Theory and Applications of Categories, Vol. 39, No. 14, 2023, pp. 444–446.

ADDENDUM TO "RANK-BASED PERSISTENCE"

MATTIA G. BERGOMI AND PIETRO VERTECHI

ABSTRACT. The Rank-based persistence framework and the generalization of topological persistence introduced in [1, 4] yield overlapping applications. We discuss the similarities and differences between the two approaches from theoretical and applied standpoints.

1. Introduction

Unbeknownst to us during the writing, review, and publication of [3], the generalization of persistent homology introduced in [1, 4] attains overlapping applications with our framework, in a different theoretical setting. Both manuscripts aim to replace a given function (associating each vector space with its dimensionality) with a more general one. The two approaches are conceptually distinct: we investigate arbitrary integer-valued functions that obey a set of axioms, whereas [4] considers a specific function valued in a potentially different group, the Grothendieck group of an Abelian category. Applications of the two approaches are closely related (the notion introduced in [4] recovers the multicolored bottleneck distance in semisimple Abelian categories). In the following section, we shall discuss in more detail the differences and similarities between the two approaches from theoretical and applied perspectives.

2. Comparison

At the core of both frameworks is the notion of a rank function. However, different assumptions are made to ensure that a *generalized rank function* leads to a suitable notion of generalized persistence.

In [3], we fixed the codomain of the rank function: it is always the Abelian group $\mathbb{Z}$. Then, given a regular category $\mathbf{C}$, we explored under what requirements a rank function $r\colon \mathrm{Obj}(\mathbf{C}) \to \mathbb{Z}$ induces a notion of stable persistence. We further introduced a stronger notion of rank function—*fiber-wise rank function [3, Def. 2.5]*—whose axioms are easier to verify compared with general rank functions. In particular, the axioms of fiber-wise rank functions become a simple condition on short exact sequences in the special case

Received by the editors 2023-03-20 and, in final form, 2023-03-20.
Transmitted by Kathryn Hess. Published on 2023-03-31.
2020 Mathematics Subject Classification: 18E10, 18A35, 55N35, 68U05.
Key words and phrases: rank, persistence, categorification, regular category, abelian category, semisimple category, classification, group action, point cloud, poset, bottleneck, interleaving.

where $\mathbf{C}$ is Abelian. However, dropping the Abelianity requirement on the category $\mathbf{C}$ leads to interesting applications, see for example [5] (where $\mathbf{C} = \mathbf{Set}$) and [6] (where $\mathbf{C} = \mathbf{Ring}$).

On the other hand, McCleary and Patel require that the category $\mathbf{C}$ is not only regular but also Abelian. They take as generalized rank function the canonical map $\mathrm{Obj}(\mathbf{C}) \to \mathcal{G}$, where $\mathcal{G}$ is the Grothendieck group of $\mathbf{C}$.

It is interesting to note that the universal property of the Grothendieck group is equivalent to the requirement of fiber-wise rank functions in Abelian categories (provided that $r(0) = 0$). In particular, in Abelian categories, fiber-wise rank functions with $r(0) = 0$, such as *length*, are coarser approximations of McCleary and Patel's generalized persistence, induced by maps $\mathcal{G} \to \mathbf{Z}$. We believe that such approximations can be advantageous in practice for computational reasons. When $\mathbf{C}$ is a semisimple Abelian category, we can fully recover McCleary and Patel's persistence via colorings [3, Sect. 4]. In the future, it will be interesting to explore whether we can weaken our requirements on the codomain of the rank function to possibly recover McCleary and Patel's persistence in the non-semisimple case.

Both notions of generalized persistence are stable, in that the bottleneck distance is bound by the interleaving distance. To study examples in which the two distances coincide, we introduce the notion of tight coloring and show a class of examples in semisimple Abelian categories. Those examples can also be recovered using McCleary and Patel's framework: the Grothendieck group of a semisimple Abelian category is a direct sum of copies of $\mathbb{Z}$.

From a practical perspective, we introduce novel applications using regular categories that are not Abelian (such as $\mathbf{Set}$) and functoriality [3, Prop. 3.3], which gives rise to examples with weighted graphs, quivers, or posets. We explore those examples both in the original manuscript and, in more detail, in [2, 5]. To show the computability of the invariants we introduced, we provide implementations at

- `https://github.com/LimenResearch/gpa`,
- `https://github.com/LimenResearch/rank_persistence`.

References

[1] Amit Patel. Generalized persistence diagrams. *Journal of Applied and Computational Topology*, 1(3-4):397–419, 2018.

[2] Mattia G Bergomi and Pietro Vertechi. Comparing neural networks via generalized persistence. In *SIS 2020 - Book of Short Papers*, pages 582–587. Springer, 2020.

[3] Mattia G Bergomi and Pietro Vertechi. Rank-based persistence. *Theory and Applications of Categories*, 35(9):228–260, 2020.

[4] Alex McCleary and Amit Patel. Bottleneck stability for generalized persistence diagrams. *Proceedings of the American Mathematical Society*, 148(733), 2020.

[5] Mattia G Bergomi, Massimo Ferri, and Antonella Tavaglione. Steady and ranging sets in graph persistence. *Journal of Applied and Computational Topology*, pages 1–24, 2022.

[6] Facundo Mémoli, Anastasios Stefanou, and Ling Zhou. Persistent cup product structures and related invariants. *arXiv preprint arXiv:2211.16642*, 2022.

Email: mattiagbergomi@gmail.com
pietro.vertechi@protonmail.com