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CENTRALITY AND THE COMMUTATIVITY OF FINITE PRODUCTS WITH COEQUALISERS

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Abstract. We study centrality of morphisms in a setting derived from that of a pointed category in which finite products commute with coequalisers. The main results of this paper show that much of the behaviour of central morphisms for unital categories [5] is retained in our setting, including categories which are (weakly) unital, but also categories outside of the unital setting.

1. Introduction

Recall that given small categories $\mathcal{A}, \mathcal{B}$ and a category $\mathbb{C}$ which has limits of shape $\mathcal{A}$ as well as colimits of shape $\mathcal{B}$, every bifunctor $F\colon \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ determines two functors $\mathrm{colim}_{\mathcal{B}}F\colon \mathcal{A} \to \mathbb{C}$ and $\lim_{\mathcal{A}}F\colon \mathcal{B} \to \mathbb{C}$ where for each object A of $\mathcal{A}$ and B of $\mathcal{B}$ we have $\mathrm{colim}_{\mathcal{B}}F(A) = \mathrm{colim}F(A,-)$ and $\lim_{\mathcal{A}}F(B) = \lim F(-,B)$. For morphisms $f\colon A \to A'$ in $\mathcal{A}$ and $g\colon B \to B'$ in $\mathcal{B}$ the corresponding natural transformations $F(f,-)\colon F(A,-) \to F(A',-)$ and $F(-,g)\colon F(-,B) \to F(-,B')$ determine the morphisms $\mathrm{colim}_{\mathcal{B}}F(f)$ and $\lim_{\mathcal{A}}F(g)$ respectively. These functors, in turn, determine a limit $\lim(\mathrm{colim}_{\mathcal{B}}F)$ and a colimit $\mathrm{colim}(\lim_{\mathcal{A}}F)$, and between them is a canonically induced morphism $\omega_F\colon \mathrm{colim}(\lim_{\mathcal{A}}F) \to \lim(\mathrm{colim}_{\mathcal{B}}F)$. Limits of shape $\mathcal{A}$ are said to *commute* with colimits of shape $\mathcal{B}$ in $\mathbb{C}$ if the morphism ω_F is an isomorphism for any bifunctor $F\colon \mathcal{A} \times \mathcal{B} \to \mathbb{C}$. It is well-known, for instance, that filtered colimits commute with all finite limits in the category Set (see [26]). That is limits of shape $\mathcal{A}$ commute with colimits of shape $\mathcal{B}$ when $\mathcal{A}$ is finite and $\mathcal{B}$ is filtered. The larger class of sifted colimits [1, 2] are such that they commute with all finite products in Set, so that reflexive coequalisers, for instance, commute with finite products in Set, and more generally in any variety of algebras. Arbitrary coequalisers, however, need not commute with finite products for general varieties of algebras. However, in many special classes of varieties (such as the variety of groups or monoids), we do have commutativity of finite products and arbitrary coequalisers (see [18] for examples).

Considered as a property of a category, the commutativity of a specified limit with a specified colimit can have significant consequences for the underlying category. We give two illustrations of this: suppose that $\mathbb{C}$ is a category with finite products and finite coproducts and suppose that binary products commute with binary coproducts. That is,

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if $X_1 \xrightarrow{x_1} X \xleftarrow{x_2} X_2$ and $Y_1 \xrightarrow{y_1} Y \xleftarrow{y_2} Y_2$ are coproduct diagrams in $\mathbb{C}$ then the diagram

$$X_1 \times Y_1 \xrightarrow{x_1 \times y_1} X \times Y \xleftarrow{x_2 \times y_2} X_2 \times Y_2$$

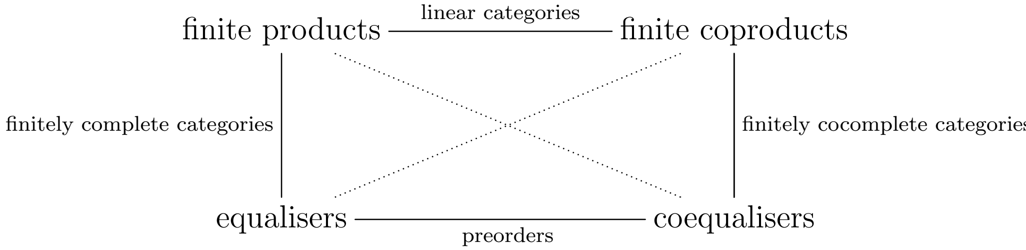
is a coproduct diagram in $\mathbb{C}$. The product of the trivial coproducts $1 \to 1 \leftarrow 0$ and $0 \to 1 \leftarrow 1$ being a coproduct gives us that $\mathbb{C}$ is pointed. Then, the product of the trivial coproducts $X \to X \leftarrow 0$ and $0 \to Y \leftarrow Y$ yield a biproduct of X and Y in $\mathbb{C}$, so that $\mathbb{C}$ is *linear* (or "half-additive" [11]). Conversely, in every linear category finite products commute with finite coproducts in $\mathbb{C}$. Now consider a category $\mathbb{C}$ with equalisers and coequalisers, and suppose that equalisers commute with coequalisers in $\mathbb{C}$, i.e., that in any diagram

$$\begin{array}{ccccc} E_1 & \longrightarrow & X_1 & \rightrightarrows & Y_1 \\ {\scriptstyle e_1}\downarrow\downarrow{\scriptstyle e_2} & & {\scriptstyle x_1}\downarrow\downarrow{\scriptstyle x_2} & & {\scriptstyle y_1}\downarrow\downarrow{\scriptstyle y_2} \\ E_2 & \longrightarrow & X_2 & \rightrightarrows & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & V & \rightrightarrows & W \end{array}$$

where the top and middle rows are equaliser diagrams and (e_1, x_1, y_1) and (e_2, x_2, y_2) are natural transformations between the equaliser diagrams, if all of the columns are coequaliser diagrams then the bottom row is an equaliser diagram (where the morphisms are all the canonically induced morphisms, making bottom right square reasonably commutative). Given two parallel morphisms $f, g\colon X \to Y$ in $\mathbb{C}$, we form

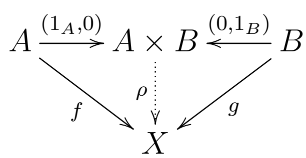
$$\begin{array}{ccccc} E & \xrightarrow{e} & X & \overset{f}{\underset{g}{\rightrightarrows}} & Y \\ {\scriptstyle fe}\downarrow\downarrow{\scriptstyle ge} & & {\scriptstyle f}\downarrow\downarrow{\scriptstyle g} & & {\scriptstyle q}\downarrow\downarrow{\scriptstyle q} \\ Y & \xrightarrow[1]{} & Y & \overset{q}{\underset{q}{\rightrightarrows}} & Q \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 1} \\ Y & \xrightarrow[q]{} & Q & \overset{1}{\underset{1}{\rightrightarrows}} & Q \end{array}$$

where q is a coequaliser of f and g, and e is their equaliser. Then the bottom row being an equaliser forces q to be an isomorphism and hence $f = g$. Thus, the commutativity of equalisers and coequalisers in $\mathbb{C}$ would force $\mathbb{C}$ to be a preorder. Let us now consider all such commutation conditions between finite products, coproducts, equalisers and coequalisers. Representing such a condition by an edge (whose vertices are the respective limit/colimit) we have the following diagram:



The labels of the solid arrows refer to class of categories determined by the respective commutation condition, while the dotted diagonal lines represent conditions which are dual to each other, namely, the commutation of finite products with coequalisers and of finite coproducts with equalisers. Perhaps it is surprising how common the commutativity of finite products with aribitrary coequalisers is in algebra: the categories of groups, rings, monoids, Boolean algebras, (non-empty) implication algebras, or congruence modular/distributive varieties with constants, but also categorical examples such as linear, regular unital (or strongly unital) categories [5] (admitting coequalisers), regular pointed majority categories [17, 19] (admitting coequalisers), as well as pointed regular Gumm or factor permutable categories (admitting coequalisers) [9, 12], all provide us with examples of categories in which finite products and coequalisers commute[1]. Hence the question of what generally, if anything, can be said of categories in which finite products commute with coequalisers. In this paper, we focus this question to pointed categories, and show that much of the behavior of central morphisms [23] in unital categories [5, 6] extends to every pointed category $\mathbb{C}$ in which binary products commute with coequalisers.

Recall that two morphisms $f\colon A \to X$ and $g\colon B \to X$ in a pointed category $\mathbb{C}$ with binary products are said to *commute* [23] (or Huq-commute) if there exists a morphism $\rho\colon A \times B \to X$ such that $\rho(1_A, 0) = f$ and $\rho(0, 1_B) = g$, where $(1_A, 0)$ and $(0, 1_B)$ are the canonical product inclusions.



Such a morphism ρ we call a *cooperator* for f and g, following the terminology of [6]. For instance, in the category Grp of groups, two subgroups $A \hookrightarrow X$ and $B \hookrightarrow X$ of a group X commute if they centralise each other, i.e., $ab = ba$ for all $a \in A$ and $b \in B$. A morphism $f\colon A \to B$ in $\mathbb{C}$ is *central* when it commutes with the identity 1_B on B, and an object A is called *commutative* if 1_A is central. Thus, a subgroup $A \hookrightarrow X$ is central in Grp if and only if it is a central subgroup, i.e., it is contained in the center $\mathsf{Z}(X)$ of X. Recall from [5] that a finitely complete pointed category $\mathbb{C}$ is *unital* if the morphisms $A \xrightarrow{(1_A,0)} A \times B \xleftarrow{(0,1_B)} B$ are jointly strongly epimorphic for any two objects A and B of $\mathbb{C}$. In the context of a unital category the class of central morphisms $\mathsf{Z}(\mathbb{C})$ in $\mathbb{C}$ forms a right ideal, and is such that between any two objects X, Y in $\mathbb{C}$ the class of central morphisms $\mathsf{Z}(X, Y)$ between X and Y forms a commutative monoid with a canonical action on $\mathbb{C}(X, Y)$. This is the so-called "additive core" of [6]. In this paper, we show that this additive core is present in any pointed category $\mathbb{C}$ admitting binary products and coequalisers and which commute. More generally, this is shown for any *centralic* category in the sense defined in section 2. In particular, every unital or weakly unital category [27] is centralic, but also contexts far from unital, such as every pointed majority category [17, 19], every pointed Gumm [9] or

[1] General varieties of universal algebras in which finite products commute with coequalisers have be syntactically described [21]. This following the corresponding characterisation for pointed varieties [18].

factor permutable category [12], all provide us with examples of centralic categories. The relation of centralic categories to the commutativity of finite products with coequalisers is that a pointed category $\mathbb{C}$ with coequalisers, whose regular epimorphisms are stable under finite products, is centralic if and only if finite products commute with coequalisers in $\mathbb{C}$. For a centralic category $\mathbb{C}$, the full subcategory $\mathsf{Com}(\mathbb{C})$ of commutative objects in $\mathbb{C}$ is linear and equivalent to the category $\mathsf{CMon}(\mathbb{C})$ of internal commutative monoids in $\mathbb{C}$. Under suitable conditions, the inclusion $\mathsf{Com}(\mathbb{C}) \to \mathbb{C}$ has a finite product preserving left-adjoint, making the category $\mathsf{Com}(\mathbb{C})$ a Birkhoff subcategory of $\mathbb{C}$. Moreover, we show that the subcategory $\mathsf{Ab}(\mathbb{C})$ of abelian objects in $\mathbb{C}$ is reflective in $\mathsf{Com}(\mathbb{C})$.

Convention and notation. Throughout this paper we will write 0 for the zero object in a given pointed category $\mathbb{C}$, as well as for zero-morphisms in $\mathbb{C}$ — provided the context is clear, and does not lead to confusion. We will also make frequent use of the language of generalised elements in what follows, and give set-theoretic arguments in proofs which involve finite limits, understanding that these arguments generalise to categories via standard techniques involving the Yoneda embedding (see Metatheorem 0.1.3 in [4]). We will also freely use general properties of regular categories [3] as can be found in [13].

2. Centralic categories

2.1. Proposition. *The following are equivalent for a pointed category with binary products:*

(i) for any $f\colon X \times X \to Y$ we have $f(x,0) = f(x',0)$ implies $f(x,y) = f(x',y)$,

(ii) for any $f\colon X \times Y \to Z$ we have $f(x,0) = f(x',0)$ implies $f(x,y) = f(x',y)$,

(iii) for any $f\colon X \times Y \to Z$ we have $f(x,0) = f(x',0)$ and $f(0,y) = f(0,y')$ implies $f(x,y) = f(x',y')$.

Proof. For $(i) \implies (ii)$, consider the morphism $\alpha\colon (X \times Y) \times (X \times Y) \to Z \times Z$ defined by $\alpha((x,y),(x',y')) = (f(x,y'), f(x',y))$. Then $\alpha((x,0),(0,0)) = \alpha((x',0),(0,0))$ so that by (i) we get $\alpha((x,0),(0,y)) = \alpha((x',0),(0,y))$ and hence $f(x,y) = f(x',y)$. For $(ii) \implies (iii)$ we have $f(x,0) = f(x',0) \implies f(x,y) = f(x',y)$ and $f(0,y) = f(0,y') \implies f(x',y) = f(x',y')$ so that $f(x,y) = f(x',y')$. Then $(iii) \implies (i)$ is trivial. ∎

2.2. Definition. *A pointed category $\mathbb{C}$ with finite products is called* centralic *if it satisfies any one of the equivalent conditions of Proposition 2.1.*

For a finitely complete category $\mathbb{C}$ we write $\mathsf{Eq}(f)$ for the kernel equivalence relation of a morphism f in $\mathbb{C}$, i.e., the equivalence relation represented by the kernel pair of f. Then the property of a finitely complete pointed category $\mathbb{C}$ to be centralic may formulated with respect to effective equivalence relations: $\mathbb{C}$ is centralic if and only if for any effective equivalence relation Θ on a product $X \times Y$ in $\mathbb{C}$ we have

$$(x,0)\Theta(x',0) \implies (x,y)\Theta(x',y).$$

This is a direct formulation of the diagrammatic condition of (3) of Proposition 2.9 in [18] so that the proposition below is just a reformulation of that proposition.

2.3. PROPOSITION. *Let $\mathbb{C}$ be a pointed finitely complete category with coequalizers, then the following are equivalent.*

1. *Binary products commute with coequalisers in $\mathbb{C}$, i.e., for any two coequalizer diagrams*

$$C_1 \underset{v_1}{\overset{u_1}{\rightrightarrows}} X_1 \xrightarrow{q_1} Q_1 \qquad C_2 \underset{v_2}{\overset{u_2}{\rightrightarrows}} X_2 \xrightarrow{q_2} Q_2,$$

in $\mathbb{C}$, the diagram

$$C_1 \times C_2 \underset{v_1 \times v_2}{\overset{u_1 \times u_2}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{q_1 \times q_2} Q_1 \times Q_2,$$

is a coequaliser diagram.

2. *For any regular epimorphism $q : X \to Y$ and any object Z in $\mathbb{C}$, the diagram*

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle (1_X,0)}\downarrow & & \downarrow{\scriptstyle (1_Y,0)} \\ X \times Z & \xrightarrow[q \times 1_Z]{} & Y \times Z \end{array}$$

is a pushout.

3. *Regular epimorphisms are stable under binary products and $\mathbb{C}$ is centralic.*

Since regular epimorphisms are stable under binary products in any regular category [3], the following corollary is immediate.

2.4. COROLLARY. *A pointed regular category with coequalisers $\mathbb{C}$ is centralic if and only if binary products commute with coequalisers.*

2.5. PROPOSITION. *Every pointed category $\mathbb{C}$ admitting binary products and coequalisers which commute is centralic.*

PROOF. Suppose that $f \colon X \times Y \to Z$ is any morphism in $\mathbb{C}$. Given $x, x' \colon S \to X$ such that $f(x, 0) = f(x', 0)$, consider a coequaliser diagram

$$S \underset{x}{\overset{x'}{\rightrightarrows}} X \xrightarrow{q} Q.$$

Since $0 \rightrightarrows Y \xrightarrow{1_Y} Y$ is trivially a coequaliser, the diagram

$$S \underset{(x,0)}{\overset{(x',0)}{\rightrightarrows}} X \times Y \xrightarrow{q \times 1_Y} Q \times Y$$

is a coequaliser, which implies that $f(x, y) = f(x', y)$ for any $y \colon S \to Y$. ∎

2.6. Proposition. *If $\mathbb{C}$ is centralic and has kernel pairs, given morphisms $f\colon X \to Y$ and $g\colon X' \to Y$ which admit a cooperator $\rho\colon X \times X' \to Y$, we have $\mathsf{Eq}(f \times g) \leqslant \mathsf{Eq}(\rho)$.*

Proof. If $(f \times g)(x, y) = (f \times g)(x', y')$ then $\rho(x, 0) = \rho(x', 0)$ and $\rho(0, y) = \rho(0, y')$, so that by Proposition 2.1 (*iii*) we get $\rho(x, y) = \rho(x', y')$, and the result follows. ∎

2.7. Examples of centralic categories. A pointed finitely complete category $\mathbb{C}$ is *weakly unital* [27] if for every two objects X and Y the product inclusions

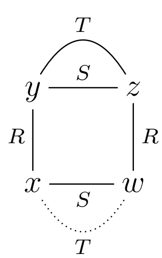
$$X \xrightarrow{(1_X,0)} X \times Y \xleftarrow{(0,1_Y)} Y$$

of any binary product diagram are jointly epimorphic. As an example, consider any pointed quasi-variety $\mathbb{V}$ of algebras which admits a binary operation $+$ satisfying $x + 0 = 0 + x$ and $x + 0 = y + 0 \implies x = y$. According to Proposition 3.2 in [15] we have that $\mathbb{V}$ is weakly unital.

2.8. Proposition. *Every weakly unital category is centralic.*

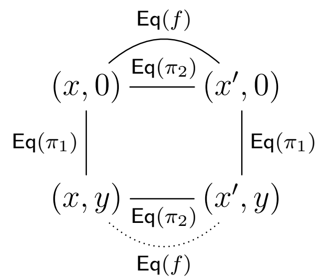
Proof. Given $x, x'\colon S \to X$, $y\colon S \to Y$, and any morphism $f\colon X \times Y \to Z$ with $f(x, 0) = f(x', 0)$ the morphisms $f(x \times y)$ and $f(x' \times y)$ are equal when composed with the canonical product inclusions $S \xrightarrow{(1_S,0)} S \times S \xleftarrow{(0,1_S)} S$. Thus they are equal, and the result follows. ∎

The notion of a *Gumm category* [9] is the categorical analogue of varieties in which Gumm's shifting lemma holds [16], i.e., congruence modular varieties. A finitely complete category $\mathbb{C}$ is a Gumm category if for any three equivalence relations R, S, T on any object X in $\mathbb{C}$ such that $R \cap S \leqslant T$, if $(x, y), (w, z) \in R$ and $(y, z), (x, w) \in S$ and $(y, z) \in T$ then we get $(x, w) \in T$. This implication of relations between the elements above is usually depicted with a diagram



2.9. Proposition. *Every pointed Gumm category is centralic.*

PROOF. Given a morphism $f\colon X \times Y \to Z$ such that $f(x,0) = f(x',0)$ then we have the diagram below where $\mathsf{Eq}(\pi_1) \cap \mathsf{Eq}(\pi_2) \leqslant \mathsf{Eq}(f)$ holds trivially.



■

A regular category $\mathbb{C}$ is said to be *factor permutable* [12] if for every object A in $\mathbb{C}$ and any equivalence relation E on A we have that $\mathsf{Eq}(p) \circ E = E \circ \mathsf{Eq}(p)$ for every product projection $p\colon A \to X$ of A. This notion is the categorical generalisation of factor permutable varieties introduced in [16].

2.10. REMARK. Applying Lemma 2.5 (the weak shifting lemma) in [12] to the same diagram as in Proposition 2.9 it will follow that every pointed factor permutable variety is centralic.

2.11. REMARK. Proposition 2.9 may also be seen as a consequence of the fact that any punctually congruence hyperextensible category in the sense of [7] is centralic.

The notion of a *majority category* has been defined in the paper [17] (see also [19]). We now recall this notion for the reader's convenience. Consider the condition on a ternary relation R between sets X, Y, Z given by

$$(x, y, z') \in R \quad \text{and} \quad (x, y', z) \in R \quad \text{and} \quad (x', y, z) \in R \implies (x, y, z) \in R.$$

Then a category $\mathbb{C}$ is a *majority category* if every internal ternary relation in $\mathbb{C}$ satisfies the above condition internalised (via the Yoneda embedding) to internal ternary relations in $\mathbb{C}$ (see [17] for the details). Majority categories capture, in a categorical way, what it means for a variety of universal algebras to admit a *majority* term, i.e., a ternary term $m(x, y, z)$ satisfying the equations

$$\begin{aligned} m(x, x, y) &= x, \\ m(x, y, x) &= x, \\ m(y, x, x) &= x. \end{aligned}$$

For instance, in the variety Lat of lattices, the term $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ or its lattice-theoretic dual both define majority terms. In a variety of rings satisfying the identity $x^n = x$ for some $n \geqslant 2$ (such as a variety of rings generated by a finite field for example), then the term $m(x, y, z) = x - (x - y)(x - z)^{n-1}$ defines a majority term.

2.12. Proposition. *Every finitely complete pointed majority category is centralic.*

Proof. Given a morphism $f\colon X \times Y \to Z$ with $f(x,0) = f(x',0)$ we define the ternary relation R between X and X and $Y \times Y$ by

$$(x, x', (y, y')) \in R \Longleftrightarrow f(x,y) = f(x', y').$$

Then we have

$$(x, x', (0,0)) \in R, \quad (x, x, (y,y)) \in R, \quad (x', x', (y,y)) \in R \implies (x, x', (y,y)) \in R,$$

and hence $f(x,y) = f(x',y)$. ■

3. Centrality in centralic categories

Throughout this section we fix a centralic category $\mathbb{C}$.

3.1. Proposition. *Given any central morphism $f\colon X \to Y$ in $\mathbb{C}$ and any morphism $g\colon X' \to Y$ in $\mathbb{C}$, there exists a unique cooperator $\rho_{f,g}\colon X \times X' \to Y$ for f and g. This cooperator is defined by $\rho_{f,g}(x,y) = \rho_f(x, g(y))$ where $\rho_f\colon X \times Y \to Y$ is the (unique) cooperator for f and 1_Y.*

Proof. Let $\rho_f\colon X \times Y \to Y$ be a cooperator for f and 1_Y, then it is easily checked that the morphism $\rho_{f,g}\colon X \times X' \to Y$ defined by $\rho_{f,g}(x,y) = \rho_f(x, g(y))$ is a cooperator for f and g. For uniqueness, suppose that $\rho : X \times X' \to Y$ is any cooperator for f and g. Consider the morphism $\alpha\colon (X \times X) \times X' \to Y$ defined by

$$\alpha((x,y),z) = \rho_f(x, \rho(y,z)).$$

Then $\alpha((x,0),0) = \alpha((0,x),0)$ since

$$\alpha((x,0),0) = \rho_f(x, \rho(0,0)) = \rho_f(x,0) = f(x) = \rho_f(0, f(x)) = \rho_f(0, \rho(x,0) = \alpha((0,x),0),$$

so that since $\mathbb{C}$ is centralic, we have $\alpha((x,0),y) = \alpha((0,x),y)$, and hence

$$\begin{aligned} \alpha((x,0),y) = \alpha((0,x),y) &\implies \\ \rho_f(x, \rho(0,y)) = \rho_f(0, \rho(x,y)) &\implies \\ \rho_f(x, g(y)) = \rho(x,y), & \end{aligned}$$

which gives us uniqueness. ■

3.2. Remark. A natural question in light of the proposition above is if every finitely complete centralic category has that any cooperator between two morphisms with the same codomain is unique, i.e., is every centralic category weakly unital? We can answer this question in the negative by considering Proposition 2.12, which states that every finitely complete pointed majority category is centralic. But not every pointed majority category is weakly unital: if $\mathbb{C}$ is a majority category then every category of points $\mathsf{Pt}_{\mathbb{C}}(X)$ (see [5] for an exact definition) is a majority category (see Example 2.15 in [17]). Thus in the variety Lat of lattices (the prototypical example of majority category) every category of points $\mathsf{Pt}_{\mathsf{Lat}}(X)$ (where X is a lattice) a finitely complete majority category. However not every category of points $\mathsf{Pt}_{\mathsf{Lat}}(X)$ is weakly unital, since if it were then Lat would be weakly Mal'tsev in the sense of [27, 28], and a variety of lattices is weakly Mal'tsev if and only if it is a variety of distributive lattices [29, 30].

The corollary below is the analogue of Theorem 3.1.13 in [23] and Theorem 4.2 in [6], in our context, and its proof is similar.

3.3. Corollary. *Given two central morphisms $f\colon X \to Y$ and $g\colon X' \to Y$ in $\mathbb{C}$ their cooperator $\rho_{f,g}\colon X \times X' \to Y$ is central.*

Proof. Let $\rho_f\colon X \times Y \to Y$ be the cooperator for f and let $\rho_g\colon X' \times Y \to Y$ be the cooperator for g and 1_Y. By Proposition 3.1 we have that the cooperator ρ for f and g is precisely given by $\rho = \rho_f(1 \times g)$. It is then easily checked that the morphism $\alpha\colon (X \times X') \times Y \to Y$ defined by $\alpha((x, x'), y) = \rho_f(x, \rho_g(x', y))$ is a cooperator for ρ. ∎

Following [5] let us write $\mathsf{Z}(\mathbb{C})$ for the class of all central morphisms in $\mathbb{C}$. Given objects X and Y in $\mathbb{C}$, we write $\mathsf{Z}(X, Y)$ for all central morphisms from X to Y (as in [5]). The below proposition is similar to Theorem 4.1 in [5] and the proof is the same.

3.4. Proposition. *The class $\mathsf{Z}(\mathbb{C})$ is a right-ideal of $\mathbb{C}$.*

Proof. If $f\colon X \to Y$ is central with cooperator ρ_f and $x\colon S \to X$ is any morphism then $\rho_f(x \times 1_Y)$ makes fx central. ∎

Note that, according to the proposition above, the class $\mathsf{Z}(\mathbb{C})$ is closed under composition.

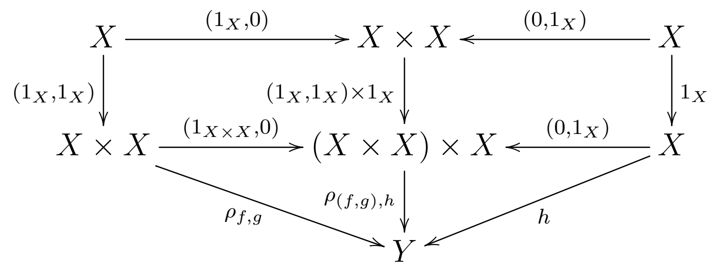
3.5. The additive core of $\mathbb{C}$. When $\mathbb{C}$ is a unital category there is a canonical commutative monoid structure on $\mathsf{Z}(X, Y)$ which acts canonically on $\mathbb{C}(X, Y)$. This is the so-called "additive core" referred to in [6]. We will now show that this additive core is present inside any centralic category $\mathbb{C}$. Consider the map (as it is defined in [6])

$$\mathsf{Z}(X, Y) \times \mathbb{C}(X, Y) \to \mathbb{C}(X, Y), \quad (f, g) \longmapsto f \star g = \rho_{f,g}(1_X, 1_X),$$

where $\rho_{f,g}$ is the unique cooperator (Proposition 3.1) of f and g. Note that we have

$$\rho_{f,g} = \rho_f(1_X \times g)$$

as the unique cooperator of f and g. The zero morphism $0\colon X \to Y$ is central with cooperator $\pi_2\colon X \times Y \to Y$, so that for any $g \in \mathbb{C}(X, Y)$ we have by the above formulas that $0 \star g = \pi_2(1_X \times g)(1_X, 1_X) = g$. Now suppose that $f, g \in \mathsf{Z}(X,Y)$ and $h \in \mathbb{C}(X,Y)$ and write $\rho_{f,g}$ for the unique cooperator between f and g (Proposition 3.1), and write $\rho_{(f,g),h}$ for the unique cooperator between $\rho_{f,g}$ and h (since $\rho_{f,g}$ is central by Corollary 3.3). Similarly, $\rho_{f,(g,h)}$ is the unique cooperator between f and $\rho_{g,h}$. The (necessarily unique) cooperator for $f \star g$ and h is just $\rho_{(f,g),h}[(1_X, 1_X) \times 1_X]$ since we have the commutative diagram



so that

$$(f \star g) \star h = \rho_{(f,g),h}((1_X, 1_X), 1_X).$$

Similarly, we also have

$$f \star (g \star h) = \rho_{f,(g,h)}(1_X, (1_X, 1_X)).$$

Let $\alpha \colon X \times (X \times X) \to (X \times X) \times X$ be the canonical associativity isomorphism. It is then routine to check that the composite $\rho_{(f,g),h}\alpha(0, 1_{X\times X})$ defines a cooperator for g and h and is therefore equal to $\rho_{g,h}$ by Proposition 3.1. Similarly, it may be checked that $\rho_{(f,g),h}\alpha(1_X, (0,0)) = f$, so that the morphism $\rho_{(f,g),h}\alpha$ defines a cooperator for f and $\rho_{g,h}$ and hence $\rho_{(f,g),h}\alpha = \rho_{f,(g,h)}$ — by Proposition 3.1. Then we have:

$$\begin{aligned} f \star (g \star h) &= \rho_{f,(g,h)}(1_X, (1_X, 1_X)) \\ &= \rho_{(f,g),h}\alpha(1_X, (1_X, 1_X)) \\ &= \rho_{(f,g),h}((1_X, 1_X), 1_X)) \\ &= (f \star g) \star h. \end{aligned}$$

Note that by Corollary 3.3 and Proposition 3.4 if $f, g \in \mathsf{Z}(X,Y)$ then $f \star g \in \mathsf{Z}(X,Y)$ so that by the above remarks the map $\star$ above restricts to a monoid operation on $\mathsf{Z}(X,Y)$ which we will denote by $+$. This monoid is commutative since if $f, g \in \mathsf{Z}(X,Y)$ then their cooperator $\rho_{f,g}$ is explicitly given by

$$\rho_g(1_X \times f) = \rho_{f,g} = \rho_f(1_X \times g).$$

Then the map $\star$ given above becomes a commutative monoid action on $\mathbb{C}(X,Y)$. Given any morphism $x \colon X' \to X$ the canonical map $\mathbb{C}(x,Y) \colon \mathbb{C}(X,Y) \to \mathbb{C}(X',Y)$ is a homomorphism of monoid actions, and the proof of Proposition 4.4 in [6] applies directly: given

$f \in \mathsf{Z}(X,Y)$ and $g \in \mathbb{C}(X,Y)$ we have

$$\begin{aligned}(f \star g)x &= \rho_{f,g}(1_X, 1_X)x \\ &= \rho_f(1_X \times g)(x \times x)(1_{X'}, 1_{X'}) \\ &= \rho_f(x \times gx)(1_{X'}, 1_{X'}) \\ &= \rho_{fx}(1_{X'} \times gx)(1_{X'}, 1_{X'}) \\ &= fx \star gx\end{aligned}$$

Moreover, restricting the canonical map $\mathbb{C}(x,Y)$ to $\mathsf{Z}(X,Y)$ produces a monoid homomorphism $\mathsf{Z}(x,Y)\colon \mathsf{Z}(X,Y) \to \mathsf{Z}(X',Y)$.

Symmetrizable morphisms. Recall from [6] the notion of a *symmetrizable morphism* (Definition 4.2 in [6]):

3.6. Definition. *A morphism $f\colon X \to Y$ in $\mathbb{C}$ is symmetrizable if it has an inverse in $\mathsf{Z}(X,Y)$. A commutative object X is called abelian if 1_X is symmetrizable. The full subcategory of abelian objects in $\mathbb{C}$ is denoted by $\mathsf{Ab}(\mathbb{C})$.*

As in [6], for two objects X, Y in $\mathbb{C}$ we write $\Sigma(X,Y)$ for all symmetrizable morphisms from X to Y. The same proofs of Proposition 4.7, 4.8, Theorem 4.3, Proposition 4.6 in [6] apply so that we have

- A central morphism $f\colon X \to Y$ is symmetrizable if and only if

$$\begin{array}{ccc} X \times Y & \xrightarrow{\rho_f} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ X & \longrightarrow & 0 \end{array}$$

  is a pullback;

- if $f\colon X \to Y$ and $g\colon X' \to Y$ are symmetrizable then their cooperator $\rho_{f,g}\colon X \times X' \to Y$ is symmetrizable;

- the class $\Sigma(\mathbb{C})$ is a right ideal and $\Sigma(X,Y)$ is an abelian group which has a canonical action on $\mathbb{C}(X,Y)$;

- an object X is abelian if and only if is the underlying object of a internal abelian group (which is necessarily unique).

3.7. Proposition. *Let $\mathbb{C}$ be a finitely complete centralic category and suppose that regular epimorphisms are stable under binary products in $\mathbb{C}$. In the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{(1_A,0)} & A \times B & \xleftarrow{(0,1_B)} & B \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_1 \times q_2} & & \downarrow{\scriptstyle q_2} \\ X & \xrightarrow{(1_X,0)} & X \times Y & \xleftarrow{(0,1_Y)} & Y \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} & \\ & & Z & & \end{array}$$

where q_1 and q_2 are regular epimorphisms, we have that f commutes with g if and only if fq_1 commutes with gq_2.

Proof. Suppose f and g commute with cooperator ρ, then $\rho(q_1 \times q_2)$ is a cooperator for fq_1 and gq_2. Conversely, suppose that $\rho\colon A \times B \to Z$ is a cooperator for fq_1 and gq_2. By Proposition 2.6 we have $\mathsf{Eq}\big((fq_1) \times (gq_2)\big) \leqslant \mathsf{Eq}(\rho)$, and since we always have $\mathsf{Eq}(q_1 \times q_2) \leqslant \mathsf{Eq}\big((fq_1) \times (gq_2)\big)$ we get $\mathsf{Eq}(q_1 \times q_2) \leqslant \mathsf{Eq}(\rho)$. Since $q_1 \times q_2$ is a regular epimorphism the dotted arrow exists making the triangle

$$\begin{array}{ccc} A \times B & \xrightarrow{q_1 \times q_2} & X \times Y \\ & \underset{\rho}{\searrow} & \vdots\, \rho' \\ & & Z \end{array}$$

commute. To see that ρ' is a cooperator for f and g, we have

$$\rho'(1_X, 0)q_1 = \rho'(q_1 \times q_2)(1_A, 0) = \rho(1_A, 0) = fq_1$$

and since q_1 is an epimorphism we have $\rho'(1_X, 0) = f$. The equality $\rho'(0, 1_Y) = g$ follows similarly. ∎

Given any finitely complete centralic category $\mathbb{C}$ whose regular epimorphisms are stable binary products, we have the following two corollaries:

3.8. Corollary. *For any two regular epimorphisms $q_1\colon A \to X$ and $q_2\colon B \to X$ in $\mathbb{C}$, the object X is commutative if and only if q_1 commutes with q_2.*

3.9. Corollary. *Let $q\colon A \to A'$ be a regular epimorphism in $\mathbb{C}$ then for any morphism $f\colon A' \to Z$ in $\mathbb{C}$ if fq is central then so is f. If $\mathbb{C}$ is regular, then fq symmetrizable implies f symmetrizable.*

Proof. If fq is central then we apply Proposition 3.7 with $q_1 = q$ and $q_2 = g = 1_Z$, so that f is central. If fq is symmetrizable, then the same argument in the proof of Proposition 4.10 in [6] works in our situation: suppose that ρ_f is the cooperator for f and 1_Z and consider the diagram below.

$$\begin{array}{ccccc} A \times Z & \xrightarrow{q \times 1_Z} & A' \times Z & \xrightarrow{\rho_f} & Z \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_1}\downarrow & & \downarrow \\ A & \xrightarrow[q]{} & A' & \longrightarrow & 0 \end{array}$$

The outer rectangle is a pullback by assumption, since $\rho_f(q \times 1_Z) = \rho_{f,q}$. The left-hand bottom horizontal morphism is a regular epimorphism, consequently the right-hand square is a pullback since $\mathbb{C}$ is a regular category (see Lemma 1.15 in [13], for instance). ∎

The proposition below is similar to Proposition 4.9 in [6].

3.10. PROPOSITION. *Let $\mathbb{C}$ be a finitely complete centralic category and suppose that regular epimorphisms in $\mathbb{C}$ are stable under binary products. Given that $f\colon A \to Y$ is central/symmetrizable in $\mathbb{C}$ and $q\colon Y \to Q$ is a regular epimorphism in $\mathbb{C}$, then qf is central/symmetrizable.*

PROOF. Let ρ_f be the cooperator of f and 1_Y, then the morphism $1_A \times q\colon A \times Y \to A \times Q$ is a regular epimorphism such that $\mathsf{Eq}(1_A \times q) \leqslant \mathsf{Eq}(q\rho_f)$. Indeed, one has the following string of implications

$$\begin{aligned} (1_X \times q)(x,y) = (1_X \times q)(x,y') &\implies \\ q(y) = q(y') &\implies \\ q\rho_f(0,y) = q\rho_f(0,y') &\implies \\ q\rho_f(x,y) = q\rho_f(x,y'), \end{aligned}$$

so that there exists a morphism $\rho\colon A \times Q \to Q$ such that $q\rho_f = \rho(1_A \times q)$. Then ρ is the desired cooperator for qf and 1_Q, since

$$\rho(1_A, 0) = \rho(1_A \times q)(1_A, 0) = q\rho_f(1_A, 0) = f$$

and

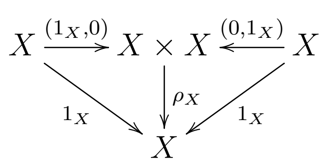
$$\rho(0, 1_Q)q = \rho(1_A \times q)(0, 1_Y) = q\rho_f(0, 1_Y) = q1_Y = 1_Q q$$

which implies $\rho(0, 1_Q) = 1_Q$ since q is a (regular) epimorphism. In the case that f is symmetrizable, the same argument in Proposition 4.9 in [6] may be applied to show that qf is symmetrizable. ∎

The following corollary is immediate.

3.11. COROLLARY. *If regular epimorphisms are stable under binary products in $\mathbb{C}$ then the subcategories $\mathsf{Com}(\mathbb{C})$ and $\mathsf{Ab}(\mathbb{C})$ are closed under regular quotients in $\mathbb{C}$, i.e., for any regular epimorphism $q\colon X \to Y$ if X is commutative/abelian then so is Y.*

3.12. COMMUTATIVE OBJECTS. Recall that an internal *unitary magma* in $\mathbb{C}$ is a pair (X, ρ_X) making the diagram below commute.



Therefore an object is commutative if and only if it has a (necessarily unique) unitary magma structure. A morphism $f\colon X \to Y$ in $\mathbb{C}$ where (X, ρ_X) and (Y, ρ_Y) are internal unitary magmas is a homorphism of internal unitary magmas if the diagram below commutes.

$$\begin{array}{ccc} X \times X & \xrightarrow{f \times f} & Y \times Y \\ {\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

3.13. Proposition. *Given any morphism $f\colon X \to Y$ in $\mathbb{C}$ where (X, ρ_X) and (Y, ρ_Y) are unitary magmas, then f is a morphism of unitary magmas.*

Proof. The morphism f is central (since 1_Y is) and the morphisms $f\rho_X$ and $\rho_Y(f \times f)$ define cooperators for f with itself, so that $f\rho_X = \rho_Y(f \times f)$ by Proposition 3.1. ■

3.14. Proposition. *Every commutative object in $\mathbb{C}$ is the underlying object of an internal commutative monoid in $\mathbb{C}$ (which is necessarily unique).*

Proof. By the classical Eckmann-Hilton argument [10] a unitary magma (X, ρ_X) is a commutative monoid if and only if its multiplication ρ_X is a morphism of the unitary magmas $(X \times X, (\rho_X \times \rho_X)m)$ and (X, ρ_X), where $m\colon (X \times X)^2 \to (X \times X)^2$ is the middle interchange isomorphism $m\colon ((x, y), (a, b)) \mapsto ((x, a), (y, b))$. Thus the result follows by Proposition 3.13. ■

Then Proposition 3.13 and Proposition 3.14 give us the following

3.15. Corollary. *The full subcategory $\mathsf{Com}(\mathbb{C})$ of commutative objects in $\mathbb{C}$ is equivalent to the category $\mathsf{CMon}(\mathbb{C})$ of internal commutative monoids in $\mathbb{C}$.*

3.16. Proposition. *For any commutative object X the diagram*

$$X \underset{(0,1_X)}{\overset{(1_X,0)}{\rightrightarrows}} X \times X \xrightarrow{\rho_X} X$$

is a coequaliser in $\mathbb{C}$.

Proof. Let $f\colon X \times X \to Y$ be any morphism such that $f(1_X, 0) = f(0, 1_X)$ then it is enough to show that the triangle in the diagram below is commutative for the existence part of the statement, where uniqueness follows since ρ_X is a split epimorphism.

$$\begin{array}{ccccc} X & \underset{(0,1_X)}{\overset{(1_X,0)}{\rightrightarrows}} & X \times X & \xrightarrow{\rho_X} & X \\ & & & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f(1_X,0)} \\ & & & & Y \end{array}$$

Consider the morphism $\alpha\colon (X \times X) \times X$ defined by $\alpha((x, y), z) = f(\rho_X(x, z), y)$ then we have

$$\alpha((y, 0), 0) = f(y, 0) = f(0, y) = \alpha((0, y), 0)$$

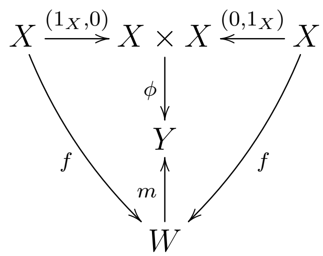
so that since $\mathbb{C}$ is centralic we have

$$\alpha((y, 0), x) = \alpha((0, y), x) \implies f(\rho_X(y, x), 0) = f(\rho_X(x, y), 0) = f(x, y).$$

■

3.17. Strongly centralic categories. In [14] the following condition was defined for a pointed category $\mathbb{C}$ with binary products (see condition 1.1.9 in [14]).

(S) for any object X and every commutative diagram



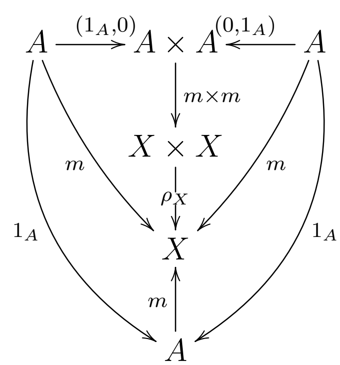
where m is a monomorphism, there exists a morphism $\psi\colon X \times X \to W$ such that $m\psi = \phi$.

For example every unital category satisfies (S) — see Proposition 1.1.10 in [14]. As we will shortly see, there are also non-unital examples of categories satisfying (S).

3.18. Definition. *A pointed category* $\mathbb{C}$ *is called* strongly centralic *if it is centralic and satisfies the property* (S).

3.19. Proposition. *Given a strongly centralic category* $\mathbb{C}$ *the subcategory* $\mathsf{Com}(\mathbb{C})$ *is closed under subobjects in* $\mathbb{C}$.

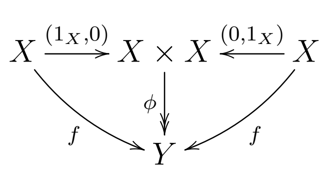
Proof. Given a monomorphism $m : A \to X$ where X is commutative, we may consider the diagram



then condition (S) gives rise to the cooperator ρ_A of 1_A with itself. ■

Consider the following condition on pointed category $\mathbb{C}$ with binary products.

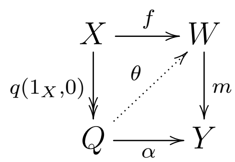
(T) for every object X in $\mathbb{C}$ and any commutative diagram



if ϕ is a regular epimorphism, then so is f.

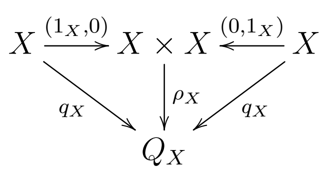
3.20. Proposition. *If $\mathbb{C}$ is a pointed category with binary products and coequalisers then* (T) *implies* (S)*. If $\mathbb{C}$ has a (regular epimorphism, monomorphism)-factorisation system, then* (S) *implies* (T)*.*

Proof. For (T) implies (S), consider the morphisms in the diagram of (S) and let $q : X \times X \to Q$ be a coequaliser of $(1_X, 0)$ and $(0, 1_X)$. Then $q(1_X, 0)$ is a regular epimorphism by (T), and there exists a morphism $\alpha \colon Q \to Y$ such that $mf = \alpha q$. Then the dotted arrow θ in the diagram below exists since regular epimorphisms are orthogonal to monomorphisms.



Then defining $\psi = \theta q$ makes the condition (S) hold. For (S) implies (T), consider the morphisms in the diagram of (T) and let $me = f$ be a (regular epimorphism, monomorphism)-factorisation of f. Then by (S) there exists $\psi \colon X \times X \to Y$ such that $m\psi = e$ and since e is regular we have that m is a strong epimorphism and is therefore an isomorphism. ∎

Commutativity of binary products and coequalisers. Throughout this section, we suppose that $\mathbb{C}$ is a category with binary products and coequalisers which commute. Further, we will suppose that $\mathbb{C}$ satisfies the condition (T). By Proposition 2.5 we have that $\mathbb{C}$ is strongly centralic. For each object X in $\mathbb{C}$ consider a coequaliser $X \xrightarrow{\rho_X} Q_X$ of $(1_X, 0)$ and $(0, 1_X)$ which gives us the diagram:



Then (T) gives us that q_X is a regular epimorphism, and hence by Corollary 3.8 Q_X is commutative. Then we can define a left adjoint $r \colon \mathbb{C} \to \mathsf{Com}(\mathbb{C})$ to the inclusion $\iota \colon \mathsf{Com}(\mathbb{C}) \to \mathbb{C}$ by defining $r(X) = Q_X$ so that q_X is a universal arrow from ι to X. To see this, suppose that we are given any morphism $f \colon X \to A$ (where A is commutative and ρ_A the cooperator for 1_A) then $\rho_A(f \times f)$ coequalises $(1_X, 0)$ and $(0, 1_X)$ so that the necessary $f' \colon r(X) \to A$ exists. Moreover, since binary products commute with coequalisers in $\mathbb{C}$ we can see that the reflection r preserves binary products as a consequence of the diagram

$$X \times Y \underset{(0,1_X)\times(0,1_Y)}{\overset{(1_X,0)\times(1_Y,0)}{\rightrightarrows}} (X \times X) \times (Y \times Y) \xrightarrow{q_X \times q_Y} r(X) \times r(Y)$$

being a coequaliser in $\mathbb{C}$. Moreover by Proposition 3.7 the inclusion $\iota \colon \mathsf{Com}(\mathbb{C}) \to \mathbb{C}$ preserves coequalisers. The proposition below summarises these remarks.

3.21. Proposition. *The subcategory* $\mathsf{Com}(\mathbb{C})$ *is a Birkhoff subcategory of* $\mathbb{C}$*, i.e., it is a reflective subcategory of* $\mathbb{C}$ *which is closed under products, subobjects and regular quotients. Moreover the reflection* $r\colon \mathbb{C} \to \mathsf{Com}(\mathbb{C})$ *preserves binary products, and the inclusion* $\iota\colon \mathsf{Com}(\mathbb{C}) \to \mathbb{C}$ *preserves coequalisers.*

We also have an inclusion $\mathsf{Ab}(\mathbb{C}) \hookrightarrow \mathsf{Com}(\mathbb{C})$ of the full subcategory of abelian objects in $\mathbb{C}$ to the full subcategory of commutative objects. The left adjoint $r\colon \mathsf{Com}(\mathbb{C}) \to \mathsf{Ab}(\mathbb{C})$ may be defined through the cokernel diagram

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{q} r(X).$$

where the unit η of the adjunction has X component $\eta_X = q(1_X, 0)$.

3.22. Proposition. *The category* $\mathsf{Ab}(\mathbb{C})$ *is a reflective subcategory of* $\mathsf{Com}(\mathbb{C})$*.*

Proof. Suppose that X is commutative and let $q\colon X \times X \to Q$ be a cokernel of the diagonal $\Delta_X\colon X \to X \times X$ then Q is commutative by Corollary 3.11. Then q is central, and if $\rho_{X\times X}$ and ρ_Q are the additions of the respective internal commutative monoid structures on $X \times X$ and Q, then q is a morphism of internal commutative monoids (Proposition 3.13). Let $i\colon X \times X \to X$ be the interchange isomorphism $(x, y) \mapsto (y, x)$ it is then easy to see that q is symmetrizable with inverse qi, so that Q is abelian by Corollary 3.9. Then we define $r\colon \mathsf{Com}(\mathbb{C}) \to \mathsf{Ab}(\mathbb{C})$ as $r(X) = Q$ and let $\eta_X\colon X \to r(X)$ be the composite $q(1_X, 0)$, then η_X is routintely checked to be universal from $\iota : \mathsf{Com}(\mathbb{C}) \to \mathsf{Ab}(\mathbb{C})$. ∎

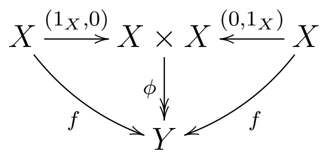
Every unital category is strongly centralic (see condition 1.1.9 in [14]), so that the above results apply to regular unital categories with coequalisers. Interestingly, there are categorical contexts far outside of the unital setting which are strongly centralic. For instance, consider a pointed variety $\mathbb{V}$ of algebras which admits a 4-ary term m satisfying the equations

$$m(x, x, y, 0) = x,$$
$$m(0, y, y, y) = y,$$
$$m(y, x, y, 0) = y.$$

For instance if $\mathbb{V}$ is a unital variety, i.e., $\mathbb{V}$ admits a binary operation $+$ satisfying $x + 0 = x = 0 + x$ then we may define $m(x, y, z, w) = x + w$. Or if $\mathbb{V}$ admits a majority term $p(x, y, z)$ (see the equations preceding Proposition 2.12), then we could define $p(x, y, z)$ then we could define $m(x, y, z, w) = p(x, y, z)$. We claim that $\mathbb{V}$ is strongly centralic. To see that it is centralic, suppose that θ is an equivalence relation on a product of algebras $X \times X$ in $\mathbb{V}$, and suppose that we are given $(x, 0)\theta(y, 0)$ then θ gives us

$$(x, 0)\theta(y, 0),$$
$$(x, z)\theta(x, z),$$
$$(y, z)\theta(y, z),$$
$$(0, z)\theta(0, z).$$

Applying the operation m component-wise above, gives us that $(x,z)\theta(y,z)$. To see that this variety satisfies (T): consider the diagram



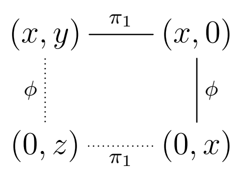
then we have that

$$\begin{aligned}\phi(x,y) &= \phi(m((0,y),(x,0),(x,y),(x,0))\\ &= m(\phi(0,y),\phi(x,0),\phi(x,y),\phi(x,0))\\ &= m(\phi(0,y),\phi(x,y),\phi(0,x),\phi(0,y)) = \phi(0,m(x,y,x,y)) = f(m(x,y,x,y)),\end{aligned}$$

so that f is surjective since ϕ is.

3.23. Proposition. *Every pointed factor permutable category is strongly centralic.*

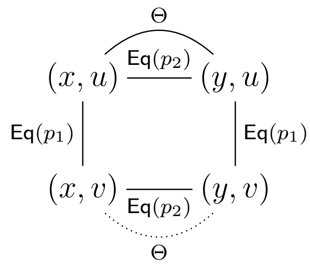
Proof. By Remark 2.10 if $\mathbb{C}$ is factor permutable then $\mathbb{C}$ is centralic. To see that it satisfies (T), consider the morphisms in the statement of (T). To see how f is surjective let $(x,y) \in X \times X$ be any element. Then factor permutability gives a $z \in X$ such that the diagram of relations:



holds. Then $f(z) = \phi(0,z) = \phi(x,y)$ — so that f is surjective. ∎

Concluding remarks

We leave the investigation of what may be called *locally (strongly) centralic* categories, i.e., finitely complete categories $\mathbb{C}$ where every category of points $\mathsf{Pt}_{\mathbb{C}}(X)$ is centralic (or strongly centralic), and what relation these categories have to the results of the papers [12, 9], for a future work. We remark here that varieties of algebras which are locally centralic in this sense have been characterised in [18] by means of H. P. Gumm's shifting lemma: a variety $\mathbb{V}$ is locally centralic if and only if for any morphisms $p : A \to X$ and $q : B \to X$ in $\mathbb{V}$ and any congruence Θ on $A \times_X B$ we have $(x,u)\Theta(y,u) \Rightarrow (x,v)\Theta(y,v)$ for any elements $(x,u),(y,u),(x,v),(y,v)$ of $A \times_X B$. This can be seen as a special case of the shifting lemma:



This observation was the main motivation behind this paper. We also remark here that some of the results in this paper took inspiration from the paper [8] — see Proposition 2.2 (and its proof) therein. Note that categorical conditions similar to some of the equivalent statements of Proposition 2.1 have been investigated in [22] as well as [20].

We also remark that the defining property of a finitely (bi)complete category to be centralic may be reformulated as an exactness property as it is defined in [24] (see section 6.3 therein). Also, the general commutativity of a specified limit with a specified colimit is an exactness property amenable to the results of the paper [25].

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