Theory and Applications of Categories, Vol. 35, No. 33, 2020, pp. 1227–1267.

SEGAL ENRICHED CATEGORIES AND APPLICATIONS

HUGO V. BACARD

ABSTRACT. In this paper we develop a theory of *Segal enriched categories*. Our motivation was to generalize the notion of *up-to-homotopy monoid* in a monoidal category, introduced by Leinster. Our formalism generalizes the classical theory of Segal categories and extends the theory of categories enriched over a 2-category. We introduce Segal dg-categories which did not exist so far. We show that the homotopy transfer problem for algebras leads directly to a Leinster–Segal algebra.

1. Introduction

Let $\mathscr{M} = (\underline{M}, \otimes, I)$ be a monoidal category. The structure of a category $\mathcal{C}$ *enriched* over $\mathscr{M}$, henceforth *an $\mathscr{M}$-category*, consists roughly speaking of a set of objects $\mathrm{Ob}(\mathcal{C}) = \{x,\ y,\ z, \cdots\}$, a set of *hom-objects* $\{\mathcal{C}(x,y) \in \mathrm{Ob}(\underline{M})\}_{(x,y)\in \mathrm{Ob}(\mathcal{C})^2}$ with a composition law $\mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \xrightarrow{c_{xyz}} \mathcal{C}(x,z)$ and an identity map $I_x : I \longrightarrow \mathcal{C}(x,x)$, satisfying the usual axioms of associativity and identity. Taking $\mathscr{M}$ equal to $(\mathrm{Set}, \times, 1)$ (resp. $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$), an $\mathscr{M}$-category is an ordinary small category (resp. a pre-additive category). The category $\mathscr{M}$ is called the *base*, as in "base of enrichment". There are notions of $\mathscr{M}$-functor and $\mathscr{M}$-natural transformation that generalize the usual ones. The reader can find a treatment of the theory of categories enriched over a monoidal category in the book of Kelly [15]. For a base $\mathscr{M}$, we denote by $\mathscr{M}$-Cat the category of $\mathscr{M}$-categories.

Bénabou [3] defined *bicategories* which are also called *weak 2-categories*, or simply 2-categories. He defined different types of morphisms between them and pointed out that a bicategory with one object is the same thing as a monoidal category. This remark gave rise to a general theory of enriched categories where the base $\mathscr{M}$ is a bicategory. We refer the reader to [16, 31] and the references therein for enrichment over a bicategory. Street [31] noticed that for a set X, an X-*polyad* of Bénabou in a bicategory $\mathscr{M}$ is the same thing as a category enriched over $\mathscr{M}$ whose set of objects is X. An X-polyad is a *lax* morphism $F : \overline{X} \longrightarrow \mathscr{M}$ of bicategories, where $\overline{X}$ is the connected 1-groupoid that contains a single isomorphism between any two elements of X. Given a polyad $F : \overline{X} \longrightarrow \mathscr{M}$, if we denote by $\mathcal{C}_F$ the corresponding $\mathscr{M}$-category, one can interpret F as *the nerve* of $\mathcal{C}_F$ and identify F with $\mathcal{C}_F$, like with Segal categories.

Recall that a classical Segal 1-category $\mathscr{A}$ is a *simplicial space* $\mathscr{A} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{sSet}$ satisfying some conditions that include the so-called *Segal conditions.* The theory of

Received by the editors 2019-09-07 and, in final form, 2020-07-24.
Transmitted by Joachim Kock. Published on 2020-07-29.
2020 Mathematics Subject Classification: 18D20, 18G30, 18N10, 18N40, 18N45, 18N60.
Key words and phrases: Segal categories, enriched categories, homotopy algebras, higher categories.

Segal categories has its roots in the paper of Segal [28] in which he proposed a solution to the *delooping problem*. The general theory starts with the works of Dwyer–Kan–Smith [10] and Schwänzl–Vogt [26]. The main development of Segal n-categories was given by Hirschowitz and Simpson [13]. The definition of Hirschowitz and Simpson is inspired by the earlier works of Tamsamani [34] and Dunn [9], who in turn followed the ideas of Segal [28]. A Segal n-category is defined by its nerve which is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathscr{M}$ satisfying the appropriate Segal conditions. The target category $\mathscr{M}$ is the model category of Segal $(n-1)$-*précats*, which is monoidal for the cartesian product. Moreover, they require the presence of *discrete objects* in $\mathscr{M}$ which play the role of "set of objects". We can interpret their approach as an enrichment over $\mathscr{M}$, even though the description "internal weak-category-object of $\mathscr{M}$" fits better. A theory of weakly enriched categories based on the same ideas was developed by Pellissier [24]. Independently, Rezk [25] followed also the ideas of Segal to define *complete Segal spaces* as weak internal categories of $(\mathrm{Top}, \times)$ and $(\mathrm{sSet}, \times)$. We refer the reader to the paper of Bergner [4] for the interactions between Segal categories, complete Segal spaces, quasicategories, and simplicial categories.

To avoid the use of discrete objects in the base $\mathscr{M}$, Simpson [30] uses a category Δ_X, introduced by Bergner, to define Segal categories as a *proper* enrichment over $\mathscr{M}$ by means of functors $(\Delta_X)^{\mathrm{op}} \longrightarrow \mathscr{M}$. Here by "proper", we simply mean that the set X, which will be the set of objects, is taken "outside" $\mathscr{M}$.

In this paper we give a "Segalic" definition of enrichment over a bicategory. Our definition generalizes the Hirschowitz–Simpson–Tamsamani approach. As expected, a strict Segal $\mathscr{M}$-category is just an ordinary category enriched over $\mathscr{M}$, thus a polyad in the sense of Bénabou. The definition of a Segal $\mathscr{M}$-category is deeply inspired by the definition of an *up-to-homotopy monoid* introduced by Leinster [19]. Our goal was to "put many objects" in the definition of Leinster. The guiding principle is the identification between monoids and enriched categories with a single object. The "many-objects" case provides, among other things, a Segal version of categories enriched over a monoidal category $\mathscr{M} = (\underline{M}, \otimes, I)$ for which the monoidal product is not the cartesian product. A typical example of such monoidal category is $(\mathrm{ChMod}_R, \otimes_R, R)$, the category of chain complexes of modules over a commutative ring R. We introduce the theory of Segal enriched categories and we show that our formalism generalizes many existing concepts.

1.1. Organization of the paper.

1. In Section 2, we construct from any 1-category $\mathcal{C}$, a strict 2-category $\mathcal{P}_{\mathcal{C}}$ called the 2-*path-category* of $\mathcal{C}$ (Proposition-Definition 2.7).

2. In Section 3, we define a *path-object* of a bicategory $\mathscr{M}$ as a colax (=oplax) morphism $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ in the sense of Bénabou (Definition 3.5). A *Segal path-object* is a path-object that satisfies the Segal conditions.

3. In Section 4, we define *Segal enriched categories* (Definition 4.4) and show that our formalism covers the following situations.

   (a) Up-to-homotopy monoids in the sense of Leinster (Proposition 4.10).

   (b) Simplicial objects (Proposition 4.10).
   (c) Classical enriched categories and the general case of enrichment over a bicategory (Proposition 4.14).
   (d) Segal n-categories in the sense of Hirschowitz–Simpson (Proposition 4.20).

4. In Section 5, we introduce Segal dg-categories (Definition 5.2). We show that the normalization functor $N : \mathrm{sAb} \longrightarrow \mathrm{Ch}^+$ in the Dold–Kan correspondence produces two types of Segal dg-categories by base change:
   (a) with the Alexander–Whitney map, any strict category $\mathcal{C}$ enriched over sAb gives a non strict Segal dg-category $N\mathcal{C}_{AW}$ (Proposition 5.3);
   (b) with the *shuffle map*, any strict category $\mathcal{C}$ enriched over sAb gives a strict Segal dg-category $N\mathcal{C}_{\nabla}$ (Proposition 5.4);
   (c) we give a preliminary result on the existence of a morphism $N\mathcal{C}_{\nabla} \longrightarrow N\mathcal{C}_{AW}$ that is a level-wise weak equivalence between Segal dg-categories (Proposition 5.5).

5. In Section 6, we weaken the definition of Segal enriched categories by introducing first nonunital Segal enriched categories and then adding a unitality constraint. This enables us to get easily a "Segalic" version of the *homotopy transfer theorem* for algebras (Theorem 6.6). As a consequence we establish that:
   (a) if (S, μ, e) is a dg-algebra and $f : S \longrightarrow A$ is a chain homotopy equivalence (resp. homotopy retraction) then there is a structure of a Leinster–Segal dg-algebra on A with a homotopy unit (resp. with a pseudo-unit) (Proposition 6.13);
   (b) the loop space of a compactly generated Hausdorff space has another presentation as a Leinster–Segal topological monoid different from the one given by Leinster [19] (Proposition 6.11).

6. In Section 7, we give the definition of morphism between path-objects and we prove a property of homotopy invariance:
   (a) We show that the shuffle map defines an $\mathscr{M}$-morphism $N\mathcal{C}_{\nabla} \longrightarrow N\mathcal{C}_{AW}$ between Segal dg-categories (Proposition 7.7) that is a level-wise weak equivalence.
   (b) We prove that if $\sigma : F \longrightarrow G$ is a level-wise weak equivalence of path-objects, then under suitable hypotheses if one of them satisfies the Segal conditions then so does the other (Proposition 7.10).

7. In Section 8, we prove that the category of path-objects $\mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ is cocomplete if $\mathscr{M}$ is (Proposition 8.1). However, computing limits is a bit technical, especially when $\mathscr{M}$ has a product $\otimes \neq \times$, and is out of the scope of the present paper.

1.2. Note. The results of Sections 2–4 go back to the author's unpublished preprint [1], essentially superseded by the present publication. However, we have not included here the notions of *Segal bimodules* and *secondary localization of a bicategory* that appear in [1]. On the other hand, the results in Sections 5–8 are new, compared to [1].

1.3. Related work. The theory of Segal $\mathscr{M}$-categories was developed by Simpson [30] when $\mathscr{M}$ is a tractable model category. In that case, $\mathscr{M}$ is a symmetric monoidal category for the cartesian product $\times$, thus a particular case of a 2-category with one object. Simpson's work is based on functors $(\Delta_X)^{\mathrm{op}} \longrightarrow \mathscr{M}$ that satisfy the Segal conditions. In our work this would be equivalent to path-objects $\mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ that satisfy the Segal conditions. The advantage of our framework is that it also applies when $\mathscr{M}$ has a monoidal product $\otimes \neq \times$. One could certainly generalize the material developed by Simpson to general path-objects $\mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$. However, the deployment of a homotopy theory for path-objects $\mathcal{P}_{\mathcal{C}} \longrightarrow (\underline{M}, \otimes, I)$ with $\otimes \neq \times$ requires a development of new techniques of Homotopy Theory since we are working with 2-categories and colax morphisms between them, whereas all other existing concepts mentioned before are based on the homotopy theory of diagrams indexed by 1-categories.

One of the most interesting directions is the study of Segal dg-categories. In this paper we have provided the foundations and some preliminary results but they surely need to be studied further, and we hope to do so in the future. The theory of dg-categories has been recognized by many experts as playing an important role in several fields of mathematics. Leinster's up-to-homotopy monoids are called *Leinster algebras* by Shoikhet [29] who used them to prove a general version of Deligne's conjecture. Tabuada [32] developed the homotopy theory of dg-categories and provided several applications to Geometry. Our first task will be to develop the homotopy theory of Segal dg-categories and compare it with that of dg-categories. We would like to understand the canonical functor dg-Cat $\hookrightarrow$ dg-SegCat with the same question as in the paper of Tamarkin [33]: "What do Segal dg-categories form?".

Segal dg-categories are to be thought of as *higher linear categories*. Other concepts of higher linear categories have been developed by Gepner and Haugseng [11], Lurie [20] and others. We would like to understand the connection between these theories. We have initiated in [2] a theory of co-Segal $\mathscr{M}$-categories, and it remains to outline the link between Segal and co-Segal $\mathscr{M}$-categories. Furthermore, there is a clear analogy between A_∞-categories and (co)-Segal dg-categories that we wish to understand. Finally, we would like to mention that the category $\mathrm{Colax}(\mathcal{P}_{\overline{X}}, \mathscr{M})$ of path-objects has a rich structure that remains elusive since it encloses several mathematical concepts. Indeed, a colax morphism $(F, \varphi) = \{F_{xy}, \varphi : F_{xz}(-_1 * -_2) \longrightarrow F_{xy}(-_1) \otimes F_{yz}(-_2)\}$ can define one of the following objects.

1. A strict category enriched over $\mathscr{M}$ if every colaxity map φ is invertible.

2. A *cocategory* in the sense of "comonoid with many objects", if the functors F_{xy} are all constant. The cocomposition and counit are given by the colaxity maps φ.

3. A Segal enriched category, which is a homotopy enriched category.
4. A simplicial object, in particular a Segal n-category, if $\mathscr{M}$ is monoidal for the cartesian product.

1.4. Acknowledgments. A large part of this paper comes from my PhD thesis, and I would like to warmly thank and express my gratitude to my former supervisor Carlos Simpson, whose encouragement, guidance and support have enabled me to develop an understanding of the subject. I would like to extend my thanks to Julia Bergner and Tom Leinster who have suggested the question of finding a Segal-like definition of enriched categories. Tom Leinster's ideas were inspirational in many ways. Many thanks to the staff of the Laboratoire J-A Dieudonné who provided me an excellent working environment. Finally, I would like to thank the anonymous referee for his/her report and the helpful comments.

Notation and conventions.

1. $\mathbb{U}$ = a universe.
2. Set = the category of $\mathbb{U}$-small sets.
3. $\mathrm{Cat}_{\leq 1}$ = the 1-category of $\mathbb{U}$-small categories.
4. 2-$\mathrm{Cat}_{\leq 1}$ = the 1-category of $\mathbb{U}$-small weak 2-categories (= bicategories) with pseudo-functors (= homomorphisms).
5. $\mathscr{B}, \mathscr{M}, \ldots$ = bicategories.
6. $\mathbb{1} = \{\diamond, \diamond \xrightarrow{Id_\diamond} \diamond\}$ = the unit category.
7. Ob$(-)$ = functor that gives the set of objects.
8. Mor$(-)$ = functor that gives the set of morphisms of a 1-category
9. $\mathrm{Mor}_1(-)$ = functor that gives the set of 1-morphisms (= 1-cells).
10. $\mathrm{Mor}_2(-)$ = functor that gives the set of 2-morphisms (= 2-cells).
11. We will use the *diagrammatic order* for the composition in $\mathscr{B}$:

$$\mathscr{B}(x,y) \times \mathscr{B}(y,z) \longrightarrow \mathscr{B}(x,z)$$
$$(x \xrightarrow{f} y, y \xrightarrow{g} z) \mapsto f * g.$$

12. We will use the symbol $\otimes$ instead of $*$ in $\mathscr{M}$: $(u,v) \mapsto u \otimes v$. The symbol $\otimes$ will be used exclusively for the letter $\mathscr{M}$, and for monoidal categories, for which the tensor product is not the cartesian product.

13. The backward infix notation $(f,g) \mapsto g \circ f$, will be used freely for 1-categories.

14. In each hom-category $\mathscr{B}(x,y)$, we will use the diagrammatic notation: $(\alpha,\beta) \mapsto \alpha \boxminus \beta$, for the vertical composition of 2-morphisms. However, since $\mathscr{B}(x,y)$ is a 1-category, we may also use the backward infix notation $(\alpha,\beta) \mapsto \beta \circ \alpha$, if there is no danger of confusion.

We assume that the reader is familiar with bicategories. However, since the notion of colax morphism is central in this paper, we have recalled the definition in the Appendix.

2. The 2-Path-category of a small 1-category

We follow the notation of Deligne [8] and denote by Δ^+ the *augmented* category of *all* finite totally ordered sets, including the empty set. We will denote by Δ the "topologists's delta", i.e., the one that does not contain the empty set, and that is commonly used to define simplicial objects. The objects of Δ^+ are ordinal numbers $n = \{0, ..., n-1\}$ and the morphisms are the nondecreasing functions $f : n \longrightarrow m$. $(\Delta^+, +, 0)$ is a non-symmetric monoidal category for the *ordinal addition*. The object 0 is an initial object and 1 is a terminal object.

Our first step is to find a tool that generalizes the monoidal category $(\Delta^+, +, 0)$, since it contains the *universal monoid* which corresponds to the object 1. More precisely, Mac Lane [22] showed that a monoid (S, μ, e) in a monoidal category $\mathscr{M}$ is equivalent to a monoidal functor $\mathscr{N}_S : (\Delta^+, +, 0) \longrightarrow \mathscr{M}$ such that $\mathscr{N}_S(1) = S$. As mentioned before, a monoid is viewed as an $\mathscr{M}$-category with one object, therefore we can consider the functor $\mathscr{N}_S$ as the *nerve* of the corresponding category whose hom-object is S. From this observation it becomes natural to find a device that will be used to "parametrize" many monoids and bimodules in $\mathscr{M}$ to form an $\mathscr{M}$-category with many objects. This led us to the notion of: 2*-path-category associated to a* 1*-category* $\mathcal{C}$.

2.1. Warning. The category $(\Delta^+, +, 0)$ considered here is simply denoted as $(\Delta, +, 0)$ by Leinster [19] and Mac Lane [22].

Let $\mathcal{C}$ be a small category. For any pair (x, y) of objects, we build a cosimplicial diagram which is similar to the one obtained from the *bar construction for algebras*:

1. If $x \neq y$:

$$\emptyset \xrightarrow{!} \mathcal{C}(x,y) \rightleftarrows \coprod \mathcal{C}(x,x_1) \times \mathcal{C}(x_1,y) \rightleftarrows \coprod \mathcal{C}(x,x_1) \times \mathcal{C}(x_1,x_2) \times \mathcal{C}(x_2,y) \cdots$$

2. If $x = y$:

$$\{x\} \cong 1 \xrightarrow{I_x} \mathcal{C}(x,x) \rightleftarrows \coprod \mathcal{C}(x,x_1) \times \mathcal{C}(x_1,x) \rightleftarrows \coprod \mathcal{C}(x,x_1) \times \mathcal{C}(x_1,x_2) \times \mathcal{C}(x_2,x) \cdots$$

The dotted arrows correspond to inserting an identity map while the solid ones correspond to applying the composition. The following remark can be found in Mac Lane [22].

2.2. Remark. The morphisms of Δ^+ are generated by the following maps under composition:

1. *cofaces* $d^i : n+1 \longrightarrow n$, $0 \leq i \leq n-1$,
2. *codegeneracies* $s^i : n \longrightarrow n+1$, $0 \leq i \leq n$.

In order to define a functor $F : \Delta^+ \longrightarrow \mathrm{Set}$, it suffices to specify the image of $F(d^i)$ and $F(s^i)$ and check that some *simplicial identities* hold (see [22, Chapter VII, Section 5]).

2.3. Definition. *Let* $\mathcal{C}$ *be a* 1*-category, and let* (x, y) *be a pair of objects of* $\mathcal{C}$*. Let* $\mathcal{P}_{xy} : \Delta^+ \longrightarrow \mathrm{Set}$ *be the functor defined by the following data.*

1. *If* $n \neq 0$, $\mathcal{P}_{xy}(n)$ *is the set of chains of* n *composable morphisms with extremal vertices* x *and* y*:*

$$\mathcal{P}_{xy}(n) = \coprod_{(x=x_0,\dots,x_n=y)} \mathcal{C}(x_0, x_1) \times \cdots \times \mathcal{C}(x_{n-1}, x_n)$$
$$\mathcal{P}_{xy}(n) \cong \{s = (x_0 \longrightarrow x_1, \cdots, x_i \longrightarrow x_{i+1}, \cdots, x_{n-1} \longrightarrow x_n)\}$$

   in particular we have: $\mathcal{P}_{xy}(1) = \mathcal{C}(x, y)$.

2. *If* $n = 0$ *then:*
   (a) $\mathcal{P}_{xy}(0) = \{(x)\}$ *is a* 1*-element set if* $x = y$,
   (b) $\mathcal{P}_{xy}(0) = \emptyset$ *is the empty set if* $x \neq y$,

3. *On morphisms, we define the image of codegeneracies and cofaces as follows.*
   (a) $\mathcal{P}_{xy}(d^i) : \mathcal{P}_{xy}(n+1) \longrightarrow \mathcal{P}_{xy}(n)$ *is the function that applies the composition at the* $(i+1)$*th vertex* $(1 \leq i+1 \leq n-1)$, *i.e., we replace the ith and* $(i+1)$*th morphisms by their composite:*

$$\mathcal{P}_{xy}(d^i) = \coprod_{(x=x_0,\dots,x_n=y)} \mathrm{Id} \times \cdots \times c(x_i, x_{i+1}, x_{i+2}) \times \cdots \mathrm{Id}, \quad \text{if } 0 \leq i \leq n-2.$$

   (b) $\mathcal{P}_{xy}(s^i) : \mathcal{P}_{xy}(n) \longrightarrow \mathcal{P}_{xy}(n+1)$ *is the function that takes a chain* s *and inserts the identity morphism of the ith vertex* $(0 \leq i \leq n)$.

It is tedious but straightforward to show that $\mathcal{P}_{xy}$ is indeed a functor. To do this, it suffices to establish the simplicial identities. These identities follow from the associativity and the identity axioms for the category $\mathcal{C}$.

2.4. Definition. [Concatenation of paths] *Given s in $\mathcal{P}_{xy}(n)$ and t in $\mathcal{P}_{yz}(m)$*

$$s = \quad x \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_i \longrightarrow x_{i+1} \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow y$$

$$t = \quad y \longrightarrow y_1 \longrightarrow \cdots \longrightarrow y_j \longrightarrow y_{j+1} \longrightarrow \cdots \longrightarrow y_{m-1} \longrightarrow z$$

we define the concatenation *of t and s to be the element of $\mathcal{P}_{xz}(n+m)$:*

$$s * t := \quad \underbrace{x \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow}_{s} y \underbrace{\longrightarrow y_1 \longrightarrow \cdots \longrightarrow y_{m-1} \longrightarrow z}_{t}.$$

It follows from the definition that for any n and for any $s \in \mathcal{P}_{xy}(n)$ we have:

- $s * (y) = s$,
- $(x) * s = s$.

The Grothendieck construction. In the following we apply the Grothendieck construction to the functors $\mathcal{P}_{xy}$.

2.5. Definition. *Let (x, y) be a pair of objects of $\mathcal{C}$. We define $\mathcal{P}_{\mathcal{C}}(x, y)$ as the* category of elements *or the* Grothendieck integral *of the functor $\mathcal{P}_{xy}$.*

1. *The objects of $\mathcal{P}_{\mathcal{C}}(x, y)$ are pairs $[n, s]$, where n is an object of Δ^+ and s is an object of $\mathcal{P}_{xy}(n)$.*
2. *A morphism $[n, s] \xrightarrow{u} [m, t]$ in $\mathcal{P}_{\mathcal{C}}(x, y)$ is a map $u : n \longrightarrow m$ of Δ^+, such that the function $\mathcal{P}_{xy}(u) : \mathcal{P}_{xy}(n) \longrightarrow \mathcal{P}_{xy}(m)$ maps $s \mapsto t$, i.e., $\mathcal{P}_{xy}(u)s = t$.*
3. *If $x \neq y$, $\mathcal{P}_{xy}(0) = \emptyset$, therefore there is no morphism of the form $[0, s]$ in $\mathcal{P}_{\mathcal{C}}(x, y)$.*

We have a forgetful functor $\mathcal{L}_{xy} : \mathcal{P}_{\mathcal{C}}(x, y) \longrightarrow \Delta^+$ that is a Grothendieck op-fibration, given by $\mathcal{L}_{xy}([n, s]) = n$ and $\mathcal{L}_{xy}([n, s] \xrightarrow{u} [m, t]) = u$. The functor $\mathcal{L}_{xy}$ will be called length.

2.6. Proposition.

1. *The concatenation of paths extends to a functor. More precisely, for each triple (x, y, z) of objects of $\mathcal{C}$ we denote by $c(x, y, z)$ the functor defined as follows:*

$$c(x, y, z) : \mathcal{P}_{\mathcal{C}}(x, y) \times \mathcal{P}_{\mathcal{C}}(y, z) \longrightarrow \mathcal{P}_{\mathcal{C}}(x, z)$$

$$\left[\begin{pmatrix} [n, s] \\ \Downarrow u \\ [m, t] \end{pmatrix}, \begin{pmatrix} [n', s'] \\ \Downarrow u' \\ [m', t'] \end{pmatrix} \right] \mapsto \begin{pmatrix} [n + n', s * s'] \\ \Downarrow u * u' \\ [m + m', t * t'] \end{pmatrix}$$

2. *The concatenation is strictly associative.*

Proof. The proof is tedious but straightforward and will be left to the reader. ■

We have now set up the necessary tools for the definition of the 2-path-category.

2.7. Proposition-Definition. *Let* $\mathcal{C}$ *be a small category. Then the following data determine a strict* 2*-category* $\mathcal{P}_{\mathcal{C}}$*, called the* 2-path-category *of* $\mathcal{C}$*.*

- *The objects of* $\mathcal{P}_{\mathcal{C}}$ *are the objects of* $\mathcal{C}$*.*
- *For each pair* (x, y) *of objects of* $\mathcal{P}_{\mathcal{C}}$*, the hom-category of* $\mathcal{P}_{\mathcal{C}}$ *is the category* $\mathcal{P}_{\mathcal{C}}(x, y)$ *described above.*
- *For each triple* (x, y, z) *of objects, the composition functor is given by* the concatenation functor *above:* $c(x, y, z) : \mathcal{P}_{\mathcal{C}}(x, y) \times \mathcal{P}_{\mathcal{C}}(y, z) \longrightarrow \mathcal{P}_{\mathcal{C}}(x, z)$.
- *For every object* x *of* $\mathcal{C}$*, we have a* strict identity *arrow* $I_x : 1 \longrightarrow \mathcal{P}_{\mathcal{C}}(x, x)$ *given by* $[0, (x)]$.
- *For each quadruple* (x, y, z, w) *of objects, the associativity natural isomorphism* $a(x, y, z, w)$ *is* the identity.
- *The left and right identities natural isomorphisms are* the identity *for each pair of objects* (x, y).

Clearly, we have:

2.8. Proposition. *If* $\mathcal{C} \cong \mathbb{1}$*, say* $\mathrm{Ob}(\mathcal{C}) = \{\diamond\}$ *and* $\mathcal{C}(\diamond, \diamond) = \{\mathrm{Id}_{\diamond}\}$*, we have a monoidal isomorphism:*

$$(\mathcal{P}_{\mathcal{C}}(\diamond, \diamond), c(\diamond, \diamond, \diamond), (\diamond)) \xrightarrow{\sim} (\Delta^+, +, 0).$$

Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. By definition, F sends composable arrows of $\mathcal{C}$ to composable arrows of $\mathcal{D}$, and respects composition and identities. We can then easily see that F induces a strict homomorphism $\mathcal{P}_F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathcal{P}_{\mathcal{D}}$. That is we have:

2.9. Proposition. *The construction* $\mathcal{C} \mapsto \mathcal{P}_{\mathcal{C}}$ *defines a functor:*

$$\mathcal{P}_{[-]} : \mathrm{Cat}_{\leq 1} \longrightarrow 2\text{-}\mathrm{Cat}_{\leq 1}$$
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longmapsto \mathcal{P}_{\mathcal{C}} \xrightarrow{\mathcal{P}_F} \mathcal{P}_{\mathcal{D}}$$

Similar "path-constructions" have been considered by Dawson, Paré and Pronk for double categories (see [7]). We can compare Example 1.2 and Remark 1.3 of their paper with the fact that here we have: $\mathcal{P}_{\mathbb{1}}$ "is" $(\Delta^+, +, 0)$. Since $\mathcal{C}$ is an arbitrary small category, we can consider geometric situations where $\mathcal{C}$ is a Grothendieck site to "transport" geometry to the context of enriched categories. In fact, if $\mathscr{M}$ is a 1-category viewed as a 2-category with only identity 2-morphisms, then a functor $\mathcal{C} \longrightarrow \mathscr{M}$ is the same thing as a (strict) 2-functor $\mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$.

2.10. Note. The 2-category $\mathcal{P}_{\mathcal{C}}$ appears in a more general adjunction due to Bénabou in unpublished work. We gave a construction of $\mathcal{P}_{\mathcal{C}}$ to have an almost self-contained paper. We shall refer the reader to a recent work of Chiche [6] for the adjunction due to Bénabou between lax morphisms and homomorphisms.

From the construction of $\mathcal{P}_{\mathcal{C}}$, we can prove directly the following result which is an application of Bénabou's adjunction in its simplest form (see [6]).

2.11. Proposition. *For any* 2*-category* $\mathscr{M}$ *we have an isomorphism of sets:*

$$\mathrm{Lax}(\mathcal{C}, \mathscr{M}) \cong \mathrm{Hom}(\mathcal{P}_{\mathcal{C}}, \mathscr{M}),$$

where the left-hand side is the set of lax morphisms from $\mathcal{C}$ *to* $\mathscr{M}$ *while the right-hand side is the set of homomorphisms in the sense of Bénabou [3].*

2.12. Basic properties. In what follows, we give some basic properties of the path-functor. They will be needed when we will study the different types of products between path-objects. Since $\mathbb{1}$ is terminal in Cat, we have by functoriality, a homomorphism $\mathcal{P}_{\mathcal{C}} \to \mathcal{P}_{\mathbb{1}}$. We may call it the "skeleton-morphism" and we will say that $\mathcal{P}_{\mathcal{C}}$ *is over* $\mathcal{P}_{\mathbb{1}}$.

2.13. Proposition. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two small categories, and let* $\mathcal{P}_{\mathcal{C}} \to \mathcal{P}_{\mathbb{1}}$, $\mathcal{P}_{\mathcal{D}} \to \mathcal{P}_{\mathbb{1}}$ *the corresponding* 2*-path-categories. Then the following hold.*

1. *We have an isomorphism of* 2*-categories:* $\mathcal{P}_{\mathcal{C} \coprod \mathcal{D}} \cong \mathcal{P}_{\mathcal{C}} \coprod \mathcal{P}_{\mathcal{D}}$.
2. *We have an isomorphism of* 2*-categories:*

$$\mathcal{P}_{\mathcal{C} \times \mathcal{D}} \xrightarrow{\cong} (\mathcal{P}_{\mathcal{C}} \times_{\mathcal{P}_{\mathbb{1}}} \mathcal{P}_{\mathcal{D}}),$$

where $(\mathcal{P}_{\mathcal{C}} \times_{\mathcal{P}_{\mathbb{1}}} \mathcal{P}_{\mathcal{D}})$ *is the strict* 2*-pullback construction.*

Proof. Assertion (1) is obvious. To prove Assertion (2) it suffices to write down the definition of $\mathcal{P}_{\mathcal{C} \times \mathcal{D}}$. A 1-morphism $[n, s]$ in $\mathcal{P}_{\mathcal{C} \times \mathcal{D}}$ is by definition the same thing as a pair of 1-morphisms $([n, s_{\mathcal{C}}], [n, s_{\mathcal{D}}])$. Moreover, a 2-morphism in $\mathcal{P}_{\mathcal{C} \times \mathcal{D}}$ is a morphism that is parametrized by a map of Δ^+, and that map parametrizes factor-wise a 2-morphism in $\mathcal{P}_{\mathcal{C}}$ and in $\mathcal{P}_{\mathcal{D}}$, which means that it is a 2-morphism of the strict 2-fiber product $\mathcal{P}_{\mathcal{C}} \times_{\mathcal{P}_{\mathbb{1}}} \mathcal{P}_{\mathcal{D}}$. Recall that $\mathcal{P}_{\mathcal{C}} \times_{\mathcal{P}_{\mathbb{1}}} \mathcal{P}_{\mathcal{D}}$ is given by the following data:

- Objects: $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$.
- Morphisms: Given $(x, y) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and $(x', y') \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ we use the length functors $\mathcal{L}_{xx'} : \mathcal{P}_{\mathcal{C}}(x, x') \longrightarrow \Delta^+$ and $\mathcal{L}_{yy'} : \mathcal{P}_{\mathcal{D}}(y, y') \longrightarrow \Delta^+$ to define:

$$(\mathcal{P}_{\mathcal{C}} \times_{\mathcal{P}_{\mathbb{1}}} \mathcal{P}_{\mathcal{D}})[(x, y), (x', y')] := \mathcal{P}_{\mathcal{C}}(x, x') \times_{\Delta^+} \mathcal{P}_{\mathcal{D}}(y, y').$$

- The composition is concatenation of chains factor-wise.

■

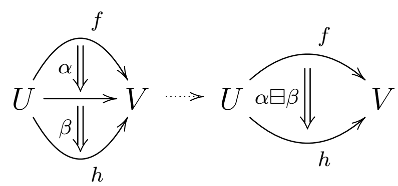
3. Path-object of a bicategory

Let $\mathscr{M}$ be a bicategory, and let $\mathscr{W}$ be a class of 2-morphisms of $\mathscr{M}$. The following definition is a generalization of the one given by Leinster [19].

3.1. DEFINITION. *The pair* $(\mathscr{M},\mathscr{W})$ *is said to be a* base of enrichment *if* $\mathscr{W}$ *has the following properties.*

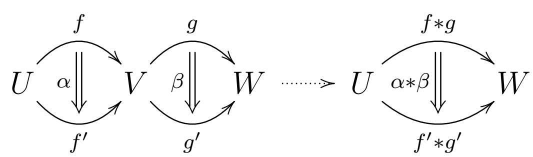
1. *Every invertible* 2*-morphism of* $\mathscr{M}$ *is in* $\mathscr{W}$*, in particular* 2*-identities, are in* $\mathscr{W}$*,*

2. $\mathscr{W}$ *has the vertical* 3*-for-*2 *property, that is, in the situation*



if 2 *elements in the set* $\{\alpha, \beta, \alpha \boxminus \beta\}$ *are in* $\mathscr{W}$*, then so is the third.*

3. $\mathscr{W}$ *is stable under* horizontal *compositions, that is, in the situation*



if α *and* β *are both in* $\mathscr{W}$*, then so is* $\alpha * \beta$.

3.2. REMARK. If $\mathscr{W} = 2\text{-Iso}$ is the class of all invertible 2-morphisms of $\mathscr{M}$, then $(\mathscr{M}, 2\text{-Iso})$ is the smallest base of enrichment since by definition every base $(\mathscr{M},\mathscr{W})$ contains $(\mathscr{M}, 2\text{-Iso})$. If $\mathscr{W}$ is the class $\mathrm{Mor}_2(\mathscr{M})$ of all of 2-morphisms, we get the largest base $(\mathscr{M}, \mathrm{Mor}_2(\mathscr{M}))$.

3.3. EXAMPLE. If $\mathscr{M} = (\underline{M}, \otimes, I)$ is a monoidal model category with $\mathscr{W}$ the class of weak equivalences as in [14] such that all objects are cofibrant, then $(\mathscr{M},\mathscr{W})$ is a base of enrichment.

3.4. DEFINITION. *Let* $\mathscr{B}$ *be a bicategory and let* $(\mathscr{M},\mathscr{W})$ *be a base of enrichment. Say that a colax morphism* $F = (F, \varphi) : \mathscr{B} \longrightarrow \mathscr{M}$ satisfies the Segal conditions *or that* F *is a* $\mathscr{W}$-colax morphism*, if the following hold.*

1. *For any pair* (f, g) *of composable* 1*-morphisms, the colaxity map* $\varphi : F(f * g) \longrightarrow F(f) \otimes F(g)$ *is in* $\mathscr{W}$*.*

2. *For any object* $x \in \mathscr{B}$*, the colaxity map* $\varphi_x : F(I_x) \longrightarrow I_{Fx}$ *is in* $\mathscr{W}$*.*

3.5. Definition. *Let $(\mathscr{M}, \mathscr{W})$ be a base of enrichment.*

1. *A* path-object *of $(\mathscr{M}, \mathscr{W})$ is a pair $(\mathcal{C}, F)$, where $\mathcal{C}$ is a small category and $F = (F, \varphi)$ a* colax *morphism: $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$.*

2. *A* Segal path-object *is a path-object that satisfies the Segal conditions, i.e., a $\mathscr{W}$-colax morphism: $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$.*

3. *A* strict Segal path-object *is a (colax)* homomorphism *of bicategories: $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$.*

3.6. Remark. The reason we consider *colax* (or oplax) morphisms, is the fact that "colax" is the appropriate replacement of *simplicial* if $\mathscr{M}$ is monoidal category with a product $\otimes$ different from the cartesian product $\times$ (see Proposition 4.9). When F satisfies the Segal conditions we may consider F as a "$\mathcal{C}$-homotopy coherent nerve".

3.7. Terminology.

1. If $Fx = U$ we will say that x is *over* U. Here we are following the geometric picture in enrichment over a bicategory that appears in [31, 35, 36].

2. Since a path-object is a sort of morphism from $\mathcal{P}_{\mathcal{C}}$ to $\mathscr{M}$ we will call it a "$\mathcal{P}_{\mathcal{C}}$-point" of $\mathscr{M}$, like in scheme theory. For short we will simply say $\mathcal{C}$-*point* of $\mathscr{M}$. We will therefore say Segal $\mathcal{C}$-point for a Segal path-object $(\mathcal{C}, F)$.

3.8. Base change. Bénabou showed in [3, Section 4.3] that lax morphisms of bicategories can be composed. If we use his arguments by reversing the direction of the laxity maps, we also see that colax morphisms of bicategories can be composed.

3.9. Definition. *Let $(\mathscr{M}_1, \mathscr{W}_1)$, $(\mathscr{M}_2, \mathscr{W}_2)$ be two bases of enrichment and let $L : \mathscr{M}_1 \longrightarrow \mathscr{M}_2$ be a colax morphism of bicategories. Say that L is a* colax morphism of bases *if the following hold.*

1. *$L(\mathscr{W}_1) \subseteq \mathscr{W}_2$.*

2. *L is a $\mathscr{W}_2$-colax morphism in the sense of Definition 3.4.*

If $(\mathcal{C}, F)$ is a path-object of $(\mathscr{M}_1, \mathscr{W}_1)$, define the base change of $(\mathcal{C}, F)$ along L*, as the path-object $(\mathcal{C}, L \circ F)$ of $(\mathscr{M}_2, \mathscr{W}_2)$. This operation is called* base change along L*.*

3.10. Proposition. *Let $(\mathcal{C}, F)$ be a path-object of $(\mathscr{M}_1, \mathscr{W}_1)$, and let $(\mathcal{C}, L \circ F)$ be the base change along $L : \mathscr{M}_1 \longrightarrow \mathscr{M}_2$. Then if F satisfies the Segal conditions, then so does $L \circ F$.*

PROOF. This is easy, it suffices to write down the colaxity maps for $L \circ F$. The colaxity maps for F are of the form $\varphi : F(s * t) \longrightarrow F(s) \otimes F(t)$. The colaxity map for $L \circ F$ is given by: $L(F(s*t)) \xrightarrow[L(\varphi)]{\sim} L(F(s) \otimes F(t)) \underset{\psi}{\xrightarrow{\sim}} L(F(s)) \otimes L(F(t))$. The map $L(\varphi)$ belongs to $\mathscr{W}_2$ because φ is in $\mathscr{W}_1$ and we have an inclusion $L(\mathscr{W}_1) \subseteq \mathscr{W}_2$. The colaxity map $\psi : L(F(s) \otimes F(t)) \xrightarrow{\sim} L(F(s)) \otimes L(F(t))$ is in $\mathscr{W}_2$ since L is a $\mathscr{W}_2$-colax morphism. By composition, we find that the map $\psi \circ L(\varphi)$ is in $\mathscr{W}_2$ as well. Similarly for any $x \in \mathcal{C}$, the composite hereafter is in $\mathscr{W}_2$:

$$LF(0,(x)) \xrightarrow[\sim]{L(\varphi_x)} L(I_{Fx}) \xrightarrow[\sim]{\psi_{Fx}} I_{LFx}.$$

■

4. Segal enriched categories

4.1. DEFINITION. *Let X be a set. Define the* coarse or indiscrete *category associated to X as the groupoid $\overline{X}$ defined as follows.*

1. $\mathrm{Ob}(\overline{X}) = X$,
2. $\mathrm{Hom}(x,y) = \{(x,y)\} \cong 1$,
3. *The morphism (x,x) is the identity morphism of x,*
4. *The composition is the unique one.*

One can easily verify the following result:

4.2. PROPOSITION. *For any 1-category $\mathcal{B}$, and for any set X, we have an isomorphism of sets:*

$$\mathrm{Hom}(\mathcal{B}, \overline{X}) \cong \mathrm{Hom}(\mathrm{Ob}(\mathcal{B}), X).$$

In other words, the assignment $X \mapsto \overline{X}$ defines a functor Set $\longrightarrow$ Cat *that is right adjoint to* Ob : Cat $\longrightarrow$ Set.

4.3. REMARK. If X has only one element, say $X = \{x\}$, then $\overline{X}$ consists of the object x with the identity 1_x, thus $\overline{X} \cong \mathbb{1}$. By Proposition 2.8, we have a monoidal isomorphism between $\mathcal{P}_{\overline{X}}(x,x)$ and $(\Delta^+, +, 0)$.

4.4. DEFINITION. *Let $(\mathscr{M}, \mathscr{W})$ be a base of enrichment.*

1. *Define an $\mathscr{M}$-*precategory *as a path-object $F : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ for some set X.*
2. *Define a* Segal $\mathscr{M}$-category *as a Segal path-object $F : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ for some set X.*
3. *Define a* normal Segal $\mathscr{M}$-category *as a Segal path-object $F : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ that is in addition a* normal colax morphism *of bicategories.*

We will show in a moment that the language of path-objects covers some classical situations.

4.5. Up-to-homotopy monoids and Simplicial objects.

4.6. Definition. *Let* $(\mathcal{N}, \otimes, I)$, $(\mathcal{N}', \otimes, I')$ *be two monoidal categories. A* colax *monoidal functor* $(Y, \xi) : (\mathcal{N}, \otimes, I) \longrightarrow (\mathcal{N}', \otimes, I')$ *consists of a functor* $Y : \mathcal{N} \longrightarrow \mathcal{N}'$ *together with colaxity maps:*

$$\xi_{AB} : Y(A \otimes B) \longrightarrow Y(A) \otimes Y(B),$$

$$\xi_0 : Y(I) \longrightarrow I',$$

$(A, B \in \mathcal{N})$*, satisfying naturality and coherence axioms.*

The following definition is due to Leinster [19].

4.7. Definition. *Let* $\mathscr{M}$ *be a monoidal category equipped with a class of* homotopy equivalences $\mathscr{W}$ *such that the pair* $(\mathscr{M}, \mathscr{W})$ *is a base of enrichment. An* up-to-homotopy monoid *in* $\mathscr{M}$ *is colax monoidal functor*

$$(Y, \xi) : (\Delta^+, +, 0) \longrightarrow \mathscr{M},$$

for which the maps ξ_0, ξ_{mn} *are in* $\mathscr{W}$ *for every* m, n *in* Δ^+*.*

4.8. Definition. *Let* $\mathscr{M}$ *be a category. A simplicial object of* $\mathscr{M}$ *is a functor* $Y : \Delta^{\mathrm{op}} \longrightarrow \mathscr{M}$*.*

The following result is due to Leinster [18].

4.9. Proposition. *Let* $(\mathscr{M}, \times, 1)$ *be a category with finite products. Then there is an isomorphism of categories*

$$\mathrm{Colax}((\Delta^+, +, 0), (\mathscr{M}, \times, 1)) \cong [\Delta^{\mathrm{op}}, \mathscr{M}].$$

4.10. Proposition. *Let* $\mathscr{M} = (\underline{M}, \otimes, I)$ *be a monoidal category.*

1. *We have an equivalence between the following data:*
   - *a* $\mathbb{1}$*-point of* $(\mathscr{M}, Iso(\mathscr{M}))$*,*
   - *a monoid of* $\mathscr{M}$*.*
2. *Assume that* $(\mathscr{M}, \mathscr{W})$ *is a base of enrichment. Then we have an equivalence between the following data:*
   - *a* $\mathbb{1}$*-point of* $(\mathscr{M}, \mathscr{W})$*,*
   - *an up-to-homotopy monoid in the sense of Leinster [19].*
3. *If* $\mathscr{M} = (\underline{M}, \times, 1)$ *is a monoidal category for the cartesian product, then we have an equivalence between the following data:*
   - *a* $\mathbb{1}$*-point of* $(\mathscr{M}, Mor(\mathscr{M}))$*,*
   - *a simplicial object of* $\mathscr{M}$*.*

4.11. Remark. When we will define the appropriate notion of a morphism of $\mathcal{C}$-points, each equivalence will be automatically an equivalence of categories.

Proof of Proposition 4.10. Let F be a $\mathbb{1}$-point of $(\mathscr{M},\mathscr{W})$. By definition F is a $\mathscr{W}$-colax morphism of bicategories $F : \mathcal{P}_{\mathbb{1}} \longrightarrow \mathscr{M}$. Since $\mathcal{P}_{\mathbb{1}}$ is a bicategory with one object, F is entirely determined by the following data:

1. a functor $F_{\diamond\diamond} : \mathcal{P}_{\mathbb{1}}(\diamond,\diamond) \longrightarrow \mathscr{M}$ which is the only component of F
2. arrows $F_{\diamond\diamond}(s*t) \xrightarrow{\varphi(\diamond,\diamond,\diamond)(s,t)} F_{\diamond\diamond}(s) \otimes F_{\diamond\diamond}(t)$ in $\mathscr{W}$, for every pair (s,t) in $\mathcal{P}_{\mathbb{1}}(\diamond,\diamond)$,
3. an arrow $F_{\diamond\diamond}([0,\diamond]) \xrightarrow{\varphi_{\diamond}} I$ in $\mathscr{W}$,
4. coherence on $\varphi(\diamond,\diamond,\diamond)(s,t)$ and $\varphi_{\diamond}$.

One can check that these data say exactly that $F_{\diamond\diamond}$ is a colax monoidal functor from $(\mathcal{P}_{\mathbb{1}}(\diamond,\diamond), c(\diamond,\diamond,\diamond), [0,(\diamond)])$ to $(\mathscr{M},\otimes,I)$. All assertions are easily seen to be true using the isomorphism $(\Delta^{+},+,0) \cong (\mathcal{P}_{\mathbb{1}}(\diamond,\diamond), c(\diamond,\diamond,\diamond), [0,(\diamond)])$ and Proposition 4.9 above. ∎

4.12. Classical enriched categories. The theory of enriched categories over a monoidal category $\mathscr{M}$ has a natural extension when $\mathscr{M}$ is a 2-category (see [31]). For completeness we recall hereafter the definition of an $\mathscr{M}$-category for a 2-category $\mathscr{M}$.

4.13. Definition. *Let $\mathscr{M}$ be a 2-category. An $\mathscr{M}$-category $\mathcal{C}$ consists of the following data:*

- *for each object U of $\mathscr{M}$, a set $\mathcal{C}_U$ of objects* over *U;*
- *If x and y are respectively over U and V, there is a 1-morphism $\mathcal{C}(x,y) : U \longrightarrow V$ in $\mathscr{M}$;*
- *for each object x over U, a 2-morphism $I_x : \mathrm{Id}_U \Longrightarrow \mathcal{C}(x,x)$ in $\mathscr{M}$;*
- *If x,y,z are over U,V,W respectively, there is a 2-morphism $c_{xyz} : \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \Longrightarrow \mathcal{C}(x,z)$ in $\mathscr{M}$;*

with the obvious three axioms of left and right identities and associativity.

The reader can immediately check that if $\mathscr{M}$ has one object, we recover the definition of enrichment over a monoidal category as in [15].

4.14. Proposition. *Let $(\mathscr{M},\mathscr{W})$ be a base of enrichment. We have an equivalence between the following data.*

1. *A strict Segal path-object $\mathcal{C} : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ of $(\mathscr{M},\mathscr{W})$, i.e., a strict $\overline{X}$-point of $(\mathscr{M},\mathscr{W})$.*
2. *A category $\mathcal{C}$ enriched over $\mathscr{M}$ such that $\mathrm{Ob}(\mathcal{C}) = X$, i.e., an X-polyad in the sense of Bénabou.*

4.15. Notation.

1. If $(f_1, \cdots, f_n)$ is a n-tuple of composable 1-morphisms of $\mathscr{M}$ we will write $f_1 \otimes \cdots \otimes f_n$ for the horizontal composition of $f_1, \cdots, f_n$ with all pairs of parentheses starting in front;

2. Similarly if $(\alpha_1, \cdots, \alpha_n)$ is a n-tuple of composable 2-morphisms we will write $\alpha_1 \otimes \cdots \otimes \alpha_n$ for the horizontal composition of the $\alpha_1, \cdots, \alpha_n$ with all pairs of parentheses starting in front;

3. For every 1-morphism $f : U \longrightarrow V$ of $\mathscr{M}$ we will write r_f (resp. l_f) for the right identity (resp. left identity) isomorphism $f \xrightarrow{\cong} f \otimes \mathrm{Id}_V$ (resp. $f \xrightarrow{\cong} \mathrm{Id}_U \otimes f$). We will write Id_f for identity 2-morphisms. Here "right" and "left" refer to the position of the identity map.

Proof of Proposition 4.14. The proof of (2) ⇒ (1) goes as follows. Let $\mathcal{C}$ be an $\mathscr{M}$-category. Denote by X the set $\coprod \mathcal{C}_U$. Our goal is to construct a (colax) homomorphism denoted again $\mathcal{C} : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$. Before proceeding, let us recall some basic facts about the 2-path-category $\mathcal{P}_{\overline{X}}$.

By definition, for each pair (x, y) of elements of X, $\mathcal{P}_{\overline{X}}(x, y)$ is the category of elements of a functor $\Delta^+ \longrightarrow \mathrm{Set}$ using the Grothendieck construction. It follows that the morphisms of $\mathcal{P}_{\overline{X}}(x, y)$ are parametrized by the morphisms of Δ^+. Since the morphisms of Δ^+ are generated by the *cofaces* $d^i : n+1 \longrightarrow n$, and the *codegeneracies* $s^i : n \longrightarrow n+1$ under composition (see [22]), then the morphisms of $\mathcal{P}_{\overline{X}}(x, y)$ are generated by the following two types of morphisms:

$$(\dagger)\qquad \begin{array}{c} [n+1, x \longrightarrow \cdots x_{i-1} \xrightarrow{(x_{i-1},x_i)} x_i \xrightarrow{(x_i,x_{i+1})} x_{i+1} \cdots \longrightarrow y] \\ \Big\downarrow d^i \\ [n, x \longrightarrow \cdots x_{i-1} \xrightarrow{(x_{i-1},x_{i+1})} x_{i+1} \cdots \longrightarrow y] \end{array}$$

and

$$(\dagger\dagger)\qquad \begin{array}{c} [n, x \longrightarrow \cdots x_i \xrightarrow{(x_i,x_{i+1})} x_{i+1} \cdots \longrightarrow y] \\ \Big\downarrow s^i \\ [n+1, x \longrightarrow \cdots x_i \xrightarrow{(x_i,x_i)} x_i \xrightarrow{(x_i,x_{i+1})} x_{i+1} \cdots \longrightarrow y] \, . \end{array}$$

A morphism of type (†) reflects the composition at the ith vertex and those of type (††) represent the insertion of the identity map of the ith vertex. With this observation we define the component $\mathcal{C}_{xy} : \mathcal{P}_{\overline{X}}(x, y) \longrightarrow \mathscr{M}$ as follows.

1. The image of $[n, x \longrightarrow \cdots x_i \xrightarrow{(x_i,x_{i+1})} x_{i+1} \cdots \longrightarrow y]$ under $\mathcal{C}_{xy}$ is the 1-morphism of $\mathscr{M}$:

$$\mathcal{C}(x, x_1) \otimes \cdots \otimes \mathcal{C}(x_i, x_{i+1}) \otimes \cdots \otimes \mathcal{C}(x_{n-1}, y).$$

2. If $x = y$ and x is over U, then the image of $[0,(x)]$ is Id_U, the unit of $\mathscr{M}(U,U)$. The image of $[0,(x)] \longrightarrow [1,(x,x)]$ is the identity map for x: $\mathrm{Id}_U \longrightarrow \mathcal{C}(x,x)$.

3. The image of a morphism of type (†) under $\mathcal{C}_{xy}$ is the composite:

$$\begin{array}{c}\mathcal{C}(x,x_1)\otimes\cdots\otimes\mathcal{C}(x_{i-1},x_i)\otimes\mathcal{C}(x_i,x_{i+1})\otimes\cdots\otimes\mathcal{C}(x_n,y)\\ \Big\downarrow\wr\\ \mathcal{C}(x,x_1)\otimes\cdots\otimes[\mathcal{C}(x_{i-1},x_i)\otimes\mathcal{C}(x_i,x_{i+1})]\otimes\cdots\otimes\mathcal{C}(x_n,y)\\ \Big\downarrow \mathrm{Id}_{\mathcal{C}(x,x_1)}\otimes\cdots\otimes c_{x_{i-1}x_ix_{i+1}}\otimes\cdots\otimes\mathrm{Id}_{\mathcal{C}(x_n,y)}\\ \mathcal{C}(x,x_1)\otimes\cdots\otimes\mathcal{C}(x_{i-1},x_{i+1})\otimes\cdots\otimes\mathcal{C}(x_n,y)\end{array}.$$

4. To define the image of a morphism of type (††) we have two cases:

   - if $i < n$, then $\mathcal{C}(s^i)$ is the composite:

$$\begin{array}{c}\mathcal{C}(x,x_1)\otimes\cdots\otimes\mathcal{C}(x_i,x_{i+1})\otimes\cdots\otimes\mathcal{C}(x_{n-1},y)\\ \wr\Big\downarrow \mathrm{Id}_{\mathcal{C}(x,x_1)}\otimes\cdots\otimes l_{\mathcal{C}(x_i,x_{i+1})}\otimes\cdots\otimes\mathrm{Id}_{\mathcal{C}(x_{n-1},y)}\\ \mathcal{C}(x,x_1)\otimes\cdots\otimes[I\otimes\mathcal{C}(x_i,x_{i+1})]\otimes\cdots\otimes\mathcal{C}(x_{n-1},y)\\ \Big\downarrow \mathrm{Id}_{\mathcal{C}(x,,x_1)}\otimes\cdots\otimes[I_{x_i}\otimes\mathrm{Id}_{\mathcal{C}(x_i,x_{i+1})}]\otimes\cdots\otimes\mathrm{Id}_{\mathcal{C}(x_{n-1},y)}\\ \mathcal{C}(x,x_1)\otimes\cdots\otimes[\mathcal{C}(x_i,x_i)\otimes\mathcal{C}(x_i,x_{i+1})]\otimes\cdots\otimes\mathcal{C}(x_{n-1},y)\\ \wr\Big\downarrow\\ \mathcal{C}(x,x_1)\otimes\cdots\otimes\mathcal{C}(x_i,x_i)\otimes\mathcal{C}(x_i,x_{i+1})\otimes\cdots\otimes\mathcal{C}(x_{n-1},y)\end{array};$$

   - if $i = n$, we take the same composition except that we replace $l_{\mathcal{C}(x_i,x_{i+1})}$ and $[I_{x_i}\otimes\mathrm{Id}_{\mathcal{C}(x_i,x_{i+1})}]$ by, respectively, $r_{\mathcal{C}(x_{n-1},x_n)}$ and $[\mathrm{Id}_{\mathcal{C}(x_{n-1},x_n)}\otimes I_{x_n}]$.

5. As mentioned before, the morphisms (†) and (††) generate all morphisms of $\mathcal{P}_{\overline{X}}(x,y)$ under composition. Furthermore, using the fact that $\otimes$ is associative and functorial in each variable, it is easy to extend the above formulae to a functor $\mathcal{C}_{xy} : \mathcal{P}_{\overline{X}}(x,y) \longrightarrow \mathscr{M}$.

The construction of the homomorphism is not complete until we say what are the colaxity maps $\varphi(x,y,z)(s,t) : \mathcal{C}_{xz}(s*t) \longrightarrow \mathcal{C}_{yz}(s)\otimes\mathcal{C}_{xy}(t)$. If $s = [n, x\cdots x_i \xrightarrow{(x_i,x_{i+1})} x_{i+1}\cdots y]$, and $t = [m, y\cdots y_j \xrightarrow{(y_j,y_{j+1})} y_{j+1}\cdots z]$, we have:

1. $s*t = [n+m, x \longrightarrow \cdots y \longrightarrow \cdots z]$,

2. $\mathcal{C}_{xz}(s*t) = \mathcal{C}(x,x_1)\otimes\cdots\otimes\mathcal{C}(x_{n-1},y)\otimes\mathcal{C}(y,y_1)\otimes\cdots\otimes\mathcal{C}(y_{m-1},z)$,

3. $\mathcal{C}_{yz}(s)\otimes\mathcal{C}_{xy}(t) = [\mathcal{C}(x,x_1)\otimes\cdots\otimes\mathcal{C}(x_{n-1},y)]\otimes[\mathcal{C}(y,y_1)\otimes\cdots\otimes\mathcal{C}(y_{m-1},z)]$.

Then $\varphi(x,y,z)(s,t)$ is the unique isomorphism from $\mathcal{C}_{xz}(s*t)$ to $\mathcal{C}_{yz}(s)\otimes\mathcal{C}_{xy}(t)$ given by the associativity of $\otimes$. This map moves the parentheses from all front to the desired places. Clearly $\varphi(x,y,z)(s,t)$ is functorial in t and s. We leave it to the reader to check that the functors $\mathcal{C}_{xy}$ together with the maps $\varphi(x,y,z)(s,t)$ and $\varphi_x = \mathrm{Id}_I$, satisfy the coherence axioms of a morphism of bicategories. Then $\mathcal{C}$ is a (colax) unitary[1] homomorphism $\mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ as desired.

The proof of (1) ⇒ (2) is shorter. Let $\mathcal{C}$ be a strict $\overline{X}$-point of $(\mathscr{M},\mathscr{W})$. We construct an $\mathscr{M}$-category denoted again $\mathcal{C}$ as follows.

1. Put $\mathrm{Ob}(\mathcal{C}) = X$.
2. For every pair (x,y) of elements of X, the hom-object is $\mathcal{C}(x,y) := \mathcal{C}_{xy}([1,(x,y)])$.
3. If $\mathcal{C}$ takes $x \in X$ to $U \in \mathrm{Ob}(\mathscr{M})$, then the component $\mathcal{C}_{xx} : \mathcal{P}_{\overline{X}}(x,x) \longrightarrow \mathscr{M}(U,U)$ is a (colax) homomorphism between monoidal categories. In particular $\varphi_x : \mathcal{C}([0,x]) \longrightarrow \mathrm{Id}_U$ is invertible. Furthermore, we have a canonical 2-morphism $[0,x] \xrightarrow{!} [1,(x,x)]$ in $\mathcal{P}_{\overline{X}}(x,x)$ whose image under $\mathcal{C}$ is a 2-morphism $\mathcal{C}([0,x]) \xrightarrow{\mathcal{C}_{xx}(!)} \mathcal{C}([1,(x,x)])$. We take the identity map Id_x to be the composite:
$$\mathrm{Id}_U \underset{\cong}{\xrightarrow{\varphi_x{}^{-1}}} \mathcal{C}([0,x]) \longrightarrow \mathcal{C}([1,(x,x)]) = \mathcal{C}(x,x).$$
4. For every triple (x,y,z) of elements of X, we construct the composition as follows. On the one hand, there is a map $d^0 : [2,(x,y,z)] \longrightarrow [1,(x,z)]$ in $\mathcal{P}_{\overline{X}}(x,z)$ whose image under $\mathcal{C}$ gives a map $\mathcal{C}(d^0) : \mathcal{C}([2,(x,y,z)]) \longrightarrow \mathcal{C}([1,(x,z)])$. On the other hand, we have a colaxity map $\varphi : \mathcal{C}([2,(x,y,z)] \xrightarrow{\cong} \mathcal{C}([1,(x,y)]) \otimes \mathcal{C}([1,(y,z)])$ that is an isomorphism by assumption. Then the composition map c_{xyz} is the composite:
$$\mathcal{C}([1,(x,y)]) \otimes \mathcal{C}([1,(y,z)]) \underset{\cong}{\xrightarrow{\varphi^{-1}}} \mathcal{C}([2,(x,y,z)] \xrightarrow{\mathcal{C}(d^0)} \mathcal{C}([1,(x,z)]).$$
5. Finally, it is not hard to check that the axioms of associativity and unity required in $\mathcal{C}$ follow directly from the axioms in the definition of the colax morphism $\mathcal{C}$.

It is clear that the above data give an $\mathscr{M}$-category. ∎

4.16. Remark. The above proof can be shortened if we use the isomorphism mentioned in Proposition 2.11:
$$\mathrm{Lax}(\overline{X},\mathscr{M}) \cong \mathrm{Hom}(\mathcal{P}_{\overline{X}},\mathscr{M}).$$
$\mathrm{Lax}(\overline{X},\mathscr{M})$ is precisely the category of $\mathscr{M}$-categories whose set of objects is X. The reason we did not present that proof in the first place is the fact that we wanted to outline the combinatorics of $\mathcal{P}_{\overline{X}}$.

[1]unitary means that φ_x is the identity for every object x.

4.17. SEGAL n-CATEGORIES. We remind the reader that Δ is the "topologists's category of simplices". Given a small category $\mathcal{C}$ we can associate functorially a simplicial set $\mathscr{N}(\mathcal{C}) : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$, called *nerve* of $\mathcal{C}$, where $\mathscr{N}(\mathcal{C})_k$ is the set of k-tuples of composable morphisms and $\mathscr{N}(\mathcal{C})_0 = \mathrm{Ob}(\mathcal{C})$. We have some natural maps, called *Segal maps*

$$\mathscr{N}(\mathcal{C})_k \xrightarrow{\cong} \mathscr{N}(\mathcal{C})_1 \times_{\mathscr{N}(\mathcal{C})_0} \cdots \times_{\mathscr{N}(\mathcal{C})_0} \mathscr{N}(\mathcal{C})_1,$$

which are isomorphisms. Hirschowitz and Simpson [13] generalized this process to define inductively Segal n-categories. Hirschowitz and Simpson defined first a category $n\,\mathrm{SePC}$ of *Segal n-précats* as follows.

1. A Segal 0-précat is a simplicial set, hence a $\mathbb{1}$-point of $(\mathrm{Set}, \times, 1)$ in our terminology.
2. For $n \geq 1$, a Segal n-précat is a functor:

$$\mathscr{A} : \Delta^{\mathrm{op}} \longrightarrow (n-1)\,\mathrm{SePC}$$

   such that $\mathscr{A}_0 = \mathscr{A}(0)$ is a *discrete* object of $(n-1)\,\mathrm{SePC}$.
3. A *morphism* of Segal n-précats is a natural transformation.

These data define the category $n\,\mathrm{SePC}$. Hirschowitz and Simpson also gave a notion of *equivalence* in $n\,\mathrm{SePC}$ and a model structure on it. In particular, $n\,\mathrm{SePC}$ is a monoidal category for the cartesian product. In the sequel, we will denote by 1 the terminal object in $n\,\mathrm{SePC}$ for every n.

4.18. DEFINITION. *A Segal n-category is a Segal n-précat $\mathscr{A} : \Delta^{\mathrm{op}} \longrightarrow (n-1)\,\mathrm{SePC}$ such that:*

1. *for every m, $\mathscr{A}_m$ is a Segal $(n-1)$-category,*
2. *for every $m \geq 1$, the canonical map*

$$\mathscr{A}_m \longrightarrow \mathscr{A}_1 \times_{\mathscr{A}_0} \cdots \times_{\mathscr{A}_0} \mathscr{A}_1$$

   is an equivalence of Segal $(n-1)$-précats.

4.19. REMARK. The above definition involves the use of *discrete* objects. A discrete object in [13], is by definition an object in the image of a fully faithful functor from $\mathrm{disc} : \mathrm{Set} \longrightarrow (n-1)\,\mathrm{SePC}$. For a Segal n-category $\mathscr{A}$ the discrete object $\mathscr{A}_0$ plays the role of "set of objects". It is important to notice that the Segal maps above is defined using fiber products over discrete objects:

$$\mathscr{A}_m \longrightarrow \mathscr{A}_1 \times_{\mathscr{A}_0} \cdots \times_{\mathscr{A}_0} \mathscr{A}_1.$$

If we do not have a notion of discrete object and we want to use a general tensor product, we need to change the construction a little bit to define generalized Segal categories. This is the motivation of this paper. The following proposition defines Segal n-categories as Segal path-objects of $(n-1)\,\mathrm{SePC}$.

4.20. Proposition. *Let $\mathscr{M} = ((n-1)\,\mathrm{SePC}, \times, 1)$ be the model category of Hirschowitz–Simpson as in [13], and denote by $\mathscr{W}$ its class of weak equivalences. For a set X we have an equivalence between the following data.*

1. *A Segal n-category $\mathscr{A}$ in the sense of Hirschowitz–Simpson such that $\mathscr{A}_0 \cong \mathrm{disc}(X)$,*
2. *A* normal *$\overline{X}$-point F of $(\mathscr{M}, \mathscr{W})$, satisfying the induction hypothesis:*
$$F(x_0, \cdots, x_m) := F[m, (x_0, \cdots, x_m)]$$
is a Segal $(n-1)$-category.

We remind the reader that being *normal* means that $F[0,(x)] = 1$, and that the colaxity maps involving the identities are natural isomorphisms.

Proof. This is not hard to prove, and we refer the reader to Simpson's book [30] for a detailed proof. A short description goes as follows. Let $\mathscr{A} : \Delta^{\mathrm{op}} \longrightarrow (n-1)\,\mathrm{SePC}$ be a Segal n-category. For every $m \geq 1$ we have a canonical map $p_m : \mathscr{A}_m \longrightarrow \mathscr{A}_0^{m+1}$ induced by the different maps $0 \longrightarrow m$ in Δ. Since $\mathscr{A}_0$ is discrete, then $\mathscr{A}_0^{m+1}$ is also discrete, and each $(x_0, ..., x_m) \in \mathscr{A}_0^{m+1}$ defines a map $\iota_{(x_0,...,x_m)} : 1 \longrightarrow \mathscr{A}_0^{m+1}$. Denote by $\mathscr{A}(x_0, ..., x_m)$ the object obtained from the strict pullback diagram:

$$\mathscr{A}_m \xrightarrow{p_m} \mathscr{A}_0^{m+1} \xleftarrow{\iota_{(x_0,...,x_m)}} 1.$$

Then $\mathscr{A}_m$ is decomposed as a coproduct $\mathscr{A}_m \cong \coprod_{(x_0,...,x_m)} \mathscr{A}(x_0, ..., x_m)$. The Segal map induces a family of Segal maps $\mathscr{A}(x_0, ..., x_m) \longrightarrow \mathscr{A}(x_0, x_1) \times \cdots \times \mathscr{A}(x_{m-1}, x_m)$, and the Segal conditions are equivalent to saying that the last map is a weak equivalence for each $(x_0, ..., x_m) \in \mathscr{A}_0^{m+1}$. It suffices to set $F[m, (x_0, \cdots, x_m)] = \mathscr{A}(x_0, ..., x_m)$ and $F[0,(x)] = 1$. For $1 \leq i \leq m-1$, we have two maps $\mathscr{A}(x_0, ..., x_m) \longrightarrow \mathscr{A}(x_0, ..., x_i)$ and $\mathscr{A}(x_0, ..., x_m) \longrightarrow \mathscr{A}(x_i, ..., x_m)$ induced by the appropriate "front face" and "back face" maps in Δ. The universal property of the product gives our colaxity map:

$$\varphi : \mathscr{A}(x_0, ..., x_m) \longrightarrow \mathscr{A}(x_0, ..., x_i) \times \mathscr{A}(x_i, ..., x_m).$$

The map φ is a weak equivalence by 3-for-2. Conversely if we are given a normal $\mathscr{W}$-colax functor $F : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$, we can build a simplicial object $\mathscr{A} : \Delta^{\mathrm{op}} \longrightarrow (n-1)\,\mathrm{SePC}$ following Leinster's recipe for the proof of Proposition 4.9. We define $\mathscr{A}_m := \mathscr{A}(m)$ by the formulae:

- $\mathscr{A}_0 = \coprod_{x \in X} 1 = \coprod_{x \in X} F[0,(x)]$,
- $\mathscr{A}_m = \coprod_{(x_0,\cdots,x_m)} F(x_0, \cdots, x_m)$, if $m \geq 1$.

For $0 < i < m$ the inner face map $d^i : \mathscr{A}_m \longrightarrow \mathscr{A}_{m-1}$ is that of F on each summand, and for $i = 0$ (resp. $i = m$) the face map is obtained from the colaxity map followed by the projection onto the first factor (resp. the second) on each summand:

$$F(x_0, \cdots, x_m) \xrightarrow{\varphi} F(x_1, \cdots, x_m) \times F(x_0, x_1) \xrightarrow{p_1} F(x_1, \cdots, x_m),$$

$$F(x_0, \cdots, x_m) \xrightarrow{\varphi} F(x_{m-1}, x_m) \times F(x_0, \cdots, x_{m-1}) \xrightarrow{p_2} F(x_0, \cdots, x_{m-1}).$$

The degeneracy maps $s^i : \mathscr{A}_m \longrightarrow \mathscr{A}_{m+1}$ are those of F on each summand if $m \geq 1$, and if $m = 0$ the map $\mathscr{A}_0 \longrightarrow \mathscr{A}_1$ is induced by the universal property of the coproduct as x runs through X:

$$1 = F[0,(x)] \longrightarrow \overbrace{F[1,(x,x)]}^{F(x,x)} \xhookrightarrow{(x,x)} \overbrace{\coprod_{(x_0,x_1)} F(x_0,x_1)}^{=\mathscr{A}_1}.$$

The map $F[0,(x)] \longrightarrow F[1,(x,x)]$ is the image under F of the 2-morphism $[0,(x)] \longrightarrow [1,(x,x)]$ in $\mathcal{P}_{\overline{X}}$, which is parametrized by the unique map $0 \longrightarrow 1$ in Δ^+. Finally, for each map $f_i : 0 \longrightarrow m$ in Δ, the corresponding map $\mathscr{A}_{f_i} : \mathscr{A}_m \longrightarrow \mathscr{A}_0$ is obtained on each summand as the composition of the (unique) map $F(x_0, \cdots, x_m) \longrightarrow 1$ followed by the inclusion $1 \overset{x_i}{\hookrightarrow} \coprod_{x \in X} 1$. It is easy to see that the simplicial identities hold and that $\mathscr{A}$ is indeed a functor. ■

5. Linear Segal categories

In this section $\mathscr{M} = (\mathrm{ChMod}_R, \otimes_R, R)$ is the monoidal category of chain complexes of R-modules, where R is a commutative ring. We have two choices for the class $\mathscr{W}$, depending on R.

1. If R is an arbitrary commutative ring, then $\mathscr{W}$ will be the class of *chain homotopy equivalences* (see [37]).

2. However, if R is a field, then $\mathscr{W}$ can be the class of *quasi-isomorphisms*, i.e., maps that induce isomorphism in homology.

5.1. Remark. As pointed out by Leinster [18], if R is an arbitrary commutative ring the class of quasi-isomorphisms may not be closed under tensor product because of the Künneth formula.

5.2. Definition. *Let X be a set and let $\mathscr{M} = (\mathrm{ChMod}_R, \otimes_R, R)$ together with $\mathscr{W}$ a suitable class of weak equivalences.*

1. *A Segal* dg*-category is an $\overline{X}$-point of $(\mathscr{M}, \mathscr{W})$, that is, a $\mathscr{W}$-colax morphism:*

$$F : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}.$$

2. *A* normal *Segal* dg*-category is a Segal* dg*-category that is in addition a normal colax morphism of bicategories.*

There is an important class of examples of Segal dg-categories obtained by base change, as we shall see in a moment. Let $\mathrm{sAb} = (\mathrm{sAb}, \otimes_{\mathbb{Z}}, \mathbb{Z}_{\bullet})$ be the monoidal category of simplicial abelian groups with level-wise tensor product over $\mathbb{Z}$, where the unit $\mathbb{Z}_{\bullet}$ is the constant simplicial abelian group of value $\mathbb{Z}$. Let $\mathrm{Ch}^{+} \hookrightarrow (\mathrm{ChMod}_{\mathbb{Z}}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ be the monoidal subcategory of nonnegatively graded chain complexes of abelian groups. Let $N : \mathrm{sAb} \longrightarrow \mathrm{Ch}^{+}$ be the normalization functor in the Dold–Kan correspondence (see [12, 21, 27, 37]).

5.3. PROPOSITION. *Let* $(\mathrm{Ch}^{+}, \mathscr{W})$ *be the base of enrichment, where* $\mathscr{W}$ *is the class of chain homotopy equivalences. Then the following hold.*

1. *The normalization functor* $(N, AW) : \mathrm{sAb} \longrightarrow \mathrm{Ch}^{+}$ *is* $\mathscr{W}$*-colax monoidal, where the colaxity map* $AW : N(A \otimes B) \longrightarrow N(A) \otimes N(B)$ *is the Alexander–Whitney map.*

2. *Any strict category* $\mathcal{C}$ *enriched over* sAb *gives a* normal *Segal* dg-*category* $N\mathcal{C}_{AW}$ *by base change along* (N, AW).

PROOF. Assertion (1) can be found in [12, 21, 27, 37]. Assertion (2) is a simple base change for colax functors. Indeed, thanks to Proposition 4.14, a category $\mathcal{C}$ enriched over sAb is equivalent to a (colax) homomorphism of bicategories denoted again $\mathcal{C} : \mathcal{P}_{\overline{X}} \longrightarrow \mathrm{sAb}$, where $X = \mathrm{Ob}(\mathcal{C})$. The Segal dg-category is the composite $(N, AW) \circ \mathcal{C} : \mathcal{P}_{\overline{X}} \longrightarrow \mathrm{Ch}^{+}$, a $\mathscr{W}$-colax morphism with colaxity maps given by: $N(\mathcal{C}(s * t)) \xrightarrow{\cong} N(\mathcal{C}(s) \otimes \mathcal{C}(t)) \xrightarrow{AW} N(\mathcal{C}(s)) \otimes N(\mathcal{C}(t))$ and the isomorphisms $N(\mathcal{C}(0, (x))) \cong N(\mathbb{Z}_{\bullet}) \cong \mathbb{Z}[0]$. We will write $N\mathcal{C}_{AW} = (N, AW) \circ \mathcal{C}$. ∎

5.4. PROPOSITION. *Let* $(\mathrm{Ch}^{+}, \mathscr{W})$ *be the base of enrichment, where* $\mathscr{W}$ *is the class of chain homotopy equivalences. Then the following hold.*

1. *The normalization functor* $(N, \nabla) : \mathrm{sAb} \longrightarrow \mathrm{Ch}^{+}$ *is* $\mathscr{W}$*-lax monoidal, where the laxity map* $\nabla : N(A) \otimes N(B) \longrightarrow N(A \otimes B)$ *is the shuffle map.*

2. *Any strict category* $\mathcal{C}$ *enriched over* sAb *gives a strict Segal* dg-*category* $N\mathcal{C}_{\nabla}$ *by base change along the normalization functor and the shuffle map.*

PROOF. Assertion (1) is classical and can be found in [23, 29.10]. To prove Assertion (2) we proceed as follows. If $\mathcal{C}$ is a category enriched over sAb with $\mathrm{Ob}(\mathcal{C}) = X$, then $\mathcal{C}$ is equivalent to a lax morphism $\mathcal{C} : \overline{X} \longrightarrow \mathrm{sAb}$. Lax morphisms of bicategories can be composed, and we find a lax functor $(N, \nabla) \circ \mathcal{C} : \overline{X} \longrightarrow \mathrm{Ch}^{+}$. This lax functor is a polyad in the sense of Bénabou, thus a strict Segal dg-category. Following Proposition 4.14 we can identify $(N, \nabla) \circ \mathcal{C}$ with a strict Segal path-object, which will be denoted by $N\mathcal{C}_{\nabla} : \mathcal{P}_{\overline{X}} \longrightarrow \mathrm{Ch}^{+}$. ∎

The following proposition will be used to establish that we have a weak equivalence between the two Segal dg-categories $N\mathcal{C}_{AW}$ and $N\mathcal{C}_{\nabla}$. A full statement will be given later (in Proposition 7.7), when we will have the definition of morphism between path-objects.

5.5. PROPOSITION. *Let $\mathcal{C}$ be a category enriched over* sAb*, and let $N\mathcal{C}_{AW}$ and $N\mathcal{C}_{\nabla}$ be the Segal* dg*-categories constructed previously. Then the following hold.*

1. *For any $(x,y) \in \mathrm{Ob}(\mathcal{C})^2$, the shuffle map induces a natural transformation $\sigma : N\mathcal{C}_{\nabla,xy} \longrightarrow N\mathcal{C}_{AW,xy}$ between elements of $\mathrm{Hom}(\mathcal{P}_{\overline{X}}(x,y), \mathrm{Ch}^+)$. Moreover, for any $s \in \mathcal{P}_{\overline{X}}(x,y)$, σ_s is a chain homotopy equivalence, that is: σ is a level-wise chain homotopy equivalence.*

2. *For every pair (s,t) of composable* 1*-morphisms, the diagram below commutes.*

$$\begin{array}{ccc} N\mathcal{C}_{\nabla}(s*t) & \xrightarrow{\sigma_{s*t}} & N\mathcal{C}_{AW}(s*t) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ N\mathcal{C}_{\nabla}(s) \otimes N\mathcal{C}_{\nabla}(t) & \xrightarrow{\sigma_s \otimes \sigma_t} & N\mathcal{C}_{AW}(s) \otimes N\mathcal{C}_{AW}(t) \end{array}$$

PROOF. We will first give the proof of Assertion (2) for which the key argument is based on the fact that $AW \circ \nabla = \mathrm{Id}$. Let us write $s = (x_0, \cdots, x_n)$ and $t = (x_n, \cdots, x_{n+m})$, so that $s * t = (x_0, \cdots, x_n, \cdots, x_{n+m})$. We will follow the same convention as in Notation 4.15. Recall that by construction we have:

- $\mathcal{C}(s) = \mathcal{C}(x_0, x_1) \otimes \cdots \otimes \mathcal{C}(x_{n-1}, x_n) = \otimes_{i=0}^{n-1} \mathcal{C}(x_i, x_{i+1})$,
- $\mathcal{C}(t) = \mathcal{C}(x_n, x_{n+1}) \otimes \cdots \otimes \mathcal{C}(x_{n+m-1}, x_{n+m}) = \otimes_{i=n}^{n+m-1} \mathcal{C}(x_i, x_{i+1})$,
- $N\mathcal{C}_{\nabla}(s) = N\mathcal{C}(x_0, x_1) \otimes \cdots \otimes N\mathcal{C}(x_{n-1}, x_n) = \otimes_{i=0}^{n-1} N\mathcal{C}(x_i, x_{i+1})$,
- $N\mathcal{C}_{\nabla}(t) = N\mathcal{C}(x_n, x_{n+1}) \otimes \cdots \otimes N\mathcal{C}(x_{n+m-1}, x_{n+m}) = \otimes_{i=n}^{n+m-1} N\mathcal{C}(x_i, x_{i+1})$,
- $N\mathcal{C}_{\nabla}(s*t) = N\mathcal{C}(x_0, x_1) \otimes \cdots \otimes N\mathcal{C}(x_{n+m-1}, x_{n+m}) = \otimes_{i=0}^{n+m-1} N\mathcal{C}(x_i, x_{i+1})$,
- $N\mathcal{C}_{AW}(s) = N[\mathcal{C}(x_0, x_1) \otimes \cdots \otimes \mathcal{C}(x_{n-1}, x_n)] = N[\otimes_{i=0}^{n-1} \mathcal{C}(x_i, x_{i+1})]$,
- $N\mathcal{C}_{AW}(t) = N[\mathcal{C}(x_n, x_{n+1}) \otimes \cdots \otimes \mathcal{C}(x_{n+m-1}, x_{n+m})] = N[\otimes_{i=n}^{n+m-1} \mathcal{C}(x_i, x_{i+1})]$,
- $N\mathcal{C}_{AW}(s*t) = N[\mathcal{C}(x_0, x_1) \otimes \cdots \otimes \mathcal{C}(x_{n+m-1}, x_{n+m})] = N[\otimes_{i=0}^{n+m-1} \mathcal{C}(x_i, x_{i+1})]$.

The colaxity map $\varphi : N\mathcal{C}_{\nabla}(s*t) \xrightarrow{\cong} N\mathcal{C}_{\nabla}(s) \otimes N\mathcal{C}_{\nabla}(t)$ is the isomorphism given by the associativity of $\otimes_{\mathbb{Z}}$ in Ch^+, so we shall write $\varphi = a_{\mathrm{Ch}^+}$. We have a similar map induced by the associativity in sAb: $\mathcal{C}(s*t) \xrightarrow[\cong]{a_{\mathrm{sAb}}} \mathcal{C}(s) \otimes \mathcal{C}(t)$. The colaxity map $\psi : N\mathcal{C}_{AW}(s*t) \longrightarrow N\mathcal{C}_{AW}(s) \otimes N\mathcal{C}_{AW}(t)$ is the composite:

$$N\mathcal{C}(s*t) \xrightarrow[\cong]{N(a_{\mathrm{sAb}})} N[\mathcal{C}(s) \otimes \mathcal{C}(t)] \xrightarrow{AW} N\mathcal{C}(s) \otimes N\mathcal{C}(t).$$

We remind the reader that the maps a_{sAb} and a_{Ch^+} have the "same shape", that is, they move the parentheses the same way: from all front to the desired places. We define the components $\sigma_s : \underbrace{N\mathcal{C}_\nabla(x_0, \cdots, x_n)}_{\otimes_{i=0}^{n-1} N\mathcal{C}(x_i, x_{i+1})} \longrightarrow \underbrace{N\mathcal{C}_{AW}(x_0, \cdots, x_n)}_{N[\otimes_{i=0}^{n-1}\mathcal{C}(x_i,x_{i+1})]}$ inductively as follows.

- if $n \leq 1$, $\sigma_s = \mathrm{Id}$,
- if $n = 2$, $\sigma_s = \nabla$ is the shuffle map,
- if $n \geq 3$, $\sigma_s = \nabla \circ (\nabla^{n-1} \otimes \mathrm{Id})$ is the "shuffle map with n variables" obtained by multiple applications of the shuffle map with 2 variables.

Now observe that since (N, ∇) is a lax monoidal functor, the shuffle map is compatible with the associativity of the tensor product in sAb and in Ch^+. Therefore by an argument similar to the one that appears in the proof of *Mac Lane's coherence Theorem* [22, Ch. VII, Sec. 2, Thm. 1], we have a commutative diagram for each (s, t):

$$
\begin{array}{ccc}
\overbrace{\otimes_{i=0}^{n+m-1} N\mathcal{C}(x_i, x_{i+1})}^{N\mathcal{C}_\nabla(s*t)} & \xrightarrow{\nabla\circ(\nabla^{n+m-1}\otimes \mathrm{Id})=\sigma_{s*t}} & \overbrace{N[\otimes_{i=0}^{n+m-1}\mathcal{C}(x_i, x_{i+1})]}^{N\mathcal{C}_{AW}(s*t)} \\
{\scriptstyle \varphi=a_{\mathrm{Ch}^+}}\downarrow\cong & & \cong\downarrow{\scriptstyle N(a_{\mathrm{sAb}})} \\
\underbrace{[\otimes_{i=0}^{n-1} N\mathcal{C}(x_i, x_{i+1})]}_{N\mathcal{C}_\nabla(s)} \otimes \underbrace{[\otimes_{i=n}^{n+m-1} N\mathcal{C}(x_i, x_{i+1})]}_{N\mathcal{C}_\nabla(t)} & \xrightarrow{\nabla\circ(\sigma_s\otimes\sigma_t)} & N[\mathcal{C}(s) \otimes \mathcal{C}(t)]
\end{array}
$$

Using the Alexander–Whitney map $N[\mathcal{C}(s)\otimes\mathcal{C}(t)] \xrightarrow{AW} N(\mathcal{C}(s))\otimes N(\mathcal{C}(t))$, we can extend this last diagram from the right lower corner to get a new commutative diagram that ends at $N(\mathcal{C}(s))\otimes N(\mathcal{C}(t))$. It is not difficult to check that this new diagram is the one appearing in Proposition 5.5 since we have:

$$AW \circ [\nabla \circ (\sigma_s \otimes \sigma_t)] = [AW \circ \nabla] \circ (\sigma_s \otimes \sigma_t) = \mathrm{Id} \circ (\sigma_s \otimes \sigma_t) = \sigma_s \otimes \sigma_t,$$

$$AW \circ N(a_{\mathrm{sAb}}) = \psi.$$

This completes the proof of Assertion (2).

To prove Assertion (1) we must show that for every morphism $u : t \longrightarrow t'$ in $\mathcal{P}_{\overline{X}}(x, y)$ we have an equality $N\mathcal{C}_{AW}(u) \circ \sigma_t = \sigma_{t'} \circ N\mathcal{C}_\nabla(u)$. As explained earlier, the morphisms in $\mathcal{P}_{\overline{X}}(x, y)$ are generated under composition by morphisms that are parametrized by the codegeneracies s^i and the cofaces d^i; therefore it suffices to prove this equality for $u = s^i$ and for $u = d^i$. Recall that if $u = s^i$ the construction of $\mathcal{C}(u), N\mathcal{C}_\nabla(u)$ and $N\mathcal{C}_{AW}(u)$ amounts to inserting the identity I_{x_i} of the ith vertex of t. If $u = d^i$ one applies instead the composition at the ith vertex (see Proposition 4.14). If we put this together with the coherence and the naturality of the shuffle map ∇^n in n variables, we have reduced to prove that the equality holds for:

- $u = s^0 : (x, y) \longrightarrow (x, x, y)$,
- $u = s^1 : (x, y) \longrightarrow (x, y, y)$,
- $u = d : (x, w, y) \longrightarrow (x, y)$.

The cases $u = s^0$ and $u = s^1$ are treated the same way, so it is enough to establish the equality for $u = s^0$ and $u = d$.

1. If $u = s^0$, the equality is given by the axiom of the unit in the definition of the lax functor (N, ∇). Indeed, we have $N\mathcal{C}_\nabla(x, y) = \mathcal{C}(x, y) = N\mathcal{C}_{AW}(x, y)$, and the map $\sigma : N\mathcal{C}_\nabla(x, y) \longrightarrow N\mathcal{C}_{AW}(x, y)$ is the identity. The axioms of the unit give the commutative diagram hereafter, from which we get the desired equality for s^0.

$$\begin{array}{ccc} \overbrace{N\mathcal{C}(x,y)}^{N\mathcal{C}_\nabla(t)} & \xrightarrow{\mathrm{Id}=\sigma_t} & \overbrace{N\mathcal{C}(x,y)}^{N\mathcal{C}_{AW}(t)} \\ {\scriptstyle N\mathcal{C}_\nabla(u)=(I_x\otimes \mathrm{Id})\circ l_{N\mathcal{C}(x,y)}}\downarrow & & \downarrow{\scriptstyle N[(I_x\otimes \mathrm{Id})\circ l_{\mathcal{C}(x,y)}]=N\mathcal{C}_{AW}(u)} \\ \underbrace{N\mathcal{C}(x,x)\otimes N\mathcal{C}(x,y)}_{N\mathcal{C}_\nabla(t')} & \xrightarrow{\nabla=\sigma_{t'}} & \underbrace{N[\mathcal{C}(x,x)\otimes \mathcal{C}(x,y)]}_{N\mathcal{C}_{AW}(t')} \end{array}$$

2. If $u = d : \underbrace{(x, w, y)}_{t} \longrightarrow \underbrace{(x, y)}_{t'}$, the map $N\mathcal{C}_\nabla(u)$ is by construction the composite:

$$\underbrace{N\mathcal{C}(x,w)\otimes N\mathcal{C}(w,y)}_{N\mathcal{C}_\nabla(t)} \xrightarrow{\nabla=\sigma_t} \underbrace{N[\mathcal{C}(x,w)\otimes \mathcal{C}(w,y)]}_{N\mathcal{C}_{AW}(t)} \xrightarrow{N[\mathcal{C}(u)]=N\mathcal{C}_{AW}(u)} \underbrace{N\mathcal{C}(x,y)}_{N\mathcal{C}_{AW}(t')}.$$

   Clearly we have our equality: $\sigma_{t'} \circ N\mathcal{C}_\nabla(u) = \mathrm{Id} \circ N\mathcal{C}_\nabla(u) = N\mathcal{C}_{AW}(u) \circ \sigma_t$.

Finally, by construction, σ_s is a chain homotopy equivalence for all s because $(\mathrm{Ch}^+, \mathscr{W})$ is a monoidal base of enrichment, that is, chain homotopy equivalences are closed under tensor product and composition. This completes the proof of Assertion (1). ∎

6. Nonunital Segal enriched categories and Applications

Definition 5.2 is a bit restrictive when we seek for examples. This is due to the axioms of the identities in a colax morphism together with the fact that a map $C \longrightarrow A \otimes B$ is not natural when $\otimes \neq \times$. What we will do is to define nonunital Segal dg-categories and add pseudo-identity morphisms. We shall produce some examples of such nonunital Segal enriched categories with pseudo-identity morphisms.

6.1. NOTATION.

1. $(\Delta^+_{\text{epi}}, +, 0) \hookrightarrow (\Delta^+, +, 0) =$ the subcategory of epimorphisms.
2. $\Delta^+_{\geq 1,\text{epi}} \hookrightarrow \Delta^+_{\text{epi}}$ the full subcategory with 0 removed.
3. $\mathcal{S}_{\mathbb{1}}$ the monoidal category $(\Delta^+_{\text{epi}}, +, 0)$ viewed as 2-category with one object "$\diamond$" and $\text{Hom}(\diamond, \diamond) = \Delta^+_{\text{epi}}$.
4. $\mathcal{S}_{\mathbb{1}} \hookrightarrow \mathcal{P}_{\mathbb{1}}$ corresponds to the previous embedding $(\Delta^+_{\text{epi}}, +, 0) \hookrightarrow (\Delta^+, +, 0)$.
5. $\mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{P}_{\mathcal{C}}$ is the 2-category obtained by taking the strict 2-pullback defined by the diagram $\mathcal{S}_{\mathbb{1}} \hookrightarrow \mathcal{P}_{\mathbb{1}} \leftarrow \mathcal{P}_{\mathcal{C}}$.
6. $\mathcal{L} : \mathcal{S}_{\mathcal{C}} \longrightarrow \mathcal{S}_{\mathbb{1}}$ is the induced functor.
7. $\mathcal{S}_{\overline{X}} =$ the 2-category $\mathcal{S}_{\mathcal{C}}$ for $\mathcal{C} = \overline{X}$.

6.2. DEFINITION. *Let $(\mathscr{M}, \mathscr{W})$ be a monoidal base of enrichment.*

1. *A nonunital* normal *Segal $\mathscr{M}$-precategory is a* normal colax *morphism $F : \mathcal{S}_{\overline{X}} \longrightarrow \mathscr{M}$*
2. *A nonunital* normal *Segal $\mathscr{M}$-category is a* normal colax *morphism $F : \mathcal{S}_{\overline{X}} \longrightarrow \mathscr{M}$ that satisfies the Segal conditions in the sense of Definition 3.4.*
3. *Say that a nonunital Segal $\mathscr{M}$-(pre)category F admits* pseudo-identity morphisms *if:*
   (a) *for every $x \in X$, there is a map $I_x : I \longrightarrow F(x, x)$,*
   (b) *for every $(x, y) \in X^2$, we have two* codegeneracies *$r_0 : F(x, y) \longrightarrow F(x, x, y)$ and $r_1 : F(x, y) \longrightarrow F(x, y, y)$ such that the composites hereafter are equal to $\text{Id}_{F(x,y)}$:*

$$F(x,y) \xrightarrow{r_0} F(x,x,y) \xrightarrow{F[(x,x,y)\longrightarrow(x,y)]} F(x,y)$$

$$F(x,y) \xrightarrow{r_1} F(x,y,y) \xrightarrow{F[(x,y,y)\longrightarrow(x,y)]} F(x,y)$$

   (c) *the two diagrams below commute for every $(x, y) \in X^2$.*

$$\begin{array}{ccc} F(x,y) & \xrightarrow[\cong]{l} & I \otimes F(x,y) \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle I_y \otimes \text{Id}} \\ F(x,y,y) & \xrightarrow[\varphi]{\sim} & F(y,y) \otimes F(x,y) \end{array} \qquad \begin{array}{ccc} F(x,y) & \xrightarrow[\cong]{r} & F(x,y) \otimes I \\ {\scriptstyle r_0}\downarrow & & \downarrow{\scriptstyle \text{Id} \otimes I_x} \\ F(x,x,y) & \xrightarrow[\varphi]{\sim} & F(x,y) \otimes F(x,x) \end{array}$$

4. *A nonunital up-to-homotopy monoid (resp. with a pseudo-unit) is a nonunital Segal $\mathscr{M}$-category with one object (resp. with a pseudo-identity morphism).*

6.3. Example. It can be immediately checked that any normal Segal $\mathscr{M}$-(pre)category gives a nonunital *normal* Segal $\mathscr{M}$-(pre)category with pseudo-identity morphisms. In particular any ordinary $\mathscr{M}$-category defines a nonunital Segal-$\mathscr{M}$-category with pseudo-identity morphisms.

6.4. Homotopy transfer.

6.5. Definition. *Let $(\mathscr{M}, \mathscr{W})$ be a monoidal base of enrichment.*

1. *Say that an object $A \in \mathscr{M}$ possesses a* Leinster–Segal algebra structure *if there is a colax monoidal functor $F : (\Delta^+, +, 0) \longrightarrow \mathscr{M}$ that satisfies the Segal conditions, and such that $F(1) = A$.*

2. *Similarly, say that $A \in \mathscr{M}$ possesses a* nonunital Leinster–Segal algebra structure *if there is a* normal colax *monoidal functor $F : (\Delta^+_{\text{epi}}, +, 0) \longrightarrow \mathscr{M}$ that satisfies the Segal conditions, and such that $F(1) = A$.*

Let (S, μ, e) be a monoid in a monoidal category $\mathscr{M} = (\underline{M}, \otimes, I)$, where $\mu : S \otimes S \longrightarrow S$ is the multiplication and $e : I \longrightarrow S$ is the unit. As mentioned before, this monoid is equivalent to a monoidal functor denoted by $F_S : (\Delta^+, +, 0) \longrightarrow \mathscr{M}$ that takes $n \mapsto S^{\otimes n}$. This functor is a strict Segal path-object. In this setting we are interested in the following problems.

1. We have map $f : S \longrightarrow A$ that is an element of $\mathscr{W}$ and we would like to transfer the monoid structure from S to A using f. This is obvious if f is an isomorphism.

2. We have a map going in the other direction $f : A \longrightarrow S$ and we are interested in the same problem.

If f is not an isomorphism but a weak equivalence, we can define a multiplication $A \otimes A \longrightarrow A$ using a weak inverse to f. In general, however, that multiplication is not strictly associative, but only associative up to homotopy. The resulting structure is a Segal algebra with a *weak unit*. The idea is simply to change one value in F_S by replacing $S = F_S(1)$ with A using the map f. This simple operation has a cost since we must remove the codegeneracies in Δ^+. We remind the reader that the codegeneracies control the unit e of S at every level. We have new colaxity maps and we shall see that they remain coherent.

6.6. Theorem. *Let $(\mathscr{M}, \mathscr{W})$ be a monoidal base of enrichment, and let (S, μ, e) be a monoid of $\mathscr{M}$. Let $f : S \longrightarrow A$ be a morphism in $\mathscr{M}$. Then the following hold.*

1. *If f is in $\mathscr{W}$, then there is an nonunital up-to-homotopy monoid $F_A : (\Delta^+_{\text{epi}}, +, 0) \longrightarrow \mathscr{M}$ such that $F_A(1) = A$.*

2. *If f is in $\mathscr{W}$, and if there exists a morphism $g : A \longrightarrow S$ such that $f \circ g = \mathrm{Id}_A$, then there is an nonunital up-to-homotopy monoid with a pseudo-unit $F_A : (\Delta^+_{\text{epi}}, +, 0) \longrightarrow \mathscr{M}$ such that $F_A(1) = A$.*

This theorem is the "one-object" version of a more general one for categories with many objects. We will discuss it in subsequent papers. For the sake of clarity we will use some lemmas as intermediate steps in the proof of the theorem. We also need some material to simplify our constructions.

6.7. Definition. *A coherent colax data in $\mathscr{M}$ consists of a set $\{m_1, m_2, m_3, m_{12}, m_{23}, m\}$ of objects of $\mathscr{M}$ together with colaxity maps:*

- $\varphi_{1,2} : m_{12} \longrightarrow m_1 \otimes m_2$,
- $\varphi_{2,3} : m_{23} \longrightarrow m_2 \otimes m_3$,
- $\varphi_{1,23} : m \longrightarrow m_1 \otimes m_{23}$,
- $\varphi_{12,3} : m \longrightarrow m_{12} \otimes m_3$,

such that $(\varphi_{1,2} \otimes \mathrm{Id}_{m_3}) \circ \varphi_{12,3} = a^{-1} \circ (\mathrm{Id}_{m_1} \otimes \varphi_{2,3}) \circ \varphi_{1,23}$ in $\mathrm{Hom}(m, (m_1 \otimes m_2) \otimes m_3)$, where $a : (m_1 \otimes m_2) \otimes m_3 \xrightarrow{\cong} m_1 \otimes (m_2 \otimes m_3)$ is the associativity.

6.8. Lemma. *Consider a coherent colax data as in Definition 6.7. Assume that we have a morphism $f_i : m_i \longrightarrow r_i$ for $i \in \{1; 2; 3\}$. Then the set $\{r_1, r_2, r_3, m_{12}, m_{23}, m\}$ with the maps hereafter define a colax coherent data:*

- $\varphi'_{1,2} = (f_1 \otimes f_2) \circ \varphi_{1,2} : m_{12} \longrightarrow r_1 \otimes r_2$,
- $\varphi'_{2,3} = (f_2 \otimes f_3) \circ \varphi_{2,3} : m_{23} \longrightarrow r_2 \otimes r_3$,
- $\varphi'_{1,23} = (f_1 \otimes \mathrm{Id}_{m_{23}}) \circ \varphi_{1,23} : m \longrightarrow r_1 \otimes m_{23}$,
- $\varphi'_{12,3} = (\mathrm{Id}_{m_{12}} \otimes f_3) \circ \varphi_{12,3} : m \longrightarrow m_{12} \otimes r_3$.

Proof. For simplicity we will adopt the following notation.

- $\alpha = (\varphi_{1,2} \otimes \mathrm{Id}_{m_3}) \circ \varphi_{12,3}$,
- $\beta = (\mathrm{Id}_{m_1} \otimes \varphi_{2,3}) \circ \varphi_{1,23}$,
- $\alpha' = (\varphi'_{1,2} \otimes \mathrm{Id}_{m_3}) \circ \varphi'_{12,3}$,
- $\beta' = (\mathrm{Id}_{m_1} \otimes \varphi'_{2,3}) \circ \varphi'_{1,23}$,
- $a_m : m_1 \otimes (m_2 \otimes m_3) \longrightarrow (m_1 \otimes m_2) \otimes m_3$ (associativity)
- $a_r : r_1 \otimes (r_2 \otimes r_3) \longrightarrow (r_1 \otimes r_2) \otimes r_3$ (associativity).

By hypothesis, we have $\alpha = a_m \circ \beta$ and the lemma will follow if we show that $\alpha' = a_r \circ \beta'$. Since $\otimes$ is functorial in each variable, we have an interchange law between $\otimes$ and $\circ$ that gives:

$$\alpha' = [(f_1 \otimes f_2) \otimes f_3] \circ \alpha,$$

$$\beta' = [f_1 \otimes (f_2 \otimes f_3)] \circ \beta.$$

On the other hand since the associativity is a natural isomorphism in three variables we have:

$$(f_1 \otimes f_2) \otimes f_3 = a_r \circ [f_1 \otimes (f_2 \otimes f_3)] \circ a_m^{-1}.$$

Then it suffices to write:

$$\alpha = a_m \circ \beta \Rightarrow [(f_1 \otimes f_2) \otimes f_3] \circ \alpha = [(f_1 \otimes f_2) \otimes f_3] \circ a_m \circ \beta = a_r \circ [f_1 \otimes (f_2 \otimes f_3)] \circ \beta = a_r \circ \beta'.$$

■

6.9. LEMMA. *Let $F : \mathcal{J} \longrightarrow \mathscr{M}$ be a functor where $\mathcal{J}$ is an* inverse Reedy 1-*category that possesses a terminal object j_0, and let $f : F(j_0) \longrightarrow A$ be a morphism in $\mathscr{M}$. Then there is a functor $G : \mathcal{J} \longrightarrow \mathscr{M}$ such that $G(j_0) = A$ with a natural transformation $\eta : F \longrightarrow G$ such that:*

1. *$\eta_j : F(j) \longrightarrow G(j)$ is the identity if $j \neq j_0$,*
2. *$\eta_{j_0} : F(j_0) \longrightarrow G(j_0)$ is f.*

PROOF. Define G by the formulas:

1. $G(j) = F(j)$ if $j \neq j_0$ and set $\eta_j = \mathrm{Id}_{F(j)}$,
2. $G(j_0) = A$ with $\eta_{j_0} = f$
3. If $\alpha : j \longrightarrow j'$ is an inverse map such that $j' \neq j_0$, define $G(\alpha) = F(\alpha)$.
4. If $\alpha : j \longrightarrow j_0$ is the unique map with $j \neq j_0$, define $G(\alpha) = f \circ F(\alpha)$
5. If $\alpha : j_0 \longrightarrow j_0$ is the identity, define $G(\alpha) = \mathrm{Id}_A$.

Recall that $\mathcal{J}$ being an inverse Reedy category means that every nonidentity map lowers the degree. Then j_0 is of smallest degree since we have a (unique) map $j \longrightarrow j_0$. It follows that there is no map $j_0 \longrightarrow j$ other than the identity Id_{j_0}, therefore the previous data define a functor with the desired properties. ■

The category $\Delta^+_{\geq 1,\text{epi}}$ is an inverse Reedy 1-category wherein 1 is the terminal object; so we can apply this lemma to functors defined on it.

Proof of Theorem 6.6. Let (S, μ, e) be a monoid and denote by $F_S : (\Delta^+, +, 0) \longrightarrow \mathscr{M}$ the corresponding (colax) monoidal functor with colaxity maps being isomorphisms. Let $F_A : (\Delta^+_{\text{epi}}, +, 0) \longrightarrow \mathscr{M}$ be the normal colax functor defined by the formulas:

1. $F_A(0) = I$ with $\varphi_0 = \text{Id}_I$
2. $F_A : \Delta^+_{\geq 1,\text{epi}} \longrightarrow \mathscr{M}$ is the functor obtained from Lemma 6.9 applied to the restriction of F_S to $\Delta^+_{\geq 1,\text{epi}}$ with respect to the morphism $f : S \longrightarrow A$. In particular $F_A(1) = A$ and we have a natural transformation between elements of $\text{Hom}(\Delta^+_{\geq 1,\text{epi}}, \mathscr{M})$:
$$\eta : F_S \longrightarrow F_A.$$
We have $\eta_1 = f$ and $\eta_n : F_S(n) \longrightarrow F_A(n)$ is the identity for $n \geq 2$. Finally we define $\eta_0 : F_S(0) \longrightarrow I$ as the isomorphism that comes in the definition of the strict monoid S. The image by F_A of the (unique) coface map $d^0 : 2 \longrightarrow 1$ is the composite $S \otimes S \xrightarrow{\mu} S \xrightarrow{f} A$, i.e., $f \circ \mu$.
3. The colaxity maps $\varphi : F_A(n+m) \longrightarrow F_A(n) \otimes F_A(m)$ are defined as follows.
   (a) If $n \geq 2$ and $m \geq 2$, then the colaxity map is that of F_S and corresponds to the isomorphism $S^{\otimes(n+m)} \xrightarrow{\cong} S^{\otimes n} \otimes S^{\otimes m}$.
   (b) If $n = m = 1$, the map $\varphi : F_A(2) \longrightarrow F_A(1) \otimes F_A(1)$ is $f \otimes f : S \otimes S \longrightarrow A \otimes A$, which is in $\mathscr{W}$ by hypothesis.
   (c) If $n = 1$ and $m \geq 2$, the colaxity map $\varphi : F_A(1+m) \longrightarrow F_A(1) \otimes F_A(m)$ is the composite: $S^{\otimes(1+m)} \xrightarrow{\cong} S \otimes (S^{\otimes m}) \xrightarrow{f \otimes \text{Id}} A \otimes (S^{\otimes m})$. By hypothesis, this map is also in $\mathscr{W}$.
   (d) Similarly, if $n \geq 2$ the map $\varphi : F_A(n+1) \longrightarrow F_A(n) \otimes F_A(1)$ is the composite:
$$S^{\otimes(n+1)} \xrightarrow{\cong} (S^{\otimes n}) \otimes S \xrightarrow{\text{Id} \otimes f} (S^{\otimes n}) \otimes A.$$
   (e) The maps $F_A(0+m) \longrightarrow F_A(0) \otimes F_A(m)$ and $F_A(n+0) \longrightarrow F_A(n) \otimes F_A(0)$ are the left and right isomorphisms obtained with $I \otimes -$ and $- \otimes I$.

In the end, we have only changed the value of F_S at the terminal object 1. As mentioned earlier, there is no map in Δ^+_{epi} whose source is 1 other than the identity. Then thanks to Lemma 6.7, we know that the new colaxity maps are coherent. It remains to show that if $u : n \longrightarrow p$ and $v : m \longrightarrow q$ are maps in Δ^+_{epi}, then the diagram $D(u, v)$ below commutes.

$$D(u,v): \quad \begin{array}{ccc} F_A(n+m) & \xrightarrow{\varphi} & F_A(n) \otimes F_A(m) \\ \downarrow{\scriptstyle F_A(u+v)} & & \downarrow{\scriptstyle F_A(u)\otimes F_A(v)} \\ F_A(p+q) & \xrightarrow[\varphi]{\sim} & F_A(p) \otimes F_A(q) \end{array}$$

If $p \neq 1$ and $q \neq 1$, the data for F_A are those of F_S, therefore $D(u,v)$ commutes. We are therefore left to prove that $D(u,v)$ commutes when either $p = 1$, or $q = 1$, or $p = q = 1$. For each case we have a commutative diagram "$D(u,v)$" for F_S that ends, respectively, at $S \otimes S^{\otimes q}$, $S^{\otimes p} \otimes S$, $S \otimes S$. If we post-compose each of the diagrams for F_S with, respectively, $f \otimes \mathrm{Id}_{S^{\otimes q}}$, $\mathrm{Id}_{S^{\otimes p}} \otimes f$, $f \otimes f$, we get a new commutative diagram that ends, respectively, at $A \otimes S^{\otimes q}$, $S^{\otimes p} \otimes A$, $A \otimes A$. This new commutative diagram is precisely the diagram $D(u,v)$. It follows that $D(u,v)$ commutes in each case.

The whole process gives a normal colax functor $F_A : (\Delta^+_{\text{epi}}, +, 0) \longrightarrow \mathscr{M}$ such that $F_A(1) = A$, with every colaxity map in $\mathscr{W}$. This means that F_A is a nonunital up-to-homotopy monoid as claimed, and Assertion (1) follows.

To prove Assertion (2) we proceed in the following manner. Consider the nonunital up-to-homotopy monoid F_A obtained from Assertion (1). The unit $e : I \longrightarrow S$ gives two maps $l_0 : S \longrightarrow S \otimes S$ and $l_1 : S \longrightarrow S \otimes S$. These maps are by construction the images under F_S of the two codegeneracies $s^i : 1 \longrightarrow 2$, for $i \in \{0,1\}$. The axiom for the unit e implies that $\mu \circ F_S(s^i) = \mathrm{Id}_S$, for $i \in \{0,1\}$. Now assume that f admits a section $g : A \longrightarrow S$, in that $f \circ g = \mathrm{Id}_A$. Let us define $e_A = f \circ e \in \mathrm{Hom}(I, A)$, $r_i = F_S(s^i) \circ g \in \mathrm{Hom}(A, S \otimes S)$, for $i \in \{0,1\}$. Then the following hold.

- $F_A(d^0) \circ r_i = f \circ \underbrace{F_S(d^0)}_{=\mu} \circ F_S(s^i) \circ g = f \circ \underbrace{\mu \circ F_S(s^i)}_{\mathrm{Id}_S} \circ g = f \circ g = \mathrm{Id}_A.$

- The diagram hereafter commutes.

$$
\begin{array}{ccc}
F_A(1) & \xrightarrow[\cong]{l} & I \otimes F_A(1) \\
{\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle e_A \otimes \mathrm{Id}} \\
F_A(2) & \xrightarrow[\varphi]{\sim} & F_A(1) \otimes F_A(1)
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \xrightarrow[\cong]{l} & I \otimes A & & \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes g} & & \\
S & \xrightarrow[\cong]{l} & I \otimes S & & \\
{\scriptstyle F(s^1)}\downarrow & & \downarrow{\scriptstyle e \otimes \mathrm{Id}} & & \\
S \otimes S & \xrightarrow{\mathrm{Id} \otimes \mathrm{Id}} & S \otimes S & \xrightarrow{f \otimes f} & A \otimes A
\end{array}
$$

  To see this, first observe that the diagram on the left is the "vertical composite" of the one on the right. If we look at the diagram on the right, the upper square is commutative because l is a natural transformation. The lower square is commutative because it is the diagram $D(u,v)$ above for F_S with respect to $u = (0 \xrightarrow{!} 1)$ and $v = (1 \xrightarrow{\mathrm{Id}} 1)$. It follows that the vertical composite of these two squares is also commutative, and if we extend it with $f \otimes f$ we find that our diagram on the left commutes, which justifies our claim.

- With a similar argument we see that the diagram below commutes.

$$\begin{array}{ccc} F_A(1) & \xrightarrow[\cong]{r} & F_A(1) \otimes I \\ {\scriptstyle r_0}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes e_A} \\ F_A(2) & \xrightarrow[\varphi]{\sim} & F_A(1) \otimes F_A(1) \end{array}$$

In the end, the normal colax morphism $F_A : (\Delta^+_{\text{epi}}, +, 0) \longrightarrow \mathscr{M}$ equipped with e_A, r_0 and r_1 is a nonunital up-to-homotopy monoid with a pseudo-unit in the sense of Definition 6.2. This completes the proof of the theorem. ■

6.10. Loop spaces. Let $(X, *)$ be a pointed compactly generated Hausdorff space and denote by $\Omega(X)$ its loop space. Let $\Omega^M(X) \subseteq \mathbb{R}^+ \times \mathrm{Top}(\mathbb{R}^+, X)$ be the space of *Moore loops* where $\mathrm{Top}(\mathbb{R}^+, X)$ carries the compact-open topology (see [5, 38]). An element of $\Omega^M(X)$ is a pair (r, γ) where $\gamma : \mathbb{R}^+ \longrightarrow X$ is such that $\gamma(0) = *$ and $\gamma(t) = *$ for $t \geq r$. We have a map $\iota : \Omega(X) \hookrightarrow \Omega^M(X)$ that identifies $\Omega(X)$ with the subspace of elements $(1, \gamma) \in \Omega^M(X)$.

6.11. Proposition. *With the above notation the following hold.*

1. *There is a map* $f : \Omega^M(X) \longrightarrow \Omega(X)$ *such that* $f \circ \iota = \mathrm{Id}_{\Omega(X)}$ *and and* $\iota \circ f$ *is homotopic to* $\mathrm{Id}_{\Omega^M(X)}$.
2. $\Omega^M(X)$ *is a strict topological monoid and* $\Omega(X)$ *is a Leinster–Segal topological monoid with a pseudo-unit.*

Proof. Assertion (1) is a well-known fact that there is a deformation retraction of $\Omega^M(X)$ to $\Omega(X)$ (see for example [5, Proposition 5.1.1]). If we unwrap the definition of a deformation retraction, we find a homotopy $H : \Omega^M(X) \times [0;1] \longrightarrow \Omega^M(X)$ such that $H(-,0) = \mathrm{Id}_{\Omega^M(X)}$, $H(u,1) \in \Omega(X)$ for all $u \in \Omega^M(X)$ and $H(-,1)_{|\Omega(X)} = \mathrm{Id}_{\Omega(X)}$. It suffices to set $f = H(-,1)$ to get Assertion (1). That $\Omega^M(X)$ is a topological monoid is also classical and we refer the reader to *loc. cit* and Whitehead [38, Chap. III]. The homotopy equivalence $f : \Omega^M(X) \longrightarrow \Omega(X)$ obtained from Assertion (1) is a retraction, and if we take $g = \iota$ we are in the hypotheses of Theorem 6.6 and we get Assertion (2). In this case, the Segal maps are $\Omega^M(X)^{\times n} \xrightarrow{f^{\times n}} \Omega(X) \times \cdots \times \Omega(X)$. ■

6.12. Homotopy transfer for DG-algebras. The results below hold in $\mathscr{M} = (\mathrm{ChMod}_R, \otimes_R, R)$, for an arbitrary commutative ring R. We will say that a map of chain complexes $f : (S, d_S) \longrightarrow (A, d_A)$ is a *chain retraction* if there is a chain map $g : (A, d_A) \longrightarrow (S, d_S)$ such that $f \circ g = \mathrm{Id}_A$ and if there is a chain homotopy $h : g \circ f \xrightarrow{\sim} \mathrm{Id}_S$. In particular f and g are both chain homotopy equivalences.

- An up-to-homotopy monoid in the base $(\mathrm{ChMod}_R, \mathscr{W})$ will be called a Leinster–Segal dg-algebra.

- Similarly an up-to-homotopy monoid with a pseudo-unit in the base $(\mathrm{ChMod}_R, \mathscr{W})$ will be called a Leinster–Segal dg-algebra with a pseudo-unit.

- A Leinster–Segal dg-algebra with a homotopy unit is a nonunital Leinster–Segal dg-algebra F : such that there is a map $I \longrightarrow F(1)$ and two codegeneracies r_i : $F(1) \longrightarrow F(2)$ $(i \in \{0; 1\})$ such that the diagrams in Definition 6.2 commute up-to-homotopy and if in addition $F(2 \longrightarrow 1) \circ r_i$ is homotopic to $\mathrm{Id}_{F(1)}$.

6.13. Proposition. *Let* (S, μ, e) *be a* dg-*algebra over* R *and let* $f : (S, d_S) \longrightarrow (A, d_A)$ *be a chain map. Then the following hold.*

1. *If* f *is an arbitrary chain homotopy equivalence, then there is a Leinster–Segal* dg-*algebra* F_A *with a homotopy unit such that* $F_A(1) = A$.

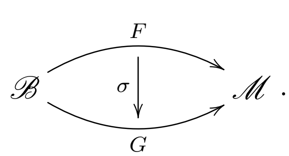
2. *If* f *is a* chain retraction *then there is a Leinster–Segal* dg-*algebra* F_A *with a pseudo-unit such that* $F_A(1) = A$.

Proof. This is simply an application of Theorem 6.6. The only thing left to check is that if f is an arbitrary chain homotopy equivalence then the nonunital Segal dg-algebra F_A admits a homotopy unit. We must then prove that the diagrams in Definition 6.2 commute up-to-homotopy. To see this, it suffices to rewrite the proof of the second assertion of Theorem 6.6. The chain homotopy $h : f \circ g \xrightarrow{\sim} \mathrm{Id}_A$ gives a chain homotopy $(e_A \otimes h) \circ l : (e_A \otimes fg) \circ l \xrightarrow{\sim} (e_A \otimes \mathrm{Id}_A) \circ l$, where l is the natural isomorphism $A \underset{l}{\xrightarrow{\cong}} I \otimes A$ and $e_A = f \circ e$. It can be easily checked that the diagram commutes up-to the chain homotopy $(e_A \otimes h) \circ l$. Finally, the composites $F_A(2 \longrightarrow 1) \circ r_i$ are both equal to $f \circ g$, and the latter map is by assumption homotopic to $\mathrm{Id}_A = \mathrm{Id}_{F_A(1)}$ via h. ■

7. Morphisms of path-objects and homotopy invariance

7.1. Morphism of path-objects. Bénabou [3] defined the notion of transformation between (co)lax morphisms of bicategories. Segal enriched categories are defined as colax morphisms between bicategories but a general transformation might not give an $\mathscr{M}$-functor. The relevant morphisms that we shall consider are called *icons* by Lack [17]. We restrict ourselves to the corresponding notion of icons for colax morphisms, which we call here *simple transformations*. We include the definition for the reader's convenience, and refer to Bénabou [3] and Lack [17] for the general definitions.

7.2. Definition. [Simple transformation] *Let* $\mathscr{B}$ *and* $\mathscr{M}$ *be two bicategories and* $F = (F, \varphi)$, $G = (G, \psi)$ *be two colax morphisms from* $\mathscr{B}$ *to* $\mathscr{M}$ *such that* $Fx = Gx$ *for every* $x \in \mathscr{B}$. *A* simple transformation $\sigma : F \longrightarrow G$



is given by the following data and axioms.
Data: *A natural transformation for each pair (x, y) of objects of $\mathscr{B}$:*

$$\mathscr{B}(x,y) \underset{G_{xy}}{\overset{F_{xy}}{\rightrightarrows}} \mathscr{M}(Fx,Fy)\ , \qquad \sigma : F_{xy} \Downarrow G_{xy}$$

thus a 2-morphism in $\mathscr{M}$, $\sigma_t : Ft \longrightarrow Gt$, for each t in $\mathscr{B}(x, y)$, natural in t.
Axioms: *The following commutes:*

$$\begin{array}{ccc} F(s*t) & \xrightarrow{\sigma_{s*t}} & G(s*t) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ F(s)\otimes F(t) & \xrightarrow{\sigma_s \otimes \sigma_t} & G(s)\otimes G(t) \end{array} \qquad \begin{array}{ccc} FI_x & \xrightarrow{\sigma_{I_x}} & GI_x \\ \downarrow{\scriptstyle \varphi_x} & & \downarrow{\scriptstyle \psi_x} \\ I_{Fx} & \overset{\mathrm{Id}}{=\!=\!=} & I_{Gx} \end{array}$$

Recall that for any category $\mathcal{C}$, by construction of $\mathcal{P}_{\mathcal{C}}$ we have $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{P}_{\mathcal{C}})$. Moreover any functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ extends to a strict homomorphism $\mathcal{P}_{\Sigma} : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathcal{P}_{\mathcal{D}}$.

7.3. Definition. *Let $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ and $G : \mathcal{P}_{\mathcal{D}} \longrightarrow \mathscr{M}$ be two path-objects of $(\mathscr{M}, \mathscr{W})$.*

1. *An $\mathscr{M}$-*premorphism *$\sigma = (\Sigma, \sigma) : F \longrightarrow G$ consists of a functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ together with a general transformation $\sigma : F \longrightarrow G \circ \mathcal{P}_{\Sigma}$ of colax morphisms of bicategories:*

$$\begin{array}{ccc} \mathcal{P}_{\mathcal{C}} & \xrightarrow{\mathcal{P}_{\Sigma}} & \mathcal{P}_{\mathcal{D}} \\ & {\scriptstyle F}\searrow \quad \overset{\sigma}{\Longrightarrow} \quad \swarrow{\scriptstyle G} & \\ & \mathscr{M} & \end{array}$$

2. *An $\mathscr{M}$-*morphism *$\sigma = (\Sigma, \sigma) : F \longrightarrow G$ consists of a functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{D}$ such that $Fx = (G \circ \mathcal{P}_{\Sigma})x$ for every $x \in \mathcal{C}$, together with a* simple transformation *of colax morphisms $\sigma : F \longrightarrow G \circ \mathcal{P}_{\Sigma}$.*

Morphisms between (nonunital) Segal $\mathscr{M}$-precategories and Segal $\mathscr{M}$-precategories with pseudo-identity morphisms are naturally defined following the above definition. We can also define a notion of *weakly unital* morphism between precategories with pseudo-identity morphisms. For the latter type of morphism, we need to say when a diagram in $\mathscr{M}$ *commutes up-to-homotopy*. Even though we have not defined any notion of such diagram, we will assume that there is a notion of homotopy between maps in $\mathscr{M}$ that allows us to say when a square commutes up-to-homotopy. If $\mathscr{M}$ is either the category of chain complexes, simplicial sets, Segal n-categories, or is a model category, we can easily define a notion of homotopy between parallel maps.

7.4. Definition. *Let* $(\mathscr{M}, \mathscr{W})$ *be a base of enrichment.*

1. *If* $F : \mathcal{P}_{\overline{X}} \longrightarrow \mathscr{M}$ *and* $G : \mathcal{P}_{\overline{Y}} \longrightarrow \mathscr{M}$ *are Segal* $\mathscr{M}$*-precategories, define an* $\mathscr{M}$*-morphism* $\sigma = (\Sigma, \sigma) : F \longrightarrow G$ *as an* $\mathscr{M}$*-morphism of path-objects in the sense of Definition 7.3.*

2. *If* $F : \mathcal{S}_{\overline{X}} \longrightarrow \mathscr{M}$ *and* $G : \mathcal{S}_{\overline{Y}} \longrightarrow \mathscr{M}$ *are nonunital Segal* $\mathscr{M}$*-precategories, an* $\mathscr{M}$*-morphism* $\sigma = (\Sigma, \sigma) : F \longrightarrow G$ *consists of:*

   (a) *a functor* $\Sigma : \overline{X} \longrightarrow \overline{Y}$*, thus a function* $\Sigma : X \longrightarrow Y$*, such that* $Fx = (G \circ \mathcal{S}_{\Sigma})x$ *for every* $x \in X$*,*

   (b) *a* simple transformation *of normal colax morphisms* $\sigma : F \longrightarrow G \circ \mathcal{S}_{\Sigma}$*.*

3. *If* $F : \mathcal{S}_{\overline{X}} \longrightarrow \mathscr{M}$ *and* $G : \mathcal{S}_{\overline{Y}} \longrightarrow \mathscr{M}$ *are nonunital Segal* $\mathscr{M}$*-precategories with pseudo-identity morphisms, an* $\mathscr{M}$*-morphism* $\sigma = (\Sigma, \sigma) : F \longrightarrow G$ *is called* strongly unital *(resp.* weakly unital*) if:*

   (a) *for every* $x \in X$ *the following commutes,*

$$\begin{array}{ccc} F(x,x) & \xrightarrow{\sigma} & G(\Sigma x, \Sigma x) \\ \uparrow{\scriptstyle I_x} & & \uparrow{\scriptstyle I_{\Sigma x}} \\ I_{Fx} & \xrightarrow{\mathrm{Id}} & I_{G\Sigma x} \end{array}$$

   (b) *and if the two diagrams hereafter are commutative (resp. commute up-to homotopy).*

$$\begin{array}{ccc} F(x,y) & \xrightarrow{\sigma} & G(\Sigma x, \Sigma y) \\ \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_1} \\ F(x,y,y) & \xrightarrow{\sigma} & G(\Sigma x, \Sigma y, \Sigma y) \end{array} \qquad \begin{array}{ccc} F(x,y) & \xrightarrow{\sigma} & G(\Sigma x, \Sigma y) \\ \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_0} \\ F(x,x,y) & \xrightarrow{\sigma} & G(\Sigma x, \Sigma x, \Sigma y) \end{array}$$

7.5. Proposition. *Let* F *and* G *be, respectively, a strict* $\overline{X}$*-point and a strict* $\overline{Y}$*-point of* $\mathscr{M}$*, and let* $\mathcal{C}_F$ *and* $\mathcal{C}_G$ *be the respective* $\mathscr{M}$*-categories they define. Then there is an equivalence between an* $\mathscr{M}$*-morphism* $\sigma : F \longrightarrow G$ *and an* $\mathscr{M}$*-functor* $\mathcal{C}_\sigma : \mathcal{C}_F \longrightarrow \mathcal{C}_G$*.*

Proof. This is an easy exercise. ∎

7.6. Definition. *Let* $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ *and* $G : \mathcal{P}_{\mathcal{D}} \longrightarrow \mathscr{M}$ *be two path-objects of* $(\mathscr{M}, \mathscr{W})$*. Say that an* $\mathscr{M}$*-morphism* $(\Sigma, \sigma) : F \longrightarrow G$ *is a* level-wise weak equivalence *if for every* 1*-morphism* s *of* $\mathcal{P}_{\mathcal{C}}$ *the component* $\sigma_s : F(s) \longrightarrow G(\Sigma s)$ *is in* $\mathscr{W}$*.*

We have a similar definition if we put $\mathcal{S}_{\overline{X}}$ and $\mathcal{S}_{\overline{Y}}$ instead of $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{D}}$, respectively. In fact, we have a notion of level-wise weak equivalence between colax morphisms indexed by arbitrary bicategories.

7.7. Proposition. *Let $\mathcal{C}$ be a category enriched over* sAb *with* $\mathrm{Ob}(\mathcal{C}) = X$*, and let* $N\mathcal{C}_{AW}$ *and* $N\mathcal{C}_{\nabla}$ *be the two Segal* dg*-categories constructed previously. Let* $\sigma_{xy} : N\mathcal{C}_{\nabla,xy} \longrightarrow N\mathcal{C}_{AW,xy}$ *be the natural transformation of Proposition 5.5. Then the family* $\{\sigma_{xy}\}_{(x,y)\in X^2}$ *determines an* $\mathscr{M}$*-morphism* $\sigma : N\mathcal{C}_{\nabla} \longrightarrow N\mathcal{C}_{AW}$ *that is a level-wise weak equivalence.*

Proof. This is clear from Proposition 5.5. ■

7.8. Remark. With the notation of Theorem 6.6, the following hold.

1. If (S, μ, e) is a monoid in $(\mathscr{M}, \mathscr{W})$ and $(f : S \longrightarrow A) \in \mathscr{W}$, then there is a canonical map $\sigma = F_S \longrightarrow F_A$ of nonunital up-to-homotopy monoids given by $\sigma = \eta$.

2. If in addition f admits a section g, then σ is a weakly unital map of up-to-homotopy monoids with a pseudo-unit. However, σ is *not* always a strongly unital map.

7.9. Homotopy invariance.

7.10. Proposition. *Let* $(\mathscr{M}, \mathscr{W})$ *be a base of enrichment and let* $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$, $G : \mathcal{P}_{\mathcal{D}} \longrightarrow \mathscr{M}$ *be two path-objects of* $\mathscr{M}$*. Let* $(\Sigma, \sigma) : F \longrightarrow G$ *be a level-wise weak equivalence. Then the following hold.*

1. *If G satisfies the Segal conditions, then so does F.*

2. *Assume that each function* $\Sigma_{xy} : \mathcal{C}(x, y) \longrightarrow \mathcal{D}(\Sigma x, \Sigma y)$ *is surjective, and that* Σ *is surjective on objects. If* F *satisfies the Segal conditions, then* G *also satisfies the Segal conditions.*

Proof. Assertion (1) will follow if we show that the colaxity maps $\varphi : F(s * t) \longrightarrow F(s) \otimes F(t)$ and $\varphi_x : F(0, (x)) \longrightarrow I$ are in $\mathscr{W}$. We give the proof for φ; the argument is the same for φ_x. The proof is based on the 3-for-2 property of maps in $\mathscr{W}$. Each colaxity map $F(s * t) \xrightarrow{\varphi} F(s) \otimes F(t)$ fits in the following diagram.

$$\begin{array}{ccc} F(s*t) & \xrightarrow[\sim]{\sigma_{s*t}} & G(\Sigma(s)\otimes\Sigma(t)) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\sim\ \psi} \\ F(s)\otimes F(t) & \xrightarrow[\sim]{\sigma_s\otimes\sigma_t} & G(\Sigma(s))\otimes G(\Sigma(t)) \end{array}$$

By hypothesis, σ_{s*t}, ψ and $\sigma_s \otimes \sigma_t$ are in $\mathscr{W}$. Since the last diagram commutes we have $(\sigma_s \otimes \sigma_t) \circ \varphi = \psi \circ (\sigma_{s*t})$, therefore $(\sigma_s \otimes \sigma_t) \circ \varphi$ is in $\mathscr{W}$, and by 3-for-2 we see that φ is in $\mathscr{W}$. It follows that F satisfies the Segal conditions, which proves Assertion (1).

Assertion (2) is proved the same way. The assumptions on Σ imply that any 1-morphism s' of $\mathcal{P}_{\mathcal{D}}$ is of the form $\Sigma(s) = s'$, where s is a 1-morphism of $\mathcal{P}_{\mathcal{C}}$ having the same length. Therefore, we can build the same type of diagram as above, where the maps σ_{s*t}, φ and $\sigma_s \otimes \sigma_t$ are in $\mathscr{W}$. By diagram chase, we also get that the map $\psi \circ (\sigma_{s*t})$ is in $\mathscr{W}$. Just like before, we deduce by 3-for-2 that ψ is in $\mathscr{W}$, which means that G satisfies the Segal conditions. This proves Assertion (2). ■

8. Cocompleteness

In the following we assume for simplicity that $\mathscr{M} = (\underline{M}, \otimes, I)$ is a symmetric monoidal closed category. Being closed implies that the tensor product $\otimes$ distributes over colimits (in each variable). For a 1-category $\mathcal{C}$, we will denote by $\mathscr{M}_{\mathcal{P}}(\mathcal{C})$ the category of path-objects $F : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$ with morphisms of the form $(\mathrm{Id}_{\mathcal{C}}, \sigma)$.

8.1. Proposition. *Let $\mathscr{M} = (\underline{M}, \otimes, I)$ be a symmetric monoidal closed category that is also cocomplete. Then for any 1-category $\mathcal{C}$, the category $\mathscr{M}_{\mathcal{P}}(\mathcal{C})$ is also cocomplete and colimits are computed level-wise.*

Proof. Let $\mathcal{J}$ be a small 1-category and consider a diagram $D : \mathcal{J} \longrightarrow \mathscr{M}_{\mathcal{P}}(\mathcal{C})$ that maps $j \mapsto F_j$, and that takes a structure map $i \xrightarrow{\alpha} j$ to an $\mathscr{M}$-morphism $\sigma_\alpha : F_i \longrightarrow F_j$. For every $(x, y) \in \mathrm{Ob}(\mathcal{C})^2$, we have a diagram $D_{xy} : \mathcal{J} \longrightarrow \mathrm{Hom}(\mathcal{P}_{\mathcal{C}}(x, y), \underline{M})$ that sends a structure map $i \xrightarrow{\alpha} j$ to the component of σ_α, which is a natural transformation: $F_{i,xy} \xrightarrow{F_{\alpha,xy}} F_{j,xy}$.

Set $F_{\infty,xy} = \mathrm{colim}\, D_{xy} \in \mathrm{Hom}(\mathcal{P}_{\mathcal{C}}(x, y), \underline{M})$, and denote by $\tau_i : F_{i,xy} \longrightarrow F_{\infty,xy}$ the canonical natural transformation for each $i \in \mathcal{J}$. For every $(s, t) \in \mathcal{P}_{\mathcal{C}}(x, y) \times \mathcal{P}_{\mathcal{C}}(y, z)$, we have a canonical map $F_{i,xy}(s) \otimes F_{i,yz}(t) \xrightarrow{\tau_i \otimes \tau_i} F_{\infty,xy}(s) \otimes F_{\infty,yz}(t)$. Given any structure map $(i \xrightarrow{\alpha} j) \in \mathrm{Mor}(\mathcal{J})$, we have a commutative diagram displayed hereafter.

$$\begin{array}{ccccc}
F_{i,xz}(s*t) & \xrightarrow{\sigma_\alpha} & F_{j,xz}(s*t) & & \\
\downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_j} & & \\
F_{i,xy}(s) \otimes F_{i,yz}(t) & \xrightarrow{\sigma_\alpha \otimes \sigma_\alpha} & F_{j,xy}(s) \otimes F_{j,yz}(t) & \xrightarrow{\tau_j \otimes \tau_j} & F_{\infty,xy}(s) \otimes F_{\infty,yz}(t)
\end{array}$$

If we connect the top horizontal morphisms σ_α in the last diagram, we get a diagram $D_{s*t} : \mathcal{J} \longrightarrow \mathscr{M}$ that maps $i \mapsto F_{i,xz}(s*t)$. This diagram is the evaluation of the diagram D_{xz} at $s*t$, i.e., $D_{s*t} = \mathrm{Ev}_{s*t} \circ D_{xz}$, where $\mathrm{Ev}_{s*t} : \mathrm{Hom}(\mathcal{P}_{\mathcal{C}}(x, z), \underline{M}) \longrightarrow \underline{M}$. Thanks to the commutative diagram above we see that the family $\{(\tau_i \otimes \tau_i) \circ \varphi_i\}_{i \in \mathrm{Ob}(\mathcal{J})}$ determines a natural transformation from D_{s*t} to the constant diagram of value $F_{\infty,xy}(s) \otimes F_{\infty,yz}(t)$. The universal property of the colimit gives a unique map:

$$\varphi_\infty : F_{\infty,xz}(s*t) \longrightarrow F_{\infty,xy}(s) \otimes F_{\infty,yz}(t),$$

that makes the whole diagram compatible. In particular, each map $(\tau_i \otimes \tau_i) \circ \varphi_i$ factors through φ_∞. Similarly, each map $\varphi_{i,x} : F_{i,xx}([0, (x)]) \longrightarrow I$ factors through a unique map $\varphi_{\infty,x} : F_{\infty,xx}([0, (x)]) \longrightarrow I$.

It is tedious but not hard to prove that the functors $F_{\infty,xy}$ along with the colaxity maps φ_∞ and $\varphi_{\infty,x}$ determine a colax morphism $F_\infty : \mathcal{P}_{\mathcal{C}} \longrightarrow \mathscr{M}$. Moreover, for each $i \in \mathcal{J}$, the natural transformations $\tau_i : F_{i,xy} \longrightarrow F_{\infty,xy}$ form together a morphism of path-objects $\tau_i : F_i \longrightarrow F_\infty$. It is not difficult to prove that F_∞, equipped with the morphisms τ_i, satisfies the universal property of the colimit for the diagram D. ∎

A. Colax morphisms of bicategories

A.1. Definition. [Colax morphism]
Let $\mathscr{B}$ and $\mathscr{M}$ be two small bicategories. A colax morphism $F = (F, \varphi) : \mathscr{B} \longrightarrow \mathscr{M}$ *is determined by the following data and axioms:*

- *A map $F : \mathrm{Ob}(\mathscr{B}) \longrightarrow \mathrm{Ob}(\mathscr{M})$, $x \mapsto Fx$*
- *Functors $F_{xy} : \mathscr{B}(x, y) \longrightarrow \mathscr{M}(Fx, Fy)$,*
$$f \mapsto Ff, \alpha \mapsto F\alpha$$
- *For each object x of $\mathscr{B}$, an arrow of $\mathscr{M}(Fx, Fx)$, i.e., a* 2-cell *of $\mathscr{M}$:*
$$\varphi_x : F(I_x) \longrightarrow I'_{Fx}$$
- *A family of natural transformations:*
$$\varphi(x, y, z) : F_{xz} \circ c(x, y, z) \longrightarrow c'(Fx, Fy, Fz) \circ (F_{xy} \times F_{yz})$$
$$\begin{array}{ccc} \mathscr{B}(x,y) \times \mathscr{B}(y,z) & \xrightarrow{c(x,y,z)} & \mathscr{B}(x,z) \\ {\scriptstyle F_{xy} \times F_{yz}} \downarrow & \Swarrow \varphi(x,y,z) & \downarrow {\scriptstyle F_{xz}} \\ \mathscr{M}(Fx,Fy) \times \mathscr{M}(Fy,Fz) & \xrightarrow[c'(Fx,Fy,Fz)]{} & \mathscr{M}(Fx,Fz) \end{array}$$
*If (f, g) is an object of $\mathscr{B}(x, y) \times \mathscr{B}(y, z)$, the (f, g)-*component *of $\varphi(x, y, z)$*
$$F(f * g) \xrightarrow{\varphi(x,y,z)(f,g)} Ff \otimes Fg$$
shall be usually abbreviated to φ_{fg} or even φ.

These data are required to satisfy the following coherence axioms:

(M.1): If (f, g, h) is an object of $\mathscr{B}(x, y) \times \mathscr{B}(y, z) \times \mathscr{B}(z, w)$ the diagram hereafter commutes, where the letters x, y, z, w have been omitted:
$$\begin{array}{ccccc} F((f * g) * h) & \xrightarrow{\varphi_{(f*g)h}} & (F(f * g)) \otimes Fh & \xrightarrow{\varphi_{fg} \otimes \mathrm{Id}} & (Ff \otimes Fg) \otimes Fh \\ {\scriptstyle Fa(f,g,h)} \downarrow & & & & \downarrow {\scriptstyle a'(Ff,Fg,Fh)} \\ F(f * (g * h)) & \xrightarrow{\varphi_{f(g*h)}} & Ff \otimes (F(g * h)) & \xrightarrow{\mathrm{Id} \otimes \varphi_{gh}} & Ff \otimes (Fg \otimes Fh) \end{array}$$

(M.2): If f is an object of $\mathscr{B}(x, y)$ the following diagrams commute:
$$\begin{array}{ccccc} F(I_x * f) & \xrightarrow[\varphi_{fI_x}]{} & FI_x \otimes Ff & \xrightarrow[\varphi_x \otimes \mathrm{Id}]{} & I'_{Fx} \otimes Ff \\ {\scriptstyle Fl(f)} \uparrow & & & & \uparrow {\scriptstyle l'(Ff)} \\ Ff & = & = & = & Ff \end{array} \qquad \begin{array}{ccccc} F(f * I_b) & \xrightarrow[\varphi_{fI_b}]{} & Ff \otimes FI_b & \xrightarrow[\mathrm{Id} \otimes \varphi_b]{} & Ff \otimes I'_{Fy} \\ {\scriptstyle Fr(f)} \uparrow & & & & \uparrow {\scriptstyle r'(Ff)} \\ Ff & = & = & = & Ff \end{array}$$

A.2. Variant.

1. We will say that $F = (F, \varphi)$ is a *normal colax* morphism if for every x the map φ_x is the identity and if φ_{fI_x} and $\varphi_{I_b f}$ are isomorphisms.

2. If $\varphi(x, y, z)$ and φ_x are *natural isomorphisms*, so that $F(f * g) \xrightarrow{\cong} Ff \otimes Fg$ and $F(I_x) \xrightarrow{\cong} I'_{Fx}$, then $F = (F, \varphi)$ is called a (colax) *homomorphism*. The operation $(F, \varphi) \mapsto (F, \varphi^{-1})$ turns a colax homomorphism into a lax homomorphism and vice versa, so we will simply say homomorphism.

3. If $\varphi(x, y, z)$ and φ_x are *identities*, so that $F(f * g) = Ff \otimes Fg$ and $F(I_x) = I'_{Fx}$, then $F = (F, \varphi)$ is called a *strict homomorphism*.

References

[1] H. Bacard. Segal Enriched Categories I. http://arxiv.org/abs/1009.3673.

[2] H. Bacard. Toward weakly enriched categories: co-Segal categories. *J. Pure Appl. Algebra*, 218(6):1130–1170, 2014.

[3] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. 1967.

[4] J. Bergner. A survey of $(\infty, 1)$-categories. In *Towards higher categories*, pages 69–83. Springer, Berlin, 2010.

[5] G. Carlsson and R.J. Milgram. Stable homotopy and iterated loop spaces. In *Handbook of algebraic topology*, pages 505–583. 1995.

[6] J. Chiche. Un théorème A de Quillen pour les 2-foncteurs lax. *Theory Appl. Categ.*, 30:49–85, 2015.

[7] R. Dawson, R. Paré, and D. Pronk. Paths in double categories. *Theory Appl. Categ.*, 16:460–521, 2006.

[8] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[9] G. Dunn. Uniqueness of n-fold delooping machines. *J. Pure Appl. Algebra*, 113(2):159–193, 1996.

[10] W. Dwyer, D. Kan, and J. Smith. Homotopy commutative diagrams and their realizations. *J. Pure Appl. Algebra*, 57(1):5–24, 1989.

[11] D. Gepner and R. Haugseng. Enriched ∞-categories via non-symmetric ∞-operads. *Adv. Math.*, 279:575–716, 2015.

[12] P. Goerss and J.F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, 1999.

[13] A. Hirschowitz and C. Simpson. Descente pour les n-champs (Descent for n-stacks). http://arxiv.org/abs/math/9807049v3.

[14] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[15] G.M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, 2005(10):1–136, 2005.

[16] G.M. Kelly, A. Labella, V. Schmitt, and R. Street. Categories enriched on two sides. *J. Pure Appl. Algebra*, 168(1):53–98, 2002.

[17] S. Lack. Icons. *Appl. Categ. Structures*, 18(3):289–307, 2010.

[18] T. Leinster. Homotopy algebras for operads. http://arxiv.org/abs/math/0002180.

[19] T. Leinster. Up-to-Homotopy Monoids. http://arxiv.org/abs/math/9912084.

[20] J. Lurie. Stable Infinity Categories. August 2006. http://arxiv.org/abs/math/0608228.

[21] S. Mac Lane. *Homology.* Springer-Verlag, Berlin, reprint of the 3rd corr. print. 1975 edition, 1995.

[22] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[23] J. P. May. *Simplicial objects in algebraic topology.* Chicago Lectures in Mathematics. The University of Chicago Press, 1992.

[24] R. Pellissier. *Catégories enrichies faibles.* Phd thesis, Université Nice Sophia Antipolis, June 2002.

[25] C. Rezk. A model for the homotopy theory of homotopy theory. *Trans. Am. Math. Soc.*, 353(3):973–1007, 2001.

[26] R. Schwänzl and R. Vogt. Homotopy homomorphisms and the hammock localization. *Bol. Soc. Mat. Mexicana (2)*, 37(1-2):431–448, 1992.

[27] S. Schwede and B. Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.

[28] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[29] B. Shoikhet. Graded Leinster monoids and generalized Deligne conjecture for 1-monoidal abelian categories. *Int. Math. Res. Not.*, 2018(19):5857–5937, 2018.

[30] C. Simpson. *Homotopy theory of higher categories*, volume 19 of *New Mathematical Monographs*. Cambridge University Press, 2012.

[31] R. Street. Enriched categories and cohomology. *Repr. Theory Appl. Categ.*, pages 1–18, 2005. Reprinted from Quaestiones Math. 6 (1983), no. 1-3, 265–283.

[32] G. Tabuada. Homotopy theory of dg categories via localizing pairs and Drinfeld's dg quotient. *Homology Homotopy Appl.*, 12(1):187–219, 2010.

[33] D. Tamarkin. What do dg-categories form? *Compos. Math.*, 143(5):1335–1358, 2007.

[34] Z. Tamsamani. Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux. *K-Theory*, 16(1):51–99, 1999.

[35] R. F. C. Walters. Sheaves and Cauchy-complete categories. *Cahiers Topol. Géom. Diff.*, 22(3):283–286, 1981.

[36] R. F. C. Walters. Sheaves on sites as Cauchy-complete categories. *J. Pure Appl. Algebra*, 24(1):95–102, 1982.

[37] C. Weibel. *An introduction to homological algebra.* Cambridge Univ. Press, 1995.

[38] G. W. Whitehead. *Elements of homotopy theory.* Springer, 1978.

Laboratoire J.A. Dieudonné, UNS

Email: `hugo.bacard@ac-versailles.fr`

This article may be accessed at `http://www.tac.mta.ca/tac/`