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Q-SYSTEM COMPLETION IS A 3-FUNCTOR

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ABSTRACT. Q-systems are unitary versions of Frobenius algebra objects which appeared in the theory of subfactors. In recent joint work with R. Hernández Palomares and C. Jones, the authors defined a notion of Q-system completion for C^*/W^* 2-categories, which is a unitary version of a higher idempotent completion in the spirit of Douglas–Reutter and Gaiotto–Johnson-Freyd. In this article, we prove that Q-system completion is a † 3-functor on the † 3-category of C^*/W^* 2-categories. We also prove that Q-system completion satisfies a universal property analogous to the universal property satisfied by idempotent completion for 1-categories.

1. Introduction

Idempotent completions for higher categories have seen tremendous recent progress. For 2-categories (which we always assume are locally idempotent complete) with enough adjoints for 1-morphisms, completing with respect to the two notions of condensation monads [GJF19] and separable monads [DR18] produces equivalent 2-categories by [GJF19, Thm. 3.3.3]. The major difference is that condensation monads are *non-unital* and include the *data* of the separating structure, while separable monads are *unital* and include only the *existence* of separating structure, the choice of which is contractible.

In the setting of C^*/W^* 2-categories (which we always assume are locally orthogonal projection complete), the analogous notion of separable monad is Longo's *Q-system* [Lon94, LR97], which was originally studied for its role in subfactor theory. In our recent joint article [CPJP21], we introduced the notion of *Q-system completion* $\mathsf{QSys}(\mathcal{C})$ for a C^*/W^* 2-category $\mathcal{C}$, which comes equipped with a canonical † 2-functor $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{QSys}(\mathcal{C})$. While we analyzed some of the general theory of Q-system completion in that article, we focused more on applications to C^*-algebra theory, showing the C^* 2-category of C^*-algebras is Q-system complete. As an application, we used Q-system completion to induce actions of unitary fusion categories on C^*-algebras, similar to the spirit of [GY20].

In this article, we study some basic formal properties of Q-system completion, and our proofs can easily be adapted to the separable monad setting. Our main results extend the treatment of idempotent completion for 2-categories in [DR18, Appendix A]. Here is our first main theorem:

1.1. THEOREM. *Q-system completion is a † 3-endofunctor on the † 3-category of* C^*/W^*

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2-categories.

In [DR18, Prop. A.6.3], Douglas and Reutter provided strong evidence towards this theorem, and they mentioned they expect such a result to be true. To prove this theorem, we introduce an *overlay* compatibility between the 2D graphical calculi for a C*/W* 2-category $\mathcal{C}$ and the C*/W* 2-category $\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ for another $\mathcal{D}$. (We show in Proposition 2.15 below that $\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ is C*/W* whenever $\mathcal{C}, \mathcal{D}$ are.) See §2.3 below for more details. By considering non-unital unitary condensation algebras (see Rem. 3.5), our proof also shows that (non-unital) unitary condensation completion $\mathsf{Kar}^\dagger$ is also a † 3-endofunctor.

Our second main theorem regards the universal property for idempotent completion for 2-categories discussed in [Déc20, §1.2], proving the best possible uniqueness statement. Given 2-categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and 2-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the 2-category of *lifts* of F to $\mathcal{E}$ along G is the homotopy fiber at F of the functor

$$- \circ G : \mathsf{Fun}(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Objects in this lift 2-category are pairs $(\widetilde{F}, \theta)$ where $\widetilde{F} : \mathcal{E} \to \mathcal{D}$ is a 2-functor and $\theta : F \Rightarrow \widetilde{F} \circ G$ is an invertible 2-transformation. We refer the reader to §4 for the rest of the unpacked definition.

1.2. Theorem. *Suppose $\mathcal{C}$ is a* C*/W* *2-category. The Q-system completion* $\mathsf{QSys}(\mathcal{C})$ *satisfies the following universal property. For any † 2-functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Q-system complete, the 2-category of lifts of F along $\iota_\mathcal{C}$ is* (-2)-truncated, *i.e., equivalent to a point. That is, $- \circ \iota_\mathcal{C} : \mathsf{Fun}^\dagger(\mathsf{QSys}(\mathcal{C}) \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ is a † 2-equivalence.*

The main idea of the proof of this theorem comes from [JMPP19, §3.1]. By a version of Grothendieck's *Homotopy Hypothesis* for 2-categories [MS93], the homotopy category of strict 2-groupoids and strict 2-functors localized at the strict equivalences is equivalent to the 1-category of homotopy 2-types. Hence the homotopy fiber of $- \circ G$ restricted to the *core* 2-groupoids

$$- \circ G : \mathrm{core}(\mathsf{Fun}(\mathcal{E} \to \mathcal{D})) \to \mathrm{core}(\mathsf{Fun}(\mathcal{C} \to \mathcal{D}))$$

is k-truncated for $-2 \leq k \leq 1$ if and only if various (essential) surjectivity properties hold for $- \circ G$. In turn, these surjectivity properties for $- \circ G$ are ensured by various levels of *dominance* for the 2-functor G. We make these notions precise in §4.

While we work in the C*/W* setting both for novelty and for applications to the world of operator algebras, we re-emphasize that these results do not depend on the dagger structure.

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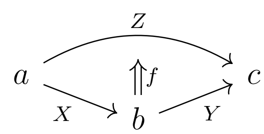
2. Preliminaries

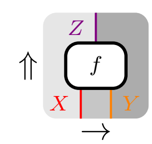
In this article, *2-category* will always mean a weak 2-category/bicategory which is locally idempotent complete, and a C*/W* 2-category will always mean a weak C*/W* 2-category which is locally orthogonal projection complete. We refer the reader to [JY20] for background on 2-categories and to [CPJP21] for background on C*/W* 2-categories. We refer the reader to [HV19] or [CPJP21] for a detailed discussion of the graphical calculus of string diagrams for 2-categories. The only 3-categories in this article are the 3-category 2Cat of 2-categories [Gur13, §5.1] and its 3-subcategories C*2Cat and W*2Cat of C*/W* 2-categories respectively.

2.1. Notation. In a 2-category $\mathcal{C}$, we refer to its objects, 1-morphisms, and 2-morphisms as *0-cells*, *1-cells*, and *2-cells* respectively. We denote 0-cells in a 2-category $\mathcal{C}$ by lowercase Roman letters a, b, c, 1-cells by uppercase Roman letters ${}_aX_b, {}_bY_c$ using bimodule notation for source (left) and target (right), and 2-cells by lowercase Roman letters later in the alphabet f, m, n, t. We write 1-composition as $\otimes$ read *left to right*, and we write 2-composition as $\star$, which is read *right to left*. In the graphical calculus of string diagrams in 2-categories, which is formally dual to the manipulation of pasting diagrams, we read 1-composition *left to right* and 2-composition *bottom to top*.

$$f : {}_aX \otimes_b Y_c \Rightarrow {}_aZ_c \qquad \rightsquigarrow$$

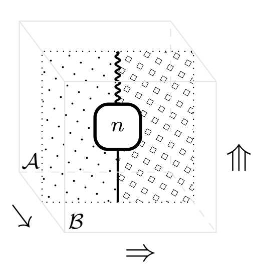


$\rightsquigarrow$



In the 3-category 2Cat of 2-categories, the object 2-categories are denoted by math calligraphic letters $\mathcal{C}, \mathcal{D}, \mathcal{E}$, the 2-functor 1-morphisms are denoted by capital Roman letters F, G, H, the 2-transformation 2-morphisms are denoted by lowercase Greek letters φ, ψ, and 2-modification 3-morphisms are denoted by lowercase Roman letters m, n. We write 1-composition of 2-functors as $\circ$, which we read *right to left*, i.e., if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, then $G \circ F : \mathcal{A} \to \mathcal{C}$. We write 2-composition of 2-transformations as $\otimes$, and we write 3-composition of 2-modification as $\star$.

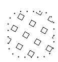
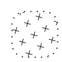
2.2. Remark. While we will not rely on any 3D string diagram graphical calculus in this article, its use for weak 3-categories can be justified using the article [Gut19]. In several locations, we provide 3D diagrams for conceptual clarity. Our conventions for 1-, 2-, and 3-composition in these 3D diagrams are indicated in the figure below.



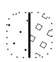
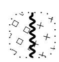
2.3. The 2-category $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ of 2-functors, 2-transformations, and 2-modifications. In this section, we first describe our graphical conventions for working with 2-functors, 2-transformations, and 2-modifications. We then use our graphical notation to unpack their definitions.

2.4. Notation. To define 2-transformations between 2-functors and 2-modifications between 2-transformations in a diagrammatic language, we overlay the 2D diagrammatic calculus for the hom 2-category $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ between 2-categories $\mathcal{A}, \mathcal{B}$ with the 2D diagrammatic calculus for $\mathcal{B}$.

For our 2D diagrammatic calculus for the hom 2-category $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$, we represent the object functors by *unshaded* regions with *textured* decorations, e.g.,

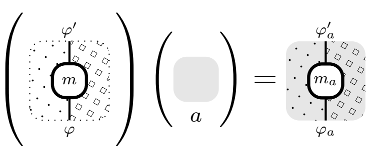
 $= F$  $= F'$  $= F''$  $= F'''$

We represent 2-transformations (see Definition 2.6 below) by *textured* strings between these textured regions, e.g.,

 $= \varphi : F \Rightarrow F'$  $= \psi : F' \Rightarrow F''$  $= \gamma : F'' \Rightarrow F'''$

We represent 2-modifications (see Definition 2.7 below) by coupons as usual.

To depict a 2-morphism in $\mathcal{B}$ in the image of F, we *overlay* the 2D string diagrammatic calculus for $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ on top of the 2D string diagrammatic calculus for $\mathcal{A}$. For example, given $F, F' : \mathcal{A} \to \mathcal{B}$, $\varphi, \varphi' : F \Rightarrow F'$, and $m : \varphi \Rrightarrow \varphi'$, we can 'overlay' the coupon for m over the shaded region for $a \in \mathcal{A}$ to obtain the 2-cell $m_a : \varphi_a \Rightarrow \varphi'_a$:



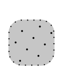
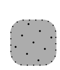
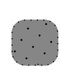
 $= F(a)$,  $= F'(a)$.

We do not attempt to formalize this 'overlay' operation, as all string diagrams can be interpreted uniquely as 2-cells in $\mathcal{B}$; see Remark 2.9 below for further discussion.

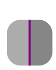
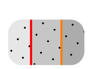
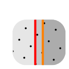
2.5. Definition. Suppose $\mathcal{A}, \mathcal{B}$ are 2-categories. We use the following conventions for the coheretors of a 2-functor $F = (F, F^2, F^1) : \mathcal{A} \to \mathcal{B}$:

$$F^2_{X,Y} \in \mathcal{B}(F(X) \otimes_{F(b)} F(Y) \Rightarrow F(X \otimes_b Y)) \qquad \text{and} \qquad F^1_a \in \mathcal{B}(1_{F(a)} \Rightarrow F(1_a)),$$

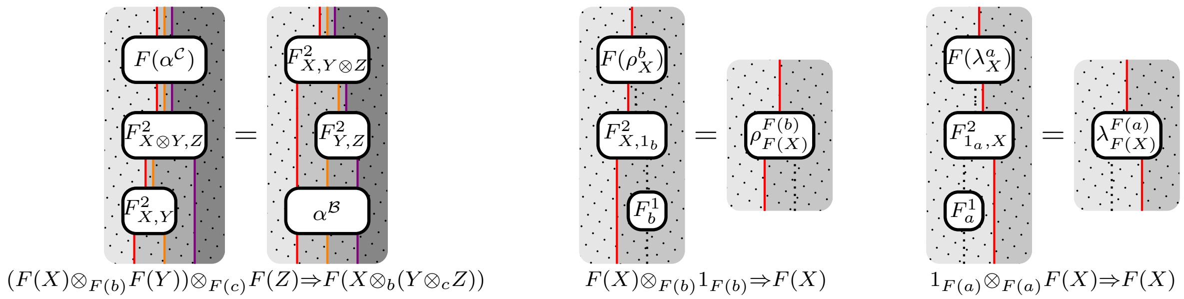
which satisfy the hexagon associativity equation and triangle unit equations. We depict these axioms below in the graphical calculus for $\mathcal{B}$. Denoting objects in $\mathcal{B}$ by the shaded regions

 $= F(a)$  $= F(b)$  $= F(c)$  $= F(d)$,

and 1-cells in $\mathcal{B}$ by shaded strands, e.g.

 $= {}_aX_b$  $= {}_bY_c$  $= {}_cZ_d$  $= F(X) \otimes_{F(b)} F(Y)$  $= F(X \otimes_b Y)$,

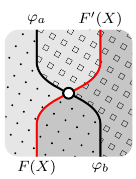
the hexagon and triangle equations are given by



Whenever possible, we will suppress the associator and unitor coheretors in our 2-categories.

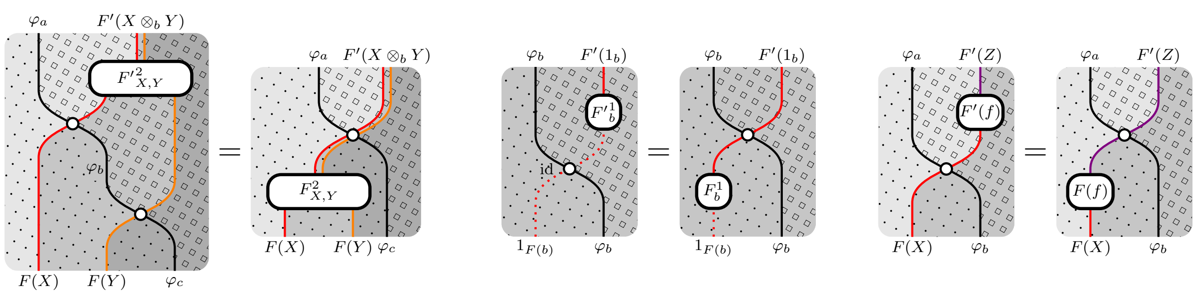
2.6. Definition. Suppose $\mathcal{A}, \mathcal{B}$ are 2-categories, $F, F' : \mathcal{A} \to \mathcal{B}$ are 2-functors. A *2-transformation* $\varphi : F \Rightarrow F'$ consists of:

- for every 0-cell $c \in \mathcal{A}$, a 1-cell $\varphi_c \in \mathcal{B}(F(c) \to F'(c))$, and
- for every 1-cell ${}_aX_b \in \mathcal{A}(a \to b)$, an invertible $F(a) - F'(b)$ bimodular 2-cell

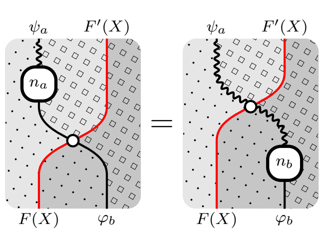


$$= \varphi_X \in \mathcal{B}(F(X) \otimes_{F(b)} \varphi_b \Rightarrow \varphi_a \otimes_{F'(a)} F'(X)).$$

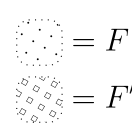
This data satisfies the following coherence properties:



2.7. Definition. Suppose $\mathcal{A}, \mathcal{B}$ are 2-categories, $F, F' : \mathcal{A} \to \mathcal{B}$ are 2-functors, and $\varphi, \psi : F \Rightarrow F'$ are 2-transformations. A *2-modification* $n : \varphi \Rrightarrow \psi$ consists of a 2-cell $n_a \in \mathcal{B}(\varphi_a \Rightarrow \psi_a)$ for all $a \in \mathcal{A}$ such that

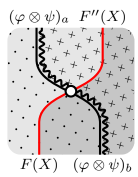


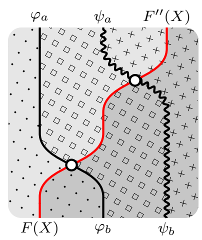
$\forall\, X \in \mathcal{A}(a \to b)$



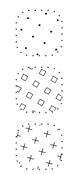
The 2-composition of 2-modifications in $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ is defined as follows. Suppose $F, F' \in \mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ and $\varphi, \varphi', \varphi''$ are 2-transformations $F \Rightarrow G$. Let $n : \varphi \Rrightarrow \varphi'$ and $n' : \varphi' \Rrightarrow \varphi''$ be 2-modifications. The 2-composition in $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$, denoted by $n' \star n : \varphi \Rrightarrow \varphi''$ is defined by $(n' \star n)_a := n'_a \star n_a$ for $a \in \mathcal{A}$ as composition of 2-cells in $\mathcal{B}$.

2.8. Definition. [1-composition in $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$] Suppose $F, F', F'' \in \mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ are 2-functors, and let $\varphi : F \Rightarrow F'$ and $\psi : F' \Rightarrow F''$ be 2-transformations. The 1-composite $\varphi \otimes \psi : F \Rightarrow F''$ of 2-transformations is defined as follows. Let $X \in \mathcal{A}(a \to b)$, we define $(\varphi \otimes \psi)_a := \varphi_a \otimes \psi_a$ as 1-composition of 1-cells in $\mathcal{B}$, and $(\varphi \otimes \psi)_X$ by

$(\varphi \otimes \psi)_X :=$ 

$:=$ 

$\forall X \in \mathcal{A}(a \to b)$


$= F$
$= F'$
$= F''$

Suppose $\varphi, \varphi' : F \Rightarrow F'$ and $\psi, \psi' : F' \Rightarrow F''$ are 2-transformations, and let $n : \varphi \Rrightarrow \varphi'$ and $t : \psi \Rrightarrow \psi'$ be 2-modifications. The 1-composite $n \otimes t : \varphi \otimes \psi \Rrightarrow \varphi' \otimes \psi'$ of 2-modifications is defined component-wise as 1-composition of 2-cells in $\mathcal{B}$ by $(n \otimes t)_a := n_a \otimes t_a$ for $a \in \mathcal{A}$.

Finally, we define the associator for 1-composition in $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ as follows. Suppose $\varphi : F \Rightarrow F'$, $\psi : F' \Rightarrow F''$: and $\gamma : F'' \Rightarrow F'''$: are 2-transformations. The associator $\alpha^{\otimes}_{\varphi,\psi,\gamma}$ is an invertible modification $(\varphi \otimes \psi) \otimes \gamma \Rrightarrow \varphi \otimes (\psi \otimes \gamma)$ which is given component-wise by

$$\left(\alpha^{\otimes}_{\varphi,\psi,\gamma}\right)_a := \alpha^{\mathcal{B}}_{\varphi(a),\psi(a),\gamma(a)}, \tag{1}$$

which is the associator in $\mathcal{B}$ between 1-cells $\varphi(a), \psi(a), \gamma(a)$. One checks that $\alpha^{\otimes}_{\varphi,\psi,\gamma}$ is a modification, and that $\alpha^{\otimes}$ satisfies the pentagon axiom.

The left/right unitors $\lambda^F_\varphi : 1_F \otimes \varphi \Rrightarrow \varphi$ and $\rho^{F'}_\varphi : \varphi \otimes 1_{F'} \Rrightarrow \varphi$ are an invertible 2-modifications which are given component-wise by

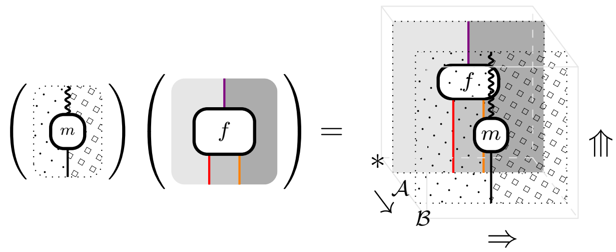
$$\left(\lambda^F_\varphi\right)_a := \lambda^{F(a)}_{\varphi(a)} \qquad \left(\rho^{F'}_\varphi\right)_a := \rho^{F'(a)}_{\varphi(a)}, \tag{2}$$

which are the unitors in $\mathcal{B}$ for 1-cell $\varphi(a)$.

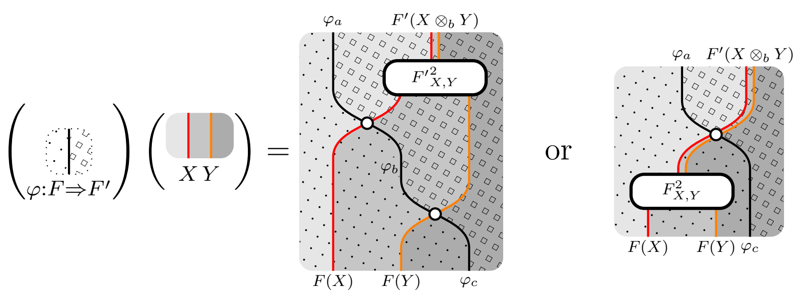
2.9. Remark. We do not attempt to formalize this overlay operation in this article, as all such string diagrams can be interpreted uniquely as a 2-cell in $\mathcal{B}$ without confusion. However, we sketch the following strategy to formalize this graphical calculus, which was communicated to us by David Reutter.

First, by [Gut19], the 3D graphical calculus for Gray-categories [BMS12, Bar14] may be applied in any 3-category, in particular, to $2\mathsf{Cat}$. Second, given a 2-category $\mathcal{A} \in 2\mathsf{Cat}$, we may identify $\mathcal{A} = \mathsf{Fun}(* \to \mathcal{A})$ where $*$ is the trivial 2-category. This identification allows us to identify the *internal* 2D string diagrammatic calculus for $\mathcal{A}$ with the *external*

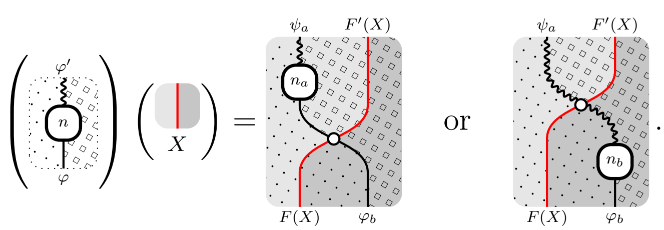
2D string diagrammatic calculus for $\mathsf{Fun}(* \to \mathcal{A})$ as a hom 2-category of $2\mathsf{Cat}$. Finally, identifying a 2-functor $F : \mathcal{A} \to \mathcal{B}$ with the 2-functor $\mathsf{Fun}(* \to \mathcal{A}) \to \mathsf{Fun}(* \to \mathcal{B})$ given by post-composition with F, and similarly for transformations and modifications, we see that our overlay graphical calculus is exactly stacking of 2D sheets in the 3D graphical calculus for $2\mathsf{Cat}$.



Now in order to interpret each diagram as a unique 2-morphism in $\mathcal{B}$, one should require the strings and coupons of our $\mathcal{A}$-diagram and our $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ diagram not overlap, except at finitely many points where strings can cross transversely. The axioms of 2-functor, 2-transformation, and 2-modification will then ensure that any two ways of resolving non-generic intersections agree. For example, we may overlay the 2-transformation $\varphi : F \Rightarrow F'$ on the identity 2-morphism $\mathrm{id}_X \otimes_b \mathrm{id}_Y$ in $\mathcal{A}$ in several ways. The equality of two such ways below produces the monoidal coherence axiom:



For another example, when we have a 2-modification between 2-transformations, we may overlay it on an identity 2-morphism id_X in many ways. The equality of two such ways below produces the modification coherence axiom:



Here, the white dots which appear may be interpreted as interchangers in $2\mathsf{Cat}$ (see Construction 2.17 below) which arise from resolving the two stacked 2D diagrams in $2\mathsf{Cat}$. (Recall that ${}_aX_b \in \mathcal{A}$ is a transformation when viewed as a 1-morphism in $\mathsf{Fun}(* \to A)$.)

We leave a rigorous proof of our formalization strategy of this 'overlay' graphical calculus to the interested reader.

2.10. The C*/W* 2-category $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ between C*/W* 2-categories. To the best of our knowledge, the notion of C* 2-category first appeared in [LR97], and the notion of W* 2-category first appeared in [Yam07]. The notion of W*-category was studied in detail in [GLR85]. We refer the reader to [CPJP21, §2.1] for an introduction to C*/W* 2-categories.

2.11. Definition. Suppose $\mathcal{A}, \mathcal{B}$ are C*/W* 2-categories. A † 2-functor $F : \mathcal{A} \to \mathcal{B}$ is a 2-functor $F = (F, F^2, F^1) : \mathcal{A} \to \mathcal{B}$ such that $F^2_{X,Y}$ and F^1_a are unitary for all composable 1-cells X, Y in $\mathcal{A}$ and all objects $a \in \mathcal{A}$. When $\mathcal{A}, \mathcal{B}$ are W*, we call a † 2-functor *normal* when each hom functor $F_{a\to b} : \mathcal{A}(a \to b) \to \mathcal{B}(F(a) \to F(b))$ is a normal † functor.

Suppose now $F, G : \mathcal{A} \to \mathcal{B}$ are †-2-functors. A †-2-transformation $\varphi : F \Rightarrow G$ consists of a 2-transformation $\varphi = (\varphi_c, \varphi_X) : F \Rightarrow G$ such that every (necessarily invertible) 2-cell $\varphi_X \in \mathcal{B}(F(X) \otimes_{F(b)} \varphi_b \Rightarrow \varphi_a \otimes_{G(a)} G(X))$ is unitary.

Given two †-2-transformations $\varphi, \psi : F \Rightarrow G$, a 2-modification $n : \varphi \Rrightarrow \psi$ is *(uniformly) bounded* if the 2-cells $n_a \in \mathcal{B}(\varphi_a \Rightarrow \psi_a)$ for all $a \in \mathcal{A}$ are uniformly bounded.

Now consider the 2-subcategory $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ of $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ consisting of † 2-functors, † 2-transformations, and uniformly bounded modifications. When $\mathcal{A}, \mathcal{B}$ are W*, we further require all † 2-functors to be normal.

2.12. Remark. It is well known (e.g., see [JY20, Thm. 7.4.1]) that a 2-functor is an equivalence if and only if it is an equivalence on hom 1-categories (fully faithful on 2-morphisms and essentially surjective on 1-morphisms) and essentially surjective on objects. Similarly, a † 2-functor is an equivalence if and only if it is a †-equivalence on hom categories (fully faithful on 2-morphisms and unitarily essentially surjective on 1-morphisms) and unitarily essentially surjective on objects.

When $F : \mathcal{C} \to \mathcal{D}$ is a † 2-functor between C* 2-categories, observe that F is a dagger equivalence if and only if the underlying 2-functor is an equivalence. Indeed, F is unitarily essentially surjective on 1-morphisms and objects if and only if it is essentially surjective on 1-morphisms and objects by the existence of polar decomposition for invertible 2-morphisms in $\mathcal{D}$.

Finally, observe that when $\mathcal{C}, \mathcal{D}$ are W*, any inverse † 2-functor will automatically be normal. This is an immediate consequence of the fact that every unital $*$-isomorphism between von Neumann algebras is automatically normal using Roberts' 2×2 trick [GLR85, Lem. 2.6] on linking algebras of hom 1-categories.

In Proposition 2.15 below, we prove that whenever $\mathcal{A}, \mathcal{B}$ are C*/W*, then so is the † 2-category $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ respectively. To prove this result, we prove Lemma 2.13 on weak* convergence in a product von Neumann algebra, which is certainly known to experts.

Suppose that $(M_i)_{i\in I}$ is a family of von Neumann algebras, and consider the product von Neumann algebra $\prod_{i\in I} M_i$, which is defined as the double commutant of the unital $*$-algebra of uniformly bounded elements (m_i) in the algebraic product of the M_i acting on the Hilbert space $\prod_{i\in I} H_i$, which consists of L^2-summable sequences of vectors. For $j \in I$, there are mutually orthogonal projections $p_j : \prod_i H_i \to H_j$ such that $\sum p_j = 1$, where the sum converges in the strong operator topology. Thus every element $m \in \prod_i M_i$ is diagonal, i.e., m may be written as $m = (m_i := p_i m p_i)_{i \in I}$.

2.13. LEMMA. *A norm-bounded net $(m_i)^j$ converges to (m_i) in the weak* topology on $\prod M_i$ if and only if every component net m_i^j converges to m_i in the weak* topology on M_i.*

PROOF. On norm-bounded sets in a von Neumann algebra, the weak* topology agrees with the weak operator topology. Suppose $\eta, \xi \in \prod_i H_i$. It is clear that $\langle (m_i)^j \eta, \xi \rangle \to \langle (m_i)\eta, \xi\rangle$ for all η, ξ implies $\langle m_i^j \eta_i, \xi_i \rangle \to \langle m_i \eta_i, \xi_i \rangle$ for all i.

For the converse, let $\varepsilon > 0$. Suppose M is the norm bound for $(m_i)^j$ and (m_i). It suffices to show $\langle (m_i)^j \eta, \xi \rangle \to \langle (m_i)\eta, \xi\rangle$ for all given $\eta, \xi \in \prod_i H_i$ with $\|\eta\|, \|\xi\| < 1$. Now choose η', ξ' in a finite product with $\|\eta'\| < 1$ and $\|\xi'\| < 1$ such that

$$\|\eta - \eta'\| < \frac{\varepsilon}{5M} \qquad \text{and} \qquad \|\xi - \xi'\| < \frac{\varepsilon}{5M}.$$

Since η', ξ' are finitely supported and $m_i^j \to m_i$ weak* for all components $i \in I$ by assumption, we can choose j_0 such that for all $j \geq j_0$,

$$|\langle [(m_i)^j - (m_i)]\eta', \xi'\rangle| < \frac{\varepsilon}{5}.$$

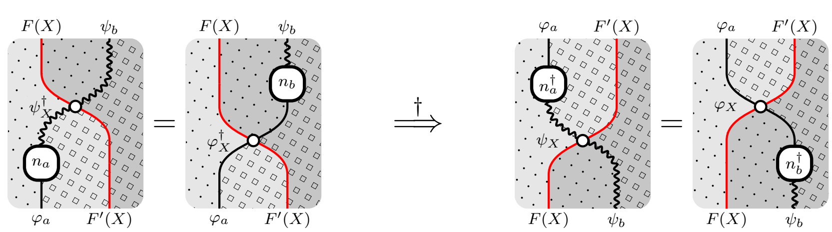
Then for all $j \geq j_0$, we have

$$\begin{aligned}
|\langle [(m_i)^j - (m_i)]\eta, \xi\rangle| &\leq |\langle (m_i)^j(\eta - \eta'), \xi\rangle| + |\langle (m_i)^j \eta', (\xi - \xi')\rangle| + |\langle [(m_i)^j - (m_i)]\eta', \xi'\rangle| \\
&\qquad + |\langle (m_i)(\eta - \eta'), \xi\rangle| + |\langle (m_i)\eta', (\xi - \xi')\rangle| \\
&\leq \|(m_i)^j\| \|\eta - \eta'\| \|\xi\| + \|(m_i)^j\| \|\eta'\| \|(\xi - \xi')\| \\
&\qquad + |\langle [(m_i)^j - (m_i)]\eta', \xi'\rangle| + \|(m_i)\| \|\eta - \eta'\| \|\xi\| \\
&\qquad + \|(m_i)\| \|\eta'\| \|\xi - \xi'\| \\
&< \varepsilon.
\end{aligned}$$

The result follows. ∎

2.14. CONSTRUCTION. We construct a †-structure on $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ (c.f. [Ver20]). Suppose $F, F' \in \mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$, $\varphi, \psi : F \Rightarrow F'$, and $n : \varphi \Rrightarrow \psi$ is a uniformly bounded modification. For each 0-cell $b \in \mathcal{B}$, we define $(n^\dagger)_b := (n_b)^\dagger$, where $(n_b)^\dagger$ is the dagger in $\mathcal{B}$.

We now verify that $n^\dagger$ is a modification $\psi \Rrightarrow \varphi$ with $\|n^\dagger\| = \|n\|$. First, note that φ_X, ψ_X are unitaries for all $X \in \mathcal{A}(a \to b)$. We compose $\psi_X^\dagger$ on the top and $\varphi_X^\dagger$ on the bottom, and apply the dagger in $\mathcal{B}$, to obtain



Thus, $n^\dagger$ is a 2-modification $\psi \Rightarrow \varphi$. Since $\dagger$ preserves the norm on all 2-cells of $\mathcal{B}$, we have $\|n_b\| = \|n_b^\dagger\|$ for all $b \in \mathcal{B}$, and thus $n^\dagger$ is uniformly bounded with $\|n^\dagger\| = \|n\|$.

We show $(n \otimes k)^\dagger = n^\dagger \otimes k^\dagger$ and $(n \star t)^\dagger = t^\dagger \star n^\dagger$, and clearly $n^{\dagger\dagger} = n$ by construction. For $a \in \mathcal{A}$,

$$(n \otimes k)_a^\dagger = ((n \otimes k)_a)^\dagger = (n_a \otimes k_a)^\dagger = n_a^\dagger \otimes k_a^\dagger = (n^\dagger)_a \otimes (k^\dagger)_a = (n^\dagger \otimes k^\dagger)_a$$
$$(n \star t)_a^\dagger = ((n \star t)_a)^\dagger = (n_a \star t_a)^\dagger = t_a^\dagger \star n_a^\dagger = (t^\dagger)_a \star (n^\dagger)_a = (t^\dagger \star n^\dagger)_a.$$

Finally, we observe that since all associators and unitors in $\mathcal{B}$ are unitary, so are the associators and unitors in $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$, as all their components are unitary by (1,2).

2.15. PROPOSITION. *When $\mathcal{A}, \mathcal{B}$ are* $\mathrm{C}^*/\mathrm{W}^*$ *2-categories, so is* $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$.

PROOF. By Construction 2.14, $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ is a $\dagger$ 2-category.

Since $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ admits direct sums of 1-morphisms, to show $\mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$ is C*, by Roberts' 2×2 trick [GLR85, Lem. 2.6], it suffices to show that for each 1-morphism/2-transformation $\varphi : F \Rightarrow G$, $\mathrm{End}_{\mathsf{Fun}^\dagger(\mathcal{A}\to\mathcal{B})}(\varphi)$ is a C* algebra. Indeed, the uniformly bounded modifications $n : \varphi \Rrightarrow \varphi$ do form a C*-algebra under the supreme norm:

$$\|n^\dagger \cdot n\| = \sup_{a\in\mathcal{A}} \|(n^\dagger \cdot n)_a\| = \sup_{a\in\mathcal{A}} \|(n^\dagger)_a \star n_a\| = \sup_{a\in\mathcal{A}} \|(n_a)^\dagger \star n_a\| = \sup_{a\in\mathcal{A}} \|n_a\|^2 = \|n\|^2.$$

Now suppose $\mathcal{A}, \mathcal{B}$ are W* 2-categories. It remains to prove $\mathrm{End}_{\mathsf{Fun}^\dagger(\mathcal{A}\to\mathcal{B})}(\varphi)$ is a W*-algebra and that 1-compositions with identity 2-transformations is a normal $\dagger$ functor on hom categories. Note that

$$n = (n_a)_{a\in\mathcal{A}} \in \mathrm{End}(\varphi : F \to G) \subset \prod_{a\in\mathcal{A}} \mathrm{End}(\varphi_a),$$

where n satisfies $\varphi_X \star (1_{F(X)} \otimes_{F(b)} n_b) = (n_a \otimes_{G(a)} 1_{G(X)}) \star \varphi_X$, for all $X \in \mathcal{A}(a \to b)$.

To prove $\mathrm{End}_{\mathsf{Fun}^\dagger(\mathcal{A}\to\mathcal{B})}(\varphi)$ is a W*-algebra, by either the Krein-Smulian or Kaplansky Density Theorems, it suffices to show the unit ball in $\mathrm{End}(\varphi)$ is weak* closed. Let $(n_j = (n_a^j))$ be a weak* convergent net in the unit ball of $\mathrm{End}(\varphi) \subset \prod_a \mathrm{End}(\varphi_a)$, a W*-algebra. By Lemma 2.13, each component net (n_a^j) converges weak* to an element n_a in the unit ball of $\prod_a \mathrm{End}(\varphi_a)$. We verify that $n := (n_a)$ is a 2-modification in $\mathrm{End}(\varphi)$. By the axioms of a W* 2-category (see (W*2') in [CPJP21, Prop. 2.4]), $1_{F(X)} \otimes_{F(b)} -$, $- \otimes_{G(a)} 1_{G(X)}$, $\varphi_X \star -$, and $- \star \varphi_X$ are normal operations on 2-cells in $\mathcal{B}$. We thus have

$$\begin{aligned}\varphi_X \star (1_{F(X)} \otimes_{F(b)} n_b) &= \lim_k \varphi_X \star (1_{F(X)} \otimes_{F(b)} (n_k)_b) \\ &= \lim_k ((n_k)_a \otimes_{G(a)} 1_{G(X)}) \star \varphi_X = (n_a \otimes_{G(a)} 1_{G(X)}) \star \varphi_X,\end{aligned}$$

which implies that n is a 2-modification $\varphi \Rrightarrow \varphi$.

We now show that 1-composition with an identity 2-transformation is normal. Let $\varphi : F \Rightarrow G$; we show $1_\varphi \otimes -$ is normal. Suppose $n^j, n : \psi \Rrightarrow \gamma$ are modifications with

$n^j \to n$ weak*. Again by Lemma 2.13, $n^j_a \to n_a$ weak* for all $a \in \mathcal{A}$. Since $1_{\varphi(a)} \otimes -$ is normal,

$$(1_\varphi \otimes n^j)_a = 1_{\varphi(a)} \otimes n^j_a \to 1_{\varphi(a)} \otimes n_a = (1_\varphi \otimes n)_a,$$

for each $a \in \mathcal{A}$, which implies $1_\varphi \otimes n_i \to 1_\varphi \otimes n$ weak* as desired. Similarly, $- \otimes 1_\varphi$ is normal. This completes the proof. ■

2.16. The 3-category of 2-categories. It is well-known that 2-categories form a 3-category $2\mathsf{Cat}$, whose hom 2-categories $2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ are given by $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$. We now explain 1-composition in this 3-category following [Gur13, §5.1]. We will then discuss the 3-subcategories $\mathrm{C}^*2\mathsf{Cat}$ and $\mathrm{W}^*2\mathsf{Cat}$.

2.17. Construction. By [Gur13, Prop. 5.1], given 2-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, there is a 2-functor

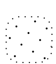
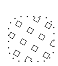
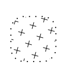
$$\circ : 2\mathsf{Cat}(\mathcal{B} \to \mathcal{C}) \times 2\mathsf{Cat}(\mathcal{A} \to \mathcal{B}) \to 2\mathsf{Cat}(\mathcal{A} \to \mathcal{C}).$$

The 2-functor $\circ$ is the 1-composition in $2\mathsf{Cat}$. We now describe its definition on 1-morphisms, 2-morphism, and 3-morphisms in $2\mathsf{Cat}$.

1-composition of 1-morphisms: For $F \in 2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ and $G \in 2\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ the 1-composite 2-functor $G \circ F \in 2\mathsf{Cat}(\mathcal{A} \to \mathcal{C})$ is given by:

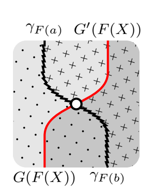
- $(G \circ F)(a) = G(F(a))$ for $a \in \mathcal{A}$, $(G \circ F)(X) = G(F(X))$ for $X \in \mathcal{A}(a \to b)$, and $(G \circ F)(f) = G(F(f))$ for $f \in \mathcal{A}(X \Rightarrow Y)$.
- $(G \circ F)^1_a := G(F^1_a) \star G^1_{F(a)} \in \mathcal{C}(1_{G(F(a))} \Rightarrow G(F(1_a)))$ for $a \in \mathcal{A}$.
- $(G \circ F)^2_{X,Y} := G(F^2_{X,Y}) \star G^2_{F(X),F(Y)} \in \mathcal{C}(G(F(X)) \otimes G(F(Y)) \Rightarrow G(F(X \otimes Y)))$ for $X \in \mathcal{A}(a \to b)$ and $Y \in \mathcal{A}(b \to c)$.

1-composition of 2-morphisms: Suppose $F, F' \in 2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ and $G, G' \in 2\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$. In the remainder of this definition, we use the following texture decorations to denote the following composite 2-functors:

 $= GF$  $= GF'$  $= G'F$  $= G'F'$.

Given 2-transformations $\varphi \in 2\mathsf{Cat}(F \Rightarrow F')$ and $\gamma \in 2\mathsf{Cat}(G \Rightarrow G')$, we define $\gamma \circ F \in 2\mathsf{Cat}(G \circ F \Rightarrow G' \circ F)$ component-wise by

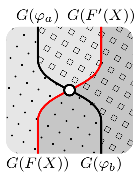
- For $a \in \mathcal{A}$, we define $(\gamma \circ F)_a := \gamma_{F(a)}$, and
- for $X \in \mathcal{A}(a \to b)$, we define

$(\gamma \circ F)_X := \gamma_{F(X)} =$ 

$\forall X \in \mathcal{A}(a \to b).$

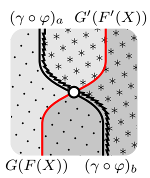
Similarly, we define $G \circ \varphi \in 2\mathsf{Cat}(G \circ F \Rightarrow G \circ F')$ by

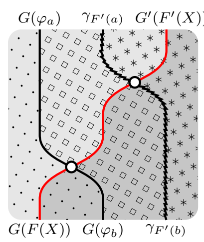
- For $a \in \mathcal{A}$, we define $(G \circ \varphi)_a := G(\varphi(a))$, and
- for $X \in \mathcal{A}(a \to b)$, we define

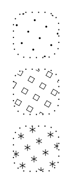
$(G \circ \varphi)_X :=$ 

$$:= \left(G(F(X)) \otimes G(\varphi_b) \xrightarrow{G^2} G(F(X) \otimes \varphi_b) \xrightarrow{G(\varphi_X)} G(\varphi_a \otimes F'(X)) \xrightarrow{(G^2)^\dagger} G(\varphi_a) \otimes G(F(X)) \right) \qquad \forall\, X \in \mathcal{A}(a \to b).$$

We then use the *cubical convention* to define the 1-composite $\gamma \circ \varphi := (G \circ \varphi) \otimes (\gamma \circ F') \in 2\mathsf{Cat}(G \circ F \Rightarrow G' \circ F')$, whose components are then given by

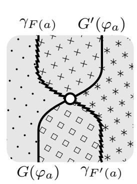


$:=$


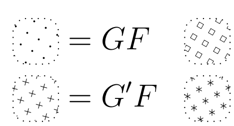

$= GF$
$= GF'$
$= G'F'$.

1-composition of 3-morphisms: Suppose $F, F' \in 2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ and $G, G' \in 2\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ are 2-functors, $\varphi, \varphi' \in 2\mathsf{Cat}(F \Rightarrow F')$ and $\gamma, \gamma' \in 2\mathsf{Cat}(G \Rightarrow G')$ are 2-transformations, and let $n \in 2\mathsf{Cat}(\varphi \Rrightarrow \varphi')$ and $k \in 2\mathsf{Cat}(\gamma \Rrightarrow \gamma')$ be 2-modifications. We define $k \circ n \in 2\mathsf{Cat}(\gamma \circ \varphi \Rrightarrow \gamma' \circ \varphi')$ component-wise at $a \in \mathcal{A}$ by $(k \circ n)_a := G(n_a) \otimes k_{F(a)}$ as 1-composition of 2-cells in $\mathcal{C}$.

Interchanger: For each pair of 1-composable 2-transformations φ, γ, there is a distinguished invertible modification $\chi^{\varphi,\gamma} : (G \circ \varphi) \otimes (\gamma \circ F') \Rrightarrow (\gamma \circ F) \otimes (G' \circ \varphi)$ between the *cubical* and *opcubical* 1-composition conventions for 2-morphisms called the *interchanger*, which is defined component-wise by

$\chi_a^{\varphi,\gamma} :=$ 

$= \gamma_{\varphi_a} \qquad \forall\, a \in \mathcal{A}$


$= GF$ $= GF'$
$= G'F$ $= G'F'$.

(Recall here that $\varphi_a \in \mathcal{B}(F(a) \to F'(a))$.) The interchanger modification is used to prove the *interchange relation* between $\circ, \otimes$. In more detail, given $\varphi \in 2\mathsf{Cat}(F \Rightarrow F')$, $\varphi' \in 2\mathsf{Cat}(F' \Rightarrow F'')$, $\psi \in 2\mathsf{Cat}(G \Rightarrow G')$, and $\psi' \in 2\mathsf{Cat}(G' \Rightarrow G'')$, the interchanger provides an invertible modification

$$(\psi \circ \varphi) \otimes (\psi' \circ \varphi') \Rrightarrow (\psi \otimes \psi') \circ (\varphi \otimes \varphi').$$

We refer the reader to [Gur13, p.88] for more details.

By [JY20, p.115], $\circ$ is strictly associative. That is, for $F \in 2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$, $G \in 2\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ and $H \in 2\mathsf{Cat}(\mathcal{C} \to \mathcal{D})$, then $(H \circ G) \circ F = H \circ (G \circ F) : \mathcal{A} \to \mathcal{D}$. By [Gur13, Props. 5.3 and 5.5], we may choose our adjoint equivalences $a : \circ(\circ \times \mathbf{1}) \Rightarrow \circ(\mathbf{1} \times \circ)$, $\ell : \circ(I_\mathcal{A} \times \mathbf{1}) \Rightarrow \mathbf{1}$, and $r : \circ(\mathbf{1} \times I_\mathcal{A}) \Rightarrow \mathbf{1}$ to be identity transformations, whose inverses are also identity transformations. Thus by [Gur13, Thm. 5.7], $2\mathsf{Cat}$ is a 3-category.

2.18. Definition. The 3-category $\mathrm{C}^*2\mathsf{Cat}$ of C^* 2-categories is the 3-subcategory of $2\mathsf{Cat}$ whose:

- objects are C^* 2-categories,
- 1-morphisms are † 2-functors,
- 2-morphisms are † 2-transformations
- 3-morphisms are bounded 2-modifications

Observe that all higher coherence data in this 3-category is unitary.

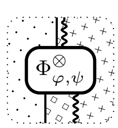
The 3-category $\mathrm{W}^*2\mathsf{Cat}$ of W^* 2-categories is the locally full 3-subcategory of $\mathrm{C}^*2\mathsf{Cat}$ whose objects are W^* 2-categories and whose 1-morphisms are normal † 2-functors.

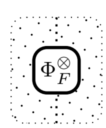
Observe that $\mathrm{C}^*2\mathsf{Cat}$ and $\mathrm{W}^*2\mathsf{Cat}$ may be equipped with †-structures making them into † 3-categories. Indeed, all hom 2-categories are $\mathrm{C}^*/\mathrm{W}^*$ by Proposition 2.15, 1-composition 2-functors are clearly compatible with the †-structure, and strictness of associativity of $\circ$ means all coheretors are inherently unitary.

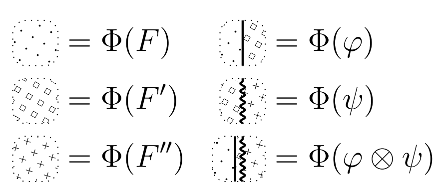
2.19. 3-endofunctors on $2\mathsf{Cat}$. In this section, we give a graphical definition of a (weak) 3-endofunctor Φ on $2\mathsf{Cat}$. The definition is considerably easier due to strictness of 1-composition $\circ$. Our treatment is adapted from [Gur13, §4.3].

Beyond an assignment of a k-morphism in $2\mathsf{Cat}$ for every k-morphism in $2\mathsf{Cat}$, Φ satisfies the following properties:

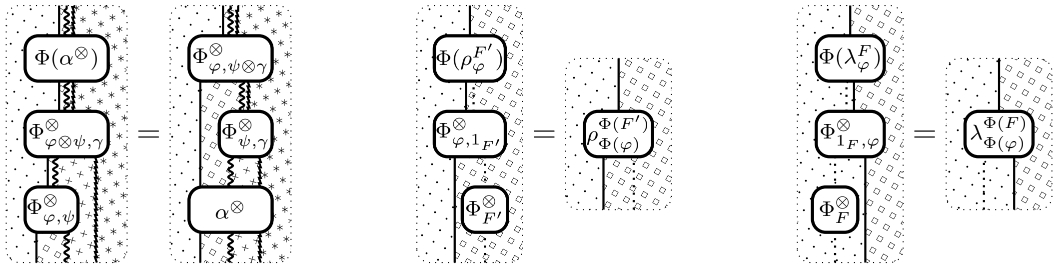
- Φ is a 2-functor on all hom 2-categories $2\mathsf{Cat}(\mathcal{A} \to \mathcal{B}) = \mathsf{Fun}(\mathcal{A} \to \mathcal{B})$ in $2\mathsf{Cat}$. That is, for all transformations $\varphi \in 2\mathsf{Cat}(F \Rightarrow F')$ and $\psi \in 2\mathsf{Cat}(F' \Rightarrow F'')$ for $F, F', F'' : \mathcal{A} \to \mathcal{B}$, there exist invertible modifications, $\Phi^{\otimes}_{\varphi,\psi} : \Phi(\varphi) \otimes \Phi(\psi) \Rrightarrow \Phi(\varphi \otimes \psi)$ and $\Phi^{\otimes}_{F} : 1_{\Phi(F)} \Rrightarrow \Phi(1_F)$, which we represent graphically by



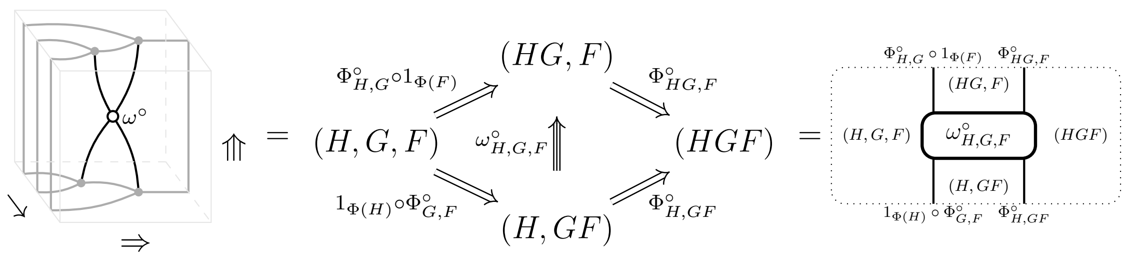




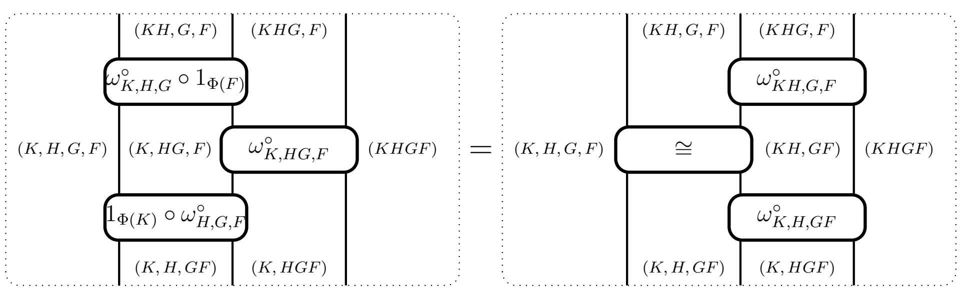
These modifications are subject to the usual associativity and unitality coherence axioms:



- We have 1-compositor adjoint equivalence transformations $\Phi^{\circ}_{G,F} : \Phi(G) \circ \Phi(F) \Rightarrow \Phi(G \circ F)$ for all $F \in 2\mathsf{Cat}(\mathcal{A} \to \mathcal{B})$ and $G \in 2\mathsf{Cat}(\mathcal{B} \to \mathcal{C})$ and $\Phi^{\circ}_{\mathcal{A}} : 1_{\Phi(\mathcal{A})} \Rightarrow \Phi(1_{\mathcal{A}})$ for all $\mathcal{A} \in 2\mathsf{Cat}$. These transformations come equipped with an invertible associator modification $\omega^{\circ}_{H,G,F}$:

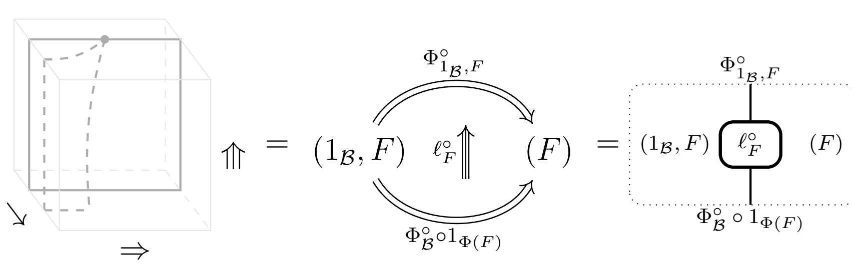


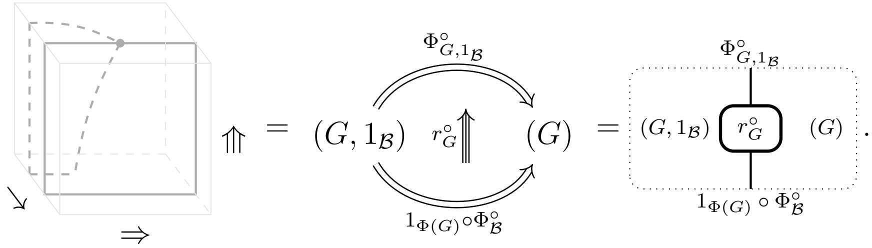
Here, we use the abbreviated notation $(GF) := \Phi(G \circ F)$ and $(G,F) := \Phi(G) \circ \Phi(F)$, so that $(K, HG, F) := \Phi(K) \circ \Phi(H \circ G) \circ \Phi(F)$ and $\Phi^{\circ}_{H,GF} := \Phi^{\circ}_{H,G\circ F} : (H,GF) \Rightarrow (HGF)$. The associator ω° satisfies the coherence axiom



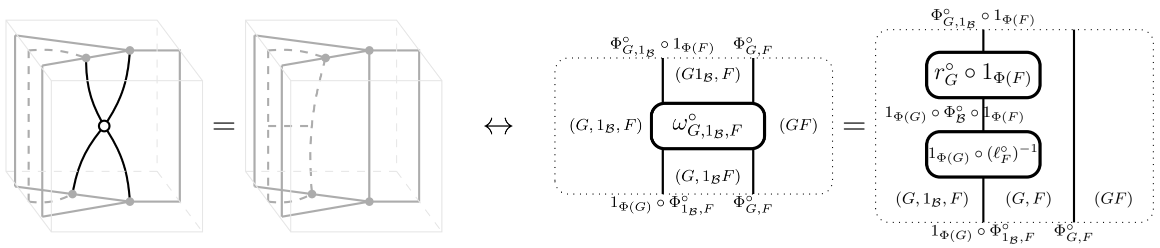
where the isomorphism on the left of the right hand side is the interchanger from Construction 2.17.

Finally, we have invertible unitor modifications ℓ°_F and r°_F:





These unitors satisfy the coherence axiom



Here, we note that $F\circ 1_\mathcal{A} = F = 1_\mathcal{B}\circ F$, so $(G1_\mathcal{B}F) = (GF)$, $(G1_\mathcal{B},F) = (G,F) = (G,1_\mathcal{B}F)$ and $(G,F) = \Phi(G)\circ 1_{\Phi(\mathcal{B})}\circ\Phi(F)$.

Given a weak 3-functor Φ on 2Cat which preserves the 3-subcategories C*2Cat and W*2Cat, we can ask whether Φ restricts to a † 3-functor. This consists of the following conditions:

- $\Phi(n^\dagger) = \Phi(n)^\dagger$ for all bounded 2-modifications n,
- the coheretors $\Phi^\otimes_{\varphi,\psi}$ and $\Phi^\otimes_F$ are unitary,
- $\Phi^\circ_{G,F}$ and $\Phi^\circ_\mathcal{A}$ are unitary adjoint equivalences, and
- the associators $\omega^\circ_{H,G,F}$ and unitors ℓ°_F, r°_F are unitary.

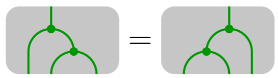
3. Q-system completion is a 3-functor

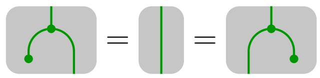
In this section, we rapidly recall the definition of Q-system completion for a C*/W* 2-category from [CPJP21, §3], and we prove Theorem 1.1 that Q-system completion is a 3-functor.

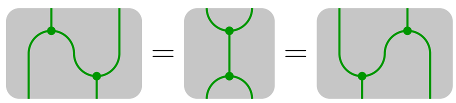
3.1. Graphical calculus for Q-systems and their bimodules. Q-systems were first defined in [Lon94], and were subsequently studied in [LR97, Zit07, BKLR15]. For this section, we fix a C*/W* 2-category $\mathcal{C}$ which we assume is locally unitarily Cauchy complete, i.e., every hom 1-category has orthogonal direct sums and all orthogonal projections split orthogonally.

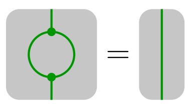
3.2. Definition. A *Q-system* in $\mathcal{C}$ consists of a triple (Q, m, i) where $Q \in \mathcal{C}(b \to b)$, $m \in \mathcal{C}(Q \otimes Q \Rightarrow Q)$, and $i \in \mathcal{C}(1_b \Rightarrow Q)$, which satisfy certain axioms. We represent b, Q, m, i and the adjoints $m^\dagger, i^\dagger$ graphically as follows:

The Q-system axioms are as follows:

(Q1) (associativity) 

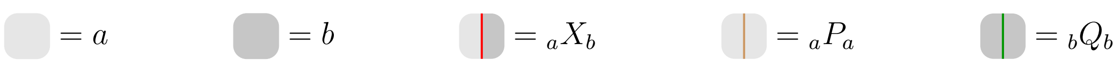
(Q2) (unitality) 

(Q3) (Frobenius) 

(Q4) (separable) 

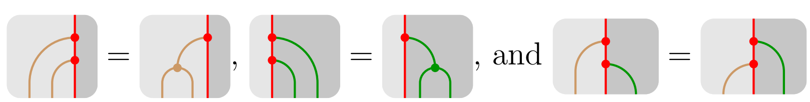
We refer the reader to [Zit07, Prop. 5.17] or [CPJP21, Facts 3.4] for various dependencies amongst these axioms.

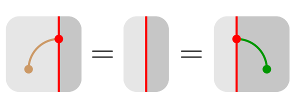
3.3. Definition. Suppose $P \in \mathcal{C}(a \to a)$ and $Q \in \mathcal{C}(b \to b)$ are Q-systems. A $P - Q$ bimodule is a triple (X, λ_X, ρ_X) consisting of $X \in \mathcal{C}(a \to b)$, $\lambda_X \in \mathcal{C}(P \otimes X \Rightarrow X)$, and $\rho_X \in \mathcal{C}(X \otimes Q \Rightarrow X)$, again satisfying certain properties. We represent a, b, X, P, Q graphically by

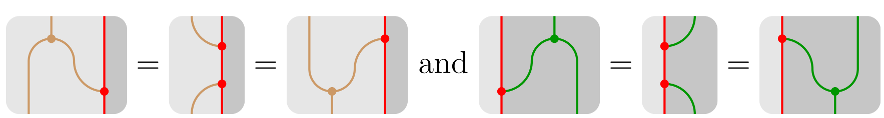


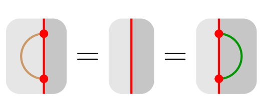
We denote λ_X, ρ_X and $\lambda_X^\dagger, \rho_X^\dagger$ by trivalent vertices:

The bimodule axioms are as follows:

(B1) (associativity) 

(B2) (unitality) 

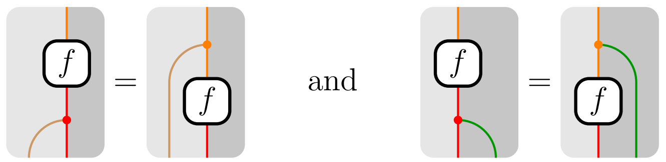
(B3) (Frobenius) 

(B4) (separable) 

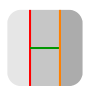
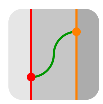
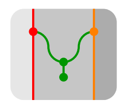
We refer the reader to [CPJP21, Facts 3.16] for various dependencies amongst these axioms.

3.4. Definition. For $\mathcal{C}$ a C*/W* 2-category, its *Q-system completion* is the C*/W* 2-category $\mathsf{QSys}(\mathcal{C})$ whose:

- 0-cells are Q-systems $(Q, m, i) \in \mathcal{C}(b \to b)$,
- 1-cells between Q-systems $P \in \mathcal{C}(a \to a)$ and $Q \in \mathcal{C}(b \to b)$ are (unital Frobenius) bimodules $({}_aX_b, \lambda_X, \rho_X) \in \mathcal{C}(a \to b)$, and
- 2-cells are bimodule intertwiners, i.e., given Q-systems ${}_aP_a, {}_bQ_b$ and $P-Q$ bimodules ${}_aX_b, {}_aY_b$, $\mathsf{QSys}(\mathcal{C})({}_PX_Q \Rightarrow {}_PY_Q)$ is the set of $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ such that



- 1-composition in $\mathsf{QSys}(\mathcal{C})$ is performed by orthogonally splitting the *separability projector*

$$p^Q_{X,Y} :=$$  $:=$  $=$  (3)

  The object ${}_aX \otimes_Q Y_b \in \mathsf{QSys}(\mathcal{C})(P \to R)$ and a $P-R$ bimodular coisometry $u^Q_{X,Y}$: $X \otimes_b Y \to X \otimes_Q Y$, unique up to canonical unitary, such that $p^Q_{X,Y} = (u^Q_{X,Y})^\dagger \star u^Q_{X,Y}$.

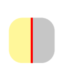
We refer the reader to [CPJP21, §3.2] for the full details that $\mathsf{QSys}(\mathcal{C})$ is a † 2-category, which is C*/W* whenever $\mathcal{C}$ is respectively.

3.5. Remark. As mentioned in passing in [CPJP21, Facts 3.16], for $\mathcal{C}$ a C*/W* 2-category, there is another C*/W* 2-category $\mathsf{Kar}^\dagger(\mathcal{C})$ called the *unitary condensation completion* whose objects are unitary condensation algebras (satisfying (Q1), (Q3), and (Q4), but not necessarily (Q2)), whose 1-morphisms are unitary condensation bimodules (satisfying (B1), (B3), and (B4), but not necessarily (B2)), and whose 2-morphisms are intertwiners. The constructions that follow in §3.7 below for the Q-system completion have obvious analogs for the unitary condensation completion. As such, we include unital constructions, but necessary verification will avoid the use of (Q2) and (B2) whenever possible.

3.6. Notation. We use the graphical notation for $\mathsf{QSys}(\mathcal{C})$ from [CPJP21, §3.3], where shaded regions for Q-systems are denoted by colored regions, but trivial Q-systems are still represented in gray-scale:

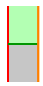
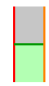
 $= P$ $= Q$ $= 1_a$  $= 1_b$.

If ${}_aP_a, {}_bQ_b \in \mathsf{QSys}(\mathcal{C})$ are Q-systems and $X \in \mathsf{QSys}(\mathcal{C})(P \to Q)$, then X may be also viewed as a $1_a - Q$, $P - 1_b$, and a $1_a - 1_b$ bimodule; we represent these four possibilities by varying the shadings:

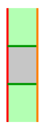
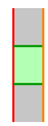
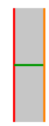
 $= {}_PX_Q$ $= {}_{1_a}X_Q$  $= {}_PX_{1_b}$  $= {}_{1_a}X_{1_b}$.

We use a similar convention for intertwiners of bimodules. We often suppress the external shading when drawing 2-cells in $\mathsf{QSys}(\mathcal{C})$; when we do so, it should be inferred that the diagram/relation depicted holds for any consistent external shading applied to the diagram(s).

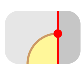
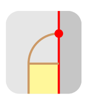
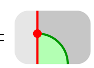
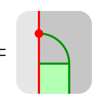
Given $X \in \mathsf{QSys}(\mathcal{C})(P \to Q)$ and $Y \in \mathsf{QSys}(\mathcal{C})(Q \to R)$, we denote the coisometry $u^Q_{X,Y}$ and its adjoint in the graphical calculus of $\mathsf{QSys}(\mathcal{C})$ by

$u^Q_{X,Y} :=$  $: X \otimes_b Y \to X \otimes_Q Y$ and $(u^Q_{X,Y})^\dagger =$  .

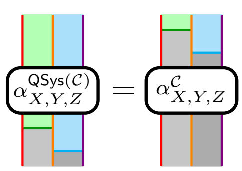
We thus get the following relations:

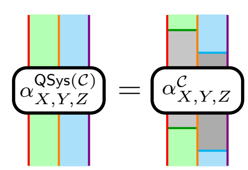
$u^Q_{X,Y} \star (u^Q_{X,Y})^\dagger =$  $=$ $= \mathrm{id}_{X\otimes_Q Y}$ $(u^Q_{X,Y})^\dagger \star u^Q_{X,Y} =$  $=$  $= p^Q_{X,Y}$.

We define canonical unitor trivalent vertices by

$\lambda^P_X =$  $:=$  $= \lambda_X \star (u^P_{P,X})^\dagger$ and $\rho^Q_X =$  $:=$  $= \rho_X \star (u^Q_{X,Q})^\dagger$.

It is straightforward to verify that λ^P_X and ρ^Q_X are unitaries (see [CPJP21, §3.3]). In this graphical notation, the associator of $\mathsf{QSys}(\mathcal{C})$ is uniquely determined by the formula on the left hand side:



$\Longrightarrow$


.

$(X\otimes_Q Y)\otimes_R Z \Rightarrow X \otimes_Q (Y \otimes_R Z)$

3.7. Constructions on 1-morphisms, 2-morphisms, and 3-morphisms in $\mathsf{2Cat}$. For this section, we fix two C*/W* 2-categories $\mathcal{C}, \mathcal{D}$.

3.8. Construction. [CPJP21, Const. 3.29] A † 2-functor $F : \mathcal{C} \to \mathcal{D}$ between C*/W* 2-categories induces a †-2-functor $\mathsf{QSys}(F) : \mathsf{QSys}(\mathcal{C}) \to \mathsf{QSys}(\mathcal{D})$.

- For $({}_bQ_b, m, i) \in \mathsf{QSys}(\mathcal{C})$, we define

$$\mathsf{QSys}(F)({}_bQ_b) := ({}_{F(b)}F(Q)_{F(b)}, F(m) \star F^2_{Q,Q}, F(i) \star F^1_b) \in \mathsf{QSys}(\mathcal{D}).$$

- For $({}_PX_Q, \lambda, \rho) \in \mathsf{QSys}(\mathcal{C})(P \to Q)$, we define

$$\mathsf{QSys}(F)({}_PX_Q) := (F(X), F(\lambda) \star F^2_{P,X}, F(\rho) \star F^2_{X,Q}) \in \mathsf{QSys}(\mathcal{D})(F(P) \to F(Q))$$

- For $f \in \mathsf{QSys}(\mathcal{C})({}_PX_Q \Rightarrow {}_PY_Q)$ we define

$$\mathsf{QSys}(F)(f) := F(f) \in \mathsf{QSys}(\mathcal{D})({}_{F(P)}F(X)_{F(Q)} \Rightarrow {}_{F(P)}F(Y)_{F(Q)}).$$

  Since F is a † 2-functor, $\mathsf{QSys}(F)$ will be as well. Moreover, when $\mathcal{A}, \mathcal{B}$ are W* and $F : \mathcal{A} \to \mathcal{B}$ is normal, so is $\mathsf{QSys}(F)$.

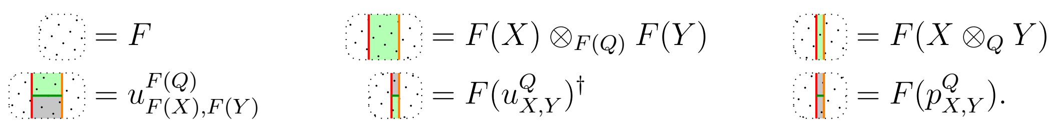
- For ${}_PX_Q \in \mathsf{QSys}(\mathcal{C})(P \to Q)$ and ${}_QY_R \in \mathsf{QSys}(\mathcal{C})(Q \to R)$, we define

$$\mathsf{QSys}(F)^2_{X,Y} := F(u_{X,Y}) \star F^2_{X,Y} \star u^\dagger_{F(X),F(Y)} \in \mathsf{QSys}(\mathcal{D})(F(X) \otimes_{F(Q)} F(Y) \Rightarrow F(X \otimes_Q Y)). \tag{4}$$

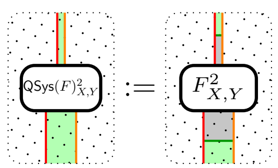
  Finally, for a Q-system $Q \in \mathcal{C}(b \to b)$, we define

$$\mathsf{QSys}(F)^1_{F(Q)} := \mathrm{id} \in \mathsf{QSys}(\mathcal{D})(1_{F(Q)} \Rightarrow F(1_Q)).$$

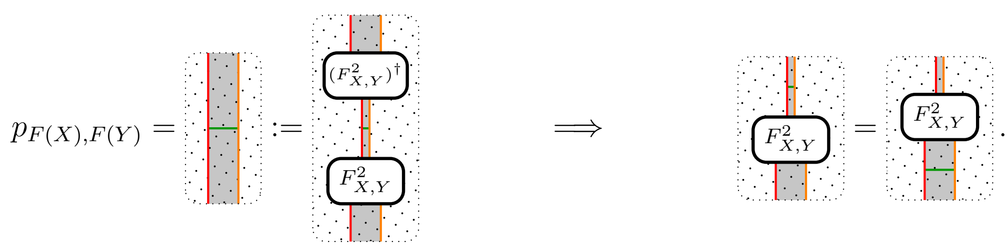
For convenience of the reader, we provide a diagrammatic proof below that $\mathsf{QSys}(F)$ is a † 2-functor. We graphically represent



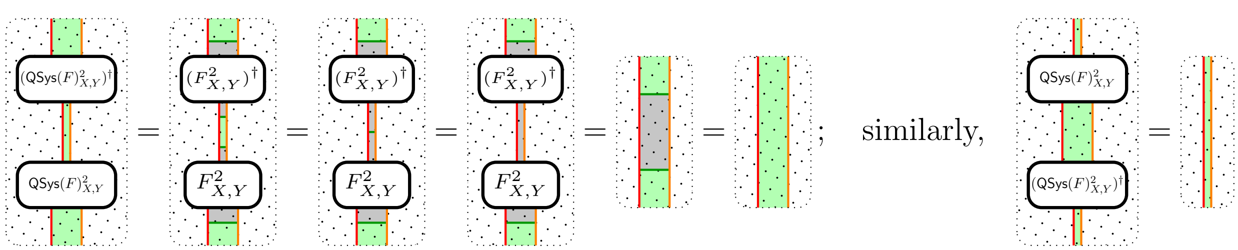
We then define



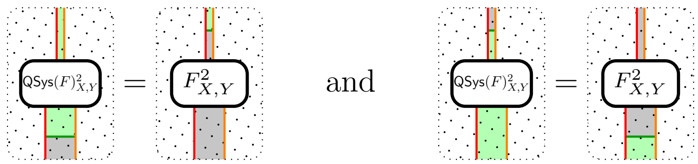
By definition of the separability projector (3) for $F(X) \otimes_{F(Q)} F(Y)$, we have



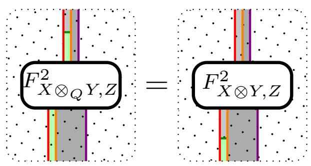
This formula for $p_{F(X),F(Y)}$ immediately implies $\mathsf{QSys}(F)^2_{X,Y}$ is unitary:



Using (4), unitarity of $\mathsf{QSys}(F)^2$, and that u is a coisometry, we have

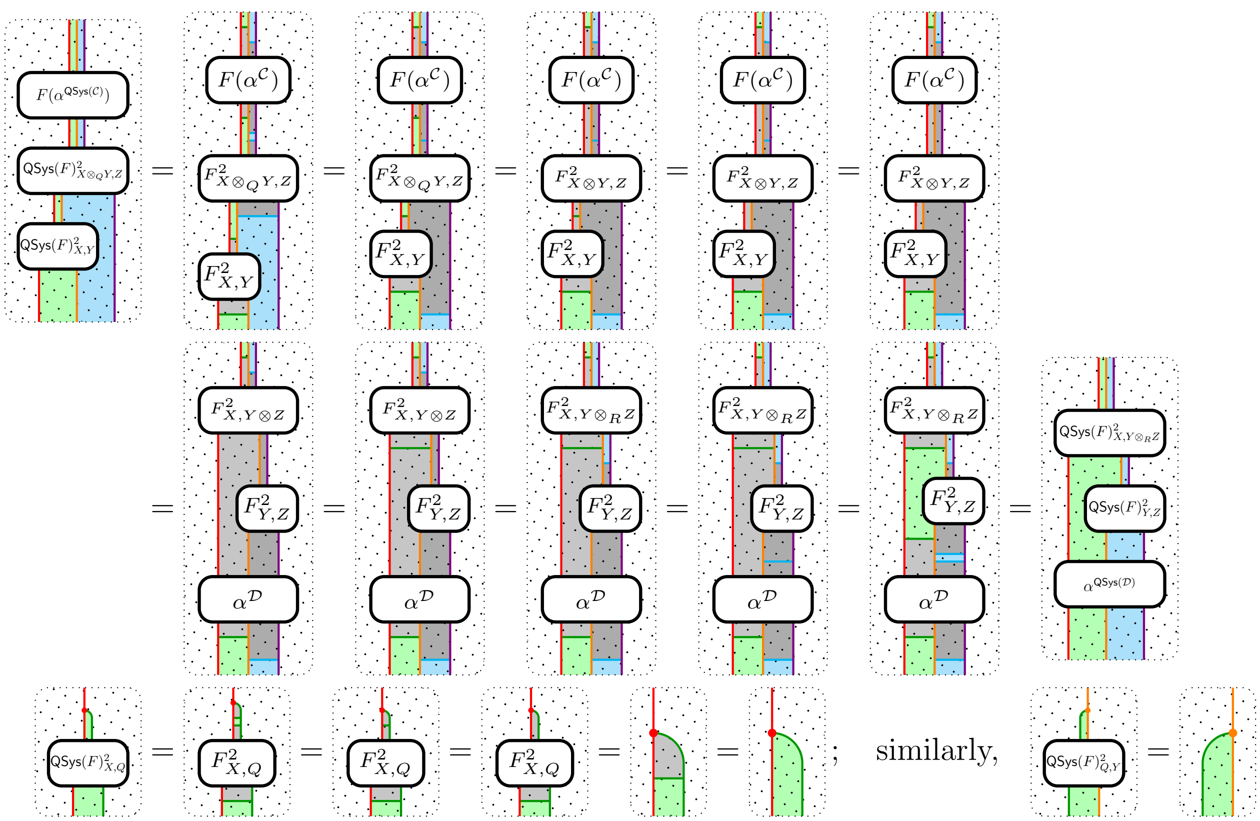


By naturality, we have



$: F(X \otimes_Q Y) \otimes F(Z) \to F(X \otimes Y) \otimes F(Z).$

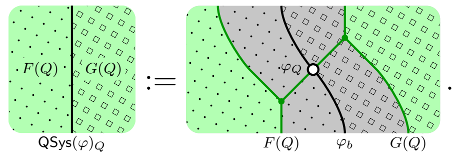
These identities are used to prove the hexagon associativity coherence for $\mathsf{QSys}(F)^2$ and the triangle unit coherences for $\mathsf{QSys}(F)^1$:



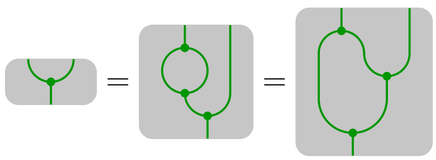
For the rest of this section, we fix two † 2-functors $F, G : \mathcal{C} \to \mathcal{D}$.

3.9. Construction. Given a †-transformation $\varphi : F \Rightarrow G$, we define a †-transformation $\mathsf{QSys}(\varphi) : \mathsf{QSys}(F) \Rightarrow \mathsf{QSys}(G)$. In the diagrams below, we suppress all coherence isomorphisms for F and G.

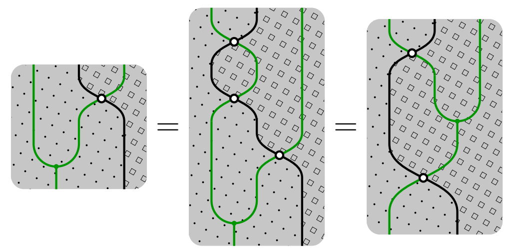
For a Q-system $({}_bQ_b, m, i) \in \mathsf{QSys}(\mathcal{C})$, we define $\mathsf{QSys}(\varphi)_Q$ by orthogonally splitting the orthgonal projection



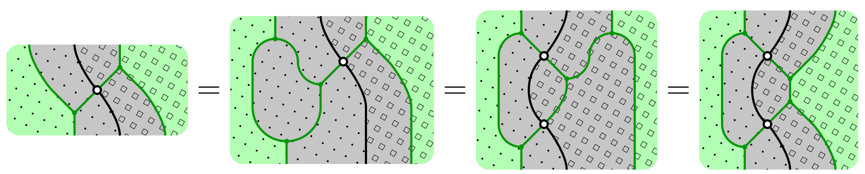
Since



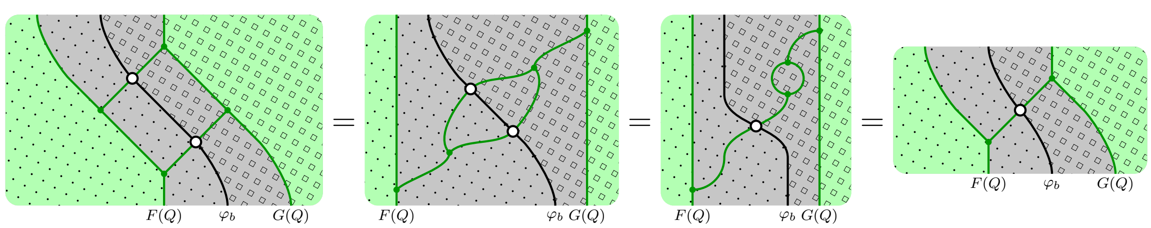
and



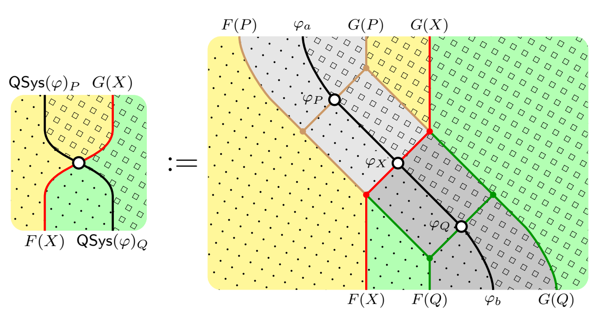
we conclude that $\mathsf{QSys}(\varphi)_Q$ is self-adjoint, as the final diagram below is self-adjoint:



To see that $\mathsf{QSys}(\varphi)_Q$ is an orthogonal projection, we calculate

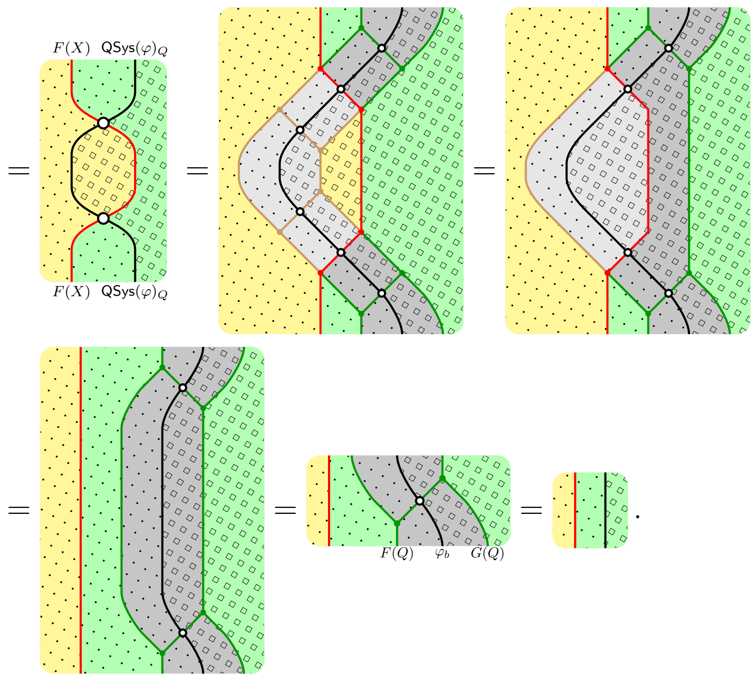


For a 1-cell $({}_PX_Q, \lambda, \rho)$, we define $\mathsf{QSys}(\varphi)_X : F(X) \otimes_{F(Q)} \mathsf{QSys}(\varphi)_Q \Rightarrow \mathsf{QSys}(\varphi)_P \otimes_{G(P)} G(X)$ by

$\mathsf{QSys}(\varphi)_X =$ 

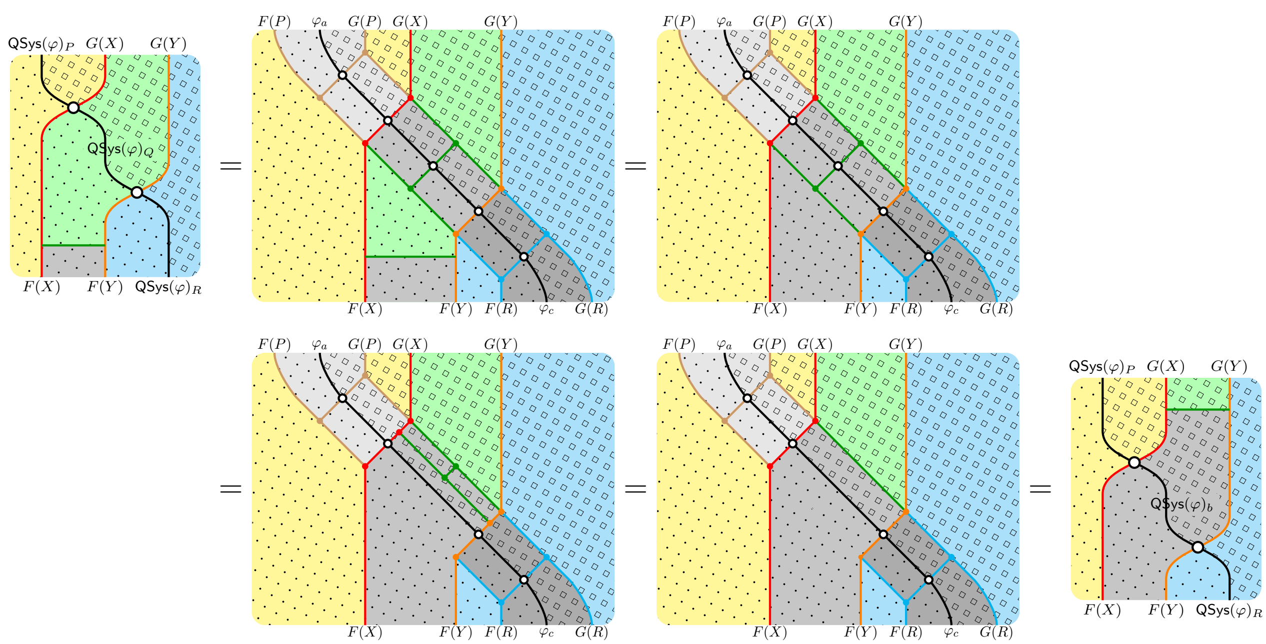
To see that $\mathsf{QSys}(\varphi)_X$ is unitary, we observe

$$\mathsf{QSys}(\varphi)_X^\dagger \star \mathsf{QSys}(\varphi)_X =$$

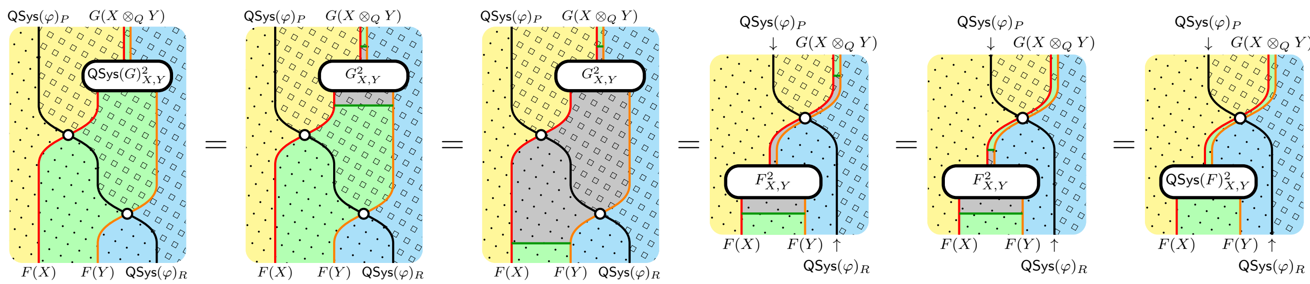


Similarly, $\mathsf{QSys}(\varphi)_X \star \mathsf{QSys}(\varphi)_X^\dagger = 1_{\mathsf{QSys}(\varphi)_P \otimes_{G(P)} G(X)}$.

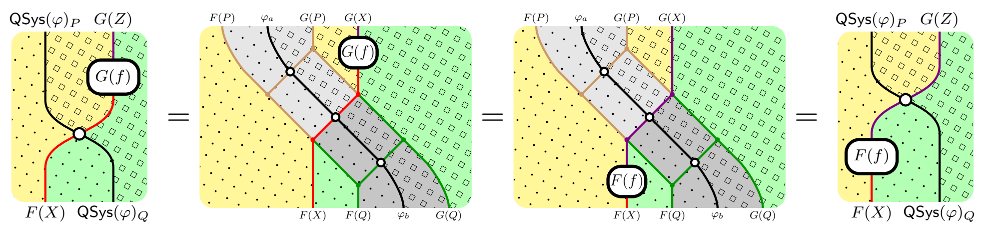
To see that $\mathsf{QSys}(\varphi) : \mathsf{QSys}(F) \Rightarrow \mathsf{QSys}(G)$ is a 2-transformation, we observe



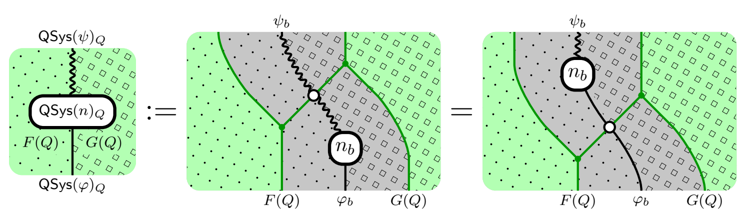
This relation implies the monoidality coherence condition:



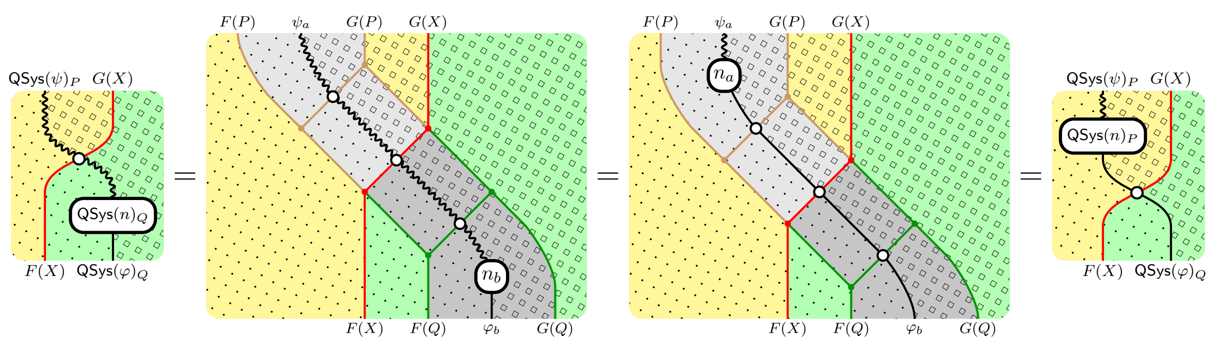
Unitality is checked similarly. Finally, to check naturality, for a 2-cell $f \in \mathcal{C}({}_PX_Q \to {}_PZ_Q)$:



3.10. Construction. Suppose $n : \varphi \Rrightarrow \psi$ is a bounded modification between †-transformations. We define a bounded modification $\mathsf{QSys}(n) : \mathsf{QSys}(\varphi) \Rrightarrow \mathsf{QSys}(\psi)$ as follows. Given a Q-system ${}_bQ_b \in \mathsf{QSys}(\mathcal{C})$, we define

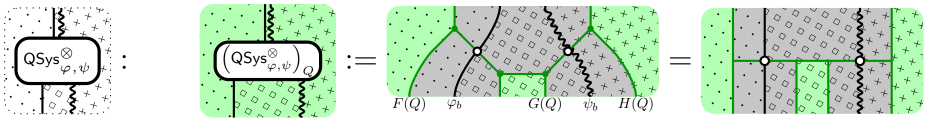


It is clear that $\mathsf{QSys}(n^\dagger) = \mathsf{QSys}(n)^\dagger$. The modification coherence axiom is verified by

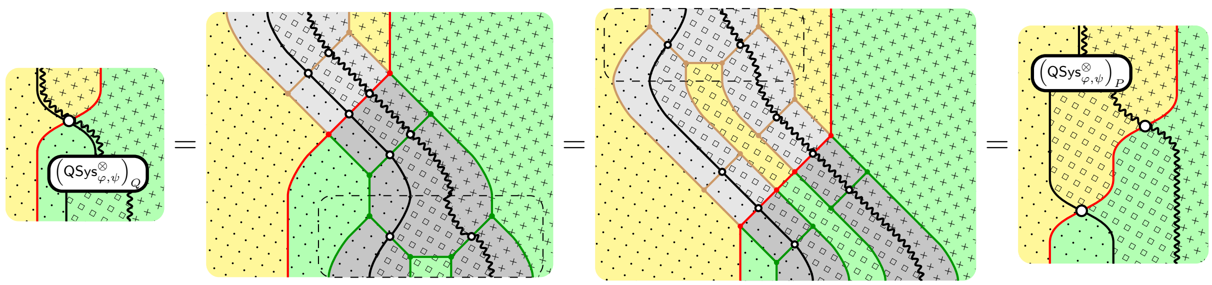


By our construction, it is clear that when $n : \varphi \Rrightarrow \psi$ is invertible, $\mathsf{QSys}(n) : \mathsf{QSys}(\varphi) \Rrightarrow \mathsf{QSys}(\psi)$ is also invertible.

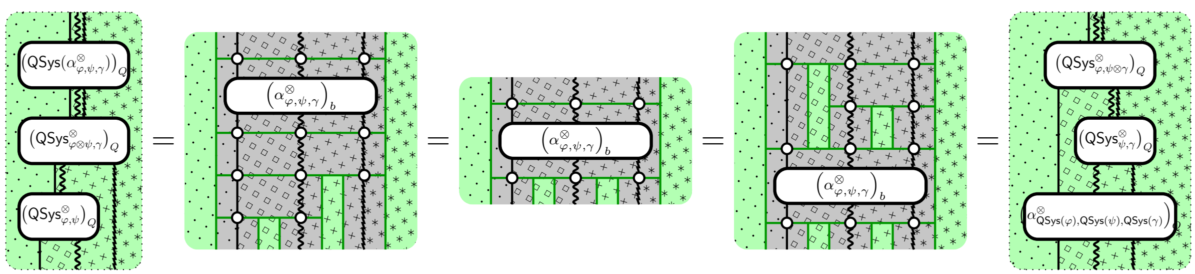
3.11. Construction. Given $\mathcal{A}, \mathcal{B} \in \mathsf{2Cat}$, $F, G, H : \mathcal{A} \to \mathcal{B}$, and $\varphi : F \Rightarrow G$, $\psi : G \Rightarrow H$, we construct $\mathsf{QSys}^{\otimes}_{\varphi,\psi} : \mathsf{QSys}(\varphi) \otimes \mathsf{QSys}(\psi) \Rrightarrow \mathsf{QSys}(\varphi \otimes \psi)$ by



It is straightforward to verify $(\mathsf{QSys}^{\otimes}_{\varphi,\psi})_Q$ is unitary. The following calculation shows $\mathsf{QSys}^{\otimes}_{\varphi,\psi}$ is a modification:



Finally, we check the monoidality coherence axiom for $\mathsf{QSys}^{\otimes}_{\bullet,\bullet}$, and we leave $\mathsf{QSys}^{\otimes}_{\bullet}$ to the reader:



Constructions 3.8, 3.9, 3.10, and 3.11 immediately imply the following proposition.

3.12. Proposition. *QSys as defined above is a † 2-functor on every hom 2-category $\mathsf{Fun}^{\dagger}(\mathcal{A} \to \mathcal{B})$.*

3.13. Lemma. *For $F \in \mathsf{2Cat}(\mathcal{A} \to \mathcal{B})$ and $G \in \mathsf{2Cat}(\mathcal{B} \to \mathcal{C})$, $\mathsf{QSys}(G) \circ \mathsf{QSys}(F) = \mathsf{QSys}(G \circ F)$.*

Proof. By Constructions 2.17 and 3.8, for a 0-cell $Q \in \mathsf{QSys}(\mathcal{A})$,

$$\mathsf{QSys}(G \circ F)(Q) = G(F(Q)) = \mathsf{QSys}(G)(\mathsf{QSys}(F)(Q)) = [\mathsf{QSys}(G) \circ \mathsf{QSys}(F)](Q),$$

for a 1-cell $X \in \mathsf{QSys}(\mathcal{A})(P \to Q)$,

$$\mathsf{QSys}(G \circ F)(X) = G(F(X)) = [\mathsf{QSys}(G) \circ \mathsf{QSys}(F)](X),$$

and for a 2-cell $f \in \mathsf{QSys}(\mathcal{A})(X \Rightarrow Y)$,

$$\mathsf{QSys}(G \circ F)(f) = G(F(f)) = [\mathsf{QSys}(G) \circ \mathsf{QSys}(F)](f).$$

For a 0-cell $Q \in \mathsf{QSys}(\mathcal{A})$, $\mathsf{QSys}(F)^1_Q = \mathrm{id}$, so $\mathsf{QSys}(G \circ F)^1_Q = \mathrm{id} = (\mathsf{QSys}(G) \circ \mathsf{QSys}(F))^1_Q$. For 1-cells $X \in \mathsf{QSys}(\mathcal{A})(P \to Q)$ and $Y \in \mathsf{QSys}(\mathcal{A})(Q \to R)$, we have

$$
\begin{aligned}
&(\mathsf{QSys}(G) \circ \mathsf{QSys}(F))^2_{X,Y} \\
&\quad = \mathsf{QSys}(G)(\mathsf{QSys}(F)^2_{X,Y}) \star \mathsf{QSys}(G)^2_{\mathsf{QSys}(F)(X),\mathsf{QSys}(F)(Y)} \\
&\quad = \mathsf{QSys}(G)(F(u^Q_{X,Y}) \star F^2_{X,Y} \star (u^{F(Q)}_{F(X),F(Y)})^\dagger) \star \mathsf{QSys}(G)^2_{F(X),F(Y)} \\
&\quad = G(F(u^Q_{X,Y})) \star G(F^2_{X,Y}) \star G((u^{F(Q)}_{F(X),F(Y)})^\dagger) \star G(u^{F(Q)}_{F(X),F(Y)}) \star G^2_{F(X),F(Y)} \star (u^{G(F(Q))}_{G(F(X)),G(F(Y))})^\dagger \\
&\quad = G(F(u^Q_{X,Y})) \star G(F^2_{X,Y}) \star G^2_{F(X),F(Y)} \star (u^{G(F(Q))}_{G(F(X)),G(F(Y))})^\dagger \\
&\quad = (G \circ F)(u^Q_{X,Y}) \star (G \circ F)^2_{X,Y} \star (u^{(G\circ F)(Q)}_{(G\circ F)(X),(G\circ F)(Y)})^\dagger \\
&\quad = \mathsf{QSys}(G \circ F)^2_{X,Y}.
\end{aligned}
$$

Hence $\mathsf{QSys}(G) \circ \mathsf{QSys}(F) = \mathsf{QSys}(G \circ F)$ as claimed. ■

PROOF PROOF OF THM. 1.1. By Lemma 3.13, we may define each $\mathsf{QSys}^\circ_{G,F} : \mathsf{QSys}(G) \circ \mathsf{QSys}(F) \Rightarrow \mathsf{QSys}(G \circ F)$ to be the identity transformation, and we may define each 1-associator modification $\omega^\circ_{H,G,F}$ to be the identity modification, as well as each unitor modification ℓ°_F and r°_G. Theorem 1.1 follows immediately, i.e., QSys is a † 3-endofunctor. ■

3.14. REMARK. The proof of Thm. 1.1 above also shows that $\mathsf{Kar}^\dagger$ is a † 3-endofunctor.

3.15. REMARK. Since 1-composition is strict in 2Cat, 2-categories and 2-functors form a 1-category where we forget all transformations and modifications. (Observe we have *not* truncated, as this would identify equivalent 2-functors.) Lemma 3.13 shows that QSys is a functor on this 1-category.

3.16. REMARK. It was pointed out to us by Thibault Décoppet and David Reutter that our † 3-endofunctor QSys on C*/W* 2Cat should be left 3-adjoint to the inclusion of the full 3-subcategory on the Q-system complete C*/W* 2-categories. We will not prove this here as it would take us too far afield. We note, however, that this would endow QSys with the structure of a symmetric lax monoidal † 3-endofunctor on C*/W* 2Cat, which we expect is strong monoidal as a 3-functor from C*/W* 2Cat to the Q-system complete C*/W* 2-categories, where the tensor product is the Q-system completion of the ordinary tensor product.

At this time, we are unaware of a definition of a symmetric monoidal structure on an algebraic tricategory, as well as a definition of symmetric (lax) monoidal 3-functor on an algebraic tricategory in sense of [Gur13]. The closest thing we are aware of is the notion of an internal bicategory [DH12]; we caution the reader that the tricategories in this latter article are expected but not known to be equivalent to those in [Gur13]. We leave this exploration to the interested reader.

4. Universal property of Q-system completion

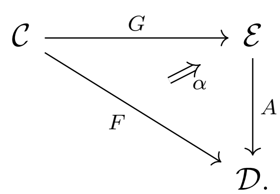
In this section, we give the strongest possible universal property which is satisfied by Q-system completion. Namely, we prove Theorem 1.2, which states that the *lift 2-category* of a † 2-functor $F : \mathcal{C} \to \mathcal{D}$ from a C*/W* 2-category $\mathcal{C}$ into a Q-system complete C*/W* 2-category $\mathcal{D}$ is (-2)-*truncated*, i.e., equivalent to a point. We now define the necessary terms to prove this theorem, and we explain the proof strategy from [JMPP19, §3.1].

4.1. Lift categories and homotopy fibers.

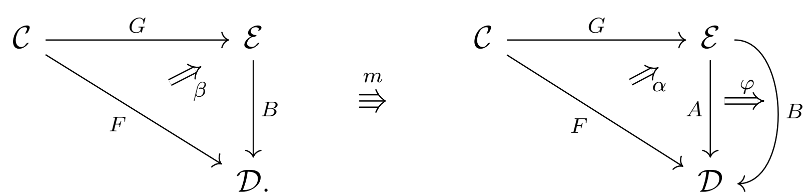
4.2. Definition. Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are C*/W* 2-categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ are † 2-functors. The *lift 2-category* of F along G is the *homotopy fiber* 2-category of the pre-composition 2-functor $- \circ G : \mathsf{Fun}^\dagger(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ at $F \in \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$. We remind the reader that the definition of $- \circ -$ in $2\mathsf{Cat}$ is detailed in Construction 2.17 above.

4.3. Remark. We now further unpack Defintion 4.2. The lift 2-category of F along G has:

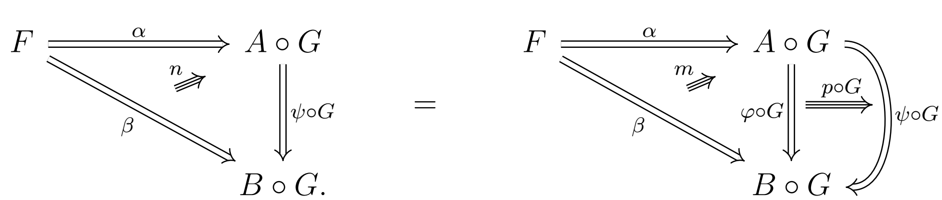
- <u>objects:</u> pairs (A, α), where $A : \mathcal{E} \to \mathcal{D}$ is a † 2-functor and $\alpha : F \Rightarrow A \circ G$ is a unitary 2-transformation.



- <u>1-morphisms:</u> pairs $(\varphi, m) : (A, \alpha) \to (B, \beta)$, where $\varphi : A \Rightarrow B$ is a † 2-transformation and $m : \beta \Rrightarrow \alpha \otimes (\varphi \circ G)$ is a unitary 2-modification:



- <u>2-morphisms:</u> $p : (\varphi, m) \Rightarrow (\psi, n)$, where $p : \varphi \Rrightarrow \psi$ is a † 2-modification such that



Recall that for a 2-category $\mathcal{C}$, its *core* is the 2-subcategory $\mathrm{core}(\mathcal{C})$ with only invertible 1-cells and invertible 2-cells. When $\mathcal{C}$ is C*/W*, its *unitary core* $\mathrm{core}^\dagger(\mathcal{C})$ is the 2-subcategory of $\mathrm{core}(\mathcal{C})$ with only unitary 2-cells. In a C*/W* 2-category, by polar decomposition for invertible 2-cells, there exists an invertible 2-cell $\mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ if and only if there exists a unitary 2-cell, so the connectivity of $\mathrm{core}(\mathcal{C})$ and $\mathrm{core}^\dagger(\mathcal{C})$ agree.

We pass to cores in order to take advantage of the notion of k-truncated 2-functor between 2-groupoids from [JMPP19, §3.1].

4.4. Definition. [cf. [JMPP19, Def. 3.3]] Suppose $\mathcal{C}, \mathcal{D}$ are 2-groupoids and $U : \mathcal{C} \to \mathcal{D}$ is a 2-functor. We call U *k-truncated* or *$(k+1)$-monic* [BS10, §5.5] if U is:

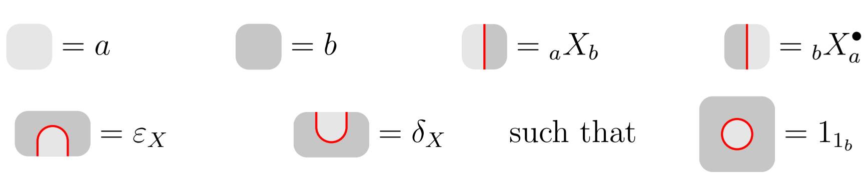
- $k = 2$: (no condition)
- $k = 1$: faithful on 2-cells
- $k = 0$: fully faithful on 2-cells
- $k = -1$: an equivalence on hom-categories
- $k = -2$: an equivalence of 2-categories.

The following proposition connects the notions of a k-truncated 2-functor between 2-groupoids and its homotopy fibers.

4.5. Proposition. [cf. [JMPP19, Prop. 3.4]] *Suppose $\mathcal{C}, \mathcal{D}$ are 2-groupoids, and $U : \mathcal{C} \to \mathcal{D}$ is a 2-functor. For every $-2 \leq k \leq 2$, U is k-truncated if and only if at each object $d \in \mathcal{D}$, the homotopy fiber $\mathsf{hoFib}_d(U)$ is k-truncated as an 2-groupoid, i.e., homotopy equivalent to a k-groupoid.*[1]

4.6. Dominance and truncation. Observe that $-\circ G : \mathsf{Fun}^\dagger(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ restricts to a † 2-functor $-\circ G : \mathrm{core}^\dagger(\mathsf{Fun}^\dagger(\mathcal{E} \to \mathcal{D})) \to \mathrm{core}^\dagger(\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D}))$. Hence, in order to apply Proposition 4.5 to the † 2-functor $-\circ G$, we need (essential) surjectivity conditions on $-\circ G$. (Being faithful on 2-morphisms is being surjective on equalities between 2-morphisms.) A suitable notion of (essential) surjectivity for a linear 2-functor is *dominance*, which we define via the notion of *condensation* in a 2-category [GJF19].

4.7. Definition. Suppose $\mathcal{C}$ is a 2-category and $a, b \in \mathcal{C}$ are 0-cells. A *condensation* $X : a \twoheadrightarrow b$ consists of 1-cells ${}_aX_b, {}_bX^\bullet_a$ and 2-cells $\varepsilon_X : {}_bX^\bullet \otimes_a X_b \to 1_b$ and $\delta_X : 1_b \to {}_bX^\bullet \otimes_a X_b$ such that $\varepsilon_X \star \delta_X = 1_{1_b}$. Graphically, we denote $X : a \twoheadrightarrow b$ by



When $\mathcal{C}$ is C*/W*, a condensation $X : a \twoheadrightarrow b$ is called a *dagger condensation* if $\delta_X = \varepsilon_X^\dagger$.

[1] We use 'negative categorical thinking' [BS10] when $k = -2, -1, 0$. That is, a 0-groupoid is a set, a (-1)-groupoid is either a point or the empty set, and a (-2)-groupoid is a point.

4.8. Definition. A 2-functor $G : \mathcal{C} \to \mathcal{E}$ is called:

- *0-dominant* if for all $e \in \mathcal{E}$, there is a condensation $G(c) \twoheadrightarrow e$ for some $c \in \mathcal{C}$,
- *locally dominant* if every hom functor $G_{a \to b} : \mathcal{C}(a \to b) \to \mathcal{E}(G(a) \to G(b))$ is dominant as a linear functor, and
- *dominant* if G is both 0-dominant and locally dominant.

When G is a † 2-functor between C*/W* 2-categories, we call G

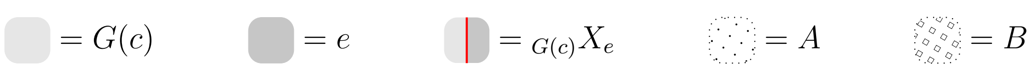
- *orthogonally* 0-dominant if for all $e \in \mathcal{E}$, there is a dagger condensation $G(c) \twoheadrightarrow e$ for some $c \in \mathcal{C}$,
- locally *orthogonally* dominant if every hom functor $G_{a \to b}$ is orthogonally dominant as a linear †-functor, i.e., every 1-cell ${}_{G(a)}Y_{G(b)} \in \mathcal{E}$ is unitarily isomorphic to an orthogonal direct summand of some ${}_{G(a)}G(X)_{G(b)}$, and
- *orthogonally* dominant if G is both orthogonally 0-dominant and locally dominant.

4.9. Remark. There is an analogous notion of k-dominance for an n-functor G between n-categories for $0 \leq k \leq n-1$: every k-morphism between two parallel $k-1$ morphisms in the image of G should admit a condensation from a source in the image of G.

For the propositions in this section, we work with algebraic 2-categories and 2-functors, and we make particular comments about the C*/W* setting.

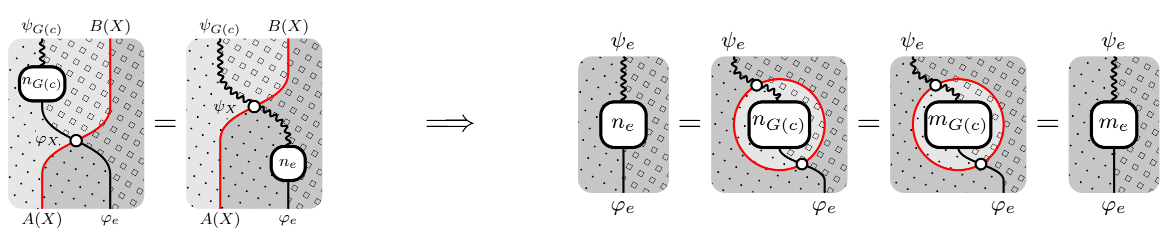
4.10. Proposition. *If a 2-functor $G : \mathcal{C} \to \mathcal{E}$ is 0-dominant, then the 2-functor $- \circ G$: $\mathsf{Fun}(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is faithful on 2-morphisms. In the* C*/W* *setting, if $G : \mathcal{C} \to \mathcal{E}$ is orthogonally 0-dominant, then $- \circ G : \mathsf{Fun}^\dagger(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ is faithful on 2-morphisms.*

Proof. Let $A, B \in \mathsf{Fun}(\mathcal{E} \to \mathcal{D})$ and $\varphi, \psi : A \Rightarrow B$. Suppose $m, n : \varphi \Rrightarrow \psi$ and $m \circ G = n \circ G$. We show $m = n$. For each $e \in \mathcal{E}$, there exists a 0-cell $c \in \mathcal{C}$ and a condensation $X : G(c) \twoheadrightarrow e$. We denote $G(c), e \in \mathcal{D}$, ${}_{G(c)}X_e \in \mathcal{D}(G(c) \to e)$, and the functors A, B graphically by



(5)

The modification axiom implies the following:

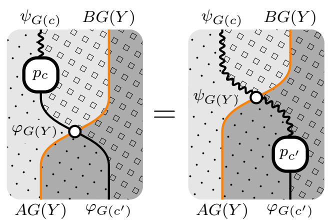


Hence $m = n$, as claimed. ■

4.11. Proposition. *If a 2-functor $G : \mathcal{C} \to \mathcal{E}$ is dominant, then the 2-functor $- \circ G$: $\mathsf{Fun}(\mathcal{C} \to \mathcal{E}) \to \mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is fully faithful on 2-morphisms. An analogous statement holds in the* C*/W* *setting.*

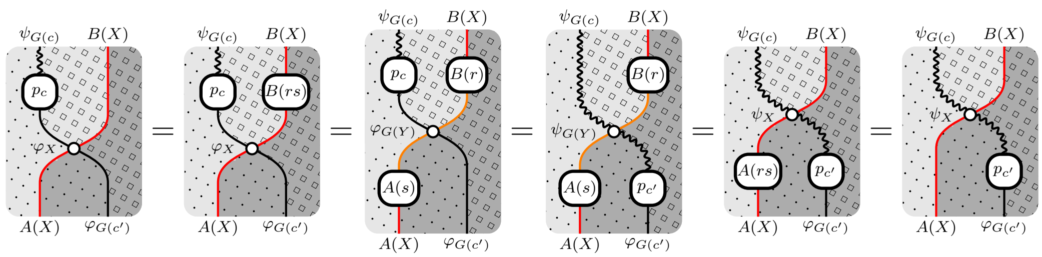
Proof. It suffices to show $- \circ G$ is full on 2-morphisms. Suppose $A, B \in \mathsf{Fun}(\mathcal{E} \to \mathcal{D})$, $\varphi, \psi : A \Rightarrow B$, and $p : \varphi \circ G \Rrightarrow \psi \circ G$. We show there exists $n : \varphi \Rrightarrow \psi$ such that $p = n \circ G$.

First, for each 1-cell $X \in \mathcal{E}(G(c) \to G(c'))$, there exists a 1-cell $Y \in \mathcal{C}(c \to c')$ such that $G(Y) \underset{s}{\overset{r}{\rightleftarrows}} X$ is a retract, i.e., $rs = 1_X$. Since $p : \varphi \circ G \Rrightarrow \psi \circ G$ is a 2-modification, building on our graphical conventions (5),



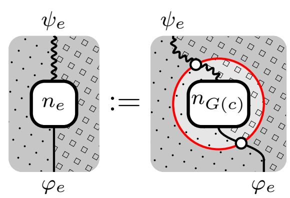
where  $= G(c')$  $= {}_{G(c)}G(Y)_{G(c')}$.

This implies that for *any* $X \in \mathcal{E}(G(c) \to G(c'))$ (and not just 1-cells in the image of G!),

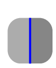
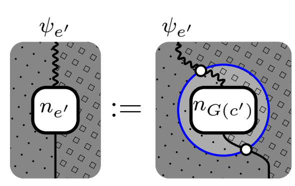


(6)

Next we construct $n : \varphi \Rrightarrow \psi$ such that $p = n \circ G$. For each $c \in \mathcal{C}$, we define $n_{G(c)} := p_c$ so that $p_c = (n \circ G)_c$, and $p = n \circ G$, provided we can extend n to a modification. For each $e \in \mathcal{E}$, there exists a 0-cell $c \in \mathcal{C}$ and a condensation $X : G(c) \twoheadrightarrow e$. We define n_e as follows

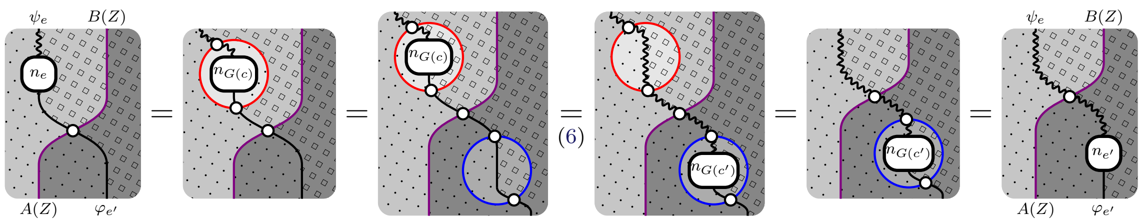


We prove n is a 2-modification $\varphi \Rrightarrow \psi$. Suppose $e' \in \mathcal{E}$ is a 0-cell and $Z \in \mathcal{E}(e \to e')$ is a 1-cell. Let $X' : G(c') \twoheadrightarrow e'$ be a condensation for some 0-cell $c' \in \mathcal{C}$. Using the graphical conventions

 $= G(c')$  $= e'$  $= X'$ 

 $= Z$,

we see that



In the third equality above, we used the fact that $X \otimes_e Z \otimes_{e'} (X')^\bullet \in \mathcal{E}(G(c) \to G(c'))$ to apply (6). This completes the proof. ∎

4.12. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. We begin by recalling the construction of the canonical inclusion $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{QSys}(\mathcal{C})$.

4.13. Construction. [CPJP21, Const. 3.24] For each $\mathcal{A} \in 2\mathsf{Cat}$, there is a canonical inclusion strict † 2-functor $\iota_{\mathcal{A}} : \mathcal{A} \to \mathsf{QSys}(\mathcal{A})$ defined as follows:

- For $a \in \mathcal{A}$, $a \mapsto 1_a$, the trivial Q-system.
- For ${}_aX_b \in \mathcal{A}(a \to b)$, X is a separable $1_a - 1_b$ bimodule, so X maps to itself.
- For $f \in \mathcal{A}(X \Rightarrow Y)$, f is automatically $1_a - 1_b$ bimodular, so f maps to itself.

4.14. Construction. Suppose $F \in \mathsf{Fun}^\dagger(\mathcal{A} \to \mathcal{B})$. We construct an invertible transformation $\psi^F : \iota_{\mathcal{B}} \circ F \Rightarrow \mathsf{QSys}(F) \circ \iota_{\mathcal{A}}$.

By Constructions 3.8 and 4.13, for a 0-cell $b \in \mathcal{A}$, we have

$$(\iota_{\mathcal{B}} \circ F)(b) = \iota_{\mathcal{B}}(F(b)) = 1_{F(b)} \qquad \text{and} \qquad (\mathsf{QSys}(F) \circ \iota_{\mathcal{A}})(b) = \mathsf{QSys}(F)(1_b) = F(1_b).$$

For a 1-cell $X \in \mathsf{QSys}(\mathcal{A})(P \to Q)$, we have an equality

$$(\iota_{\mathcal{B}} \circ F)(X) = \iota_{\mathcal{B}}(F(X)) = F(X) = \mathsf{QSys}(F)(X) = (\mathsf{QSys}(F) \circ \iota_{\mathcal{A}})(X),$$

as well as for a 2-cell $f \in \mathsf{QSys}(\mathcal{A})(X \Rightarrow X')$:

$$(\iota_{\mathcal{B}} \circ F)(f) = \iota_{\mathcal{B}}(F(f)) = F(f) = \mathsf{QSys}(F)(f) = (\mathsf{QSys}(F) \circ \iota_{\mathcal{A}})(f).$$

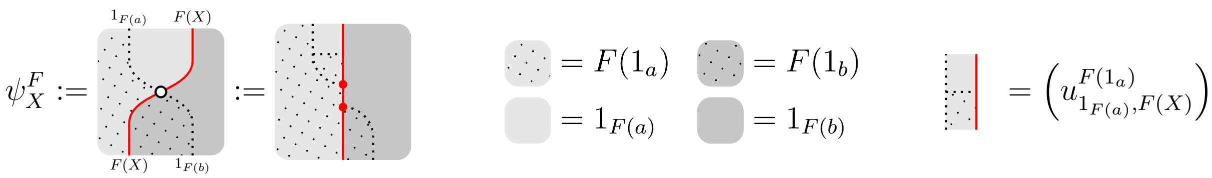
Now $F(1_b)$ is equivalent to the trivial Q-system $1_{F(b)}$, and thus for every $X \in \mathcal{A}(a \to b)$,

$$u^{F(1_b)}_{F(X), 1_{F(b)}} : F(X) \otimes_{1_{F(b)}} 1_{F(b)} \Rightarrow F(X) \otimes_{F(1_b)} 1_{F(b)}$$

from (3.6) is unitary; similarly, $u^{F(1_a)}_{1_{F(a)}, F(X)}$ is a unitary. We define:

- For 0-cell $a, b \in \mathcal{A}$ and 1-cell $X \in \mathcal{A}(a \to b)$, we define $\psi^F_b := 1_{F(b)}$ as an $F(1_b) - 1_{F(b)}$ bimodule, which is clearly invertible.

- For ${}_aX_b \in \mathcal{A}$, we define



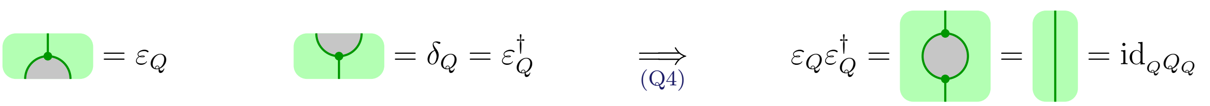
  Clearly ψ^F_X is unitary.

We leave the verification that ψ^F is a 2-transformation to the reader.

Suppose now $\mathcal{C}, \mathcal{D}$ are C*/W* 2-categories with $\mathcal{D}$ Q-system complete. We apply the propositions from §4.6 in the case that $\mathcal{E} = \mathsf{QSys}(\mathcal{C})$ and $G = \iota_\mathcal{C}$.

4.15. Lemma. *$\iota_\mathcal{C}$ is dominant.*

Proof. For each 0-cell/Q-system ${}_bQ_b \in \mathsf{QSys}(\mathcal{C})$ where $b \in \mathcal{C}$, $Q : \iota_\mathcal{C}(b) = 1_b \twoheadrightarrow Q$ is a dagger condensation when equipped with the 1-cells ${}_bQ_Q =$ , ${}_QQ_b^\bullet := {}_QQ_b =$ , and the 2-cells



The result now follows as $\iota_\mathcal{C}$ is a local equivalence on hom categories by definition. ∎

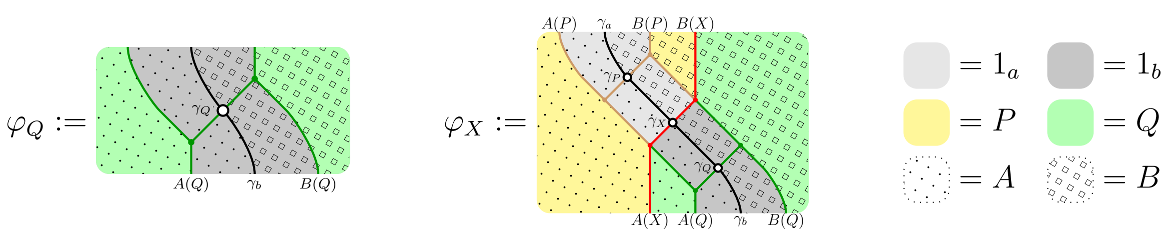
4.16. Proposition. $-\circ\iota_\mathcal{C} : \mathsf{Fun}^\dagger(\mathsf{QSys}(\mathcal{C}) \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ *is a dagger equivalence on hom categories.*

Proof. By Lemma 4.15, $\iota_\mathcal{C}$ is dominant, so by Proposition 4.11, $-\circ\iota_\mathcal{C}$ is fully faithful on 2-morphism. To prove $-\circ\iota_\mathcal{C}$ is a dagger equivalence on hom categories, it remains to prove $-\circ\iota_\mathcal{C}$ is unitarily essentially surjective on 1-morphisms, i.e., for all $A, B \in \mathsf{Fun}^\dagger(\mathsf{QSys}(\mathcal{C}) \to \mathcal{D})$ and each 1-morphism $\gamma : A \circ \iota_\mathcal{C} \Rightarrow B \circ \iota_\mathcal{C}$, there exists $\varphi : A \Rightarrow B$ such that $\gamma \cong \varphi \circ \iota_\mathcal{C}$.

For 0-cells/Q-systems ${}_aP_a, {}_bQ_b \in \mathsf{QSys}(\mathcal{C})$ and a 1-cell ${}_PX_Q \in \mathsf{QSys}(\mathcal{C})(P \to Q)$, we define $\varphi_Q \in \mathcal{D}(A(Q) \to B(Q))$ and

$$\varphi_X \in \mathcal{D}({}_{A(P)}A(X) \otimes_{A(Q)} \varphi_Q{}_{B(Q)} \Rightarrow {}_{A(P)}\varphi_P \otimes_{B(P)} B(X)_{B(Q)})$$

by

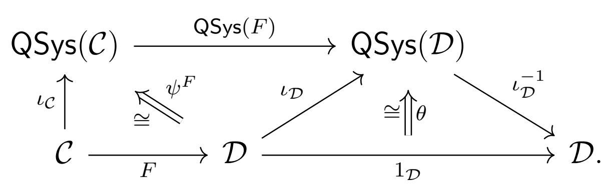


Then for each 0-cell $b \in \mathcal{C}$, and 1-cell ${}_aX_b \in \mathcal{C}(a \to b)$, by Construction 2.17,

$$(\varphi \circ \iota_{\mathcal{C}})_b = \varphi_{\iota_{\mathcal{C}}(b)} = \varphi_{1_b} = \gamma_b \qquad \text{and} \qquad (\varphi \circ \iota_{\mathcal{C}})_X = \varphi_{\iota_{\mathcal{C}}(X)} = \varphi_X = \gamma_X$$

where the latter is viewed as $1_a - 1_b$ bimodular. Therefore $\varphi \circ \iota_{\mathcal{C}} \cong \gamma$ as desired, so $- \circ \iota_{\mathcal{C}}$ is gives a dagger functor on hom 1-categories whose underlying functor is an equivalence. Since $\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ is C*, $- \circ \iota_{\mathcal{C}}$ is a dagger equivalence on hom 1-categories by polar decomposition as discussed in Remark 2.12. ∎

Proof of Theorem 1.2. By Propositions 4.5 and 4.16, $- \circ \iota_{\mathcal{C}}$ is (-1)-truncated when restricted to unitary cores, i.e., the homotopy fiber at each $F \in \mathrm{core}^\dagger(\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D}))$ is either empty or equivalent to a point. By Constructions 3.8 and 4.14, the homotopy fiber of $- \circ \iota_{\mathcal{C}}$ is non-empty at each $F \in \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$. Indeed, since $\mathcal{D}$ is Q-system complete, $\iota_{\mathcal{D}}$ is invertible, so there exists a † 2-functor $\iota_{\mathcal{D}}^{-1} : \mathsf{QSys}(\mathcal{D}) \to \mathcal{D}$ together with an invertible † 2-transformation $\theta_{\mathcal{D}} : 1_{\mathcal{D}} \Rightarrow \iota_{\mathcal{D}}^{-1} \circ \iota_{\mathcal{D}}$. Thus $\iota_{\mathcal{D}}^{-1} \circ \mathsf{QSys}(F)$ provides the desired lift together with the composite invertible transformation



(7)

Thus the homotopy fiber of $- \circ \iota_{\mathcal{C}}$ at each $F \in \mathrm{core}^\dagger(\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D}))$ is equivalent to a point. By Proposition 4.5, $- \circ \iota_{\mathcal{C}}$ is (-2)-truncated when restricted to unitary cores. This implies $- \circ \iota_{\mathcal{C}} : \mathsf{Fun}^\dagger(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ is essentially surjective on objects. Again by Remark 2.12, Proposition 4.16, and [JY20, Thm. 7.4.1], $- \circ \iota_{\mathcal{C}} : \mathsf{Fun}^\dagger(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ is a †-equivalence of C*/W* 2-categories. ∎

4.17. Remark. Observe that we did not really need to pass to (unitary) cores, nor use Proposition 4.5. Indeed, $- \circ \iota_{\mathcal{C}}$ is an equivalence on hom categories by Proposition 4.16 and essentially surjective on objects by (7), and thus an equivalence by [JY20, Thm. 7.4.1] and Remark 2.12.

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