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COMPLETELY DISTRIBUTIVE ENRICHED CATEGORIES ARE NOT ALWAYS CONTINUOUS

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ABSTRACT. In contrast to the fact that every completely distributive lattice is necessarily continuous in the sense of Scott, it is shown that complete distributivity of a category enriched over the closed category obtained by endowing the unit interval with a continuous t-norm does not imply its continuity in general. Necessary and sufficient conditions for the implication are presented.

1. Introduction

Preordered sets are often viewed as thin categories, and the other way around, categories have also been studied as "generalized ordered structures". Illuminating examples include the study of continuous categories in [1, 13] and that of completely (totally) distributive categories in [22, 23]. A bit more generally, categories enriched over a monoidal closed category can be viewed as "ordered sets" with *truth-values* taken in that closed category [21]. This point of view has led to a theory of *quantitative domains*, of which the core objects are categories enriched in a quantale, see e.g. [3, 6, 10, 11, 17, 32].

Continuous dcpos [7] are characterized by the relation between a poset P and the poset $\mathsf{Idl}(P)$ of ideals of P. For all $p \in P$, $\downarrow p = \{x \in P \mid x \leq p\}$ defines an embedding $\downarrow : P \longrightarrow \mathsf{Idl}(P)$. A poset P is directed complete if $\downarrow$ has a left adjoint

$$\sup : \mathsf{Idl}(P) \longrightarrow P$$

and is continuous if there is a string of adjunctions

$$\Downarrow \dashv \sup \dashv \downarrow : P \longrightarrow \mathsf{Idl}(P).$$

In a locally small category $\mathcal{E}$, ind-objects (or equivalently, filtered colimits of representable presheaves) play the role of ideals in posets. Let $\mathsf{Ind}\text{-}\mathcal{E}$ be the category of all filtered colimits of representable presheaves on $\mathcal{E}$. Then, $\mathcal{E}$ has filtered colimits if the Yoneda embedding $\mathsf{y} : \mathcal{E} \longrightarrow \mathsf{Ind}\text{-}\mathcal{E}$ has a left adjoint

$$\mathrm{colim} : \mathsf{Ind}\text{-}\mathcal{E} \longrightarrow \mathcal{E}$$

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and it is further continuous if there is a string of adjunctions

$$\mathsf{w} \dashv \mathrm{colim} \dashv \mathsf{y} : \mathcal{E} \longrightarrow \mathsf{Ind}\text{-}\mathcal{E}.$$

For categories enriched in a commutative and unital quantale Q, forward Cauchy weights (i.e., presheaves generated by forward Cauchy nets, see Definition 4.1) play the role of ideals. For each Q-category A, let $\mathcal{C}A$ be the Q-category of all forward Cauchy weights of A. Then, A is forward Cauchy cocomplete if the Q-functor $\mathsf{e}_A : A \longrightarrow \mathcal{C}A$, which is obtained by restricting the codomain of the enriched Yoneda embedding, has a left adjoint

$$\sup_A : \mathcal{C}A \longrightarrow A;$$

and A is continuous if it is *separated* (for definition see below) and there is a string of adjoint Q-functors

$$\mathsf{t}_A \dashv \sup_A \dashv \mathsf{e}_A : A \longrightarrow \mathcal{C}A.$$

If we replace, in the definition of continuous lattice, $\mathsf{Idl}(P)$ by the poset of all lower sets, then we obtain the concept of (constructively) completely distributive lattices [5, 34]. Similarly, if we replace, respectively, the category of ind-objects and the Q-category of forward Cauchy weights by the category of all small presheaves and the Q-category of all weights, then we obtain the concepts of completely distributive categories [23, Remark 4.7] and completely distributive Q-categories [29, 25].

{0,1}	Q	Set
preordered sets	Q-categories	categories
lower sets	weights	small presheaves
directed lower sets	forward Cauchy weights	ind-objects
continuous dcpos	continuous Q-categories	continuous categories
c.d. lattices	c.d. Q-categories	c.d. categories

It is well-known in order theory that a completely distributive lattice is necessarily continuous, see e.g. [7]. This paper investigates whether there is an enriched version of this conclusion. It turns out that it depends on the structure of the *truth-values*, i.e., the structure of the closed category. The main result, Theorem 6.4, shows that if Q is the interval $[0,1]$ equipped with a continuous t-norm $\&$, then all completely distributive Q-categories are continuous if and only if the corresponding implication (see below) $\rightarrow: [0,1]^2 \longrightarrow [0,1]$ is continuous at every point off the diagonal $\{(x,x) \mid x \in [0,1]\}$.

2. Complete quantale-enriched categories

This section recalls some basic notions and results about quantale-enriched categories (which are a special case of quantaloid-enriched categories [27]) that will be needed.

A commutative and unital quantale (a quantale for short) [26] is a small, complete and symmetric monoidal closed category. Explicitly, a quantale

$$\mathsf{Q} = (\mathsf{Q}, \&, k)$$

is a commutative monoid with k being the unit, such that the underlying set Q is a complete lattice and the multiplication & distributes over arbitrary joins. The unit k need not be the top element in Q. If it happens that k is the top element, then Q is said to be *integral*. The multiplication & determines a binary operator $\rightarrow$, often called the implication corresponding to &, via the adjoint property:

$$p \,\&\, q \leq r \iff q \leq p \rightarrow r.$$

Let $\mathsf{Q} = (\mathsf{Q}, \&, k)$ be quantale. A Q-*category* consists of a set A and a map $a : A \times A \longrightarrow \mathsf{Q}$ such that

$$k \leq a(x, x) \quad \text{and} \quad a(y, z) \,\&\, a(x, y) \leq a(x, z)$$

for all $x, y, z \in A$. We often write A for the pair (A, a) and $A(x, y)$ for $a(x, y)$ if no confusion would arise.

For a Q-category A, the underlying preorder $\sqsubseteq$ on A refers to the preorder given by

$$x \sqsubseteq y \iff k \leq A(x, y).$$

Two elements x, y in a Q-category A are said to be isomorphic if $A(x, y) \geq k$ and $A(y, x) \geq k$. A Q-category A is *separated* if isomorphic elements of A are necessarily identical. It is clear that A is separated if and only if $(A, \sqsubseteq)$ is a partially ordered set.

2.1. Example. (Q, d_L) is a separated Q-category, where $d_L(x, y) = x \rightarrow y$ for all $x, y \in \mathsf{Q}$.

A Q-*functor* $f : A \longrightarrow B$ between Q-categories is a map $f : A \longrightarrow B$ such that

$$A(x, y) \leq B(f(x), f(y))$$

for all $x, y \in A$. With the pointwise order between Q-functors inherited from B, i.e.,

$$f \leq g : A \longrightarrow B \iff \forall x \in A,\ f(x) \sqsubseteq g(x),$$

Q-categories and Q-functors constitute a locally ordered category

$$\mathsf{Q}\text{-}\mathsf{Cat}.$$

A Q-*distributor* $\phi : A \to\!\!\!\!\circ\; B$ between Q-categories is a map $\phi : A \times B \longrightarrow \mathsf{Q}$ such that

$$B(y, y') \,\&\, \phi(x, y) \,\&\, A(x', x) \leq \phi(x', y')$$

for all $x, x' \in A$ and $y, y' \in B$. The composition of Q-distributors $\phi : A \to\!\!\!\!\circ\; B$ and $\psi : B \to\!\!\!\!\circ\; C$ is given by

$$\psi \circ \phi : A \to\!\!\!\!\circ\; C, \quad \psi \circ \phi(x, z) = \bigvee_{y \in B} \psi(y, z) \,\&\, \phi(x, y).$$

Q-categories and Q-distributors constitute a locally ordered category

$$\mathsf{Q}\text{-}\mathsf{Dist}$$

with local order inherited from Q.

Each Q-functor $f : A \longrightarrow B$ induces distributors

$$f_*(x, y) = B(f(x), y) : A \mathrel{\circ\!\!\!\rightarrow} B \quad \text{and} \quad f^*(y, x) = B(y, f(x)) : B \mathrel{\circ\!\!\!\rightarrow} A,$$

called respectively the *graph* and *cograph* of f.

Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be Q-functors. We say f is left adjoint to g (or, g is right adjoint to f), and write $f \dashv g$, if

$$1_A \sqsubseteq g \circ f \quad \text{and} \quad f \circ g \sqsubseteq 1_B.$$

It is easily seen that f is left adjoint to g if and only if

$$f_* = g^*,$$

i.e., $B(f(x), y) = A(x, g(y))$ for all $x \in A$ and $y \in B$.

With $\star$ denoting the singleton Q-category with only one object and $a(\star, \star) = k$, Q-distributors of the form $\phi : A \mathrel{\circ\!\!\!\rightarrow} \star$ are called *weights* (or, *presheaves*) of A. The weights of A constitute a Q-category $\mathcal{P}A$ with

$$\mathcal{P}A(\phi, \rho) = \bigwedge_{x \in A} \phi(x) \rightarrow \rho(x).$$

Dually, Q-distributors of the form $\psi : \star \mathrel{\circ\!\!\!\rightarrow} A$ are called *coweights* (or, *copresheaves*) of A. The coweights of A constitute a Q-category $\mathcal{P}^\dagger A$ with

$$\mathcal{P}^\dagger A(\psi, \sigma) = \bigwedge_{x \in A} \sigma(x) \rightarrow \psi(x).$$

For any Q-category A, the underlying order on $\mathcal{P}A$ coincides with the local order in $\mathsf{Q}\text{-}\mathsf{Dist}$, while the underlying order on $\mathcal{P}^\dagger A$ is *opposite* to the local order in $\mathsf{Q}\text{-}\mathsf{Dist}$, i.e.,

$$\psi \sqsubseteq \sigma \text{ in } \mathcal{P}^\dagger A \iff \sigma \leq \psi \text{ in } \mathsf{Q}\text{-}\mathsf{Dist}.$$

Each Q-functor $f : A \longrightarrow B$ induces two Q-functors

$$\mathcal{P}f : \mathcal{P}A \longrightarrow \mathcal{P}B, \quad \mathcal{P}f(\phi) = \phi \circ f^*$$

and

$$\mathcal{P}^\dagger f : \mathcal{P}^\dagger A \longrightarrow \mathcal{P}^\dagger B, \quad \mathcal{P}^\dagger f(\psi) = f_* \circ \psi.$$

Both $\mathcal{P}$ and $\mathcal{P}^\dagger$ are endofunctors on $\mathsf{Q}\text{-}\mathsf{Cat}$. The functor $\mathcal{P}$ can be made into a monad $(\mathcal{P}, \mathsf{s}, \mathsf{y})$, called the *presheaf monad*, with unit given by the Yoneda embedding

$$\mathsf{y}_A : A \longrightarrow \mathcal{P}A, \quad \mathsf{y}_A(x) = A(-, x)$$

and multiplication given by

$$\mathsf{s}_A : \mathcal{P}\mathcal{P}A \longrightarrow \mathcal{P}A, \quad \mathsf{s}_A(\Phi) = \Phi \circ (\mathsf{y}_A)_* : A \multimap \mathcal{P}A \multimap \star\,.$$

Similarly, the functor $\mathcal{P}^\dagger$ can also be made into a monad $(\mathcal{P}^\dagger, \mathsf{s}^\dagger, \mathsf{y}^\dagger)$, called the *copresheaf monad*, with unit $\mathsf{y}^\dagger$ given by the co-Yoneda embedding

$$\mathsf{y}^\dagger_A : A \longrightarrow \mathcal{P}^\dagger A, \quad \mathsf{y}^\dagger_A(x) = A(x, -)$$

and multiplication $\mathsf{s}^\dagger$ given by

$$\mathsf{s}^\dagger_A : \mathcal{P}^\dagger\mathcal{P}^\dagger A \longrightarrow \mathcal{P}^\dagger A, \quad \mathsf{s}^\dagger_A(\Psi) = (\mathsf{y}^\dagger_A)^* \circ \Psi : \star \multimap \mathcal{P}^\dagger A \multimap A.$$

The presheaf monad $(\mathcal{P}, \mathsf{s}, \mathsf{y})$ is a KZ-doctrine and the copresheaf monad $(\mathcal{P}^\dagger, \mathsf{s}^\dagger, \mathsf{y}^\dagger)$ is a co-KZ-doctrine on the locally ordered category Q-Cat.[1]

Let A be a Q-category and ϕ be a weight of A. An element x of A is called a *supremum* of ϕ if for all $y \in A$,

$$A(x, y) = \mathcal{P}A(\phi, \mathsf{y}(y)) = \bigwedge_{z\in A} (\phi(z) \rightarrow A(z, y)).$$

In the terminology of category theory, the element x is a *colimit* of the identity functor $A \longrightarrow A$ weighted by ϕ. However, following the tradition of order theory, we call it a *supremum of* ϕ and denote it by $\sup_A \phi$. Supremum of a weight ϕ, when exists, is unique up to isomorphism. We say that A is *cocomplete* [27] if the Yoneda embedding $\mathsf{y}_A : A \longrightarrow \mathcal{P}A$ has a left adjoint, $\sup_A : \mathcal{P}A \longrightarrow A$, which sends each weight ϕ to its supremum. Dually, we say that a Q-category A is *complete* if the co-Yoneda embedding $\mathsf{y}^\dagger_A : A \longrightarrow \mathcal{P}^\dagger A$ has a right adjoint, $\inf_A : \mathcal{P}^\dagger A \longrightarrow A$, which sends each $\psi \in \mathcal{P}^\dagger A$ to its *infimum.*

2.2. Proposition. ([27, Proposition 5.10]) *A Q-category A is complete if and only if it is cocomplete.*

Since $(\mathcal{P}, \mathsf{s}, \mathsf{y})$ is a KZ-doctrine, a Q-category A is a $\mathcal{P}$-algebra if and only if $\mathsf{y}_A : A \longrightarrow \mathcal{P}A$ has a left inverse (hence A is separated), and in this case the left inverse is necessarily a left adjoint of y_A, see e.g. [8, Theorem 2.4]. A *$\mathcal{P}$-homomorphism* $f : A \longrightarrow B$ between $\mathcal{P}$-algebras A and B is a Q-functor $f : A \longrightarrow B$ such that

$$\sup_B \circ \mathcal{P}f = f \circ \sup_A,$$

[1] A monad $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ on a locally ordered category X is a KZ-doctrine (co-KZ-doctrine, resp.) [9, 16, 35] if $\mathcal{T}$ is a 2-functor, and for each object A of X, there is a string of adjoint arrows

$$\mathcal{T}\mathsf{e}_A \dashv \mathsf{m}_A \dashv \mathsf{e}_{\mathcal{T}A} : \mathcal{T}A \longrightarrow \mathcal{T}\mathcal{T}A \quad (\mathcal{T}\mathsf{e}_A \vdash \mathsf{m}_A \vdash \mathsf{e}_{\mathcal{T}A} : \mathcal{T}A \longrightarrow \mathcal{T}\mathcal{T}A, \text{ resp.}).$$

The latter condition is equivalent to

$$\mathcal{T}\mathsf{e}_A \leq \mathsf{e}_{\mathcal{T}A} \quad (\mathcal{T}\mathsf{e}_A \geq \mathsf{e}_{\mathcal{T}A}, \text{ resp.})$$

for each object A of X.

which is equivalent to f being a left adjoint. Therefore, the category of $\mathcal{P}$-algebras and $\mathcal{P}$-homomorphisms is just the category

$$\mathsf{Q}\text{-}\mathsf{Sup}$$

of separated cocomplete Q-categories and left adjoint Q-functors. Dually, since $(\mathcal{P}^{\dagger}, \mathsf{s}^{\dagger}, \mathsf{y}^{\dagger})$ is a co-KZ-doctrine, a $\mathcal{P}^{\dagger}$-algebra is exactly a separated complete Q-category; a $\mathcal{P}^{\dagger}$-homomorphism $f : A \longrightarrow B$ between $\mathcal{P}^{\dagger}$-algebras is a right adjoint Q-functor. Thus, the category of $\mathcal{P}^{\dagger}$-algebras and $\mathcal{P}^{\dagger}$-homomorphisms is just the category

$$\mathsf{Q}\text{-}\mathsf{Inf}$$

of separated complete Q-categories and right adjoint Q-functors.

For each subset C of a Q-category A, we write $\bigvee C$ for a join of C (which is unique up to isomorphism) in $(A, \sqsubseteq)$; likewise we write $\bigwedge C$ for a meet of C in $(A, \sqsubseteq)$.

2.3. Proposition. ([4, 28]) *A Q-category A is cocomplete if and only if it satisfies the following conditions:*

(1) *A is tensored in the sense that for all $p \in \mathsf{Q}$, $x \in A$, there is an element $p \otimes x \in A$, called the tensor of p with x, such that for any $y \in A$,*

$$A(p \otimes x, y) = p \to A(x, y);$$

(2) *Every subset C of A has a join in $(A, \sqsubseteq)$ and for all $x \in A$,*

$$A(\bigvee C, x) = \bigwedge_{y \in C} A(y, x).$$

It is not hard to check that for each weight ϕ of a cocomplete Q-category A,

$$\sup_A \phi = \bigvee_{x \in X} \phi(x) \otimes x.$$

2.4. Proposition. ([4, 28]) *A Q-category A is complete if and only if it satisfies the following conditions:*

(1) *A is cotensored in the sense that for all $p \in \mathsf{Q}$, $x \in A$, there is an element $p \multimap x \in A$, called the cotensor of p with x, such that for any $y \in A$,*

$$A(y, p \multimap x) = p \to A(y, x);$$

(2) *Every subset C of A has a meet in $(A, \sqsubseteq)$ and for all $x \in A$,*

$$A(x, \bigwedge C) = \bigwedge_{y \in C} A(x, y).$$

2.5. PROPOSITION. ([28]) *Let $f: A \longrightarrow B$ be* Q*-functor between complete* Q*-categories.*

(1) *f is a left adjoint if and only if $f(p \otimes x) = p \otimes f(x)$ for all $p \in \mathsf{Q}$, $x \in A$ and $f(\bigvee C) = \bigvee f(C)$ for all $C \subseteq A$.*
(2) *f is a right adjoint if and only if $f(p \multimap x) = p \multimap f(x)$ for all $p \in \mathsf{Q}$, $x \in A$ and $f(\bigwedge C) = \bigwedge f(C)$ for all $C \subseteq A$.*

2.6. EXAMPLE. (1) For each Q-category A, $\mathcal{P}A$ is complete, in which

$$(p \otimes \phi)(x) = p \,\&\, \phi(x) \quad \text{and} \quad (p \multimap \phi)(x) = p \to \phi(x)$$

for all $p \in \mathsf{Q}$ and $\phi \in \mathcal{P}A$.

(2) For each Q-category A, $\mathcal{P}^\dagger A$ is complete, in which

$$(p \multimap \psi)(x) = p \,\&\, \psi(x) \quad \text{and} \quad (p \otimes \psi)(x) = p \to \psi(x)$$

for all $p \in \mathsf{Q}$ and $\psi \in \mathcal{P}^\dagger A$.

3. Completely distributive quantale-enriched categories

A *saturated class of weights* [2, 14, 30] is a full submonad $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ of the presheaf monad $(\mathcal{P}, \mathsf{s}, \mathsf{y})$ on $\mathsf{Q}\text{-}\mathsf{Cat}$. Explicitly, a saturated class of weights is a triple $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ subject to:

- $\mathcal{T}$ is a subfunctor of $\mathcal{P}: \mathsf{Q}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Q}\text{-}\mathsf{Cat}$;
- all inclusions $\varepsilon_A : \mathcal{T}A \longrightarrow \mathcal{P}A$ are fully faithful;
- all ε_A form a natural transformation such that

$$\mathsf{s} \circ (\varepsilon * \varepsilon) = \varepsilon \circ \mathsf{m} \quad \text{and} \quad \varepsilon \circ \mathsf{e} = \mathsf{y}.$$

Said differently, a saturated class of weights is a functor $\mathcal{T} : \mathsf{Q}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Q}\text{-}\mathsf{Cat}$ such that $\mathcal{T}A$ is a full sub-Q-category of $\mathcal{P}A$ through which the Yoneda embedding $\mathsf{y}_A : A \longrightarrow \mathcal{P}A$ factors, and that for each $\Phi \in \mathcal{T}\mathcal{T}A$, the supremum of

$$\Phi \circ \varepsilon_A^* : \mathcal{P}A \,{-\!\!\circ\!\!\!\rightarrow}\, \mathcal{T}A \,{-\!\!\circ\!\!\!\rightarrow}\, \star$$

in $\mathcal{P}A$ belongs to $\mathcal{T}A$.

Since $(\mathcal{P}, \mathsf{s}, \mathsf{y})$ is a KZ-doctrine, then so is every saturated class of weights $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ on $\mathsf{Q}\text{-}\mathsf{Cat}$. Thus, for each saturated class of weights $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ on $\mathsf{Q}\text{-}\mathsf{Cat}$, a $\mathcal{T}$*-algebra* A is a Q-category A such that

$$\mathsf{e}_A : A \longrightarrow \mathcal{T}A$$

has a left inverse (which is necessarily a left adjoint of e_A)

$$\sup_A : \mathcal{T}A \longrightarrow A.$$

Said differently, a $\mathcal{T}$-algebra is a separated Q-category A such that every $\phi \in \mathcal{T}A$ has a supremum. A *$\mathcal{T}$-homomorphism* $f : A \longrightarrow B$ between $\mathcal{T}$-algebras is a Q-functor such that

$$f \circ \sup_A = \sup_B \circ \mathcal{T}f.$$

The category of $\mathcal{T}$-algebras and $\mathcal{T}$-homomorphisms is denoted by

$$\mathcal{T}\text{-}\mathsf{Alg}.$$

For the largest saturated class of weights $\mathcal{P}$, the category $\mathcal{P}$-Alg is just the category Q-Sup of separated cocomplete Q-categories and left adjoint Q-functors.

It is clear that every $\mathcal{P}$-algebra is a $\mathcal{T}$-algebra and every $\mathcal{P}$-homomorphism is a $\mathcal{T}$-homomorphism, so the category $\mathcal{P}$-Alg is a subcategory of $\mathcal{T}$-Alg.

3.1. PROPOSITION. *Let $\mathcal{T}$ be a saturated class of weights on Q-Cat. Then, every retract of a $\mathcal{T}$-algebra in Q-Cat is a $\mathcal{T}$-algebra.*

PROOF. Suppose that B is a $\mathcal{T}$-algebra; $s : A \longrightarrow B$ and $r : B \longrightarrow A$ are Q-functors such that $r \circ s = 1_A$. Let $\sup_A$ be the composite

$$\mathcal{T}A \xrightarrow{\mathcal{T}s} \mathcal{T}B \xrightarrow{\sup_B} B \xrightarrow{r} A.$$

Then

$$\sup_A \circ \mathsf{e}_A = r \circ \sup_B \circ \mathcal{T}s \circ \mathsf{e}_A = r \circ \sup_B \circ \mathsf{e}_B \circ s = r \circ s = 1_A,$$

so, $\sup_A$ is a left inverse of e_A and consequently, A is a $\mathcal{T}$-algebra. ∎

3.2. DEFINITION. Let $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ be a saturated class of weights on Q-Cat. A Q-category is said to be $\mathcal{T}$-continuous if it is a $\mathcal{T}$-continuous $\mathcal{T}$-algebra; that is, if A is separated and there is a string of adjoint Q-functors

$$\mathsf{t}_A \dashv \sup_A \dashv \mathsf{e}_A : A \longrightarrow \mathcal{T}A.$$

3.3. PROPOSITION. *Let $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ be a saturated class of weights on Q-Cat. Then, for every Q-category A, the Q-category $\mathcal{T}A$ is $\mathcal{T}$-continuous.*

PROOF. Since $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ is saturated, it follows that for every Q-category A, there is a string of adjoint Q-functors

$$\mathcal{T}\mathsf{e}_A \dashv \mathsf{m}_A \dashv \mathsf{e}_{\mathcal{T}A} : \mathcal{T}A \longrightarrow \mathcal{T}\mathcal{T}A,$$

which entails that $\mathcal{T}A$ is $\mathcal{T}$-continuous. ∎

3.4. Proposition. *Let $\mathcal{T}$ be a saturated class of weights on* Q-Cat*. Then, in the category* $\mathcal{T}$-Alg*, every retract of a $\mathcal{T}$-continuous $\mathcal{T}$-algebra is $\mathcal{T}$-continuous.*

Proof. Suppose that B is a $\mathcal{T}$-continuous $\mathcal{T}$-algebra; $s : A \longrightarrow B$ and $r : B \longrightarrow A$ are $\mathcal{T}$-homomorphisms such that $r \circ s = 1_A$. We claim that $\mathsf{t}_A := \mathcal{T}r \circ \mathsf{t}_B \circ s$ is left adjoint to $\sup_A$, hence A is $\mathcal{T}$-continuous. On one hand,

$$\begin{aligned}\sup_A \circ\, \mathsf{t}_A &= \sup_A \circ\, \mathcal{T}r \circ \mathsf{t}_B \circ s \\ &= r \circ \sup_B \circ\, \mathsf{t}_B \circ s && (r \text{ is a } \mathcal{T}\text{-homomorphism}) \\ &= r \circ s \\ &= 1_A.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathsf{t}_A \circ \sup_A &= \mathcal{T}r \circ \mathsf{t}_B \circ s \circ \sup_A \\ &= \mathcal{T}r \circ \mathsf{t}_B \circ \sup_B \circ\, \mathcal{T}s && (s \text{ is a } \mathcal{T}\text{-homomorphism}) \\ &\sqsubseteq \mathcal{T}r \circ \mathcal{T}s && (\mathsf{t}_B \dashv \sup_B) \\ &= 1_{\mathcal{T}A}.\end{aligned}$$

Thus, t_A is left adjoint to $\sup_A$, as desired. ∎

3.5. Corollary. *Let $\mathcal{T}$ be a saturated class of weights. Then, a $\mathcal{T}$-algebra A is $\mathcal{T}$-continuous if and only if it is a retract of $\mathcal{T}A$ in* $\mathcal{T}$-Alg*.*

Letting $\mathcal{T} = \mathcal{P}$ in Definition 3.2 we obtain the notion of completely distributive Q-categories. Explicitly,

3.6. Definition. [29] A Q-category A is said to be completely distributive (or, totally continuous) if it is a $\mathcal{P}$-continuous $\mathcal{P}$-algebra; that is, if A is separated and there exists a string of adjoint Q-functors

$$\mathsf{t}_A \dashv \sup_A \dashv \mathsf{y}_A : A \longrightarrow \mathcal{P}A.$$

3.7. Proposition. *A complete* Q*-category A is completely distributive if and only if it is a retract of some power of* (Q, d_L) *in* Q-Sup*.*

Proof. For each set X, the power $(\mathsf{Q}, d_L)^X$ (see Example 2.1) in Q-Sup is clearly the Q-category $\mathcal{P}X$ when X is viewed as a discrete Q-category (defined in an evident way). So, sufficiency follows from propositions 3.3 and 3.4. Necessity follows from the observation that a completely distributive Q-category A is a retract of $\mathcal{P}A$ which is a retract of $\mathcal{P}|A|$, where $|A|$ is the discrete Q-category with the same objects as those of A. ∎

3.8. Definition. [18] A separated complete Q-category A is completely co-distributive if there exists a string of adjoint Q-functors:

$$\mathsf{y}^\dagger_A \dashv \inf_A \dashv \mathsf{t}^\dagger_A : A \longrightarrow \mathcal{P}^\dagger A.$$

It is not hard to see that a Q-category A is completely co-distributive if and only if A^{op}, the opposite of A given by $A^{\mathrm{op}}(x,y) = A(y,x)$, is completely distributive. Since $(\mathcal{P}^\dagger, \mathsf{s}^\dagger, \mathsf{y}^\dagger)$ is a co-KZ-doctrine on Q-Cat, for each Q-category A, the Q-category $\mathcal{P}^\dagger A$ is easily verified to be completely co-distributive. It is known in lattice theory that the notion of complete distributivity is self dual, i.e., a complete lattice is completely distributive if and only if so is its opposite, see e.g. [12, VII.1.10]. But, this is not always true for Q-categories. In fact, it is proved in [18, Theorem 8.2] that for an integral quantale Q, every completely distributive Q-category is completely co-distributive if and only if Q satisfies the law of double negation. So, complete distributivity and complete co-distributivity are no longer equivalent concepts for quantale-enriched categories.

4. Continuous quantale-enriched categories

In order to define continuous Q-categories, the first step is to find for Q-categories an analogue of ideals (= directed lower sets) in a partially ordered set and/or ind-objects in a locally small category. Forward Cauchy weights will play the role.

Let A be a Q-category. A net in A is a map x from a directed set $(E, \leq)$ to A. It is customary to write x_λ for $x(\lambda)$ and to write $\{x_\lambda\}_\lambda$ for the net.

4.1. Definition. [3, 6, 33] Let A be a Q-category. A net $\{x_\lambda\}_\lambda$ in A is called forward Cauchy if

$$\bigvee_\lambda \bigwedge_{\gamma\geq\mu\geq\lambda} A(x_\mu, x_\gamma) \geq k.$$

A weight $\phi : A \xrightarrow{\circ} \star$ is called forward Cauchy if

$$\phi = \bigvee_\lambda \bigwedge_{\lambda\leq\mu} A(-, x_\mu)$$

for some forward Cauchy net $\{x_\lambda\}_\lambda$ in A.

Let $\{x_\lambda\}_\lambda$ be a forward Cauchy net in a Q-category A. An element $x \in A$ is called a *liminf* (a.k.a. Yoneda limit) [3, 33] of $\{x_\lambda\}_\lambda$, if for all $y \in A$,

$$A(x, y) = \bigvee_\lambda \bigwedge_{\mu\geq\lambda} A(x_\mu, y).$$

We say that a Q-category A is *forward Cauchy cocomplete* (a.k.a. Yoneda complete) if every forward Cauchy net has a liminf. The following conclusion, which is proved in [6, Lemma 46] when Q is a *value quantale* (see [6, Definition 6]) and in [20, Theorem 5.13] for the general case, implies that a Q-category A is forward Cauchy cocomplete if and only if every forward Cauchy weight of A has a supremum.

4.2. Proposition. *Let $\{x_\lambda\}_\lambda$ be a forward Cauchy net in a* Q*-category A. An element x of A is a liminf of $\{x_\lambda\}_\lambda$ if and only if x is a supremum of the weight*

$$\phi = \bigvee_{\lambda} \bigwedge_{\mu \geq \lambda} A(-, x_\mu).$$

We do not know whether assigning to each Q-category A the Q-category of forward Cauchy weights of A gives a saturated class of weights, however, there is an easy-to-check sufficient condition which is presented in [6, 20].

A quantale is said to be *continuous* if its underlying complete lattice is continuous. The following conclusion is proved in [6, Proposition 13] when Q is a value quantale (which is necessarily integral and continuous) and in [20, Theorem 6.5] for the version stated below.

4.3. Proposition. *Let* Q *be an integral and continuous quantale. Then, assigning to each* Q*-category A the* Q*-category*

$$\mathcal{C}A := \{\phi \in \mathcal{P}A \mid \phi \textit{ is forward Cauchy}\}$$

defines a saturated class of weights on $\mathsf{Q}\text{-}\mathsf{Cat}$*, which is denoted by $\mathcal{C}$.*

4.4. Convention. When talking about forward Cauchy weights, if not otherwise specified, we always assume that $(\mathsf{Q}, \&, k)$ is continuous, commutative and integral. For such a quantale, the class $\mathcal{C}$ of forward Cauchy weights is saturated and the category of $\mathcal{C}$-algebras and $\mathcal{C}$-homomorphisms is exactly the category of separated and forward Cauchy cocomplete Q-categories and Q-functors that preserve liminf of forward Cauchy nets.

Letting $\mathcal{T} = \mathcal{C}$ in Definition 3.2 we obtain the notion of continuous Q-categories. Explicitly,

4.5. Definition. [17] A Q-category A is said to be continuous if it is a $\mathcal{C}$-continuous $\mathcal{C}$-algebra; that is, if A is separated and there is a string of adjoint Q-functors:

$$\mathsf{t}_A \dashv \sup\nolimits_A \dashv \mathsf{e}_A : A \longrightarrow \mathcal{C}A.$$

When Q is the two-element Boolean algebra $\{0, 1\}$, a continuous Q-category is exactly a continuous dcpo. It is well-known that a completely distributive lattice is necessarily a continuous lattice, so it is natural to ask:

4.6. Question. Is every completely distributive Q-category continuous in the sense of Definition 4.5?

As we shall see in Section 6, the answer depends on the structure of the truth-values, i.e., the structure of the quantale Q. A sufficient and necessary condition will be given when Q is the interval $[0, 1]$ equipped with a continuous t-norm.

4.7. Proposition. *The following statements are equivalent:*

(1) *Every completely distributive* Q*-category is continuous.*

(2) $\mathcal{P}A$ *is continuous for every* Q*-category* A*.*

Proof. That (1) implies (2) is trivial. Conversely, let A be a completely distributive Q-category. From Corollary 3.5 it follows that A is a retract of $\mathcal{P}A$ in Q-Sup, hence a retract of $\mathcal{P}A$ in the category of $\mathcal{C}$-algebras. Since $\mathcal{P}A$ is continuous by assumption, then so is A by Proposition 3.4. ∎

Given a cocomplete Q-category A, denote the set of all ideals of the complete lattice $(A, \sqsubseteq)$ by $\mathsf{Idl}(A)$. Since each ideal D of $(A, \sqsubseteq)$ can be seen as a forward Cauchy net of A,

$$\Lambda(D) := \bigvee_{d \in D} A(-, d)$$

is then a forward Cauchy weight. Conversely, given a forward Cauchy weight ϕ of A,

$$\Gamma(\phi) := \{x \in A \mid \phi(x) \geq k\}$$

is an ideal of $(A, \sqsubseteq)$.

4.8. Proposition. *Let* A *be a separated complete* Q*-category. Then*

$$\Lambda : (\mathsf{Idl}(A), \subseteq) \longrightarrow (\mathcal{C}A, \leq)$$

is a left adjoint and a left inverse of

$$\Gamma : (\mathcal{C}A, \leq) \longrightarrow (\mathsf{Idl}(A), \subseteq).$$

Moreover, $\sup_A \phi = \bigvee \Gamma(\phi)$ *for each* $\phi \in \mathcal{C}A$*.*

Proof. Suppose D is an ideal in $(A, \sqsubseteq)$ and ϕ is a forward Cauchy weight of A. Then

$$D \subseteq \Gamma(\phi) \iff \forall d \in D, \phi(d) \geq k \iff \Lambda(D) \leq \phi,$$

which implies $\Lambda \dashv \Gamma$.

Now we check that for each forward Cauchy weight ϕ of A, $\Lambda\Gamma(\phi) = \phi$. On one hand, since Λ is left adjoint to Γ, it follows that $\Lambda\Gamma(\phi) \leq \phi$. On the other hand, by assumption there is a forward Cauchy net $\{x_\lambda\}_{\lambda \in E}$ in A such that

$$\phi(x) = \bigvee_{\lambda \in E} \bigwedge_{\lambda \leq \mu} A(x, x_\mu).$$

Let

$$D_\phi := \Big\{ \bigwedge_{\mu \geq \lambda} x_\mu \mid \lambda \in E \Big\},$$

where $\bigwedge_{\mu\geq\lambda} x_\mu$ denotes the meet of $\{x_\mu \mid \mu \geq \lambda\}$ in the complete lattice $(A, \sqsubseteq)$. Then D_ϕ is a directed subset of $(A, \sqsubseteq)$ and $\phi(x) = \bigvee_{d\in D_\phi} A(x, d)$, so $D_\phi \subseteq \Gamma(\phi)$, and consequently, $\phi \leq \Lambda\Gamma(\phi)$.

Finally, we check that $\sup_A \phi = \bigvee \Gamma(\phi)$ for each $\phi \in \mathcal{C}A$. Since $\sup_A \phi = \bigvee_{x\in A} \phi(x)\otimes x$, it follows that $\sup_A \phi \geq \bigvee \Gamma(\phi)$. Conversely, since

$$\sup_A\phi = \sup_A\Big(\bigvee_{d\in D_\phi} A(-, d)\Big) = \bigvee D_\phi,$$

then $\sup_A \phi \leq \bigvee \Gamma(\phi)$. ■

4.9. Corollary. *Let A, B be separated complete Q-categories and $f : A \longrightarrow B$ be a Q-functor. Then, f preserves liminf of forward Cauchy nets if and only if*

$$f : (A, \sqsubseteq) \longrightarrow (B, \sqsubseteq)$$

is Scott continuous (i.e., preserves directed joins).

4.10. Lemma. *For each complete Q-category A, the set $\mathcal{C}A$ of forward Cauchy weights of A is closed in Q^A (pointwise ordered) under meets and directed joins.*

Proof. Let $\{\phi_i\}_{i\in I}$ be a subset of $\mathcal{C}A$. Then, for all $x \in A$, by continuity of Q we have

$$\bigwedge_{i\in I}\phi_i(x) = \bigwedge_{i\in I}\bigvee_{d\in\Gamma(\phi_i)} A(x, d) = \bigvee_{s\in\prod\limits_{i\in I}\Gamma(\phi_i)}\bigwedge_{i\in I} A(x, s(i)) = \bigvee_{s\in\prod\limits_{i\in I}\Gamma(\phi_i)} A\Big(x, \bigwedge_{i\in I} s(i)\Big).$$

Since

$$\Big\{\bigwedge_{i\in I} s(i) \mid s \in \prod_{i\in I}\Gamma(\phi_i)\Big\}$$

is a directed set of $(A, \sqsubseteq)$, it follows that $\bigwedge_{i\in I}\phi_i$ is a forward Cauchy weight of A, hence belongs to $\mathcal{C}A$.

Let $\{\phi_i\}_i$ be a directed set of $(\mathcal{C}A, \leq)$. Then $\{\Gamma(\phi_i)\}_i$ is a directed family in $(\mathsf{Idl}(A), \subseteq)$ and $D = \bigcup_i \Gamma(\phi_i)$ is an ideal in $(A, \sqsubseteq)$. Since

$$\bigvee_{d\in D} A(-, d) = \bigvee_i \phi_i,$$

it follows that $\bigvee_i \phi_i \in \mathcal{C}A$.

Therefore, $\mathcal{C}A$ is closed in Q^A under meets and directed joins. ■

4.11. Proposition. *A separated complete* Q*-category* A *is continuous if and only if*

(1) $(A,\sqsubseteq)$ *is a continuous lattice;*

(2) *for each* $x\in A$ *and each forward Cauchy weight* ϕ *of* A,

$$A(x,\sup_A\phi)=\bigwedge_{y\ll x}\phi(y),$$

where $\ll$ *denotes the way below relation (see [7] for definition) in* $(A,\sqsubseteq)$.

Proof. Sufficiency follows from the fact that, under the assumption, the assignment $x\mapsto\bigvee_{y\ll x}\mathsf{y}(y)$ defines a left adjoint of $\sup_A:\mathcal{C}A\longrightarrow A$.

Now we turn to the necessity. Since $\mathcal{C}A$ is closed in Q^A under meets and directed joins, it is a continuous lattice since so is Q by our convention. Since $(A,\sqsubseteq)$ is a retract of $(\mathcal{C}A,\sqsubseteq)$ in the category of dcpos by Corollary 4.9, it follows that $(A,\sqsubseteq)$ is also a continuous lattice. This proves (1). As for (2), it suffices to check that if A is continuous, then the left adjoint $\mathsf{t}_A:A\longrightarrow\mathcal{C}A$ of $\sup_A:\mathcal{C}A\longrightarrow A$ is given by $\mathsf{t}_A(x)=\bigvee_{y\ll x}\mathsf{y}(y)$. On one hand, since $x=\sup_A\bigvee_{y\ll x}\mathsf{y}(y)$, it follows that $\mathsf{t}_A(x)\leq\bigvee_{y\ll x}\mathsf{y}(y)$. On the other hand, since $\bigvee\Gamma(\mathsf{t}_A(x))=\sup_A\mathsf{t}_A(x)=x$ by Proposition 4.8, it follows that every element way below x in $(A,\sqsubseteq)$ is contained in $\Gamma(\mathsf{t}_A(x))$, so $\bigvee_{y\ll x}\mathsf{y}(y)\leq\mathsf{t}_A(x)$. ■

4.12. Proposition. *Let* A *be a complete* Q*-category. If* $(A,\sqsubseteq)$ *is a continuous lattice and for all* $p\in\mathsf{Q}$*, the cotensor* $p\multimap -:(A,\sqsubseteq)\longrightarrow(A,\sqsubseteq)$ *is Scott continuous, then* A *is continuous.*

Proof. Since $(A,\sqsubseteq)$ is a continuous lattice, A is necessarily separated. So it remains to show that for each $x\in A$ and each forward Cauchy weight ϕ of A,

$$A(x,\sup_A\phi)=\bigwedge_{y\ll x}\phi(y),$$

where $\ll$ denotes the way below relation in $(A,\sqsubseteq)$.

On one hand, since $\{y\in A\mid y\ll x\}$ is a directed set with join x, it follows that $\bigvee_{y\ll x}\mathsf{y}(y)$ is a forward Cauchy weight with supremum x. Thus,

$$A(x,\sup_A\phi)\geq\mathcal{C}A\Big(\bigvee_{y\ll x}\mathsf{y}(y),\phi\Big)=\bigwedge_{y\ll x}\phi(y).$$

On the other hand, for all $p \in \mathsf{Q}$,

$$\begin{aligned}
p \leq A(x, \sup_A \phi) &\Longrightarrow x \leq p \multimap \sup_A \phi \\
&\Longrightarrow x \leq \bigvee_{d \in \Gamma(\phi)} (p \multimap d) && (\sup_A \phi = \bigvee \Gamma(\phi)) \\
&\Longrightarrow \forall y \ll x, \exists d \in \Gamma(\phi), y \leq p \multimap d && ((A, \sqsubseteq) \text{ is continuous}) \\
&\Longrightarrow \forall y \ll x, \exists d \in \Gamma(\phi), p \leq A(y, d) \\
&\Longrightarrow \forall y \ll x, p \leq \bigvee_{d \in \Gamma(\phi)} A(y, d) \\
&\Longrightarrow \forall y \ll x, p \leq \phi(y) \\
&\Longrightarrow p \leq \bigwedge_{y \ll x} \phi(y),
\end{aligned}$$

hence $A(x, \sup_A \phi) \leq \bigwedge_{y \ll x} \phi(y)$. ∎

4.13. Corollary. *Let* $(\mathsf{Q}, \&, k)$ *be an integral quantale such that the underlying lattice is completely distributive and that for all* $p \in \mathsf{Q}$*,* $p \,\&\, - : \mathsf{Q} \longrightarrow \mathsf{Q}$ *preserves filtered meets. Then, for every* Q*-category* A*, the* Q*-category* $\mathcal{P}^\dagger A$ *is continuous.*

Proof. This follows immediately from a combination of Example 2.6 (2), Proposition 4.12, the fact that the underlying order of $\mathcal{P}^\dagger A$ is opposite to that inherited from Q^A (pointwise ordered) and that $\mathcal{P}^\dagger A$ is closed in Q^A under joins and meets. ∎

Not all completely distributive Q-categories are continuous even when the underlying lattice of Q is the interval $[0, 1]$, as we shall see in Section 6, so the following conclusion is a bit unexpected.

4.14. Corollary. *Let* $(\mathsf{Q}, \&, k)$ *be an integral quantale such that the underlying lattice is completely distributive and that for all* $p \in \mathsf{Q}$*,* $p \,\&\, - : \mathsf{Q} \longrightarrow \mathsf{Q}$ *preserves filtered meets. Then, every completely co-distributive* Q*-category is continuous.*

Proof. Since $\inf : \mathcal{P}^\dagger A \longrightarrow A$ has both a left adjoint and a right adjoint, A is a retract of $\mathcal{P}^\dagger A$, which implies that A is continuous because so is $\mathcal{P}^\dagger A$ by the above corollary. ∎

5. Relation to distributive law

Related to Question 4.6 a general one is:

5.1. Question. Let $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ be a saturated class of weights on $\mathsf{Q}\text{-}\mathsf{Cat}$. Is every completely distributive Q-category $\mathcal{T}$-continuous?

The answer depends on whether the copresheaf monad $(\mathcal{P}^\dagger, \mathsf{s}^\dagger, \mathsf{y}^\dagger)$ *distributes over* the monad $(\mathcal{T}, \mathsf{m}, \mathsf{e})$. We would like to note that in this section the quantale Q is not assumed to be a continuous one as we have agreed in 4.4 when talking about forward Cauchy weights; that is to say, the results apply to any commutative quantale.

By a *lifting* of $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ through the forgetful functor $U : \mathsf{Q\text{-}Inf} \longrightarrow \mathsf{Q\text{-}Cat}$ we mean a monad $(\widetilde{\mathcal{T}}, \widetilde{\mathsf{m}}, \widetilde{\mathsf{e}})$ on $\mathsf{Q\text{-}Inf}$ such that

$$U \circ \widetilde{\mathcal{T}} = \mathcal{T} \circ U, \quad U \circ \widetilde{\mathsf{m}} = \mathsf{m} \circ U, \quad U \circ \widetilde{\mathsf{e}} = \mathsf{e} \circ U.$$

$$\begin{tikzcd} \mathsf{Q\text{-}Inf} \arrow[r, "\widetilde{\mathcal{T}}"] \arrow[d, "U"'] & \mathsf{Q\text{-}Inf} \arrow[d, "U"] \\ \mathsf{Q\text{-}Cat} \arrow[r, "\mathcal{T}"'] & \mathsf{Q\text{-}Cat} \end{tikzcd}$$

It is clear that such a lifting of $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ exists if and only if for each separated and complete Q-category A, both $\mathsf{e}_A : A \longrightarrow \mathcal{T}A$ and $\mathsf{m}_A : \mathcal{T}\mathcal{T}A \longrightarrow \mathcal{T}A$ are $\mathsf{Q\text{-}Inf}$ morphisms. Furthermore, such a lifting, when it exists, is necessarily unique since the functor U is injective on objects.

A *distributive law* of the monad $\mathcal{P}^\dagger$ over $\mathcal{T}$ is a natural transformation $\delta : \mathcal{P}^\dagger\mathcal{T} \longrightarrow \mathcal{T}\mathcal{P}^\dagger$ satisfying certain conditions, see e.g. [9, II.3.8]. Since $\mathsf{Q\text{-}Inf}$ is the category of Eilenberg-Moore algebras of the copresheaf monad $\mathcal{P}^\dagger$, it follows from [9, II.3.8.2] that distributive laws of $\mathcal{P}^\dagger$ over $\mathcal{T}$ correspond bijectively to liftings of $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ through the forgetful functor U. Therefore, distributive laws of $\mathcal{P}^\dagger$ over $\mathcal{T}$, when they exist, are unique. So in this case we simply say that $\mathcal{P}^\dagger$ *distributes over* $\mathcal{T}$. The main result in this section asserts that for a saturated class of weights $\mathcal{T}$ on $\mathsf{Q\text{-}Cat}$, to require that every completely distributive Q-category is $\mathcal{T}$-continuous is to require that $\mathcal{P}^\dagger$ distributes over $\mathcal{T}$.

5.2. Theorem. *For a saturated class of weights $\mathcal{T}$ on $\mathsf{Q\text{-}Cat}$, the following statements are equivalent:*

(1) *Every completely distributive Q-category is $\mathcal{T}$-continuous.*

(2) *The copresheaf monad $\mathcal{P}^\dagger$ distributes over $\mathcal{T}$.*

A lemma first.

5.3. Lemma. *Let $\mathcal{T}$ be a saturated class of weights on $\mathsf{Q\text{-}Cat}$. Then, the following statements are equivalent:*

(1) *The copresheaf monad $\mathcal{P}^\dagger$ distributes over $\mathcal{T}$.*

(2) *The composite $\mathcal{T}\mathcal{P}^\dagger$ is a monad on $\mathsf{Q\text{-}Cat}$.*

(3) *$\mathcal{T}$ has a lifting through the forgetful functor $U : \mathsf{Q\text{-}Inf} \longrightarrow \mathsf{Q\text{-}Cat}$.*

(4) *For every separated complete Q-category A, $\mathcal{T}A$ is a complete Q-category.*

(5) *For every separated complete Q-category A, the inclusion $\mathcal{T}A \longrightarrow \mathcal{P}A$ has a left adjoint.*

PROOF. The equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows immediately from [9, II.3.8.2] and the fact that $\mathsf{Q}\text{-}\mathsf{Inf}$ is the category of Eilenberg-Moore algebras of the monad $\mathcal{P}^\dagger$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) By Proposition 2.5, it suffices to check that $\mathcal{T}A$ is closed in $\mathcal{P}A$ with respect to cotensors and meets. For $p \in \mathsf{Q}$ and $\phi \in \mathcal{T}A$, let $p \multimap \phi$ be the cotensor of p and ϕ in $\mathcal{T}A$. Then, for all $x \in A$,

$$(p \multimap \phi)(x) = \mathcal{T}A(\mathsf{e}_A(x), p \multimap \phi) = p \to \mathcal{T}A(\mathsf{e}_A(x), \phi) = p \to \phi(x)$$

by Proposition 2.4 (1), hence $\mathcal{T}A$ is closed in $\mathcal{P}A$ with respect to cotensors. If ϕ is the meet of a family ϕ_i in $\mathcal{T}A$, then for all $x \in A$,

$$\phi(x) = \mathcal{T}A(\mathsf{e}_A(x), \phi) = \bigwedge_i \mathcal{T}A(\mathsf{e}_A(x), \phi_i) = \bigwedge_i \phi_i(x)$$

by Proposition 2.4 (2), hence $\mathcal{T}A$ is closed in $\mathcal{P}A$ with respect to meets.

(5) $\Rightarrow$ (3) For each object A in $\mathsf{Q}\text{-}\mathsf{Inf}$, since $\mathcal{P}A$ is complete and $\mathcal{T}A$ is a retract of $\mathcal{P}A$ in $\mathsf{Q}\text{-}\mathsf{Cat}$, it follows from Proposition 3.1 that $\mathcal{T}A$ is complete. For each morphism $f : A \longrightarrow B$ in $\mathsf{Q}\text{-}\mathsf{Inf}$, let $g : B \longrightarrow A$ be the left adjoint of f. Then, $\mathcal{P}f : \mathcal{P}A \longrightarrow \mathcal{P}B$ is right adjoint to $\mathcal{P}g : \mathcal{P}B \longrightarrow \mathcal{P}A$, so $\mathcal{T}f : \mathcal{T}A \longrightarrow \mathcal{T}B$ is a right adjoint because $\mathcal{T}$ is a subfunctor of $\mathcal{P}$. Therefore, the assignment

$$\widetilde{\mathcal{T}}A := \mathcal{T}A$$

gives rise to an endofunctor on $\mathsf{Q}\text{-}\mathsf{Inf}$. To see that $\widetilde{\mathcal{T}}$ is a lifting of $\mathcal{T}$ through the forgetful functor U, it remains to check that for each separated complete Q-category A, both $\mathsf{e}_A : A \longrightarrow \mathcal{T}A$ and $\mathsf{m}_A : \mathcal{T}\mathcal{T}A \longrightarrow A$ are right adjoints. First, since A is separated and cocomplete, it is a $\mathcal{T}$-algebra, so e_A is right adjoint to $\sup_A : \mathcal{T}A \longrightarrow A$. Second, since $\mathcal{T}$ is a KZ-doctrine, it follows that m_A is both a left and a right adjoint. ∎

PROOF OF THEOREM 5.2. (1) $\Rightarrow$ (2) By Lemma 5.3, it suffices to show that for each complete Q-category A, the inclusion functor $\mathcal{T}A \longrightarrow \mathcal{P}A$ has a left adjoint. Since $\mathcal{P}A$ is completely distributive, then, by assumption, the left adjoint $\sup_{\mathcal{P}A} : \mathcal{T}\mathcal{P}A \longrightarrow \mathcal{P}A$ of $\mathsf{e}_{\mathcal{P}A} : \mathcal{P}A \longrightarrow \mathcal{T}\mathcal{P}A$ has a left adjoint, say, $\mathsf{t}_{\mathcal{P}A} : \mathcal{P}A \longrightarrow \mathcal{T}\mathcal{P}A$. Since A is cocomplete, the Yoneda embedding $\mathsf{y}_A : A \longrightarrow \mathcal{P}A$ has a left adjoint $\mathsf{s}_A : \mathcal{P}A \longrightarrow A$. Since any 2-functor preserves adjunctions, it follows that $\mathcal{T}\mathsf{s}_A : \mathcal{T}\mathcal{P}A \longrightarrow \mathcal{T}A$ is left adjoint to $\mathcal{T}\mathsf{y}_A : \mathcal{T}A \longrightarrow \mathcal{T}\mathcal{P}A$. Thus, $\mathcal{T}\mathsf{s}_A \circ \mathsf{t}_{\mathcal{P}A} : \mathcal{P}A \longrightarrow \mathcal{T}A$ is left adjoint to $\sup_{\mathcal{P}A} \circ \mathcal{T}\mathsf{y}_A : \mathcal{T}A \longrightarrow \mathcal{P}A$. Since $\mathcal{T}$ is a submonad of $\mathcal{P}$, then

$$\sup_{\mathcal{P}A} \circ \mathcal{T}\mathsf{y}_A(\phi) = \mathsf{s}_{\mathcal{P}A} \circ \mathcal{P}\mathsf{y}_A(\phi) = \phi$$

for all $\phi \in \mathcal{T}A$. Therefore, the inclusion functor $\mathcal{T}A \longrightarrow \mathcal{P}A$, which coincides with $\sup_{\mathcal{P}A} \circ \mathcal{T}\mathsf{y}_A$, has a left adjoint, given by $\mathcal{T}\mathsf{s}_A \circ \mathsf{t}_{\mathcal{P}A}$.

(2) $\Rightarrow$ (1) Let A be a completely distributive Q-category. Since $\mathcal{P}^\dagger$ distributes over $\mathcal{T}$, it follows from Lemma 5.3 that the inclusion $\mathcal{T}A \longrightarrow \mathcal{P}A$ has a left adjoint. Then, the composite of the left adjoint of $\mathsf{s}_A : \mathcal{P}A \longrightarrow A$ with the left adjoint of the inclusion $\mathcal{T}A \longrightarrow \mathcal{P}A$ is a left adjoint of $\sup_A : \mathcal{T}A \longrightarrow A$, so A is $\mathcal{T}$-continuous. ∎

5.4. Remark. Putting $\mathcal{T} = \mathcal{P}$ in Theorem 5.2 one obtains that $\mathcal{P}^\dagger$ distributes over $\mathcal{P}$, as has already been pointed out in [19, 31].

5.5. Proposition. *If $\mathcal{T}$ is a saturated class of weights over which $\mathcal{P}^\dagger$ distributes, then for each separated Q-category A, the following statements are equivalent:*

(1) *A is a $\widetilde{\mathcal{T}}$-algebra, where $\widetilde{\mathcal{T}}$ is the lifting of $\mathcal{T}$ through the forgetful functor $\mathsf{Q}\text{-}\mathsf{Inf}\longrightarrow\mathsf{Q}\text{-}\mathsf{Cat}$.*

(2) *A is a $\mathcal{T}\mathcal{P}^\dagger$-algebra.*

(3) *A is a complete and $\mathcal{T}$-continuous Q-category.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is a special case of a general result in category theory, see e.g. [9, II.3.8.4]. It remains to check (1) $\Leftrightarrow$ (3). If A is a $\widetilde{\mathcal{T}}$-algebra, then $\mathsf{e}_A : A\longrightarrow\mathcal{T}A$ has a left adjoint $\sup_A : \mathcal{T}A \longrightarrow A$ which is also a morphism in $\mathsf{Q}\text{-}\mathsf{Inf}$. This means that $\sup_A$ has a left adjoint, so A is $\mathcal{T}$-continuous. Conversely, let A be a complete and $\mathcal{T}$-continuous Q-category. Then, $\mathcal{T}A$ is complete by Lemma 5.3, so the string of adjoint Q-functors

$$\mathsf{t}_A \dashv \sup_A \dashv \mathsf{e}_A : A \longrightarrow \mathcal{T}A,$$

ensures that $\sup_A$ is a morphism in $\mathsf{Q}\text{-}\mathsf{Inf}$ and consequently, A is a $\widetilde{\mathcal{T}}$-algebra. ■

5.6. Corollary. *For each saturated class of weights $\mathcal{T}$ on $\mathsf{Q}\text{-}\mathsf{Cat}$, let $\mathcal{T}\text{-}\mathsf{Cont}$ denote the category that has as objects complete and $\mathcal{T}$-continuous Q-categories and has as morphisms those Q-functor that are right adjoints and $\mathcal{T}$-homomorphisms. If $\mathcal{P}^\dagger$ distributes over $\mathcal{T}$, then $\mathcal{T}\text{-}\mathsf{Cont}$ is monadic over $\mathsf{Q}\text{-}\mathsf{Cat}$.*

6. The main result

A continuous t-norm [15] is a continuous map $\& : [0,1]^2 \longrightarrow [0,1]$ that makes $([0,1], \&, 1)$ into a commutative quantale. Given a continuous t-norm $\&$, the quantale $\mathsf{Q} = ([0,1], \&, 1)$ is clearly integral and continuous. We record here a simple fact for later use: for any x and y in $[0,1]$, x is way below y (i.e., $x \ll y$) if either $x = 0$ or $x < y$.

6.1. Example. Some basic continuous t-norms and their implications:

(1) The Gödel t-norm:

$$x \,\&\, y = \min\{x, y\}, \quad x \to y = \begin{cases} 1, & x \le y, \\ y, & x > y. \end{cases}$$

The implication $\to$ of the Gödel t-norm is continuous except at (x, x), $x < 1$.

(2) The product t-norm:

$$x \mathbin{\&_P} y = xy, \quad x \to y = \begin{cases} 1, & x \leq y, \\ y/x, & x > y. \end{cases}$$

The implication $\to$ of the product t-norm is continuous except at $(0,0)$. The quantale $([0,1], \&_P, 1)$ is isomorphic to Lawvere's quantale $([0,\infty]^{\mathrm{op}}, +, 0)$ [21].

(3) The Łukasiewicz t-norm:

$$x \mathbin{\&_{\L}} y = \max\{0, x+y-1\}, \quad x \to y = \min\{1-x+y, 1\}.$$

The implication $\to$ of the Łukasiewicz t-norm is continuous on $[0,1]^2$.

Let $\&$ be a continuous t-norm. An element $p \in [0,1]$ is said to be idempotent if $p \mathbin{\&} p = p$.

6.2. Proposition. ([15, Proposition 2.3]) *Let $\&$ be a continuous t-norm on $[0,1]$ and p be an idempotent element of $\&$. Then $x \mathbin{\&} y = \min\{x,y\}$ whenever $x \leq p \leq y$.*

It follows immediately that $y \to x = x$ whenever $x < p \leq y$ for some idempotent p. Another consequence of Proposition 6.2 is that for any idempotent elements p, q with $p < q$, the restriction of $\&$ to $[p,q]$, which is also denoted by $\&$, makes $[p,q]$ into a commutative quantale with q being the unit element. The following theorem is of fundamental importance in the theory of continuous t-norms.

6.3. Theorem. ([15, 24]) *Let $\&$ be a continuous t-norm. If $a \in [0,1]$ is non-idempotent, then there exist idempotent elements $a^-, a^+ \in [0,1]$ such that $a^- < a < a^+$ and that the quantale $([a^-, a^+], \&, a^+)$ is isomorphic to either $([0,1], \&_{\L}, 1)$ or $([0,1], \&_P, 1)$. Conversely, for each set of disjoint open intervals $\{(a_n, b_n)\}_n$ of $[0,1]$, the binary operator*

$$x \mathbin{\&} y := \begin{cases} a_n + (b_n - a_n) T_n\Big(\dfrac{x-a_n}{b_n-a_n}, \dfrac{y-a_n}{b_n-a_n}\Big), & (x,y) \in [a_n, b_n]^2, \\ \min\{x,y\}, & \text{otherwise} \end{cases}$$

is a continuous t-norm, where each T_n is a continuous t-norm on $[0,1]$.

Let $\&$ be a continuous t-norm and $\mathsf{Q} = ([0,1], \&, 1)$. Then from Proposition 4.3 we obtain that the class $\mathcal{C}$ of forward Cauchy weights is saturated. Now we present the main result in this paper.

6.4. Theorem. *Let $\&$ be a continuous t-norm and $\mathsf{Q} = ([0,1], \&, 1)$. Then the following statements are equivalent:*

(1) *Every completely distributive Q-category is continuous.*

(2) *The Q-category $([0,1], d_L)$ is continuous.*

(3) *For each non-idempotent element* $a \in [0,1]$*, the quantale* $([a^-, a^+], \&, a^+)$ *is isomorphic to* $([0,1], \&_P, 1)$ *whenever* $a^- > 0$.

(4) *The implication* $\rightarrow\colon [0,1]^2 \longrightarrow [0,1]$ *is continuous at every point off the diagonal* $\{(x,x) \mid x \in [0,1]\}$.

(5) *For each* $p \in (0,1]$*, the map* $p \rightarrow - : [0,1] \longrightarrow [0,1]$ *is Scott continuous on* $[0,p)$.

(6) *For every complete* Q*-category* A*, the inclusion* $\mathcal{C}A \longrightarrow \mathcal{P}A$ *has a left adjoint.*

(7) $\mathcal{P}^\dagger$ *distributes over* $\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) By Proposition 4.11, if the Q-category $([0,1], d_L)$ is continuous, then for each $x \in [0,1]$ and each forward Cauchy weight ϕ of $([0,1], d_L)$,

$$x \rightarrow \sup\phi = \bigwedge_{y \ll x} \phi(y).$$

Now, suppose on the contrary that there exist idempotent elements $p, q > 0$ such that $([p,q], \&, q)$ is isomorphic to $([0,1], \&_{\L}, 1)$. Let ϕ be the forward Cauchy weight $\bigvee_{r<p} \mathsf{y}(r)$. Then for all $x \in (p,q)$,

$$\begin{aligned} \bigwedge_{y \ll x} \phi(y) &= \bigwedge_{y<x} \bigvee_{r<p} (y \rightarrow r) && (x > 0) \\ &= \bigwedge_{p \leq y < x} \bigvee_{r<p} (y \rightarrow r) && (y \rightarrow r = 1 \text{ whenever } y \leq r) \\ &= \bigvee_{r<p} r && (y \rightarrow r = r \text{ since } r < p \leq y) \\ &= p. \end{aligned}$$

But, since $([p,q], \&, q)$ is isomorphic to $([0,1], \&_{\L}, 1)$, it follows that

$$x \rightarrow \sup\phi = x \rightarrow p > p$$

whenever $p < x < q$, a contradiction.

(3) $\Rightarrow$ (4) Routine verification.

(4) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (6) It suffices to show that for every complete Q-category A, $\mathcal{C}A$ is closed in $\mathcal{P}A$ under meets and cotensors. That $\mathcal{C}A$ is closed in $\mathcal{P}A$ under meets is ensured by Lemma 4.10. To see that $\mathcal{C}A$ is closed in $\mathcal{P}A$ under cotensors, for $p \in [0,1]$ and $\phi \in \mathcal{C}A$, set

$$D := \{d \in A \mid p \leq \phi(d)\}.$$

Then D is a directed set of $(A,\sqsubseteq)$. We claim that $\{p \multimap y \mid y\in\Gamma(\phi)\}\subseteq D$. In fact, since $\phi=\bigvee_{z\in\Gamma(\phi)}A(-,z)$, it follows that for all $y\in\Gamma(\phi)$,

$$\begin{aligned} p\rightarrow\phi(p\multimap y) &= p\rightarrow\bigvee_{z\in\Gamma(\phi)}A(p\multimap y,z)\\ &\geq p\rightarrow A(p\multimap y,y)\\ &= A(p\multimap y,p\multimap y)\\ &=1, \end{aligned}$$

hence $p\multimap y\in D$.

Let

$$\rho:=\bigvee_{d\in D}A(-,d).$$

Since ρ is a forward Cauchy weight, it suffices to show that $p\rightarrow\phi=\rho$. That $\rho\leq p\rightarrow\phi$ is clear. It remains to check that $p\rightarrow\phi(x)\leq\rho(x)$ for all $x\in A$. If $p\leq\phi(x)$, then $x\in D$ and

$$\rho(x)=\bigvee_{d\in D}A(x,d)\geq A(x,x)=1.$$

If $p>\phi(x)$, then

$$\begin{aligned} p\rightarrow\phi(x) &= p\rightarrow\bigvee_{y\in\Gamma(\phi)}A(x,y)\\ &=\bigvee_{y\in\Gamma(\phi)}(p\rightarrow A(x,y)) && \Big(\bigvee_{y\in\Gamma(\phi)}A(x,y)=\phi(x)<p\Big)\\ &=\bigvee_{y\in\Gamma(\phi)}A(x,p\multimap y)\\ &\leq\bigvee_{d\in D}A(x,d)\\ &=\rho(x). \end{aligned}$$

(6) $\Rightarrow$ (7) Lemma 5.3.

(7) $\Rightarrow$ (1) Theorem 5.2. ■

By the ordinal sum decomposition theorem, the map $\&:[0,1]^2\longrightarrow[0,1]$, given by

$$x\,\&\,y:=\begin{cases}1/2+\max\{x+y-3/2,0\}, & (x,y)\in[1/2,1]^2,\\ \min\{x,y\}, & \text{otherwise},\end{cases}$$

is a continuous t-norm. Since the restriction of $\&$ on $[1/2,1]$ is isomorphic to the Łukasiewicz t-norm, it follows that for the quantale $\mathsf{Q}=([0,1],\&,1)$, the completely distributive Q-category $([0,1],d_L)$ is not continuous.

6.5. Remark. A continuous t-norm & is said to be Archimedean if it has no idempotent elements other than 0 and 1 [15]. It is known that an Archimedean continuous t-norm is isomorphic to either the product t-norm or the Łukasiewicz t-norm. If $\mathsf{Q} = ([0,1], \&, 1)$ with & being a continuous Archimedean t-norm, then the converse conclusion of Proposition 4.12 is also true. That means, if a complete Q-category A is continuous, then for each $p \in [0,1]$, the map

$$p \multimap - : (A, \sqsubseteq) \longrightarrow (A, \sqsubseteq)$$

is Scott continuous. Given a directed set D of $(A, \sqsubseteq)$, let

$$\phi := \bigvee_{d \in D} A(-, d).$$

Since & is Archimedean, the map

$$p \rightarrow - : [0,1] \longrightarrow [0,1]$$

is continuous for all $p \in [0,1]$. It follows from the argument of (5) $\Rightarrow$ (6) in Theorem 6.4 that $p \rightarrow \phi \in \mathcal{C}A$ and it is the cotensor of p with ϕ in $\mathcal{C}A$. Therefore,

$$\begin{aligned} p \multimap \bigvee D &= p \multimap \sup{}_A \phi \\ &= \sup{}_A (p \rightarrow \phi) && (\sup{}_A \text{ is a right adjoint}) \\ &= \sup{}_A \bigvee_{d \in D} (p \rightarrow A(-, d)) && (p \rightarrow - \text{ is continuous}) \\ &= \sup{}_A \bigvee_{d \in D} A(-, p \multimap d) && (\text{definition of cotensor}) \\ &= \bigvee_{d \in D} p \multimap d. \end{aligned}$$

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