n)}(l_7)(a''))] = [(n+1, \mathcal{C}(C,-)^{(n+1)}(h)(x))]$$

which is equivalent to

$$[(n, \mathcal{C}(C,-)^{(n)}(l_7)(a''))] = [(n+1, [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, hg, a')])]$$

or, in other words, to

$$\lambda_{\mathcal{C}(C,-)^{(n)},D}(\mathcal{C}(C,-)^{(n)}(l_7)(a'')) = [(n', (e_i)_i, (f_i)_i, \overline{f}, \overline{e}, hg, a')].$$

Using Step 21, this equality is attested by the span

$$A' \xleftarrow{\;l_1\;} A'' \xrightarrow{\;l_7\;} D$$

together with the element $a'' \in \mathcal{C}(C,-)^{(n)}(A'')$, the pullback (53) and the equalities (51) and (54).

STEP 42. CONCLUSION OF THE PROOF OF THEOREM 2.2. We are now able to construct a small full subcategory $\mathcal{D}$ of $\mathsf{Set}^{\mathcal{C}}$ satisfying properties (i), (ii) and (iii) described in Step 2. We define in this case $\mathcal{D}$ to be the full subcategory of $\mathsf{Set}^{\mathcal{C}}$ made of the functors $\overline{\mathcal{C}(C,-)}$ and $\overline{\overline{\mathcal{C}(C,-)}^{C}_{\iota_{\mathcal{C}(C,-),C}(1_C)}}$ for each object $C \in \mathcal{C}$. Since $\mathcal{C}$ is small, so is $\mathcal{D}$. Property (i), requiring that each functor $F \in \mathcal{D}$ preserves finite limits, has been proved in Step 25. Property (ii), requiring that for each $e \in E$ and each $F \in \mathcal{D}$ the function $F(e)$ is surjective, follows from Step 26. For each object $C \in \mathcal{C}$, the natural transformation $\iota_C \colon \mathcal{C}(C,-) \to F_C$ mentioned in property (iii) is given by $\iota_{\mathcal{C}(C,-)} \colon \mathcal{C}(C,-) \rightarrowtail \overline{\mathcal{C}(C,-)}$. Step 28 shows that property (iii)(a) holds, i.e., that $\iota_{\mathcal{C}(C,-)}$ is a monomorphism in $\mathsf{Set}^{\mathcal{C}}$. By the Yoneda Lemma, property (iii)(b) can be reformulated exactly as Step 29. Property (iii)(d) is satisfied in view of Step 27. Given an object $C \in \mathcal{C}$, let $c = \iota_{\mathcal{C}(C,-),C}(1_C) \in \overline{\mathcal{C}(C,-)}(C)$. By Step 41,

$$\mathcal{C}(C,-) \overset{\iota_{\mathcal{C}(C,-)}}{\rightarrowtail} \overline{\mathcal{C}(C,-)} \underset{\xi_2^C}{\overset{\xi_1^C}{\rightrightarrows}} \overline{\mathcal{C}(C,-)}^{C}_{c}$$

is an equalizer diagram in $\mathsf{Set}^{\mathcal{C}}$. By Step 25, $\iota_{\overline{\mathcal{C}(C,-)}^{C}_{c}} \colon \overline{\mathcal{C}(C,-)}^{C}_{c} \rightarrowtail \overline{\overline{\mathcal{C}(C,-)}^{C}_{c}}$ is a monomorphism. Therefore

$$\mathcal{C}(C,-) \overset{\iota_{\mathcal{C}(C,-)}}{\rightarrowtail} \overline{\mathcal{C}(C,-)} \underset{\iota_{\overline{\mathcal{C}(C,-)}^{C}_{c}}\xi_2^C}{\overset{\iota_{\overline{\mathcal{C}(C,-)}^{C}_{c}}\xi_1^C}{\rightrightarrows}} \overline{\overline{\mathcal{C}(C,-)}^{C}_{c}}$$

is also an equalizer diagram in $\mathsf{Set}^{\mathcal{C}}$, proving that property (iii)(c) holds. ∎

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