Theory and Applications of Categories, Vol. 35, No. 49, 2020, pp. 1823–1832.

METRIC SPACES OF EXTREME POINTS

ERNIE MANES

ABSTRACT. It is shown that any compact metric space of diameter at most 2 embeds isometrically as a linearly independent set of extreme points of the unit ball of a separable Banach space. The proof illustrates how category theory can play a useful role in a problem of functional analysis.

The well-known Arens-Eells embedding theorem [7] asserts that an arbitrary metric space may be isometrically embedded as a set of linearly independent vectors in a Banach space. We use elementary category theory to prove the following stronger result for compact metric spaces.

MAIN THEOREM Given a compact metric space (X, d) of diameter ≤ 2, there exists a separable real Banach space $F(X, d)$ in which (X, d) may be isometrically embedded with image a linearly independent set of extreme points each of norm 1.

This result is counterintuitive. For example, consider the case with (X, d) the unit interval. The isometry of the theorem provides a continuous curve in the "surface" of the unit sphere. Hence the image of such a curve can be a linearly independent set of extreme points.

In the first section of the paper we will review some basic definitions and facts. In the second section we introduce free Banach spaces. The third section proves the main theorem using the free Banach space generated by a metric space.

There is some literature concerned with categories of Banach spaces (see [9, 8, 2] and the references cited there). These develop interesting concepts which have little intersection with mainstream functional analysis. This paper attempts to demonstrate that such an intersection is possible.

We thank the referee for helpful suggestions.

1. The Role of Extreme Points

See [6] for a wealth of theory and examples concerning Banach spaces. We outline here some basic definitions and facts which we need.

Received by the editors 2017-04-13 and, in final form, 2020-10-30.
Transmitted by Rick Blute. Published on 2020-10-30.
2020 Mathematics Subject Classification: 46B04, 46B50, 18A40.
Key words and phrases: free Banach space, Arens-Eells embedding, extreme point.

Let X be a real vector space. A **norm** on X is a function $\| \cdot \| : X \to [0, \infty)]$ satisfying $\| x + y \| \leq \| x \| + \| y \|$, $\| \lambda x \| = |\lambda| \, \| x \|$ for all $\lambda \in \mathbb{R}$ and $\| x \| = 0 \Leftrightarrow x = 0$. If $\| \cdot \|$ is a norm then (X, d) is a metric space if $d(x, y) = \| x - y \|$. We then define the **ball** of radius r to be $B_r = \{x \in X : \| x \| \leq r\}$ and the **sphere** of radius r to be $S_r = \{x \in X : \| x \| = r\}$. B_1 is the **unit ball** and S_1 is the **unit sphere**.

For $a, b \in X$ we define the **line segment** $[a, b]$ to be $\{\lambda\, a + (1 - \lambda)\, b : 0 \leq \lambda \leq 1\}$. Here a, b are the **endpoints** of the segment and all other points in the segment are its **interior points**. A nonempty subset $C \subset X$ is **convex** if $[a, b] \subset C$ whenever $a, b \in C$. For example, B_r is convex for every norm. A subset $B \subset X$ is **radial at** 0 if $0 \in B$ and if for every $0 \neq x \in X$, there exists $\lambda > 0$ such that $[0, \lambda x] \subset B$. For every norm, each B_r is radial at 0.

A subset B of X is the unit ball of some norm if and only if it is convex and radial at 0. This is easy to prove: $\| x \| = (\bigvee\{\lambda \in \mathbb{R} : [0, \lambda x] \subset B\})^{-1}$. This opens up the possibility of "generating a norm" by generating B_1. Now any intersection of convex sets is convex, so the **convex hull** of a set is the smallest convex set containing it. In $\mathbb{R}^2$, any square centered on the origin is convex and radial at 0 and it is finitely generated being the convex hull of its four corners.

For any subset $A \subset X$, $e \in A$ is an **extreme point** of A if no line segment contained in A has e as interior point. In the example above with B_1 a square centered at the origin in the plane, the extreme points of B_1 are its four corners and B_1 is the convex hull of its extreme points. For the Euclidean norm $\| (x, y) \| = \sqrt{x^2 + y^2}$ in the plane, the extreme points of B_1 is all of S_1, a circle of radius 1, and it is still true that B_1 is the convex hull of its extreme points.

A **(real) Banach space** is a real vector space with a norm whose metric is complete, that is, every Cauchy sequence converges. All finite dimensional normed spaces are Banach spaces and any two are homeomorphic [6, Corollary 1.4.17].

We are led to consider the question of whether the unit ball of a Banach space is, in some appropriate sense, generated by its extreme points. This cannot always happen, since there exist Banach spaces whose unit ball has no extreme points [6, Example 2.10.4]. Even so, there are large classes of Banach spaces possessing a more affirmative answer as we now discuss.

An important construction is the **dual** $X^\star$ of a normed space X which is the vector space of all of the continuous linear maps $f : X \to \mathbb{R}$ with norm $\| f \| = \bigvee\{|fx| : \| x \| = 1\}$. Even if X is not complete, $X^\star$ is always a Banach space.

In a normed linear space, a subset is **bounded** if it is contained in B_r for some $r > 0$.

Now, any normed space X is a **topological vector space**, that is, addition and scalar multiplication are continuous $X \times X \to X$, $\mathbb{R} \times X \to X$. Every normed space is Hausdorff. In any topological vector space, the closure of a convex set is convex so the smallest closed and convex set containing a subset A is the closure of the convex hull of A. A topological vector space is **locally convex** if every neighborhood of the origin contains a convex neighborhood of the origin. Any normed space is locally convex. Thus the following theorem applies:

Krein-Milman Theorem [6, Theorem 2.10.6] In any locally convex, Hausdorff topological vector space, every non-empty compact convex set is the closed convex hull of its extreme points.

For a finite-dimensional Banach space, the unit ball is closed and bounded so is compact by the Heine-Borel theorem. For any infinite dimensional normed space, the unit ball, while closed and bounded, is not compact [1]. For a dual space $X^\star$ there is a way around this problem. If (Y_i) is a family of topological vector spaces and $f_i : X \to Y_i$ are linear maps, the smallest topology on X rendering the f_i continuous makes X a topological vector space. If X is a normed space, we may apply this construction to the continuous linear maps $\{X^\star \xrightarrow{ev_x} X : x \in X\}$ where $ev_x(f) = fx$. In this way, $X^\star$ is a topological vector space which, in general, has fewer open sets than it has in the norm topology, but the space is still Hausdorff by the Hahn-Banach theorem stated in Theorem 2.3. This new topology is called the **weak-$\star$** topology on $X^\star$. We have

Banach-Alaoglu Theorem [6, Theorem 2.6.18] For any normed space X, the unit ball of $X^\star$ is compact in the weak-$\star$ topology.

Thus the Krein-Milman theorem gives that, in the Banach space $X^\star$, the unit ball is the weak-$\star$ closure of the convex hull of its extreme points (and, in particular, the unit ball has extreme points).

For any normed space X, the canonical map $ev : X \to X^{\star\star}$, $x \mapsto ev_x$, is a linear isometry. If this map is surjective, X is said to be **reflexive**. In that case, X is linearly isometric to a dual space so must be a Banach space. Every finite-dimensional Banach space is reflexive [6, Theorem 1.11.9].

Two sections of [6] are devoted to characterizations of reflexive Banach spaces. One of the most basic results was proved by Smullyan in 1939 and is as follows:

Theorem [6, Theorem 1.13.9] A Banach space is reflexive if and only if each descending sequence of non-empty closed and bounded convex sets has non-empty intersection.

2. Free Banach Spaces

To define a category of Banach spaces we must settle on a notion of morphism. One very natural choice would be continuous linear maps. There is some danger that respecting the topology as opposed to the norm, we will lose information. The following shows that this is not really true.

2.1. Proposition. *Let X, Y be Banach spaces and let $\psi : X \to Y$ be a linear map. Then ψ is continuous if and only if ψ is* ***bounded****, that is, there exists $M > 0$ such that $\| fx \| \leq M \ \| x \|$ for all $x \in X$.*

Proof. If ψ is continuous then $\psi^{-1}(0)$ is a neighborhood of 0 so there exists $r > 0$ with $\psi(B_r) \subset B_1$, i.e., $\| x \| \leq r \Rightarrow \| \psi x \| \leq 1$. Let $\lambda = \frac{r}{\|x\|}$. As $\| \lambda x \| = r$, $\lambda \| \psi x \| \leq 1$ so $\| \psi x \| \leq \frac{1}{r} \ \| x \|$. Conversely, let ψ be bounded and let x_n be a sequence converging to

x in X. Then $\| \psi x - \psi x_n \| = \| \psi(x - x_n) \| \leq M \| x - x_n \|$ and this converges to 0, so ψx_n converges to ψx and ψ is continuous. ∎

When ψ as above is a continuous linear map, the **norm** $\| \psi \|$ of ψ is the infimum of all M as in the Proposition. With this norm, the vector space $[X, Y]$ of all continuous linear maps $X \to Y$ is a Banach space. It is well known that there is a notion of tensor product (see [2]) with respect to which Banach spaces and continuous maps becomes a closed monoidal category with internal hom $[X, Y]$.

Even so, we reject continuous linear maps as our choice of morphism. This is because the category of Banach spaces and continuous linear maps does not have infinite products. To see this, suppose $\pi_n : P \to \mathbb{R}$ $(n \in \mathbb{N})$ were a countably infinite power of the scalar field. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the continuous linear map $f_n \lambda = \| \pi_n \| \, n \, \lambda$. By the universal property there exists continuous linear $\psi : \mathbb{R} \to P$ with $\pi_n \psi = f_n$ for all n. Then

$$\| \pi_n \| \; n \; = \; |f_n 1| \; = \; |\pi_n \psi 1| \; \leq \| \pi_n \| \| \psi \|$$

We can assume no $\pi_n = 0$, so this gives the contradiction that $\| \psi \| \geq n$ for all n.

Our choice of morphism is as follows:

2.2. DEFINITION. *The category* **Ban** *has Banach spaces as objects and norm-decreasing linear maps as morphisms.*

Thus the morphisms have been chosen as those continuous linear ψ with $\| \psi \| \leq 1$. That this forms a category is routine.

We note that the isomorphisms $\psi : X \to Y$ of **Ban** are those linear isomorphisms which are also isometries, $\| \psi x - \psi y \| = \| x - y \|$. To see this,

$$\begin{aligned} \| x - y \| \; &= \; \| \psi^{-1}(\psi x - \psi y) \| \; \leq \; \| \psi x - \psi y \| \\ &= \; \| \psi(x - y) \| \; \leq \; \| x - y \| \end{aligned}$$

Before stating the main theorem of this section, we recall the **Hahn-Banach theorem** [6, 1.9.6]

2.3. THEOREM. *Let X be a normed vector space and let A be a vector subspace of X. Then every $\mathbb{R}$-valued continuous linear function on A extends to a continuous linear function on X with the same norm.*

2.4. THEOREM. *Let $\mathcal{C}$ be a locally small category. Every functor U :* **Ban** $\to \mathcal{C}$ *which preserves limits has a left adjoint.*

PROOF. We check all the conditions of the special adjoint functor theorem. If $f, g : X \to Y$ then $E = \{x \in X : fx = gx\}$ is a closed vector subspace of X and so is a Banach space since a closed subset of a complete metric space is complete. It is easy to show that the inclusion of E provides the equalizer of f, g in **Ban**. Let $(X_i : i \in I)$ be a family of Banach spaces. Let P be the set of all I-tuples (x_i) with $x_i \in X_i$ such that $\bigvee \| x_i \| < \infty$ and let $\pi_i : P \to X_i$ map (x_j) to x_i, a norm-decreasing map. That this forms a product in **Ban** is immediate. Thus **Ban** is a locally small category with small limits. Let $\psi : X \to Y$ be monic in **Ban** and let $x, y \in X$ with $x \neq y$. Let $\phi : \mathbb{R} \to X$ be the morphism $\phi\lambda = \lambda\frac{x-y}{\|x-y\|}$. If $\psi x = \psi y$ then $\psi\phi = \psi 0$. As $\phi \neq 0$, this contradicts ψ being monic. It follows that monics are injective functions in **Ban**. In particular, **Ban** is well-powered, that is, every object has a small set of subobjects. To complete the proof, we show $\mathbb{R}$ is a cogenerator, that is, if $\phi, \psi : X \to Y$ with $\phi \neq \psi$ then there exists $\gamma : Y \to \mathbb{R}$ with $\gamma\phi \neq \gamma\psi$. If $\phi x \neq \psi x$, $S = \mathbb{R}(\phi x - \psi x)$ is a nonzero subspace of Y. Let $h_o : S \to \mathbb{R}$ be the linear map sending $\frac{\phi x - \psi x}{\|\phi x - \psi x\|}$ to 1. Then $\| h_o \| = 1$. By the Hahn-Banach theorem, there exists a norm-1 extension $\gamma : Y \to \mathbb{R}$ so γ is a morphism in **Ban** and $\gamma\phi \neq \gamma\psi$. ■

For (X, d) a metric space, $f : X \to \mathbb{R}$ is a **Lipschitz** function if there exists $M \geq 0$ with $|fx - fy| \leq Md(x, y)$ for all $x, y \in X$. Bounded Lipschitz functions comprise a Banach space $\mathrm{Lip}(X, d)$ with $\| f \|$ the maximum of $\bigvee(|fx| : x \in X)$ and the infimum of all M above.

We will take a closer look at $Lip(X, d)$ in the next section.

3. Free Banach Spaces Over a Metric Space

In this section, we prove the main theorem as well as Theorem 3.8 by studying the left adjoint of a particular functor **Ban** $\to$ **Met** where

3.1. Definition. **Met** *is the category of metric spaces and maps* $f : (X, d) \to (Y, e)$ *which satisfy* $e(fa, fb) \leq d(a, b)$ *for all* $a, b \in X$*. Such maps are said to be* ***distance-decreasing***

That **Met** is a category is routine. Since distance is a function of two variables as opposed to norm which is a function of only one, the device used to define products in **Ban** does not work in **Met**. On the other hand, let (X_i, d_i) be a family of metric spaces with each of diameter $\leq M$, that is, $d_i(a, b) \leq M$ for all $a, b \in X_i$. Then the usual product set $\prod X_i$ with metric $d((x_i), (y_i)) = \bigvee d_i(x_i, y_i)$ provides a product in **Met**. The proof is routine.

For the balance of the paper, let U : **Ban** $\to$ **Met** be the unit ball functor mapping the Banach space X to its unit Ball $UX = B_1$ considered as a metric space. Because a map in **Ban** has norm ≤ 1 it maps the unit ball into the unit ball. Since maps in **Ban** are distance-decreasing $X \to Y$ their restriction $UX \to UY$ remains distance-decreasing.

3.2. Theorem. U : ***Ban*** $\to$ **Met** *has a left adjoint.*

Proof. While a product in **Ban** is not built on the whole cartesian product set, the unit ball of the product *is* the cartesian product of the unit balls. Thus U preserves products based on the discussion of products of metric spaces of diameter ≤ 2 above. That U preserves equalizers is routine. Now use Theorem 2.4. ■

To spell out what the previous theorem says, given an arbitrary metric space (X, d) there exists a Banach space $F(X, d)$ called the **free Banach space generated by** (X, d) together with a distance-decreasing map $\eta : (X, d) \to UF(X, d)$ possessing the universal mapping property summed up by the following diagram:

$$\begin{tikzcd}
(X,d) \arrow[r, "\eta"] \arrow[dr, "f"'] & UF(X,d) \arrow[d, "U\psi"] & F(X,d) \arrow[d, "\psi"] \\
 & UE & E
\end{tikzcd}$$

In detail, for every Banach space E and distance-decreasing map $f : (X, d) \to UE$ there exists a unique linear, norm-decreasing map $\psi : F(X, d) \to E$ such that the triangle commutes.

The next series of lemmas studies the properties of η. We begin with a preliminary result which is due to [7].

3.3. Lemma. *Let (X, d) be any metric space. Then there exists a Banach space C and an isometry $h : (X, d) \to C$, i.e. $\| hx - hy \| = d(x, y)$ for all $x, y \in X$.*

Proof. This is trivial if X has 2 or fewer elements. Otherwise, let $y_o \neq z_o \in X$. Let $Y = X \cup \{x_o\}$ where $x_o \notin X$ and extend d to a metric e on Y by

$$e(x, x_o) \;=\; \frac{1}{2}(\, d(y_o, x) + d(z_o, x)\,)$$

That e is a metric is routine. In [7] is is shown that the subspace of L of functions in $\mathrm{Lip}(Y, e)$ with $f(x_0) = 0$ provided with a modified Lipschitz norm is such that (Y, e) is isometrically embedded in the dual space $L^\star$ via $y \mapsto ev_y$. ∎

3.4. Lemma. *Let (X, d) be a metric space of diameter ≤ 2. Then $\eta : (X, d) \to F(X, d)$ is an isometry.*

Proof. Let $Y = X \cup \{x_o\}$ with $x_o \notin X$ and extend d to a metric e on Y by $e(x, x_o) = 1$ for all $x \in X$. That this is a metric is trivial. Applying Lemma 3.3, there exists a Banach space C and an isometry $g : (Y, e) \to C$. Define $h : (Y, e) \to UC$ by $hy = gy - gx_0$. We have

$$\| hy \| = \| gy - gx_0 \| = \; e(y, x_o) \;\in \{1, 0\}$$

so this is well defined. Also, h is an isometry because $z \mapsto z - gx_0$ is. Let $t : (X, d) \to UC$ be the restriction of h to (X, d). Induce φ by the universal mapping property $(U\varphi)\eta = t$. It follows that

$$\begin{aligned} d(x, y) \;&=\; \| tx - ty \| = \| \varphi\eta x - \varphi\eta y \| \leq \| \varphi \| \, \| \eta x - \eta y \| \\ &\leq\; \| \eta x - \eta y \| \leq \; d(x, y) \end{aligned}$$

the last because η is distance-decreasing. ∎

The next result shows that any metric space of diameter ≤ 2 may be isometrically embedded in the unit sphere of a Banach space.

3.5. LEMMA. *For any metric space (X,d), $\eta : (X,d) \to UF(X,d)$ maps all elements of X to unit vectors.*

PROOF. The universal property of freeness induces a diagram

$$\begin{tikzcd} (X,d) \arrow[r, "\eta"] \arrow[dr, "1"'] & UF(X,d) \arrow[d, "U\psi"] & F(X,d) \arrow[d, "\psi"] \\ & U\mathbb{R} & \mathbb{R} \end{tikzcd}$$

where 1 is the constant function with value 1. As $\psi\eta x = 1$,

$$1 \;=\; |\psi\eta x| \;\leq \|\,\psi\,\|\,\|\,\eta x\,\| \leq \|\,\eta x\,\| \leq\; 1.$$

∎

3.6. LEMMA. *Let (X,d) be a metric space of diameter ≤ 2. Then the image of $\eta : (X,d) \to UF(X,d)$ is a linearly independent set.*

PROOF. For $a \in X$, $d_a(x) = d(a,x)$ is a distance-decreasing function $(X,d) \to \mathbb{R}$ since $d(a,x) \leq d(a,y)+d(y,x) \;\Rightarrow\; |d(a,x)-d(a,y)| \leq d(x,y)$. A routine calculation shows that if $f_1,\ldots,f_n$ are distance-decreasing then $f_1 \wedge \cdots \wedge f_n$ also is. Now suppose $x_1,\ldots,x_{n+1}$ are distinct elements of X and let $f : (X,d) \to [0,1]$ be the distance-decreasing map defined by

$$fx \;=\; \frac{1}{4}(\,d(x_1,x) \wedge \cdots \wedge d(x_n,x)\,)$$

By the universal property, there exists $\psi : F(X,d) \to \mathbb{R}$ in **Ban** with $(U\psi)\,\eta = f$. If $\eta\, x_{n+1} = \lambda_1(\eta x_1) + \cdots + \lambda_n(\eta x_n)$ we obtain the contradiction

$$0 \;\neq\; f(x_{n+1}) \;=\; \psi\,\eta\,x_{n+1} \;=\; \sum \lambda_i\, f(x_i) \;=\; 0$$

This shows that $\eta\, x_1,\ldots,\eta\, x_{n+1}$ cannot be dependent. ∎

3.7. LEMMA. *$F(X,d)$ is the closed linear span of $\eta(X)$.*

PROOF. Let E be the closed linear span of $\eta(X)$ with inclusion $i : E \to F(X,d)$. Since E is a Banach space, there exists unique $\psi : F(X,d) \to E$ in **Ban** with $(U\psi)\eta = \eta_0 : X \to UE$, the range-restriction of η. Then $i\psi : F(X,d) \to F(X,d)$ in **Ban** satisfies $U(i\psi)\,\eta = \eta$ so $i\psi = id_{F(X,d)}$ and $i : E \to F(X,d)$ is surjective as desired. ∎

We turn now to some applications of the theory so far. It is always interesting to ask if a given Banach space is the dual of some other Banach space. This would prove, for instance, that the unit ball would have extreme points.

[10, Section 2] was devoted to showing if (X, d) is the unit cube in $\mathbb{R}^n$ with an appropriate metric, then $\mathrm{Lip}(X, d)$ is a dual space. Two years later, [3, Theorem 4.1] showed that this was true for all metric spaces. The following, proved by the author in 1974, showed that $\mathrm{Lip}(X, d)$ is the dual of the free Banach space $F(X, d)$. This leads to:

3.8. THEOREM. *Let (X, d) be a metric space and let $\mathcal{T}$ be the smallest topology making the linear maps $Lip(X, d) \to \mathbb{R}$, $f \mapsto fx$ (for $x \in X$) continuous. Then the unit ball of $Lip(X, d)$ is the $\mathcal{T}$-closed convex hull of its extreme points.*

PROOF. $\mathrm{Lip}(X, d) \cong F(X, d)^\star$ by [5, Example 1.1 and Corollary 3.2]. That the weak-$\star$ topology transfers as stated to $\mathrm{Lip}(X, d)$ is obvious. ∎

The way one proves that $\mathrm{Lip}(X, d) \cong F(X, d)^\star$ in the above proof is to scale down a Lipschitz function to map $f : (X, d) \to UF(X, d)$ to yield a corresponding element of $F(X, d)^\star$ using the universal property. The cited proof takes care of the details. The result is a good example of how the existence of the free Banach space has specific consequences without needing to determine its detailed structure.

That being said, the next result determines an aspect of the structure of $F(X, d)$.

3.9. THEOREM. *Let (X, d) be a metric space of diameter at most 2, and let $D \subset X$ be dense. Define a set $\hat{X}$ of unit vectors in $F(X, d)$ by*

$$\hat{X} \;=\; \{\frac{\eta a - \eta b}{\parallel \eta a - \eta b \parallel} : a \neq b \in D\} \;\cup\; \{\eta x : x \in D\} \;\cup\; \{-\eta x : x \in D\}$$

Then the closed convex hull of $\hat{X}$ is all of B_1.

PROOF. Let C be the closed convex hull of $\hat{X}$. Suppose $\parallel u \parallel \leq 1$ in $F(X, d)$ with $u \notin C$. We seek a contradiction. As $\hat{X} = -\hat{X}$, it follows from the theorem of strong separation [4, Corollary 14.4] that there exists a linear functional $f : F(X, d) \to \mathbb{R}$ with

$$\bigvee\{|fy| : y \in C\} \;<\; 1 \;<\; fu$$

Noting that $\eta(X) \subset C$, let g be the restriction of f to $\eta(X)$. For $x, y \in X$ with $x \neq y$ there exist sequences a_n, b_n in D with $a_n \neq b_n$, $a_n \rightharpoonup x$, $b_n \rightharpoonup y$ (where $\rightharpoonup$ denotes convergence). We have

$$\frac{|g\eta x - g\eta y|}{\parallel \eta x - \eta y \parallel} \;=\; \lim_n f(\frac{|\eta a_n - \eta b_n|}{\parallel \eta a_n - \eta b_n \parallel}) \;\leq\; 1$$

so that $g : (X, d) \to [-1, 1]$ is distance-decreasing and so has a linear extension of norm ≤ 1 by the universal mapping property. By Lemma 3.7, any two continuous linear functionals on $F(X, d)$ agreeing on $\eta(X)$ must be equal. But f is an extension of g with norm > 1, the desired contradiction. ∎

A Banach space is **separable** if it is the closed linear span of a countable set.

PROOF OF THE MAIN THEOREM. Let (X, d) be compact. In particular, (X, d) has a countable dense set. We will show that the free Banach space $F(X, d)$ has the desired properties. By Theorem 3.9, the unit ball is the closed convex hull of a countable set so $F(X, d)$ is the closed span of a countable set, hence is separable. It remains to show that $\eta(X)$ consists of extreme points, since then η is the desired embedding. We note the obvious fact that if X is a vector subspace of a vector space Y and if $a \in A \subset X$ then if a is an extreme point of A in Y it is also an extreme point of A in X. Recall from Theorem 3.8 that $\mathrm{Lip}(X, d) \cong F(X, d)^{\star}$. Now using compactness of (X, d), it is proved in [10, Lemma 1.2] that the evaluation function $ev_x : \mathrm{Lip}(X, d) \to \mathbb{R}$ is an extreme point of the unit ball of $(\mathrm{Lip}(X, d))^{\star}$. Putting these together, for $x \in X$, ηx is an extreme point of the unit ball of $(F(X, d))^{\star\star}$ so is a fortiori an extreme point of the unit ball in $F(X, d)$. ∎

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Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003-9305
Email: manes@umass.edu