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ON THE TERNARY COMMUTATOR, I: EXACT MAL'TSEV CATEGORIES

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ABSTRACT. In this first article on the Bulatov commutator, we introduce a ternary commutator of equivalence relations in the context of an exact Mal'tsev category with binary coproducts. We prove that, for Mal'tsev varieties, our notion is a particular case (where $n = 3$) of the n-fold commutator introduced (originally in the context of Mal'tsev algebras) by A. Bulatov. We study its basic stability properties as well as the relationship with the (binary) Smith-Pedicchio commutator.

In a forthcoming second article, we restrict the context to algebraically coherent semi-abelian categories, where we prove that the commutator introduced here corresponds to the ternary Higgins commutator of M. Hartl and the second author.

1. Introduction

In his study of *polynomially inequivalent algebras*, A. Bulatov introduced in [11] a *higher-order (n-ary) commutator operator* of congruences in Mal'tsev algebras, which extends the binary Smith commutator [30] and is based on a generalisation of the so-called *term condition*. In the article [1], this higher-order commutator theory has been further developed by E. Aichinger and N. Mudrinski in the context of Mal'tsev varieties. In their approach, they found analogues for the higher-order commutator of certain properties known to be valid for the (binary) Smith commutator such as monotonicity, stability with respect to joins, stability with respect to restriction, etc. In [26], J. Opršal introduced a relational description of the higher-order commutator in Mal'tsev varieties. He studied the connection between the term condition and a certain n-fold relation, called the *algebra of* 2^n*-matrices*, which can be seen as a higher-order version of the double relation $\Delta_{R,S}$ (from [30], written in M. C. Pedicchio's notation [27]; in the present article, we write $\Delta(R, S)$). From it, he derived properties of the higher-order commutator.

The theory of higher commutators has recently been extended to varieties that are no longer Mal'tsev. In [23], A. Moorhead used the term condition as a basis for higher-order commutator theory in congruence modular varieties. In [24] he introduced a concept of higher centrality based on matrix constructions, which led to a connection with the term condition in congruence modular varieties and a characterisation of the ternary

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$$
\begin{tikzcd}
 & \Delta(R,S,T) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & \Delta(S,T) \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] \\
\Delta(R,S) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & S \arrow[dd, shift right, "s_1"'] \arrow[dd, shift left, "s_2"] & \\
 & \Delta(R,T) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, shift left] \arrow[dl, shift right] & & T \arrow[dl, shift right, "t_1"'] \arrow[dl, shift left, "t_2"] \\
R \arrow[rr, shift left, "r_1"] \arrow[rr, shift right, "r_2"'] & & X &
\end{tikzcd}
$$

Figure 1: The 3-fold Δ-equivalence relation $\Delta(R,S,T)$

commutator in terms of a three-fold relation denoted $\Delta_{R,S,T}$ which extends the double relation $\Delta_{R,S}$. Further development of higher commutator properties outside congruence modular varieties was made in [31] by A. Wires. Note that several ways of extending centrality of two equivalence relations to more than two equivalence relations exist in the literature that happen to be equivalent in the Mal'tsev context—see for instance [25].

In this article we make a categorical analysis of what we would call the *Bulatov commutator*, extending the work of M. C. Pedicchio [27] and results of F. Borceux, D. Bourn and M. Gran [7, 3] in the binary case. In parallel with A. Moorhead's work, we introduce a construction, valid in the context of exact Mal'tsev categories with binary coproducts, of a three-fold equivalence relation $\Delta(R,S,T)$ based on Pedicchio's $\Delta(R,S)$. In turn, we introduce 3-*fold* Δ-*equivalence relations*, the concept of a 3-*dimensional connector*, and a *ternary Bulatov commutator*.

Given three equivalence relations R, S and T on an object X, the 3-fold Δ-equivalence relation $\Delta(R,S,T)$ in Figure 1 is the initial 3-fold equivalence relation on R, S and T. It may be obtained as the kernel pair of the coequaliser of $\big(\Delta_R{\circ}\sigma_R{\circ}t_1, \Delta_R{\circ}\sigma_R{\circ}t_2\big)$ where $\sigma_R\colon X \to R$ and $\Delta_R\colon R \to \Delta(R,S)$ are the canonical morphisms that come with the reflexivity of the relations R and $\Delta(R,S)$. Here, "Δ-equivalence relation" means that the front, right and bottom double equivalence relations in the cube are all instances of the double equivalence relation introduced by Pedicchio.

The relations R, S and T are *centralising* when the cube of first projections in Figure 1 is a limit. This happens when a so-called 3-*dimensional connector* exists for R, S and T (Definition 5.18). The *ternary Bulatov commutator* $[R,S,T]^{\mathrm{B}}$ measures how far a triple (R,S,T) is from being centralising (Definition 6.1). It has some convenient stability properties, in parallel with what happens in the binary case, namely:

1. $[R,S,T]^{\mathrm{B}} \leqslant R \wedge S \wedge T$;
2. if $T \leqslant T'$, then $[R,S,T]^{\mathrm{B}} \leqslant [R,S,T']^{\mathrm{B}}$

3. $[R,S,T]^{\mathrm{B}} \leqslant [R,S]^{\mathrm{S}}$, the Smith-Pedicchio commutator;
4. $[R,S,T]^{\mathrm{B}}$ is independent of the order of R, S and T;
5. $f([R,S,T]^{\mathrm{B}}) = [f(R), f(S), f(T)]^{\mathrm{B}}$;
6. $[R,S,T \vee T']^{\mathrm{B}} = [R,S,T]^{\mathrm{B}} \vee [R,S,T']^{\mathrm{B}}$,

for any regular epimorphism $f\colon X \to Y$ and any equivalence relations R, S, T and T' on X.

The text is structured as follows. In Section 2 we recall the basic definitions and properties of internal equivalence relations and their higher-order variations. Section 3 is devoted to a detailed analysis of Pedicchio's double equivalence $\Delta(R,S)$, so that we have a solid foundation for its extension to three equivalence relations $\Delta(R,S,T)$ in Section 4. Section 5 introduces centrality for triples of equivalence relations, and compares the categorical approach with universal-algebraic versions of the concept. For instance, Theorem 5.21 treats the case of Mal'tsev varieties. The second half of the section focuses on stability properties of $\Delta(R,S,T)$, which are useful in Section 6 where the commutator is introduced and the above-mentioned properties are proved (Theorem 6.4 and the ensuing propositions). Section 7 concludes this article with an overview of work-in-progress and some open questions.

2. Equivalence relations, and their double and three-fold versions

In what follows, we let $\mathbb{X}$ be a regular category. We recall certain concepts and notations having to do with equivalence relations, as well as double and three-fold equivalence relations in this context.

2.1. Equivalence relations. An **equivalence relation** R on an object X of $\mathbb{X}$ is a reflexive, symmetric and transitive relation and will be denoted as in the left part of the diagram

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] & X \arrow[l, "\sigma_R" description] \arrow[r, two heads, "q_R"] & Q_R, \end{tikzcd}$$

where r_1 and r_2 are respectively the first and the second projection and $\sigma_R\colon X \to R$ is such that $r_1 \circ \sigma_R = 1_X = r_2 \circ \sigma_R$. When $\mathbb{X}$ is a Barr exact category, R is an **effective** equivalence relation, which means that the coequaliser $q_R\colon X \to Q_R$ of r_1 and r_2 exists and R is its kernel relation $X \times_{Q_R} X$.

We denote by $\mathsf{ERel}(\mathbb{X})$ the category where objects are pairs (X,R) where X is an object of $\mathbb{X}$ and R is an equivalence relation on X. An arrow (f, f_R) between (X,R) and

(Y, R') in $\mathsf{ERel}(\mathbb{X})$ is a commutative diagram of the form

$$
\begin{tikzcd}
R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] \arrow[d, "f_R"'] & X \arrow[l, "\sigma_R" description] \arrow[d, "f"] \\
R' \arrow[r, shift left=2, "r'_2"] \arrow[r, shift right=2, "r'_1"'] & Y. \arrow[l, "\sigma'_R" description]
\end{tikzcd}
$$

Here "commutative" means that $f \circ r_1 = r'_1 \circ f_R$, $f \circ r_2 = r'_2 \circ f_R$ and $\sigma'_R \circ f = f_R \circ \sigma_R$.

A finitely complete category $\mathbb{X}$ is a **Mal'tsev category** when every reflexive relation in $\mathbb{X}$ is an equivalence relation. If $\mathbb{X}$ is a regular Mal'tsev category, then so is $\mathsf{ERel}(\mathbb{X})$: see [7]. The category $\mathsf{ERel}(\mathbb{X})$ has those limits and colimits $\mathbb{X}$ has. Regular epimorphisms in $\mathsf{ERel}(\mathbb{X})$ are level-wise. Also limits in $\mathsf{ERel}(\mathbb{X})$ are computed level-wise, so that a morphism (f, f_R) is a monomorphism in $\mathsf{ERel}(\mathbb{X})$ if and only if f and f_R are monomorphisms.

2.2. Double equivalence relations. A **double equivalence relation** in a regular category $\mathbb{X}$ is an object of $\mathsf{ERel}(\mathsf{ERel}(\mathbb{X}))$. Such a double equivalence relation may be pictured as in the commutative diagram

$$
\begin{tikzcd}
D \arrow[r, shift left=2, "\pi_2^S"] \arrow[r, shift right=2, "\pi_1^S"'] \arrow[d, shift right=2, "\pi_1^R"'] \arrow[d, shift left=2, "\pi_2^R"] & S \arrow[l, "\sigma_S^D" description] \arrow[d, shift right=2, "s_1"'] \arrow[d, shift left=2, "s_2"] \\
R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] \arrow[u, "\sigma_R^D" description] & X \arrow[l] \arrow[u]
\end{tikzcd}
\qquad (\mathbf{A})
$$

where $D \rightrightarrows R$, $D \rightrightarrows S$, $R \rightrightarrows X$ and $S \rightrightarrows X$ are objects of $\mathsf{ERel}(\mathbb{X})$. D is also called a **double equivalence relation on** R **and** S, which are relations on a common object X. "Commutative" here means that the following diagrams are arrows in $\mathsf{ERel}(\mathbb{X})$:

$$
\begin{tikzcd}
D \arrow[r, shift left=2, "\pi_2^S"] \arrow[r, shift right=2, "\pi_1^S"'] \arrow[d, "\pi_1^R"'] & S \arrow[l, "\sigma_S^D" description] \arrow[d, "s_1"] \\
R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] & X \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, shift left=2, "\pi_2^S"] \arrow[r, shift right=2, "\pi_1^S"'] \arrow[d, "\pi_2^R"'] & S \arrow[l, "\sigma_S^D" description] \arrow[d, "s_2"] \\
R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] & X \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, shift left=2, "\pi_2^S"] \arrow[r, shift right=2, "\pi_1^S"'] & S \arrow[l, "\sigma_S^D" description] \\
R \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] \arrow[u, "\sigma_R^D"] & X \arrow[l] \arrow[u, "\sigma_S"']
\end{tikzcd}
$$

We write $\mathsf{ERel}^2(\mathbb{X})$ instead of $\mathsf{ERel}(\mathsf{ERel}(\mathbb{X}))$ for the category whose objects are double equivalence relation in $\mathbb{X}$. As above, limits in $\mathsf{ERel}^2(\mathbb{X})$ are level-wise, so that monomorphisms there are level-wise monomorphisms. $\mathsf{ERel}^2(\mathbb{X})$ is a regular Mal'tsev category as soon as $\mathbb{X}$ is.

$$
\begin{tikzcd}
R\square S \arrow[r, shift left=2, "\pi_2^S"] \arrow[r, shift right=2, "\pi_1^S"'] \arrow[d, shift right=3, "\pi_1^R"'] \arrow[d, shift left=3, "\pi_2^R"] & S \arrow[l] \arrow[d, shift right=3, "s_1"'] \arrow[d, shift left=3, "s_2"] \\
R \arrow[u] \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
R\square S \arrow[r, "\langle \pi_1^S, \pi_2^S \rangle"] \arrow[d, "\langle \pi_1^R, \pi_2^R \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \times S \arrow[d, "\langle s_1, s_2 \rangle \times \langle s_1, s_2 \rangle"] \\
R \times R \arrow[r, "\langle r_1 \times r_1, r_2 \times r_2 \rangle"'] & X^4
\end{tikzcd}
$$

Figure 2: The double equivalence relation $R\square S$

The largest double equivalence relation on two given equivalence relations R and S on a common object X always exists; it is denoted as on the left in Figure 2 and may be constructed via the pullback on the right. An "element" of $R\square S$ is a quadruple

$$
\begin{tikzcd}
c \arrow[r, dash, dotted, "R"] \arrow[d, dash, dotted, "S"'] & b \arrow[d, dash, dotted, "S"] \\
d \arrow[r, dash, dotted, "R"'] & a
\end{tikzcd}
$$

of elements of X where aRd, bRc, aSb and dSc,

Given $I \subseteq 2 = \{0, 1\}$, we write $\boxdot^I(R, S) = R\boxdot^I S$ for the pullback of $r_{\delta_I(0)}$ and $s_{\delta_I(1)}$, where $\delta_I(i)$ is 1 if $i \notin I$ and 2 if $i \in I$. So the subset I of 2 chooses which pair of projections to take a pullback of. Actually, up to isomorphism, that pullback is independent of the chosen set I: the symmetry of the relations R and S allows us to easily prove that $\boxdot^I(R, S) \cong \boxdot^J(R, S)$ for any choice of I, $J \subseteq 2$. For this reason we shall suppress the index I from the notation, writing $\boxdot(R, S) = R\boxdot S = R \times_X S$. We take $I = \varnothing$ when a choice needs to be made, as in the pullback of split epimorphisms

$$
\begin{tikzcd}
R\boxdot S \arrow[r, shift left, "\pi_S^{\boxdot}"] \arrow[d, shift left, "\pi_R^{\boxdot}"] & S \arrow[l, shift left, "\sigma_S^{\boxdot}"] \arrow[d, shift left, "s_1"] \\
R \arrow[u, shift left, "\sigma_R^{\boxdot}"] \arrow[r, shift left, "r_1"] & X. \arrow[l, shift left, "\sigma_R"] \arrow[u, shift left, "\sigma_S"]
\end{tikzcd}
\tag{B}
$$

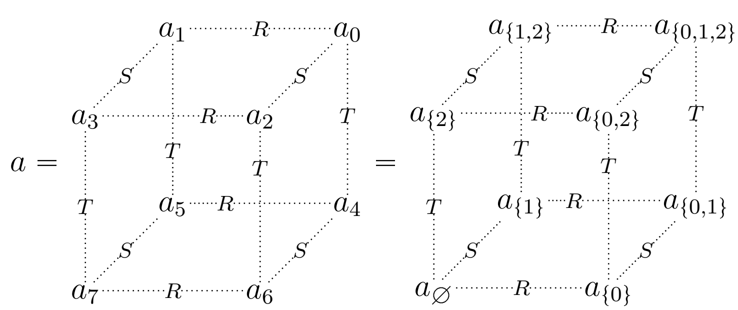
2.3. Three-fold equivalence relations. Inductively, n-**fold equivalence relations** are defined in any regular category, and $\mathsf{ERel}^n(\mathbb{X})$ denotes the category of n-fold equivalence relations in $\mathbb{X}$—see Section 3 in [29] for more details. Here we are interested in the case $n = 3$. So, a **three-fold equivalence relation** on an object X of a regular category may be viewed as a diagram as in Figure 3 where all of the cube's faces are double equivalence relations in $\mathbb{X}$ and $D_\varnothing = X$. When $R = D_{\{0\}}$, $S = D_{\{1\}}$ and $T = D_{\{2\}}$, we say that D is a **three-fold equivalence relation on** R**,** S **and** T. $\mathsf{ERel}^3(\mathbb{X})$ denotes the

$$
\begin{tikzcd}
 & D_3 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, dotted, shift left] \arrow[dd, dotted, shift right] & & D_{\{1,2\}} \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dd, shift left] \arrow[dd, shift right] \\
D_{\{0,1\}} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & D_{\{1\}} \arrow[dd, shift left] \arrow[dd, shift right] & \\
 & D_{\{0,2\}} \arrow[rr, dotted, shift left] \arrow[rr, dotted, shift right] \arrow[dl, dotted, shift left] \arrow[dl, dotted, shift right] & & D_{\{2\}} \arrow[dl, shift left] \arrow[dl, shift right] \\
D_{\{0\}} \arrow[rr, shift left] \arrow[rr, shift right] & & D_{\varnothing} &
\end{tikzcd}
$$

Figure 3: A three-fold equivalence relation

category of three-fold equivalence relations in $\mathbb{X}$. When $\mathbb{X}$ is a regular Mal'tsev category, $\mathsf{ERel}^3(\mathbb{X})$ is again a regular Mal'tsev category.

The largest three-fold equivalence relation on R, S and T, denoted $\square(R,S,T)$, was studied in detail in Section 3.1 of [29]. Its top object, also denoted $\square(R,S,T)$, consists of $2 \times 2 \times 2$ matrices



where the elements are related by R, S and T as indicated. We can either number those elements using subsets $I \subseteq 3$ where 0, 1 or 2 in I or not in I indicates whether the element is in the second or in the first factor of R, S or T, respectively. We can also number them as on the left, and the two points of view correspond via $I \mapsto \sum_{i \in 3 \setminus I} 2^i$.

Any choice of a subset $I \subseteq 3$ corresponds to one of the commutative three-cubes in the three-fold equivalence relation D, namely the cube whose diagonal is the I-th projection, which sends a three-dimensional matrix a to the element a_I. We shall denote this cube D^I. There are three "initial ribs" in this cube, each of which is a morphism with domain the initial object D_3 of the cube. The one in the direction i is given by the projection $\pi^{D_3}_{D_{3\setminus\{i\}},\delta_I(i)} \colon D_3 \to D_{3\setminus\{i\}}$.

2.4. The object $\boxdot D$. Given any three-fold equivalence relation D on R, S and T, and any subset $I \subseteq 3$, the object $\boxdot^I D$ is the limit $\mathrm{L}(D^I)$ of the diagram D^I from which the initial object D_3 is removed. Its elements are $2 \times 2 \times 2$ matrices as in $\square(R,S,T)$ with one element (indexed by I) missing. That limit comes with projections $\pi^{\boxdot}_{D_{3\setminus\{i\}}} \colon \boxdot^I D \to D_{3\setminus\{i\}}$, split by monomorphisms $\sigma^{\boxdot}_{D_{3\setminus\{i\}}} \colon D_{3\setminus\{i\}} \to \boxdot^I D$ for all $i < 3$. We write $\pi^I \colon D_3 \to \boxdot^I D$

$$\begin{tikzcd}
& X \arrow[rr, "p"] \arrow[dd, "\beta"'] & & Y \arrow[dd, "p_*(\beta)"] \\
X \arrow[ur, equal] \arrow[rr, "p"] \arrow[dd, "\alpha"'] & & Y \arrow[ur, equal] \arrow[dd, "p_*(\alpha)"] & \\
& B \arrow[rr] & & Y +_X B \\
A \arrow[ur, "f"] \arrow[rr] & & Y +_X A \arrow[ur, "p_*(f)"'] &
\end{tikzcd}$$

Figure 4: The cobase change functor

for the canonical projection (induced by the universal property of the limit) which forgets the I-face. In what follows, when the I is suppressed from the notation, we mean that $I = \varnothing$. In particular, this gives us the object $\boxdot D$ with its projection $\pi\colon D_3 \to \boxdot D$.

3. The double equivalence relation $\Delta(R, S)$

3.1. Cobase change for arrows. Let $\mathbb{X}$ be a category. For an object X in $\mathbb{X}$, the **coslice category** $(X \downarrow \mathbb{X})$ is the category whose objects are pairs (A, α), where A is an object of $\mathbb{X}$ and $\alpha\colon X \to A$ a morphism whose domain is X; a morphism $f\colon (A, \alpha) \to (B, \beta)$ is an arrow $f\colon A \to B$ such that $\beta = f \circ \alpha$.

We assume that $\mathbb{X}$ is finitely cocomplete. Then any morphism $p\colon X \to Y$ induces a **cobase change functor**

$$p_*\colon (X \downarrow \mathbb{X}) \to (Y \downarrow \mathbb{X})\colon (A, \alpha) \mapsto (Y +_X A, p_*(\alpha))$$

defined on morphisms as in Figure 4. From the first statement in the following well-known result we immediately deduce Lemma 3.3.

3.2. Lemma. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "\alpha"'] & Y \arrow[r, "q"] \arrow[d, "\beta"'] & Z \arrow[d, "\gamma"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C.
\end{tikzcd}$$

1. *If both squares are pushouts, then the outer rectangle is a pushout as well.*

2. *If the outer rectangle is a pushout and either β or f is an epimorphism, then the right hand side square is a pushout.*

3.3. Lemma. *Given morphisms $p\colon X \to Y$, $q\colon Y \to Z$ in a finitely cocomplete category, we have $(q\circ p)_* = q_*\circ p_*\colon (X \downarrow \mathbb{X}) \to (Z \downarrow \mathbb{X})$.*

3.4. Arrows versus equivalence relations. Let $\mathbb{X}$ be a category in which every arrow has a kernel pair and every reflexive graph has a coequaliser. We consider the basic coequaliser/kernel pair adjunction

$$\begin{tikzcd} \mathsf{RGrph}(\mathbb{X}) \arrow[r, shift left, "\mathrm{Coeq}"] \arrow[r, phantom, "\perp"] & \mathsf{Arr}(\mathbb{X}) \arrow[l, shift left, "\mathrm{Eq}"] \end{tikzcd}$$

between the category $\mathsf{RGrph}(\mathbb{X})$ of reflexive graphs and the category $\mathsf{Arr}(\mathbb{X})$ of arrows in $\mathbb{X}$. The left adjoint sends a reflexive graph (G, d, c, e)

$$\begin{tikzcd} G \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X \arrow[l, "e" description] \end{tikzcd} \qquad\qquad d\circ e = 1_X = c\circ e$$

to the coequaliser $\mathrm{Coeq}(d, c)$ of d and c, while the right adjoint sends an arrow $f\colon X \to Y$ to its kernel relation $\mathrm{Eq}(f) = (X \times_Y X, \pi_1, \pi_2, \sigma)$.

We shall be concerned with restricting this adjunction to an adjoint equivalence. By definition, an arrow is a regular epimorphism if and only if it is the coequaliser of some parallel pair of maps. Equivalently, it is the coequaliser of its kernel pair. So the image of the left adjoint Coeq is the full subcategory $\mathsf{Reg}(\mathbb{X})$ of $\mathsf{Arr}(\mathbb{X})$ determined by the regular epimorphisms. On the other hand, a reflexive graph is in the image of the functor Eq precisely when it is an effective equivalence relation. Writing $\mathsf{EERel}(\mathbb{X})$ for the category of effective equivalence relations in $\mathbb{X}$, the above adjunction restricts to an equivalence of categories

$$\begin{tikzcd} \mathsf{EERel}(\mathbb{X}) \arrow[r, shift left, "\mathrm{Coeq}"] \arrow[r, phantom, "\simeq"] & \mathsf{Reg}(\mathbb{X}). \arrow[l, shift left, "\mathrm{Eq}"] \end{tikzcd} \tag{$\mathbf{C}$}$$

When $\mathbb{X}$ is Barr exact, we may take the category $\mathsf{ERel}(\mathbb{X})$ of equivalence relations in $\mathbb{X}$ on the left; and when $\mathbb{X}$ is, moreover, a Mal'tsev category, we may take the category $\mathsf{RRel}(\mathbb{X})$ of reflexive relations instead. When $\mathbb{X}$ is a regular category, the existence of the respective equivalence characterises when $\mathbb{X}$ is Barr exact (equivalence relations are effective) or Barr exact Mal'tsev (reflexive relations are effective equivalence relations). Note that in a Barr exact Mal'tsev category, any reflexive graph has a coequaliser, since this coequaliser may be computed as the coequaliser of the support of the graph. Given a reflexive graph (G, d, c, e), its support (R, r_1, r_2, σ_R), which is a reflexive relation, hence an effective equivalence relation, is obtained via the factorisation

$$\begin{tikzcd} & R \arrow[dr, shift right, "r_1"'] \arrow[dr, shift left=2, "r_2"] & \\ G \arrow[ur, two heads, "p"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X \arrow[l, "e" description] \arrow[ul, "\sigma_R" description] \arrow[r, two heads, "q_R"] & Q_R \end{tikzcd} \tag{$\mathbf{D}$}$$

$$
\begin{tikzcd}
R \arrow[r, "\overline{p}"] \arrow[d, shift right=2, "r_2"'] \arrow[d, shift left=2, "r_1"] & p_\bullet(R) \arrow[d, shift right=2, "r_2^\bullet"'] \arrow[d, shift left=2, "r_1^\bullet"] \\
X \arrow[u] \arrow[r, "p"] \arrow[d, "q_R"'] & Y \arrow[u] \arrow[d, "p_*(q_R)"] \\
Q_R \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Y +_X Q_R.
\end{tikzcd}
$$

Figure 5: Cobase change

of $\langle d, c\rangle\colon G \to X \times X$ into a regular epi p and a mono $\langle r_1, r_2\rangle\colon R \to X \times X$. The coequaliser q_R of r_1 and r_2 is also the coequaliser of d and c since p is an epimorphism. (An alternative viewpoint, which follows from this analysis, is that the universal comparison p from G to the kernel pair R of the coequaliser q_R of d and c is always a regular epimorphism.) We may conclude that existence of binary coproducts suffices for the category to be finitely cocomplete. For this reason, we shall be working mainly in the context of a Barr exact Mal'tsev category with binary coproducts.

3.5. Cobase change for equivalence relations. Given an object X in $\mathbb{X}$, we write $\mathsf{ERel}_X(\mathbb{X})$ for the category of equivalence relations on X. Combining the construction in 3.1 with the equivalence in 3.4, we see that if $\mathbb{X}$ is a finitely cocomplete Barr exact category, then each morphism $p\colon X \to Y$ induces a **cobase change functor**

$$p_\bullet\colon \mathsf{ERel}_X(\mathbb{X}) \to \mathsf{ERel}_Y(\mathbb{X})\colon R \mapsto p_\bullet(R) \coloneqq \mathrm{Eq}(p_*(\mathrm{Coeq}(R)))$$

which takes an equivalence relation R on X and sends it to the kernel pair of $p_*(q_R)$, where q_R is the coequaliser of the two projections of R as in Figure 5.

3.6. Lemma. *Given morphisms $p\colon X \to Y$ and $q\colon Y \to Z$ in a finitely cocomplete Barr exact category, we have $(q\circ p)_\bullet = q_\bullet \circ p_\bullet\colon \mathsf{ERel}_X(\mathbb{X}) \to \mathsf{ERel}_Z(\mathbb{X})$.*

Proof. This follows immediately from Lemma 3.3 and the fact that (**C**) is an equivalence of categories. ∎

3.7. The double equivalence relation $\Delta(R, S)$. Let $\mathbb{X}$ be a Barr exact category with finite coproducts and X an object of $\mathbb{X}$. Given any two equivalence relations R and S on X, in the articles [27, 28], Pedicchio defines a double equivalence relation $\Delta(R, S)$ on R and S as in Figure 6. It is constructed by first taking the coequaliser $q_{RS}\colon R \to Q_{RS}$ of the pair $(\sigma_R\circ s_1, \sigma_R\circ s_2)$ and then the kernel pair $\Delta(R, S) = \mathrm{Eq}(q_{RS})$ of the result. S being a kernel pair, by the universal property of S, this construction induces a reflexive graph $(\Delta(R, S), d_1^S, d_2^S)$ on S, which is clearly a reflexive relation. Hence it is an (effective)

$$\begin{tikzcd}[row sep=large, column sep=huge]
\Delta(R,S) \arrow[r, dotted, shift left=2, "d_2^S"] \arrow[r, dotted, shift right=2, "d_1^S"'] \arrow[d, shift right=2, "d_2^R"'] \arrow[d, shift left=2, "d_1^R"] & S \arrow[l, dotted, "\Delta_S" description] \arrow[r, dotted, two heads, "q_{SR}"] \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "s_1"] & Q_{SR} \arrow[d, dotted, shift right=2] \arrow[d, dotted, shift left=2] \\
R \arrow[u, "\Delta_R" description] \arrow[r, shift left=2, "r_2"] \arrow[r, shift right=2, "r_1"'] \arrow[d, two heads, "q_{RS}"'] & X \arrow[u, "\sigma_S" description] \arrow[l, "\sigma_R" description] \arrow[r, dotted, two heads, "q_R"'] \arrow[d, dotted, two heads, "q_S"] & Q_R \arrow[u, dotted] \arrow[d, dotted, two heads, "q'"] \\
Q_{RS} \arrow[r, dotted, shift left=2] \arrow[r, dotted, shift right=2] & Q_S \arrow[l, dotted, "\sigma" description] \arrow[r, dotted, two heads, "q"'] & Q
\end{tikzcd}$$

Figure 6: Constructing the double equivalence relation $\Delta(R,S)$

equivalence relation, so that the top left part of the diagram forms a double equivalence relation on R and S.

As we shall see, $\Delta(R,S)$ is the smallest double equivalence relation on R and S. We first analyse its construction in terms of a cobase change.

3.8. Proposition. *Let R and S be equivalence relations on an object X of an exact category with finite colimits. Complete the diagram in Figure 6 with the coequaliser q_S and the induced bottom left reflexive graph induced by the universal property. The thus obtained square*

$$\begin{tikzcd}
R & X \arrow[l, "\sigma_R"'] \arrow[d, two heads, "q_S"] \\
Q_{RS} \arrow[from=u, two heads, "q_{RS}"'] & Q_S \arrow[l, "\sigma"]
\end{tikzcd}$$

is a pushout.

Proof. In order to check the universal property, we assume that $u\colon R \to Z$, $v\colon Q_S \to Z$ are morphisms such that $u\circ\sigma_R = v\circ q_S$. Since q_{RS} is the coequaliser of $(\sigma_R\circ s_1, \sigma_R\circ s_2)$, the equalities $u\circ\sigma_R\circ s_1 = v\circ q_S\circ s_1 = v\circ q_S\circ s_2 = u\circ\sigma_R\circ s_2$ tell us that there exists a unique morphism $\phi\colon Q_{RS} \to Z$ for which $\phi\circ q_{RS} = u$. We also have $\phi\circ\sigma\circ q_S = \phi\circ q_{RS}\circ\sigma_R = u\circ\sigma_R = v\circ q_S$. Since q_S is a (regular) epimorphism, it follows that $\phi\circ\sigma = v$. ∎

3.9. Corollary. *Let R and S be equivalence relations on an object X of an exact category with finite colimits. Then $\Delta(R,S) = (\sigma_R)_\bullet(S)$ as a relation on R.*

3.10. Proposition. *[27, 28] In an exact Mal'tsev category with binary coproducts, if R and S are equivalence relations on an object X, then $\Delta(R,S) \cong \Delta(S,R)$.*

Proof. Let us complete the diagram in Figure 6 with the coequaliser q_R and $q_{SR} = (\sigma_S)_*(q_R)$. In order to show that $\Delta(R,S) = (\sigma_S)_\bullet(R)$, which yields the claimed isomorphism, we only need to prove that q_{SR} is the coequaliser of the effective equivalence relation $(\Delta(R,S), d_1^S, d_2^S, \Delta_S)$. Suppose $t\colon S \to T$ is such that $t\circ d_1^S = t\circ d_2^S$. Then $t\circ\sigma_S\circ r_1 = t\circ d_1^S\circ\Delta_R = t\circ d_2^S\circ\Delta_R = t\circ\sigma_S\circ r_2$. Hence $t\circ\sigma_S$ factors through q_R via a unique morphism $u\colon Q_R \to T$ such that $u\circ q_R = t\circ\sigma_S$. The pushout property of Q_{SR} now yields the needed $v\colon Q_{SR} \to T$ such that $v\circ q_{SR} = t$. ■

The rest of the diagram in Figure 6 is obtained by taking the coequaliser q, and then completing with the top right reflexive graph. Since q_{RS} is an epimorphism, the induced bottom right square is a pushout.

3.11. Universal-algebraic interpretation. Let us recall from [16] that in a congruence modular variety $\mathbb{V}$, given congruences R and S on an algebra X in $\mathbb{V}$, the double relation $M(R,S)$ is the subalgebra of X^4 generated by matrices of the form

$$\begin{tikzcd} b_0 \arrow[r, dash, dotted] \arrow[d, dash, dotted] & a_0 \arrow[d, dash, dotted, "S"] \\ b_0 \arrow[r, dash, dotted, "R"'] & a_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b_1 \arrow[r, dash, dotted] \arrow[d, dash, dotted] & b_1 \arrow[d, dash, dotted, "S"] \\ a_1 \arrow[r, dash, dotted, "R"'] & a_1 \end{tikzcd}$$

where $a_0 R b_0$ and $a_1 S b_1$. More precisely, $M(R,S)$ consists of all matrices

$$\begin{tikzcd} t(b^0,b^1) \arrow[r, dash, dotted] \arrow[d, dash, dotted] & t(a^0,b^1) \arrow[d, dash, dotted, "S"] \\ t(b^0,a^1) \arrow[r, dash, dotted, "R"'] & t(a^0,a^1) \end{tikzcd}$$

where a^0, b^0 are m-tuples and a^1, b^1 are n-tuples of elements of X such that $a^0 R b^0$ and $a^1 S b^1$, whereas t is a term of arity $m+n$. On the other hand, $\Delta(R,S)$ is the congruence on R generated by matrices of the form

$$\begin{tikzcd} b \arrow[r, dash, dotted] \arrow[d, dash, dotted] & b \arrow[d, dash, dotted, "S"] \\ a \arrow[r, dash, dotted, "R"'] & a \end{tikzcd}$$

where aSb. Since $M(R,S)$ is always a double reflexive and symmetric relation, in general, $\Delta(R,S)$ is the transitive closure of $M(R,S)$; hence in the Mal'tsev setting, they coincide. Proposition 3.29 gives a categorical expression of this fact.

3.12. Pairs of connected equivalence relations. In an exact Mal'tsev category with binary coproducts, two equivalence relations R and S on an object X are **connected** when they admit a **centralising double equivalence relation**, which is a double equivalence relation (**A**) on R and S such that any of the commutative squares of projections forms a pullback.

3.13. Proposition. *[27] In an exact Mal'tsev category with binary coproducts, if two equivalence relations R and S on an object X admit a centralising double equivalence relation C, then $C \cong \Delta(R,S)$.*

In other words, R and S are connected if and only if any of the top left downward pointing commutative squares in Figure 6 is a pullback.

Recall the object $\boxdot(R,S)$ from diagram (**B**). Its "elements" may be viewed as triples (x,y,z) where xRy and ySz. It is well known—see, for instance, [8, 7, 27, 28]—that the existence of a centralising double equivalence relation C on R and S is equivalent to the existence of a unique morphism (called a **connector**) $p\colon \boxdot(R,S) \to X$, such that $p(x,y,y) = x$ and $p(x,x,y) = y$.

There is a universal way of making equivalence relations connected. The colimit $\odiv(R,S)$ of the outer solid arrows of the left hand diagram below can be obtained as the pushout in the right hand diagram.

$$
\begin{tikzcd}
 & R \arrow[dl, "\langle 1_R, \sigma_S \circ r_2 \rangle"'] \arrow[d, dotted] \arrow[dr, "r_1"] & \\
\boxdot(R,S) \arrow[r, dotted, "\varphi_X"] & \odiv(R,S) & X \arrow[l, dotted, "\psi_X"'] \\
 & S \arrow[ul, "\langle \sigma_R \circ s_1, 1_S \rangle"] \arrow[u, dotted] \arrow[ur, "s_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R + S \arrow[r, "(r_1, s_2)"] \arrow[d, two heads] & X \arrow[d, two heads, "\psi_X"] \\
\boxdot(R,S) \arrow[r, "\varphi_X"'] & \odiv(R,S) \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}
$$

The morphism ψ_X is an extremal epimorphism. It is universal for making the equivalence relations $\psi_X(R)$ and $\psi_X(S)$ connected.

3.14. Definition. *Let $\mathbb{X}$ be an exact Mal'tsev category with finite colimits. For any two equivalence relations R, S on an object X of $\mathbb{X}$, their **Smith-Pedicchio commutator** $[R,S]^{\mathrm{S}}$ is the kernel pair of the morphism ψ_X.*

For Mal'tsev varieties, this is exactly the well-known Smith commutator introduced in [30]. Moreover, the following properties hold in any finitely cocomplete exact Mal'tsev category [3, 5]:

1. $[R,S]^{\mathrm{S}} = [S,R]^{\mathrm{S}}$;
2. $[R,S]^{\mathrm{S}} \leqslant R \wedge S$;
3. if $R \leqslant R'$ and $S \leqslant S'$, then $[R,S]^{\mathrm{S}} \leqslant [R',S']^{\mathrm{S}}$;
4. $f([R,S]^{\mathrm{S}}) = [f(R), f(S)]^{\mathrm{S}}$ for any regular epimorphism $f\colon X \to Y$.

3.15. Regular pushouts, higher extensions. Our aim is to obtain similar properties for the ternary commutator which we shall introduce below. As a first step towards these results, in Subsection 3.21 we need to recall certain stability properties of the objects $\Delta(R, S)$. Our approach depends on the concept of a *higher extension.* We start by recalling the definition of a *regular pushout.*

3.16. Proposition. [4, 2] *In a regular Mal'tsev category, a commutative square of regular epimorphisms*

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "x"'] & Y \arrow[d, two heads, "y"] \\ X' \arrow[r, two heads, "f'"'] & Y' \end{tikzcd}$$

is called a ***regular pushout*** *when any of the following equivalent conditions hold:*

(i) *the comparison morphism* $\langle x, f\rangle \colon X \to X' \times_{Y'} Y$ *is a regular epimorphism;*

(ii) *the induced morphism* $\overline{x} \colon X \times_Y X \to X' \times_{Y'} X'$ *is a regular epimorphism;*

(iii) *the induced morphism* $\overline{f} \colon X \times_{X'} X \to Y \times_{Y'} Y$ *is a regular epimorphism.*

Proposition 5.6 in [12] says that a regular category is exact Mal'tsev precisely when any two regular epimorphisms with a common domain admit a pushout, and this pushout is a regular pushout. So in a Barr exact Mal'tsev category, when the morphism p in Figure 5 is a regular epimorphism, the morphism $\overline{p}$ is also a regular epimorphism. This implies that the relation $p_\bullet(R)$ may be obtained as the regular image of R along p, via the image factorisation of the morphism $(p \times p) \circ \langle r_1, r_2\rangle \colon R \to Y \times Y$.

An n**-fold arrow** in $\mathbb{X}$ is a contravariant functor $F \colon (2^n)^{\mathrm{op}} \to \mathbb{X}$ where 2^n is the power-set of $n \coloneqq \{0, \ldots, n-1\}$ considered as a small category. Morphisms between two n-fold arrows F and G are natural transformations $f \colon F \to G$. We write $\mathsf{Arr}^n(\mathbb{X}) = \mathsf{Fun}((2^n)^{\mathrm{op}}, \mathbb{X})$ for the functor category of n-fold arrows in $\mathbb{X}$. Limits and colimits in $\mathsf{Arr}^n(\mathbb{X})$ are computed pointwise in $\mathbb{X}$, so it is an exact Mal'tsev category as soon as $\mathbb{X}$ is. See [14] for further details on this category, and on the following use we make of it:

3.17. Definition. *Let $\mathbb{X}$ be a regular category. A 0-fold extension is an object of $\mathbb{X}$ and a 1-fold extension is a regular epimorphism in $\mathbb{X}$. An* n***-fold extension*** *in $\mathbb{X}$ with $n \geqslant 2$ is a commutative diagram in $\mathsf{Arr}^{n-2}(\mathbb{X})$ as in Figure 7 where all arrows are $(n-1)$-fold extensions. The n-fold extensions in $\mathbb{X}$ forms a full subcategory $\mathsf{Ext}^n(\mathbb{X})$ of $\mathsf{Arr}^n(\mathbb{X})$.*

3.18. Example. In any regular Mal'tsev category $\mathbb{X}$, a 2-fold extension is a regular pushout in $\mathbb{X}$. In any exact Mal'tsev category, a 2-fold extension is a pushout square of regular epimorphisms.

$$
\begin{tikzcd}
A_1 \arrow[drr, bend left, "f_1"] \arrow[dr, dotted, "\langle a, f_1\rangle"] \arrow[ddr, bend right, "a"'] & & \\
& A_0 \times_{B_0} B_1 \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, "b"] \\
& A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}
$$

Figure 7: The outer square is an n-fold extension when all arrows in the diagram are $(n-1)$-fold extensions

3.19. Example. It easily follows from Proposition 3.16 that in any regular Mal'tsev category, a split epimorphism of split epimorphisms, viewed as a commutative square, is always a double extension. More generally, any n-fold split epimorphism gives rise to an n-fold extension. A concrete situation where this happens is pictured in Figure 3: any of the commutative three-cubes induced by choosing projections in a three-fold equivalence relation is a three-fold extension, as a split epimorphism between double extensions.

Recall that a family $(f_i\colon A_i \to B)_{i\in I}$ of morphisms in $\mathbb{X}$ with the same codomain is said to be **jointly extremal-epimorphic** when any monomorphism $m\colon M \to B$ through which all the f_i factor is an isomorphism.

3.20. Theorem. *Let E be a three-fold split epimorphism in a regular Mal'tsev category. Then the induced comparison arrows $\sigma_i\colon E_{3\setminus\{i\}} \to \mathrm{L}(E)$ form a jointly extremal-epimorphic family $(\sigma_i)_{i\in 3}$.*

Proof. Recall the definition of the limit $\mathrm{L}(E)$ from 2.4. Consider a monomorphism $m\colon M \to \mathrm{L}(E)$ through which each σ_i factors as an arrow $\tau_i\colon E_{3\setminus\{i\}} \to M$ such that $m\circ\tau_i = \sigma_i$. Then we obtain a three-fold split epimorphism F by putting $F_3 = M$ and $F_I = E_I$ for $I \subsetneq 3$, with the obvious arrows which are either the arrows of the diagram E, arrows induced by composition with m, or one of the τ_i. Then F is a three-fold extension, which implies that m is a regular epimorphism, hence an isomorphism. It follows that the σ_i form a jointly extremal-epimorphic family. ■

3.21. Stability properties. We start with preservation of $\Delta(R,S)$ under direct images, where we can immediately apply the above. Let $\mathbb{X}$ be a regular Mal'tsev category. Recall from [7] the construction of the category 2-$\mathsf{Eq}(\mathbb{X})$ whose objects are triples (X,R,S) where R and S are equivalence relations on X. A morphism

$$(f, f_R, f_S)\colon (X,R,S) \to (Y,R',S')$$

consists of morphisms $f\colon X \to Y$, $f_R\colon R \to R'$, $f_S\colon S \to S'$ in $\mathbb{X}$ making (f,f_R) and (f,f_S) morphisms of equivalence relations. As explained in [7], limits and regular epimorphisms in this category are pointwise.

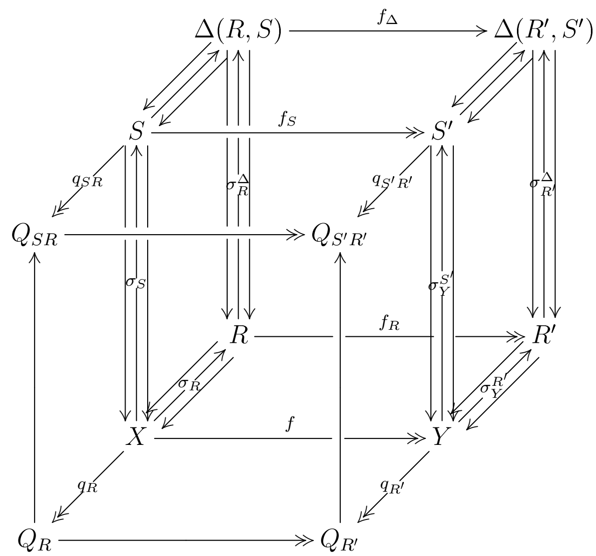


Figure 8: Preservation of Δ under direct images

3.22. Proposition. *In an exact Mal'tsev category with binary coproducts $\mathbb{X}$, consider a regular epimorphism $(f, f_R, f_S)\colon (X, R, S) \to (Y, R', S')$ in $2\text{-}\mathsf{Eq}(\mathbb{X})$. Then the induced morphism $f_\Delta\colon \Delta(R, S) \to \Delta(R', S')$ in $\mathbb{X}$ is a regular epimorphism as well, so that $f(\Delta(R, S)) = \Delta(f(R), f(S))$.*

Proof. The commutative diagram in Figure 8 leads us to diagram

$$\begin{tikzcd}
X \arrow[r, "\sigma_S"] \arrow[d, two heads, "q_R"'] & S \arrow[r, two heads, "f_S"] \arrow[d, two heads, "q_{SR}"'] & S' \arrow[d, two heads, "q_{S'R'}"] \\
Q_R \arrow[r] & Q_{SR} \arrow[r, two heads] & Q_{S'R'}
\end{tikzcd}$$

in which the outer rectangle is a pushout. This outer rectangle is a pushout square because in the front commutative cube of Figure 8, the bottom square is a pushout square, since f_R is a regular epimorphism. Moreover, in the definition of $\Delta(R', S')$, Proposition 3.8 tells us that the right-hand square pointing up is also a pushout square. Hence, the rectangle is a pushout square. By Lemma 3.2, the right-hand square is a pushout square of regular epimorphisms, which in our (Barr-exact) context is a regular pushout. Hence by Proposition 3.16, f_Δ is a regular epimorphism, so that $f(\Delta(R, S)) = \Delta(f(R), f(S))$. ■

3.23. Preservation of joins. The **join** $R \vee S$ of two equivalence relations R and S on an object X is the smallest equivalence relation that contains them both. It is well known that in the context of a regular Mal'tsev category, it may be obtained as the composite $R \circ S = S \circ R$ of R and S. In the exact Mal'tsev case [12], there is a characterisation in terms of the induced quotients of R and S: the diagonal in their pushout is precisely the quotient induced by $R \vee S$.

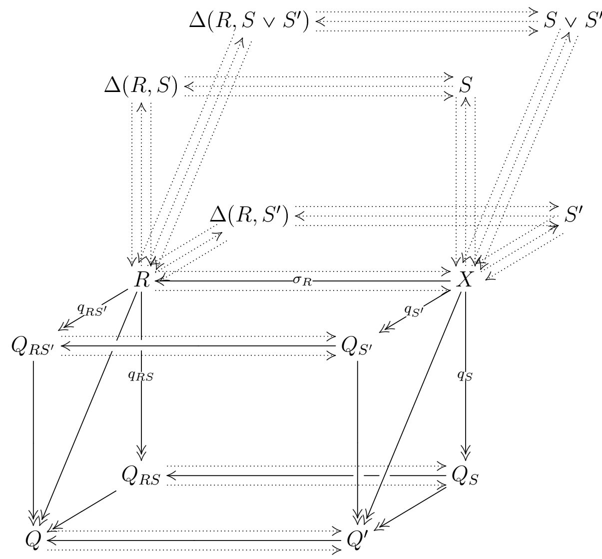


Figure 9: Preservation of joins by $\Delta(R, -)$

3.24. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S and S' be equivalence relations on an object X. Then as equivalence relations on R, we have $\Delta(R, S \vee S') = \Delta(R, S) \vee \Delta(R, S')$.*

Proof. Let us consider the diagram in Figure 9, where $\Delta(R, S)$, $\Delta(R, S')$ and $\Delta(R, S \vee S')$ are constructed. The solid diagonal square is a pushout, hence so is the bottom square in the cube of solid arrows. By Lemma 3.2, it follows that the left-hand square in this cube is a pushout as well, which implies $\Delta(R, S \vee S') = \Delta(R, S) \vee \Delta(R, S')$. ■

In what follows, when R and S are connected, we can always consider $\Delta(R, S)$ as the double centralising equivalence relation on R and S. This apparently simple consideration will allows us to provide alternative proofs for some useful results which already appeared in the literature, see for instance [8, 7, 27, 28], and which will help generalising the theory to ternary commutators.

3.25. Lemma. *[17] In a regular category, consider the commutative diagram*

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, two heads, "a"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, "g"] \arrow[d, two heads, "b"] \arrow[dr, phantom, "(2)"] & C \arrow[d, "c"] \\
A' \arrow[r, two heads, "f'"'] & B' \arrow[r, "g'"'] & C'.
\end{tikzcd}$$

If the outer rectangle (1) + (2) *is a pullback and the left hand square* (1) *is a regular pushout, then both squares* (1) *and* (2) *are pullbacks.*

3.26. Lemma. *[15] Suppose that the following three-cube is an extension.*

$$\begin{tikzcd}[row sep=small, column sep=small]
& A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] \\
A \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & B \arrow[ur] & \\
& C' \arrow[rr] & & D' \\
C \arrow[ur] \arrow[rr] & & D \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

If the top square is a pullback, then so is the bottom square.

Proof. Assume that the top square is a pullback. Taking pullbacks on the top and on the bottom squares, we obtain the comparison morphism

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, two heads] & A' \times_{B'} B \arrow[d, two heads] \\
C \arrow[r, two heads] & C' \times_{D'} D
\end{tikzcd}$$

which is a double extension, hence a pushout square. Therefore, the comparison morphism of the bottom square is an isomorphism. ∎

3.27. Proposition. *[7, 8, 27, 28] In an exact Mal'tsev category with binary coproducts, let R, S and S' be equivalence relations on an object X. Then the following are equivalent:*

(i) *R and S are connected, and R and S' are connected;*

(ii) *R and $S \vee S'$ are connected.*

Proof. (i) ⇒ (ii) If we assume that R and S, respectively R and S' are connected, then in Figure 9, the double equivalence relations $\Delta(R, S)$ and $\Delta(R, S')$ are centralising double equivalence relations. We must prove that also $\Delta(R, S \vee S')$ is a centralising double equivalence relation. By the Barr–Kock Theorem [3, Lemma A.5.8], the top and the back square in the cube determined by first projections pointing to the right are pullbacks. Since any split epimorphism between double extensions determines a three-fold extension, Lemma 3.26 implies that the bottom square of first projections pointing to the right is a pullback. It follows that the diagonal square of first projections is a pullback, so that $\Delta(R, S \vee S')$ is a centralising double equivalence relation.

(ii) ⇒ (i) Now suppose that $\Delta(R, S \vee S')$ is a centralising double equivalence relation on R and $S \vee S'$. Then the diagonal square of first projections, pointing to the right in Figure 9, is a pullback square. Using Lemma 3.25, it is easy to see that $\Delta(R, S)$ and $\Delta(R, S')$ are centralising double equivalence relations. ∎

We end this section with the characterisation of $\Delta(R,S)$ as the initial double equivalence relation on R and S.

3.28. Lemma. *In an exact Mal'tsev category with binary coproducts, consider a commutative diagram*

$$\begin{tikzcd} \mathrm{Eq}(f) \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] \arrow[d, "u"'] & A \arrow[l, "\sigma" description] \arrow[r, two heads, "f"] \arrow[d, "v"] & B \arrow[d, "w"] \\ \mathrm{Eq}(f') \arrow[r, shift left=2, "\pi_1'"] \arrow[r, shift right=2, "\pi_2'"'] & A' \arrow[l, "\sigma'" description] \arrow[r, two heads, "f'"'] & B' \end{tikzcd}$$

where the rows are exact forks. The right hand square is a pushout if and only if u and σ' are jointly extremal-epimorphic.

Proof. Assume that u and σ' are jointly extremal-epimorphic. Let $a\colon A' \to M$ and $b\colon B \to M$ be morphisms such that $a\circ v = b\circ f$. We then have

$$a\circ\pi_1'\circ u = a\circ v\circ\pi_1 = b\circ f\circ\pi_1 = b\circ f\circ\pi_2 = a\circ v\circ\pi_2 = a\circ\pi_2'\circ u$$

and $a\circ\pi_1'\circ\sigma' = a = a\circ\pi_2'\circ\sigma'$. Since u and σ' are jointly extremal-epimorphic, we have $a\circ\pi_1' = a\circ\pi_2'$. Hence there exists a unique morphism $\varphi\colon B' \to M$ such that $\varphi\circ f' = a$. It remains to be shown that $\varphi\circ w = b$. That follows because f is a regular epimorphism and $b\circ f = a\circ v = \varphi\circ f'\circ v = \varphi\circ w\circ f$.

For the converse, assume now that the right hand square is a pushout. Let us compute the pushout of v and σ and write ϕ for the induced morphism $\mathrm{Eq}(f) +_A A' \to \mathrm{Eq}(f')$. We have to prove that it is an extremal epimorphism. In the current, Barr-exact Mal'tsev context, from the reasoning involving Diagram (**D**) we may deduce that this is equivalent to saying that f' is still the coequaliser of $\pi_1'\circ\phi$ and $\pi_2'\circ\phi$. Let us consider this coequaliser and denote it by $q\colon A' \to Q$. It is easily seen that a unique morphism $\lambda\colon Q \to B'$ exists such that $f' = \lambda\circ q$. We also have

$$q\circ v\circ\pi_1 = q\circ\pi_1'\circ u = q\circ\pi_1'\circ\phi\circ\bar{v} = q\circ\pi_2'\circ\phi\circ\bar{v} = q\circ\pi_2'\circ u = q\circ v\circ\pi_2,$$

so there is a unique $\alpha\colon B \to Q$ for which $\alpha\circ f = q\circ v$. The right hand square being pushout implies the existence of a unique $\lambda'\colon B' \to Q$ such that $\lambda'\circ f' = q$ and $\lambda'\circ w = \alpha$. Hence since q and f' are epimorphisms, the morphism λ is an isomorphism. The Barr-exactness of $\mathbb{X}$ implies that ϕ is an extremal epimorphism. Now since $\bar{\sigma}$ and $\bar{v}$ are jointly extremal-epimorphic, we conclude that $u = \phi\circ\bar{v}$ and $\sigma' = \phi\circ\bar{\sigma}$ are jointly extremal-epimorphic. ∎

3.29. Proposition. *In an exact Mal'tsev category with binary coproducts, the morphisms Δ_R and Δ_S in Figure 6 are jointly extremal-epimorphic.*

Proof. This follows immediately from the construction of the diagram in Figure 6 combined with Lemma 3.28. ∎

This implies that $\Delta(R,S)$ is minimal amongst double equivalence relations on R and S, since any double equivalence relation included in $\Delta(R,S)$ is isomorphic to it. It is actually a minimum, as follows from Proposition 3.31. We may also use this to prove the next result; however, a more general proof, valid in regular Mal'tsev categories, is already known—see [3].

3.30. Proposition. *In a regular Mal'tsev category, the splittings $\sigma_R^{\boxdot}$ and $\sigma_S^{\boxdot}$ in the pullback* (**B**) *are jointly extremal-epimorphic.*

3.31. Proposition. *In an exact Mal'tsev category with binary coproducts, consider equivalence relations R and S on a common object X. The double equivalence relation $\Delta(R,S)$ is an initial object in the category of double equivalence relations on R and S.*

Proof. This follows from the construction of $\Delta(R,S)$ as a kernel pair of a coequaliser. Let (**A**) be a double equivalence relation on R and S, and write $q_D\colon R \to Q_D$ for the induced coequaliser of π_1^R and π_2^R. Since $q_D \circ \sigma_R \circ s_1 = q_D \circ \sigma_R \circ s_2$, there is a unique comparison morphism $t\colon Q_{RS} \to Q_S$ such that $t \circ q_{RS} = q_D$, which induces the needed morphism $\Delta(R,S) \to D$. ■

3.32. Corollary. *In an exact Mal'tsev category with binary coproducts, a double equivalence relation such as* (**A**) *is $\Delta(R,S)$ if and only if the morphisms σ_R^D and σ_S^D are jointly extremal-epimorphic.*

Proof. Given a double equivalence relation (**A**) on R and S such that σ_R^D and σ_S^D are jointly extremal-epimorphic, since the comparison morphism from $\Delta(R,S)$ to D, induced by Proposition 3.31 is a monomorphism (as a morphism of equivalence relations on a common object), it is an isomorphism. Hence the given double equivalence relation is isomorphic to $\Delta(R,S)$. ■

4. The three-fold equivalence relation $\Delta(R,S,T)$

Our aim is now to extend the definition of $\Delta(R,S)$ to a similar notion for three equivalence relations: the object $\Delta(R,S,T)$, which is the initial vertex of the initial three-fold equivalence relation on R, S and T.

Given equivalence relations R, S and T on a common object X, let us consider the diagram in Figure 10, where $\Delta(R,S), \Delta(R,T)$ and $\Delta(S,T)$ are the equivalence relations defined as in 3.7 and $q_{\Delta\Delta}$ is the coequaliser of the pair $\big(\Delta_R \circ \sigma_R \circ t_1, \Delta_R \circ \sigma_R \circ t_2\big)$. By the commutativity of the sub-diagrams that determine $\Delta(R,S)$, $\Delta(R,T)$ and $\Delta(S,T)$, this coequaliser can be obtained in several equivalent ways; for instance, it is also the coequaliser of the pair $(\Delta_R \circ d_1^R \circ \Delta_T, \Delta_R \circ d_2^R \circ \Delta_T)$.

4.1. Definition. *In an exact category with finite colimits $\mathbb{X}$, consider equivalence relations R, S and T on an object X. We let $\Delta(R,S,T)$ be the kernel pair of the morphism $q_{\Delta\Delta}$ defined above. When $\mathbb{X}$ is Mal'tsev, the construction in Figure 10 (with functorially induced arrows between the kernel pairs) determines a three-fold equivalence relation*

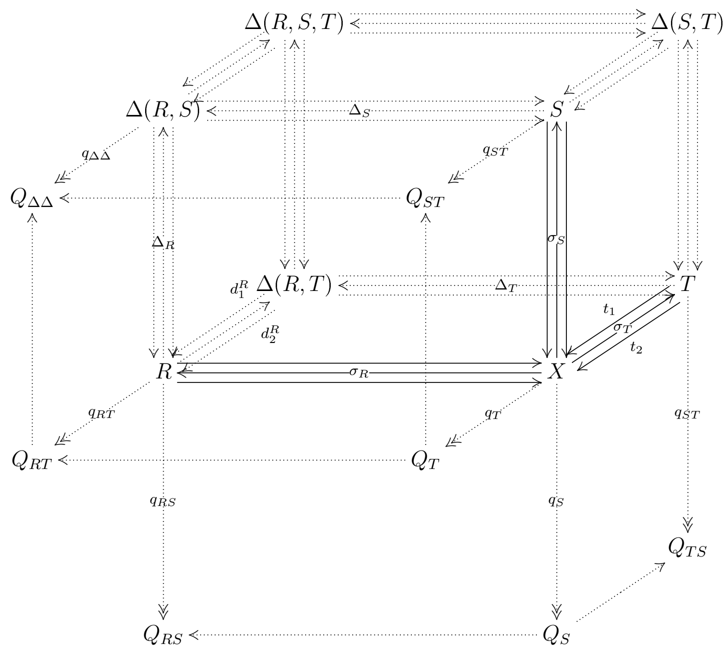


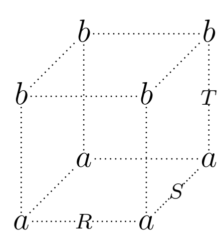
Figure 10: The construction of $\Delta(R,S,T)$

$\Delta(R,S,T)$ on R, S and T. When $\mathbb{X}$ is exact Mal'tsev, all reflexive graphs in the diagram are effective equivalence relations.

4.2. Proposition. *In an exact Mal'tsev category with binary coproducts, consider equivalence relations R, S and T on an object X. $\Delta(R,S,T)$ is the initial three-fold equivalence relation on R, S and T.*

Proof. Consider a three-fold equivalence relation as in Figure 3, where $X = D_{\varnothing}$, $R = D_{\{0\}}$, $S = D_{\{1\}}$ and $T = D_{\{2\}}$. By Proposition 3.31, we already find universally induced arrows $\Delta(R,S) \to D_{\{0,1\}}$, $\Delta(R,T) \to D_{\{0,2\}}$ and $\Delta(S,T) \to D_{\{1,2\}}$. The needed arrow $\Delta(R,S,T) \to D_3$ is now obtained either via an argument as in the proof of Proposition 3.31, or as a consequence of Proposition 4.8. ∎

4.3. Example. In any congruence modular variety, the object $\Delta(R,S,T)$ happens to be a congruence on $\Delta(R,S)$, generated by cubes of the form

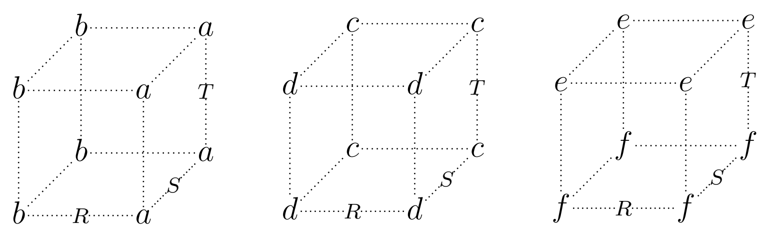


$\in \square(R,S,T).$

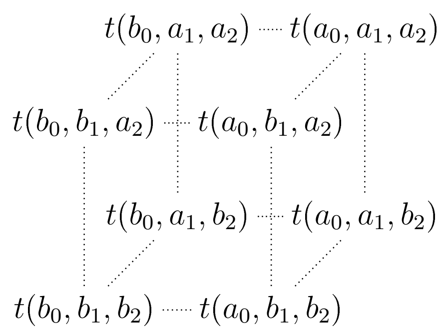
Hence the construction in Definition 4.1 is a categorical conceptualisation of the relation $\Delta(R,S,T)$ introduced in [24, 26].

In the Mal'tsev context this can be further worked out, because reflexivity is enough for a relation to be an equivalence relation:

4.4. Example. In a congruence modular variety $\mathbb{V}$, the object $M(R,S,T)$ introduced in [26] is generated as a subalgebra of $\square(R,S,T) \leqslant X^8$ by elements of the form



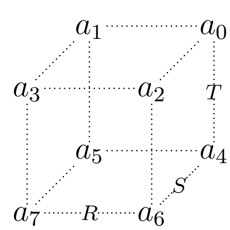
where aRb, cSd and eTf. Hence a general element of $M(R,S,T)$ is of the form



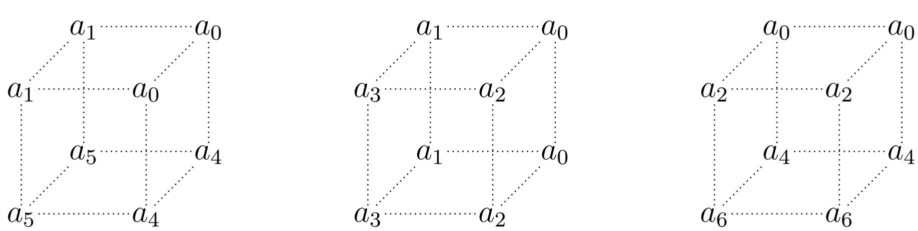
for some $(k+n+l)$-ary term t in the theory of $\mathbb{V}$ and for all vectors a_0, $b_0 \in X^k$, a_1, $b_1 \in X^n$, and a_2, $b_2 \in X^l$ such that a_0Rb_0, a_1Sb_1 and a_2Tb_2.

It is clear that $M(R,S,T) \leqslant \Delta(R,S,T)$; Theorem 5.4 shows that the two coincide in the context of a Mal'tsev variety.

4.5. Remark. In any Mal'tsev variety, let R, S and T be equivalence relations on a common algebra X. If



is in $\Delta(R,S,T)$, then for instance also



are in $\Delta(R,S,T)$.

In order to study this relation's properties, we first simplify its construction. To do so, we depend on the following lemma.

4.6. LEMMA. *In a category with coequalisers, we consider the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & C \arrow[r, "g"] & D. \end{tikzcd}$$

If $\mathrm{Coeq}(u,v) = \mathrm{Coeq}(u\circ f, v\circ f)$, *then* $\mathrm{Coeq}(g\circ u, g\circ v) = \mathrm{Coeq}(g\circ u\circ f, g\circ v\circ f)$.

PROOF. Note that $\mathrm{Coeq}(u,v) = \mathrm{Coeq}(u\circ f, v\circ f)$ precisely when for each $h\colon C \to Z$,

$$h\circ u\circ f = h\circ v\circ f \quad \Leftrightarrow \quad h\circ u = h\circ v.$$

In particular, for any $j\colon D \to Z$ we have

$$j\circ g\circ u\circ f = j\circ g\circ v\circ f \quad \Leftrightarrow \quad j\circ g\circ u = j\circ g\circ v.$$

It follows that $\mathrm{Coeq}(g\circ u\circ f, g\circ v\circ f) = \mathrm{Coeq}(g\circ u, g\circ v)$, because these two coequalisers have the same universal property. ∎

4.7. LEMMA. *In an exact category with finite colimits, consider equivalence relations R, S and T on an object X. Then $\Delta(R,S,T) = (\Delta_R\circ\sigma_R)_\bullet(T)$ as equivalence relations on $\Delta(R,S)$.*

PROOF. First of all, we notice that $(\Delta_R\circ\sigma_R)_\bullet(T) = (\Delta_R)_\bullet((\sigma_R)_\bullet(T)) = (\Delta_R)_\bullet(\Delta(R,T))$. So, we must prove that $q_{\Delta\Delta} = \mathrm{Coeq}(\Delta_R\circ d_1^R, \Delta_R\circ d_2^R)$. Since

$$q_{RT} = \mathrm{Coeq}(d_1^R, d_2^R) = \mathrm{Coeq}(\sigma_R\circ t_1, \sigma_R\circ t_2) = \mathrm{Coeq}\left(d_1^R\circ\Delta_T, d_2^R\circ\Delta_T\right),$$

by Lemma 4.6 it follows that $\mathrm{Coeq}(\Delta_R\circ d_1^R, \Delta_R\circ d_2^R) = \mathrm{Coeq}(\Delta_R\circ d_1^R\circ\Delta_T, \Delta_R\circ d_2^R\circ\Delta_T)$. Now $q_{\Delta\Delta} = \mathrm{Coeq}(\Delta_R\circ d_1^R, \Delta_R\circ d_2^R)$, and $\Delta(R,S,T) = (\Delta_R)_\bullet(\Delta(R,T)) = (\Delta_R\circ\sigma_R)_\bullet(T)$ by Corollary 3.9. ∎

4.8. PROPOSITION. *In an exact category with finite colimits, consider equivalence relations R, S and T on an object X. Then $\Delta(R,S,T) = \Delta(\Delta(R,S),\Delta(R,T))$ as equivalence relations on $\Delta(R,S)$.*

PROOF. By Lemma 4.7 we have $\Delta(R,S,T) = (\Delta_R\circ\sigma_R)_\bullet(T)$, equal to $(\Delta_R)_\bullet((\sigma_R)_\bullet(T)) = (\Delta_R)_\bullet(\Delta(R,T)) = \Delta(\Delta(R,S),\Delta(R,T))$. ∎

4.9. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. Then*

$$\Delta(R,S,T) = \Delta(R,T,S) = \Delta(S,R,T) = \Delta(S,T,R) = \Delta(T,R,S) = \Delta(T,S,R).$$

Proof. Without loss of generality, we may show that $\Delta(R,S,T) = \Delta(S,R,T)$.

$$\begin{aligned}\Delta(R,S,T) &= (\Delta_{R}\circ\sigma_R)_\bullet(T) = (\Delta_{S}\circ\sigma_S)_\bullet(T)\\ &= (\Delta_S)_\bullet((\sigma_S)_\bullet(T)) = (\Delta_S)_\bullet(\Delta(S,T))\\ &= \Delta(\Delta(R,S),\Delta(S,T)) = \Delta(\Delta(S,R),\Delta(S,T))\\ &= \Delta(S,R,T),\end{aligned}$$

using Proposition 4.8, Proposition 3.10 and the fact that all equivalence relations in Figure 10 are effective. ∎

4.10. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S, T and T' be equivalence relations on an object X. As equivalence relations on $\Delta(R,S)$, we have $\Delta(R,S,T\vee T') = \Delta(R,S,T)\vee\Delta(R,S,T')$.*

Proof. The equalities

$$\begin{aligned}\Delta(R,S,T\vee T') &= \Delta(\Delta(R,S),\Delta(R,T\vee T'))\\ &= \Delta(\Delta(R,S),\Delta(R,T)\vee\Delta(R,T'))\\ &= \Delta(\Delta(R,S),\Delta(R,T))\vee\Delta(\Delta(R,S),\Delta(R,T'))\\ &= \Delta(R,S,T)\vee\Delta(R,S,T')\end{aligned}$$

follow from Proposition 3.24 and Proposition 4.8. ∎

Recall the notation 2-$\mathsf{Eq}(\mathbb{X})$ introduced in 3.21. Given a regular Mal'tsev category $\mathbb{X}$, we now write 3-$\mathsf{Eq}(\mathbb{X})$ for the category whose objects are quadruples (X,R,S,T) where R, S and T are equivalence relations on a common object X and arrows in 3-$\mathsf{Eq}(\mathbb{X})$ are quadruples (f,f_R,f_S,f_T) making the diagram

$$\begin{tikzcd}
R \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, "f_R"'] & & X \arrow[dd, "f"] & & S \arrow[ll, shift left] \arrow[ll, shift right] \arrow[dd, "f_S"] \\
& T \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dd, "f_T"'] & & & \\
R' \arrow[rr, shift left] \arrow[rr, shift right] & & X' & & S' \arrow[ll, shift left] \arrow[ll, shift right] \\
& T' \arrow[ur, shift left] \arrow[ur, shift right] & & &
\end{tikzcd}$$

commute. In other words, any arrow $f\colon X\to X'$ in $\mathbb{X}$ such that $f(R)\leqslant R'$, $f(S)\leqslant S'$ and $f(T)\leqslant T'$ determines and arrow $f\colon (X,R,S,T)\to(X',R',S',T')$ in 3-$\mathsf{Eq}(\mathbb{X})$. The category 3-$\mathsf{Eq}(\mathbb{X})$ is again a regular Mal'tsev category. In particular, limits in 3-$\mathsf{Eq}(\mathbb{X})$ are levelwise limits in $\mathbb{X}$ and regular epimorphisms in 3-$\mathsf{Eq}(\mathbb{X})$ are determined by regular epimorphisms $f\colon X\to X'$ in $\mathbb{X}$ such that $f(R)=R'$, $f(S)=S'$ and $f(T)=T'$.

4.11. Proposition. *Let $\mathbb{X}$ be an exact Mal'tsev category with binary coproducts. For any regular epimorphism $(f, f_R, f_S, f_T)\colon (X, R, S, T) \to (Y, R', S', T')$ in $3\text{-}\mathsf{Eq}(\mathbb{X})$, the induced morphism $f_\Delta\colon \Delta(R, S, T) \to \Delta(R', S', T')$ is a regular epimorphism as well, so that $f(\Delta(R, S, T)) = \Delta(f(R), f(S), f(T))$.*

Proof. This combines Proposition 4.8 with Proposition 3.22. ■

5. Triples of centralising relations

5.1. Ternary Bulatov centrality in varieties of universal algebra. We recall from [1, 11, 26] the definition of *Bulatov centrality* for triples of equivalence relations on a common object in a variety of universal algebras based on the so-called *term condition* (Proposition 5.3):

5.2. Definition. *In a variety of universal algebras $\mathbb{V}$, let X be an algebra and let R, S and T be congruences on X. We say that R, S* ***centralise*** *T and we write $C(R, S; T)$ when for every $(k + n + l)$-ary term t in the theory of $\mathbb{V}$ and for all vectors a_0, $b_0 \in X^k$, a_1, $b_1 \in X^n$, and a_2, $b_2 \in X^l$ such that a_0Rb_0, a_1Sb_1 and a_2Tb_2, if the condition*

$$t(x_0, x_1, a_2) = t(x_0, x_1, b_2)$$

holds for all $(x_0, x_1) \in \bigl(\{a_0, b_0\} \times \{a_1, b_1\}\bigr)\backslash\{(b_0, b_1)\}$, then $t(b_0, b_1, a_2) = t(b_0, b_1, b_2)$.

We are going to reinterpret this in terms of $\Delta(R, S, T)$. We first write $M(R, S, T)$ for the set of all three-dimensional matrices

$$m = (m_I)_{I\subseteq 3} \in \square(R, S, T)$$

such that $m_I = t(x_0, x_1, x_2)$ where

$$x_i = \begin{cases} a_i & \text{if } i \in I \\ b_i & \text{if } i \notin I. \end{cases}$$

Then the following characterisation follows immediately from Definition 5.2:

5.3. Proposition. *R, S centralise T when for every $m \in M(R, S, T)$, the condition $m_{I\cup\{2\}} = m_I$ for all $\varnothing \neq I \subseteq 2$ implies $m_{\{2\}} = m_\varnothing$.*

The following now clarifies the link with the previous section.

5.4. Theorem. *In a Mal'tsev variety, let R, S and T be congruences on a common object. Then $M(R, S, T) = \Delta(R, S, T)$.*

Proof. We view both as congruences on $M(R, S) = \Delta(R, S)$: for $M(R, S, T)$ this follows from Lemma 3.4 (iii) in [26], which also shows that the generators of $\Delta(R, S, T)$ are in $M(R, S, T)$. It is clear that also $M(R, S, T) \leqslant \Delta(R, S, T)$, because the generators of $M(R, S, T)$ are all contained in $\Delta(R, S, T)$—see Example 4.4.

An alternative proof follows from a result in [25], which shows that in a congruence modular variety, $\Delta(R, S, T)$ is the transitive closure of $M(R, S, T)$. This closure operation is not necessary in the Mal'tsev context. ■

5.5. Remark. Note how the symmetry in the variables R, S, T in the three-fold relation $\Delta(R,S,T)$ established in Proposition 4.9 implies that in any Mal'tsev variety, also the condition $C(R,S;T)$ and the one in Proposition 5.3 must be symmetric in those variables. When this condition holds, we may say that R**,** S **and** T **centralise each other**.

Theorem 5.4 is a first step towards a purely categorical approach to Bulatov centrality. The categorical interpretation of the relation $M(R,S,T)$ now being clear, we next aim to express the condition that appears in Proposition 5.3 in categorical terms. In particular, we wish to characterise Bulatov centrality in two equivalent ways: (1) through the concept of a *centralising three-fold equivalence relation* and (2) via the existence of some kind of *three-dimensional connector* (cf. 3.12 for the two-dimensional case).

5.6. Seven out of eight. In a Mal'tsev variety, triples of equivalence relations that centralise each other may be characterised as follows.

5.7. Lemma. *In a Mal'tsev variety, let R, S and T be equivalence relations on an algebra X such that R, S centralise T in the sense of Definition 5.2. If q and r are both in $\Delta(R,S,T)$ and $q_I = r_I$ for all $\varnothing \neq I \subseteq 3$, then $q_\varnothing = r_\varnothing$.*

Proof. This follows form Proposition 5.3. Let p be a Mal'tsev operation. In the matrix $m \in \Delta(R,S,T)$ defined by $m = p(q,r,s)$ where $s_I = r_{I\setminus\{2\}}$, we have $m_\varnothing = p(q_\varnothing, r_\varnothing, r_\varnothing) = q_\varnothing$ while $m_{\{2\}} = p(q_{\{2\}}, r_{\{2\}}, r_\varnothing) = r_\varnothing$ and

$$\begin{aligned} m_{I\cup\{2\}} &= p(q_{I\cup\{2\}}, r_{I\cup\{2\}}, s_{I\cup\{2\}}) = p(q_{I\cup\{2\}}, r_{I\cup\{2\}}, r_I) \\ &= r_I = p(q_I, r_I, r_I) = p(q_I, r_I, s_I) = m_I \end{aligned}$$

for all $\varnothing \neq I \subseteq 2$. ■

This naturally leads to the object $\Delta^I(R,S,T) \coloneqq \boxdot^I\Delta(R,S,T)$ of "three-dimensional matrices $x \in \Delta(R,S,T)$ where the entry x_I is removed".

Let R, S and T be equivalence relations on an object X of a finitely cocomplete regular Mal'tsev category $\mathbb{X}$. Any choice of a subset $I \subseteq 3$, for $3 \coloneqq \{0,1,2\}$ corresponds to one of the commutative three-cubes which appear in the three-fold equivalence relation $\Delta(R,S,T)$; namely the three-fold extension denoted by $\Delta(R,S,T)^I$ and displayed as in Figure 11. The projections are determined by the characteristic function δ_I defined on page 383. Let us consider the three-fold extension $\Delta(R,S,T)^I$, remove its top object $\Delta(R,S,T)^I_3 = \Delta(R,S,T)$, then take the limit $\mathrm{L}(\Delta(R,S,T)^I)$ of the remaining diagram as in 2.4. It consists of an object $\boxdot^I\Delta(R,S,T)$ with projections as in Figure 12.

5.8. Lemma. *Given any equivalence relations R, S and T on an object X of a finitely cocomplete exact Mal'tsev category, for each choice of I, $J \subseteq 3$ we have*

$$\boxdot^I\Delta(R,S,T) = \mathrm{L}(\Delta(R,S,T)^I) \cong \mathrm{L}(\Delta(R,S,T)^J) = \boxdot^J\Delta(R,S,T).$$

Proof. This follows immediately from the symmetry of the equivalence relations of which the three-fold equivalence relation $\Delta(R,S,T)$ consists. ■

$$
\begin{tikzcd}
& \Delta(R,S,T) \arrow[rr, "d^{\Delta(S,T)}_{\delta_I(0)}"] \arrow[dl, "d^{\Delta(R,S)}_{\delta_I(2)}"'] \arrow[dd, "d^{\Delta(R,T)}_{\delta_I(1)}" near start] & & \Delta(S,T) \arrow[dl, "d^{S}_{\delta_I(2)}"'] \arrow[dd, "d^{T}_{\delta_I(1)}"] \\
\Delta(R,S) \arrow[rr, "d^{S}_{\delta_I(0)}" near end] \arrow[dd, "d^{R}_{\delta_I(1)}"'] & & S \arrow[dd, "s_{\delta_I(1)}"' near end] & \\
& \Delta(R,T) \arrow[rr, "d^{T}_{\delta_I(0)}" near start] \arrow[dl, "d^{R}_{\delta_I(2)}"] & & T \arrow[dl, "t_{\delta_I(2)}"] \\
R \arrow[rr, "r_{\delta_I(0)}"'] & & X &
\end{tikzcd}
$$

Figure 11: A choice of projections determined by $I \subseteq 3$ in the three-fold equivalence relation $\Delta(R,S,T)$

$$
\begin{tikzcd}
& \boxdot^I \Delta(R,S,T) \arrow[rr, "\pi^{\boxdot}_{ST}"] \arrow[dl, "\pi^{\boxdot}_{RS}"'] \arrow[dd, "\pi^{\boxdot}_{RT}" near start] & & \Delta(S,T) \arrow[dl, "d^{S}_{\delta_I(2)}"'] \arrow[dd, "d^{T}_{\delta_I(1)}"] \\
\Delta(R,S) \arrow[rr, "d^{S}_{\delta_I(0)}" near end] \arrow[dd, "d^{R}_{\delta_I(1)}"'] & & S \arrow[dd, "s_{\delta_I(1)}"' near end] & \\
& \Delta(R,T) \arrow[rr, "d^{T}_{\delta_I(0)}" near start] \arrow[dl, "d^{R}_{\delta_I(2)}"] & & T \arrow[dl, "t_{\delta_I(2)}"] \\
R \arrow[rr, "r_{\delta_I(0)}"'] & & X &
\end{tikzcd}
$$

Figure 12: The three-fold extension $\Delta^I(R,S,T)$ for $I \subseteq 3$

This symmetry allows us to fix a choice of $I \subseteq 3$ once and for all, and thus make our notations somewhat less heavy. We choose $I = \varnothing$ and write $\Delta(R, S, T)$ for $\boxdot^{\varnothing}\Delta(R, S, T)$.

5.9. Proposition. *In any Mal'tsev variety, let R, S and T be congruences on an algebra X. For $a \in \Delta(R, S, T)$ as on the left,*

$$a = \begin{tikzcd}[row sep=small, column sep=small] & y \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\ t \arrow[rr, dash, dotted] & & z \arrow[dd, dash, dotted] & \\ & v \arrow[rr, dash, dotted] & & u \arrow[dl, dash, dotted] \\ & & w & \end{tikzcd} \qquad\qquad \begin{tikzcd}[row sep=small, column sep=small] & y \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\ t \arrow[rr, dash, dotted] \arrow[dd, dash, dotted] & & z \arrow[dd, dash, dotted] & \\ & v \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] & & u \arrow[dl, dash, dotted] \\ t_3(a) \arrow[rr, dash, dotted] & & w & \end{tikzcd}$$

we write $t_3(a) = p(p(x, y, z), t, p(u, v, w))$, where $p(x, y, z) = m(y, x, z)$ and m is a Mal'tsev term. Then the $2 \times 2 \times 2$ matrix on the right is an element of $\Delta(R, S, T)$.

Proof. First, let us observe that the term t_3 satisfies the equations

$$\begin{aligned} t_3(x, y, z, t, x, y, z) &= p(p(x, y, z), t, p(x, y, z)) = t, \\ t_3(x, x, z, z, u, u, w) &= p(p(x, x, z), z, p(u, u, w)) = p(z, z, w) = w, \\ t_3(x, y, x, y, u, v, u) &= p(p(x, y, x), y, p(u, v, u)) = p(y, y, v) = v. \end{aligned}$$

In view of Remark 4.5, it is not difficult to prove that the matrices A–G in the diagram of Figure 13 are all in $\Delta(R, S, T)$. Since $\Delta(R, S, T)$ is an algebra, it follows that $t_3(A, B, C, D, E, F, G)$ is an element of $\Delta(R, S, T)$. It is, however, easy to see that this element is equal to the the $2 \times 2 \times 2$ matrix on the right in the statement of the proposition. ■

So in a Mal'tsev variety, each element of $\Delta(R, S, T)$ may be completed to an element of $\Delta(R, S, T)$, and when R, S and T centralise each other, this element is uniquely determined. In other words, the cube in Figure 11 given by the choice of projections $I = \varnothing$ does actually coincide with the cube in Figure 12. That is to say, it is a limit cube. This property happens to be characteristic of triples of centralising relations, and naturally leads to the following definitions.

5.10. Definition. *In an exact Mal'tsev category with binary coproducts, consider equivalence relations R, S and T on an object X. A three-fold equivalence relation C on R, S and T is a* ***three-fold Δ-equivalence relation****, when the sub–two-fold equivalence relation of C determined by the relations R and S (respectively R and T, or S and T) is $\Delta(R, S)$ (respectively $\Delta(R, T)$, or $\Delta(S, T)$)—see Figure 1.*

A three-fold Δ-equivalence relation C on R, S and T is a ***centralising*** *three-fold Δ-equivalence relation when for all $I \subseteq 3$, the three-fold extension consisting of the projections determined by the set I is a limit cube.*

We say that R, ***S and T centralise each other*** *when they admit a centralising three-fold Δ-equivalence relation.*

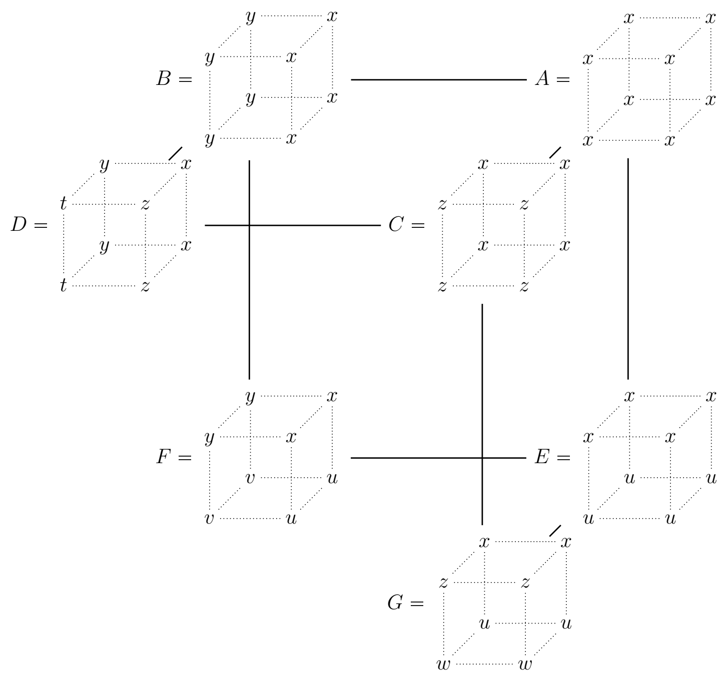


Figure 13: $2 \times 2 \times 2$ matrices

5.11. Remark. Note that we may reason as in Lemma 5.8 to show that when any of the cubes, obtained from C by choosing projections determined by a set $I \subseteq 3$, is a limit, then all of them are such. In particular, we may always choose $I = \varnothing$.

5.12. Proposition. *In an exact Mal'tsev category with binary coproducts, whenever three equivalence relations R, S and T on an object X centralise each other, the three-fold Δ-equivalence relation $\Delta(R,S,T)$ on R, S and T is centralising.*

Proof. Let C be a centralising three-fold Δ-equivalence relation on R, S and T. Then the comparison morphism $\Delta(R,S,T) \to C_3 \cong \Delta(R,S,T)$ is a regular epimorphism, since the cube of projections determined by $\varnothing$ is a three-fold extension. The result then follows, because as equivalence relations on $\Delta(R,S)$, we have $\Delta(R,S,T) \leqslant C_3$. ∎

5.13. Lemma. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. Then the following are equivalent:*

(i) *$\Delta(R,S)$ and $\Delta(R,T)$ admit a centralising double equivalence relation, as relations on R;*

(ii) *S and T admit a centralising double equivalence relation, as relations on X.*

Proof. We must prove that in the diagram below, the left hand side vertical square is a pullback if and only if so is the middle one.

$$
\begin{tikzcd}
& \Delta(R,S,T) \arrow[rr, two heads, shift left] \arrow[rr, two heads, shift right] \arrow[dl, two heads, "d_1^{\Delta(R,S)}"'] \arrow[dd, two heads, "d_2^{\Delta(R,T)}"] & & \Delta(S,T) \arrow[rr, two heads, "q_{\Delta\Delta}"] \arrow[dl, two heads, "d_1^S"'] \arrow[dd, two heads, "d_2^T"] & & Q_{\Delta\Delta} \arrow[dl, two heads] \arrow[dd, two heads] \\
\Delta(R,S) \arrow[rr, two heads, shift left] \arrow[rr, two heads, shift right] \arrow[dd, two heads, "d_2^R"'] & & S \arrow[rr, two heads, "q_{SR}"] \arrow[dd, two heads, "s_2"] & & Q_{SR} \arrow[dd, two heads] & \\
& \Delta(R,T) \arrow[rr, two heads, shift left] \arrow[rr, two heads, shift right] \arrow[dl, two heads, "d_1^R"'] & & T \arrow[rr, two heads, "q_{TR}"] \arrow[dl, two heads, "t_1"'] & & Q_{TR} \arrow[dl, two heads] \\
R \arrow[rr, two heads, shift left] \arrow[rr, two heads, shift right] & & X \arrow[rr, two heads, "q_R"'] & & Q_R &
\end{tikzcd}
$$

One implication is immediate from Lemma 3.26; if the left hand side vertical square is a pullback, then so is the middle one.

Now assume that the middle vertical square is a pullback. Then, by Lemma 3.26 again, the right hand side vertical square is a pullback. The result now follows, since kernel pairs commute with pullbacks. ∎

5.14. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. If S and T centralise each other, then so do R, S and T.*

Proof. Use Lemma 5.13: in the diagram of the proof, when the left vertical square is a pullback, the cubes on the left are limit cubes. ∎

5.15. Example. The discrete equivalence relation Δ_X centralises any pair of equivalence relations S, T, because S and Δ_X centralise each other.

5.16. Three-dimensional connectors. The projections in Figure 12 are split epimorphisms, and come with canonical splittings induced by the ones expressing the reflexivity of the relations present in the diagram. They are denoted

$$\sigma^{\boxdot}_{RS}\colon \Delta(R,S) \to \Delta(R,S,T), \qquad \sigma^{\boxdot}_{RT}\colon \Delta(R,T) \to \Delta(R,S,T)$$

and $\sigma^{\boxdot}_{ST}\colon \Delta(S,T) \to \Delta(R,S,T)$ as in Figure 14.

5.17. Proposition. *Given any equivalence relations R, S and T on an object X of a finitely cocomplete exact Mal'tsev category, the inclusions $\sigma^{\boxdot}_{RS}$ and $\sigma^{\boxdot}_{RT}$ are jointly extremal-epimorphic. In particular, so are the three canonical inclusions defined above.*

Proof. This is a consequence of Proposition 3.30 combined with Proposition 4.8. ∎

$$
\begin{tikzcd}
 & \Delta(R,S,T) \arrow[dddr, dotted, "\theta"] & & \Delta(S,T) \arrow[ll, "\sigma^{\boxdot}_{ST}"'] \arrow[dl, "d_2^S"'] \arrow[dd, "d_2^T"] \\
\Delta(R,S) \arrow[ur, "\sigma^{\boxdot}_{RS}"] \arrow[rr, "d_2^S"] \arrow[dd, "d_2^R"'] & & S \arrow[dd, "s_2"] & \\
 & \Delta(R,T) \arrow[uu, "\sigma^{\boxdot}_{RT}"'] \arrow[rr, "d_2^T"] \arrow[dl, "d_2^R"] & & T \arrow[dl, "t_2"] \\
R \arrow[rr, "r_2"'] & & X &
\end{tikzcd}
$$

Figure 14: The diagram $\mathrm{Cube}^{\varnothing}(R, S, T)$: the dotted morphism exists when R, S and T are Bulatov connected

5.18. Definition. *Consider equivalence relations R, S and T on an object X of a finitely cocomplete exact Mal'tsev category. They are said to be **(Bulatov) connected** if a morphism $\theta\colon \Delta(R, S, T) \to X$ exists for which the diagram in Figure 14 commutes. When such a θ exists, it is necessarily unique (by Proposition 5.17) and called a **(three-dimensional) connector on** R**,** S **and** T.*

Commutativity of the diagram in Figure 14 means that the squares induced by the construction of $\Delta(R, S)$, $\Delta(R, T)$ and $\Delta(S, T)$ commute and the morphism θ is such that $\theta\circ\sigma^{\boxdot}_{RS} = r_2\circ d_2^R$, $\theta\circ\sigma^{\boxdot}_{RT} = t_2\circ d_2^T$ and $\theta\circ\sigma^{\boxdot}_{ST} = s_2\circ d_2^S$.

Note that the projections here are *second* projections, whereas $\Delta(R, S, T)$ was computed as a limit from the diagram of *first* projections.

Generalising the construction in Figure 14, in what follows, we write $\mathrm{Cube}^I(D)$ for the diagram $\Delta^{3\setminus I}D$ in which the initial object and the arrows pointing towards it are replaced by the ones of the limit $\Delta^I D$.

5.19. Theorem. *In an exact Mal'tsev category with binary coproducts, consider equivalence relations R, S and T on an object X. The following conditions are equivalent:*

(i) *R, S and T are Bulatov connected (Definition 5.18);*

(ii) *R, S and T centralise each other (Definition 5.10).*

Proof. (i) $\Rightarrow$ (ii) Let $\theta\colon \Delta(R, S, T) \to X$ be a connector between R, S and T. Let us consider $\Delta(R, S, T)$ as the reflexive relation

$$
\begin{tikzcd}
\Delta(R,S,T) \arrow[r, shift left=2, "\pi^{\boxdot}_{RT}"] \arrow[r, shift right=2, "p_{\Delta(R,T)}"'] & \Delta(R,T) \arrow[l]
\end{tikzcd}
$$

on $\Delta(R,T)$ where $\pi^{\boxdot}_{RT}$ is one of the projections induced by the limit, together with its canonical splitting, and $p_{\Delta(R,T)}$ is the morphism defined by

$$
a = \begin{tikzcd}[row sep=small, column sep=small]
 & a_1 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & a_0 \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
a_3 \arrow[rr, dash, dotted] & & a_2 \arrow[dd, dash, dotted] & \\
 & a_5 \arrow[rr, dash, dotted] & & a_4 \arrow[dl, dash, dotted] \\
 & & a_6 &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
a_3 \arrow[r, dash, dotted] \arrow[d, dash, dotted] & a_2 \arrow[d, dash, dotted] \\
\theta(a) \arrow[r, dash, dotted] & a_6.
\end{tikzcd}
$$

Similarly, we view $\Delta(R,S,T)$ as an equivalence relation on $\Delta(R,S)$ and $\Delta(S,T)$. This makes it into a centralising three-fold Δ-equivalence relation on R, S and T.

(ii) $\Rightarrow$ (i) If there exists a centralising three-fold Δ-equivalence relation C on R, S and T, then the composite of second projections

$$r_2 \circ d_2^R \circ d_2^{\Delta(R,S)} \colon \Delta(R,S,T) \cong C_3 \to X$$

is a connector on R, S and T. ■

5.20. Example. In the context of a Mal'tsev variety, similarly to the presentation of $\Delta(R,S,T)$ in Example 4.4, the object $\Delta(R,S,T)$ is generated as a subalgebra of X^7 by elements of the form

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & b \arrow[rr, dash, dotted, "R"] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & a \arrow[dl, dash, dotted] \arrow[dd, dash, dotted, "T"] \\
b \arrow[rr, dash, dotted] & & a \arrow[dd, dash, dotted] & \\
 & b \arrow[rr, dash, dotted] & & a \arrow[dl, dash, dotted, "S"] \\
 & & a &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & c \arrow[rr, dash, dotted, "R"] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & c \arrow[dl, dash, dotted] \arrow[dd, dash, dotted, "T"] \\
d \arrow[rr, dash, dotted] & & d \arrow[dd, dash, dotted] & \\
 & c \arrow[rr, dash, dotted] & & c \arrow[dl, dash, dotted, "S"] \\
 & & d &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & e \arrow[rr, dash, dotted, "R"] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & e \arrow[dl, dash, dotted] \arrow[dd, dash, dotted, "T"] \\
e \arrow[rr, dash, dotted] & & e \arrow[dd, dash, dotted] & \\
 & f \arrow[rr, dash, dotted] & & f \arrow[dl, dash, dotted, "S"] \\
 & & f &
\end{tikzcd}
$$

Thus, given a morphism of algebras $\theta \colon \Delta(R,S,T) \to X$, the commutativity of the diagram in Definition 5.18 can be reformulated element-wise by asking that for any

$$
x = \begin{tikzcd}[row sep=small, column sep=small]
 & x_1 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x_0 \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
x_3 \arrow[rr, dash, dotted] & & x_2 \arrow[dd, dash, dotted] & \\
 & x_5 \arrow[rr, dash, dotted] & & x_4 \arrow[dl, dash, dotted] \\
 & & x_6 &
\end{tikzcd}
\in \Delta(R,S,T)
$$

the equalities

$$
\begin{aligned}
\theta(x_0, x_1, x_2, x_3, x_0, x_1, x_2) &= x_3 \\
\theta(x_0, x_1, x_0, x_1, x_4, x_5, x_4) &= x_5 \\
\theta(x_0, x_0, x_2, x_2, x_4, x_4, x_6) &= x_6
\end{aligned}
$$

hold.

For instance, $t_3(a) = p(p(x,y,z), t, p(u,v,w))$, for $p(x,y,z) = m(y,x,z)$ where m is a Mal'tsev term, satisfies these equalities.

5.21. Theorem. *In any Mal'tsev variety $\mathbb{V}$, Definition 5.2 and Definition 5.18 are equivalent.*

Proof. Assume that R and S centralise T as in Proposition 5.3. We must then prove that the map $\theta = t_3\colon \Delta(R,S,T) \to X$ is a homomorphism of algebras. Let t be an l-ary term in the theory of $\mathbb{V}$ and let x^i for $i = 1, \dots, l$ be elements of $\Delta(R,S,T)$. Without loss of generality, we may assume that $l = 2$ and that x^1 and x^2 are respectively given by the cubes

$$\begin{tikzcd}[column sep=small, row sep=small]
& x_1^1 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x_0^1 \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
x_3^1 \arrow[rr, dash, dotted] & & x_2^1 \arrow[dd, dash, dotted] & \\
& x_5^1 \arrow[rr, dash, dotted] & & x_4^1 \arrow[dl, dash, dotted] \\
& & x_6^1 &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small, row sep=small]
& x_1^2 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x_0^2 \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
x_3^2 \arrow[rr, dash, dotted] & & x_2^2 \arrow[dd, dash, dotted] & \\
& x_5^2 \arrow[rr, dash, dotted] & & x_4^2 \arrow[dl, dash, dotted] \\
& & x_6^2 &
\end{tikzcd}$$

The element $t(x^1, x^2)$, which also belongs to $\Delta(R,S,T)$, is the matrix on the left below.

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& t(x_1^1,x_1^2) \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & t(x_0^1,x_0^2) \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
t(x_3^1,x_3^2) \arrow[rr, dash, dotted] & & t(x_2^1,x_2^2) \arrow[dd, dash, dotted] & \\
& t(x_5^1,x_5^2) \arrow[rr, dash, dotted] & & t(x_4^1,x_4^2) \arrow[dl, dash, dotted] \\
& & t(x_6^1,x_6^2) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny, row sep=small]
& t(x_1^1,x_1^2) \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & t(x_0^1,x_0^2) \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
t(x_3^1,x_3^2) \arrow[rr, dash, dotted] \arrow[dd, dash, dotted] & & t(x_2^1,x_2^2) \arrow[dd, dash, dotted] & \\
& t(x_5^1,x_5^2) \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] & & t(x_4^1,x_4^2) \arrow[dl, dash, dotted] \\
\theta(t(x^1,x^2)) \arrow[rr, dash, dotted] & & t(x_6^1,x_6^2) &
\end{tikzcd}$$

Proposition 5.9 tells us that the matrix on the right, which we shall call q, is in $\Delta(R,S,T)$. When we apply again Proposition 5.9 to x^1 and x^2, it follows that the elements $\bar{x}^1$ and $\bar{x}^2$ given respectively by the cubes

$$\begin{tikzcd}[column sep=small, row sep=small]
& x_1^1 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x_0^1 \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
x_3^1 \arrow[rr, dash, dotted] \arrow[dd, dash, dotted] & & x_2^1 \arrow[dd, dash, dotted] & \\
& x_5^1 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] & & x_4^1 \arrow[dl, dash, dotted] \\
\theta(x^1) \arrow[rr, dash, dotted] & & x_6^1 &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small, row sep=small]
& x_1^2 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] & & x_0^2 \arrow[dl, dash, dotted] \arrow[dd, dash, dotted] \\
x_3^2 \arrow[rr, dash, dotted] \arrow[dd, dash, dotted] & & x_2^2 \arrow[dd, dash, dotted] & \\
& x_5^2 \arrow[rr, dash, dotted] \arrow[dl, dash, dotted] & & x_4^2 \arrow[dl, dash, dotted] \\
\theta(x^2) \arrow[rr, dash, dotted] & & x_6^2 &
\end{tikzcd}$$

are in $\Delta(R,S,T)$. So, the element $r \coloneqq t(\bar{x}^1, \bar{x}^2)$ is then also an element of $\Delta(R,S,T)$, because $\Delta(R,S,T)$ is an algebra.

Now, q and r are elements of $\Delta(R,S,T)$ such that $q_I = r_I$ for all $\varnothing \neq I \subseteq 3$ and $q_\varnothing = \theta(t(x^1,x^2))$ and $r_\varnothing = t(\theta(x^1), \theta(x^2))$. Since R and S centralise T, it follows by Lemma 5.7 that $q_\varnothing = r_\varnothing$. Therefore $\theta(t(x^1,x^2)) = t(\theta(x^1), \theta(x^2))$.

For the converse, we assume that R, S and T are Bulatov connected, so that we have a morphism of algebras $\theta\colon \Delta(R,S,T) \to X$. Let m be an element of $\Delta(R,S,T)$ such that $m_{I\cup\{2\}} = m_I$ for all $\varnothing \neq I \subseteq 2$. We must then show that $m_\varnothing$ equals $m_{\{2\}}$. Since $\Delta(R,S,T) = M(R,S,T)$, it follows that m can be written in the form

$$m = \begin{tikzcd}[sep=small] & t(b_0,a_1,a_2) \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & t(a_0,a_1,a_2) \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ t(b_0,b_1,a_2) \arrow[rr, dotted, dash] \arrow[dd, dotted, dash] & & t(a_0,b_1,a_2) \arrow[dd, dotted, dash] & \\ & t(b_0,a_1,b_2) \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] & & t(a_0,a_1,b_2) \arrow[dl, dotted, dash] \\ t(b_0,b_1,b_2) \arrow[rr, dotted, dash] & & t(a_0,b_1,b_2) & \end{tikzcd}$$

for some $(k+n+l)$-ary term t in the theory of $\mathbb{V}$ and for all vectors a_0, $b_0 \in X^k$, a_1, $b_1 \in X^n$, and a_2, $b_2 \in X^l$ such that $a_0 R b_0$, $a_1 S b_1$ and $a_2 T b_2$. Hence

$$\begin{aligned} t(b_0,b_1,b_2) &= t\left(\theta\left(\begin{tikzcd}[sep=tiny] & b_0 \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & a_0 \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ b_0 \arrow[rr, dotted, dash] & & a_0 \arrow[dd, dotted, dash] & \\ & b_0 \arrow[rr, dotted, dash] & & a_0 \arrow[dl, dotted, dash] \\ & & a_0 & \end{tikzcd}\right), \theta\left(\begin{tikzcd}[sep=tiny] & a_1 \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & a_1 \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ b_1 \arrow[rr, dotted, dash] & & b_1 \arrow[dd, dotted, dash] & \\ & a_1 \arrow[rr, dotted, dash] & & a_1 \arrow[dl, dotted, dash] \\ & & b_1 & \end{tikzcd}\right), \theta\left(\begin{tikzcd}[sep=tiny] & a_2 \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & a_2 \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ a_2 \arrow[rr, dotted, dash] & & a_2 \arrow[dd, dotted, dash] & \\ & b_2 \arrow[rr, dotted, dash] & & b_2 \arrow[dl, dotted, dash] \\ & & b_2 & \end{tikzcd}\right)\right) \\ &= \theta\left(t\left(\begin{tikzcd}[sep=tiny] & b_0 \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & a_0 \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ b_0 \arrow[rr, dotted, dash] & & a_0 \arrow[dd, dotted, dash] & \\ & b_0 \arrow[rr, dotted, dash] & & a_0 \arrow[dl, dotted, dash] \\ & & a_0 & \end{tikzcd}, \begin{tikzcd}[sep=tiny] & a_1 \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & a_1 \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ b_1 \arrow[rr, dotted, dash] & & b_1 \arrow[dd, dotted, dash] & \\ & a_1 \arrow[rr, dotted, dash] & & a_1 \arrow[dl, dotted, dash] \\ & & b_1 & \end{tikzcd}, \begin{tikzcd}[sep=tiny] & a_2 \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & a_2 \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ a_2 \arrow[rr, dotted, dash] & & a_2 \arrow[dd, dotted, dash] & \\ & b_2 \arrow[rr, dotted, dash] & & b_2 \arrow[dl, dotted, dash] \\ & & b_2 & \end{tikzcd}\right)\right) \\ &= \theta\left(\begin{tikzcd}[sep=small] & t(b_0,a_1,a_2) \arrow[rr, dotted, dash] \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] & & t(a_0,a_1,a_2) \arrow[dl, dotted, dash] \arrow[dd, dotted, dash] \\ t(b_0,b_1,a_2) \arrow[rr, dotted, dash] & & t(a_0,b_1,a_2) \arrow[dd, dotted, dash] & \\ & t(b_0,a_1,b_2) \arrow[rr, dotted, dash] & & t(a_0,a_1,b_2) \arrow[dl, dotted, dash] \\ & & t(a_0,b_1,b_2) & \end{tikzcd}\right) \end{aligned}$$

which is equal to $t(b_0,b_1,a_2)$ by the identities of a three-dimensional connector and the fact that $m_{I\cup\{2\}} = m_I$ for all $\varnothing \neq I \subseteq 2$. ∎

5.22. Stability properties. We extend the stability properties of $\Delta(R,S)$ to the threefold equivalence relation $\Delta(R,S,T)$. We first need to recall a well-known technical result:

5.23. Lemma. *[18] In a regular Mal'tsev category, consider the diagram*

$$
\begin{tikzcd}
& W \times_D C \arrow[dl] \arrow[rr, dotted, "v"] \arrow[dd, shift left, "k"] & & Y \times_B A \arrow[dl] \arrow[dd, shift left, "h"] \\
C \arrow[rr, two heads, "c"] \arrow[dd, shift left, "g"] & & A \arrow[dd, shift left, "f"] & \\
& W \arrow[uu, shift left, "j"] \arrow[dl] \arrow[rr, two heads, "w"] & & Y \arrow[uu, shift left, "i"] \arrow[dl] \\
D \arrow[uu, shift left, "t"] \arrow[rr, two heads, "d"'] & & B \arrow[uu, shift left, "s"] &
\end{tikzcd}
$$

where c, d and w are regular epimorphisms and the downward pointing arrows are split epimorphisms, with upward-pointing splittings. Then the comparison morphism v is a regular epimorphism and both front and back faces are regular pushouts.

5.24. Proposition. *In an an exact Mal'tsev category with binary coproducts, consider a regular epimorphism $f\colon X \to Y$. Then for equivalence relations R, S and T on X, the comparison morphism $f_{\Delta(R,S,T)}\colon \Delta(R,S,T) \to \Delta(f(R), f(S), f(T))$ is a regular epimorphism.*

Proof. By Proposition 3.22, the regular epimorphism f induces regular epimorphisms $f_{\Delta(R,S)}\colon \Delta(R,S) \to \Delta(f(R), f(S))$, $f_{\Delta(R,T)}\colon \Delta(R,T) \to \Delta(f(R), f(T))$ and

$$f_{\Delta(S,T)}\colon \Delta(S,T) \to \Delta(f(S), f(T)).$$

By Lemma 5.23, the morphism

$$\boxdot(\Delta(R,S), \Delta(R,T)) \to \boxdot(\Delta(f(R), f(S)), \Delta(f(R), f(T)))$$

is a regular epimorphism. We may now use Lemma 5.23 on the diagram

$$
\begin{tikzcd}
& \Delta(R,S,T) \arrow[rr, "f_{\Delta(R,S,T)}"] \arrow[dl, two heads] \arrow[dd, two heads] & & \Delta(f(R), f(S), f(T)) \arrow[dl, two heads] \arrow[dd, two heads] \\
\Delta(S,T) \arrow[rr, two heads, "f_{\Delta(S,T)}"] \arrow[dd, two heads] & & \Delta(f(S), f(T)) \arrow[dd, two heads] & \\
& \boxdot(\Delta(R,S), \Delta(R,T)) \arrow[rr, two heads] \arrow[dl, shift left] & & \boxdot(\Delta(f(R), f(S)), \Delta(f(R), f(T))) \arrow[dl, shift left] \\
\boxdot(S,T) \arrow[ur, shift left] \arrow[rr, two heads, "f_{\boxdot}"'] & & \boxdot(f(S), f(T)) \arrow[ur, shift left] &
\end{tikzcd}
$$

to see that $f_{\Delta(R,S,T)}$ is a regular epimorphism as well. ∎

5.25. Lemma. *In an exact Mal'tsev category with binary coproducts, let C be a three-fold Δ-equivalence relation on equivalence relations R, S and T on an object X, let $f\colon X \to Y$ be a regular epimorphism, and let D be a three-fold Δ-equivalence relation on $f(R), f(S)$ and $f(T)$ such that $f\colon C \to D$ is a (levelwise) regular epimorphism.*

If $C^{\varnothing}$ is a limit cube, then so is $D^{\varnothing}$. Moreover, if C is a centralising three-fold Δ-equivalence relation, then so is D.

Proof. The four-fold arrow $f^{\varnothing}\colon C^{\varnothing} \to D^{\varnothing}$ is a four-fold extension, as a regular epimorphism of three-fold split epimorphisms. Hence the comparison morphism

$$
\begin{tikzcd}
 & C_3 \arrow[rr, shift left, two heads] \arrow[dl, two heads, "f_3"'] \arrow[dd, two heads] & & \Delta(S,T) \arrow[ll, shift left] \arrow[dl, two heads, "f_{\Delta(R,S)}"] \arrow[dd, two heads] \\
D_3 \arrow[rr, shift left, two heads] \arrow[dd, two heads] & & \Delta(f(S),f(T)) \arrow[ll, shift left] \arrow[dd, two heads] & \\
 & \boxdot(\Delta(R,S),\Delta(R,T)) \arrow[rr, shift left, two heads] \arrow[dl, two heads, "f_{\boxdot}"] & & \boxdot(S,T) \arrow[ll, shift left] \arrow[dl, two heads, "f_{\boxdot}"] \\
\boxdot\big(\Delta(f(R),f(S)),\Delta(f(R),f(T))\big) \arrow[rr, shift left, two heads] & & \boxdot(f(S),f(T)) \arrow[ll, shift left] &
\end{tikzcd}
$$

is a three-fold extension. By Lemma 3.26, if the left-hand square is a pullback, then the right-hand square is also a pullback. ■

5.26. Proposition. *In an exact Mal'tsev category with binary coproducts, consider equivalence relations R, S and T on an object X and $f\colon X \to Y$ a regular epimorphism. If R, S and T are Bulatov connected, then $f(R)$, $f(S)$ and $f(T)$ are Bulatov connected.*

Proof. Use Theorem 5.19 and Lemma 5.25. ■

5.27. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S, T and T' be equivalence relations on an object X. The following are equivalent:*

(i) *R, S and T are Bulatov connected, and R, S and T' are Bulatov connected;*

(ii) *R, S and $T \vee T'$ are Bulatov connected.*

Proof. We consider two copies of Figure 10, one induced by T, and another induced by T'. In each case, we view the three-fold equivalence relation as a down-left pointing equivalence relation of double equivalence relations, which a coequaliser we shall denote q and q', respectively. Choosing first projections in q and q', we now see them as three-fold extensions with a common domain. We first take their pushout, then the pullback of the induced square. Since the pushout of q and q' is a four-fold extension as a double split epimorphism of double extensions (pushouts of regular epimorphisms in an exact Mal'tsev category), the comparison

$$
\begin{tikzcd}
 & \Delta(R,S) \arrow[rr, two heads] \arrow[dl, two heads] \arrow[dd, two heads] & & Q_{\Delta(R,S)\Delta(S,T')} \arrow[dl, two heads] \arrow[dd, two heads] \\
Q_{\Delta(R,S)\Delta(S,T)} \arrow[rr, two heads] \arrow[dd, two heads] & & Q_{\Delta(R,S)\Delta(S,T)} +_{\Delta(R,S)} Q_{\Delta(R,S)\Delta(S,T')} \arrow[dd, two heads] & \\
 & \boxdot(R,S) \arrow[rr, two heads] \arrow[dl, two heads] & & \boxdot(Q_{R,T'},Q_{S,T'}) \arrow[dl, two heads] \\
\boxdot(Q_{RT},Q_{ST}) \arrow[rr, two heads] & & \boxdot(Q_{RT} +_R Q_{R,T'}, Q_{S,T} +_S Q_{ST'}) &
\end{tikzcd}
$$

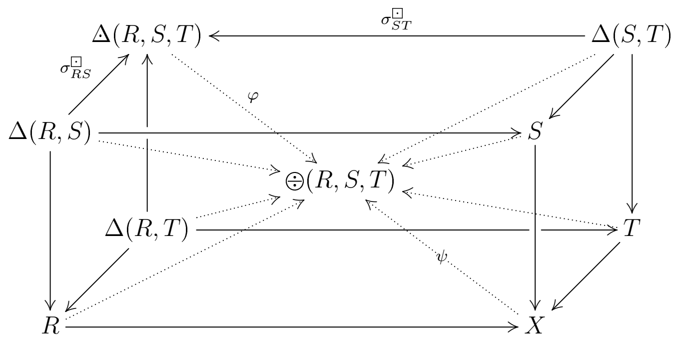


Figure 15: Definition of the Bulatov commutator

to the pullback is a three-fold extension.

If we assume that the assertion (i) holds, then the left-hand square and the back square are pullbacks. According to Lemma 3.26, the vertical faces of the cube are pullbacks, so that the right hand square of the diagram

$$\begin{tikzcd}
\Delta(R,S,T\vee T') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & \Delta(R,S) \arrow[r, two heads] \arrow[d, two heads] & Q_{\Delta(R,S)\Delta(S,T)} +_{\Delta(R,S)} Q_{\Delta(R,S)\Delta(S,T')} \arrow[d, two heads] \\
\boxdot(\Delta(R,T\vee T'),\Delta(S,T\vee T')) \arrow[r, shift left] \arrow[r, shift right] & \boxdot(R,S) \arrow[r, two heads] & \boxdot(Q_{RT} +_R Q_{R,T'}, Q_{S,T} +_S Q_{ST'})
\end{tikzcd}$$

is also a pullback. Conversely, if we assume (ii), then both squares in the left-hand side of the above diagram are pullbacks, hence the right-hand side will also be a pullback. According to Lemma 3.25, the left and the back square of the above cube will be pullbacks, so that (i) holds. ∎

6. The ternary Bulatov commutator

In a finitely cocomplete exact Mal'tsev category, consider equivalence relations R, S and T on a common object X. In Figure 15, we construct the colimit $\odiv(R,S,T)$ of the outer part of the diagram of Definition 5.18. By Proposition 5.17, the inclusions into $\Delta(R,S,T)$ are jointly extremal-epimorphic. This implies that the comparison morphism $\psi\colon X\to\odiv(R,S,T)$ is always an extremal (= regular) epimorphism, since it can be computed as in the pushout

$$\begin{tikzcd}
\Delta(R,S)+\Delta(R,T)+\Delta(S,T) \arrow[r, "\zeta"] \arrow[d, two heads, "\upsilon"'] \arrow[dr, phantom, "(*)"] & X \arrow[d, dotted, two heads, "\psi"] \\
\Delta(R,S,T) \arrow[r, dotted, "\varphi"'] & \odiv(R,S,T)
\end{tikzcd}$$

in the category in $\mathbb{X}$, where the morphisms

$$\zeta = \lgroup r_2 \circ d_2^R, s_2 \circ d_2^S, t_2 \circ d_2^T \rgroup \colon \Delta(R,S) + \Delta(R,T) + \Delta(S,T) \to X$$

and

$$\upsilon = \lgroup \sigma^{\boxdot}_{RS}, \sigma^{\boxdot}_{RT}, \sigma^{\boxdot}_{ST} \rgroup \colon \Delta(R,S) + \Delta(R,T) + \Delta(S,T) \to \Delta(R,S,T)$$

are induced by the universal property of coproducts. Moreover, the morphism υ is a strong epimorphism (= regular epimorphism) since the family $(\sigma^{\boxdot}_{RS}, \sigma^{\boxdot}_{RT}, \sigma^{\boxdot}_{ST})$ is a jointly strongly epimorphic family—see Proposition 5.17.

In fact, if we compute the object $\oplus(R,S,T)$ as in the colimit of Figure 15 and we assume that $a \colon \Delta(R,S,T) \to Z$ and $b \colon X \to Z$ are morphisms such that $a \circ \upsilon = b \circ \zeta$, then $b \circ r_2 \circ d_2^R = a \circ \sigma^{\boxdot}_{RS}$, $b \circ s_2 \circ d_2^S = a \circ \sigma^{\boxdot}_{ST}$ and $b \circ t_2 \circ d_2^T = a \circ \sigma^{\boxdot}_{RT}$, so that we obtain a cocone

$$\begin{tikzcd}
& \Delta(R,S,T) \arrow[ddr, dotted, "a"] & & & \Delta(S,T) \arrow[lll, "\sigma^{\boxdot}_{ST}"'] \arrow[dl] \arrow[ddd] \arrow[ddll, dotted] \\
\Delta(R,S) \arrow[ur, "\sigma^{\boxdot}_{RS}"] \arrow[rrr] \arrow[ddd] \arrow[drr, dotted] & & & S \arrow[ddd] \arrow[dl, dotted] & \\
& & Z & & \\
& \Delta(R,T) \arrow[uuu] \arrow[rrr] \arrow[dl] \arrow[ur, dotted] & & & T \arrow[dl] \arrow[ull, dotted] \\
R \arrow[rrr] \arrow[uurr, dotted] & & & X \arrow[uul, dotted, "b"] &
\end{tikzcd}$$

By the universal property of the colimit, a unique morphism $\phi \colon \oplus(R,S,T) \to Z$ exists such that $\phi \circ \varphi = a$ and $\phi \circ \psi = b$. This proves that $(*)$ is a pushout square.

Conversely, if we assume that the object $\oplus(R,S,T)$ is computed as in the pushout $(*)$, then the equality $\varphi \circ \upsilon = \psi \circ \zeta$ gives a cocone as in Figure 15. If we further assume that there is another cocone over an object Z as above, then the universal property of pushout will induce a unique morphism $\phi \colon \oplus(R,S,T) \to Z$ making everything commutative.

6.1. Definition. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. The* ***ternary Bulatov commutator*** $[R,S,T]^{\mathrm{B}}$ *of R, S and T is the kernel pair* $\mathrm{Eq}(\psi)$ *of the morphism ψ.*

6.2. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. The following conditions are equivalent:*

(i) *(R,S,T) are Bulatov connected (Definition 5.18);*

(ii) $[R,S,T]^{\mathrm{B}} = \Delta_X$.

Proof. First note that $[R,S,T]^{\mathrm{B}} = \Delta_X$ if and only if the arrow ψ constructed above is an isomorphism.

To see that (i) implies (ii), assume that R, S and T are Bulatov connected. Then there is $\theta\colon \Delta(R,S,T) \to X$ which makes the diagram

$$\begin{tikzcd}
\Delta(R,S)+\Delta(R,T)+\Delta(S,T) \arrow[r] \arrow[d, two heads] & X \arrow[d, two heads, "\psi"] \arrow[ddr, equal, bend left] & \\
\Delta(R,S,T) \arrow[r, "\varphi"'] \arrow[drr, bend right, "\theta"'] & \odiv(R,S,T) \arrow[dr, dotted, "\lambda"] & \\
& & X
\end{tikzcd}$$

of solid arrows commute. Since the square is a pushout, we have the dotted arrow λ in the diagram. Hence ψ is an isomorphism and $[R,S,T]^{\mathrm{B}} = \Delta_X$.

(ii) implies (i) because if ψ is an isomorphism, then the composite

$$\Delta(R,S,T) \to \odiv(R,S,T) \cong X$$

is a (three-dimensional) connector on R, S and T. ∎

6.3. Lemma. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. The regular images $\psi(R), \psi(S)$ and $\psi(T)$ under the morphism ψ are Bulatov connected.*

Proof. By Lemma 5.24 the factorisation $\psi_\Delta\colon \Delta(R,S,T) \to \Delta(\psi(R),\psi(S),\psi(T))$ is a regular epimorphism. It easily follows that φ factors through $\Delta(\psi(R),\psi(S),\psi(T))$ to produce a connector $\theta\colon \Delta(\psi(R),\psi(S),\psi(T)) \to \odiv(R,S,T)$. ∎

6.4. Theorem. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. The induced map ψ is an extremal epimorphism, universal for making the equivalence relations $\psi(R)$, $\psi(S)$ and $\psi(T)$ connected.*

Proof. This follows immediately from the universal property of colimits and the above Lemma 6.3. ∎

By 3-$\mathsf{Conn}(\mathbb{X})$ we denote the category whose objects are quadruples (X,R,S,T), where X is an object of $\mathbb{X}$ and R, S and T are equivalence relations on X such that (R,S,T) are Bulatov connected. 3-$\mathsf{Conn}(\mathbb{X})$ is a full and replete subcategory of the category 3-$\mathsf{Eq}(\mathbb{X})$ defined on page 401.

6.5. Theorem. *Let $\mathbb{X}$ be an exact Mal'tsev category with binary coproducts. The category* 3-$\mathsf{Conn}(\mathbb{X})$ *is a reflective subcategory of* 3-$\mathsf{Eq}(\mathbb{X})$.

Proof. By Theorem 6.4, the functor $F\colon$ 3-$\mathsf{Eq}(\mathbb{X}) \to$ 3-$\mathsf{Conn}(\mathbb{X})$ defined by

$$F(X,R,S,T) = (\odiv(R,S,T), \psi(R), \psi(S), \psi(T))$$

is the needed reflector. ∎

The following property can be shown directly from the definition of the ternary Bulatov commutator.

6.6. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X.*

1. *For all (R, S, T) and (R', S', T') such that $R \leqslant R'$ on X, we have*
$$[R, S, T]^{\mathrm{B}} \leqslant [R', S, T]^{\mathrm{B}};$$
2. *$[R, S, T]^{\mathrm{B}} = [R, T, S]^{\mathrm{B}} = \cdots = [T, S, R]^{\mathrm{B}}$.*

6.7. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S and T be equivalence relations on an object X. Then $[R, S, T]^{\mathrm{B}} \leqslant R \wedge S \wedge T$.*

Proof. Let us show that $[R, S, T]^{\mathrm{B}} \leqslant R$. First take the quotient $q_R \colon X \to X/R$ of X by R. Then $q_R(R)$ is the discrete relation on the quotient X/R. As in Example 5.15, the triple $(q_R(R), S', T')$ is connected for all equivalence relations S', T'. In particular, $(q_R(R), q_R(S), q_R(T))$ is connected. By Theorem 6.4, there is a factorisation $\overline{q}_R \colon F(X, R, S, T) \to X/R$, so that $[R, S, T]^{\mathrm{B}} \leqslant R$. ■

Recall the construction of the binary Smith-Pedicchio commutator (Definition 3.14).

6.8. Proposition. *In an exact Mal'tsev category with binary coproducts, let R, S, T and T' be equivalence relations on an object X. Then*

1. *$[R, S, T]^{\mathrm{B}} \leqslant [R, S]^{\mathrm{S}}$;*
2. *$[R, S, T \vee T']^{\mathrm{B}} = [R, S, T]^{\mathrm{B}} \vee [R, S, T']^{\mathrm{B}}$.*

Proof. 1. According to Proposition 5.14, if $[R, S] = \Delta_X$, then there exists a centralising three-fold equivalence relation on R, S and T. By Proposition 6.2, it follows that $[R, S, T]^{\mathrm{B}} = \Delta_X$.

2. According to Proposition 6.2 and Proposition 5.27, $[R, S, T \vee T']^{\mathrm{B}} = \Delta_X$ if and only if $[R, S, T]^{\mathrm{B}} \vee [R, S, T']^{\mathrm{B}} = \Delta_X$. ■

6.9. Proposition. *In an exact Mal'tsev category with binary coproducts, consider equivalence relations R, S and T on an object X. For every regular epimorphism $f \colon X \to Y$, we have that $f([R, S, T]^{\mathrm{B}}) = [f(R), f(S), f(T)]^{\mathrm{B}}$.*

Proof. By Proposition 4.11 and Proposition 5.24, we have a three-cube in $\mathsf{Arr}(\mathbb{X})$ given by the diagram in Figure 16, where $f_{\odiv} \colon \odiv(R, S, T) \to \odiv(f(R), f(S), f(T))$ is its colimit. The morphism $f_{(\psi_X)} \colon f \to f_{\odiv}$ in $\mathsf{Arr}(\mathbb{X})$ is the commutative square

$$\begin{tikzcd}
X \arrow[rr, two heads, "f"] \arrow[d, two heads, "\psi_X"'] & & Y \arrow[d, two heads, "\psi_Y"] \\
\odiv(R, S, T) \arrow[rr, two heads, "f_{\odiv}"'] & & \odiv(f(R), f(S), f(T))
\end{tikzcd}$$

$$
\begin{tikzcd}
 & f_{\Delta} \arrow[ddr, dotted, "f_{(\varphi_X)}"] & & & f_{\Delta(S,T)} \arrow[lll] \arrow[dl] \arrow[ddd] \arrow[ddll, dotted] \\
f_{\Delta(R,S)} \arrow[ur] \arrow[rrr] \arrow[ddd] \arrow[drr, dotted] & & & f_S \arrow[ddd] \arrow[dl, dotted] & \\
 & & f_{\oplus} & & \\
 & f_{\Delta(R,T)} \arrow[uuu] \arrow[rrr] \arrow[dl] \arrow[ur, dotted] & & & f_T \arrow[dl] \arrow[ull, dotted] \\
f_R \arrow[rrr] \arrow[uurr, dotted] & & & f \arrow[uul, dotted, "f_{(\psi_X)}"'] &
\end{tikzcd}
$$

Figure 16: Functoriality of the commutator construction

$$
\begin{tikzcd}
\boxdot(\mathrm{Eq}(f_R), \mathrm{Eq}(f_S), \mathrm{Eq}(f_T)) \arrow[r, shift left=2, "f_{\Delta,1}"] \arrow[r, shift right=2, "f_{\Delta,2}"'] \arrow[d, shift left, "\pi^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_S)}"] & \Delta(R,S,T) \arrow[l, dotted, "\sigma_{f_\Delta}"' description] \arrow[r, two heads, "f_\Delta"] \arrow[d, shift left, "\pi^{\boxdot}_{RS}"] & \Delta(f(R), f(S), f(T)) \arrow[d, shift left, "\pi^{\boxdot}_{f(R)f(S)}"] \\
\square(\mathrm{Eq}(f_R), \mathrm{Eq}(f_S)) \arrow[u, shift left, "\sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_S)}"] \arrow[r, shift left=2, "f_{\Delta(R,S),1}"] \arrow[r, shift right=2, "f_{\Delta(R,S),2}"'] \arrow[d, shift left, "d_2^{\mathrm{Eq}(f_R)}"] & \Delta(R,S) \arrow[u, shift left, "\sigma^{\boxdot}_{RS}"] \arrow[l, dotted, "\Delta_{RS}" description] \arrow[r, two heads, "f_{\Delta(R,S)}"'] \arrow[d, shift left, "d_2^R"] & \Delta(f(R), f(S)) \arrow[u, shift left, "\sigma^{\boxdot}_{f(R)f(S)}"] \arrow[d, shift left, "d_2^{f(R)}"] \\
\mathrm{Eq}(f_R) \arrow[u, shift left, "\sigma_{\mathrm{Eq}(f_R)}"] \arrow[r, shift left=2, "f_{R,1}"] \arrow[r, shift right=2, "f_{R,2}"'] \arrow[d, shift left, "\bar{r}_2"] & R \arrow[u, shift left, "\Delta_R"] \arrow[l, dotted, "\sigma_{f_R}" description] \arrow[r, two heads, "f_R"'] \arrow[d, shift left, "r_2"] & f(R) \arrow[u, shift left, "\Delta_{f(R)}"] \arrow[d, shift left, "r_2'"] \\
\mathrm{Eq}(f) \arrow[u, shift left, "\Delta_{\mathrm{Eq}(f)}"] \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & X \arrow[u, shift left, "\sigma^R_X"] \arrow[l, dotted, "\sigma_f" description] \arrow[r, two heads, "f"'] & Y \arrow[u, shift left, "\sigma_Y^{f(R)}"]
\end{tikzcd}
$$

Figure 17: $u \circ \varphi_X$ coequalises $(f_{\Delta,1}, f_{\Delta,2})$

in $\mathbb{X}$. By Proposition 5.17, the morphisms ψ_X and ψ_Y are regular epimorphisms, so that $f_{\oplus}$ is also a regular epimorphism. Hence the above square is a commutative square of regular epimorphisms. In our context, its remains to prove that it is a pushout square, so that $f([R,S,T]^S) = [f(R), f(S), f(T)]^S$ by Proposition 3.16. To do so, let us assume that $u\colon \oplus(R,S,T) \to Z$ and $v\colon Y \to Z$ are morphisms in $\mathbb{X}$ such that $u \circ \psi_X = v \circ f$.

First we are going to prove that $u \circ \varphi_X$ coequalises $(f_{\Delta,1}, f_{\Delta,2})$. Here $(\mathrm{Eq}(f_\Delta), f_{\Delta,1}, f_{\Delta,2})$ denotes the kernel pair of the morphism f_Δ. Let us consider the diagram in Figure 17. Then

$$
\begin{aligned}
u \circ \varphi_X \circ f_{\Delta,1} \circ \sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_S)} &= u \circ \varphi_X \circ \sigma^{\boxdot}_{RS} \circ f_{\Delta(R,S),1} = u \circ \psi_X \circ r_2 \circ d_2^R \circ f_{\Delta(R,S),1} \\
&= v \circ f \circ r_2 \circ f_{R,1} \circ d_2^{\mathrm{Eq}(f_R)} = v \circ f \circ f_1 \circ \bar{r}_2 \circ d_2^{\mathrm{Eq}(f_R)} \\
&= v \circ f \circ f_2 \circ \bar{r}_2 \circ d_2^{\mathrm{Eq}(f_R)} = u \circ \psi_X \circ r_2 \circ f_{R,2} \circ d_2^{\mathrm{Eq}(f_R)} \\
&= u \circ \psi_X \circ r_2 \circ d_2^R \circ f_{\Delta(R,S),2} = u \circ \varphi_X \circ \sigma^{\boxdot}_{RS} \circ f_{\Delta(R,S),2} \\
&= u \circ \varphi_X \circ f_{\Delta,2} \circ \sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_S)}.
\end{aligned}
$$

By a similar argument

$$u \circ \varphi_X \circ f_{\Delta,1} \circ \sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_T)} = u \circ \varphi_X \circ f_{\Delta,2} \circ \sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_T)},$$
$$u \circ \varphi_X \circ f_{\Delta,1} \circ \sigma^{\boxdot}_{\mathrm{Eq}(f_S)\mathrm{Eq}(f_T)} = u \circ \varphi_X \circ f_{\Delta,2} \circ \sigma^{\boxdot}_{\mathrm{Eq}(f_S)\mathrm{Eq}(f_T)}.$$

Since by Theorem 3.20, the splittings $\sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_S)}, \sigma^{\boxdot}_{\mathrm{Eq}(f_R)\mathrm{Eq}(f_T)}$ and $\sigma^{\boxdot}_{\mathrm{Eq}(f_S)\mathrm{Eq}(f_T)}$ are jointly extremal-epimorphic, we have the equality $u \circ \varphi_X \circ f_{\Delta,1} = u \circ \varphi_X \circ f_{\Delta,2}$.

It follows that there is a unique morphism ϕ from $\Delta(f(R), f(S), f(T))$ to Z such that $\phi \circ f_\Delta = u \circ \varphi_X$.

Now, we are going to use the universal property of the colimit cube in order to get our result. The morphisms u, v and ϕ are such that

$$\begin{aligned} v \circ r'_2 \circ d_2^{f(R)} \circ f_{\Delta(R,S)} &= v \circ r'_2 \circ f_R \circ d_2^R = v \circ f \circ r_2 \circ d_2^R = u \circ \psi_X \circ r_2 \circ d_2^R \\ &= u \circ \varphi_X \circ \sigma^{\boxdot}_{RS} = \phi \circ f_\Delta \circ \sigma^{\boxdot}_{RS} = \phi \circ \sigma^{\boxdot}_{f(R)f(S)} \circ f_{\Delta(R,S)}. \end{aligned}$$

Since $f_{\Delta(R,S)}$ is a regular epimorphism, it follows that $v \circ r'_2 \circ d_2^{f(R)} = \phi \circ \sigma^{\boxdot}_{f(R)f(S)}$. Similarly, $v \circ s'_2 \circ d_2^{f(S)} = \phi \circ \sigma^{\boxdot}_{f(S)f(T)}$ and $v \circ t'_2 \circ d_2^{f(T)} = \phi \circ \sigma^{\boxdot}_{f(R)f(T)}$, so

$$\begin{tikzcd}
 & \Delta(f(R),f(S),f(T)) \arrow[ddr, dotted, "\phi"] & & & \Delta(f(S),f(T)) \arrow[lll] \arrow[dl] \arrow[ddd] \arrow[ddll, dotted] \\
\Delta(f(R),f(S)) \arrow[ur] \arrow[rrr] \arrow[ddd] \arrow[drr, dotted] & & & f(S) \arrow[ddd] \arrow[dl, dotted] & \\
 & & Z & & \\
 & \Delta(f(R),f(T)) \arrow[uuu] \arrow[rrr] \arrow[dl] \arrow[ur, dotted] & & & f(T) \arrow[dl] \arrow[ull, dotted] \\
f(R) \arrow[rrr] \arrow[uurr, dotted] & & & Y \arrow[uul, dotted, "v"] &
\end{tikzcd}$$

is a cocone. The universal property of the colimit $\oplus(f(R), f(S), f(T))$ tells us that there is a unique morphism $\gamma \colon \oplus(f(R), f(S), f(T)) \to Z$ such that $\gamma \circ \varphi_Y = \phi$ and $\gamma \circ \psi_Y = v$. In order to prove that $\gamma \circ f_\oplus = u$, let us consider the cocone

$$\begin{tikzcd}
 & \Delta(R,S,T) \arrow[ddr, dotted, "\phi \circ f_\Delta"] & & & \Delta(S,T) \arrow[lll] \arrow[dl] \arrow[ddd] \arrow[ddll, dotted] \\
\Delta(R,S) \arrow[ur] \arrow[rrr] \arrow[ddd] \arrow[drr, dotted] & & & S \arrow[ddd] \arrow[dl, dotted] & \\
 & & Z & & \\
 & \Delta(R,T) \arrow[uuu] \arrow[rrr] \arrow[dl] \arrow[ur, dotted] & & & T \arrow[dl] \arrow[ull, dotted] \\
R \arrow[rrr] \arrow[uurr, dotted] & & & X \arrow[uul, dotted, "v \circ f"] &
\end{tikzcd}$$

The universal property of the colimit $\oplus(R, S, T)$ gives a unique $\lambda \colon \oplus(R, S, T) \to Z$ such that $\lambda \circ \varphi_X = \phi \circ f_\Delta$ and $\lambda \circ \psi_X = v \circ f$. By the above results, $u \circ \varphi_X = \phi \circ f_\Delta$ and $u \circ \psi_X = v \circ f$, while $(\gamma \circ f_\oplus) \circ \varphi_X = \gamma \circ \varphi_Y \circ f_\Delta = \phi \circ f_\Delta$ and $(\gamma \circ f_\oplus) \circ \psi_X = \gamma \circ \psi_Y \circ f = v \circ f$. Hence the uniqueness of λ implies that $u = \lambda = \gamma \circ f_\oplus$. ■

7. Conclusion

The aim of this article is to give a categorical description of the Bulatov commutator in the context of exact Mal'tsev categories, and to show that it has many of the convenient properties of its universal-algebraic counterparts. In a forthcoming second article, we restrict the context to algebraically coherent [13] semi-abelian [21] categories, where we prove that the commutator introduced here corresponds to the ternary Higgins commutator of M. Hartl and the second author [19], which extends the original definition of [20, 22]. This answers the question, what kind of universal property characterises that commutator.

It is quite clear that what we do in this article can in principle be done for higher orders—involving n-fold equivalence relations where $n \geqslant 4$.

Another line of investigation which we are currently pursuing is to generalise the binary Smith-Pedicchio commutator to a higher-order version, necessarily different from the Bulatov commutator, which may be used to characterise higher central extensions in the sense of [14]. This answers a question in [29], namely how to appropriately define *higher-order pregroupoids*, in such a way that they can be used in the description of cohomology groups.

Some open questions remain. First of all, the availability of the commutator in congruence modular varieties [23] suggests that its categorical counterpart might be extended beyond the exact Mal'tsev context. Doing so would involve replacing certain arguments which are typical for exact Mal'tsev categories, such as those based on the use of three-fold extensions, by more general ones which stay valid in, say, *exact Gumm categories*—a context introduced in [9, 10, 6] which seems suitable for this kind of considerations.

Another open question concerns the relationship between 2-nilpotency defined in terms of the commutator considered here (the condition that $[\nabla_X, \nabla_X, \nabla_X]$ vanishes) and the 2-*folded objects* of Berger-Bourn [2]. This is related to the main question of [25] on the relation between so-called *supernilpotency* (defined in terms of a higher-order commutator) and nilpotency (defined in terms of binary commutators).

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