Theory and Applications of Categories, Vol. 38, No. 9, 2022, pp. 257–310.

ON DUALIZABLE OBJECTS IN MONOIDAL BICATEGORIES

PIOTR PSTRĄGOWSKI

Abstract. We prove coherence theorems for dualizable objects in monoidal bicategories and for fully dualizable objects in symmetric monoidal bicategories, describing *coherent dual pairs* and *coherent fully dual pairs*. These are structures one can attach to an object which we show are property-like and equivalent to, respectively, dualizability and full dualizability. In the latter case, our work reduces the two-dimensional Cobordism Hypothesis of Baez-Dolan to a comparison problem between two explicitly defined bicategories.

Contents

1. Introduction

In this paper we prove coherence theorems for dualizable objects in monoidal bicategories and fully dualizable objects in symmetric monoidal bicategories. These coherence results are modeled on the classical fact that any equivalence of categories can be promoted to an adjoint equivalence, and that such an adjoint equivalence is unique up to a unique isomorphism.

We define a collection of data, that of a *coherent dual pair*, see Definition 3.3, Definition 3.11, that one can attach to an object in a monoidal bicategory. Intuitively, this data consists of an algebraic proof that a given object is dualizable, so that it consists of another object, co(unit) morphisms, and invertible 2-cells witnessing the triangle identities, subject to appropriate coherence equations, one of which we describe informally below. We then prove the following result.

1.1. Theorem. [3.16] *Let $\mathcal{M}$ be a monoidal bicategory. The forgetful homomorphism*

$$\mathcal{C}\mathrm{ohDualPair}(\mathcal{M}) \to (\mathcal{M}^d)^{\cong}$$

Received by the editors 2017-05-24 and, in final form, 2022-01-25.
Transmitted by Ezra Getzler. Published on 2022-02-01.
2020 Mathematics Subject Classification: 18N10, 18N20.
Key words and phrases: monoidal bicategories, dualizable objects, fully dualizable objects.

between, respectively, the bicategory of coherent dual pairs and the groupoid of dualizable objects, is a surjective on objects equivalence.

In particular, an object can be completed to a coherent dual pair if and only if it is dualizable and any two coherent dual pairs living over a given object are canonically equivalent. In the language of Kelly and Lack [KL97], the theorem implies that a coherent dual pair is a property-like structure equivalent to the property of dualizability.

We then move on to the case of fully dualizable objects in symmetric monoidal bicategories. Again, we describe a collection of data, a *coherent fully dual pair*, that one can attach to an object, which intuitively consists of an algebraic proof that the object in question is fully dualizable, subject to its own coherence equations.

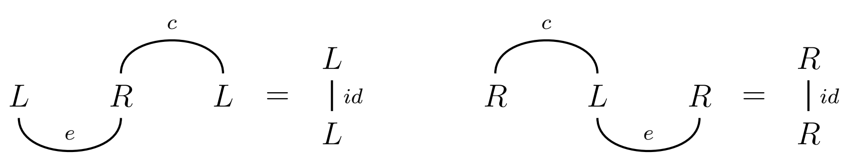
1.2. THEOREM. [4.27] *Let $\mathcal{M}$ be a symmetric monoidal bicategory. The forgetful homomorphism*

$$\mathcal{C}\mathrm{ohFullyDualPair}(\mathcal{M}) \to (\mathcal{M}^{fd})^{\cong}$$

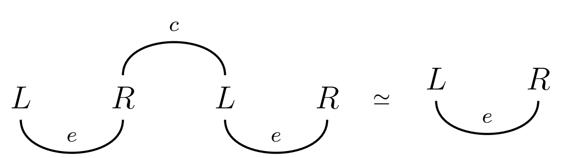
between, respectively, the bicategory of coherent fully dual pairs and the groupoid of fully dualizable objects, is a surjective on objects equivalence.

In both the case of dual and fully dual pairs, the additional coherence equations we impose are inspired by the generating relations of the oriented bordism bicategory, as described by Schommer-Pries [SP11]. To give the reader some idea of what these look like, let us describe informally one of these equations, the so-called *Swallowtail*, whose formal description is given in Definition 3.11.

In a monoidal category, a dual pair consists of objects L, R, an *evaluation* map $e : L \otimes R \to I$ and a *coevaluation* map $c : I \to R \otimes L$, which are subject to the *triangle equations* of the form



In the case of monoidal *bicategories*, each of the above equalities needs to be replaced by an invertible 2-cell, and each of these 2-cells in turn induces an a priori different isomorphism



The *Swallowtail equation*, which is a part of the definition of a coherent dual pair, asserts that these two invertible 2-cells are in fact the same, and Theorem 1.1 implies that this — and an analogous equation concering the coevaluation morphism — can always be arranged, and that dual pairs satisfying these conditions are essentially uniquely equivalent.

The current work is strongly connected to the Cobordism Hypothesis of Baez-Dolan [BD95] [Lur09], as Theorem 1.2 can be interpreted as saying that the free symmetric monoidal bicategory on a coherent fully dual pair satisfies the conjectural universal property of the framed bordism bicategory. This reduces the two-dimensional Cobordism Hypothesis to comparing these two explicitly-defined symmetric monoidal bicategories.

1.3. Summary of contents. Section 1 is introductory and illustrates the main ideas of the current work in the familiar setting of monoidal categories.

Section 2 concerns the theory of dualizable objects in monoidal bicategories. We define a collection of data one can attach to a dualizable object, a *dual pair*, whose existence is by definition equivalent to dualizability. We then describe additional equations on the components of a dual pair and term those pairs that satisfy them *coherent*. We then prove a strictification result that any dualizable object can be completed to a coherent dual pair and a coherence theorem which says that the forgetful homomorphisms from the bicategory of coherent dual pairs into the 2-groupoid of dualizable objects is an equivalence.

Section 3 develops the theory of fully dualizable objects in symmetric monoidal bicategories. We start by recalling the notion of a Serre autoequivalence, which is a canonical automorphism one can attach to any fully dualizable object in a symmetric monoidal bicategory, and verify its basic properties. We then introduce the notions of fully dual pairs and coherent fully dual pairs, and prove strictification and coherence theorems analogous to the dualizable case.

In the Appendix A we write down a variant on the theory of monoidal generated by a set of data, modeled on computadic symmetric monoidal bicategories of Christopher Schommer-Pries [SP11]. The variant described has the advantage of allowing generating 1-cells and generating 2-cells whose sources and targets are only consequences of other generating data, and are not necessarily generating themselves. We also prove a couple of technical results concerning the behaviour of these bicategories when some of the generating data is omitted.

1.4. Notation and terminology. We will assume that all of the objects we consider are small with respect to some fixed universe; this has no effect on our considerations as all of our constructions are determined by a finite list of data.

A *monoidal bicategory* is by definition a tricategory with one unnamed object. We use the definition of a tricategory as given in [Gur13a], which differs from the one of [GPS95] by the fact that it is fully algebraic in the sense that all functors that are postulated to be equivalences come in form of adjoint equivalences. The related notions of *symmetric monoidal bicategories*, *(symmetric) monoidal homomorphisms*, *transformations* and *modifications* can be found in [SP11].

We will refer to functors between bicategories as *homomorphisms*; they are always assumed to be strong, that is, their constraint 2-cells are isomorphisms, but not necessarily identities. If they are identities, we will talk about *strict homomorphisms*. The word *functor* itself will be reserved for ordinary functors between categories.

If $\mathcal{M}$ is a monoidal bicategory, we will denote its monoidal product by $\otimes$ and by I its monoidal unit. If it doesn't lead to confusion, we will also denote the monoidal product by juxtaposition. The associator will be denoted by a and if $\mathcal{M}$ is symmetric, we will denote the symmetry by σ.

We will usually use the name of the object in question to denote identity one-cells, constraint 2-cells witnessing naturality of a homomorphism of bicategories will be denoted by the name of the homomorphism.

If $\mathcal{M}$ is a bicategory, by $\mathcal{M}^{\cong}$ we denote the *underlying 2-groupoid*; that is, the bicategory obtained by disregarding all the morphisms that are not equivalences and all the 2-cells that are not isomorphisms. By $\mathrm{h}(\mathcal{M})$ we will denote its *homotopy category*, which is obtained by identifying isomorphic 1-cells. The categories of morphisms will be denoted by $\mathcal{M}(-,-)$ and referred to as Hom-categories.

When working with **Gray**-monoids, see below for an explanation, we will use the "first the maps on the left" convention. More specifically, when writing $f_1 \otimes \cdots \otimes f_n$ we will always mean $(1 \otimes \ldots 1 \cdots f_n) \circ (1 \otimes \cdots f_{n-1} \otimes 1) \circ \cdots \circ (f_1 \otimes 1 \otimes \cdots \otimes 1)$. We will denote the interchange isomorphism via $\Sigma_{f,g} : (1 \otimes g)(f \otimes 1) \Rightarrow (f \otimes 1)(1 \otimes g)$. We reserve the right to suppress it in the presence of a different 2-cell if it is clear it should be inserted to make the pasting diagram well-formed.

1.5. Coherence issues. By the coherence theorem for bicategories, see [Gur13a], [SP11], a pasting diagram of 2-cells in a bicategory has a unique value once a choice of association has been made for the source and target. As a consequence, we will not name or draw any constraint 2-cells coming from the structure of a bicategory.

There are two related coherence results in the theory of monoidal bicategories which we will use. The first one is a classical theorem of Robert Gordon, Anthony Power and Ross Street.

1.6. Theorem. [GPS95] *Any monoidal bicategory is equivalent to a **Gray**-monoid.*

A **Gray**-monoid is a monoid object in the category of 2-categories and strict homomorphisms equipped with the Gray tensor product. It can be identified with a particularly strict kind of a monoidal bicategory, which is strict as a bicategory, its associator, left and right units are given by the identities, and whose tensor product homomorphism is *cubical*, in particular strict in each variable separately.

Related to the fact that not every monoidal bicategory can be fully strictified is the issue that diagrams consisting only of constraint 2-cells need not always commute [Gur13a, 10.3]. To alleviate this, one has the following result of Nick Gurski.

1.7. Theorem. [Gur13a, 10.6] *Let E be a locally discrete category-enriched graph. Then, any diagram of 2-cells in $\mathbb{F}(E)$, the free monoidal bicategory generated by E, commutes.*

This implies that in any monoidal bicategory, if we have two isomorphic composites of morphisms which are both in the image of some strict homomomorphism out of a freely generated monoidal bicategory as above (ie. can be written in terms of the same set of generators), then there is a preferred composite of constraint 2-cells between them.

We will use two coherence results for symmetric monoidal bicategories. The first one, due to Nick Gurski and Angélica Osorno, is a more restrictive analogue of Theorem 1.7 and is used in the same way .

1.8. Theorem. [GO13] *In a free symmetric monoidal bicategory* $\mathbb{F}(X)$ *on a set of objects* X*, any diagram of* 2*-cells commutes.*

The second result is due to Christopher Schommer-Pries, and is an analogue of the strictification of monoidal bicategories of Gordon, Power and Street.

1.9. Theorem. [SP11, 2.96] *Any symmetric monoidal bicategory is equivalent to a quasi-strict one.*

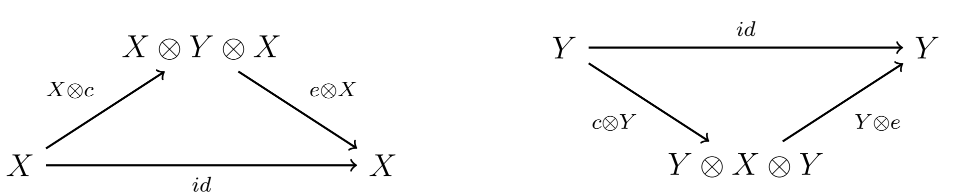
A quasi-strict symmetric monoidal bicategory is a partially strict kind of symmetric monoidal bicategories which has, among other things, a **Gray**-monoid as an underlying monoidal bicategory.

1.10. Acknowledgements. This paper is around one half of the author's Master's thesis, written at Bonn University. I would like to thank Christopher Schommer-Pries, who supervised this project, for his guidance, patience and many helpful comments. I would also like to thank the anonymous referee, whose comments greatly improved the current work.

2. Recollections on dualizable objects in monoidal categories

In this short section we review the basic theory of dualizable objects in monoidal categories. Everything here is classical, and our main goal is to state the coherence result in a way which generalizes to the case of monoidal bicategories.

2.1. Definition. *Let* $(\mathcal{M}, \otimes, I)$ *be a monoidal category. A* dual pair (X, Y, e, c) *in* $\mathcal{M}$ *consists of an object* X*, which we call the* left dual*, an object* Y*, which we call the* right dual*, together with* evaluation $e : X \otimes Y \to I$ *and* coevaluation $c : I \to Y \otimes X$ *maps that satisfy* triangle equations *pictured below.*



We say that an object $X \in \mathcal{M}$ *is* left dualizable *if it can be completed into a dual pair as a left dual.*

Figure 1: Triangle equations in string diagram form

2.2. Remark. Similarly, one defines an object to be right dualizable if it can be completed to a dual pair as a right dual. Their theory is formally analogous and so we will focus exclusively on left dualizability, referring to it simply as *dualizability*.

The triangle equations can be drawn using string diagrams, which reveal the connection with topology of 1-manifolds. If we denote evaluation with the right elbow and coevaluation with the left elbow, we can picture the equations as in Figure 1.

Observe that being dualizable is a *property*. More specifically, it entails existence of some amount of data, but does not specify it. A given dualizable object can be in general completed to many different dual pairs and thus it is a valid question to ask whether all such pairs are in fact in some sense the same. This is indeed the case, as we will show below in Theorem 2.5.

Using the language of Lack and Kelly, we say that the structure of a dual pair is *property-like* [KL97]. This means that an appropriately defined space (in the sense of homotopy theory) of dual pairs over a given object is either empty or contractible, depending on whether the object in question is dualizable. The terminology is motivated by the fact that in this case the space-valued association

$$X \mapsto (\text{Collection of dual pairs over } X)$$

takes only two values, and so is uniquely determined by a subset of X.

2.3. Definition. *If $\mathcal{M}$ is a monoidal category, then a* morphisms of dual pairs

$$(X_1, Y_1, e_1, c_1) \to (X_2, Y_2, e_2, c_2)$$

consists of arrows $x : X_1 \to X_2$, $y : Y_1 \to Y_2$ that are natural with respect to (co)evaluation maps; that is, we have $e_2 \circ (x \otimes y) = e_1$ and $c_2 = (y \otimes x) \circ c_1$. We will denote the category of dual pairs by $\mathcal{D}\text{ualPair}(\mathcal{M})$

Another possible approach to making dual pairs into a category, one which generalizes well to higher-dimension, is to consider an appropriate free monoidal category.

2.4. Proposition. *There exists a monoidal category $\mathbb{F}_d$, the* free monoidal category on a dual pair*, such that for any other monoidal category there is a bijection*

$$\mathbf{MonCat}_{strict}(\mathbb{F}_d, \mathcal{M}) \simeq \{ \text{ Dual pairs in } \mathcal{M} \ \}$$

between strict monoidal homomorphisms $\mathbb{F}_d \to \mathcal{M}$ and the set of dual pairs in $\mathcal{M}$, natural with respect to strict homomorphisms.

By Yoneda lemma, such a monoidal category is unique up to a canonical invertible strict homomorphism. Observe that the above natural bijection endows the set of dual pairs with a natural structure of a category, with morphisms given by natural transformations. It is not difficult to check that this structure coincides with the one given in Definition 2.3.

The question of whether the structure of a dual pair is property-like can now be answered in affirmative in the form of the following theorem.

2.5. THEOREM. [Coherence for dualizable objects in monoidal categories] *Suppose*

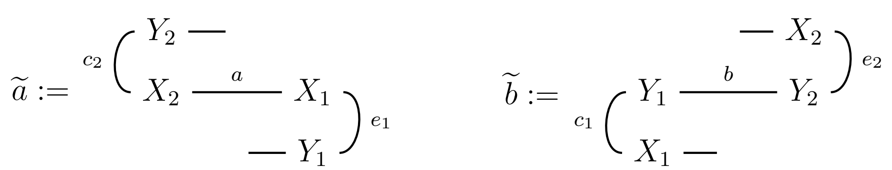
$$(x, y) : (X_1, Y_1, e_1, c_1) \to (X_2, Y_2, e_2, c_2)$$

is a morphism of dual pairs. Then $x : X_1 \to X_2$ is an isomorphism and, conversely, for any such isomorphism there is a unique $y : Y_1 \to Y_2$ that completes it to a map of dual pairs. More concisely, the forgetful functor

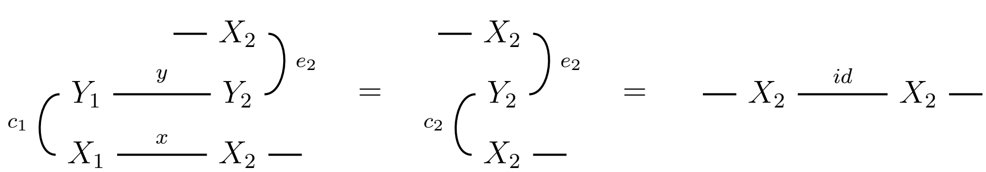
$$\begin{aligned} \mathcal{D}\text{ualPair}(\mathcal{M}) &\to (\mathcal{M}^d)^{\cong} \\ (X, Y, e, c) &\mapsto X \end{aligned}$$

from the category of dual pairs into the groupoid of dualizable objects in $\mathcal{M}$ is a surjective on objects equivalence of categories.

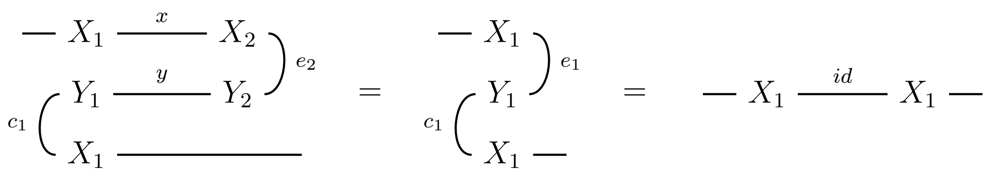
PROOF. If $a : X_2 \to X_1$ and $b : Y_1 \to Y_2$ are morphisms, then we define their duals $\widetilde{a}, \widetilde{b}$ using the string diagrams



The triangle equations imply that this establishes inverse bijections $\mathcal{M}(Y_1, Y_2) \simeq \mathcal{M}(X_2, X_1)$. We claim that $\widetilde{y}$ is inverse to x. Indeed, one composite is equal to identity by the easy computation



where we have used the fact that $y \otimes x$ commutes with coevaluation. For the other we compute



,

using that $x \otimes y$ commutes with evaluation. This also shows that there is at most one such y, as it must then be necessarily equal to $\widetilde{x^{-1}}$, as taking dual twice recovers the morphism.

Conversely, one verifies easily that if $x : X_1 \to X_2$ is invertible, then setting $y = \widetilde{x^{-1}}$ does indeed complete it to a morphism of dual pairs. ∎

2.6. Remark. One of our main results is an analogue of Theorem 2.5 for dualizable objects in *monoidal bicategories*. In this context, the most naive definition of dual pair will *not* have the needed property and we will have to require additional coherence equations.

3. Dualizable objects in monoidal bicategories

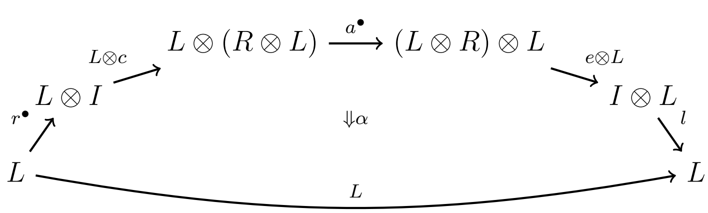
This section is devoted to the theory of dualizable objects in monoidal bicategories. Our main result is a description of a property-like structure that one can attach to an object in a monoidal bicategory, namely that of a *coherent dual pair*, equivalent to the property of being dualizable.

3.1. Definition. *Let $\mathcal{M}$ be a monoidal bicategory. We say that an object $L \in \mathcal{M}$ is* left dualizable *if it is left dualizable as an object of the homotopy category* $\mathrm{h}(\mathcal{M})$*.*

3.2. Remark. As in the case of monoidal categories, there is an analogous notion of right dualizability. We will exclusively focus on left dualizability and refer to it simply as *dualizability*.

In the case of monoidal bicategories, like in the classical case, the property of being dualizable can be rephrased as requiring existence of some auxiliary data. It is rather easy to specify the latter, as it arises as a direct categorification of the notion of a dual pair introduced in Definition 2.1, where the triangle equations are now replaced by invertible 2-cells.

3.3. Definition. *A* dual pair *in a monoidal bicategory $\mathcal{M}$ is a tuple $(L, R, e, c, \alpha, \beta)$, where $L, R \in \mathcal{M}$ are objects, $e : L \otimes R \to I$, $c : I \to R \otimes L$ are 1-cells and α, β are isomorphisms*



$$\begin{array}{ccccccc} & & (R\otimes L)\otimes R & \xrightarrow{a} & R\otimes(L\otimes R) & & \\ & \nearrow^{c\otimes R} & & & & \searrow^{R\otimes e} & \\ I\otimes R & & & \Downarrow\beta & & & R\otimes I \\ \uparrow^{l^\bullet} & & & & & & \downarrow^{r} \\ R & & & \xrightarrow[R]{} & & & R \end{array}.$$

We will refer to e as evaluation*, to c as* coevaluation *and to α, β as* cusp isomorphisms*.*

3.4. Remark. The terminology of referring to α, β as *cusp isomorphisms* comes from the oriented bordism bicategory, whose presentation was obtained by Schommer-Pries, see [SP11]. In this bicategory one has a dual pair consisting of a positive and negative point. The (co)evaluation maps are given by appropriate elbows and cusp isomorphisms are given by surfaces that exhibit a type of cusp.

Note that if $\mathcal{M}$ has only identity 2-cells, so that it can be identified with an ordinary monoidal category, then the cusp isomorphisms are necessarily unique and equal to the identity, so that the above definition reduces to Definition 2.1.

3.5. Notation. We will often refer to the whole dual pair in the sense of Definition 3.3 just by referring to the underlying objects, using the notation $\langle L, R\rangle_d := (L, R, e, c, \alpha, \beta)$.

3.6. Proposition. *An object $L \in \mathcal{M}$ is dualizable if and only if it can be made part of a dual pair $\langle L, R\rangle_d$ for some choice of R, (co)evaluation maps and cusp isomorphisms.*

Proof. Clearly, if an object is a part of a dual pair then it is dualizable. Conversely, assume that L is dualizable, so that one can complete it to a dual pair in the homotopy category $h(\mathcal{M})$, giving us the right dual and (co)evaluation maps. The fact that the triangle equations hold in the homotopy category implies existence of witnessing invertible 2-cells, which we can take to be the cusp isomorphisms α, β. ∎

To organize dual pairs into a bicategory, we will use the language of *computadic monoidal bicategories* and of *P-shapes* developed in [SP11], which we also recall in Appendix A. Informally, a monoidal bicategory is *computadic* if it admits an explicit presentation P in which one allows generating objects, 1-cells and 2-cells, and where the sources and targets of generating 1-cells and 2-cells are not necessarily generators themselves, but only their consequences. The only allowed relations are at the level of 2-cells.

If P is such a presentation and $\mathcal{M}$ is an auxiliary monoidal bicategory, then a P-shape in M consists of a compatible association of objects, morphisms and 2-cells of M to each generator in P, subject to appropriate relations, see Definition A.11. These shapes can be assembled into a bicategory $P(\mathcal{M})$ which is by construction isomorphic to the bicategory of strict homomorphisms out of the computadic monoidal bicategory generated by P.

3.7. Definition. *Let G_d be a generating datum for a monoidal bicategory consisting of*

1. *two generating objects L, R,*

2. *two generating morphisms* $e : L \otimes R \to I$ *and* $c : I \to R \otimes L$,

3. *four generating* 2*-cells* $\alpha, \beta, \alpha^{-1}$ *and* β^{-1}, *where* α, β *have sources and targets exactly like the cusp isomorphisms and* α^{-1}, β^{-1} *the opposite.*

Let $\mathcal{R}_d$ *consist of relations that* α, α^{-1} *and* β, β^{-1} *are inverse to each other. We call the pair* $P_d = (G_d, \mathcal{R}_d)$ *the* presentation of the free monoidal bicategory on a dual pair.

The bicategory of dual pairs can be now be conveniently described as a bicategory of shapes of the above type.

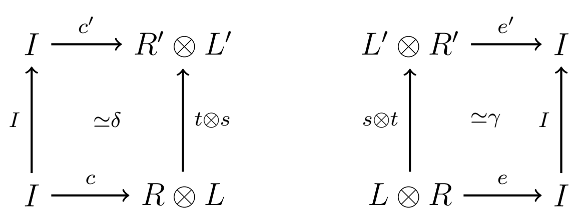
3.8. Definition. *If* $\mathcal{M}$ *is a monoidal bicategory, we define the* bicategory of dual pairs *in* $\mathcal{M}$ *as*

$$\mathcal{D}\mathrm{ualPair}(\mathcal{M}) := P_d(\mathcal{M}),$$

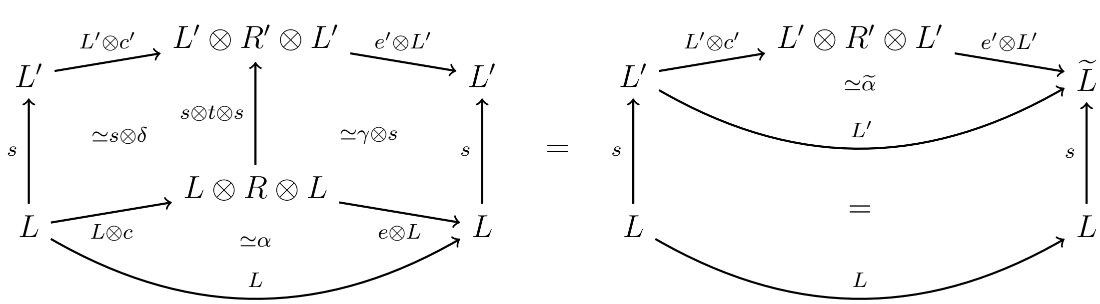
the bicategory of shapes in $\mathcal{M}$ *of type* P_d *in the sense of Definition A.11.*

Below, we give an explicit description of the bicategory of dual pairs, both for the convenience of the reader and also to fix notation. We will assume that $\mathcal{M}$ is **Gray**-monoid to make drawing diagrams manageable, but after adding appropriate coherence constraints the description given below holds in the general case.

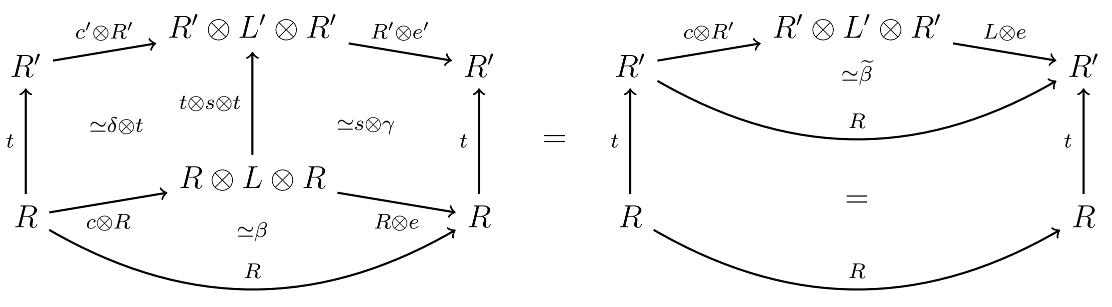
3.9. Notation. The objects of $\mathcal{D}\mathrm{ualPair}(\mathcal{M})$ are precisely the dual pairs in the sense of Definition 3.3. A morphism $(s,t)_d : \langle L, R\rangle_d \to \langle L', R'\rangle_d$ of dual pairs consists of data of 1-cells $s : L \to L'$, $t : R \to R'$ and constraint invertible 2-cells γ, δ of type



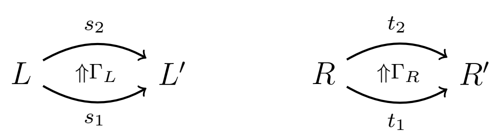
whose purpose is to witness naturality of s, t with respect to (co)evaluation maps. Additionally, these constraint isomorphisms are required to satisfy further naturality condition with respect to cusp isomorphisms, namely that



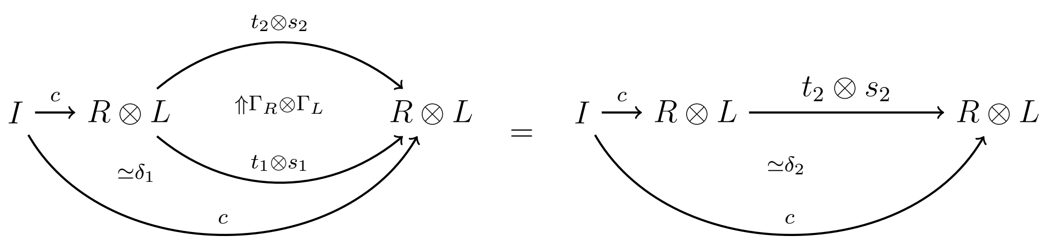
and



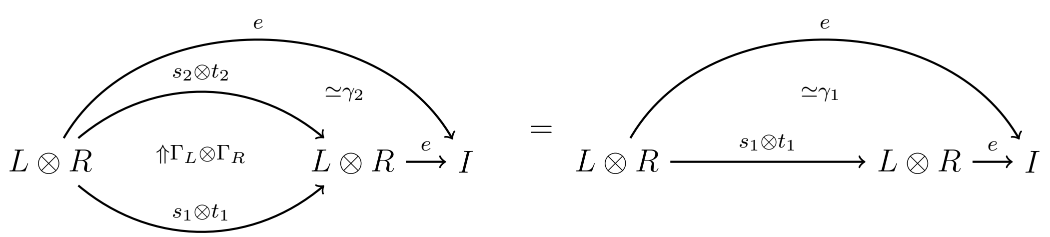
If $(s_1, t_1)_d, (s_2, t_2)_d$ are parallel morphisms, then a 2-cell $\Gamma : (s_1, t_1)_d \to (s_2, t_2)_d$ consists of data of 2-cells



that satisfy naturality equations with respect to isomorphisms witnessing naturality of maps $(s_1, t_1)_d, (s_2, t_2)_d$ of the form



and



We now proceed to establish a basic property of bicategories of dual pairs, namely that they are groupoids. This is analogous to the similar result for dual pairs in ordinary monoidal categories, which we have discussed in the introduction.

3.10. Proposition. *The bicategory* $\mathcal{D}\mathrm{ualPair}(\mathcal{M})$ *of dual pairs is a 2-groupoid; that is, all of its morphisms are equivalences and all of its 2-cells are isomorphisms.*

Proof. By construction, we can identify $\mathcal{D}\mathrm{ualPair}(\mathcal{M})$ with the bicategory

$$\mathbf{MonBicat}_{strict}(\mathbb{F}(P_d), \mathcal{M})$$

of strict monoidal homomorphisms out of the computadic monoidal bicategory generated by the presentation P_d of Definition 3.7. By the cofibrancy result, see Theorem A.17, the inclusion

$$\mathbf{MonBicat}_{strict}(\mathbb{F}(P_d), \mathcal{M}) \hookrightarrow \mathbf{MonBicat}(\mathbb{F}(P_d), \mathcal{M})$$

into the bicategory of all monoidal homomorphisms, is an equivalence. Since the latter clearly only depends on $\mathcal{M}$ up to equivalence, to prove the result we can assume that $\mathcal{M}$ is a **Gray**-monoid.

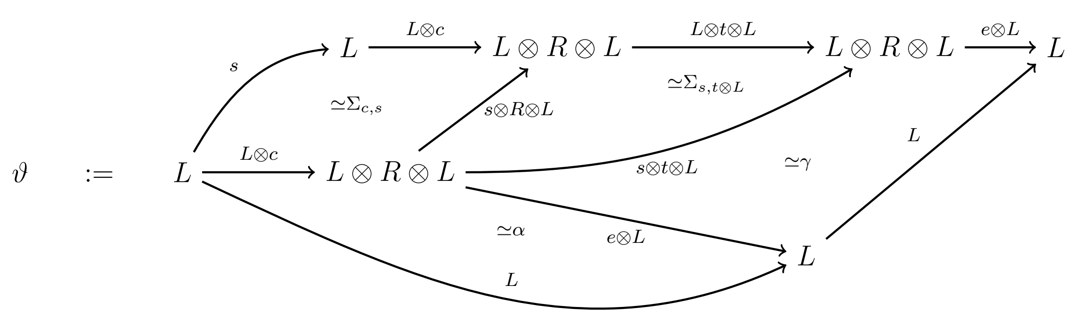
We first show that all morphisms are equivalences. Observe that a 1-cell $(s,t)_d = (s,t,\gamma,\delta)$ is an equivalence if and only if s,t are, this follows from the corresponding statement in the homomorphism bicategory. Any map of dual pairs in a monoidal bicategory induces a map of the corresponding dual pairs in the homotopy category $\mathrm{h}(\mathcal{M})$, whose both components must be isomorphisms by Theorem 2.5. It follows that s,t are equivalences.

We are now left with showing that all 2-cells are isomorphisms, by the part above it is enough to do so for 2-cells between autoequivalences. We will do so by exhibiting for any endomorphism $(s,t)_d : \langle L, R\rangle_d \to \langle L, R\rangle_d$ a "canonical" pseudoinverse to the component $s : L \to L$, together with "canonical" witnessing isomorphisms between the respective composites and the identity. By "canonical" we mean that this structure will be natural with respect to maps of such endomorphisms.

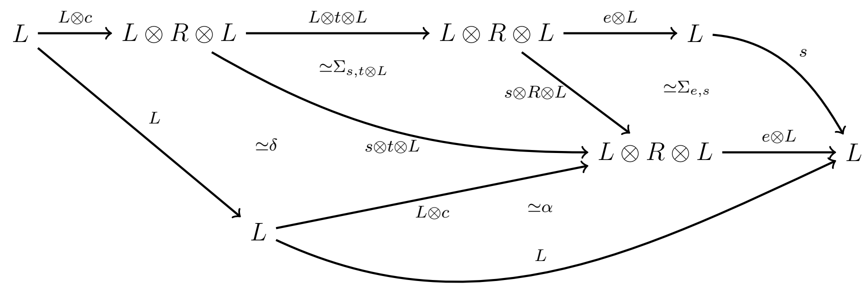
We construct it as follows. The pseudoinverse of s is given by

$$s^{-1} := (e \otimes L) \circ (L \otimes t \otimes L) \circ (L \otimes c),$$

that is, the the dual of t. It is easy to observe that the witnessing isomorphisms $\theta : ss^{-1} \simeq id_L$ and $\vartheta : s^{-1}s \simeq id_L$ pictured below

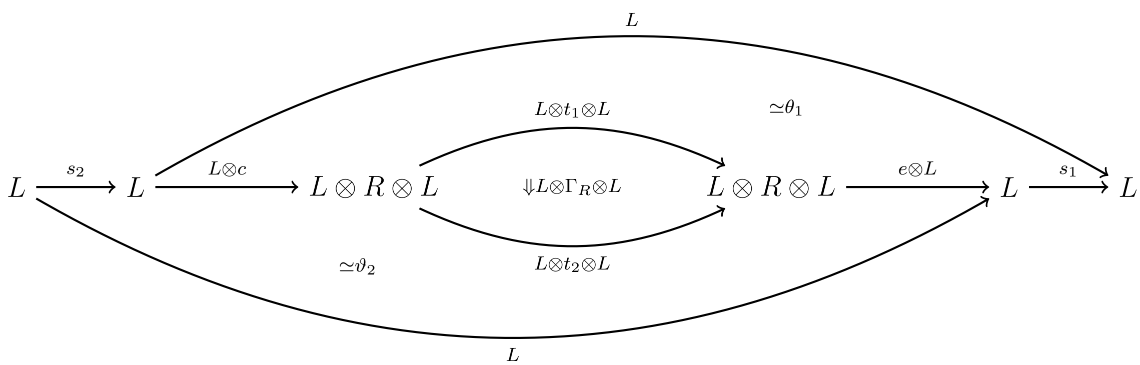


$\theta \quad :=$



are natural with respect to maps between such endomorphisms. This follows from the naturality properties of the interchanger and the naturality of γ, δ constraint 2-cells with respect to such maps.

Once we know that the component $s : L \to L$ of an endomorphism of a dual pair is canonically exhibited as a witnessed equivalence, the fact that such 2-cells must be invertible follows by a variation on the argument from Theorem 2.5. In detail, suppose we are now given two endomorphisms $(s, t)_d = (s, t, \gamma, \delta)$ and $(s', t')_d = (s', t', \gamma', \delta')$ of $\langle L, R\rangle_d$. Consider the composite



.

If we paste Γ_L from below, which amounts to postcomposition, by the naturality of θ the resulting pasting diagram will be an isomorphism involving θ_2, ϑ_2. Similarly, if we paste Γ_L from above, which amounts to precomposition, the naturality of ϑ implies that the resulting diagram will be an isomorphism involving θ_1, ϑ_1. It follows that Γ_L is an isomorphism itself.

The argument for Γ_R is analogous, where we would now exhibit the t-component of a map of dual pairs as a witnessed equivalence with pseudoinverse given by the dual of s. ■

As observed before in Proposition 3.6, it is almost tautological that an object is dualizable if and only if it can be completed to a dual pair. However, the naive bicategorical analogue of Theorem 2.5 fails — the forgetful homomorphism $\mathcal{D}\mathrm{ualPair}(\mathcal{M}) \to (\mathcal{M}^d)^{\cong}$ onto the groupoid of dualizable objects is usually *not* an equivalence of bicategories. In particular, a given object can in general be completed to many non-equivalent dual pairs.

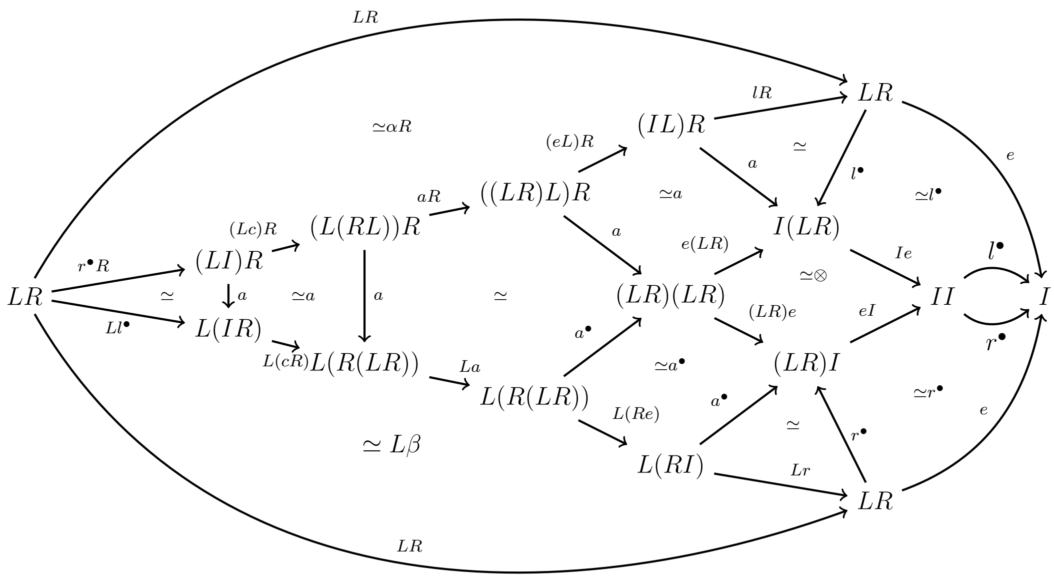


Figure 2: Swallowtail composite (E)

To fix this defect, we will introduce a class of dual pairs which satisfy additional coherence equations, and which we we will simply call *coherent*. Our main result of this section, namely Theorem 3.16, will be that any object can be completed to an essentially unique coherent dual pair.

3.11. Definition. *We say a dual pair* $(L, R, e, c, \alpha, \beta)$ *is* coherent *if the* Swallowtail composites *depicted in Figure 2 and Figure 3 are both equal to the identity.*

3.12. Remark. The Swallowtail equations were first introduced by Dominic Verity in [Ver11] in the context of **Gray**-categories and later generalized to tricategories by Nick Gurski, see [Gur12]. One way to motivate them geometrically is that they hold for a natural choice of cusp isomorphisms in the bordism bicategory [SP11], and are an ingredient in the proof of the categorified Riemann-Roch theorem of Hoyois, Safronov, Scherotzke and Sibilla [HSSS18].

We will first prove a straightforward strictification theorem, which says that given a fixed choice of L, R and candidate (co)evaluation maps e, c, if the cusp isomorphisms α, β can be chosen at all, then one can also make a choice that would make the resulting dual pair coherent.

We begin with the following straighforward lemma, which is the obvious categorification of the usual result on bijections between certain Hom-sets for dual pairs in ordinary monoidal categories.

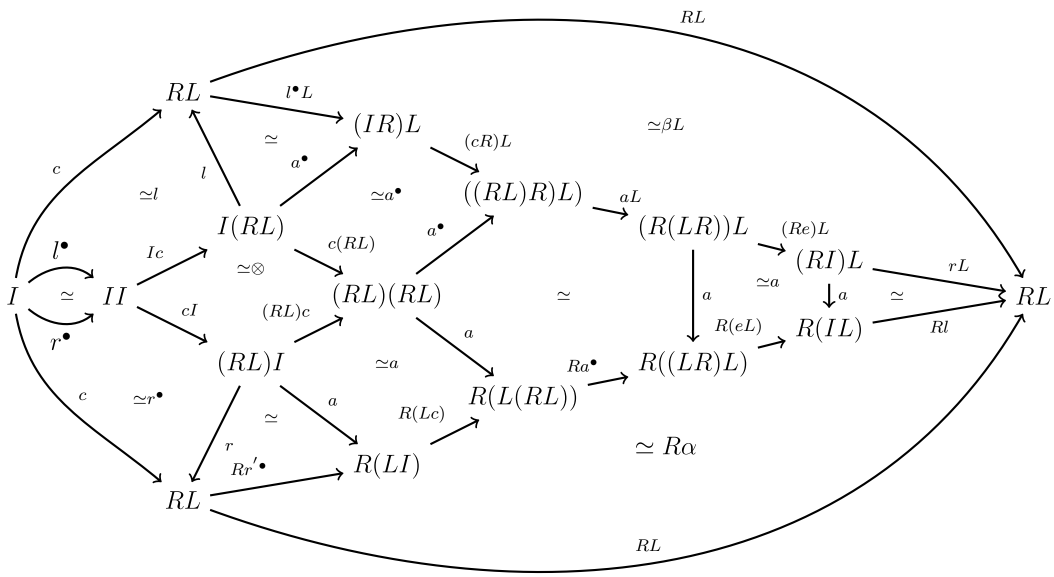


Figure 3: Swallowtail composite (C)

3.13. Lemma. *Let $A, B \in \mathcal{M}$ be objects and let $\langle L, R\rangle_d$ be a dual pair in a monoidal bicategory $\mathcal{M}$. Then, the functor $\mathcal{M}(A\otimes R, B) \to \mathcal{M}(A, B\otimes L)$ given by $f \mapsto (f\otimes L)\circ(A\otimes c)$ is an equivalence of categories with an explicit inverse given by $g \mapsto (B \otimes e) \circ (g \otimes R)$.*

Proof. The needed natural isomorphisms between the relevant composites and identities are induced by the cusp isomorphisms. ■

Observe that in the statement of Lemma 3.13 we have supressed the obvious associators and unitors that should be included in the definition of the functors so that they are well-formed. By coherence result of Gurski which we stated as Theorem 1.7, any choices one would have to make do not matter, as all the different composites of associators and unitors are uniquely isomorphic.

3.14. Theorem. [Strictification for dual pairs] *If a dual pair $\langle L, R\rangle_d$ satisfies either of the Swallowtail equations, then it satisfies both of them.*

Moreover, if we keep the objects and (co)evaluation maps fixed, then for any choice of a cusp isomorphism α there is a unique cusp isomorphism β such that together they satisfy both Swallowtail equations. In particular, any dual pair can be made coherent by only a change of β.

3.15. Remark. The strictification result was first proven by Nick Gurski in [Gur12] under the additional assumption that the (co)evaluation maps e, c are equivalences.

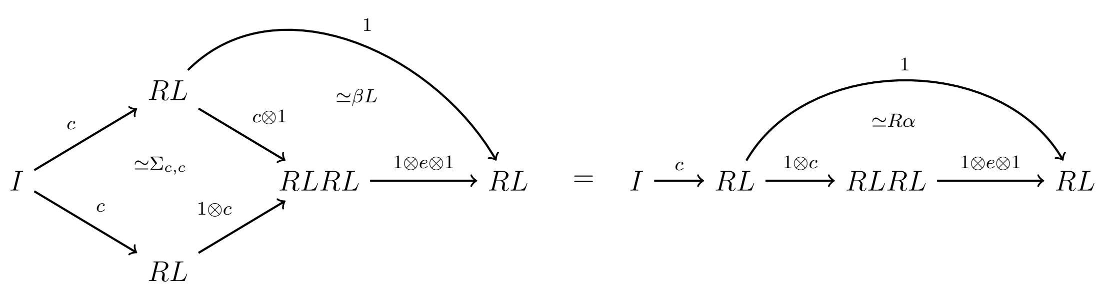
Proof. We first claim that it is enough to prove the result under the additional assumption that $\mathcal{M}$ is a **Gray**-monoid.

Indeed, let $\mathcal{M}$ be an arbitrary monoidal bicategory and let $\widetilde{\mathcal{M}}$ be its computadic ("cofibrant") replacement in the sense of Definition A.5 and Definition A.9. In more detail, let $\widetilde{\mathcal{M}}$ be the computadic monoidal bicategory freely generated by all the objects, 1-cells and 2-cells on $\mathcal{M}$, subject to the relation that two 2-cells are equal if and only if their images under the obvious strict homomorphism $\widetilde{\mathcal{M}} \to \mathcal{M}$ are equal.

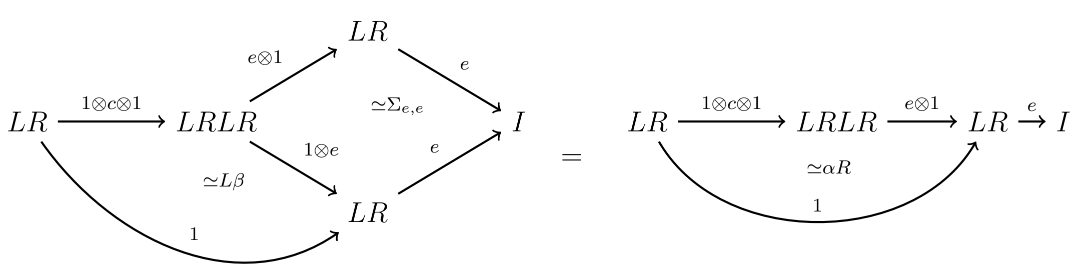
The strict homomorphism in question is an equivalence, as it is surjective on objects, morphisms and locally bijective on 2-cells by construction. Since any dual pair in $\mathcal{M}$ is an image of a dual pair in $\widetilde{\mathcal{M}}$, local bijectivity on 2-cells allows us to deduce that the theorem holds for $\mathcal{M}$ if and only if it holds for its computadic replacement.

By coherence for tricategories, which we stated as Theorem 1.6, we can choose an equivalence $\widetilde{\mathcal{M}} \to \mathbb{G}$, where $\mathbb{G}$ is a **Gray**-monoid. By the cofibrancy theorem for computadic monoidal bicategories, see Theorem A.17, we can assume that it is in fact a strict homomorphism. Since $\widetilde{\mathcal{M}} \to \mathbb{G}$ is locally bijective on 2-cells and we already know that the theorem holds for all dual pairs in $\mathbb{G}$, we deduce that the same is true for $\widetilde{\mathcal{M}}$, ending the argument.

Thus, let us assume that $\mathcal{M}$ is a **Gray**-monoid. In this case the Swallowtail equations take a particularly simple form. Namely, identity (C) reads

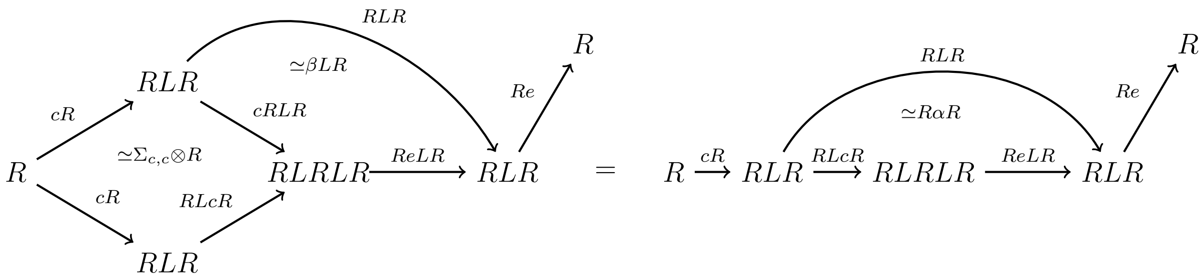


and similarly identity (E) is



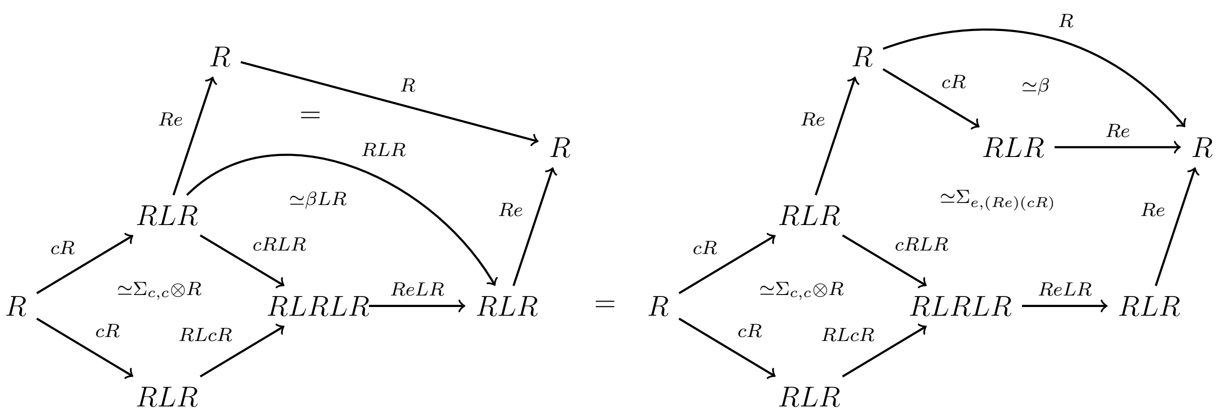
.

We will show that equation (E) implies equation (C), the other case is analogous. By Lemma 3.13, it is enough to show that



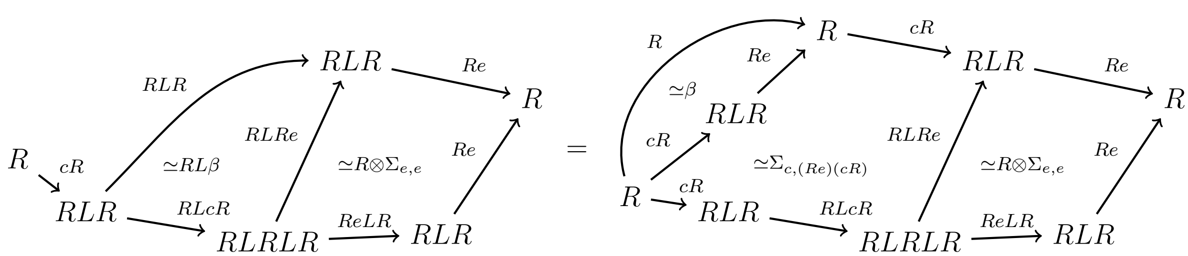
,

as the functor $(R\otimes e)\circ(-\otimes R)$ is an equivalence on the respective Hom-categories. Using the naturality of the interchanger we can rewrite the left hand side of the equation as

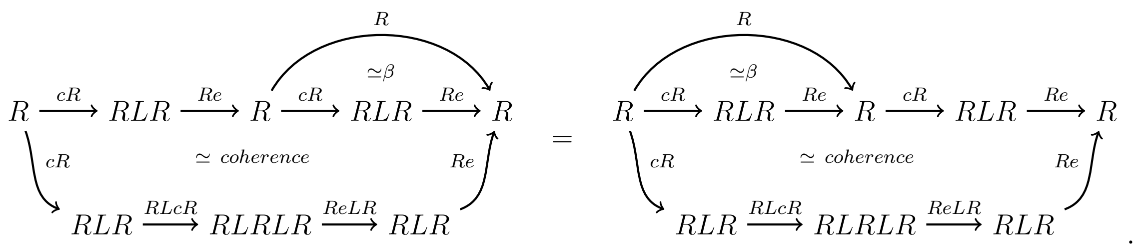


.

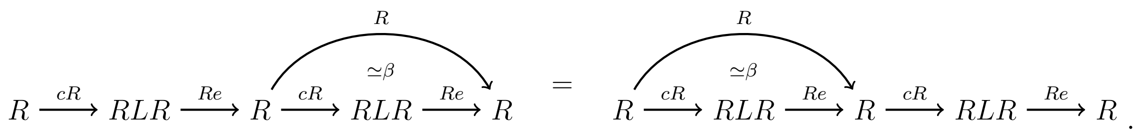
Similarly, the right hand side can be rewritten as



by first using the Swallowtail identity (E) to replace $R\alpha R$ by $RL\beta$ and some coherence and then using the naturality of the interchanger exactly like we did during the manipulation of the left hand side. Thus, we have shown that under the assumption of (E), equation (C) is equivalent to one of the form

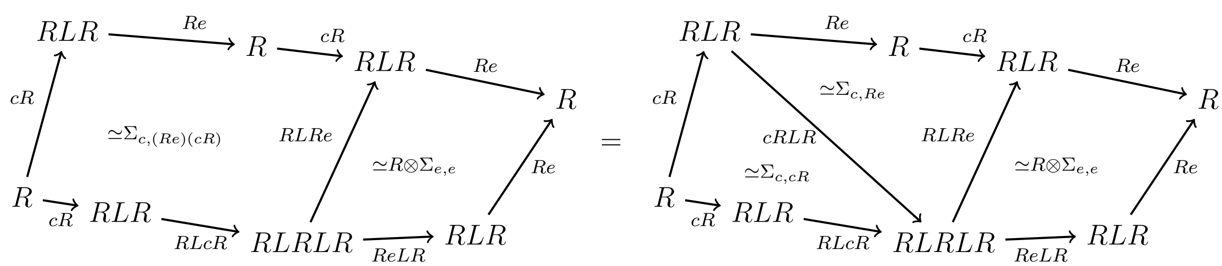


We claim the above equation always holds. The equality of

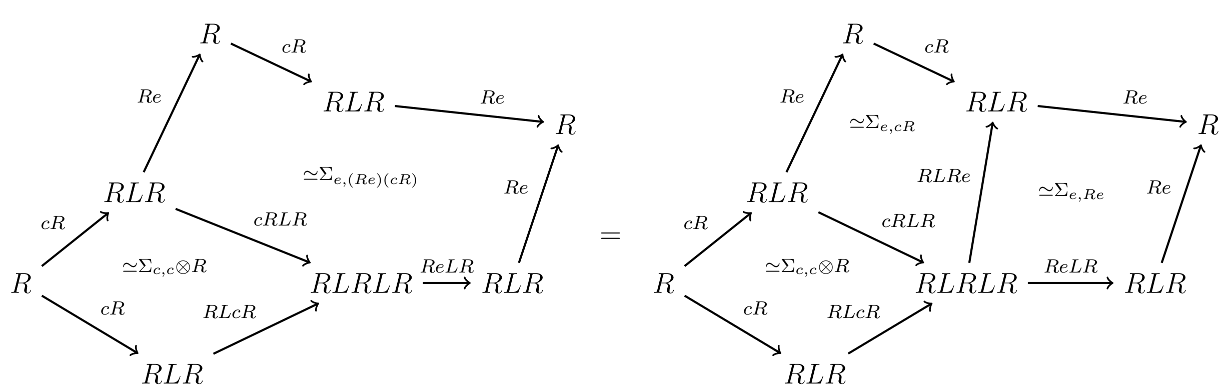


follows from the simple fact that in any ordinary monoidal category, if $A \to I$ is an isomorphism with the monoidal unit, the composite $A \simeq A \otimes I \to A \otimes A$ is equal to $A \simeq I \otimes A \to A \otimes A$. Here, we apply it to the the isomorphism β in the endomorphism monoidal category $\mathcal{M}(R, R)$.

To see that the two coherence regions are also equal, one would like to apply coherence for monoidal bicategories, but one cannot do so directly, as the relevant composites involve non-trivially the unit. Instead, we rewrite



and similarly



using the compatibility of the interchanger with composition. The coherence result of Nick Gurski, which we stated as Theorem 1.7, now applies to any of the three inside regions of the right hand sides, implying that the coherence cells inside must coincide.

We now move to the second part, namely that choosing one of the cusp isomorphisms uniquely determines the other subject to the Swallowtail equations. To see this, notice that the right hand side of the identity (C) is the image of α under the equivalence of categories of Lemma 3.13. Thus, a choice of β uniquely determines α. Conversely, the left hand side of the identity is the image of β under an equivalence of Lemma 3.13 together with pasting an invertible 2-cell. Thus, similarly a choice of α uniquely determines β. This ends the argument. ∎

In particular, Theorem 3.14 implies that any dualizable object can be made part of a coherent dual pair. The next theorem, which is the main result of this section, shows that this coherent dual pair is essentially unique.

3.16. Theorem. [Coherence for dualizable objects] *Let $\mathcal{M}$ be a monoidal bicategory. Then, the forgetful homomorphism*

$$\pi : \mathcal{C}\mathrm{ohDualPair}(\mathcal{M}) \to (\mathcal{M}^d)^{\cong}, \\ \langle L, R\rangle_d \mapsto L$$

between, respectively, the bicategory of coherent dual pairs in $\mathcal{M}$ and the 2-groupoid of dualizable objects in $\mathcal{M}$, is a surjective on objects equivalence of bicategories.

3.17. Remark. Observe that we know that the image of π must land in the maximal subgroupoid of $\mathcal{M}^d$, since $\mathcal{C}\mathrm{ohDualPair}(\mathcal{M})$ is a groupoid itself by Proposition 3.10.

The result should be understood as saying that the notion of a coherent dual pair is a *property-like structure* equivalent to the property of being dualizable. This means, in particular, that an object is dualizable if and only if it can be completed to a coherent dual pair, and any two coherent dual pairs living over the same dualizable object are equivalent. In fact, even more is true — a suitably defined space of coherent dual pairs over any given object is necessarily contractible.

We will now proceed with the proof, which will take the remainder of this section. Notice that by Theorem 3.14, we already know that the functor π is surjective on objects. Hence, we are left with showing that π is essentially surjective on morphisms and locally fully faithful, as these are exactly the conditions for a homomorphism of bicategories to be an equivalence.

We will first reduce to the case of $\mathcal{M}$ being a **Gray**-monoid and later prove essential surjectivity and local fully faithfulness directly as Lemma 3.19 and Lemma 3.20. This will end the proof.

3.18. Lemma. *Let P be a property of homomorphisms of bicategories that is stable under composition with equivalences. Then $\pi_{\mathcal{M}} : \mathcal{C}\mathrm{ohDualPair}(\mathcal{M}) \to (\mathcal{M}^d)^{\cong}$ satisfies P if and only if $\pi_{\mathbb{G}} : \mathcal{C}\mathrm{ohDualPair}(\mathbb{G}) \to (\mathbb{G}^d)^{\cong}$ does, where $\mathbb{G}$ is any **Gray**-monoid equivalent to $\mathcal{M}$.*

Proof. Arguing as in in the beginning of the proof of Theorem 3.14, let $\widetilde{\mathcal{M}}$ denote the computadic replacement and choose a strict homomorphism $\widetilde{\mathcal{M}} \to \mathbb{G}$ into a **Gray**-monoid which is also an equivalence of monoidal bicategories. In the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}\mathrm{ohDualPair}(\mathcal{M}) & \leftarrow & \mathcal{C}\mathrm{ohDualPair}(\widetilde{\mathcal{M}}) & \rightarrow & \mathcal{C}\mathrm{ohDualPair}(\mathbb{G}) \\ \downarrow \pi_{\mathcal{M}} & & \downarrow \pi_{\widetilde{\mathcal{M}}} & & \downarrow \pi_{\mathbb{G}} \\ (\mathcal{M}^d)^{\cong} & \longleftarrow & (\widetilde{\mathcal{M}}^d)^{\cong} & \longrightarrow & (\mathbb{G}^d)^{\cong} \end{array}$$

of bicategories and strict homomorphisms the horizontal maps are all equivalences, from which we immediately conclude that $\pi_{\mathbb{G}}$ satisfies P if and only if $\pi_{\mathcal{M}}$ does. ■

3.19. Lemma. *The homomorphism $\pi : \mathcal{C}\mathrm{ohDualPair}(\mathcal{M}) \to (\mathcal{M}^d)^{\cong}$ is essentially surjective on morphisms. In other words, any equivalence in $\mathcal{M}^d$ is isomorphic to one that is an image under π of a morphism in $\mathcal{C}\mathrm{ohDualPair}(\mathcal{M})$.*

Proof. By Lemma 3.18 we may assume that $\mathcal{M}$ is a **Gray**-monoid, so that we may use explicit description of the bicategory of dual pairs following Definition 3.8, as well as the corresponding notation. We will prove that in this case *every* equivalence in $\mathcal{M}$ may be lifted to one in the bicategory of coherent dual pairs.

Let $\langle L, R\rangle_d$, $\langle L', R'\rangle_d$ be coherent dual pairs and suppose we are given an equivalence $s : L \to L'$. We have to complete it to a 1-cell $(s,t)_d : \langle L, R\rangle_d \to \langle L', R'\rangle_d$ in $\mathcal{C}\mathrm{ohDualPair}(\mathcal{M})$ with components $(s,t)_d = (s, t, \gamma, \delta)$.

We first pass to the homotopy category of $\mathcal{M}$. Observe that by coherence for dual pairs in monoidal categories, which we stated as Theorem 2.5, there is a morphism t, unique up to an invertible 2-cell, that will complete s to a map of dual pairs in $h(\mathcal{M})$. In fact, we see from the proof of the aforementioned result that it is given by the dual of any pseudoinverse of s. We fix any such morphism to be the component t. We are now left with choosing the needed constraint isomorphisms.

The fact that s, t commute with (co)evaluation maps in the homotopy category implies that we can choose the constraint isomorphisms γ, δ *in some way*. However, we need to be careful and choose them in a way that makes them natural with respect to the cusp isomorphisms α, β.

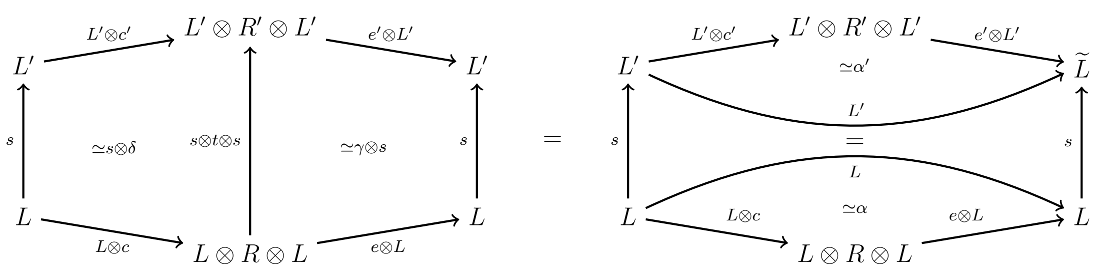
The first step in showing that this can be done is to prove that for any choice of γ, δ, naturality with respect to α implies naturality with respect to β. This can be done in several ways, one of which is a straightforward manipulation with diagrams, which is rather lengthy. This is, luckily, unnecessary. Instead, we will leverage the work put into Theorem 3.14, which shows that for coherent dual pairs, any cusp isomorphism uniquely determines the other.

Suppose we have chosen some γ, δ that are natural with respect to the cusp isomorphism α. The data of (s, t, γ, δ) then already defines an equivalence $(L, R, e, c) \to (L', R', e', c')$ in the bicategory $G'_d(\mathcal{M})$ of 1-truncated G_d-shapes, where G_d is as in Definition 3.7. Explicitly, the objects of this shape bicategory consist of a pair of objects together with candidate (co)evaluation maps, but no choice of cusp isomorphisms.

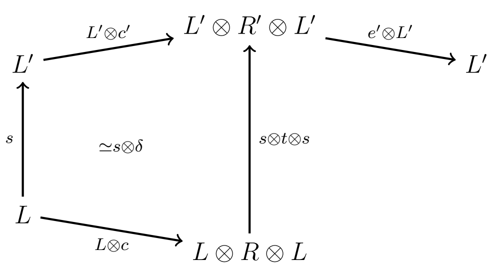
Using Lemma A.20, we can transport the structure of an (untruncated) G_d-shape along the chosen equivalence from $\langle L, R\rangle_d$ to (L', R', e', c'). The new G_d-shape, which we will denote by $\langle L', R'\rangle_d^\circ$, will be a coherent dual pair by Lemma A.23, as it is equivalent to one. Note that the chosen tuple (s, t, γ, δ) defined an equivalence $\langle L, R\rangle_d \simeq \langle L', R'\rangle_d$ of dual pairs if and only if we have $\langle L', R'\rangle_d^\circ = \langle L, R\rangle_d$, which amounts to agreeing on cusp isomorphisms, as the two dual pairs agree on objects and (co)evaluation maps by construction.

As (s, t, γ, δ) was assumed natural with respect to it, the dual pairs $\langle L', R'\rangle_d^\circ$ and $\langle L, R\rangle_d$ share the same cusp isomorphism α. However, they are both coherent, Theorem 3.14, implies that they are equal. This ends the first step, showing that we only need to construct a tuple (s, t, γ, δ) which is natural with respect to one of the cusp isomorphisms.

To finish the proof of the lemma, we are left with showing that one can choose constraint isomorphisms γ, δ such that they satisfy the equation of α-naturality, which can be rewritten as

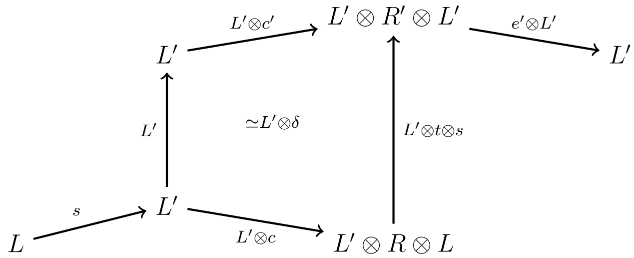


By adding the inverse of $\gamma\otimes s$ to the bottom right corner we now see the equation becomes



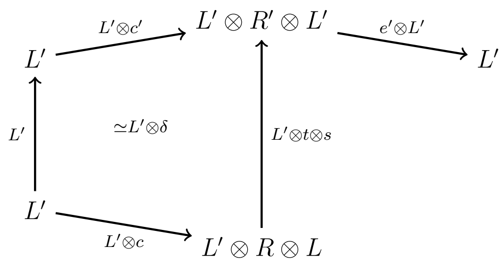
$=$ { *isomorphism only depending on* γ } ,

where we do not redraw the right hand side, as for our purpose it is not needed to know its exact value, as long as it is an isomorphism completely definable in terms of only the two dual pairs in question and γ. We now apply interchangers to the equation above, to the maps in the bottom right to move the left s from the cell $s\otimes t\otimes s = (L'\otimes t\otimes s)(s\otimes R\otimes L)$ past the $L\otimes c$ before it, so that we obtain



$= \{$ *isomorphism only depending on* γ $\}$.

As s is an equivalence, this equation is equivalent to one of the form



$= \{$ *isomorphism only depending on* γ $\}$

and Lemma 3.13 implies that it has a unique solution, ending the proof. ∎

We will now finish the proof of Theorem 3.16 by showing that π is locally bijective on 2-cells. This holds for arbitrary dual pairs, not necessarily coherent, hence we state the result in this slightly greater generality.

3.20. Lemma. *The homomorphism* $\pi : \mathcal{D}\mathrm{ualPair}(\mathcal{M}) \to (\mathcal{M}^d)^{\cong}$ *is locally bijective on 2-cells.*

Proof. As in the proof of Lemma 3.18, we will assume $\mathcal{M}$ is a **Gray**-monoid and use the corresponding explicit description of the bicategory of dual pairs.

Suppose we have morphisms $(s_i, t_i)_d : \langle L, R \rangle_d \to \langle L', R' \rangle_d$ of dual pairs for $t = 1, 2$, with components $(s_i, t_i)_d = (s_i, t_i, \gamma_i, \delta_i)$. For simplicity, we may assume that $\langle L, R \rangle_d = \langle L', R' \rangle_d$. Indeed, the two relevant bicategories are both groupoids, so that π is bijective on all 2-cells if and only if it is locally bijective on 2-cells between endomorphisms.

Let $\Gamma_L : s_1 \to s_2$ be an isomorphism between the images of these two endomorphisms under π. We have to show that Γ_L lifts uniquely to an invertible 2-cell in the bicategory of dual pairs, which amounts to finding an invertible 2-cell $\Gamma_R : t_1 \to t_2$ such that together they satisfy γ and δ-naturality equations.

As in the proof of Lemma 3.18, we will first establish that one of the naturality equations implies the other by using transport of structure.

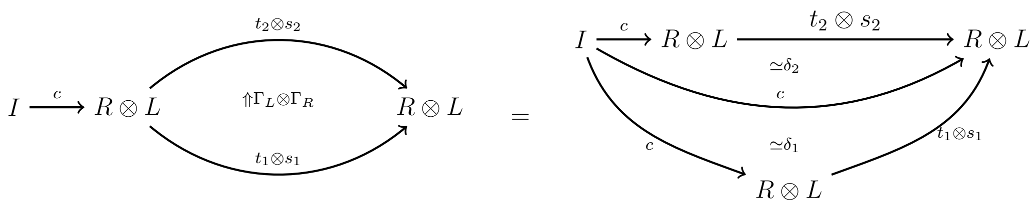
Suppose we have chosen some isomorphisms Γ_L, Γ_R that are natural with respect to the constraint isomorphism δ. This data alone defines an invertible 2-cell $\Gamma : (s_1, t_1) \to (s_2, t_2)$ between morphisms of 0-truncated G_d-shapes. Here, explicitly, a 0-truncated G_d-shape is just a pair of objects with no choice of (co)evaluation maps or cusp isomorphisms, and a morphism of such is just a pair of morphisms in $\mathcal{M}$.

By Lemma A.25, we can transport the structure of a map of 1-truncated G_d-shapes from $(s_1, t_1)_d$ to (s_2, t_2) along the above isomorphism Γ. This way, we obtain a new equivalence $(s_2, t_2)^\circ_d$ which, being isomorphic to a map of dual pairs, is a morphism of dual pairs too; in other words, it is compatible with the cusp isomorphisms.

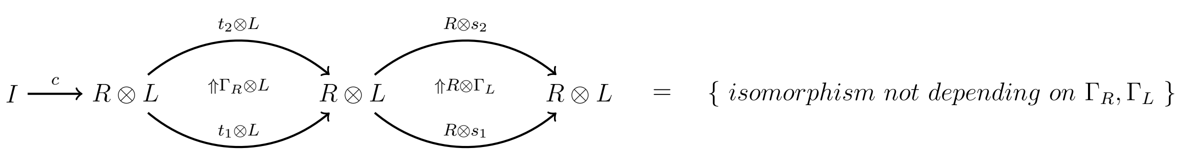
Observe that both $(s_2, t_2)_d$ and $(s_2, t_2)^\circ_d$ are equivalences of dual pairs which agree on their morphism components by construction and moreover agree on the constraint isomorphism δ, as we assumed that Γ was natural with respect to it. However, from the proof of Lemma 3.19 above we know that for any given equivalence of dual pairs, the constraint isomorphism δ determines the constraint isomorphism γ.

This implies that we have $(s_2, t_2)_d = (s_2, t_2)^\circ_d$, so that the isomorphisms Γ_R, Γ_L were natural with respect to both γ, δ to begin with. This ends the first step, showing that for any choice of Γ, naturality with respect to δ implies naturality with respect to γ.

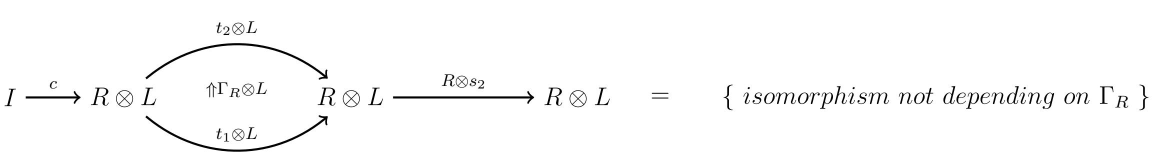
We will now prove that δ-naturality makes Γ_L uniquely determine Γ_R, this will end the proof of the lemma. We can redraw the relevant equation as



or perhaps more simply as



since, again, we do not need to know precisely what is the right hand side. Note that to obtain an equation of this form from the one just above we needed to apply interchangers to the top and the bottom, as according to our convention $(t \otimes s) = (t \otimes L)(r \otimes R)$ and here we want it the other way around. We can now append $R \otimes \Gamma_L^{-1}$ to the left hand side to get



.

This equation clearly has a unique solution Γ_R. Indeed, postcomposition with $R \otimes s_2$ is an equivalence on Hom-categories, since the morphism itself is an equivalence, as is tensoring via $- \otimes L$ followed by precomposition with c by Lemma 3.13. This ends the argument. ∎

4. Fully dualizable objects in symmetric monoidal bicategories

In this section we focus on the theory of fully dualizable objects in symmetric monoidal bicategories. Our main goal is a coherence result analogous to the one we proved for dualizable objects, namely description of a property-like structure equivalent to full dualizability.

4.1. Serre autoequivalence and fully dual pairs. We will first define full dualizability and show that every fully dualizable object in a symmetric monoidal bicategory admits a canonical up to isomorphism autoequivalence, *the Serre autoequivalence* of Lurie [Lur09]. We then use it to identify minimal conditions for an object to be fully dualizable, which we then package into a notion of a *fully dual pair*.

The property of full dualizability is one of the possible ways to strengthen dualizability by putting some conditions on the (co)evaluation maps. In an ordinary monoidal category, essentially the only thing one can do is to require them to be isomorphisms, obtaining the notion of a monoidal inverse.

However, in the bicategorical case there is a property of morphisms which is weaker then being an equivalence, but has some of the same consequences: the property of having an adjoint. In the strongest version we might require the (co)evaluation maps to *have all adjoints*, that is, to have both left and right adjoints and also that these left/right adjoints should have both adjoints on their own, and so on. This is what we will call full dualizability.

4.2. Definition. *If $\mathcal{M}$ is a bicategory, let $\mathcal{M}^{adj}$ be the maximal subbicategory with the property that all its* 1*-cells admit both adjoints.*

The bicategory $\mathcal{M}^{adj}$ can be obtained by an iterated process of removing morphisms that do not have a left or a right adjoint. Namely, if $\mathcal{M}$ is a bicategory, let $\mathcal{M}^{(1)} \subseteq \mathcal{M}$ be the subbicategory with the same objects, only those morphisms that admit both a left and right adjoint in $\mathcal{M}$, and all 2-cells between them. One sees easily that $\mathcal{M}^{adj} = \bigcap_{i \geq 1} \mathcal{M}^{(i)}$, where $\mathcal{M}^{(i+1)} = (\mathcal{M}^{(i)})^{(1)}$.

4.3. Proposition. *If $\mathcal{M}$ is symmetric monoidal, then the symmetric monoidal structure restricts to give one on $\mathcal{M}^{adj}$.*

Proof. It is enough to show that the tensor product preserves the property of having adjoints, as then one can naively restrict the monoidal structure to $\mathcal{M}^{adj}$. However, if we have $f_1 \dashv g_1$, $f_2 \dashv g_2$, then $f_1 \otimes f_2 \dashv g_1 \otimes g_2$. ■

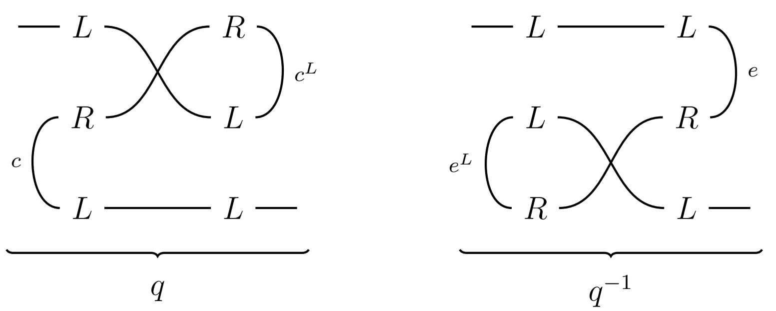
4.4. Definition. *We say an object $L \in \mathcal{M}$ in a symmetric monoidal bicategory is* fully dualizable *if it is dualizable as an object of $\mathcal{M}^{adj}$.*

In other words, an object $L \in \mathcal{M}$ is fully dualizable if and only if it can be completed to a dual pair $\langle L, R \rangle_d$ in the sense of Definition 3.3 with (co)evaluation maps admitting all adjoints.

4.5. Remark. A vigilant reader will point out that full dualizability as we defined it makes sense even in a non-symmetric monoidal bicategory. We will not work in this generality and restrict to the symmetric case; one reason to do is that to define the Serre autoequivalence, see Definition 4.6 below, one needs at least some weak form of symmetry.

We start by deriving some basic properties of fully dualizable objects. Note that, a priori, to show that an object is fully dualizable one needs an infinite amount of data, as one has to witness an existence of infinitely many adjoints. Our goal will be to describe a *finite* set of data whose existence witnesses full dualizability.

4.6. Definition. *Let $\langle L, R\rangle_d$ be a dual pair in a symmetric monoidal bicategory $\mathcal{M}$ and suppose we have adjunctions $e^L \dashv e, c^L \dashv c$. We define the* Serre autoequivalence *q of L and the* pseudoinverse to Serre autoequivalence *q^{-1} to be the 1-cells*



Note that, a priori, the Serre autoequivalence depends on the choice of a dual pair and the adjoints to (co)evaluation morphisms. We will see below, in Corollary 4.11, that at least up to invertible 2-cells, the Serre autoequivalence is intrinsic to the object L.

4.7. Remark. The terminology of *Serre equivalence* is due to Jacob Lurie, see [Lur09, 4.2.4]. It is motivated by a particular case of a suitably defined $(\infty, 2)$-category of cocomplete dg-categories, where the endofunctor described by Serre autoequivalence has traditionally been called the *Serre functor*.

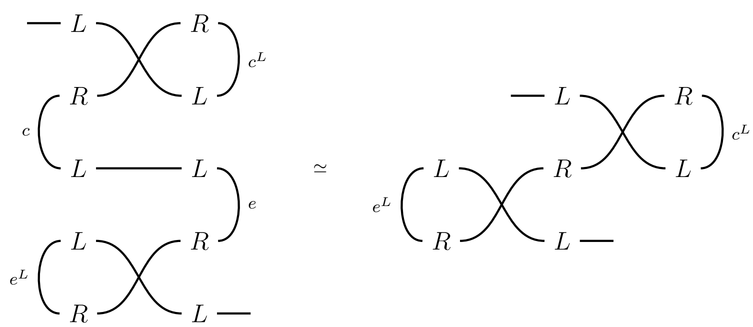
4.8. Remark. There is a more general, geometric definition of the Serre autoequivalence which rests on the Cobordism Hypothesis [Lur09, 2.4.14]. Namely, if $\mathcal{M}$ is a symmetric monoidal (∞, n)-category, its space of fully dualizable objects is predicted to be equivalent to the space of n-dimensional framed topological field theories with values in $\mathcal{M}$, and so should acquire an action of the Lie group $O(n)$, which acts via change of framing.

If $n \geq 1$, then the group $\pi_1(O(n))$ is cyclic, generated by a "full twist". Under the action on the space of fully dualizable objects, this generator corresponds to a natural endotransformation of the identity. The component of this transformation at any given fully dualizable object is precisely its Serre autoequivalence.

We will proceed by establishing some basic properties of the Serre autoequivalence, using the definition in terms of string diagrams given above. Note that most of these properties would follow trivially if we defined the Serre autoequivalence as the component of some invertible endotransformation of the identity homomorphism, but doing so would require us to assume the Cobordism Hypothesis. We will instead proceed directly using Definition 4.6.

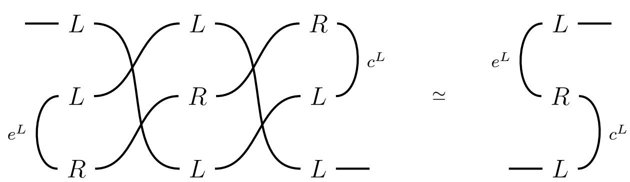
4.9. PROPOSITION. *The Serre autoequivalence q and its pseudoinverse q^{-1} are, in fact, pseudoinverse to each other. In other words, $[q] \circ [q^{-1}] = [q^{-1}] \circ [q] = [id_L]$ in the homotopy category $h(\mathcal{M})$.*

PROOF. Since the homotopy category is an ordinary symmetric monoidal category, we can perform a calculation with classical string diagrams. The class of $[q^{-1}] \circ [q]$ can be represented by



,

where we obtain the right hand side from the left hand side by using the triangle equations, which are witnessed by cusp isomorphisms. Since in a symmetric monoidal category the composition of braidings depends only on the underlying permutation, we can rewrite the right hand side as

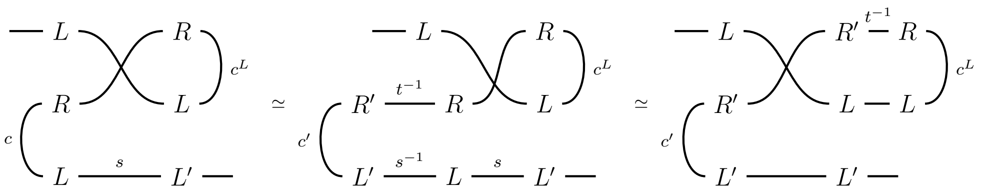


.

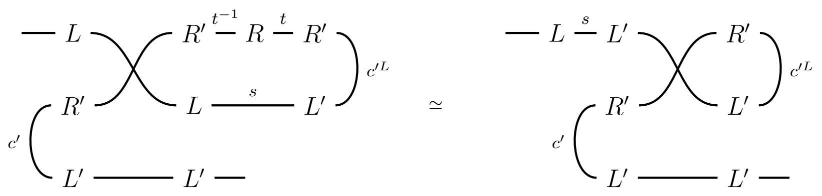
However, the right hand side is a left adjoint to the corresponding composite of e, c and so it is the identity by the triangle equations, which ends the proof that $[q^{-1}] \circ [q] = [id_L]$. The case of $[q] \circ [q^{-1}] = [id_L]$ is completely analogous. ∎

4.10. PROPOSITION. *Suppose that $\langle L, R\rangle_d$, $\langle L', R'\rangle_d$ are dual pairs and $s : L \to L'$ is an equivalence. Then, $[s] \circ [q] = [q'] \circ [s]$ in the homotopy category.*

PROOF. Again, we can argue using classical string diagrams. Choose a map $t : R \to R'$ that completes it into a map of dual pairs in the homotopy category, which can be done by Theorem 2.5. One computes that $[s] \circ [q]$ can be represented by



The right hand side of the above is



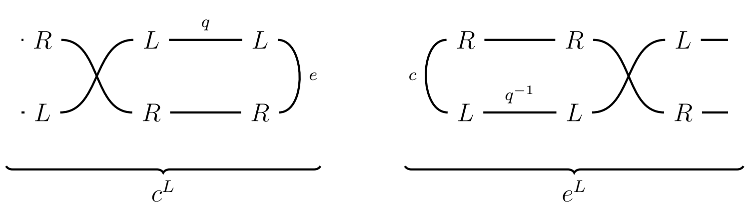
as one sees by observing that $c'^L \circ (t \otimes s)$ is isomorphic to c^L, as they are both left adjoint to c. This ends the proof, as the last composite represents $[q'] \circ [s]$.

To deduce the corollary, apply the proposition to the identity morphism of L with different choices of (co)evaluation maps and their left adjoints on both sides. ∎

4.11. Corollary. *Up to invertible 2-cells, the Serre autoequivalence of L does not depend on the choice of its dual, (co)evaluation maps or their left adjoints. Moreover, it descends to a natural automorphism of the identity on the homotopy category of dualizable objects in $\mathcal{M}$ and equivalences.*

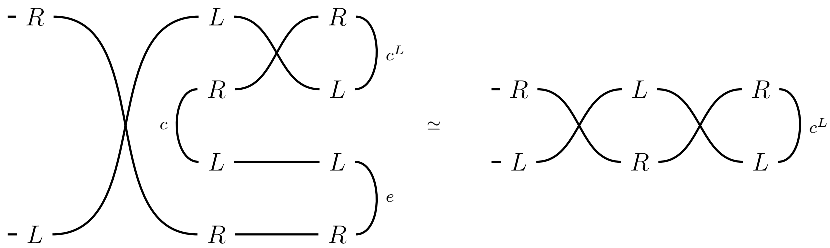
Proof. This is immediate from Proposition 4.10, with the first part following from the case $s = id_L$ for a choice of two different dual pairs completing L. ∎

4.12. Proposition. *Let $\langle L, R\rangle_d$ be a dual pair and suppose we have adjunctions $c^L \dashv c, e^L \dashv e$. Then, in the homotopy category $h(\mathcal{M})$, the classes of c^L and e^L can be represented by*



In particular, up to invertible 2-cells, the left adjoints to (co)evaluation maps are definable in terms of Serre autoequivalence and its pseudoinverse alone.

Proof. We only prove the theorem for c^L, as the proof for e^L is basically the same. In this case, the relevant composite is



,

where the isomorphism pictured is induced by the cusp. The right hand side is isomorphic to c^L, as that composite of braidings corresponds to the identity permutation. ∎

4.13. Theorem. [Minimal conditions for full dualizability] *Let $\langle L, R\rangle_d$ be a dual pair and suppose that the (co)evaluation morphisms admit left adjoints, so that L admits autoequivalences q and q^{-1}. Then, for all $n \in \mathbb{Z}$ we have adjunctions*

1. $(q^{-n-1} \otimes R) \circ \sigma \circ c \dashv e \circ (q^n \otimes R)$,
2. $(R \otimes q^{-n-1}) \circ c \dashv e \circ \sigma \circ (R \otimes q^n)$,
3. $e \circ \sigma \circ (R \otimes q^{n+1}) \dashv (R \otimes q^{-n}) \circ c$,
4. $e \circ (q^{n+1} \otimes R) \dashv (q^{-n} \otimes R) \circ \sigma \circ c$,

where $\sigma : R \otimes L \to L \otimes R$ is the symmetry. In particular, L is already fully dualizable.

Proof. The families (1) and (2) correspond to each other through the symmetry, as do (3) and (4), so it is enough to prove only one from each pair.

For $n = 0$, the adjunctions (1) and (3) are exactly Proposition 4.12. All the other adjunctions can be obtained by composing with $q \dashv q^{-1}$ or $q^{-1} \dashv q$ and using the fact that the composite of adjoints is adjoint to the composite. ∎

As discussed above, Theorem 4.13 is very convenient, as the property of full dualizability is a priori difficult to verify, as it assures the existence of an infinite strings of adjoints. However, it follows that it is enough to verify that only *two* of the needed adjoints exist, as all of the others can be defined in terms of the Serre autoequivalence.

This suggests the following compact, fully dualizable analogue of the notion of a dual pair in a monoidal bicategory introduced in Definition 3.3.

4.14. Definition. *A* fully dual pair *in a symmetric monoidal bicategory $\mathcal{M}$ is a tuple*

$$(L, R, e, c, q, q^{-1}, \alpha, \beta, \mu_e, \epsilon_e, \mu_c, \epsilon_c, \psi, \phi)$$

which consists of

1. a dual pair $\langle L, R\rangle_d$,

2. morphisms $q, q^{-1} : L \to L$ together with isomorphisms $\psi : qq^{-1} \simeq id_L$, $\phi : q^{-1}q \simeq id_L$,

3. 2-cells $\mu_e : id_I \to e \circ e^L$ and $\epsilon_e : e^L \circ e \to id_{L\otimes R}$, where $e^L = \sigma \circ (R \otimes q^{-1}) \circ c$, which satisfy triangle equations which make them into a unit and counit of an adjunction $e^L \dashv e$,

4. 2-cells $\mu_c : id_{R\otimes L} \to c \circ c^L$ and $\epsilon_c : c^L \circ c \to id_I$, where $c^L = e \circ (q \otimes R) \circ \sigma$, which satisfy triangle equations which make them into a unit and counit of an adjunction $c^L \dashv c$

Here, $\sigma : R \otimes L \to L \otimes R$ is the symmetry.

4.15. Notation. By abuse of language, we will sometimes refer to the whole fully dual pair just by referring to the underlying objects, in which case we will denote it by $\langle L, R\rangle_{fd}$ to distinguish it from the similar notation for dual pairs.

The intuition about this definition is that by Theorem 4.13, to witness full dualizability it is enough to give the left adjoints of (co)evaluation maps. Thus, one way to define a "fully dualizable pair" would be to include these left adjoints as part of the structure.

However, for technical reasons, we take a slightly different route. Instead of adjoints, we postulate the Serre autoequivalence as part of the data of a fully dual pair and only express the left adjoints in terms of it using the formula given by Proposition 4.12. Observe that this forces q, q^{-1} that are given as part of a structure of a fully dual pair to be, up to isomorphism, Serre autoequivalence and its pseudoinverse of L in the sense of Definition 4.6.

4.16. Proposition. *An object $L \in \mathcal{M}$ in a symmetric monoidal bicategory is fully dualizable if and only if it can be completed to a fully dual pair $\langle L, R\rangle_{fd}$.*

Proof. Clearly any object L that is a part of a fully dual pair is fully dualizable by Theorem 4.13, as its (co)evaluation maps have left adjoints.

Conversely, if L is a fully dualizable object, then in particular it can be completed to a dual pair $\langle L, R\rangle_d$ for which there exist left adjoints e^L, c^L to the chosen (co)evaluation maps. In terms of these maps we can define q, q^{-1} using Definition 4.6.

We are now only left with giving the missing 2-cells. We can find the invertible 2-cells $\psi : qq^{-1} \simeq id_L$ and $\phi : q^{-1}q \simeq id_L$ as the relevant maps are pseudoinverse by Proposition 4.9. Finally, by Proposition 4.12, the composites $(q^{-1} \otimes R) \circ \sigma \circ c$ and $e \circ \sigma \circ (R \otimes q)$ are left adjoint to (co)evaluation maps and this allows us to define the necessary (co)unit cells. ∎

As in the non-fully dual case given in as in Definition 3.7, to organize fully dual pairs into a bicategory, we will again use the language of computadic (symmetric) monoidal bicategories.

4.17. Definition. *Let G_{fd} be a generating datum for a symmetric monoidal bicategory consisting of*

1. *two generating objects L, R,*
2. *four generating morphisms $e : L \otimes R \to I$, $c : I \to R \otimes L$, $q : L \to L$ and $q^{-1} : L \to L$, and*
3. *twelve generating 2-cells $\alpha, \alpha^{-1}, \beta, \beta^{-1}, \phi, \phi^{-1}, \psi, \psi^{-1}, \epsilon_e, \mu_e, \epsilon_c$ and μ_c, whose sources and targets are exactly as in Definition 4.14.*

Let $\mathcal{R}_{fd}$ consist of relations that α, α^{-1}, β, β^{-1}, ϕ, ϕ^{-1} and ψ, ψ^{-1} are inverse to each other and that $\epsilon_e, \mu_e, \epsilon_c, \mu_c$ satisfy triangle equations.

4.18. Definition. *We define the* bicategory of fully dual pairs *in $\mathcal{M}$ as*

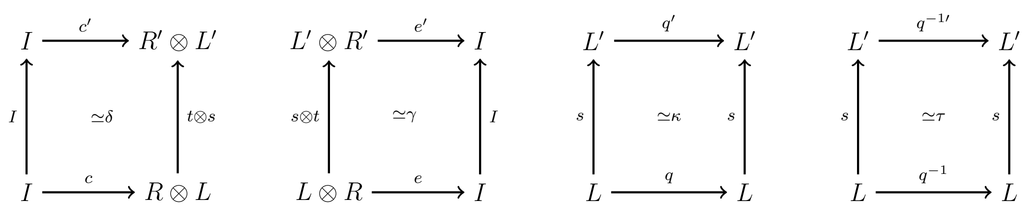
$$\mathfrak{F}\text{ullyDualPair}(\mathcal{M}) := P_{fd}(\mathcal{M}),$$

the bicategory of shapes in $\mathcal{M}$ of type $P_{fd} = (G_{fd}, R_{fd})$.

The above compact definition can be made explicit, and we do so below.

4.19. Notation. If $\mathcal{M}$ is a symmetric monoidal bicategory, then the objects of the bicategory $\mathfrak{F}\text{ullyDualPair}(\mathcal{M})$ are precisely the fully dual pairs in the sense of Definition 4.14.

A morphism $(s, t)_{fd} : \langle L, R\rangle_{fd} \to \langle L', R'\rangle_{fd}$ consists of data of 1-cells $s : L \to L'$, $t : R \to R'$ and constraint 2-cells $\gamma, \delta, \kappa, \tau$ of the form



These constraint isomorphisms are assumed to satisfy naturality equations with respect to cusp isomorphisms α, β, (co)units $\epsilon_e, \mu_e, \epsilon_c, \mu_c$ and witnessing isomorphisms ψ and ϕ. We will not draw these, as they are completely analogous to the ones appearing in Notation 3.9.

A 2-cell $\Gamma : (s_1, t_1)_{fd} \to (s_2, t_2)_{fd}$ consists of data 2-cells Γ_L, Γ_R in $\mathcal{M}$ of the form

These are required to satisfy naturality with respect to constraint isomorphisms γ, δ, κ and τ.

An important property of the bicategory of dual pairs, namely that it is a groupoid, implies the same property for bicategories of fully dual pairs, as the following shows.

4.20. PROPOSITION. *The bicategory* $\mathcal{F}\mathrm{ullyDualPair}(\mathcal{M})$ *is a 2-groupoid.*

PROOF. A map of fully dual pairs induces a map of underlying dual pairs and similarly for 2-cells. Moreover, comparing Notation 3.9 and Notation 4.19 we see that between dual pairs and fully dual pairs, morphisms and 2-cells differ only in constraint cells, which are necessarily invertible. Since non-constraint cells of a morphism or a 2-cell in $\mathcal{D}\mathrm{ualPair}(\mathcal{M})$ are invertible by Proposition 3.10, the same is true in the case of fully dual pairs and the result follows. ■

4.21. COHERENCE FOR FULLY DUALIZABLE OBJECTS. As in the dualizable case, a fully dualizable object in a symmetric monoidal bicategory can in general be completely to many non-equivalent fully dual pairs. We will remedy this by adding additional coherence equations at the level of 2-cells.

Our main result will be an analogue of Theorem 3.16, showing that any fully dualizable object can be completed to an essentially unique fully dual pair which is *coherent* in the following sense.

4.22. DEFINITION. *We say a fully dual pair* $\langle L, R\rangle_{fd}$ *is* coherent *if*

1. *it is coherent as a dual pair,*
2. *witnessing isomorphisms* ϕ, ψ *make* q, q^{-1} *into an adjoint equivalence, and*
3. *the cusp-counits equation* $(CC1) = (CC2)$ *holds, where the composites* $(CC1)$, $(CC2)$ *are given by pasting diagrams pictured below in Figure 4 and Figure 5.*

4.23. REMARK. In Figure 5 defining the cusp-counits composite $(CC2)$, there are some unmarked regions. Each of these has a unique way to be filled out with a composite of constraint 2-cells, by coherence results of Nick Gurski and Angélica Osorno which we stated as Theorem 1.7 and Theorem 1.8, and this is the unmarked cell.

4.24. REMARK. The cusp-counits equation is inspired by geometry of two-dimensional manifolds; more precisely, it is a form of the cusp flip relation in the presentation of the oriented bordism bicategory due to Schommer-Pries [SP11].

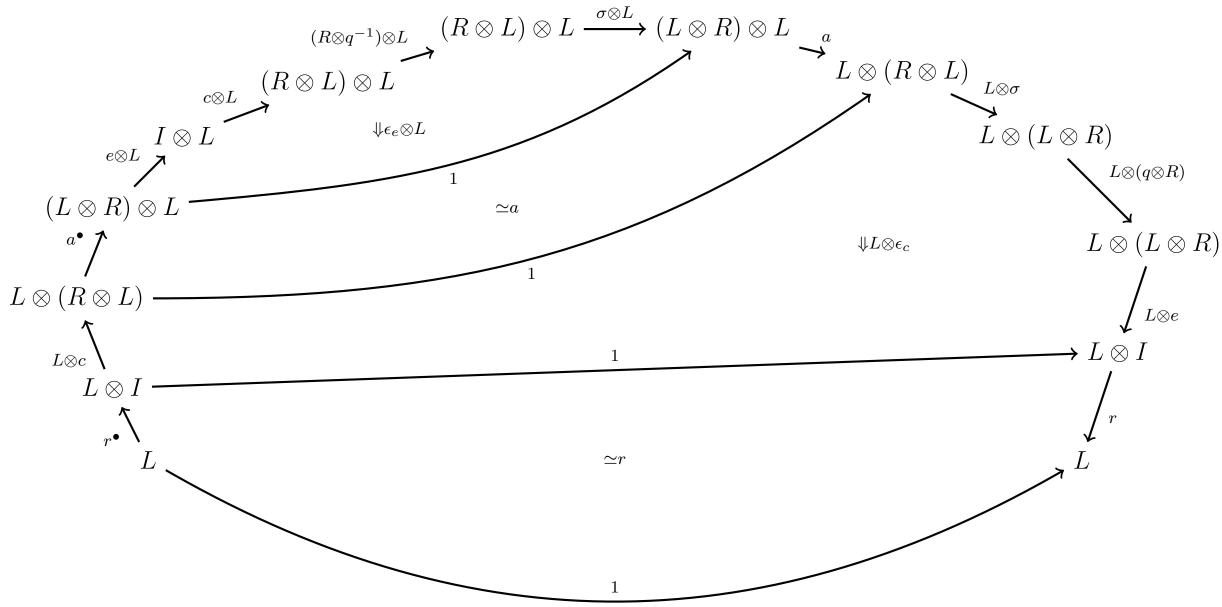


Figure 4: Cusp-counits composite (CC1)

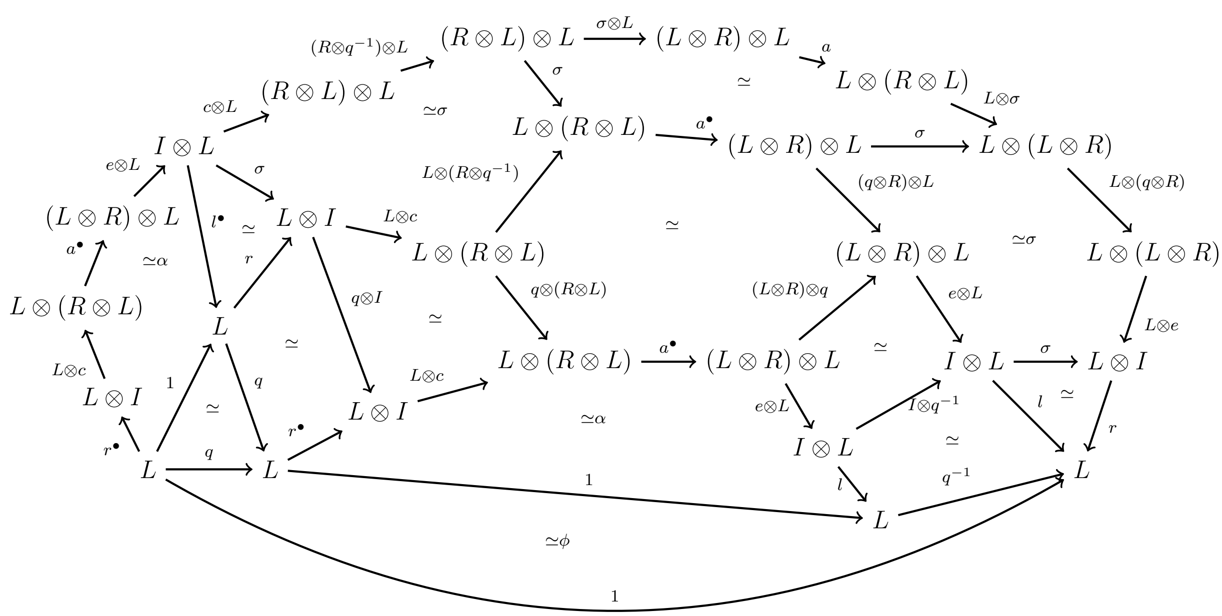


Figure 5: Cusp-counits composite (CC2)

4.25. Remark. We feel like we owe the reader some explanation as to why requiring an identity as complicated as the cusp-counits equation is a reasonable thing to do. For one thing, as one sees in the first diagram in the proof of Proposition 4.26, at least the composite ($CC1$) is not that complicated once one removes coherence cells from the picture, for example by working with a **Gray**-monoid, as it consists of a repeated application of two adjunction counits.

The composite ($CC2$) is more complex, but its only property which matter in practice is that it does not depend at the adjunction (co)units at all. Thus, requiring $(CC1) = (CC2)$ has the effect of asking for the two adjunctions to be compatible with each other by fixing the value of the composite.

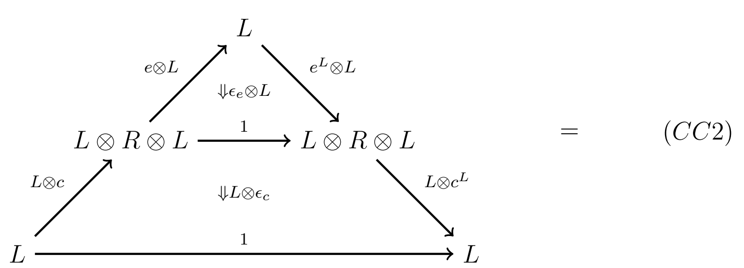
We now begin the proof of the coherence result, starting with strictification.

4.26. Proposition. *Any fully dualizable object can be completed to a coherent fully dual pair.*

Proof. In the same way as in the proof of Theorem 3.14, we can use strictification results of Schommer-Pries to reduce to the case where the symmetric monoidal bicategory $\mathcal{M}$ has an underlying **Gray**-monoid. Indeed, any symmetric monoidal bicategory can be related by a span of strict homomorphisms to a *quasi-strict* one, see [SP11][2.96]. The latter are partially strict symmetric monoidal bicategories that have, among other things, underlying **Gray**-monoids, as needed.

Let L be fully dualizable. By Proposition 4.16 we can complete it to a not-necessarily coherent fully dual pair and by strictification for dual pairs, that is, Theorem 3.14, we can make the chosen data satisfy Swallowtail identities by only a change of the cusp isomorphism β. Since q, q^{-1} are pseudoinverse, one can choose a unique ϕ such that together with the already chosen ψ they form an adjoint equivalence.

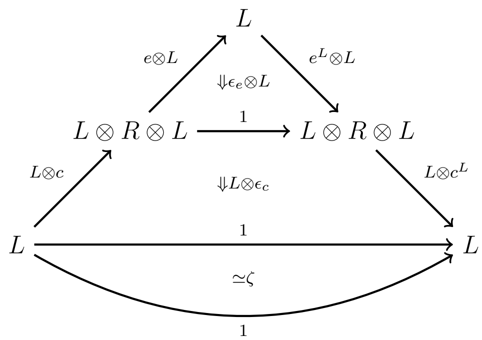
We are now only left with enforcing the cusps-counits equation. Under our assumption of $\mathcal{M}$ having an underlying **Gray**-monoid, we can draw it as



Here, the precise value of the right hand side is not needed; it is only important to observe that it is an isomorphism and that it does not depend on the adjunction (co)units ϵ, ν.

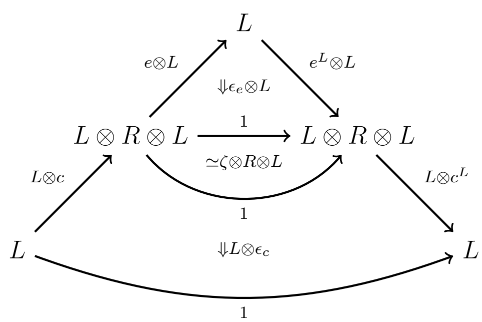
The left hand side is also an isomorphism. Indeed, it is the counit of the induced adjunction $(L \otimes c^L) \circ (e^L \otimes L) \dashv (e \otimes L) \circ (L \otimes c)$ and since the right adjoint, being isomorphic to the identity, is an equivalence, the adjunction itself must be an adjoint equivalence and so both (co)units are isomorphisms.

It follows that in the cusp-counits equation, the difference between the right and left hand is at most some automorphism $\zeta : id_L \simeq id_L$, in the sense that



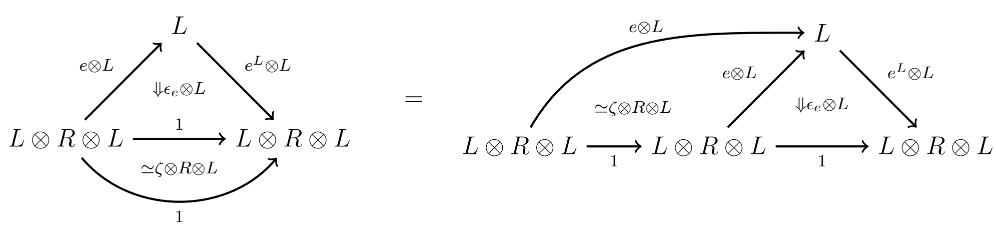
$= \qquad (CC2)$

already holds. The idea is to "absorb" this ζ into one of the counits, this will enforce the cusps-counits equation. To do so, we first commute ζ with ϵ_c to obtain that



$= \qquad (CC2)$.

We claim that the upper part of the diagram on the left hand side is already of the form $\epsilon \otimes L$ for some different counit of $e^L \dashv e$. This finishes the proof, as replacing the original ϵ_e, μ_e by these new (co)units we obtain a fully dual pair that is coherent. We have



,

as ζ is an automorphism of the identity. However, the right hand side is precisely obtained by applying $- \otimes L$ to the counit of the induced adjunction $e^L \dashv e$ obtained by transferring the original one along the isomorphism $e \circ (\zeta \otimes R) : e \simeq e$ and the identity of e^L. ∎

We will devote the rest of this section to the proof of the coherence theorem for fully dualizable objects, which identifies the notion of a coherent fully dual pair as a property-like structure equivalent to full dualizability.

4.27. THEOREM. *Let $\mathcal{M}$ be a symmetric monoidal bicategory. The forgetful homomorphism*

$$\pi : \mathcal{C}\mathrm{ohFullyDualPair}(\mathcal{M}) \to (\mathcal{M}^{fd})^{\cong},$$
$$\langle L, R\rangle_{fd} \mapsto L$$

between, respectively, the bicategory of coherent fully dual pairs and the groupoid of fully dualizable objects, is a surjective on objects equivalence.

4.28. REMARK. The statement of Theorem 4.27 has a strong connection with the Cobordism Hypothesis of Baez-Dolan [BD95], as we now explain.

The bicategory of coherent fully dual pairs can be identified with the bicategory of strict homomorphisms $\mathbb{F}(P_{cfd}) \to \mathcal{M}$ from the computadic symmetric monoidal bicategory generated by the presentation of Definition 4.18 together with the coherence relations. In turn, the cofibrancy results of Schommer-Pries allows us to identify it with the bicategory of *all* symmetric monoidal homomorphisms [SP11].

Then, Theorem 4.27 implies that a homomorphisms out of the explicitly constructed symmetric monoidal bicategory $\mathbb{F}(P_{cfd})$ is essentially uniquely determined by the image of the distinguished object L, which can take any fully dualizable value. This is exactly the universal property which the Cobordism Hypothesis predicts is satisfied by a suitably defined bicategory of framed cobordisms.

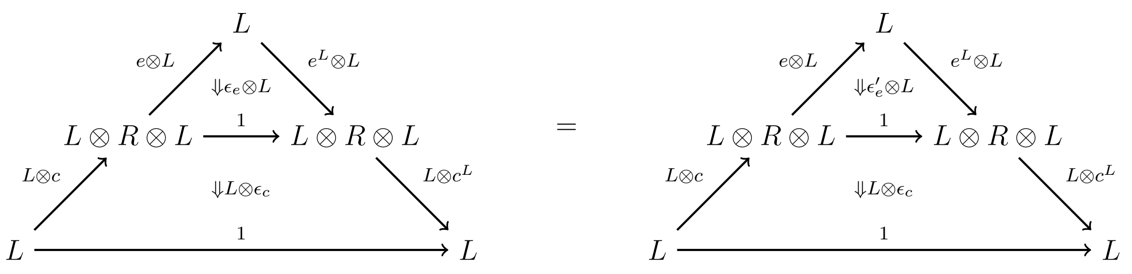
This precise statement of the Cobordism Hypothesis is due to Lurie, who also sketches a proof of the general case in the setting of (∞, n)-categories [Lur09]. The results of this paper give a more direct approach to the two-dimensional case by reducing it to the question of comparing the framed bordism bicategory with $\mathbb{F}(P_{cfd})$. This is an essentially computational task which can be attacked using Morse-theoretic techniques of Schommer-Pries, see [Pst14][Ch.8].

The rest of this section will be devoted to the proof of Theorem 4.27. We will fix a symmetric monoidal bicategory $\mathcal{M}$; arguing as in the proofs of Proposition 4.26 and Lemma 3.18 we may assume that the underlying monoidal bicategory is a **Gray**-monoid.

We have already established that the forgetful functor π is surjective on objects. Thus, we are left with essential surjectivity on morphisms, which we prove as Lemma 4.30, and local bijectivity on 2-cells, which is Lemma 4.31.

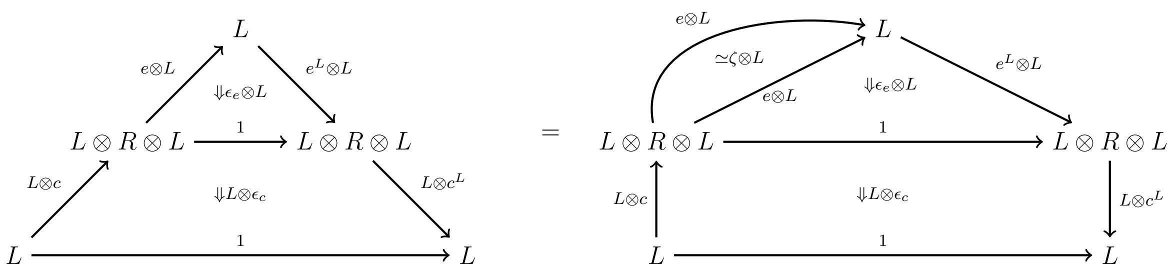
4.29. LEMMA. *Suppose $\langle L, R\rangle_{fd}$ and $\langle L', R'\rangle_{fd}$ are fully dual pairs which share the same objects, same morphisms and same 2-cells except possibly for the (co)units of the adjunction $e^L \dashv e$. Then, if they both satisfy the cusps-counits equation, they are equal.*

PROOF. As both fully dual pairs in question satisfy cusp-counits equation, we also have



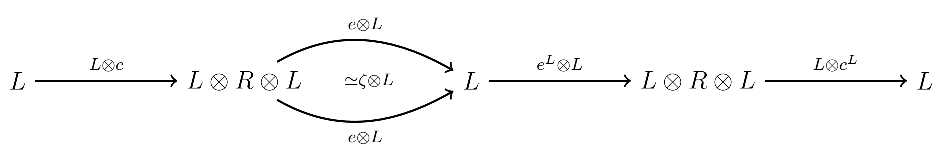
as these are exactly the $(CC1)$ composites of the dual pairs in question, and so must be equal to $(CC2)$, which doesn't depend at all on the counits. As observed before, both sides are invertible.

Since both ϵ_e and ϵ'_e are counits of an adjunction $e^L \dashv e$, they can at most differ by an automorphism of any of the components. Let $\zeta : e \simeq e$ be the difference between them, so that we have

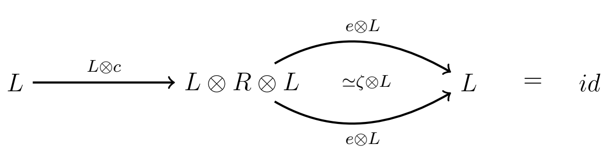


We want to show that this equation implies that $\zeta = id_e$, as then we will know that $\epsilon_e = \epsilon'_e$.

Using the fact that the big triangle composite is an isomorphism, the above equation is equivalent to



being the identity, as one sees from pasting the inverse to the triangle from below. The composite of the last two morphisms is left adjoint to one of the cusp composites, so it is an equivalence, and thus the equation above is equivalent to



By Lemma 3.13, this equation implies that ζ is the identity, ending the argument. ■

4.30. Lemma. *The homomorphism* $\pi : \mathcal{C}\mathrm{ohFullyDualPair}(\mathcal{M}) \to (\mathcal{M}^{fd})^{\cong}$ *is essentially surjective on morphisms.*

Proof. We will in fact show that given coherent $\langle L, R\rangle_{fd}$ any $\langle L', R'\rangle_{fd}$, *any* equivalence $s : L \to L'$ can be lifted to one of fully dual pairs.

To do so, we have to choose the component $t : R \to R'$ and constraint isomorphisms $\gamma, \delta, \kappa, \tau$ as spelled out in Notation 4.19; that is, in a way that where the latter will be natural with respect to all the 2-cells that are part of the structure of a fully dual pair.

Since both pairs are coherent, among these 2-cells we have three (co)unit pairs, namely ϵ_e, μ_e witnessing the adjunction $e^L \dashv e$, ϵ_c, μ_c witnessing $c^L \dashv c$ and ψ, ϕ witnessing $q \dashv q^{-1}$. We claim that it is enough to choose the constraints so that they are natural with respect to counits, and that this will already imply also the naturality with respect to the units.

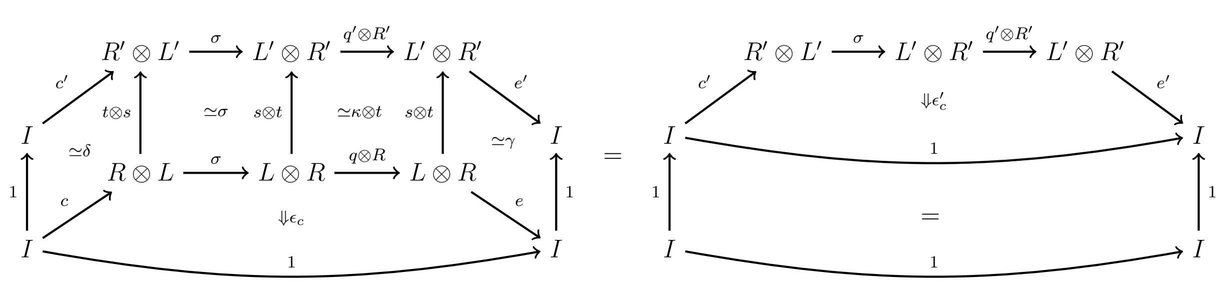
To see this, suppose we have chosen constraint isomorphisms such that they are natural with respect to the counits. This already determines an equivalence $\langle L, R\rangle_{fd} \to \langle L'R'\rangle_{fd}$ of G'_{fd}-shapes, where G_{fd} is the 1-truncation of G_{fd}.

By Lemma A.20 we can use this equivalence to transport the structure of a (non-truncated) G_{fd}-shape to obtain a new fully dual pair $\langle L', R'\rangle^{\circ}$ which differs from $\langle L', R'\rangle$ only in 2-cells. Since the equivalence we constructed was assumed to be natural with respect to the counits, these two fully dual pairs must also have the same counits. Since in any adjunction, (co)units determine each other, the original and transported structure must also agree on units, proving naturality also in this case.

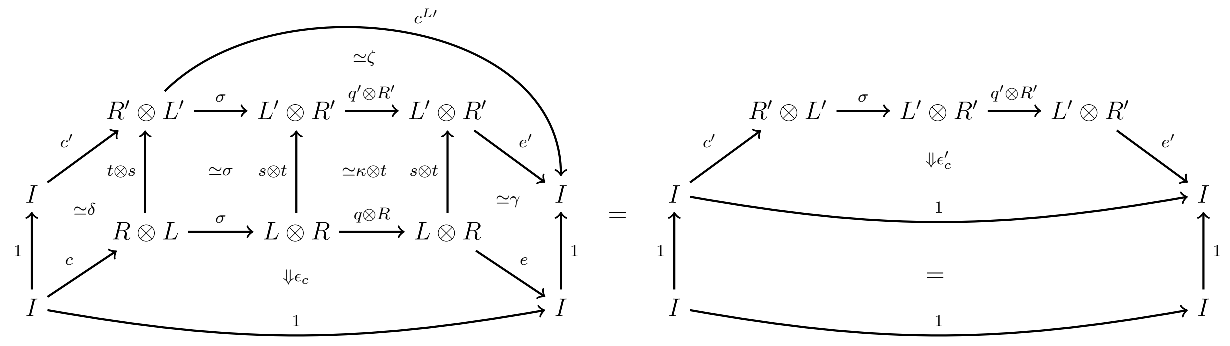
By the same argument and Lemma 4.29, we similarly see that if we choose constraint isomorphisms that are natural with respect to all structural 2-cells except maybe ϵ_e, μ_e, then they will already be natural with respect to all of them.

We now begin the argument proper. By Lemma 3.19, s can be completed to an equivalence of dual pairs, which allows us to choose t and constraint γ, δ which are natural with respect to the cusp isomorphisms α, β. By Proposition 4.10, the Serre autoequivalence is natural up to invertible 2-cells, so that we can choose *some* invertible constraint κ.

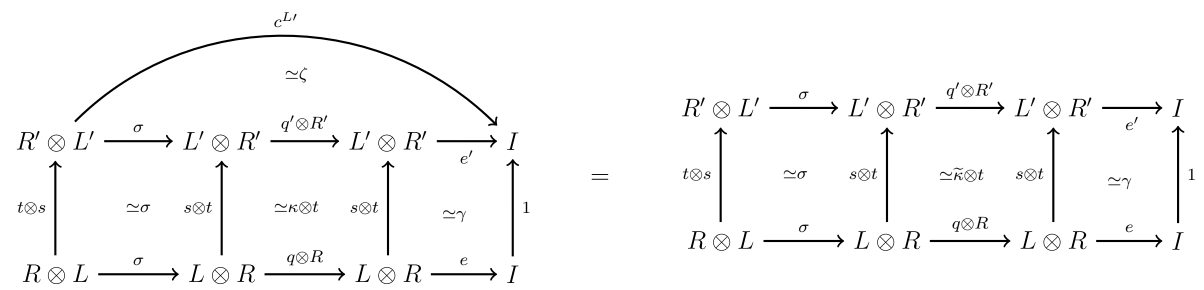
We would like to choose a κ which satisfies, in particular, naturality with respect to ϵ_c; in other words, such that



Observe that the left hand side is the counit ϵ_c of the pair $\langle L, R\rangle_{fd}$ transported to the pair $\langle L', R'\rangle_{fd}$, so it differs from ϵ'_c at most by an automorphism of $c^{L\prime}$, as they are both counits of the same adjunction. In other words, we can find some invertible $\zeta : c^{L'} \simeq c^{L'}$ such that

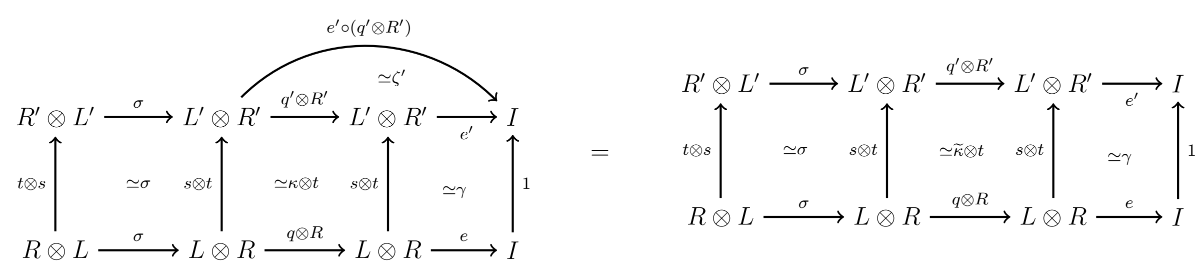


already holds. The idea, very similar to the one we used in the proof of Lemma 4.29, is to "absorb" this ζ into the isomorphism κ. This new, "corrected" $\widetilde{\kappa}$, together with the γ, δ we already have, will then satisfy naturality with respect to ϵ_c. Thus, our goal is to show that there exists $\widetilde{\kappa}$ such that

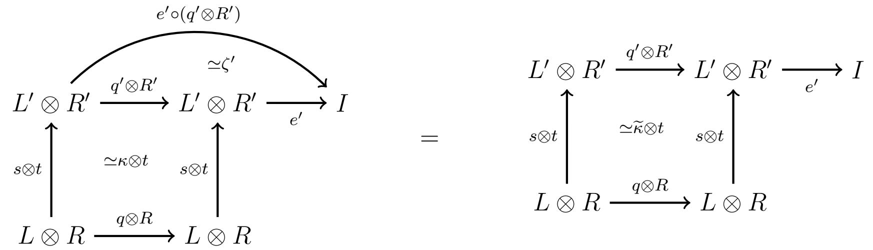


In fact, we will see that such $\widetilde{\kappa}$ must be unique, this will be important later on, in the proof of Lemma 4.31.

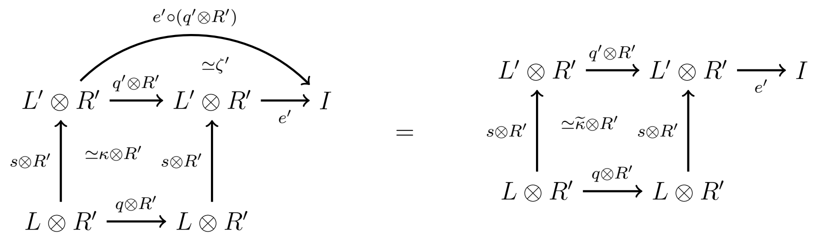
As the symmetry σ is an equivalence, there exists a unique automorphism ζ' of $e' \circ (q' \otimes R')$ which has the property that it reduces to ζ under whiskering by σ. Thus, an equation equivalent to the one given above is



As the two outer squares on both sides are invertible, we can omit them to obtain an equivalent equation. One can then remove σ is well, as it is an equivalence. In the end, we are left with showing that the equation



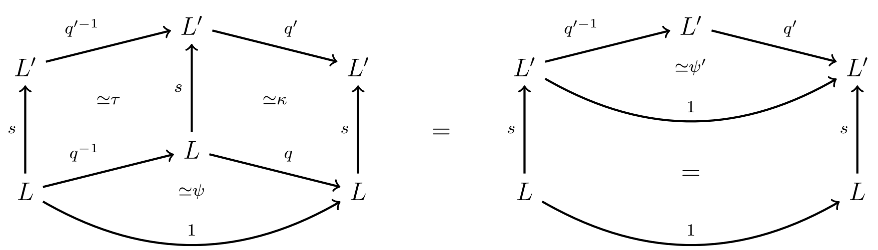
has a unique solution $\widetilde{\kappa}$. By adding the necessary interchangers to the left bottom square we can move the $t\otimes(-)$ component to the front and whisker t out, obtaining an equivalent equation. We can then leave out t altogether, as it is an equivalence, so that the are left with



This has a unique invertible solution $\widetilde{\kappa}$ by Lemma 3.13, as the left hand side is an isomorphism not depending on $\widetilde{\kappa}$.

Replacing the first choice of κ by $\widetilde{\kappa}$, we see that we can choose constraint isomorphisms γ, δ, κ which satisfy naturality with respect to cusp isomorphisms and ϵ_c. By our reasoning above using transport of structure, this also implies that they are natural with respect to μ_c, and thus also ϵ_e, μ_e.

We are then left with choosing τ, which has to satisfy the equation of ψ-naturality



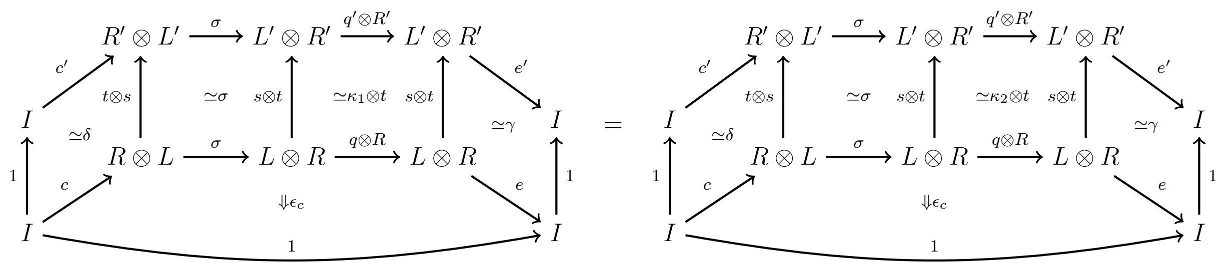
This clearly has a unique invertible solution, as everything in sight is either invertible or an equivalence. As observed above, this τ will also then satisfy ϕ-naturality, as ψ, ϕ form a (co)unit pair. This ends the proof. ∎

4.31. Lemma. *The homomorphism* $\pi : \mathcal{C}\mathrm{ohFullyDualPair}(\mathcal{M}) \to (\mathcal{M}^{fd})^{\cong}$ *is locally bijective on* 2*-cells.*

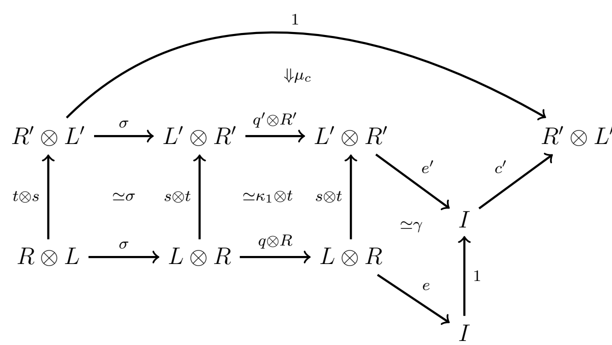
Proof. Consulting Notation 4.19, we see that a 2-cell between morphisms of fully dual pairs has the same components as a 2-cell between morphisms of dual pairs, since it consists only of components indexed on objects, the only difference is that it is bound by more equations. This will allow us to reduce most of the argument to the case of dual pairs Lemma 3.20.

Suppose that $(s_1, t_1)_{fd}, (s_2, t_2)_{fd} : \langle L, R\rangle_{fd} \to \langle L', R'\rangle_{fd}$ are parallel equivalences of fully dual pairs which agree on their morphism components as well as constraint isomorphisms γ, δ. We claim that they must be in fact equal.

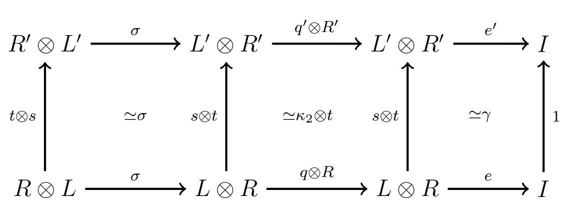
To see this, observe that since both satisfy naturality with respect to ϵ_c, we must have



as both sides are equal to ϵ'_c. If we paste



to the right hand side along their common boundary $c' = c' \otimes 1$, then after an application of the naturality of $(s_2, t_2)_{fd}$ with respect to μ_c and zig-zag equations for μ_c, ϵ_c, we will be left with



Similarly, doing the same pasting to the right hand side we will obtain

$$\begin{CD}
R' \otimes L' @>{\sigma}>> L' \otimes R' @>{q' \otimes R'}>> L' \otimes R' @>{e'}>> I \\
@A{t \otimes s}AA \simeq\sigma \quad @A{s \otimes t}AA \simeq\kappa_1 \otimes t \quad @A{s \otimes t}AA \simeq\gamma \quad @AA{1}A \\
R \otimes L @>{\sigma}>> L \otimes R @>{q \otimes R}>> L \otimes R @>{e}>> I
\end{CD} .$$

so these two diagrams must be equal.

However, we've already seen in the proof of the Lemma 4.30 that these composites uniquely determine κ, so we deduce that $\kappa_1 = \kappa_2$. Since constraints κ and τ uniquely determine each other via ψ or ϕ-naturality, it follows that also $\tau_1 = \tau_2$. This ends the proof that an equivalence of fully dual pairs is uniquely determined by its morphisms- and γ, δ-components.

Now suppose that $(s_1, t_1)_{fd}, (s_2, t_2)_{fd} : \langle L, R \rangle_{fd} \to \langle L', R' \rangle_{fd}$ are arbitrary equivalences and that we have an isomorphism $\Gamma_L : L \to L'$. We want to lift it to an isomorphism of morphisms of fully dual pairs.

As we've shown in Lemma 3.20, there is a unique lift to a 2-cell $\Gamma = (\Gamma_L, \Gamma_R)$ between morphisms of dual pairs, in other words, Γ is natural with respect to the constraint isomorphisms γ, δ. We will argue that it is also natural with respect to κ, τ.

By Lemma A.25, we can transport the structure of a morphism of G_{fd}-shapes from $(s_1, t_1)_{fd}$ along the isomorphism Γ to obtain a new morphism $(s_2, t_2)^{\circ}_{fd}$ of G_{fd}-shapes. Since it is isomorphic to one, it is also a morphism of fully dual pairs, and it agrees with $(s_2, t_2)_{fd}$ on its components on objects by construction.

Since Γ was natural with respect to γ, δ, $(s_2, t_2)^{\circ}_{fd}$ and $(s_2, t_2)_{fd}$ also agree on these two constraint isomorphisms. Thus, by what we've shown above they must in fact be equal, showing that Γ is a 2-cell between morphisms of fully dual pairs. This ends the argument. ■

A. Computadic monoidal bicategories and shapes

In this section we give an overview of the theory of *computadic monoidal bicategories*. Informally, this is a specific class of freely generated monoidal bicategories, where the presentation is allowed to have generating datum of all levels, but relations only at the level of 2-cells. This approach is completely analogous to the theory of *computadic symmetric monoidal bicategories* developed by Christopher Schommer-Pries [SP11].

We also prove a few technical results related to *bicategories of P-shapes*, which give an explicit description of bicategories of strict homomorphisms out of computadic monoidal bicategories. These are rather simple-minded in nature and are used in several places of the current work to simplify bookkeeping.

A.1. Computadic monoidal bicategories. There are different kinds of data one can use to generate a monoidal bicategory, like sets, category-enriched graphs and bicategories themselves. Many of these have been studied in the literature, see [Gur13b].

These approaches are, however, not quite sufficient for our purposes. We would like to allow the generating 1-cells to have domains and codomains that are only *consequences* of the generating data at the level of objects and similarly for 2-cells. This can make the presentations of bicategories defined using such data smaller and more readable.

A.2. Definition. *A* 0-truncated generating datum *(for a monoidal bicategory) consists of a set G_0. We inductively define the set $BW(G_0)$ of* binary words in G_0 *by declaring that*

1. *the symbol I is a binary word,*
2. *the symbol X is a binary word for all $X \in G_0$,*
3. *if X, Y are binary words, then so is $(X \otimes Y)$.*

In what follows we will talk and construct objects with *source* and *target* maps into some other kind of objects. We will then use the function notation to talk about these, even when there is no category in sight, so that for example "$f : A \to B$" will just mean "f has source A and target B".

A.3. Definition. *A* 1-truncated generating datum *consists of a tuple of sets (G_0, G_1) together with* source *and* target *maps $s, t : G_1 \to BW(G_0)$.*

If (G_0, G_1) is a 1-truncated generating datum, then we inductively define the set $BW(G_1)$ of binary words in G_1, *also with source and target in $BW(G_0)$, by declaring that*

1. *if X is a binary word in G_0, then the symbol id_X is a binary word in G_1 with source and target X,*
2. *If X, Y, Z are binary words in G_0 and x is a symbol taken from Table 1, then x is a binary word in G_1 with source and target given according to the table,*
3. *if X, Y, Z are binary words in G_0 and x is a symbol taken from Table 1, then $x^\bullet$ is a binary word in G_1 with source and target opposite to the one given in the table,*
4. *If $f \in G_1$, then the symbol f is a binary word with source $s(f)$ and target $t(f)$.*

Moreover, we inductively define the set $BS(G_1)$ of binary sentences in G_1, *again with source and target in $BW(G_0)$, by declaring that*

1. *If $f \in BW(G_1)$ is a binary word, then f is a binary sentence with the same source and target,*

2. *If $f, g \in BS(G_1)$ are binary sentences such that the target of f matches the source of g, then $(g) \circ (f)$ is a binary sentence with source $s(f)$ and target $t(g)$,*

3. *If $f, g \in BW(G_1)$ are binary sentences, then $f \otimes g$ is a binary sentence with source $s(f) \otimes s(g)$ and target $t(f) \otimes t(g)$.*

A.4. Definition. *A* generating datum *consists of a tuple (G_0, G_1, G_2), where (G_0, G_1) is a 1-truncated generating datum, together with* source *and* target *maps $s, t : G_2 \to BS(G_1)$ satisfying the globularity condition $s(s(\zeta)) = s(t(\zeta))$ and $t(s(\zeta)) = t(t(\zeta))$ for all $\zeta \in G_2$.*

If (G_0, G_1, G_2) is a generating datum, we inductively the set of binary words in G_2 *with source and target in $BS(G_1)$, by declaring that*

1. *if X, Y, Z are binary words in G_0 and x is a symbol taken from Table 1, then μ_x is a binary word in G_2 with source $id_{s(x)}$ and target $x^\bullet \circ x$ and ϵ_x is a binary word in G_2 with source $x \circ x^\bullet$ and target $id_{t(x)}$,*

2. *if $f : A \to B$, $f' : B \to C$, $f'' : C \to D$ are binary sentenes in G_1 and x is a bicategorical constraint symbol from Table 2, then x is a binary word in G_2 with source and target as given in the table and x^{-1} is a binary word with source and target opposite to the given ones,*

3. *if $f : B \to C$, $f' : A \to B$, $g : Y \to Z$, $g' : X \to Y$ and $h : P \to Q$ are binary sentences in G_1 and x is a monoidal morphism constraint symbol from Table 3, then x is also a binary word in G_2 with source and target as given in the table and x^{-1} is a binary word with source and target opposite to the given ones,*

4. *if A, B, C, D are binary words in G_0 and x is a monoidal object constraint symbol from Table 4, then x is also a binary word in G_2 with source and target as given in the table and x^{-1} is a binary word with source and target opposite to the given ones,*

5. *if ζ in G_2, then the symbol ζ is binary word.*

Consequently, we inductively define the set $BS(G_2)$ of binary sentences in G_2 *with source and target maps in $BS(G_1)$ by declaring that*

1. *if α is a binary word in G_2, then it is also a binary sentence with the same source and target,*

2. *if α, β are binary sentences in G_2 such that $t(t(\alpha)) = s(s(\beta))$, then $\beta * \alpha$ is a binary sentence in G_2 with source $s(\beta) \circ s(\alpha)$ and target $t(\beta) \circ t(\alpha)$,*

3. *if u, v are binary sentences in G_2, then $u \otimes v$ is binary sentence with source $s(u) \otimes s(v)$ and target $t(u) \otimes t(v)$,*

4. *if $\alpha_0, \alpha_1, \ldots, \alpha_k$ is a composable sequence of sentences, that is, we have $s(\alpha_i) = t(\alpha_{i-1})$ for all $1 \leq i \leq k$, then $p_k p_{k-1} \ldots p_0 = p_k \circ p_{k-1} \circ \ldots \circ p_0$ is a binary sentence in G_2 with target $t(p_k)$ and source $s(p_0)$.*

Table 1: Symbols for binary words in G_1

Symbol	**Source**	**Target**
l_X	$I \otimes X$	X
r_X	$X \otimes I$	X
$a_{X,Y,Z}$	$(X \otimes Y) \otimes Z$	$X \otimes (Y \otimes Z)$

Table 2: Bicategorical constraint symbols for binary words in G_2

Symbol	**Source**	**Target**
id_f	f	f
$a^c_{f,f',f''}$	$(f \circ f') \circ f''$	$f \circ (f' \circ f'')$
r^c_f	$f \circ id_A$	f
l^c_f	$id_B \circ f$	f

Table 3: Morphism constraint symbols for binary words in G_2

Symbol	**Source**	**Target**
$\phi^{\otimes}_{(f,g),(f',g')}$	$(f \otimes g) \circ (f' \circ g')$	$(f \circ f') \otimes (g \circ g')$
$\alpha_{f,g,h}$	$a_{C,Z,Q} \circ (f \otimes g) \otimes h$	$f \otimes (g \otimes h) \circ a_{B,Y,P}$
l_f	$l_C \circ (id_I \otimes f)$	$f \circ l_B$
r_f	$r_C \circ (f \otimes id_I)$	$f \circ l_B$

Table 4: Object constraint symbols for binary words in G_2

Symbol	**Source**	**Target**
$\phi^{\otimes}_{(A,B)}$	$id_{A\otimes B}$	$id_A \otimes id_B$
$\pi_{A,B,CD}$	$((id_A \otimes a_{B,C,D}) \circ a_{A,B\otimes C,D}) \circ (a_{A,B,C} \otimes id)$	$a_{A,B,C\otimes D} \circ a_{A\otimes B,C,D}$
$\mu_{A,B}$	$((id_A \otimes l_B) \circ a_{A,I,B}) \circ (r_A \otimes id_B)$	$id_{A\otimes B}$
$\lambda_{A,B}$	$l_A \otimes id_B$	$l_{A\otimes B} \circ_{1,A,B}$
$\rho_{A,B}$	$id_A \otimes r_B$	$a_{A,B,1} \circ_{A\otimes B}$

We are now ready to define a computadic monoidal bicategory generated by a free generating datum.

A.5. Definition. *Let $G = (G_0, G_1, G_2)$ be a generating datum. We define $\mathbb{F}(G)$, the* computadic monoidal bicategory *generated by G, as follows.*

1. *The objects of $\mathbb{F}(G)$ are precisely the binary words in G_0.*

2. *The morphisms of $\mathbb{F}(G)$ are precisely the binary sentences in G_1, with source and target as defined.*

3. *The 2-cells of $\mathbb{F}(G)$ are* equivalence classes *of sentences in G_2, with source and target as defined.*

The equivalence relation $\sim$ on binary sentences is the smallest equivalence relation such that

1. *if x is a symbol from Table 2, Table 3 or Table 4, then $xx^{-1} \sim id_{t(x)}$ and $x^{-1}x \sim id_{s(x)}$,*

2. *the* 2*-cells $a^c, r^c, l^c, \phi^{\otimes}_{(f,g),(f',g')}, \phi^{\otimes}_{(X,Y)}, a_{f,g,h}, l_f, r_f$ are components of a natural transformation, that is, in the relevant naturality pasting diagrams the two different composites are equivalent*

3. *the axioms of a monoidal bicategory hold and*

4. *the equivalence relation is closed under the tensor product $\otimes$, horizontal composition $*$ and vertical composition $\circ$.*

The structure of a monoidal bicategory is defined formally. More specifically, the composite of two binary words f and f' is $f \circ f'$, the horizontal composite of equivalence *classes α, β is $\beta * \alpha$, their vertical composite is $\beta \circ \alpha$. Similarly, one defines the monoidal product of objects, morphisms and 2-cells using the symbol $\otimes$. All the required axioms that this structure has to satisfy are enforced by the equivalence relation on binary sentences in G_2.*

We will now discuss relations on monoidal bicategories, the only kind we will consider is of relations between paralell 2-cells.

A.6. Definition. *A* class of relations $\mathcal{R}$ *on a monoidal bicategory* $\mathcal{M}$ *consists of a relation on the set of its 2-cells such that if $\alpha \sim_{\mathcal{R}} \beta$, then the source and target of α, β coincide. If $\mathcal{R}$ is a class of relations, then we define its* closure $c(\mathcal{R})$ *to be the smallest equivalence class of relations that contains $\mathcal{R}$ and also*

1. *it is closed under vertical composition, that is, if $\alpha \sim_{c(\mathcal{R})} \alpha'$ and $\beta \sim_{c(\mathcal{R})} \beta'$ and α, β are vertically composable, then $\beta\alpha \sim_{c(\mathcal{R})} \beta'\alpha'$,*
2. *it is closed under whiskering, that is, if $\alpha \sim_{c(\mathcal{R})} \alpha'$, then also $f * \alpha \sim_{c(\mathcal{R})} f * \alpha'$ and $\alpha * g \sim_{c(\mathcal{R})} \alpha' * f$ whenever this makes sense, ie. when f and $s(f)$ or $s(f)$ and g are composable, and*
3. *it is closed under tensor product, that is, if $\alpha \sim_{c(\mathcal{R})} \alpha'$ then also $f \otimes \alpha \sim_{c(\mathcal{R})} f \otimes \alpha'$ and $\alpha \otimes f \sim_{c(\mathcal{R})} \alpha' \otimes f$.*

A.7. Definition. *Suppose $\mathcal{M}$ is a monoidal bicategory and $\mathcal{R}$ is a class of relations on it. We define the* quotient monoidal bicategory ${}^{\mathcal{M}}\!/_{\mathcal{R}}$ *to have the same objects and morphisms as $\mathcal{M}$ and with 2-cells given by equivalence classes of 2-cells of $\mathcal{M}$ under the closure $c(\mathcal{R})$.*

Observe that there is a canonical strict quotient homomorphism $\pi_{\mathcal{R}} : \mathcal{M} \to \mathcal{M}/\mathcal{R}$ given by identity on objects and morphisms and by passing to equivalence classes on 2-cells. We will now establish its universal property.

A.8. Proposition. *For any monoidal bicategory $\mathcal{N}$, the precomposition homomorphism*

$$\pi^* : \mathbf{MonBicat}({}^{\mathcal{M}}\!/_{\mathcal{R}}, \mathcal{N}) \to \mathbf{MonBicat}(\mathcal{M}, \mathcal{N})$$

between bicategories of monoidal homomorphisms identifies the source with the full subbicategory of the target spanned by those homomorphisms $\phi : \mathcal{M} \to \mathcal{N}$ satisfying $\phi(\alpha) = \phi(\beta)$ whenever $\alpha \sim_{\mathcal{R}} \beta$.

Proof. Clearly any homomorphism coming from precomposition with π will have this property, as we have $\pi(\alpha) = \pi(\beta)$ if $\alpha \sim_{\mathcal{R}} \beta$. Conversely, any such ϕ must also necessarily satisfy $\phi(\alpha) = \phi(\beta)$ if $\alpha \sim_{c(\mathcal{R})} \beta$, where $c(\mathcal{R})$ is the closure. Thus, it factors through the quotient, showing that π^* is an inclusion onto the right subbicategory.

The homomorphism π^* is fully faithful by direct inspection of the definitions of a monoidal transformation and modification, which have no components related to 2-cells at all. ∎

We can now define a generating datum for a monoidal bicategory, which will consists of a free generating datum together with a class of relations. The associated computadic monoidal bicategory will be the obvious quotient.

A.9. Definition. *A* presentation of a monoidal bicategory *consists of a tuple* $P = (G, \mathcal{R})$*, where* $G = (G_0, G_1, G_2)$ *is a generating datum and* $\mathcal{R}$ *is a class of relations on* $\mathbb{F}(G)$*. The corresponding* computadic monoidal bicategory $\mathbb{F}(P)$ *is the quotient* $\mathbb{F}(G)/\mathcal{R}$.

It is not too difficult to see that any monoidal bicategory is equivalent to a computadic one through a strict homomorphism. However, computadic monoidal bicategories are special as they have the following two properties which make them convenient to work with:

1. for any monoidal bicategory $\mathcal{M}$, the bicategory of strict homomorphisms $\mathbb{F}(P) \to \mathcal{M}$ admits a compact description as bicategory of P-shapes, which we describe below and
2. any monoidal homomorphism $\mathbb{F}(P) \to \mathcal{M}$ is equivalent to a strict one.

To expand on the first property, one sees by inspecting Definition A.5 that the cells of a computadic monoidal bicategory are freely generated by G_i under the operations available in the structure of a monoidal bicategory. It follows that a strict homomorphism out of such a bicategory is uniquely determined by the list of its values on the generators, this list is what we will call a *P-shape*. Thus, a P-shape S will be given by a triple of maps $S_i : G_i \to \mathcal{M}_i$, where $\mathcal{M}_i$ is the set of i-cells of $\mathcal{M}$, the triple is required to be globular, that is, respect source and targets.

To make sense of this, one needs to observe that given such a triple there is a canonical extension of S_0 to all of $BW(G_0)$, the set of binary words in generating objects of G, given by inductively declaring that $S_0(I) = I_{\mathcal{M}}$, the unit of $\mathcal{M}$ and that $S_0(X \otimes Y) = S_0(X) \otimes_{\mathcal{M}} S_0(Y)$. Similarly, there are extensions of S_1 to the set $BS(G_1)$ of binary sentences in G_1 and of X_2 to the set $BS(G_2)$ of binary sentences in G_2 given by evaluation of expressions using the structure of the monoidal bicategory $\mathcal{M}$.

Additionally, if the set of relations $\mathcal{R}$ is non-empty, there are additional conditions that are needed to ensure that the homomorphism will in fact factor through the quotient $\mathbb{F}(P) = \mathbb{F}(G)/\mathcal{R}$.

A.10. Definition. *Let G be a generating datum for a monoidal bicategory. A triple S of maps $S_i : G_i \to \mathcal{M}_i$, where $\mathcal{M}_i$ is the set of i-cells of $\mathcal{M}$ is a* G-shape with values $\mathcal{M}$ *if the globularity conditions*

1. $s(S_1(f)) = S_0(s(f))$, $t(S_1(f)) = S_0(t(f))$ *for all* $f \in G_1$

2. $s(S_2(\alpha)) = S_1(s(\alpha))$, $t(S_2(\alpha)) = S_1(t(\alpha))$ *for all* $\alpha \in G_2$

are satisfied after the canonical extension of P_0 to the set of binary words in G_0, of P_1 to the set of binary sentences in G_1 and of P_2 to the set of binary sentences in G_2.

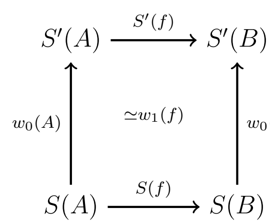
If S is a G-shape with values in $\mathcal{M}$, then the associated strict homomorphism $\mathbb{F}(G) \to \mathcal{M}$, *which we will also denote by S, is the unique strict homomorphism given on i-cells by S_i.*

A.11. Definition. *If $P = (G, \mathcal{R})$ is a presentation for a monoidal bicategory, then a G-shape is called a* P-shape *if and only if the associated strict homomorphism factors through $\mathcal{F}(P)$; that is, when we have $P(\alpha) = P(\beta)$ whenever $\alpha \sim_{\mathcal{R}} \beta$.*

A.12. Remark. Observe that being a P-shape is a *property*, rather than additional structure, on a given G-shape S.

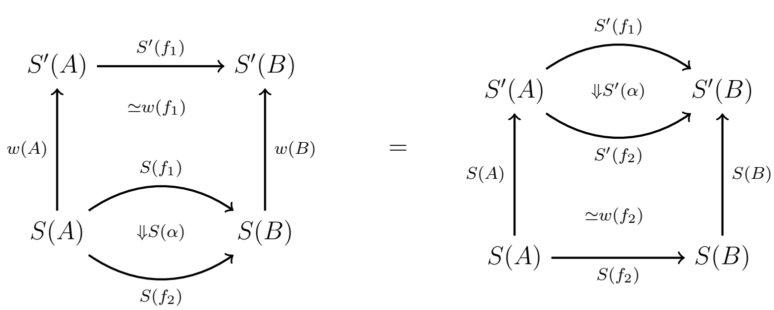
We will now proceed to define a bicategory of P-shapes with values in a fixed monoidal bicategory $\mathcal{M}$. As the reader can guess, since a P-shape encodes a homomorphism, cells between them will encode natural transformations and modifications.

A.13. Definition. *A* morphism $w : S \to S'$ *of G-shapes with values in $\mathcal{M}$ consists of a tuple of maps $w_0 : G_0 \to \mathcal{M}_1$ and $w_1 : G_1 \to \mathcal{M}_2$, where $w_0(X) : S(X) \to S'(X)$ for all $X \in G_0$ and $w_1(f)$ is an isomorphism fitting the diagram*



.

These are required to satisfy naturality with respect to all $\alpha \in G_2$; in other words, that for all such $\alpha : f_1 \to f_2$ we have



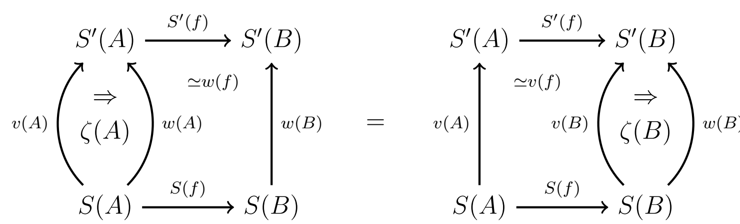
.

Here, w_1 was uniquely extended to all binary paragraphs in G_1 in such a way that this family of 2-cells plays a role of constraint isomorphisms for a natural transformation between the associated homomorphisms, the associated natural transformation.

Observe that a natural transformation associated to a morphism of shapes is partially strict. More precisely, it is not necessarily strict as a natural transformation of homomorphisms between monoidal bicategories — it would be so precisely when $S(f) = id$ for all $f \in G_1$ — but the additional data making it into a *monoidal* transformation is trivial. This is in line with the notion of P-shape itself, which encodes a strict homomorphism, rather than an arbitrary one.

A.14. REMARK. The theme of working with strict homomorphisms together with non-strict transformations is a recurrent one in category theory. For example, one often considers the natural enrichment of the category 2-Cat of strict 2-categories in itself, using 2-categories of strict functors and all transformations. This leads to the **Gray** tensor product, which is already "weak enough" to model all tricategories [GPS95].

A.15. DEFINITION. *If $w, v : S \to S'$ are morphisms of G-shapes, then a* transformation *$\zeta : w \to v$ consists of a map $\zeta_0 : G_0 \to \mathcal{M}_2$ such that for all $X \in G_0$ we have $\zeta_0(X) : w(X) \to v(X)$. Additionally, we require naturality with respect to all $f \in G_1$, ie. that for all such $f : A \to B$ we have*



,

Here, the function ζ is implicitly extended to the set of all binary words in G_0 so that its components give the associated modification *between natural transformations presented by w, v.*

As expected, together with the above definitions, G-shapes in a given monoidal bicategory can be assembled into a bicategory of their own.

A.16. DEFINITION. *Let $\mathcal{M}$ be a monoidal bicategory and $P = (G, \mathcal{R})$ be a presentation. The* bicategory of P-shapes*, denoted by $\mathcal{M}(P)$, has objects given by P-shapes in $\mathcal{M}$, 1-cells given by their morphisms and 2-cells given by transformations.*

As observed above, any P-shape determines a strict homomorphism $\mathbb{F}(P) \to \mathcal{M}$ from the computadic monoidal bicategory generated by P. Similarly, morphisms between shapes determine natural tranformations between the associated homomorphisms, and 2-cells determine modifications. This construction assembles to a homomorphism

$$\mathcal{M}(P) \to \mathbf{MonBicat}(\mathbb{F}(P), \mathcal{M})$$

which identifies the source as the subbicategory spanned by strict homomorphisms, monoidally strict natural transformations and all modifications.

The convenient property of computadic monoidal bicategories is that this inclusion is in fact an equivalence.

A.17. Theorem. [Cofibrancy theorem] *The inclusion* $\mathcal{M}(P) \hookrightarrow \mathbf{MonBicat}(\mathbb{F}(P), \mathcal{M})$ *is an equivalence of bicategories. In particular, any monoidal homomorphism out of a computadic monoidal bicategory is equivalent to a strict one.*

Proof. A detailed account in the symmetric monoidal setting is given in [SP11][2.78]. Since the monoidal case is identical, we just sketch the argument here.

If $\phi : \mathbb{F}(P) \to \mathcal{M}$ is an arbitrary monoidal homomorphism, not necessarily strict, then an appropriate restriction of ϕ to the generators of $\mathbb{F}(P)$ will yield a P-shape. This gives a homomorphism $\mathbf{MonBicat}(\mathbb{F}(P), \mathcal{M}) \to \mathcal{M}(P)$ going in the other direction. Of the two relevant composites, one is the identity and the other can be shown to be equivalent to it by an explicit natural transformation. ∎

A.18. Promotion between bicategories of shapes. In this section we will present a few results concerning truncations of G-shapes, by which we mean that the data of higher morphisms is ommited. We will be mainly interested in singling out favourable conditions under which such a truncated set of data can be "promoted" to more complete one. These are mainly used to simplify bookkeeping, see Remark A.22 for explanation.

Let G be a generating datum for a monoidal category and let G' be its 1-truncation in the sense that we have $G'_0 = G_0, G'_1 = G_1$ and $G'_2 = \emptyset$. The inclusions $G'_i \subseteq G_i$ yield a strict homomorphism $i : \mathbb{F}(G') \to \mathbb{F}(G)$ which is, as one sees immediately from the construction, bijective on objects and morphisms.

Dually, if $\mathcal{M}$ is any monoidal category, then any G-shaped diagram in $\mathcal{M}$ yields a G'-shaped by neglecting data of higher cells, similarly one can "truncate" homomorphisms and transformations. This assembles to a strict forgetful homomorphism $\tau_{\leq 1} : \mathcal{M}(G) \to \mathcal{M}(G')$ which in our description of the diagrams as encoding a monoidal homomorphism corresponds to precomposition with the above inclusion i.

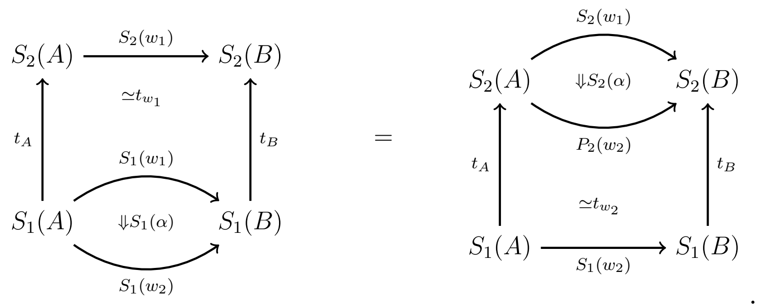
A.19. Lemma. *The forgetful homomorphism* $\tau_{\leq 1} : \mathcal{M}(G) \to M(G')$ *is locally on* Hom-*categories an inclusion of a full subcategory closed under isomorphisms.*

Proof. One sees directly from the definition of the shape bicategories that if $S_1, S_2 \in \mathcal{M}(G)$, then to give a map $S_1 \to S_2$ of G-shapes is exactly the same data as to give a map of G'-shapes, as it concerns only objects and 1-cells of the generating datum, only the conditions are different. In different words, to be a map of G-shapes — compared to being a map of G'-shapes — is a property rather than a structure. This shows that $\tau_{\leq 1}$ is locally injective on morphisms.

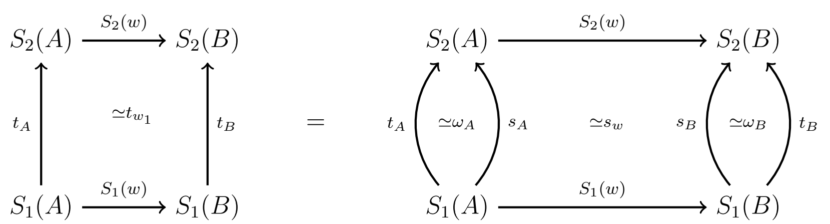
Similarly, directly from the definitions one sees that a 2-cell in the bicategory of shapes is given by data concerning only the objects of the generating datum and axioms only concerning the 1-cells. These both coincide for G and G', which yields that $\tau_{\leq 1}$ is locally bijective on 2-cells.

We now only have to verify closure under isomorphisms. Let $s : S_1 \to S_2$ be a map of G-shapes, $s : S_1 \to S_2$ be a map of G'-shapes and suppose we have an isomorphism $\omega : s \simeq t$. We have to show that then $s : S_1 \to S_2$ is also a map in $\mathcal{M}(G)$, which amounts to verifying that it satisfies naturality with respect to 2-cells in G_2.

Let $\alpha : w_1 \to w_2$ be an element of G_2, where w_i are sentences in G_1 with $w_i : A \to B$. We have to verify that



However, since ω is an isomorphism of maps of G'-shapes, we have for each sentence w in G_1

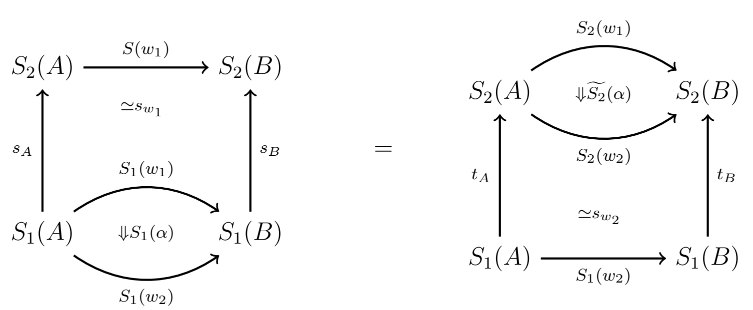


Pasting these decompositions of t_{w_i} in terms of s_{w_i} into the naturality equation above we see that it reduces to naturality for s, which we assumed.

■

A.20. Lemma. [Promoting equivalences of G'-shapes] *Let $S_1 \in \mathcal{M}(G)$, $S_2 \in \mathcal{M}(G')$ and suppose we are given an equivalence $s : \pi_{\leq 1}(S_1) \to S_2$. Then there is a unique G-shape $\widetilde{S_2}$ such that there exists an equivalence $\widetilde{s} : S_1 \to \widetilde{S_2}$ of G-shapes with $\tau_{\leq 1}(\widetilde{s}) = s$.*

Proof. To promote S_2 to a G-shape we have to define it on each 2-cell $\alpha \in G_2$. To say that $s : S_1 \to \widetilde{S_2}$ is an equivalence of G-shapes is to say that for each such α the naturality equation



holds. However, since s is an equivalence, the vertical maps are all equivalences and we see that this equation uniquely defines $\widetilde{S_2}(\alpha)$. ■

A.21. Corollary. *If S_1, S_2 are G-shapes and $s : \tau_{\leq 1}(S_1) \to \tau_{\leq 1}(S_2)$ is an equivalence of G'-shapes, then it can be lifted to an equivalence of G-shapes if and only if $S_2 = \widetilde{\tau_{\leq 1}(S_2)}$, where the latter is as in Lemma A.20.*

Proof. This follows immediately from uniqueness. ∎

A.22. Remark. We feel we owe the reader some explanation as to a how a result as straightforward as Lemma A.20 could be useful towards anything. The key here is Corollary A.21, which allows one to reduce checking that a given equivalence of underlying G'-shapes is actually an equivalence G-shapes, which asks for more naturality equations to hold, to checking that two G-shapes are equal. This allows one to reapply any coherence results already proven for *objects* to the case of *equivalences* of such, see Lemma 4.30 for a typical application.

A.23. Lemma. *Let $P = (G, \mathcal{R})$ be a presentation for a monoidal bicategory and let $S_1, S_2 \in \mathcal{M}$ be equivalent G-shapes. Then, if S_1 is a P-shape if and only if S_2 is.*

Proof. Recall that a G-shape S is a P-shape if and only if the associated homomorphism $F(G) \to \mathcal{M}$ factors through the quotient $\mathbb{F}(P) = {}^{\mathbb{F}(G)}/_{\mathcal{R}}$. If S_1 is a P-shape, then for any relation $\alpha \sim \beta$ in $\mathcal{R}$ we have $S_1(\alpha) = S_1(\beta)$. Then, the naturality equation for some chosen equivalence $s : S_1 \to S_2$ as in the proof of Lemma A.20 forces $S_2(\alpha) = S_2(\beta)$, as needed. ∎

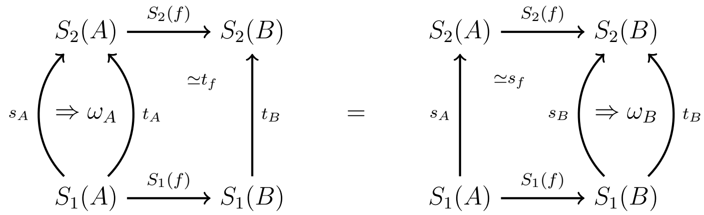
Let us now move to the case of the 0-*truncation* G'', by which we mean that $G''_0 = G_0$, $G''_1 = \emptyset$ and $G''_2 = \emptyset$. In other words, G'' has the same generating objects as G and has no generating 1-cells or 2-cells.

A.24. Proposition. *The forgetful homomorphism $\tau_{\leq 0} : \mathcal{M}(G') \to \mathcal{M}(G'')$ is locally faithful on 2-cells.*

Proof. Observe that to give a 2-cell in the bicategory of shapes is to give data concerning only the objects of the generating datum. The two agree for G' and G'' and the conclusion follows. ∎

A.25. Lemma. [Promotion of invertible 2-cells] *Let $S_1, S_2 \in \mathcal{M}(G')$ and suppose we are given a homomorphism $s : S_1 \to S_2$ of G'-shapes, a homomorphism $t : \tau_{\leq 0}(S_1) \to \tau_{\leq 0}(S_2)$ of G''-shapes and an invertible 2-cell $\omega : \tau_{\leq 0}(s) \to t$. Then, there is a unique map of G'-shapes $\widetilde{t}$ with $\tau_{\leq 0}(\widetilde{t}) = t$ such that there exists an invertible 2-cell $\widetilde{\omega} : s \to \widetilde{t}$ with $\tau_{\leq 0}(\widetilde{\omega}) = \omega$.*

Proof. To promote t to a map of G'-shapes we have to define the constraint isomorphisms for all 1-cells in $G'_1 = G_1$. Let $f : A \to B$ be one such, where A, B are some words in G_0. If we want ω to become an isomorphism 2-cell in $\mathcal{M}(G'')$, then for each such f the naturality equation



must hold. However, since the components of ω are isomorphisms, this equation clearly has a unique solution t_f. This allows one to define the needed constraint 2-cells and shows that they are unique. ∎

A.26. Corollary. *If $s, t : S_1 \to S_2$ are both maps of G'-shapes and $\omega : \tau_{\leq 0}(s) \to \tau_{\leq 0}(t)$ is an isomorphism of maps of G''-shapes, then it is also an isomorphism in $\mathcal{M}(G')$ if and only if we have $t = \widetilde{\tau_{\leq 0}(t)}$, where the latter is as in Lemma A.25.*

Proof. This follows immediately from uniqueness. ∎

A.27. Remark. By the closure under isomorphisms part of Lemma A.19, if S_1, S_2 are in fact G-shapes and s is a map of G-shapes, the promotion $\widetilde{t}$ of Lemma A.25 will also be.

References

[BD95] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105. MR 1355899 (97f:18003)

[GO13] Nick Gurski and Angélica M. Osorno, *Infinite loop spaces, and coherence for symmetric monoidal bicategories*, Adv.Math. **246** (2013), 1–32.

[GPS95] R. Gordon, A. J. Power, and Ross Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. **117** (1995), no. 558, vi+81. MR 1261589 (96j:18002)

[Gur12] Nick Gurski, *Biequivalences in tricategories*, Theory Appl. Categ. **26** (2012), No. 14, 349–384. MR 2972968

[Gur13a] ______, *Coherence in three-dimensional category theory*, vol. 201, Cambridge University Press, 2013.

[Gur13b] ______, *Coherence in three-dimensional category theory*, Cambridge Tracts in Mathematics, vol. 201, Cambridge University Press, Cambridge, 2013. MR 3076451

[HSSS18] Marc Hoyois, Pavel Safronov, Sarah Scherotzke, and Nicolò Sibilla, *The categorified Grothendieck-Riemann-Roch theorem*, arXiv preprint arXiv:1804.00879 (2018).

[KL97] G. M. Kelly and Stephen Lack, *On property-like structures*, Theory Appl. Categ. **3** (1997), No. 9, 213–250 (electronic). MR 1476422 (98h:18005)

[Lur09] Jacob Lurie, *On the classification of topological field theories*, http://arxiv.org/abs/0905.0465, 2009.

[Pst14] Piotr Pstrągowski, *On dualizable objects in monoidal bicategories, framed surfaces and the cobordism hypothesis*, arXiv preprint arXiv:1411.6691 (2014).

[SP11] Christopher J. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, http://arxiv.org/abs/1112.1000, 2011.

[Ver11] Dominic Verity, *Enriched categories, internal categories and change of base*, Repr. Theory Appl. Categ. (2011), no. 20, 1–266. MR 2844536

Harvard University

This article may be accessed at http://www.tac.mta.ca/tac/