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ON ROTA-BAXTER LIE 2-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of Rota-Baxter Lie 2-algebras, which is a categorification of Rota-Baxter Lie algebras. We prove that the category of Rota-Baxter Lie 2-algebras and the category of 2-term Rota-Baxter L_∞-algebras are equivalent. We introduce the notion of a crossed module of Rota-Baxter Lie algebras, and show that there is a one-to-one correspondence between strict 2-term Rota-Baxter L_∞-algebras and crossed modules of Rota-Baxter Lie algebras. At last, as applications of the crossed modules of Rota-Baxter Lie algebras, we give constructions of crossed modules of pre-Lie algebras and crossed modules of Lie algebras from them.

1. Introduction

The notion of a Rota-Baxter algebra originated from the 1960 paper [1] of G. Baxter in his probability study to understand Spitzer's identity in fluctuation theory. Soon afterwards, this concept attracted the attention of well-known mathematicians such as P. Cartier and G.-C. Rota whose fundamental papers [10, 19] around 1970 brought the subject into the areas of algebra and combinatorics. Rota-Baxter algebras have broad connections with mathematical physics, including the application in Connes-Kreimer's algebraic approach to the renormalization in perturbative quantum field theory [12]. Rota-Baxter algebras also lead to the splitting of operads [6, 22], and are closely related to noncommutative symmetric functions and Hopf algebras [13, 15, 17, 29]. We refer the reader to [14] for more details about Rota-Baxter algebras.

In the Lie algebra context, a Rota-Baxter operator was introduced independently in the 1980s as the operator form of the classical Yang-Baxter equation, named after the physicists C.-N. Yang and R. Baxter [8, 28], whereas the classical Yang-Baxter equation plays important roles in mathematics and mathematical physics such as integrable systems and quantum groups [11, 24]. A Lie algebra equipped with a Rota-Baxter operator, called a Rota-Baxter Lie algebra, naturally gives rise to a pre-Lie algebra or a post-Lie algebra which has its origin in a study of operads [27] as a special case of the splitting of Lie algebras [6]. Recently, as an integration and geometrization of Rota-Baxter Lie algebras, the notions of Rota-Baxter Lie groups and Rota-Baxter Lie algebroids were introduced

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in [16]. Besides, cohomologies, deformations, extensions and homotopy theory of Rota-Baxter Lie algebras were well studied in [18, 21, 26].

Motivated by the study of string theory, there has been great attention to higher categorical structures. One way to obtain higher categorical structures is by categorifying existing mathematical concepts. One of the simplest higher structures is a 2-vector space, which is a categorification of a vector space. If we further put Lie algebra structures on 2-vector spaces, then we obtain Lie 2-algebras [2]. L_∞-algebras, sometimes called strongly homotopy Lie algebras, were introduced in [20] as a model for Lie algebras that satisfy Jacobi identity up to all higher homotopies. It is well-known that the category of Lie 2-algebras is equivalent to the category of 2-term L_∞-algebras. The structure of Lie 2-algebras appears in many areas such as string theory [4], higher symplectic geometry [3], and Courant algebroids [23].

In this paper, we provide a categorification of Rota-Baxter Lie algebras, called Rota-Baxter Lie 2-algebra. Rota-Baxter operators on 2-term L_∞-algebras were first introduced in [25] as a tool to study 2-graded classical Yang-Baxter equations, which could naturally generate examples of Lie 2-bialgebras [7]. Soon afterwards, Rota-Baxter operators on L_∞-algebras were given and studied in [21]. We prove that the category of Rota-Baxter Lie 2-algebras and category of 2-term Rota-Baxter L_∞-algebras are equivalent. Here a 2-term Rota-Baxter L_∞-algebra consists of a 2-term L_∞-algebra and a Rota-Baxter operator on it. The notion of crossed modules of Rota-Baxter Lie algebras is also introduced and we prove that there is a one-to-one correspondence between strict 2-term Rota-Baxter L_∞-algebras and crossed modules of Rota-Baxter Lie algebras. We show that a crossed module of Rota-Baxter Lie algebras gives a crossed module of pre-Lie algebras and thus gives a crossed module of Lie algebras naturally.

The paper is organized as follows. In Section 2, we recall Rota-Baxter Lie algebras and their representations, 2-vector spaces and 2-term chain complexes. In Section 3, we first give the notion of Rota-Baxter Lie 2-algebras, which is the categorification of Rota-Baxter Lie algebras. Then we introduce the category of Rota-Baxter Lie 2-algebras and the category of 2-term Rota-Baxter L_∞-algebras and show that they are equivalent. In Section 4, we introduce the notion of crossed modules of Rota-Baxter Lie algebras and show that there is a one-to-one correspondence between strict 2-term Rota-Baxter L_∞-algebras and crossed modules of Rota-Baxter Lie algebras. We show that the underlying algebraic structure of a crossed module of Rota-Baxter Lie algebras is a crossed module of pre-Lie algebras and then a new crossed module of Lie algebras is constructed.

In this paper, all the vector spaces are over an algebraically closed field $\mathbb{K}$ of characteristic 0, and finite dimensional.

2. Preliminaries

2.1. Rota-Baxter Lie algebras and their representations.

2.2. Definition. *Let* $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ *be a Lie algebra. A linear operator* $T:\mathfrak{g}\longrightarrow\mathfrak{g}$ *is called a* **Rota-Baxter operator** *if*

$$[T(x),T(y)]_\mathfrak{g}=T\big([T(x),y]_\mathfrak{g}+[x,T(y)]_\mathfrak{g}\big),\quad \forall x,y\in\mathfrak{g}.$$

Moreover, a Lie algebra $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ *with a Rota-Baxter operator* T *is called a* **Rota-Baxter Lie algebra**. *We denote it by* $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},T)$.

2.3. Definition. *A* **pre-Lie algebra** *is a pair* $(\mathfrak{g},*_\mathfrak{g})$, *where* $\mathfrak{g}$ *is a vector space and* $*_\mathfrak{g}$ *is a bilinear multiplication on* $\mathfrak{g}$ *satisfying that the associator* $(x,y,z)=(x*_\mathfrak{g}y)*_\mathfrak{g}z-x*_\mathfrak{g}(y*_\mathfrak{g}z)$ *is symmetric in* x,y, *i.e.*

$$(x,y,z)=(y,x,z),\ \text{ or equivalently, }\ (x*_\mathfrak{g}y)*_\mathfrak{g}z-x*_\mathfrak{g}(y*_\mathfrak{g}z)=(y*_\mathfrak{g}x)*_\mathfrak{g}z-y*_\mathfrak{g}(x*_\mathfrak{g}z).$$

Let $(\mathfrak{g},*_\mathfrak{g})$ be a pre-Lie algebra. The commutator $[x,y]_\mathfrak{g}=x*_\mathfrak{g}y-y*_\mathfrak{g}x$ defines a Lie algebra structure on $\mathfrak{g}$, which is called the **sub-adjacent Lie algebra** of $(\mathfrak{g},*_\mathfrak{g})$ and denoted by $\mathfrak{g}^c$. Furthermore, $L:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ defined by

$$L_xy=x*_\mathfrak{g}y,\quad \forall x,y\in\mathfrak{g} \tag{1}$$

gives a representation of $\mathfrak{g}^c$ on $\mathfrak{g}$. See [5, 9] for more details.

The following proposition reviews the well-known transformation from a Rota-Baxter Lie algebra to a pre-Lie algebra.

2.4. Proposition. *Let* $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$ *be a Lie algebra and* $T:\mathfrak{g}\longrightarrow\mathfrak{g}$ *a Rota-Baxter operator. Define a new operation on* $\mathfrak{g}$ *by*

$$x*y=[T(x),y]_\mathfrak{g}.$$

Then $(\mathfrak{g},*)$ *is a pre-Lie algebra and* T *is a homomorphism from the sub-adjacent Lie algebras* $(\mathfrak{g},[\cdot,\cdot]_T)$ *to* $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g})$, *where* $[x,y]_T=x*y-y*x$.

2.5. Definition. *Let* $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},T)$ *and* $(\mathfrak{h},[\cdot,\cdot]_\mathfrak{h},S)$ *be two Rota-Baxter Lie algebras. A* **Rota-Baxter Lie algebra homomorphism** *from* $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},T)$ *to* $(\mathfrak{h},[\cdot,\cdot]_\mathfrak{h},S)$ *is a linear map* $\phi:\mathfrak{g}\to\mathfrak{h}$ *such that* ϕ *a Lie algebra homomorphism and satisfies* $\phi\circ T=S\circ\phi$.

A **Rota-Baxter Lie subalgebra** (resp., **Rota-Baxter Lie ideal**) of a Rota-Baxter Lie algebra $(\mathfrak{g},T)$ is a Lie subalgebra (resp., a Lie ideal) I of $\mathfrak{g}$ such that $T(I)\subseteq I$. Let $f:(\mathfrak{g},T)\to(\mathfrak{h},S)$ be a Rota-Baxter Lie algebra homomorphism. Then $\ker f$ is a Rota-Baxter Lie ideal of the Rota-Baxter Lie algebra $(\mathfrak{g},T)$.

2.6. Definition. ([18]) *A* **representation of a Rota-Baxter Lie algebra** $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},T)$ *on a vector space* V *with respect to a linear map* $\mathfrak{T}\in\mathfrak{gl}(V)$ *is a representation* ρ *of the Lie algebra* $\mathfrak{g}$ *on* V, *satisfying*

$$\rho(T(x))\circ\mathfrak{T}=\mathfrak{T}\circ\rho(T(x))+\mathfrak{T}\circ\rho(x)\circ\mathfrak{T},\quad \forall x\in\mathfrak{g}. \tag{2}$$

We denote the above representation by $(V;\rho,\mathfrak{T})$, and give some examples as follows.

2.7. Example. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$ be a Rota-Baxter Lie algebra. Then $(\mathfrak{g}; \mathrm{ad}, T)$ is a representation, which is called the **adjoint representation** of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$.

2.8. Proposition. ([18]) *Let $(V; \mathcal{T}, \rho)$ be a representation of a Rota-Baxter Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$. Then $(V^*; \rho^*, -\mathcal{T}^*)$ is also a representation of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$, which is called the* **dual representation**.

2.9. Example. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$ be a Rota-Baxter Lie algebra. Then $(\mathfrak{g}^*; \mathrm{ad}^*, -T^*)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$, which is called the **coadjoint representation**.

2.10. Proposition. *Let $(V; \rho, \mathcal{T})$ be a representation of a Rota-Baxter Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, T)$. Then $(\mathfrak{g}\oplus V, [\cdot,\cdot]_\ltimes, \mathfrak{T})$ is a Rota-Baxter Lie algebra, where $[\cdot,\cdot]_\ltimes$ is the semidirect product Lie bracket given by*

$$[x+u, y+v]_\ltimes = [x,y]_\mathfrak{g} + \rho(x)v - \rho(y)u, \quad \forall x, y \in \mathfrak{g}, u, v \in V,$$

and $\mathfrak{T} : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ is a linear map given by

$$\mathfrak{T}(x+u) = T(x) + \mathcal{T}(u), \quad \forall x \in \mathfrak{g}, u \in V.$$

2.11. 2-vector spaces. Let **Vect** be the category of vector spaces. Vector spaces can be categorified to 2-vector spaces. A good introduction for this subject is [2].

2.12. Definition. *A* **2-vector space** *is an internal category in the category Vect.*

Thus, a 2-vector space $V = (V_1, V_0, s, t, i, \circ)$ is a category with a vector space of objects V_0 and a vector space of morphisms V_1, such that the source and target maps $s, t : V_1 \to V_0$, the identity-assigning map $i : V_0 \to V_1$, and the composition map $\circ : V_1 \times_{V_0} V_1 \to V_1$ satisfy the specified category laws.

Given a morphism $f : x \to y \in V_1$, define the **arrow part** of f, denoted as $\overrightarrow{f}$, by

$$\overrightarrow{f} = f - i(x).$$

Furthermore, we identify $f : x \to y$ with the ordered pair $(x, \overrightarrow{f})$. It was shown in [2] that the composition map $\circ : V_1 \times V_1 \to V_1$ is uniquely determined by

$$f \circ g = (x, \overrightarrow{f} + \overrightarrow{g}), \qquad f = (x, \overrightarrow{f}), g = (y, \overrightarrow{g}) \in V_1. \tag{3}$$

Thus the structure of a 2-vector space is completely determined by the vector spaces V_0 and V_1 together with the source, target and identity-assigning maps.

Let V and W be two 2-vector spaces. Recall that a **linear functor** $F : V \to W$ is an internal functor in Vect.

Let **2Vect** denote the category consisting of 2-vector spaces and linear functors between them. There is a category, denoted as **2Term**, whose objects are 2-term chain complexes and whose morphisms are chain maps.

It is well known that the categories **2Vect** and **2Term** are equivalent. Roughly speaking, given a 2-vector space $V = (V_1, V_0, s, t, i, \circ)$,

$$\ker(s) \xrightarrow{t} V_0$$

is a 2-term complex. Conversely, the 2-term complex of vectors $C_1 \xrightarrow{d} C_0$ gives a 2-vector space of which the set of objects is C_0, the set of morphism is $C_0 \oplus C_1$, the identity-assigning map is given by $i(x) = (x, 0)$ for any $x \in C_0$, the source map s is given by $s(x, \overrightarrow{f}) = x$ and the target map t is given by $t(x, \overrightarrow{f}) = x + d\overrightarrow{f}$ for all $(x, \overrightarrow{f}) \in C_0 \oplus C_1$.

2.13. Definition.

1. *Given two linear functors $F, G : V \to W$ between 2-vector spaces, a* **linear natural transformation** $\alpha : F \Rightarrow G$ *is a natural transformation in Vect.*
2. *Given two chain maps $\varphi, \psi : C \to C'$ of 2-term chain complexes, a* **chain homotopy** $\tau : \varphi \Rightarrow \psi$ *is a linear map $\tau : C_0 \to C_1'$ satisfying $d'\tau = \psi_0 - \varphi_0$ and $\tau d = \psi_1 - \varphi_1$.*

Let **2Vect** denote the 2-category of 2-vector spaces, linear functors and linear natural transformations. Also let **2Term** be the 2-category of 2-term chain complexes, chain maps, and chain homotopies.

Furthermore, we have

2.14. Proposition. ([2]) *The 2-category 2Vect is 2-equivalent to the 2-category 2Term.*

3. Rota-Baxter Lie 2-algebras and 2-term Rota-Baxter L_∞-algebras

In this section, we first introduce the notion of a Rota-Baxter Lie 2-algebra which is a Lie 2-algebra with a linear functor satisfying the Rota-Baxter identity up to a natural isomorphism. Then we introduce the notion of a 2-term Rota-Baxter L_∞-algebra. Finally, we show that the category of Rota-Baxter Lie 2-algebras and the category of 2-term Rota-Baxter L_∞-algebras are equivalent.

3.1. Rota-Baxter Lie 2-algebras . We begin by reviewing the concept of a Lie 2-algebra given in [2].

3.2. Definition.

1. *A* **Lie 2-algebra** *is a 2-vector space L together with a skew-symmetric bilinear functor $[\cdot,\cdot] : L \times L \to L$ and a completely antisymmetric trilinear natural isomorphism, the* **Jacobiator**,
$$J_{x,y,z} : [[x, y], z] \to [x, [y, z]] + [[x, z], y],$$

satisfying the identity:

$$\begin{aligned} &([w, J_{x,y,z}] + 1)([J_{w,y,z}, x] + 1)(J_{[w,y],x,z} + J_{w,[x,y],z})[J_{w,x,y}, z] \\ = \;& (J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]})([J_{w,x,z}, y] + 1)J_{[w,x],y,z}. \end{aligned}$$

A Lie 2*-algebra is called* **strict** *if the Jacobiator is the identity isomorphism.*

2. *Given two Lie* 2*-algebras* L *and* L'*, a* **homomorphism** $F = (F_0, F_1, F_2) : L \to L'$ *consists of a linear functor* (F_0, F_1) *from the underlying* 2*-vector space of* L *to that of* L'*, and a skew-symmetric bilinear natural transformation*

$$F_2[x, y] : [F_0(x), F_0(y)] \to F_0[x, y]$$

satisfying

$$(F_1(J_{x,y,z}))F_2[F_2, 1] = (F_2 + F_2)([1, F_2] + [F_2, 1])J_{F_0(x),F_0(y),F_0(z)}.$$

In the following, we give the main definition in this paper.

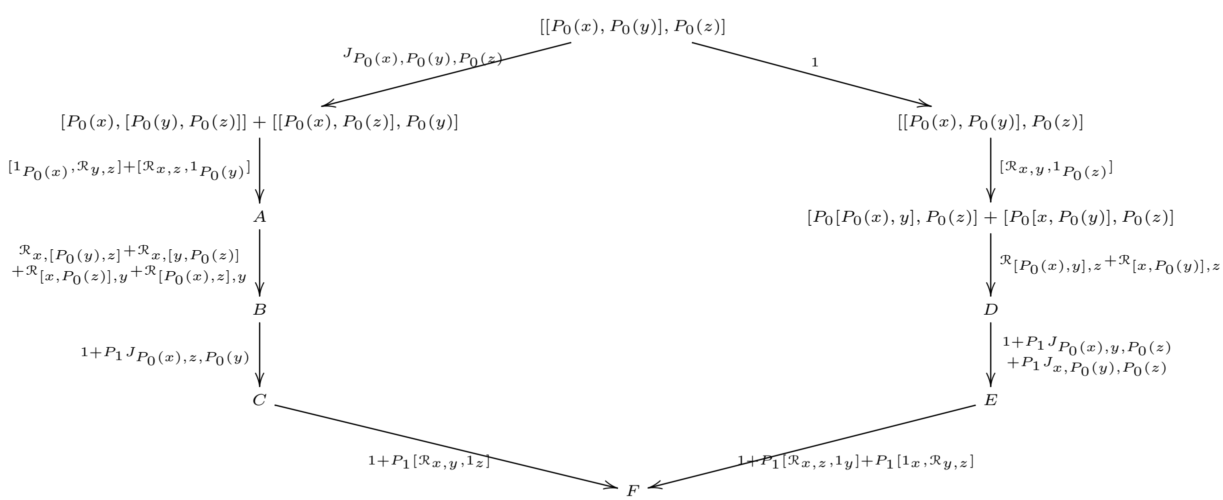
3.3. Definition. *A* **Rota-Baxter Lie 2-algebra** *is a triple* $((L, [\cdot,\cdot]), P, \mathcal{R})$*, where* $(L, [\cdot,\cdot])$ *is a Lie* 2*-algebra,* $P = (P_0, P_1) : L \to L$ *is a linear functor and for* $x, y \in L$*,* $\mathcal{R}_{x,y}$ *is an antisymmetric bilinear natural isomorphism given by*

$$\mathcal{R}_{x,y} : [P_0(x), P_0(y)] \to P_0[P_0(x), y] + P_0[x, P_0(y)],$$

such that the following Rota-Baxter relation is satisfied,

$$\begin{aligned} & (1 + P_1[\mathcal{R}_{x,y}, 1_z])\left(1 + P_1 J_{P_0(x),z,P_0(y)}\right)\left(\mathcal{R}_{x,[P_0(y),z]} + \mathcal{R}_{x,[y,P_0(z)]} + \mathcal{R}_{[x,P_0(z)],y} + \mathcal{R}_{[P_0(x),z],y}\right) \\ & \left([1_{P_0(x)}, \mathcal{R}_{y,z}] + [\mathcal{R}_{x,z}, 1_{P_0(y)}]\right) J_{P_0(x),P_0(y),P_0(z)} \\ = \;& (1 + P_1[\mathcal{R}_{x,z}, 1_y] + P_1[1_x, \mathcal{R}_{y,z}])\left(1 + P_1 J_{P_0(x),y,P_0(z)} + P_1 J_{x,P_0(y),P_0(z)}\right) \\ & \left(\mathcal{R}_{[P_0(x),y],z} + \mathcal{R}_{[x,P_0(y)],z}\right)[\mathcal{R}_{x,y}, 1_{P_0(z)}], \end{aligned}$$

which can be showed as the following commutative diagram



where

$$\begin{array}{rcl}
A & = & [P_0(x), P_0[P_0(y), z] + P_0[y, P_0(z)]] + [P_0[x, P_0(z)], P_0(y)] + [P_0[P_0(x), z], P_0(y)]; \\
B & = & P_0[P_0(x), [P_0(y), z]] + P_0[x, P_0[P_0(y), z]] + P_0[P_0(x), [y, P_0(z)]] + P_0[x, P_0[y, P_0(z)]] \\
 & & +P_0[P_0[x, P_0(z)], y] + P_0[[x, P_0(z)], P_0(y)] + P_0[P_0[P_0(x), z], y] + P_0[[P_0(x), z], P_0(y)]; \\
C & = & P_0[P_0(x), [P_0(y), z]] + P_0[x, P_0[P_0(y), z]] + P_0[P_0(x), [y, P_0(z)]] + P_0[x, P_0[y, P_0(z)]] \\
 & & +P_0[P_0[x, P_0(z)], y] + P_0[[x, P_0(z)], P_0(y)] + P_0[P_0[P_0(x), z], y] + P_0[P_0(x), [z, P_0(y)]] \\
 & & +P_0[[P_0(x), P_0(y)], z]; \\
D & = & P_0[P_0[P_0(x), y], z] + P_0[P_0[x, P_0(y)], z] + P_0[[P_0(x), y], P_0(z)] + P_0[[x, P_0(y)], P_0(z)]; \\
E & = & P_0[P_0[P_0(x), y], z] + P_0[P_0[x, P_0(y)], z] + P_0[P_0(x), [y, P_0(z)]] + P_0[[x, P_0(z)], P_0(y)] \\
 & & +P_0[[P_0(x), P_0(z)], y] + P_0[x, [P_0(y), P_0(z)]]; \\
F & = & P_0[P_0[P_0(x), y], z] + P_0[P_0[x, P_0(y)], z] + P_0[P_0(x), [y, P_0(z)]] + P_0[[x, P_0(z)], P_0(y)] \\
 & & +P_0[P_0[P_0(x), z], y] + P_0[P_0[x, P_0(z)], y] + P_0[x, P_0[P_0(y), z]] + P_0[x, P_0[y, P_0(z)]].
\end{array}$$

A Rota-Baxter Lie 2-algebra is called **strict** *if* $(L, [\cdot, \cdot])$ *is a strict Lie 2-algebra and the natural isomorphism* $\mathcal{R}_{x,y}$ *is the identity isomorphism.*

3.4. Definition. *Let* $(L, P, \mathcal{R})$ *and* $(L', P', \mathcal{R}')$ *be two Rota-Baxter Lie 2-algebras. A* **homomorphism of Rota-Baxter Lie 2-algebras** $F : L \to L'$ *consists of a homomorphism of Lie 2-algebras* $(F_0, F_1, F_2) : L \to L'$ *and a natural linear transformation*

$$F_3(x) : P'_0(F_0(x)) \to F_0(P_0(x))$$

such that the following equation holds

$$\begin{array}{rl}
 & (F_3[P_0(x), y] + F_3[x, P_0(y)])(P'_1F_2(P_0(x), y) + P'_1F_2(x, P_0(y))) \\
 & (P'_1[F_3(x), 1_{F_0(y)}] + P'_1[1_{F_0(x)}, F_3(y)])\mathcal{R}_{F_0(x),F_0(y)} \\
= & F_1(\mathcal{R}_{x,y})F_2(P_0(x), P_0(y))[F_3(x), F_3(y)],
\end{array}$$

or, in terms of commutative diagram,

$$\begin{array}{ccccc}
[P'_0(F_0(x)), P'_0(F_0(y))] & \xrightarrow{\mathcal{R}_{F_0(x),F_0(y)}} & \begin{array}{c}P'_0[P'_0(F_0(x)),F_0(y)] \\ +P'_0[F_0(x),P'_0(F_0(y))]\end{array} & \xrightarrow{\begin{array}{c}P'_1[F_3(x),1_{F_0(y)}] \\ +P'_1[1_{F_0(x)},F_3(y)]\end{array}} & \begin{array}{c}P'_0[F_0(P_0(x)),F_0(y)] \\ +P'_0[F_0(x),F_0(P_0(y))]\end{array} \\
\Big\downarrow{\scriptstyle [F_3(x),F_3(y)]} & & & & \Big\downarrow{\scriptstyle P'_1F_2(P_0(x),y)+P'_1F_2(x,P_0(y))} \\
[F_0(P_0(x)), F_0(P_0(y))] & & & & P'_0(F_0[P_0(x), y]) + P'_0(F_0[x, P_0(y)]) \\
\Big\downarrow{\scriptstyle F_2(P_0(x),P_0(y))} & & & & \Big\downarrow{\scriptstyle F_3[P_0(x),y]+F_3[x,P_0(y)]} \\
F_0[P_0(x), P_0(y)] & & \xrightarrow[F_1(\mathcal{R}_{x,y})]{} & & F_0(P_0[P_0(x), y]) + F_0(P_0[x, P_0(y)]).
\end{array}$$

Let $(L,P,\mathcal{R})$, $(L',P',\mathcal{R}')$ and $(L'',P'',\mathcal{R}'')$ be Rota-Baxter Lie 2-algebras. Let $F:(L,P,\mathcal{R})\to(L',P',\mathcal{R}')$ and $G:(L',P',\mathcal{R}')\to(L'',P'',\mathcal{R}'')$ be homomorphisms of Rota-Baxter Lie 2-algebras. We define the composite functor $G\circ F:(L,P,\mathcal{R})\to(L'',P'',\mathcal{R}'')$ to be the usual composite of the underlying 2-vector space functor: $L\xrightarrow{F}L'\xrightarrow{G}L''$, while letting $(G\circ F)_2$ and $(G\circ F)_3$ be defined as the following composite

$$\begin{aligned}(G\circ F)_2[(G\circ F)_0(x),(G\circ F)_0(y)] &= (G\circ F_2)\,(G_2[G_0(F_0(x)),G_0(F_0(y))])=(G\circ F)_0[x,y],\\ (G\circ F)_3\,(P_0''((G\circ F)_0(x))) &= (G\circ F_3)\,(G_3(P_0''(G_0(F_0(x)))))=(G\circ F)_0\,(P_0(x)),\end{aligned}$$

where $G\circ F_2$(resp. $G\circ F_3$) is the result of whiskering the functor G by the natural transformation F_2(resp. F_3). The identity homomorphism 1_L has the identity functor as its underlying functor, together identity natural transformations $(1_L)_2$ and $(1_L)_3$. It is straightforward to obtain

3.5. PROPOSITION. *There is a category, which we denote by* **RBLie2Alg***, with Rota-Baxter Lie 2-algebras as objects, Rota-Baxter Lie 2-algebra homomorphisms as morphisms.*

3.6. 2-TERM ROTA-BAXTER L_∞-ALGEBRAS. The notion of an L_∞-algebra was introduced by Stasheff in [20]. We begin by reviewing the concept of a 2-term L_∞-algebra.

3.7. DEFINITION. *A* **2-term L_∞-algebra** *on a graded vector space $\mathcal{G}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ consists of the following data:*

- *a complex of vector spaces:* $\mathfrak{g}_1\xrightarrow{l_1}\mathfrak{g}_0$,
- *a skew-symmetric bilinear map* $l_2:\mathfrak{g}_i\otimes\mathfrak{g}_j\longrightarrow\mathfrak{g}_{i+j}$, *where* $0\le i+j\le 1$, *which we denote more suggestively as* $[\cdot,\cdot]$,
- *a skew-symmetric trilinear map* $l_3:\wedge^3\mathfrak{g}_0\longrightarrow\mathfrak{g}_1$,

such that for all $x_i,x,y,z\in\mathfrak{g}_0$ and $u,v\in\mathfrak{g}_1$, the following equalities are satisfied:

(a) $l_1l_2(x,u)=l_2(x,l_1(u)),\quad l_2(l_1(u),v)=l_2(u,l_1(v))$,

(b) $l_1l_3(x,y,z)=l_2(x,l_2(y,z))+l_2(z,l_2(x,y))+l_2(y,l_2(z,x))$,

(c) $l_3(x,y,l_1(u))=l_2(x,l_2(y,u))+l_2(u,l_2(x,y))+l_2(y,l_2(u,x))$,

(d) *the Jacobiator identity:*

$$\sum_{i=1}^{4}(-1)^{i+1}l_2(x_i,l_3(x_1,\cdots,\hat{x_i},\cdots,x_4))+\sum_{i<j}(-1)^{i+j}l_3(l_2(x_i,x_j),x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_4)=0.$$

We usually denote a 2-term L_∞-algebra by $(\mathfrak{g}_1,\mathfrak{g}_0,l_1,l_2,l_3)$, or simply by $\mathcal{G}$. A 2-term L_∞-algebra is called **strict** if $l_3=0$.

3.8. Definition. *Let* $\mathcal{G} = (\mathfrak{g}_1, \mathfrak{g}_0, l_1, l_2, l_3)$ *and* $\mathcal{G}' = (\mathfrak{g}'_1, \mathfrak{g}'_0, l'_1, l'_2, l'_3)$ *be 2-term* L_∞*-algebras. An* L_∞**-homomorphism** $\phi : \mathcal{G} \to \mathcal{G}'$ *consists of:*

- *a chain map* $\phi : \mathcal{G} \to \mathcal{G}'$ *which consists of linear maps* $\phi_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0$ *and* $\phi_1 : \mathfrak{g}_1 \to \mathfrak{g}'_1$ *preserving the differential:* $l'_1\phi_1 = \phi_0 l_1$,
- *a skew-symmetric bilinear map* $\phi_2 : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}'_1$,

such that the following equations hold for all $x, y, z \in \mathfrak{g}_0$, $u \in \mathfrak{g}_1$:

$$\begin{aligned} l'_1(\phi_2(x,y)) &= \phi_0[x,y] - [\phi_0(x), \phi_0(y)], \\ \phi_2(x, l_1(u)) &= \phi_1[x,u] - [\phi_0(x), \phi_1(u)], \end{aligned}$$

and

$$\begin{aligned} & [\phi_2(x,y), \phi_0(z)] + \phi_2([x,y],z) + \phi_1(l_3(x,y,z)) \\ = \; & l_3(\phi_0(x), \phi_0(y), \phi_0(z)) + [\phi_0(x), \phi_2(y,z)] + [\phi_2(x,z), \phi_0(y)] + \phi_2(x,[y,z]) + \phi_2([x,z],y). \end{aligned}$$

There is a category $\mathbf{2TermL_\infty}$ with 2-term L_∞-algebras as objects and L_∞-homomorphisms as morphisms.

The Rota-Baxter operators on L_∞-algebras are introduced in [21]. In the following, we give the notion of Rota-Baxter operators on 2-term L_∞-algebras.

3.9. Definition. *Let* $\mathcal{G} = (\mathfrak{g}_1, \mathfrak{g}_0, l_1, l_2, l_3)$ *be a 2-term* L_∞*-algebra. A triple* $\mathfrak{R} = (R_0, R_1, R_2)$, *where* $R_0 : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$, $R_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_1$ *is a chain map, and* $R_2 : \wedge^2\mathfrak{g}_0 \longrightarrow \mathfrak{g}_1$ *is a linear map, is called a* **Rota-Baxter operator** *on* $\mathcal{G}$ *if for all* $x, y, x_1, x_2, x_3 \in \mathfrak{g}_0$ *and* $u \in \mathfrak{g}_1$, *the following conditions are satisfied:*

1. $R_0\big(l_2(R_0x, y) + l_2(x, R_0y)\big) - l_2(R_0x, R_0y) = l_1R_2(x,y)$,
2. $R_1\big(l_2(R_1u, x) + l_2(u, R_0x)\big) - l_2(R_1u, R_0x) = R_2(l_1(u), x)$,
3. $$\begin{aligned} & l_2(R_0x_1, R_2(x_2,x_3)) + l_2(R_0x_3, R_2(x_1,x_2)) + l_2(R_0x_2, R_2(x_3,x_1)) \\ & +R_2\big(x_3, l_2(R_0x_1, x_2) - l_2(R_0x_2, x_1)\big) + R_2\big(x_2, l_2(R_0x_3, x_1) - l_2(R_0x_1, x_3)\big) \\ & +R_2\big(x_1, l_2(R_0x_2, x_3) - l_2(R_0x_3, x_2)\big) + R_1\big(l_2(R_2(x_2,x_3), x_1) - l_3(R_0x_2, R_0x_3, x_1)\big) \\ & +R_1\big(l_2(R_2(x_1,x_2), x_3) - l_3(R_0x_1, R_0x_2, x_3)\big) + R_1\big(l_2(R_2(x_3,x_1), x_2) - l_3(R_0x_3, R_0x_1, x_2)\big) \\ & +l_3(R_0x_1, R_0x_2, R_0x_3) = 0. \end{aligned}$$

Moreover, a 2-term L_∞*-algebra* $\mathcal{G}$ *with a triple* $\mathfrak{R} = (R_0, R_1, R_2)$ *is called a* **2-term Rota-Baxter** L_∞**-algebra**. *We denote a 2-term Rota-Baxter* L_∞*-algebra by* $(\mathcal{G}, \mathfrak{R})$. *A 2-term Rota-Baxter* L_∞*-algebra is called* **strict** *if* $l_3 = 0$ *and* $R_2 = 0$.

3.10. Definition. *Let* $(\mathcal{G},\mathfrak{R})$ *and* $(\mathcal{G}',\mathfrak{R}')$ *be* 2*-term Rota-Baxter* L_∞*-algebras. A* **Rota-Baxter** L_∞**-homomorphism** $\phi=(\phi_0,\phi_1,\phi_2,\phi_3):(\mathcal{G},\mathfrak{R})\to(\mathcal{G}',\mathfrak{R}')$ *consists of a homomorphism* (ϕ_0,ϕ_1,ϕ_2) *from the* 2*-term* L_∞*-algebra* $\mathcal{G}$ *to the* 2*-term* L_∞*-algebra* $\mathcal{G}'$ *and a linear map* $\phi_3:\mathfrak{g}_0\to\mathfrak{g}_1'$*, such that, for all* $x,y\in\mathfrak{g}_0$*,* $u\in\mathfrak{g}_1$*, the following equations hold*

$$\begin{aligned}
&l_1'(\phi_3(x))=-R_0'(\phi_0(x))+\phi_0(R_0(x));\\
&\phi_3(l_1(u))=\phi_1(R_1(u))-R_1'(\phi_1(u));\\
&R_2'(\phi_0(x),\phi_0(y))+R_1'[\phi_3(x),\phi_0(y)]+R_1'[\phi_0(x),\phi_3(y)]\\
&+R_1'(\phi_2(R_0(x),y))+R_1'(\phi_2(x,R_0(y)))+\phi_3(R_0(x),y)+\phi_3(x,R_0(y))\\
=\ &[\phi_3(x),\phi_3(y)]+\phi_2(R_0(x),R_0(y))+\phi_1(R_2(x,y)).
\end{aligned}$$

Let $\phi:(\mathcal{G},\mathfrak{R})\to(\mathcal{G}',\mathfrak{R}')$ and $\psi:(\mathcal{G}',\mathfrak{R}')\to(\mathcal{G}'',\mathfrak{R}'')$ be a pair of Rota-Baxter L_∞-homomorphisms. The composite $\psi\phi:(\mathcal{G},\mathfrak{R})\to(\mathcal{G}'',\mathfrak{R}'')$ is the usual chain map while defining $(\psi\phi)_2$ and $(\psi\phi)_3$ as follows:

$$\begin{aligned}
(\psi\phi)_2(x,y)&=\psi_2(\phi_0(x),\phi_0(y))+\psi_1(\phi_2(x,y)),\\
(\psi\phi)_3(x)&=\psi_3(\phi_0(x))+\psi_1(\phi_3(x)).
\end{aligned}$$

The identity homomorphism $1_{(\mathcal{G},\mathfrak{R})}:(\mathcal{G},\mathfrak{R})\to(\mathcal{G},\mathfrak{R})$ has the identity chain map as its underlying map, together with $(1_{(\mathcal{G},\mathfrak{R})})_2=0$ and $(1_{(\mathcal{G},\mathfrak{R})})_3=0$.

With these definitions, it is straightforward to obtain

3.11. Proposition. *There is a category* $\mathbf{2TermRBL_\infty}$ *with* 2*-term Rota-Baxter* L_∞*-algebras as objects and Rota-Baxter* L_∞*-homomorphisms as morphisms.*

3.12. The equivalence of Rota-Baxter Lie 2-algebras and 2-term Rota-Baxter L_∞-algebras . The well-known fact between Lie 2-algebras and 2-term L_∞-algebras is given as follows.

3.13. Theorem. ([2]) *The categories* **Lie2Alg** *and* $\mathbf{2TermL_\infty}$ *are equivalent.*

In order to prove the main Theorem 3.14, we first recall the construction of the equivalence between **Lie2Alg** and $\mathbf{2TermL_\infty}$ in the following.

The functor from **Lie2Alg** to $\mathbf{2TermL_\infty}$ is denoted as

$$S:\mathbf{Lie2Alg}\to\mathbf{2TermL_\infty}. \tag{4}$$

Suppose that L is a Lie 2-algebra. The corresponding 2-term L_∞-algebra $S(L)=(\mathfrak{g}_1,\mathfrak{g}_0,l_1,l_2,l_3)$ is given by

$$\begin{aligned}
&\mathfrak{g}_0=L_0,\quad \mathfrak{g}_1=\ker(s)\subseteq L_1,\\
&l_1(u)=t(u),\qquad u\in\mathfrak{g}_1,\\
&l_2(x,y)=[x,y],\qquad x,y\in\mathfrak{g}_0,\\
&l_2(x,u)=-l_2(u,x)=[1_x,u],\qquad x\in\mathfrak{g}_0,u\in\mathfrak{g}_1,\\
&l_2(u,v)=0,\qquad u,v\in\mathfrak{g}_1,\\
&l_3(x,y,z)=\overrightarrow{J_{x,y,z}},\qquad x,y,z\in\mathfrak{g}_0.
\end{aligned}$$

Let L and L' be two Lie 2-algebras. Let $S(L) = (\mathfrak{g}_1, \mathfrak{g}_0, l_1, l_2, l_3)$ and $S(L') = (\mathfrak{g}'_1, \mathfrak{g}'_0, l'_1, l'_2, l'_3)$ be the corresponding 2-term L_∞-algebras. Assume that $F : L \to L'$ is a Lie 2-algebra homomorphism. The corresponding L_∞-homomorphism $\phi = S(F) : S(L) \to S(L')$ is given by

$$\phi_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0, \tag{5}$$
$$\phi_1 : \mathfrak{g}_1 \to \mathfrak{g}'_1, \tag{6}$$
$$\phi_2 : V_0 \times V_0 \to V'_1, \tag{7}$$

where $\phi_0(x) = F_0(x)$, $\phi_1(u) = F_1|_{\ker(s)}(u)$ and $\phi_2(x, y) = \overrightarrow{F_2(x, y)}$.

The functor from $\mathbf{2TermL_\infty}$ to $\mathbf{Lie2Alg}$ is denoted as

$$T : \mathbf{2TermL_\infty} \to \mathbf{Lie2Alg}. \tag{8}$$

Given a 2-term L_∞-algebra $\mathcal{G} = (\mathfrak{g}_1, \mathfrak{g}_0, l_1, l_2, l_3)$, we have a Lie 2-algebra $T(\mathcal{G}) = L$, where the object $L_0 = \mathfrak{g}_0$, the morphism $L_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the source, target, identity-assigning and composite maps are given by

$$\begin{aligned} s(f) &= x, \quad f = (x, \overrightarrow{f}) \in L_1, \\ t(f) &= x + l_1(\overrightarrow{f}), \quad f = (x, \overrightarrow{f}) \in L_1, \\ i(y) &= (y, 0), \quad y \in L_0, \\ f \circ g &= (x, \overrightarrow{f} + \overrightarrow{g}), \quad f = (x, \overrightarrow{f}), g = (y, \overrightarrow{g}) \in L_1. \end{aligned}$$

Then we see $t(f) - s(f) = l_1(\overrightarrow{f})$. The bracket functor $[\cdot, \cdot] : L \times L \to L$ is given by

$$[x, y] = l_2(x, y), \tag{9}$$
$$[f, g] = (l_2(x, z), l_2(\overrightarrow{f}, z) + l_2(y, \overrightarrow{g})) = (l_2(x, z), l_2(x, \overrightarrow{g}) + l_2(\overrightarrow{f}, w)) \tag{10}$$

for arbitrary objects $x, y \in L_0$, and arbitrary morphisms $f : x \to y, g : z \to w \in L_1$. Note that the identity

$$l_2(\overrightarrow{f}, z) + l_2(y, \overrightarrow{g}) = l_2(x, \overrightarrow{g}) + l_2(\overrightarrow{f}, z)$$

holds since $l_2(l_1(\overrightarrow{f}), \overrightarrow{g}) = l_2(\overrightarrow{f}, l_1(\overrightarrow{g}))$. The Jacobiator for L is given by

$$J_{x,y,z} = ([[x, y], z], l_3(x, y, z)).$$

For each L_∞-homomorphism $\phi : \mathcal{G} \to \mathcal{G}'$, we let $T(\mathcal{G}) = L$ and $T(\mathcal{G}') = L'$. The Lie 2-algebra homomorphism $T(\phi) = F : L \to L'$ is defined as follows

$$\begin{aligned} &F_0 : L_0 \to L'_0, \qquad F_0(x) = \phi_0(x), \\ &F_1 : L_1 \to L'_1, \qquad F_1(f) = F_1(x, \overrightarrow{f}) = (\phi_0(x), \phi_1(\overrightarrow{f})), \\ &F_2 : L_0 \times L_0 \to L'_1, \qquad F_2(x, y) = ([\phi_0(x), \phi_0(y)], \phi_2(x, y)). \end{aligned}$$

Finally, the natural isomophisms $\alpha : TS \Longrightarrow 1_{\mathbf{Lie2Alg}}$ and $\beta : ST \Longrightarrow 1_{\mathbf{2TermL_\infty}}$ imply the equivalence between $\mathbf{Lie2Alg}$ and $\mathbf{2TermL_\infty}$.

As a generalization of Theorem 3.13, we have

3.14. THEOREM. *The categories* **RBLie2Alg** *an* $\mathbf{2TermRB}L_\infty$ *are equivalent.*

PROOF. First we construct a functor $S^{\mathrm{RB}}:\mathbf{RBLie2Alg}\to\mathbf{2TermRB}L_\infty$ which 'lifts' the functor S in (4) as the following commutative diagram shows

$$\begin{array}{ccc} \mathbf{RBLie2Alg} & \xrightarrow{S^{\mathrm{RB}}} & \mathbf{2TermRB}L_\infty \\ \Big\downarrow{\scriptstyle U_{\mathbf{RBLie2Alg}}} & & \Big\downarrow{\scriptstyle U_{\mathbf{2TermRB}L_\infty}} \\ \mathbf{Lie2Alg} & \xrightarrow[S]{} & \mathbf{2TermL_\infty} \end{array}$$

where $U_{\mathbf{RBLie2Alg}}$ and $U_{\mathbf{2TermRB}}L_\infty$ are forgetful functors.

Given a Rota-Baxter Lie 2-algebra (L,P), we obtain a 2-term Rota-Baxter L_∞ algebra $S^{\mathrm{RB}}(L,P)=(\mathcal{G},\mathfrak{R})$. Here $\mathcal{G}=(\mathfrak{g}_1,\mathfrak{g}_0,l_1,l_2,l_3)$ is the 2-term L_∞ algebra $S(P)$, and $\mathfrak{R}=(R_0,R_1,R_2)$ on $\mathcal{G}$ is given by

$$\begin{aligned} &R_0:\mathfrak{g}_0\to\mathfrak{g}_0, \qquad R_0(x)=P_0(x),\\ &R_1:\mathfrak{g}_1\to\mathfrak{g}_1, \qquad R_1(u)=P_1(u),\\ &R_2:\mathfrak{g}_0\times\mathfrak{g}_0\to\mathfrak{g}_1', \qquad R_2(x,y)=\overrightarrow{\mathcal{R}_{x,y}}. \end{aligned}$$

In the following, we first show that the conditions (1), (2) and (3) in Definition 3.9 hold.

The condition (1) holds since

$$R_0\left(l_2(R_0x,y)+l_2(x,R_0y)\right)-l_2(R_0x,R_0y)=(t-s)\mathcal{R}_{x,y}=t\overrightarrow{\mathcal{R}_{x,y}}=l_1R_2(x,y).$$

The naturality of $\mathcal{R}_{x,y}$ implies that for any $f:x\to z$, we have the identity

$$\mathcal{R}_{x,y}\left(P_1[P_1(f),1_y]+P_1[f,1_{P_0(y)}]\right)=[P_1(f),1_{P_0(y)}]\mathcal{R}_{z,y}, \tag{11}$$

Taking the arrow parts of both sides of the above Eq. (11), we have

$$\overrightarrow{\mathcal{R}_{x,y}}+\left(\overrightarrow{P_1[P_1(f),1_y]}+\overrightarrow{P_1[f,1_{P_0(y)}]}\right)=\overrightarrow{[P_1(f),1_{P_0(y)}]}+\overrightarrow{\mathcal{R}_{z,y}},$$

which implies that

$$P_1\left([P_1(\overrightarrow{f}),1_y]+[\overrightarrow{f},1_{P_0(y)}]\right)-[P_1(\overrightarrow{f}),1_{P_0(y)}]=\overrightarrow{\mathcal{R}_{z-x,y}}. \tag{12}$$

Thus we have

$$R_1\left(l_2(R_1(\overrightarrow{f}),y)+l_2(\overrightarrow{f},R_0(y))\right)-l_2(R_1(\overrightarrow{f}),R_0(y))=R_2(l_1(\overrightarrow{f}),y). \tag{13}$$

This implies that the conditon (2) holds.

It is straightforward to check that the Rota-Baxter relation in Definition 3.3 is equivalent to

$$[1_{P_0(x)},\overrightarrow{\mathcal{R}_{y,z}}]+\overrightarrow{\mathcal{R}_{z,[P_0(x),y]+[x,P_0(y)]}}+P_1\left([\overrightarrow{\mathcal{R}_{y,z}},1_x]-\overrightarrow{J_{P_0(y),P_0(z),x}}\right)+c.p.+\overrightarrow{J_{P_0(x),P_0(y),P_0(z)}}=0,$$

which implies that

$$\begin{aligned}&l_2(R_0x, R_2(y,z)) + R_2\left(g(z, l_2(R_0x,y) - l_2(R_0y,x)\right)\\&+R_1\left(l_2(R_2(y,z),x) - l_3(R_0y,R_0z,x)\right) + c.p. + l_3(R_0x,R_0y,R_0z) = 0.\end{aligned}$$

This implies that the conditon (3) holds.

Next we construct a Rota-Baxter L_∞-homomorphism from a Rota-Baxter Lie 2-algebra homomorphism. Let $F : (L,P,\mathcal{R}) \to (L',P',\mathcal{R}')$ be a Rota-Baxter Lie 2-algebra homomorphism. Let $(\mathcal{G},\mathfrak{R}) = S^{\mathrm{RB}}(L,P,\mathcal{R})$ and $(\mathcal{G},\mathfrak{R}) = S^{\mathrm{RB}}(L',P',\mathcal{R}')$. Then we obtain an L_∞-homomorphism $\phi = S(F) : \mathcal{G} \to \mathcal{G}'$ of $S^{\mathrm{RB}}(F)$ as in (5-7). Define a map $\phi_3 : V_0 \to V_1'$ by

$$\phi_3(x) = \overrightarrow{F_3(x)} : 0 \to -P_0'(\phi_0(x)) + \phi_0(P_0(x)).$$

By the following identity

$$l_1'(\phi_3(x)) = t(\overrightarrow{F_3(x)}) = -P_0'(\phi_0(x)) + \phi_0(P_0(x)) = -R_0'(\phi_0(x)) + \phi_0(R_0(x)),$$

we have the first equation in Definition 3.10.

By the naturality of F_3, for every morphism $f : x \to y$, we obtain

$$\phi_1(P_1(f))F_3(x) = F_3(y)P_1'(\phi_1(f)).$$

Furthermore, we have

$$\overrightarrow{F_3(x)} + \phi_1(P_1(\overrightarrow{f})) = \overrightarrow{\phi_1(P_1(f))F_3(x)} = \overrightarrow{F_3(y)P_1'(\phi_1(f))} = P_1'(\phi_1(\overrightarrow{f})) + \overrightarrow{F_3(y)},$$

which implies that

$$\phi_3(l_1(f)) = \overrightarrow{F_3(y-x)} = \overrightarrow{F_3(y)} - \overrightarrow{F_3(x)} = \phi_1(P_1(\overrightarrow{f})) - P_1'(\phi_1(\overrightarrow{f})).$$

Thus for any $u \in \mathfrak{g}_1$, we have

$$\phi_3(l_1(u)) = \phi_1(P_1(u)) - P_1'(\phi_1(u)) = \phi_1(R_1(u)) - R_1'(\phi_1(u)).$$

This implies that the second equation in Definition 3.10 holds.

It is straightforward to check that the coherence law in Definition 3.4 is equivalent to the following equation

$$\begin{aligned}&(\overrightarrow{F_3[P_0(x),y]} + \overrightarrow{F_3[x,P_0(y)]})(P_1'\overrightarrow{F_2(P_0(x),y)} + P_1'\overrightarrow{F_2(x,P_0(y))})\\&(P_1'[\overrightarrow{F_3(x)},1_{F_0(y)}] + P_1'[1_{F_0(x)},\overrightarrow{F_3(y)}])\overrightarrow{\mathcal{R}_{F_0(x),F_0(y)}}\\=\;&F_1(\overrightarrow{\mathcal{R}_{x,y}})\overrightarrow{F_2(P_0(x),P_0(y))}[\overrightarrow{F_3(x)},\overrightarrow{F_3(y)}],\end{aligned}$$

which implies that the third equation in Definition 3.10 holds.

One can also deduce that S^{RB} preserves the identity homomorphisms and the composition of homomorphisms. Thus S^{RB} is a functor from **RBLie2Alg** to **2TermRB**L_∞.

Conversely, we construct a functor $T^{\mathrm{RB}} : \mathbf{2TermRB}L_\infty \to \mathbf{RBLie2Alg}$ as a 'lifting' of the functor T in (8) in terms of the following commutative diagram

$$\begin{array}{ccc} \mathbf{2TermRB}L_\infty & \xrightarrow{T^{\mathrm{RB}}} & \mathbf{RBLie2Alg} \\ {\scriptstyle U_{\mathbf{2TermRB}L_\infty}}\downarrow & & \downarrow{\scriptstyle U_{\mathbf{RBLie2Alg}}} \\ \mathbf{2TermL}_\infty & \xrightarrow[T]{} & \mathbf{Lie2Alg} \end{array}$$

where $U_{\mathbf{2TermRB}}L_\infty$ and $U_{\mathbf{RBLie2Alg}}$ are corresponding forgetful functors.

Let $(\mathcal{G}, \mathfrak{R})$ be a 2-term Rota-Baxter L_∞ algebra, where $\mathcal{G} = (\mathfrak{g}_0, \mathfrak{g}_1, l_1, l_2, l_3)$ is a 2-term L_∞ algebra and $\mathfrak{R} = (R_0, R_1, R_2)$ is a Rota-Baxter operator on $\mathcal{G}$. Then we have a Lie 2-algebra $T(\mathcal{G}) = L$ with $L_0 = \mathfrak{g}_0$ and $L_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Define a linear functor $P : L \to L$ by

$$\begin{aligned} &P_0 : L_0 \to L_0, \quad P_0(x) = R_0(x) \quad \forall\, x \in L_0, \\ &P_1 : L_1 \to L_1, \quad P_1(y, u) = (R_0(y), R_1(u)), \quad \forall\, y \in \mathfrak{g}_0, u \in \mathfrak{g}_1. \end{aligned}$$

The natural isomorphism $\mathcal{R}_{x,y} : [P_0(x), P_0(y)] \to P_0([P_0(x), y] + [x, P_0(y)])$ is defined by

$$\mathcal{R}_{x,y} = ([P_0(x), P_0(y)], R_2(x, y)).$$

Thus, we obtain a Rota-Baxter Lie 2-algebra $(L, P) = T^{\mathrm{RB}}(\mathcal{G}, \mathfrak{R})$ from a 2-term Rota-Baxter L_∞ algebra $(\mathcal{G}, \mathfrak{R})$.

For any Rota-Baxter L_∞-homomorphism $\phi : (\mathcal{G}, \mathfrak{R}) \to (\mathcal{G}', \mathfrak{R}')$, we next construct a Rota-Baxter Lie 2-algebra homomorphism $F = T^{\mathrm{RB}}(\phi)$ from $T^{\mathrm{RB}}(\mathcal{G}, \mathfrak{R})$ to $T^{\mathrm{RB}}(\mathcal{G}', \mathfrak{R}')$.

The underlying Lie 2-algebra homomorphism is given by

$$\begin{aligned} &F_0 = \phi_0 : L_0 \to L_0', \\ &F_1 = \phi_0 \oplus \phi_1 : L_1 \to L_1', \\ &F_2 : L_0 \times L_0 \to L_1', \quad F_2(x, y) = ([\phi_0(x), \phi_0(y)], \phi_2(x, y)). \end{aligned}$$

The natural transformation $F_3(x) : P_0'(F_0(x)) \to F_0(P_0(x))$ is defined by

$$F_3(x) = (P_0'(F_0(x)), \phi_3(x)).$$

Applying the correspondence between the composition of morphisms and the addition of their arrow parts, the second equation in Definition 3.10 implies the naturality of F_3. The coherence law in Definition 3.4 also holds by the third equation. Thus F is a Rota-Baxter Lie 2-algebra homomorphism. Furthermore, T^{RB} preserves the identity homomorphisms and the composition of homomorphisms. Therefore, T^{RB} is a functor from $\mathbf{2TermRB}L_\infty$ to $\mathbf{RBLie2Alg}$.

We are left to show that there are natural isomorphisms

$$\alpha^{\mathrm{RB}} : T^{\mathrm{RB}} S^{\mathrm{RB}} \Rightarrow 1_{\mathbf{RBLie2Alg}}, \qquad \beta^{\mathrm{RB}} : S^{\mathrm{RB}} T^{\mathrm{RB}} \Rightarrow 1_{\mathbf{2TermRB}L_\infty}.$$

For any Rota-Baxter Lie 2-algebra $(L, P, \mathcal{R})$, we obtain a 2-term Rota-Baxter L_∞ algebra

$$S^{\mathrm{RB}}(L, P, \mathcal{R}) = (\mathcal{G}, \mathfrak{R}) = ((\mathfrak{g}_0, \mathfrak{g}_1, l_1, l_2, l_3), (R_0, R_1, R_2)),$$

where $S(L) = \mathcal{G}$, and $R_0 = P_0$, $R_1 = P_1|_{\mathfrak{g}_1}$, $R_2(x, y) = \overrightarrow{\mathcal{R}_{x,y}}$. Applying the functor T^{RB} to $(\mathcal{G}, \mathfrak{R})$, we obtain a Rota-Baxter Lie 2-algebra, denoted by $(L', P', \mathcal{R}')$. Here $L' = T(S(L))$, and for all $x \in L'_0$ and $(y, u) \in L'_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, one has

$$P'_0(x) = R_0(x), P'_1(y, u) = (R_0(y), R_1(u)), \mathcal{R}'_{x,y} = ([P_0(x), P_0(y)], R_2(x, y)) = \mathcal{R}_{x,y}.$$

By the isomorphism $\alpha_L : L' \to L$ of semistrict Lie 2-algebras: $(\alpha_L)_0(x) = x$ and $(\alpha_L)_1(y, u) = i(y) + u$, we have $P'_0(x) = P_0(x)$, and

$$P_1((\alpha_L)_1(y, u)) = P_1(i(y)+u) = i(P_0(y))+P_1(u) = (\alpha_L)_1(P_0(y), P_1(u)) = (\alpha_L)_1(P'_1(y, u)).$$

Thus $\alpha^{\mathrm{RB}} : (L', P', \mathcal{R}') \to (L, P, \mathcal{R})$ is an isomorphism of Rota-Baxter Lie 2-algebras. Also by the naturality of α, we see that α^{RB} is a natural isomorphism.

For a 2-term Rota-Baxter L_∞ algebra $(\mathcal{G}, \mathfrak{R}) = ((\mathfrak{g}_0, \mathfrak{g}_1, l_1, l_2, l_3), (R_0, R_1, R_2))$, applying the functor T^{RB} to $(\mathcal{G}, \mathfrak{R})$, we obatin a Rota-Baxter Lie 2-algebra $(L, P, \mathcal{R})$, where $L_0 = \mathfrak{g}_0$, $L_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $P_0(x) = R_0(x)$, $P_1(y, u) = R_0(y)+R_1(u)$ and $\mathcal{R}_{x,y} = ([P_0(x), P_0(y)], R_2(x, y))$ for all $x \in L_0$ and $(y, u) \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Applying S^{RB} to (L, P), we have a 2-term Rota-Baxter L_∞ algebra $(\mathcal{G}', \mathfrak{R}')$, where $\mathcal{G}' = S(T(\mathcal{G}))$, $R'_0(x) = P_0(x)$, $R'_1(u) = P_1(u) = R_1(h)$ and $R'_2(x, y) = \overrightarrow{\mathcal{R}_{x,y}} = R_2(x, y)$ for any $x, y \in \mathfrak{g}_0$ and $u \in \mathfrak{g}_1$. Thus we obtain the 2-term Rota-Baxter L_∞-algebra isomorphism $\beta^{\mathrm{RB}} : (\mathcal{G}', \mathfrak{R}') \to (\mathcal{G}, \mathfrak{R})$. The naturality of β^{RB} follows that of β. Then we obtain a natural isomorphism β^{RB}. ■

For strict Rota-Baxter Lie 2-algebras, there is a category **SRBLie2Alg** with strict Rota-Baxter Lie 2-algebra as objects and Rota-Baxter Lie 2-algebra homomorphisms as morphisms, which is a subcategory of **RBLie2Alg**.

For strict 2-term Rota-Baxter L_∞-algebras, there is a category **SRB2TermL**$_\infty$ with strict 2-term Rota-Baxter L_∞-algebras as objects and Rota-Baxter L_∞-homomorphisms as morphisms, which is a subcategory of **2TermRB**L_∞.

It is straightforward to check that

3.15. Corollary. *The categories SRBLie2Alg and SRB2TermL*$_\infty$ *are equivalent.*

4. Strict 2-term Rota-Baxter L_∞-algebras and crossed modules of Rota-Baxter Lie algebras

In this section, we study the relations between strict 2-term Rota-Baxter L_∞-algebras and crossed modules of Rota-Baxter Lie algebras.

First, we recall the definition of crossed modules of Lie algebras.

4.1. DEFINITION. *A* **crossed module of Lie algebras** *is a quadruple* $((\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1}),(\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0}),\mathrm{d},\rho)$*, where* $(\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1})$ *and* $(\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0})$ *are Lie algebras,* $\mathrm{d}:\mathfrak{g}_1\to\mathfrak{g}_0$ *is a Lie algebra homomorphism and* $\rho:\mathfrak{g}_0\to\mathrm{Der}(\mathfrak{g}_1)$ *is an action of Lie algebra* $\mathfrak{g}_0$ *on Lie algebra* $\mathfrak{g}_1$ *as a derivation, such that*

$$\mathrm{d}(\rho(x)(u))=[x,\mathrm{d}u]_{\mathfrak{g}_0},\quad \rho(\mathrm{d}u)(v)=[u,v]_{\mathfrak{g}_1},\quad \forall\, x\in\mathfrak{g}_0,u,v\in\mathfrak{g}_1. \tag{14}$$

Relations between strict Lie 2-algebras and crossed modules of Lie algebras are described in the following theorem.

4.2. THEOREM. ([2]) *There is a one-to-one corresponding between strict Lie* 2*-algebras and crossed modules of Lie algebras.*

4.3. DEFINITION. *A* **crossed module of Rota-Baxter Lie algebras** *is a quadruple*

$$((\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1},T_1),(\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0},T_0),\mathrm{d},\rho),$$

where $(\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1},T_1)$ *and* $(\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0},T_0)$ *are Rota-Baxter Lie algebras,* $\mathrm{d}:\mathfrak{g}_1\to\mathfrak{g}_0$ *is a Rota-Baxter Lie algebra homomorphism and* $(\rho,T_1):\mathfrak{g}_0\to\mathrm{Der}(\mathfrak{g}_1)$ *is an action of Rota-Baxter Lie algebra* $(\mathfrak{g}_0,T_0)$ *on Lie algebra* $\mathfrak{g}_1$ *as a derivation of the Lie algebra, such that*

$$\mathrm{d}(\rho(x)(u))=[x,\mathrm{d}u]_{\mathfrak{g}_0},\quad \rho(\mathrm{d}u)(v)=[u,v]_{\mathfrak{g}_1},\quad \forall\, x\in\mathfrak{g}_0,u,v\in\mathfrak{g}_1. \tag{15}$$

It is obvious that $((\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0}),(\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1}),\mathrm{d},\rho)$ is a crossed module of Lie algebras.

4.4. EXAMPLE. *Let* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},T)$ *be a Rota-Baxter Lie algebra and* $\mathfrak{h}$ *a Rota-Baxter Lie ideal of* $(\mathfrak{g},T)$*. Then* $(\mathfrak{g},\mathfrak{h},\mathrm{d}=\imath,\rho=\mathrm{ad})$ *is a crossed module for Rota-Baxter Lie algebras, where* $\imath:\mathfrak{h}\longrightarrow\mathfrak{g}$ *is the inclusion.*

4.5. EXAMPLE. *For any Rota-Baxter Lie algebra homomorphism* $f:\mathfrak{g}\longrightarrow\mathfrak{h}$*,* $(\mathfrak{g},\ker f,\imath,\mathrm{ad})$ *is a crossed module of Rota-Baxter Lie algebras.*

4.6. PROPOSITION. *Let* $((\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0},T_0),(\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1},T_1),\mathrm{d},\rho)$ *be a crossed module of Rota-Baxter Lie algebras. Then there is a Rota-Baxter Lie algebra structure on* $\mathfrak{g}_0\oplus\mathfrak{g}_1$ *given by*

$$[x+u,y+v] = [x,y]_{\mathfrak{g}_0}+\rho(x)v-\rho(y)u+[u,v]_{\mathfrak{g}_1}, \tag{16}$$
$$T(x+u) = T_0(x)+T_1(u),\quad \forall\, x,y\in\mathfrak{g}_0,u,v\in\mathfrak{g}_1. \tag{17}$$

PROOF. Since $((\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0}),(\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1}),\mathrm{d},\rho)$ is a crossed module of Lie algebras, we have a Lie algebra $(\mathfrak{g}_0\oplus\mathfrak{g}_1,[\cdot,\cdot])$.

Furthermore, it is straightforward to check that T is a Rota-Baxter operator on the Lie algebra $\mathfrak{g}\oplus\mathfrak{h}$ if and only if T_0 is a Rota-Baxter operator on the Lie algebra $\mathfrak{g}_0$, T_1 is a Rota-Baxter operator on the Lie algebra $\mathfrak{g}_1$ and the following equation holds:

$$T_1(\rho(T_0x)u+\rho(x)T_1u)=\rho(T_0x)T_1u,\quad \forall\, x\in\mathfrak{g}_0,u\in\mathfrak{g}_1,$$

which follows from that (ρ,T_1) is a representation of the Rota-Baxter Lie algebra $(\mathfrak{g}_0,T_0)$ on $\mathfrak{g}_1$. ∎

4.7. THEOREM. *There is a one-to-one corresponding between strict 2-term Rota-Baxter L_∞-algebras and crossed modules of Rota-Baxter Lie algebras.*

PROOF. Let $(\mathfrak{g}_0, \mathfrak{g}_1, l_1, l_2, l_3 = 0; R_0, R_1, R_2 = 0)$ be a strict 2-term Rota-Baxter L_∞-algebra. Define the brackets $[\cdot,\cdot]_{\mathfrak{g}_0}$ and $[\cdot,\cdot]_{\mathfrak{g}_1}$ by

$$[x,y]_{\mathfrak{g}_0} = l_2(x,y), \quad [u,v]_{\mathfrak{g}_1} = l_2(l_1(u),v), \quad \forall\, x,y \in \mathfrak{g}_0, u,v \in \mathfrak{g}_1.$$

Define $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_1)$ by

$$\rho(x)u = l_2(x,u), \quad \forall\, x \in \mathfrak{g}_0, u \in \mathfrak{g}_1.$$

Then $((\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}), (\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}), \mathrm{d} = l_1, \rho)$ is a crossed module of Lie algebras.

Set $T_0 = R_0$ and $T_1 = R_1$. By condition (1) in Definition 3.9, T_0 is a Rota-Baxter operator on the Lie algebra $\mathfrak{g}_0$. By condition (2) in Definition 3.9 and condition (a) in Definition 3.7, for $u, v \in \mathfrak{g}_1$, we have

$$\begin{aligned} & T_1([T_1(u),v]_{\mathfrak{g}_1} + [u, T_1(v)]_{\mathfrak{g}_1}) - [T_1(u), T_1(v)]_{\mathfrak{g}_1} \\ = \; & R_1\big(l_2(l_1R_1(u),v) + l_2(l_1(u),R_1(v)\big) - l_2(l_1R_1(u),R_1(v))) \\ = \; & R_1\big(l_2(R_1(u),l_1(v)) + l_2(u,R_0l_1(v)\big) - l_2(R_1(u),R_0l_1(v))) \\ = \; & 0, \end{aligned}$$

which implies that T_1 is a Rota-Baxter operator on the Lie algebra $\mathfrak{g}_1$. By the fact that d is a Lie algebra homomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_0$ and $l_1 \circ R_1 = R_0 \circ l_1$, d is a Rota-Baxter Lie algebra homomorphism from $(\mathfrak{g}_1, T_1)$ to $(\mathfrak{g}_0, T_0)$. By condition (2) in Definition 3.9, the map $(\rho, T_1) : \mathfrak{g}_0 \to \mathrm{Der}(\mathfrak{g}_1)$ is an action of Rota-Baxter Lie algebra $(\mathfrak{g}_0, T_0)$ on Lie algebra $\mathfrak{g}_1$. Therefore, $((\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}, T_0), (\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}, T_1), \mathrm{d} = l_1, \rho)$ is a crossed module of Rota-Baxter Lie algebras.

Conversely, let $((\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}, T_0), (\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}, T_1), \mathrm{d}, \rho)$ be a crossed module of Rota-Baxter Lie algebras, and then we have a strict 2-term Rota-Baxter L_∞-algebra

$$(\mathfrak{g}_0, \mathfrak{g}_1, l_1 = \mathrm{d}, l_2, l_3 = 0; R_0 = T_0, R_1 = T_1, R_2 = 0),$$

where $l_2 : \mathfrak{g}_i \wedge \mathfrak{g}_j \to \mathfrak{g}_{i+j},\ 0 \le i+j \le 1$ is given by

$$l_2(x,y) = [x,y]_{\mathfrak{g}_0}, \quad l_2(x,u) = \rho(x)u, \quad \forall\, x,y \in \mathfrak{g}_0, u,v \in \mathfrak{g}_1.$$

The conditions in crossed module of Rota-Baxter Lie algebras give various conditions for a strict 2-term Rota-Baxter L_∞-algebra. We omit the details. ■

Let $(\mathfrak{g}, *)$ be a pre-Lie algebra and V a vector space. A **representation** of $\mathfrak{g}$ on V consists of a pair (l, r), where $l : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{g}^c$ on V and $r : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a linear map satisfying

$$r_x l_y - r_y l_x = r_{x*y} - r_y r_x, \quad \forall\, x,y \in \mathfrak{g}. \tag{18}$$

Recall that a **crossed module of pre-Lie algebras** is a quadruple $((\mathfrak{g}_0,*_0),(\mathfrak{g}_1,*_1),\delta,(l,r))$, where $(\mathfrak{g}_0,*_0)$ and $(\mathfrak{g}_1,*_1)$ are pre-Lie algebras, $\delta:\mathfrak{g}_1\to\mathfrak{g}_0$ is a homomorphism of pre-Lie algebras, and (l,r) is a representation of the pre-Lie algebra $(\mathfrak{g}_0,*_0)$ on $\mathfrak{g}_1$, such that for $x\in\mathfrak{g}_0$ and $u,v\in\mathfrak{g}_1$,the following equalities are satisfied:

$$\delta(l_xu) = x*_0\delta u,\quad \delta(r_xu)=(\delta u)*_0x, \tag{19}$$
$$l_{\delta u}v = r_{\delta v}u=u*_1v. \tag{20}$$

4.8. Proposition. *Let $((\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0},T_0),(\mathfrak{g}_1,[\cdot,\cdot]_{\mathfrak{g}_1},T_1),\mathrm{d},\rho)$ be a crossed module of Rota-Baxter Lie algebras. Define $*_0:\mathfrak{g}_0\otimes\mathfrak{g}_0\to\mathfrak{g}_0$, $*_1:\mathfrak{g}_1\otimes\mathfrak{g}_1\to\mathfrak{g}_1$ and $l,r:\mathfrak{g}_0\to\mathfrak{gl}(\mathfrak{g}_1)$ by*

$$\begin{aligned} x*_0y &= [T_0x,y]_{\mathfrak{g}_0},\quad u*_1v=[T_1u,v]_{\mathfrak{g}_1},\\ l_xu &= \rho(T_0x)u,\quad r_xu=-\rho(x)T_1(u),\quad \forall\ x,y\in\mathfrak{g}_0,u,v\in\mathfrak{g}_1.\end{aligned}$$

Then $((\mathfrak{g}_0,*_0),(\mathfrak{g}_1,*_1),\mathrm{d},(l,r))$ is a crossed module of pre-Lie algebras.

Proof. Since T_0 is a Rota-Baxter operator on the Lie algebra $(\mathfrak{g}_0,[\cdot,\cdot]_{\mathfrak{g}_0})$, $(\mathfrak{g}_0,*_0)$ is a pre-Lie algebra. Similarly, $(\mathfrak{g}_1,*_1)$ is also a pre-Lie algebra. By the fact that d is a Rota-Baxter Lie algebra homomorphism, we have

$$\begin{aligned}\mathrm{d}(u*_1v) &= \mathrm{d}[T_1u,v]_{\mathfrak{g}_1}=[\mathrm{d}(T_1u),\mathrm{d}v]_{\mathfrak{g}_1}\\ &= [T_0(\mathrm{d}u),\mathrm{d}]_{\mathfrak{g}_1}=(\mathrm{d}u)*_0(\mathrm{d}v),\end{aligned}$$

which implies that d is a pre-Lie algebra homomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_0$.

By the fact that ρ is a representation of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_1$ and T_0 is a Rota-Baxter operator on $\mathfrak{g}_0$, we have

$$\begin{aligned} l_{[x,y]_{T_0}} &= \rho(T_0([x,y]_{T_0}))=\rho([T_0x,T_0y]_{\mathfrak{g}_0})\\ &= [\rho(T_0x),\rho(T_0y)]=[l_x,l_y],\end{aligned}$$

which implies that l is a representation of the sub-adjacent Lie algebra $\mathfrak{g}_0^c$ on $\mathfrak{g}_1$. Furthermore, by Eq. (2) in the representation of the Rota-Baxter Lie algebra, we have

$$\begin{aligned} & l_x(r_yu)-r_y(l_xu)-r_{x*_0y}u+r_y(r_xu)\\ =\ & -\rho(T_0x)\rho(y)(T_1u)+\rho(y)T_1(\rho(T_0x)u)+\rho([T_0x,y]_{\mathfrak{g}_0})u+\rho(y)T_1\rho(x)(T_1u)\\ =\ & \rho(y)T_1(\rho(T_0x)u)-\rho(y)\rho(T_0x)u+\rho(y)T_1\rho(x)(T_1u)=0.\end{aligned}$$

Thus (l,r) is a representation of the pre-Lie algebra $(\mathfrak{g}_0,*_0)$ on $\mathfrak{g}_1$.

Furthermore, the condition $\mathrm{d}(\rho(x)(u))=[x,\mathrm{d}u]_{\mathfrak{g}_0}$ implies that

$$\mathrm{d}(l_xu)=x*_0\mathrm{d}u,\quad \mathrm{d}(r_xu)=(\mathrm{d}u)*_0x$$

hold and the condition $\rho(\mathrm{d}u)(v)=[u,v]_{\mathfrak{g}_1}$ implies that

$$l_{\mathrm{d}u}v=r_{\mathrm{d}v}u=u*_1v$$

hold. Therefore we obtain a crossed module of pre-Lie algebras $((\mathfrak{g}_0,*_0),(\mathfrak{g}_1,*_1),\mathrm{d},(l,r))$. ∎

4.9. Proposition. ([25]) *Let* $((\mathfrak{g}_0, *_0), (\mathfrak{g}_1, *_1), \mathrm{d}, (l, r))$ *be a crossed module of pre-Lie algebras. Then* $((\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}), (\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}), \mathrm{d}, \rho = l - r)$ *is a crossed module of Lie algebras, where the brackets* $[\cdot,\cdot]_{\mathfrak{g}_0}$ *and* $[\cdot,\cdot]_{\mathfrak{g}_1}$ *are given by*

$$[x, y]_{\mathfrak{g}_0} = x *_0 y - y *_0 x, \quad [u, v]_{\mathfrak{g}_1} = u *_1 v - v *_1 u \tag{21}$$

for $x, y \in \mathfrak{g}_0, u, v \in \mathfrak{g}_1$.

Let $((\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}), (\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}), \mathrm{d}_\mathfrak{g}, \rho_\mathfrak{g})$ and $((\mathfrak{h}_1, [\cdot,\cdot]_{\mathfrak{h}_1}), (\mathfrak{h}_0, [\cdot,\cdot]_{\mathfrak{h}_0}), \mathrm{d}_\mathfrak{h}, \rho_\mathfrak{h})$ be two crossed modules of Lie algebras. Recall that a **homomorphism** from $(\mathfrak{g}_0, \mathfrak{g}_1, \mathrm{d}_\mathfrak{g}, \rho_\mathfrak{g})$ to $(\mathfrak{h}_0, \mathfrak{h}_1, \mathrm{d}_\mathfrak{h}, \rho_\mathfrak{h})$ is a pair (ψ_0, ψ_1), such that $\psi_0 : \mathfrak{g}_0 \to \mathfrak{h}_0$ is a Lie algebra homomorphism and $\psi_1 : \mathfrak{g}_1 \to \mathfrak{h}_1$ is a Lie algebra homomorphism satisfying

$$\mathrm{d}_\mathfrak{h} \circ \psi_1 = \psi_0 \circ \mathrm{d}_\mathfrak{g}, \quad \psi_1(\rho_\mathfrak{g}(x)v) = \rho_\mathfrak{h}(\psi_0(x))\psi_1(v), \quad \forall\, x \in \mathfrak{g}_0, v \in \mathfrak{g}_1. \tag{22}$$

4.10. Proposition. *Let* $((\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}, T_0), (\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}, T_1), \mathrm{d}, \rho)$ *be a crossed module of Rota-Baxter Lie algebras. Then* $((\mathfrak{g}_0, [\cdot,\cdot]_{T_0}), (\mathfrak{g}_1, [\cdot,\cdot]_{T_1}), \mathrm{d}, \rho_T)$ *is a crossed module of Lie algebras, where* $[\cdot,\cdot]_{T_0}$, $[\cdot,\cdot]_{T_1}$ *and* ρ_T *are given by*

$$\begin{aligned} [x, y]_{T_0} &= [T_0 x, y]_{\mathfrak{g}_0} - [T_0 y, x]_{\mathfrak{g}_0}, \\ [u, v]_{T_1} &= [T_1 u, v]_{\mathfrak{g}_1} - [T_1 v, u]_{\mathfrak{g}_1}, \\ \rho_T(x)u &= \rho(T_0 x)u + \rho(x)T_1(u), \quad \forall\, x, y \in \mathfrak{g}_0, u, v \in \mathfrak{g}_1. \end{aligned}$$

Furthermore,

$$(T_0, T_1) : ((\mathfrak{g}_0, [\cdot,\cdot]_{T_0}), (\mathfrak{g}_1, [\cdot,\cdot]_{T_1}), \mathrm{d}, \rho_T) \to ((\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0}, T_0), (\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1}, T_1), \mathrm{d}, \rho)$$

is a homomorphism of crossed modules of Lie algebras.

Proof. The first conclusion follows from Proposition 4.8 and 4.9.

Since T_0 is a Rota-Baxter operator on $\mathfrak{g}_0$, T_0 is a Lie algebra homomorphism from $(\mathfrak{g}_0, [\cdot,\cdot]_{T_0})$ to $(\mathfrak{g}_0, [\cdot,\cdot]_{\mathfrak{g}_0})$. Similarly, T_1 is a Lie algebra homomorphism from $(\mathfrak{g}_1, [\cdot,\cdot]_{T_1})$ to $(\mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}_1})$. By the fact that d is a Rota-Baxter Lie algebra homomorphism, we have $\mathrm{d} \circ T_1 = T_0 \circ \mathrm{d}$ and furthermore, by the fact that ρ is a representation of the Rota-Baxter Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_1$, we have

$$T_1(\rho_T(x)u) - \rho(T_0(x))T_1(u) = T_1(\rho(T_0 x)u + \rho(x)T_1(u)) - \rho(T_0(x))T_1(u) = 0.$$

Thus the second conclusion follows. ∎

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