Theory and Applications of Categories, Vol. 34, No. 28, 2019, pp. 854–882.

ON SPANS WITH RIGHT FIBRED RIGHT ADJOINTS

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Abstract. We introduce a new condition on an abstract span of categories which we refer to as having right fibred right adjoints, RFRA for short. We show that:

(a) the *span of split extensions* of a semi-abelian category $\mathbb{C}$ has RFRA if and only if $\mathbb{C}$ is action representable;

(b) the *reversed span* to the one considered in (a) has RFRA if and only if $\mathbb{C}$ is locally algebraically cartesian closed;

(c) the span of split extensions of the category of morphisms of $\mathbb{C}$ has RFRA if and only if $\mathbb{C}$ is action representable and has normalizers;

(d) the reversed span to the one considered in (c) has RFRA if and only if $\mathbb{C}$ is locally algebraically cartesian closed.

We also examine our condition for the span of monoid actions (of monoids in a monoidal category $\mathbf{C}$ on objects in a given category on which $\mathbf{C}$ acts), and for various other spans.

1. Introduction

It is well known that the category **Grp** of groups is very far from being cartesian closed. Nevertheless it admits two constructions that can be seen as special types of exponents. Specifically, given a group G one can form:

(a) the automorphism group $\mathrm{Aut}(G)$;

(b) for any group X the G-group X^G.

Each of these two constructions can be defined purely-categorically, and their existence in **Grp** is expressed by saying that **Grp** is action representable [5] and is locally algebraically cartesian closed ([13], [14], [8]), respectively. There are also a few other examples of categories that are action representable ([5], [6], [3], [4]) and of categories which are locally algebraically cartesian closed ([14], [15], [8]).

The main purpose of this paper is threefold:

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number 109203)

Received by the editors 2018-06-28 and, in final form, 2019-09-17.

Transmitted by Tim Van der Linden. Published on 2019-09-20.

2010 Mathematics Subject Classification: 18A05, 18A22, 18A25, 18A40, 18A99, 18B99, 18D99.

Key words and phrases: action representable, locally algebraically cartesian closed, semi-abelian, split extension, normalizer, prefibration, right fibred right adjoints, regular span.

I. To show that action representability and locally-cartesian-closedness, expressed in the language of the span of split extensions, in a given semi-abelian category $\mathbb{C}$, become symmetric to each other. More precisely, we introduce a condition, RFRA from the Abstract, on an abstract span $\underline{\mathbf{S}}$, so that:

(a) when $\underline{\mathbf{S}}$ is

$$\mathbb{C} \xleftarrow{\text{codomain}} \text{Split extensions in } \mathbb{C} \xrightarrow{\text{kernel}} \mathbb{C} \tag{1.1}$$

it has RFRA if and only if $\mathbb{C}$ is action representable;

(b) when $\underline{\mathbf{S}}$ is the reversed span

$$\mathbb{C} \xleftarrow{\text{kernel}} \text{Split extensions in } \mathbb{C} \xrightarrow{\text{codomain}} \mathbb{C} \tag{1.2}$$

it has RFRA if and only if $\mathbb{C}$ is locally algebraically cartesian closed.

II. To replace $\mathbb{C}$ with its category $\mathbb{C}^{\mathbf{2}}$ of morphisms and show that:

(c) when $\underline{\mathbf{S}}$ is

$$\mathbb{C}^{\mathbf{2}} \xleftarrow{\text{codomain}} \text{Split extensions in } \mathbb{C}^{\mathbf{2}} \xrightarrow{\text{kernel}} \mathbb{C}^{\mathbf{2}} \tag{1.3}$$

it has RFRA if and only if $\mathbb{C}$ is action representable and has normalizers;

(d) when $\underline{\mathbf{S}}$ is the reversed span

$$\mathbb{C}^{\mathbf{2}} \xleftarrow{\text{kernel}} \text{Split extensions in } \mathbb{C}^{\mathbf{2}} \xrightarrow{\text{codomain}} \mathbb{C}^{\mathbf{2}} \tag{1.4}$$

it has RFRA if and only if $\mathbb{C}$ is locally algebraically cartesian closed.

In addition to these facts, which essentially follow from I(a) and I(b) together with Theorem 4.8 of [16] and Proposition 4.8 below, we show that:

(e) $\mathbb{C}$ is action representable and has normalizers if and only if for each split extension, and each pair of morphisms θ and ϕ as displayed using solid arrows in the diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\underline{\kappa}"] \arrow[d, equal] & \underline{A} \arrow[r, dashed, shift left, "\underline{\alpha}"] \arrow[dd, dashed, "\underline{\psi}"] & \underline{B} \arrow[l, dashed, shift left, "\underline{\beta}"] \arrow[d, dashed, "g"] \\
X \arrow[d, "\theta"'] & & B \arrow[d, "\phi"] \\
Z \arrow[r, "\sigma"] & C \arrow[r, shift left, "\gamma"] & D, \arrow[l, shift left, "\delta"]
\end{tikzcd} \tag{1.5}$$

there exists a universal lifting as displayed with dotted arrows;

(f) $\mathbb{C}$ is locally algebraically cartesian closed if and only if for each split extension, and each pair of morphisms θ and ϕ as displayed using solid arrows in the diagram

$$\begin{tikzcd}
\bar{X} \arrow[r, dashed, "\bar{\kappa}"] \arrow[d, dashed, "e"'] & \bar{A} \arrow[r, dashed, shift left, "\bar{\alpha}"] \arrow[dd, dashed, "\bar{\psi}"] & B \arrow[l, dashed, shift left, "\bar{\beta}"] \arrow[d, equal] \\
X \arrow[d, "\theta"'] & & B \arrow[d, "\phi"] \\
Z \arrow[r, "\sigma"] & C \arrow[r, shift left, "\gamma"] & D \arrow[l, shift left, "\delta"]
\end{tikzcd} \tag{1.6}$$

there exists a universal lifting as displayed with dotted arrows.

III. To examine having RFRA for various other spans, namely the following ones:

$$\mathbb{C} \xleftarrow{\ \otimes\ } \mathbb{C} \times \mathbb{C} \xrightarrow{\ \pi_2\ } \mathbb{C} \tag{1.7}$$

where $\mathbb{C}$ and $\otimes$ are part of the structure of a monoidal category $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ and π_2 is the second projection;

$$\mathbb{C} \xleftarrow{\ \mathrm{dom}\ } \mathbb{C}^{\mathbf{2}} \xrightarrow{\ \mathrm{cod}\ } \mathbb{C} \tag{1.8}$$

where $\mathbb{C}$ is an arbitrary category, and dom and cod are the functors sending a morphism to its domain and codomain, respectively;

$$\mathbb{C} \xleftarrow{\ P\ } \mathrm{Span}(\mathbb{C}) \xrightarrow{\ Q\ } \mathbb{C} \tag{1.9}$$

where $\mathrm{Span}(\mathbb{C})$ is the category of spans in an arbitrary category $\mathbb{C}$, and P and Q are the functors sending a span $A \xleftarrow{p} S \xrightarrow{q} B$ to A and B, respectively;

$$\mathbf{Mon}(\mathbf{C}) \xleftarrow{\ P\ } \mathbf{Act}(\mathcal{A}) \xrightarrow{\ Q\ } \mathbb{X} \tag{1.10}$$

where $\mathbf{Mon}(\mathbf{C})$ is the category of monoids in a monoidal category $\mathbf{C}$, $\mathbf{Act}(\mathcal{A})$ is the category of monoid actions defined with respect to an action $\mathcal{A}$ of $\mathbf{C}$ on $\mathbb{X}$, and P and Q are forgetful functors; the various "dual" spans of 1.8, 1.9 and 1.10

The paper is organized as follows. First we introduce the condition of having RFRA in Section 2. After that examining what it means for a span $\underline{\mathbf{S}}$ to have RFRA, for various induced or special spans, we consider the cases of:

- several "dual" spans $\underline{\mathbf{S}}^{\mathrm{op}}$, $\underline{\mathbf{S}}_{\circ}$, etc. (Section 2);
- "functor span" $\underline{\mathbf{S}}^{\mathbb{I}}$ (Section 3);
- $\underline{\mathbf{S}}$ being right regular (Section 4);
- $\underline{\mathbf{S}}$ being the span (1.10) (Section 5, where it is denoted by $\underline{\mathbf{Act}(\mathcal{A})}$);
- $\underline{\mathbf{S}}$ being the span of split extensions in a pointed finitely complete category $\mathbb{C}$ under various conditions on $\mathbb{C}$ (Section 6, where it is denoted by $\underline{\mathbf{SE}(\mathbb{C})}$).

2. Right fibred right adjoints

In this section we introduce and study several conditions on a span of categories. Before we do so let us introduce some notation and terminology.

Let $F : \mathbb{C} \to \mathbb{X}$ be a functor. For an object X in $\mathbb{X}$ we will denote by $F^{-1}(X)$, the fibre of F above X, that is the subcategory of $\mathbb{C}$ consisting of those objects and morphisms which are mapped by F to X and 1_X, respectively. For a span of categories $\underline{\mathbf{S}} =$

$$\mathbb{A} \xleftarrow{P} \mathbb{S} \xrightarrow{Q} \mathbb{B} \tag{2.1}$$

and for each A in $\mathbb{A}$ and B in $\mathbb{B}$, we will write $Q_A : P^{-1}(A) \to \mathbb{B}$ and $P_B : Q^{-1}(B) \to \mathbb{A}$ for the composite of the inclusion of $P^{-1}(A)$ in $\mathbb{S}$ with Q, and the composite of the inclusion of $Q^{-1}(B)$ in $\mathbb{S}$ with P, respectively.

2.1. Definition. *We will say that the span* $\underline{\mathbf{S}}$ *has* right fibred right adjoints (RFRA) *if for each* B *in* $\mathbb{B}$ *the functor* $P_B : Q^{-1}(B) \to \mathbb{A}$ *has a right adjoint.*

For the span of categories $\underline{\mathbf{S}}$ there are three *dual* spans $\underline{\mathbf{S}}^{\mathrm{op}} =$

$$\mathbb{A}^{\mathrm{op}} \xleftarrow{P^{\mathrm{op}}} \mathbb{S}^{\mathrm{op}} \xrightarrow{Q^{\mathrm{op}}} \mathbb{B}^{\mathrm{op}}, \tag{2.2}$$

$\underline{\mathbf{S}}_\circ =$

$$\mathbb{B} \xleftarrow{Q} \mathbb{S} \xrightarrow{P} \mathbb{A} \tag{2.3}$$

and $\underline{\mathbf{S}}_\circ^{\mathrm{op}} =$

$$\mathbb{B}^{\mathrm{op}} \xleftarrow{Q^{\mathrm{op}}} \mathbb{S}^{\mathrm{op}} \xrightarrow{P^{\mathrm{op}}} \mathbb{A}^{\mathrm{op}}. \tag{2.4}$$

Translating what it means for the spans $\underline{\mathbf{S}}^{\mathrm{op}}$, $\underline{\mathbf{S}}_\circ$, and $\underline{\mathbf{S}}_\circ^{\mathrm{op}}$ to have RFRA into the language of $\underline{\mathbf{S}}$, we will say that the span $\underline{\mathbf{S}}$ has

- right fibred left adjoints RFLA, when for each B in $\mathbb{B}$ the functor $P_B : Q^{-1}(B) \to \mathbb{A}$ has a left adjoint;
- left fibred right adjoints LFRA, when for each A in $\mathbb{A}$ the functor $Q_A : P^{-1}(A) \to \mathbb{B}$ has a right adjoint;
- left fibred left adjoints LFLA, when for each A in $\mathbb{A}$ the functor $Q_A : P^{-1}(A) \to \mathbb{B}$ has a left adjoint.

Let us begin by seeing how the condition of having RFRA fits in with some general conditions in category theory. It is easy to observe that:

2.2. Remark. *For the span $\underline{\mathbf{S}}$:*

(i) if $\mathbb{A} = \mathbf{1}$ is the one object and one morphism category, then $\underline{\mathbf{S}}$ having RFRA is equivalent to the functor Q having terminal objects in its fibres;

(ii) if $\mathbb{A}$ has a terminal object, then $\underline{\mathbf{S}}$ having RFRA implies that the functor Q has terminal objects in its fibres;

(iii) if $\mathbb{B} = \mathbf{1}$, then $\underline{\mathbf{S}}$ having RFRA is equivalent to the functor P having a right adjoint.

For the span $\underline{\mathbf{S}}$ we will write $\pi_1 : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{S}$, $\pi_2 : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{A}$, $\pi_1 : \mathbb{B} \times \mathbb{A} \to \mathbb{B}$ and $\pi_2 : \mathbb{B} \times \mathbb{A} \to \mathbb{A}$ for the respective projections of the categories $(P \downarrow 1_{\mathbb{A}})$ and $\mathbb{B} \times \mathbb{A}$. We will denote by

$$\langle Q\pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{B} \times \mathbb{A} \tag{2.5}$$

the unique functor with $\pi_1\langle Q\pi_1, \pi_2\rangle = Q\pi_1$ and $\pi_2\langle Q\pi_1, \pi_2\rangle = \pi_2$. It straightforward to check that:

2.3. Lemma. *For the span of categories $\underline{\mathbf{S}}$ and for each A in $\mathbb{A}$ and B in $\mathbb{B}$, the categories $(P_B \downarrow A)$ and $\langle Q\pi_1, \pi_2\rangle^{-1}(B, A)$ are isomorphic.*

As an immediate corollary we see that having RFRA can be thought of as a certain functor having terminal objects in its fibres.

2.4. Corollary. *The span $\underline{\mathbf{S}}$ has RFRA if and only if the functor $\langle Q\pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{B} \times \mathbb{A}$ has terminal objects in its fibres.*

Combining Remark 2.2 (iii) and Corollary 2.4 we obtain the following simple fact:

2.5. Proposition. *A functor $F : \mathbb{C} \to \mathbb{X}$ has a right adjoint if and only if the second projection $\pi_2 : (F \downarrow 1_{\mathbb{X}}) \to \mathbb{X}$ has terminal objects in its fibres.*

We will also need the following fact:

2.6. Proposition. *A functor $F : \mathbb{C} \to \mathbb{X}$ is a prefibration if and only if for each B in $\mathbb{C}$ the functor $F^B : (\mathbb{C} \downarrow B) \to (\mathbb{X} \downarrow F(B))$, which sends (A, f) to $(F(A), F(f))$, has terminal objects in its fibres.*

To end this section we will give some examples of spans having RFRA which will not be of central interest in the rest of the paper.

2.7. Proposition. *Let $\mathbf{C} = (\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ be a monoidal category. The span*

$$\mathbb{C} \xleftarrow{\otimes} \mathbb{C} \times \mathbb{C} \xrightarrow{\pi_2} \mathbb{C} \tag{2.6}$$

has RFRA if and only if $\mathbf{C}$ is (left) monoidal closed.

Proof. The statement of the proposition follows from the fact that for each B in $\mathbb{C}$ the functor $\otimes_B : \pi_2^{-1}(B) \to \mathbb{C}$ is essentially the same as $- \otimes B : \mathbb{C} \to \mathbb{C}$. ∎

Throughout the rest of the paper we will denote by $\mathbf{2}$ the category with two objects 0 and 1 and one non-identity morphism $0 \to 1$. For a category $\mathbb{C}$ we will identify the category $\mathbb{C}^{\mathbf{2}}$ with the category of morphisms in $\mathbb{C}$ and write its objects as triples (A, B, f) where A and B are objects and $f : A \to B$ is a morphism in $\mathbb{C}$. A morphism $(A, B, f) \to (A', B', f')$ in $\mathbb{C}^{\mathbf{2}}$ will be written as a pair (u, v) where $u : A \to A'$ and $v : B \to B'$ are morphisms in $\mathbb{C}$ with $f'u = vf$.

Throughout the rest of this section let $\mathbb{C}$ be a fixed category. Let us consider the span $\underline{\mathrm{Mor}(\mathbb{C})} =$

$$\mathbb{C} \xleftarrow{\mathrm{dom}} \mathbb{C}^{\mathbf{2}} \xrightarrow{\mathrm{cod}} \mathbb{C} \tag{2.7}$$

where dom and cod are the domain and codomain functors.

Since for each A and B in $\mathbb{C}$ the category $(\mathrm{dom}_B \downarrow A)$ is isomorphic to the category of cones over the discrete diagram consisting of A and B we obtain:

2.8. Proposition. *The span* $\underline{\mathrm{Mor}(\mathbb{C})}$ *has* RFRA *if and only if* $\mathbb{C}$ *has binary products.*

On the other hand.

2.9. Proposition. *The span* $\underline{\mathrm{Mor}(\mathbb{C})}$ *has* LFRA *if and only if* $\mathbb{C}$ *is indiscrete (i.e. there is a unique morphism between any two objects).*

Proof. Trivially if there is a unique morphism $f : A \to B$ for each A and B in $\mathbb{C}$, then the terminal object in $(\mathrm{cod}_A \downarrow B)$ is $((A, B, f), 1_B)$. Conversely, suppose that $((A, \bar{B}, f), e)$ is the terminal object in $(\mathrm{cod}_A \downarrow B)$. For each morphism $g : A \to B$ there exists a unique morphism $u : B \to \bar{B}$ making the two squares in the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, equal] & B \arrow[r, "1_B"] \arrow[d, "u"] & B \arrow[d, equal] \\ A \arrow[r, "f"'] & \bar{B} \arrow[r, "e"'] & B \end{tikzcd} \tag{2.8}$$

commute. This however means that $g = eug = ef$ and so $\hom(A, B) = \{ef\}$. ∎

Applying Propositions 2.8 and 2.9 to $\mathbb{C}^{\mathrm{op}}$, and using the fact that there is an isomorphism (the vertical arrow below) making the diagram

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[d, equal] & (\mathbb{C}^{\mathbf{2}})^{\mathrm{op}} \arrow[l, "\mathrm{cod}^{\mathrm{op}}"'] \arrow[r, "\mathrm{dom}^{\mathrm{op}}"] \arrow[d] & \mathbb{C}^{\mathrm{op}} \arrow[d, equal] \\ \mathbb{C}^{\mathrm{op}} & (\mathbb{C}^{\mathrm{op}})^{\mathbf{2}} \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & \mathbb{C}^{\mathrm{op}} \end{tikzcd} \tag{2.9}$$

commute, we obtain:

2.10. Proposition.

1. *The span* $\underline{\mathrm{Mor}(\mathbb{C})}$ *has* LFLA *if and only if* $\mathbb{C}$ *has binary coproducts.*

2. *The span* $\underline{\mathrm{Mor}(\mathbb{C})}$ *has* RFLA *if and only if* $\mathbb{C}$ *is indiscrete.*

Next we consider the span $\underline{\mathrm{Span}(\mathbb{C})} =$

$$\mathbb{C} \xleftarrow{P} \mathrm{Span}(\mathbb{C}) \xrightarrow{Q} \mathbb{C} \tag{2.10}$$

where $\mathrm{Span}(\mathbb{C})$ is the category of spans in $\mathbb{C}$, and P and Q are the functors sending a span

$$A \xleftarrow{p} S \xrightarrow{q} B \tag{2.11}$$

to A and B, respectively. Note that for objects A and B in $\mathbb{C}$ an object in $(P_B \downarrow A)$ will be represented by a diagram

$$A \xleftarrow{e} \tilde{A} \xleftarrow{p} S \xrightarrow{q} B$$

in $\mathbb{C}$ with the expected interpretation. It is easy to observe that:

2.11. Lemma. *The functor sending each span* $A \xleftarrow{p} S \xrightarrow{q} B$ *in* $\mathrm{Span}(\mathbb{C})$ *to its reverse span* $B \xleftarrow{q} S \xrightarrow{p} A$ *is an isomorphism making the diagram*

$$\begin{tikzcd} \mathbb{C} \arrow[d, equal] & \mathrm{Span}(\mathbb{C}) \arrow[l, "P"'] \arrow[r, "Q"] \arrow[d] & \mathbb{C} \arrow[d, equal] \\ \mathbb{C} & \mathrm{Span}(\mathbb{C}) \arrow[l, "Q"] \arrow[r, "P"'] & \mathbb{C} \end{tikzcd} \tag{2.12}$$

commute.

We obtain:

2.12. Proposition. *The following conditions are equivalent:*

(a) the span $\underline{\mathrm{Span}(\mathbb{C})}$ *has* RFRA*;*

(b) the span $\underline{\mathrm{Span}(\mathbb{C})}$ *has* LFRA*;*

(c) $\mathbb{C}$ *has binary products.*

PROOF. The equivalence of (a) and (b) follows from Lemma 2.11. On the other hand, it is easy to check that if $\mathbb{C}$ has binary products, then for A and B in $\mathbb{C}$ the diagram

$$A \xleftarrow{1_A} A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B \tag{2.13}$$

is the terminal object in $(P_B \downarrow A)$. Now suppose that $\underline{\mathrm{Span}(\mathbb{C})}$ has RFRA. We will show that $\mathbb{C}$ has binary products. If, for A and B in $\mathbb{C}$, the diagram

$$A \xleftarrow{e} \bar{A} \xleftarrow{p} S \xrightarrow{q} B \tag{2.14}$$

is the terminal object in $(P_B \downarrow A)$, then there exist unique morphisms u and v making the lower part (and hence the whole) of the diagram

$$\begin{tikzcd}
A \arrow[d, equal] & \bar{A} \arrow[l, "e"'] \arrow[d, "e"'] & S \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "1_S"] & B \arrow[d, equal] \\
A \arrow[d, equal] & A \arrow[l, "1_A"'] \arrow[d, "u"'] & S \arrow[l, "ep"'] \arrow[r, "q"] \arrow[d, "v"] & B \arrow[d, equal] \\
A & \bar{A} \arrow[l, "e"'] & S \arrow[l, "p"'] \arrow[r, "q"] & B
\end{tikzcd} \tag{2.15}$$

commute. It follows that e is an isomorphism and hence the diagram

$$A \xleftarrow{1_A} A \xleftarrow{ep} S \xrightarrow{q} B \tag{2.16}$$

is also a terminal object in $(P_B \downarrow A)$. One easily concludes that the span on the right in (2.16) is a product. ■

The following proposition which seems to be interesting in its own right, will be useful in analysing what having RFLA means for the span $\underline{\mathrm{Span}(\mathbb{C})}$.

2.13. PROPOSITION. *If $\mathbb{C}$ is non-empty, then the following conditions are equivalent:*

(a) the induced functor $\langle P, Q\rangle : \mathrm{Span}(\mathbb{C}) \to \mathbb{C} \times \mathbb{C}$, where P and Q are the functors forming part of $\underline{\mathrm{Span}(\mathbb{C})}$, has initial objects in its fibres, i.e. for each pair of objects A and B in $\mathbb{C}$ the category of cones over the discrete diagram consisting of A and B has an initial object;

(b) $\mathbb{C}$ has an initial object.

PROOF. If $\mathbb{C}$ has an initial object, then trivially

$$A \longleftarrow 0 \longrightarrow B \tag{2.17}$$

is the initial object in $\langle P, Q\rangle^{-1}(A, B)$. This proves that (b) $\Rightarrow$ (a). If $\mathbb{C}$ has a terminal object 1, then $\langle P, Q\rangle^{-1}(1, 1) \cong \mathbb{C}$ and hence (a) implies (b). However we will see that this implication holds even if $\mathbb{C}$ doesn't have a terminal object. Suppose that the functor $\langle P, Q\rangle$ has initial objects in its fibres, and for some A in $\mathbb{C}$ let

$$A \xleftarrow{p} S \xrightarrow{q} A \tag{2.18}$$

be an initial object in $\langle P, Q\rangle^{-1}(A, A)$. We will show that S is an initial object in $\mathbb{C}$. For an object B in $\mathbb{C}$, let

$$B \xleftarrow{r} T \xrightarrow{s} A \tag{2.19}$$

be an initial object in $\langle P, Q\rangle^{-1}(B, A)$. Accordingly, there is a unique morphism making the diagram

$$\begin{tikzcd} A \arrow[d, equal] & S \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\theta"] & A \arrow[d, equal] \\ A & T \arrow[l, "s"] \arrow[r, "s"'] & A \end{tikzcd} \tag{2.20}$$

commute, meaning that $p = q = s\theta$. Furthermore, since $r\theta$ is a morphism from S to B, and B was arbitrary, S is a weak initial object in $\mathbb{C}$. Now suppose that $f, g : S \to B$ are arbitrary morphisms in $\mathbb{C}$. By assumption there is a unique morphism from the initial object in $\langle P, Q\rangle^{-1}(B, B)$ as shown in the diagram

$$\begin{tikzcd} B \arrow[d, equal] & S' \arrow[l, "p'"'] \arrow[r, "p'"] \arrow[d, "v"] & B \arrow[d, equal] \\ B & S \arrow[l, "f"] \arrow[r, "g"'] & B. \end{tikzcd} \tag{2.21}$$

However since S is initial in $\langle P, Q\rangle^{-1}(A, A)$ any morphism into it is a split epimorphism and hence $f = g$. ■

2.14. Proposition. *If $\mathbb{C}$ is non-empty, then the following conditions are equivalent:*

(a) the span $\underline{\mathrm{Span}(\mathbb{C})}$ has RFLA*;*

(b) the span $\underline{\mathrm{Span}(\mathbb{C})}$ has LFLA*;*

(c) $\mathbb{C}$ has an initial object.

Proof. As in Proposition 2.12 the equivalence of (a) and (b) follows from Lemma 2.11. On the other hand, it is easy to check that if $\mathbb{C}$ has an initial object 0, then for A and B in $\mathbb{C}$ the diagram

$$A \xrightarrow{1_A} A \longleftarrow 0 \longrightarrow B \tag{2.22}$$

is the initial object in $(A \downarrow P_B)$. Now suppose that $\underline{\mathrm{Span}(\mathbb{C})}$ has RFLA. We will show that $\langle P, Q\rangle : \mathrm{Span}(\mathbb{C}) \to \mathbb{C} \times \mathbb{C}$ has initial objects in its fibres. For A and B in $\mathbb{C}$, suppose the diagram

$$A \xrightarrow{\bar{n}} \bar{A} \xleftarrow{\bar{p}} \bar{S} \xrightarrow{\bar{q}} B \tag{2.23}$$

is the initial object in $(A \downarrow P_B)$. We will show that $\bar{n}$ is an isomorphism. To do so let

$$\begin{tikzcd} A \arrow[r, "\tilde{n}"] & \tilde{A} & \tilde{S} \arrow[l, "\tilde{p}"'] \arrow[r, "\tilde{q}"] & \bar{A} \end{tikzcd} \tag{2.24}$$

be the initial object in $(A \downarrow P_{\bar{A}})$. Accordingly, we obtain unique morphisms

$$\begin{tikzcd} A \arrow[r, "\tilde{n}"] \arrow[d, equal] & \tilde{A} \arrow[d, "u"] & \tilde{S} \arrow[l, "\tilde{p}"'] \arrow[r, "\tilde{q}"] \arrow[d, "v"] & \bar{A} \arrow[d, equal] \\ A \arrow[r, "\bar{n}"] & \bar{A} & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{p}"] & \bar{A} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\bar{n}"] \arrow[d, equal] & \bar{A} \arrow[d, "i"] & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{q}"] \arrow[d, "j"'] & B \arrow[d, equal] \\ A \arrow[r, "\tilde{n}"] & \tilde{A} & \tilde{S} \arrow[l, "\tilde{p}"'] \arrow[r, "\bar{q}v"] & B. \end{tikzcd} \tag{2.25}$$

It follows that the left hand diagram commutes

$$\begin{tikzcd} A \arrow[r, "\bar{n}"] \arrow[d, equal] & \bar{A} \arrow[d, "i"] & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{q}"] \arrow[d, "j"'] & B \arrow[d, equal] \\ A \arrow[r, "\tilde{n}"] \arrow[d, equal] & \tilde{A} \arrow[d, "u"'] & \tilde{S} \arrow[l, "\tilde{p}"'] \arrow[r, "\bar{q}v"] \arrow[d, "v"'] & B \arrow[d, equal] \\ A \arrow[r, "\bar{n}"] & \bar{A} & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{q}"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\tilde{n}"] \arrow[d, equal] & \tilde{A} \arrow[d, "u"] & \tilde{S} \arrow[l, "\tilde{p}"'] \arrow[r, "\tilde{q}"] \arrow[d, "v"] & \bar{A} \arrow[d, equal] \\ A \arrow[r, "\bar{n}"] \arrow[d, equal] & \bar{A} \arrow[d, "i"] & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{p}"] \arrow[d, "j"] & \bar{A} \arrow[d, equal] \\ A \arrow[r, "\tilde{n}"] & \tilde{A} & \tilde{S} \arrow[l, "\tilde{p}"'] \arrow[r, "\tilde{q}"] & \bar{A} \end{tikzcd} \tag{2.26}$$

and so we must have that $ui = 1_{\bar{A}}$ and $vj = 1_{\bar{S}}$. This means that $\tilde{q}j = \bar{p}vj = \bar{p}$ and hence the right hand diagram above commutes. It follows that $iu = 1_{\tilde{A}}$ and $jv = 1_{\tilde{S}}$, and so u and v are isomorphisms, and

$$\begin{tikzcd} A \arrow[r, "\bar{n}"] & \bar{A} & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{p}"] & \bar{A} \end{tikzcd} \tag{2.27}$$

is an initial object in $(A \downarrow P_{\bar{A}})$. Accordingly, there exist unique morphisms making the upper part of the diagram on the left

$$\begin{tikzcd} A \arrow[r, "\bar{n}"] \arrow[d, equal] & \bar{A} \arrow[d, "e"'] & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{p}"] \arrow[d, "f"] & \bar{A} \arrow[d, equal] \\ A \arrow[r, "1_A"] \arrow[d, equal] & A \arrow[d, "\bar{n}"'] & A \arrow[l, "1_A"'] \arrow[r, "\bar{n}"] \arrow[d, "\bar{n}"'] & \bar{A} \arrow[d, equal] \\ A \arrow[r, "\bar{n}"] & \bar{A} & \bar{A} \arrow[l, "1_{\bar{A}}"'] \arrow[r, "1_{\bar{A}}"] & \bar{A} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\bar{n}"] \arrow[dd, equal] & \bar{A} \arrow[dd, "1_{\bar{A}}"'] & \bar{S} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{p}"] \arrow[dd, "\bar{p}"] & \bar{A} \arrow[dd, equal] \\ & & & \\ A \arrow[r, "\bar{n}"] & \bar{A} & \bar{A} \arrow[l, "1_{\bar{A}}"'] \arrow[r, "1_{\bar{A}}"] & \bar{A} \end{tikzcd} \tag{2.28}$$

commute. This means that $e\bar{n} = 1_A$ and $\bar{n}f = \bar{p}$. Therefore since the entire diagram on the left as well as the diagram on the right commute it follows $\bar{n}e = 1_{\bar{A}}$ (and $\bar{n}f = \bar{p}$) so that $\bar{n}$ is an isomorphism, and

$$\begin{tikzcd} A \arrow[r, "1_A"] & A & \bar{S} \arrow[l, "\bar{n}^{-1}\bar{p}"'] \arrow[r, "\bar{q}"] & B \end{tikzcd} \tag{2.29}$$

is an initial object in $(A \downarrow P_B)$. From this it easily follows that for each pair of objects A and B in $\mathbb{C}$ the category $\langle P, Q\rangle^{-1}(A, B)$ has an initial object, and hence so does $\mathbb{C}$ by the previous proposition. ∎

Combining Propositions 2.12 and 2.14 we immediately obtain:

2.15. Corollary. *If $\mathbb{C}$ is non-empty, then the span $\underline{\mathrm{Span}(\mathbb{C})}$ and all its duals have* RFRA *if and only if $\mathbb{C}$ has binary products and an initial object.*

Using the fact that $\underline{\mathrm{CoSpan}(\mathbb{C})}$, the span of cospans in $\mathbb{C}$, can be defined by $\underline{\mathrm{CoSpan}(\mathbb{C})} = \underline{\mathrm{Span}(\mathbb{C}^{\mathrm{op}})}^{\mathrm{op}}$, translating Propositions 2.12 and 2.14, and Corollary 2.15 we obtain:

2.16. Corollary.

1. *The following conditions are equivalent:*
   - *(a) the span $\underline{\mathrm{CoSpan}(\mathbb{C})}$ has* RFLA*;*
   - *(b) the span $\underline{\mathrm{CoSpan}(\mathbb{C})}$ has* LFLA*;*
   - *(c) $\mathbb{C}$ has binary coproducts.*
2. *If $\mathbb{C}$ non-empty, then the following conditions are equivalent:*
   - *(a) the span $\underline{\mathrm{CoSpan}(\mathbb{C})}$ has* RFRA*;*
   - *(b) the span $\underline{\mathrm{CoSpan}(\mathbb{C})}$ has* LFRA*;*
   - *(c) $\mathbb{C}$ has a terminal object.*
3. *If $\mathbb{C}$ is non-empty, then the span $\underline{\mathrm{CoSpan}(\mathbb{C})}$ and all its duals have* RFRA *if and only if $\mathbb{C}$ has binary coproducts and a terminal object.*

3. Right fibred right adjoints for induced spans between functor categories

In this section we study the relationship between the span $\underline{\mathbf{S}}$ having RFRA and, for some category $\mathbb{I}$, the induced span $\underline{\mathbf{S}}^{\mathbb{I}} =$

$$\mathbb{A}^{\mathbb{I}} \xleftarrow{P^{\mathbb{I}}} \mathbb{S}^{\mathbb{I}} \xrightarrow{Q^{\mathbb{I}}} \mathbb{B}^{\mathbb{I}} \tag{3.1}$$

having RFRA.

The following example shows that $\underline{\mathbf{S}}$ having RFRA does not in general imply that $\underline{\mathbf{S}}^{\mathbb{I}}$ has RFRA.

3.1. Example. *For $\mathbb{I} = \mathbf{2}$, according to Remark 2.2 (i), it is sufficient to find a functor F with terminal objects in its fibres, but such that $F^{\mathbf{2}}$ has at least one fibre with no terminal object. It is easy to see that such functors exist, for instance if $\mathbb{C}$ is the discrete category with objects 0 and 1 and $F : \mathbb{C} \to \mathbf{2}$ is the functor which is identity on objects, then F has terminal objects in its fibres but the functor $F^{\mathbf{2}}$ does not have a terminal object in the fiber above $0 \to 1$.*

The following easy lemma and proposition will allow us to translate the problem of comparing $\underline{\mathbf{S}}^{\mathbb{I}}$ having RFRA, and $\underline{\mathbf{S}}$ having RFRA, into a problem of comparing, for some functor $F : \mathbb{C} \to \mathbb{X}$, the existence of terminal objects in fibres of $F^{\mathbb{I}}$ and the existence of terminal objects in the fibres of F.

3.2. Lemma. *For the span of categories* $\underline{\mathbf{S}}$ *and for a category* $\mathbb{I}$, *the canonical isomorphisms of categories (the vertical functors below) are such that the diagram*

$$\begin{tikzcd} (P^{\mathbb{I}} \downarrow 1_{\mathbb{A}^{\mathbb{I}}}) \arrow[r, "\langle Q^{\mathbb{I}}\pi_1, \pi_2\rangle"] \arrow[d] & \mathbb{B}^{\mathbb{I}} \times \mathbb{A}^{\mathbb{I}} \arrow[d] \\ (P \downarrow 1_{\mathbb{A}})^{\mathbb{I}} \arrow[r, "\langle Q\pi_1, \pi_2\rangle^{\mathbb{I}}"'] & (\mathbb{B} \times \mathbb{A})^{\mathbb{I}} \end{tikzcd} \tag{3.2}$$

commutes.

3.3. Proposition. *For the span of categories* $\underline{\mathbf{S}}$ *and for a category* $\mathbb{I}$, *the following conditions are equivalent:*

(a) the span $\underline{\mathbf{S}}^{\mathbb{I}}$ *has* RFRA;

(b) the functor $\langle Q^{\mathbb{I}}\pi_1, \pi_2\rangle : (P^{\mathbb{I}} \downarrow 1_{\mathbb{A}^{\mathbb{I}}}) \to \mathbb{B}^{\mathbb{I}} \times \mathbb{A}^{\mathbb{I}}$ *has terminal objects in its fibres;*

(c) the functor $\langle Q\pi_1, \pi_2\rangle^{\mathbb{I}} : (P \downarrow 1_{\mathbb{A}})^{\mathbb{I}} \to (\mathbb{B} \times \mathbb{A})^{\mathbb{I}}$ *has terminal objects in its fibres.*

Proof. Just combine the previous lemma with Corollary 2.4. ∎

Note that throughout this paper we will say that a functor weakly creates a limit dropping the uniqueness requirement from the definition of creation of limits by a functor.

For a functor $F : \mathbb{C} \to \mathbb{X}$, the relationship between the existence of terminal objects in the fibres of F, the existence of terminal objects in the fibres of $F^{\mathbb{I}}$ for some finite $\mathbb{I}$, and F being a prefibration was studied in [17]. In order to apply Theorem 2.24 of [17] we need the following lemma.

3.4. Lemma. *For the span of categories* $\underline{\mathbf{S}}$ *and for a diagram* $D : \mathbb{G} \to (P \downarrow 1_{\mathbb{A}})$, *the functor* $\langle Q\pi_1, \pi_2\rangle$ *weakly creates the limit of* D *as soon as* Q *weakly creates the limit of* $\pi_1 D$.

Proof. Since the functor $(Q \times 1_{\mathbb{A}}) : \mathbb{S} \times \mathbb{A} \to \mathbb{B} \times \mathbb{A}$ weakly creates the limit of D as soon as Q weakly creates the limit of $\pi_1 D$, the statement of the lemma follows from the fact that $\langle Q\pi_1, \pi_2\rangle = (Q \times 1_{\mathbb{A}}) \circ \langle \pi_1, \pi_2\rangle$ and the functor $\langle \pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{S} \times \mathbb{A}$ creates limits. ∎

3.5. Theorem. *For the span* $\underline{\mathbf{S}}$, *if the categories* $\mathbb{A}$ *and* $\mathbb{B}$ *have finite limits, and* Q *weakly creates finite limits, then the following conditions are equivalent:*

(a) the span of categories $\underline{\mathbf{S}}^{\mathbf{2}}$ *has* RFRA;

(b) for each finite category $\mathbb{I}$ *the span of categories* $\underline{\mathbf{S}}^{\mathbb{I}}$ *has* RFRA;

(c) the functor $\langle Q^{\mathbf{2}}\pi_1, \pi_2\rangle : (P^{\mathbf{2}} \downarrow 1_{\mathbb{A}^{\mathbf{2}}}) \to \mathbb{B}^{\mathbf{2}} \times \mathbb{A}^{\mathbf{2}}$ *has terminal objects in its fibres;*

(d) for each finite category $\mathbb{I}$ *the functor* $\langle Q^{\mathbb{I}}\pi_1, \pi_2\rangle : (P^{\mathbb{I}} \downarrow 1_{\mathbb{A}^{\mathbb{I}}}) \to \mathbb{B}^{\mathbb{I}} \times \mathbb{A}^{\mathbb{I}}$ *has terminal objects in its fibres;*

(e) the functor $\langle Q\pi_1, \pi_2\rangle^{\mathbf{2}} : (P \downarrow 1_{\mathbb{A}})^{\mathbf{2}} \to (\mathbb{B} \times \mathbb{A})^{\mathbf{2}}$ *has terminal objects in its fibres;*

(f) for each finite category $\mathbb{I}$ *the functor* $\langle Q\pi_1, \pi_2\rangle^{\mathbb{I}} : (P \downarrow 1_{\mathbb{A}})^{\mathbb{I}} \to (\mathbb{B} \times \mathbb{A})^{\mathbb{I}}$ *has terminal objects in its fibres;*

(g) the functor $\langle Q\pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{B} \times \mathbb{A}$ *is a prefibration;*

(h) for each finite category $\mathbb{I}$ *the functor* $\langle Q\pi_1, \pi_2\rangle^{\mathbb{I}} : (P \downarrow 1_{\mathbb{A}})^{\mathbb{I}} \to (\mathbb{B}\times\mathbb{A})^{\mathbb{I}}$ *is a prefibration.*

Proof. The equivalences $(a) \Leftrightarrow (c) \Leftrightarrow (e)$, and $(b) \Leftrightarrow (d) \Leftrightarrow (f)$ follow from Proposition 3.3. The equivalences $(e) \Leftrightarrow (f) \Leftrightarrow (g) \Leftrightarrow (h)$ follow via Lemma 3.4 from Theorem 2.24 of [17] applied to the functor $\langle Q\pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to \mathbb{B} \times \mathbb{A}$. ∎

It is worth explaining explicitly what Condition 3.5 (g) means. To do so we shall use the convention: for a functor $F : \mathbb{C} \to \mathbb{X}$ and two morphisms $f : A \to B$ in $\mathbb{C}$ and $\theta : X \to Y$ in $\mathbb{X}$, a display of the form

$$\begin{tikzcd} A \arrow[r, mapsto, "F"] \arrow[d, "f"'] & X \arrow[d, "\theta"] \\ B \arrow[r, mapsto, "F"'] & Y \end{tikzcd} \tag{3.3}$$

will be called a commutative diagram, if $F(f) = \theta$. Furthermore, we will do the same for displays containing such parts and commutative diagrams in the usual sense. Now Condition 3.5 (g) means that, for objects A_1, A_2 in $\mathbb{A}$, B_1 in $\mathbb{B}$, S_2 in $\mathbb{S}$ and morphisms $\alpha : A_1 \to A_2$ and $e_2 : P(S_2) \to A_2$ in $\mathbb{A}$, and $\beta : B_1 \to Q(S_2)$ in $\mathbb{B}$, the category with objects all commutative diagrams of the form

$$\begin{tikzcd} A_1 \arrow[d, "\alpha"'] & P(S_1) \arrow[l, "e_1"'] \arrow[d, "P(\sigma)"] & S_1 \arrow[l, mapsto, "P"'] \arrow[r, mapsto, "Q"] \arrow[d, "\sigma"] & B_1 \arrow[d, "\beta"] \\ A_2 & P(S_2) \arrow[l, "e_2"] & S_2 \arrow[l, mapsto, "P"] \arrow[r, mapsto, "Q"'] & Q(S_2) \end{tikzcd} \tag{3.4}$$

and morphisms all commutative diagrams of the form

$$\begin{tikzcd} A_1 \arrow[dr, equal] \arrow[dd, "\alpha"'] & & P(S_1') \arrow[ll, "e_1'"'] \arrow[dr, "P(u)"] \arrow[dd, "P(\sigma')"' near end] & & S_1' \arrow[ll, mapsto, "P"'] \arrow[rr, mapsto, "Q"] \arrow[dr, "u"] \arrow[dd, "\sigma'" near end] & & B_1 \arrow[dr, equal] \arrow[dd, "\beta"' near start] & \\ & A_1 \arrow[dd, "\alpha" near start] & & P(S_1) \arrow[ll, "e_1"'] \arrow[dd, "P(\sigma)"' near start] & & S_1 \arrow[ll, mapsto, "P"] \arrow[rr, mapsto, "Q"'] \arrow[dd, "\sigma" near start] & & B_1 \arrow[dd, "\beta"] \\ A_2 \arrow[dr, equal] & & P(S_2) \arrow[ll, "e_2" near end] \arrow[dr, equal] & & S_2 \arrow[ll, mapsto, "P"] \arrow[rr, mapsto, "Q"] \arrow[dr, equal] & & Q(S_2) \arrow[dr, equal] & \\ & A_2 & & P(S_2) \arrow[ll, "e_2"] & & S_2 \arrow[ll, mapsto, "P"] \arrow[rr, mapsto, "Q"'] & & Q(S_2) \end{tikzcd} \tag{3.5}$$

has a terminal object. However, each of these categories is isomorphic to the category defined in the same way where A_2 is replaced by $P(S_2)$, α by the projection $\pi_2 : A_1 \times_{A_2} P(S_2) \to P(S_2)$ of the pullback of α and e_2, and e_2 by the identity morphism $1_{P(S_2)}$ i.e. the category with objects of the form

$$\begin{tikzcd}
A_1 \times_{A_2} P(S_2) \arrow[d, "\pi_2"'] & P(S_1) \arrow[l, "\bar{e}_1"'] \arrow[d, "P(\sigma)"] & S_1 \arrow[l, "P"'] \arrow[r, "Q"] \arrow[d, "\sigma"] & B_1 \arrow[d, "\beta"] \\
P(S_2) & P(S_2) \arrow[l, "1_{P(S_2)}"] & S_2 \arrow[l, "P"] \arrow[r, "Q"'] & Q(S_2)
\end{tikzcd} \tag{3.6}$$

and morphisms defined in the expected way. The existence of terminal objects in categories of the last form can easily be seen to be equivalent to the condition:

3.6. CONDITION. *For each object S in $\mathbb{S}$, the induced span*

$$(\mathbb{A} \downarrow P(S)) \xleftarrow{P^S} (\mathbb{S} \downarrow S) \xrightarrow{Q^S} (\mathbb{B} \downarrow Q(S)), \tag{3.7}$$

where P^S and Q^S are the functors which send (S', σ) to $(P(S'), P(\sigma))$ and $(Q(S'), Q(\sigma))$, respectively, has RFRA.

Note that this means that this condition is a further equivalent condition of the above theorem. Furthermore, since comma categories of the form $(\mathbb{C} \downarrow C)$, where C is an object in a category $\mathbb{C}$, always have terminal objects, it follows by Remark 2.2 (ii), that Q^S has terminal objects in its fibres, and hence by Proposition 2.6 the functor Q is a prefibration. Therefore, we have proved:

3.7. PROPOSITION. *For the span $\underline{\mathbf{S}}$, suppose that the categories $\mathbb{A}$ and $\mathbb{B}$ have finite limits, and that Q weakly creates finite limits. If any (and hence all) of the Conditions 3.5 (a) - (h), Condition 3.6 hold, then Q is a prefibration.*

4. Right/left fibred right adjoints for right regular spans

In this section we study a condition on a span $\underline{\mathbf{S}}$ which, for most of the examples below, does not hold for all of the dual spans $\underline{\mathbf{S}}^{\mathrm{op}}$, $\underline{\mathbf{S}}_\circ$ or $\underline{\mathbf{S}}_\circ^{\mathrm{op}}$. This condition might be thought of as the source of the non-symmetric (or non-dual) descriptions and properties of action representability and locally-algebraically-cartesian closedness. Recall that G. Janelidze called a span $\underline{\mathbf{S}}$ right regular [19] (splitting the definition of N. Yoneda's regular span [22] into two parts - the other part being the same condition for the span $\underline{\mathbf{S}}_\circ^{\mathrm{op}}$) if for each A in $\mathbb{A}$, S in $\mathbb{S}$ and $\alpha : A \to P(S)$ in $\mathbb{A}$, there exists a P-cartesian morphism $\sigma : \bar{S} \to S$ in $\mathbb{S}$ such that $P(\sigma) = \alpha$ and $Q(\sigma) = 1_{Q(S)}$.

The following lemma is part of the folklore but we couldn't find an explicit reference for it.

4.1. Lemma. *Let $F : \mathbb{C} \to \mathbb{X}$ be a functor. Suppose that $\mathbb{X}$ has binary products and $\mathbb{C}$ has a terminal object T. Let $F^T : (\mathbb{C} \downarrow T) \to (\mathbb{X} \downarrow F(T))$ be the functor between comma categories induced by F. The category $(F^T \downarrow (X \times F(T), \pi_2))$ is isomorphic to the category $(F \downarrow X)$, and hence $(F^T \downarrow (X \times F(T), \pi_2))$ has a terminal object if and only if $(F \downarrow X)$ does. As a consequence, under the conditions above, if F is a fibration or more generally if F^T has a right adjoint, then F has a right adjoint.*

Proof. To see why these two categories are isomorphic note that for an object $((C, C \to T), e)$ in $(F^T \downarrow (X \times F(T), \pi_2))$ the morphism e is completely determined by $\pi_1 e$ since $\pi_2 e$ must be the image under F of the unique morphism $C \to T$. ■

It is easy to check that if the span $\underline{\mathbf{S}}$ is right regular, then for each B in $\mathbb{B}$ the functor P_B is a fibration.

4.2. Proposition. *If the span $\underline{\mathbf{S}}$ is right regular (or more generally for each B in $\mathbb{B}$ the functor P_B is a fibration) and $\mathbb{A}$ has finite products, then the span $\underline{\mathbf{S}}$ has RFRA if and only if for each B in $\mathbb{B}$ the category $Q^{-1}(B)$ has a terminal object.*

Proof. The "if" part follows from Lemma 4.1 applied to the functor $P_B : Q^{-1}(B) \to \mathbb{A}$, while the "only if" part follows trivially from the fact that right adjoints preserve limits. ■

The following lemma follows from Remark 2.14 of [17] and should also be compared to Lemma 2.7 of [9]. To make the paper more self contained, we include a proof.

4.3. Lemma. *Let $F : \mathbb{C} \to \mathbb{X}$ be a functor weakly creating pullbacks and let $\theta : X \to Y$ be a morphism in $\mathbb{X}$. If the fibers $F^{\mathbf{2}^{-1}}(X, Y, \theta)$ and $F^{-1}(Y)$ have terminal objects, then F admits precartesian liftings of θ.*

Proof. By Lemma 2.6 of [17] we know that the morphism $f : A \to B$ forming part of the terminal object in $F^{\mathbf{2}^{-1}}(X, Y, \theta)$ has codomain terminal in $F^{-1}(Y)$. Now if v is the unique morphism from an object B' to B in $F^{-1}(Y)$ and the diagram on the left

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "u"'] & B' \arrow[d, "v'"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "1_X"'] & Y \arrow[d, "1_Y"] \\ X \arrow[r, "\theta"'] & Y \end{tikzcd}$$

is the pullback whose image under F is the pullback on the right (F weakly creates pullbacks), then one easily checks that $f' : A' \to B'$ is an F-precartesian lifting of θ to B'. ■

It is straightforward to verify that:

4.4. Lemma. *For the span $\underline{\mathbf{S}}$, for each object B in $\mathbb{B}$, $(Q^{-1}(B))^{\mathbf{2}} = Q^{\mathbf{2}^{-1}}(B, B, 1_B)$ and $(P_B)^{\mathbf{2}} = P^{\mathbf{2}}_{(B,B,1_B)}$.*

As a corollary we obtain:

4.5. Lemma. *If the span $\underline{\mathbf{S}}$ has RFRA and the functor Q weakly creates pullbacks, then for each morphism $\alpha : A_1 \to A_2$ in $\mathbb{A}$ and for each B in $\mathbb{B}$, the functor $\langle Q\pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to (\mathbb{B} \times \mathbb{A})$ admits precartesian liftings of $(1_B, \alpha) : (B, A_1) \to (B, A_2)$.*

Proof. Let $\alpha : A_1 \to A_2$ be a morphism in $\mathbb{A}$ and B an object in $\mathbb{B}$. Since by assumption the functor P_B has a right adjoint, it follows that the functor $(P_B)^{\mathbf{2}} = P^{\mathbf{2}}_{(B,B,1_B)}$ has a right adjoint. Therefore, according to Lemmas 2.3 and 3.2, the functors $\langle Q\pi_1, \pi_2\rangle : (P \downarrow 1_{\mathbb{A}}) \to (\mathbb{B} \times \mathbb{A})$ and $\langle Q\pi_1, \pi_2\rangle^{\mathbf{2}} : (P \downarrow 1_{\mathbb{A}})^{\mathbf{2}} \to (\mathbb{B} \times \mathbb{A})^{\mathbf{2}}$ have terminal objects in the fibres above (B, A_2) and $((B, A_1), (B, A_2), (1_B, \alpha))$ respectively. The claim now follows from Lemmas 3.4 and 4.3. ∎

4.6. Lemma. *If the span $\underline{\mathbf{S}}$ is right regular, then for each $\alpha : A_1 \to A_2$ in $\mathbb{A}$ and for each B in $\mathbb{B}$ the functor $\langle P\pi_1, \pi_2\rangle : (Q \downarrow 1_{\mathbb{B}}) \to \mathbb{A} \times \mathbb{B}$ admits cartesian liftings of $(\alpha, 1_B) : (A_1, B) \to (A_2, B)$.*

Proof. Let $\alpha : \bar{A} \to A$ be a morphism in $\mathbb{A}$, let B be an object in $\mathbb{B}$, let S be an object in $\mathbb{S}$ such that $P(S) = A$, and let $f : Q(S) \to B$ be a morphism in $\mathbb{B}$. Since the span $\underline{\mathbf{S}}$ is right regular, there exists a P-cartesian morphism $\sigma : \bar{S} \to S$ such that $P(\sigma) = \alpha$ and $Q(\sigma) = 1_{Q(S)}$. It follows that the morphism $(\sigma, 1_B) : (\bar{S}, B, f) \to (S, B, f)$ is a morphism in $(Q \downarrow 1_{\mathbb{B}})$ such that $\langle P\pi_1, \pi_2\rangle(\sigma, 1_B) = (\alpha, 1_B)$. We will show that $(\sigma, 1_B)$ is $\langle P\pi_1, \pi_2\rangle$-cartesian. Let $(\sigma', \beta') : (S', B', f') \to (S, B, f)$ be a morphism in $(Q \downarrow 1_{\mathbb{B}})$ and let $(\theta, \phi) : (P(S'), B') \to (\bar{A}, B)$ be a morphism in $\mathbb{A} \times \mathbb{B}$ such that $\langle P\pi_1, \pi_2\rangle(\sigma', \beta') = (\alpha, 1_B)(\theta, \phi)$. This means that $fQ(\sigma') = \beta' f'$, $P(\sigma') = \alpha\theta$ and $\beta' = \phi$ and hence the diagram

$$\begin{tikzcd}
P(S') \arrow[dr, "\theta"] \arrow[dd, "\alpha\theta"'] & & S' \arrow[ll, mapsto, "P"'] \arrow[rr, mapsto, "Q"] \arrow[dr, dashed, "u"] \arrow[dd, "\sigma'"'] & & Q(S') \arrow[rr, "f'"] \arrow[dr, "Q(\sigma')"] \arrow[dd, "Q(\sigma')"] & & B' \arrow[dr, "\phi"] \arrow[dd, "\beta'"'] & \\
& \bar{A} \arrow[dd, "\alpha"'] & & \bar{S} \arrow[ll, mapsto, "P"'] \arrow[rr, mapsto, "Q"] \arrow[dd, "\sigma"] & & Q(\bar{S}) \arrow[rr, "f"] \arrow[dd, equal] & & B \arrow[dd, equal] \\
A \arrow[dr, equal] & & S \arrow[ll, mapsto, "P"] \arrow[rr, mapsto, "Q"] \arrow[dr, equal] & & Q(S) \arrow[rr, "f"] \arrow[dr, equal] & & B \arrow[dr, equal] & \\
& A & & S \arrow[ll, mapsto, "P"] \arrow[rr, mapsto, "Q"'] & & Q(S) \arrow[rr, "f"] & & B
\end{tikzcd} \tag{4.1}$$

commutes (in the sense described in the discussion immediately after Theorem 3.5). Accordingly there exists a unique morphism $u : S' \to \bar{S}$ such that $\sigma' = \sigma u$ and $P(u) = \theta$. Since $Q(u) = 1_{Q(S)}Q(u) = Q(\sigma u) = Q(\sigma')$ it follows that $fQ(u) = fQ(\sigma') = \beta' f'$ and hence $(u, \beta') : (S', B', f') \to (\bar{S}, B, f)$ is the desired unique morphism in $\mathbb{S}$ such that $(\sigma', \beta') = (\sigma, 1_B)(u, \beta')$ and $\langle P\pi_1, \pi_2\rangle(u, \beta') = (\theta, \phi)$. ∎

Recall the following well-known fact:

4.7. Lemma. *Let $F : \mathbb{C} \to \mathbb{X}$ be a functor and let $f : A \to B$ and $g : B \to C$ be morphisms in $\mathbb{C}$. If f is F-precartesian and g is F-cartesian, then the composite $g \circ f$ is F-precartesian.*

4.8. Proposition. *If the span $\underline{\mathbf{S}}$ is right regular and has LFRA, and P weakly creates pullbacks, then the span $\underline{\mathbf{S}}^{\mathbf{2}}$ has LFRA.*

Proof. Since each morphism $(\alpha, \beta) : (A_1, B_1) \to (A_2, B_2)$ in $\mathbb{A} \times \mathbb{B}$ factors as $(\alpha, \beta) = (\alpha, 1_{B_2})(1_{A_1}, \beta)$ it follows from Lemmas 4.5, 4.6 and 4.7 that the functor $\langle P\pi_1, \pi_2\rangle : (Q \downarrow 1_{\mathbb{B}}) \to \mathbb{A} \times \mathbb{B}$ is a prefibration. Therefore, since by Corollary 2.4 it has terminal objects in its fibres, it follows from Proposition 2.7 in [17] that $\langle P\pi_1, \pi_2\rangle^{\mathbf{2}} : (Q \downarrow 1_{\mathbb{B}})^{\mathbf{2}} \to (\mathbb{A} \times \mathbb{B})^{\mathbf{2}}$ has terminal objects in its fibres and hence by Proposition 3.3 the span $\underline{\mathbf{S}}^{\mathbf{2}}$ has LFRA. ∎

5. General monoid actions

In the paper [5] F. Borceux, G. Janelidze and G. M. Kelly gave an extremely informative exposition on monoidal actions in general, and introduced and studied the notion of representable monoid actions. We recall from [5] that an action of a monoidal category $\mathbf{C} = (\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ on a category $\mathbb{X}$ can be defined as a monoidal functor (F, θ, ϕ) from $\mathbf{C}$ to the strict monoidal category $\mathbf{End}(\mathbb{X}) = (\mathrm{End}(\mathbb{X}), 1_{\mathbb{X}}, \circ)$ or equivalently as a triple $(\bullet, \theta, \gamma)$ where $\bullet$ is a functor $\mathbb{C} \times \mathbb{X} \to \mathbb{X}$, and $(\theta_X : X \to I \bullet X)_{X\in\mathbb{X}}$ and $(\gamma_{A,B,X} : A \bullet (B \bullet X) \to (A \otimes B) \bullet X)_{A,B\in\mathbb{C},X\in\mathbb{X}}$ are natural transformations making the diagrams

$$\begin{tikzcd}
A \bullet (B \bullet (C \bullet X)) \arrow[r, equal] \arrow[d, "1_A \bullet \gamma_{B,C,X}"'] & A \bullet (B \bullet (C \bullet X)) \arrow[d, "\gamma_{A,B,C\bullet X}"] \\
A \bullet ((B \otimes C) \bullet X) \arrow[d, "\gamma_{A,B\otimes C,X}"'] & (A \otimes B) \bullet (C \bullet X) \arrow[d, "\gamma_{A\otimes B,C,X}"] \\
(A \otimes (B \otimes C)) \bullet X \arrow[r, "\alpha_{A,B,C} \bullet 1_X"'] & ((A \otimes B) \otimes C) \bullet X
\end{tikzcd} \tag{5.1}$$

$$\begin{tikzcd}
A \bullet X \arrow[r, equal] \arrow[ddr, "\lambda_A \bullet 1_X"'] & A \bullet X \arrow[d, "\theta_{A\bullet X}"] \\
& I \bullet (A \bullet X) \arrow[d, "\gamma_{I,A,X}"] \\
& (I \otimes A) \bullet X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \bullet X \arrow[r, equal] \arrow[d, "1_A \bullet \theta_X"'] & A \bullet X \arrow[ddl, "\rho_A \bullet 1_X"] \\
A \bullet (I \bullet X) \arrow[d, "\gamma_{A,I,X}"'] & \\
(A \otimes I) \bullet X &
\end{tikzcd} \tag{5.2}$$

commute. The two descriptions of an action are related by $F(A)(X) = A \bullet X$, $\phi_{A,B_X} = \gamma_{A,B,X}$ and θ is the same for both. Such an action is called strong when θ and ϕ (or equivalently θ and γ) are isomorphisms. A monoidal functor (F, θ, ϕ) as above determines, as usual, a functor from the category $\mathbf{Mon}(\mathbf{C})$ of monoids in $\mathbf{C}$ to the category $\mathbf{Mon}(\mathbf{End}(\mathbb{X}))$ of monoids in $\mathbf{End}(\mathbb{X})$. Using the fact that monoids in $\mathbf{Mon}(\mathbf{End}(\mathbb{X}))$ are monads, a monoid action, with respect to $\mathcal{A} = (\mathbf{C}, \mathbb{X}, \bullet, \theta, \gamma)$ can be defined as a pair $((M, e, m), (X, h))$ where (M, e, m) is a monoid in $\mathbf{C}$, (X, h) is an algebra for the monad $M \bullet -$. An algebra for this monad can be more explicitly described as a pair (X, h) where

X is in $\mathbb{X}$ and $h : M \bullet X \to X$ is a morphism making the diagram

$$
\begin{tikzcd}
M \bullet (M \bullet X) \arrow[r, "\gamma_{M,M,X}"] \arrow[d, "1_M \bullet h"'] & (M \otimes M) \bullet X \arrow[r, "m \bullet 1_X"] & M \bullet X \arrow[d, "h"] & I \bullet X \arrow[l, "e \bullet 1_X"'] & X \arrow[l, "\theta_X"'] \arrow[dll, equal] \\
M \bullet X \arrow[rr, "h"'] & & X & &
\end{tikzcd}
\tag{5.3}
$$

commute. A morphism of monoid actions with domain $((M,e,m),(X,h))$ and codomain $((M',e',m'),(X',h'))$ is a pair (f,g) where $f : M \to M'$ and $g : X \to X'$ are morphisms in $\mathbb{C}$ and $\mathbb{X}$ respectively, such that f is a monoid morphism $(M,e,m) \to (M',e',m')$ and $h'(f \bullet g) = gh$. For the monoidal action $\mathcal{A} = (\mathbf{C}, \mathbb{X}, \bullet, \theta, \gamma)$, let us denote by $\mathbf{Act}(\mathcal{A})$ the category of monoid actions and by P and Q the forgetful functors to $\mathbf{Mon}(\mathbf{C})$ and $\mathbb{X}$ respectively. These data together forms a span $\underline{\mathbf{Act}(\mathcal{A})} =$

$$
\mathbf{Mon}(\mathbf{C}) \xleftarrow{P} \mathbf{Act}(\mathcal{A}) \xrightarrow{Q} \mathbb{X}. \tag{5.4}
$$

Recall also that

5.1. Definition. *The monoidal action $\mathcal{A}$ has representable monoid actions on an object X in $\mathbb{X}$, if the functor*

$$
\mathrm{Act}(-,X) : \mathbf{Mon}(\mathbf{C})^{\mathrm{op}} \to \mathbf{Set} \tag{5.5}
$$

which sends a monoid in $\mathbf{C}$ to the set of its actions on X is representable.

5.2. Theorem. *For the monoidal action $\mathcal{A}$. If $\mathbb{C}$ has finite products, then the span $\underline{\mathbf{Act}(\mathcal{A})}$ has RFRA if and only if $\mathcal{A}$ has representable monoid actions.*

Proof. Note that for an object X in $\mathbb{X}$ the category of elements of the functor $\mathrm{Act}(-,X) : \mathbf{Mon}(\mathbf{C})^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to $Q^{-1}(X)$ and hence $\mathrm{Act}(-,X)$ is representable if and only if the fibre $Q^{-1}(X)$ has a terminal object. For an object X in $\mathbb{X}$, it is easy to see that $P_X : Q^{-1}(X) \to \mathbf{Mon}(\mathbf{C})$ is a discrete fibration, in fact this is used to make $\mathrm{Act}(-,X)$ into a functor. Therefore, Proposition 4.2 implies that the span $\underline{\mathbf{Act}(\mathcal{A})}$ has RFRA if and only if $\mathcal{A}$ has representable monoid actions. ∎

5.3. Corollary. *If $\mathcal{A}$ is strong monoidal and $\mathbb{C}$ has finite products, then the span $\underline{\mathbf{Act}(\mathcal{A})}$ has RFRA as soon as, for each X in $\mathbb{X}$, the functor $- \bullet X : \mathbb{C} \to \mathbb{X}$ has a right adjoint.*

Proof. F. Borceux, G. Janelidze and G. M. Kelly explained in [5] that under the above assumptions the action $\mathcal{A}$ has representable monoid actions. The statement of the corollary now follows from Theorem 5.2. ∎

5.4. REMARK. *Note that the condition, for each X in $\mathbb{X}$ the functor $-\bullet X : \mathbb{C} \to \mathbb{X}$ has a right adjoint, from the previous corollary could be stated as the span*

$$\mathbb{X} \xleftarrow{\bullet} \mathbb{C} \times \mathbb{X} \xrightarrow{\pi_2} \mathbb{X} \tag{5.6}$$

has RFRA.

5.5. DEFINITION. *The monoidal action $\mathcal{A}$ has exponentiable monoid actions for a monoid (M, e, m), if the forgetful functor from the category of algebras over the induced monad $M \bullet -$ to the category $\mathbb{X}$ has a right adjoint.*

Since for the monoidal action $\mathcal{A}$ and each monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$ the functor

$$Q_{(M,e,m)} : P^{-1}((M, e, m)) \to \mathbb{X} \tag{5.7}$$

induced by the span $\underline{\mathbf{Act}(\mathcal{A})}$ is essentially the forgetful functor from the category of algebras over the monad $M \bullet -$ to $\mathbb{X}$ we obtain essentially by definition:

5.6. THEOREM. *The span $\underline{\mathbf{Act}(\mathcal{A})}$ has LFRA if and only if $\mathcal{A}$ has exponentiable monoid actions.*

Recall the following well-known fact (see [11]).

5.7. LEMMA. *Let $\mathbf{T} = (T, \eta, \mu)$ be a monad in $\mathbb{X}$. The functor T has a right adjoint if and only if the forgetful functor $U^{\mathbf{T}}$ from the category of $\mathbf{T}$-algebras to $\mathbb{X}$ has a right adjoint.*

5.8. THEOREM. *The span $\underline{\mathbf{Act}(\mathcal{A})}$ has LFRA, if and only if for each monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$ the functor $M \bullet -$ has a right adjoint.*

PROOF. Since, as explained above, for each monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$ the functor $Q_{(M,e,m)} : P^{-1}((M, e, m)) \to \mathbb{X}$ is essentially the forgetful functor from the category of $M \bullet -$ algebras to $\mathbb{X}$, it follows easily from Lemma 5.7 that the two conditions are equivalent. ∎

5.9. REMARK. *It is sometimes of interest to have a more explicit description of the right adjoint above. Using the notation from the previous theorem and writing $-^M$ for the right adjoint of $M \bullet -$, the right adjoint of $Q_{(M,e,m)}$ sends X to $((M, e, m), (X^M, h))$ where h is the unique morphism making the diagram*

$$\begin{tikzcd}
M \bullet (M \bullet X^M) \arrow[r, "\gamma_{M,M,X^M}"] \arrow[d, "1_M \bullet h"'] & (M \otimes M) \bullet X^M \arrow[r, "m \bullet 1_{X^M}"] & M \bullet X^M \arrow[d, "\epsilon_X"] \\
M \bullet X^M \arrow[rr, "\epsilon_X"'] & & X,
\end{tikzcd} \tag{5.8}$$

in which ϵ is the counit of the adjunction $M \bullet - \dashv -^M$, commute.

For the monoidal action $\mathcal{A}$ and a monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$, let us write R_M for the right adjoint of the forgetful functor from the category of algebras over the induced monad $M \bullet -$ to the category $\mathbb{X}$ when it exists. Let us add a column to the table on pages 242 and 243 of [5] in order to give some examples. Recall that $[X]$ in the table below denotes the representing object for $\mathrm{Act}(-, X)$.

Monoidal category $\mathbf{C}$	Action of $\mathbf{C}$ on $\mathbb{X}$	Monoid M in $\mathbf{C}$	M-action on an object in $\mathbb{X}$	$[X]$	$R_M(X)$
An ordinary monoid M	An ordinary M-action on a set $\mathbb{X}$	e, the identity element of the monoid	Every object has a unique action	e	X
$\mathbf{End}(\mathbb{X})$	The (strict) evaluation action of $\mathbf{End}(\mathbb{X})$ on $\mathbb{X}$ defined by $A \bullet X = A(X)$	A monad $\mathbf{T}$ on $\mathbb{X}$	An algebra over the monad $\mathbf{T}$	$\langle X, X \rangle$ the right Kan extension of X (considered as a functor from the category $\mathbf{1}$ to $\mathbb{X}$) along itself provided that Kan extension does exist	The right adjoint of the forgetful functor from the category of T-algebras to $\mathbb{X}$ applied to X (provided it exists)
A category $\mathbb{C}$ with finite products regarded as a monoidal category with $\otimes = \times$	$\mathbf{C}$ canonically acting on $\mathbb{C}$; so that $A \bullet X = A \times X$	An internal monoid (= monoid object) M in $\mathbb{C}$ in the usual sense	An internal M-action in $\mathbb{C}$ (= M-object in $\mathbb{C}$) in the usual sense	X^X provided $\mathbb{C}$ is cartesian closed	X^M, with algebra structure as in Remark 5.9, provided $\mathbb{C}$ is cartesian closed
As above, but with $\mathbb{C} = \mathbf{Set}$	As above, but with $\mathbb{X} = \mathbb{C} = \mathbf{Set}$	An ordinary monoid M	An ordinary M-action in $\mathbf{Set}$ (= M-set)	X^X	X^M
A category $\mathbb{C}$ with finite coproducts regarded as a monoidal category with $\otimes = +$	$\mathbf{C}$ canonically acting on $\mathbb{C}$ so that $A \bullet X = A + X$	Every object M in $\mathbb{C}$ has a unique monoid structure given by the unique morphism $0 \to M$ and the codiagonal $M + M \to M$	An object in $\mathbb{C}$ equipped with a morphism from M to it	X	Only exists if M is an initial object, in which case it is X

Ab, the category of abelian groups with $\otimes$ the ordinary tensor product	**C** canonically acting on $\mathbb{C}$; so that $A \bullet X = A \otimes X$	A ring M (with 1)	An M-module	The ring $\mathrm{Hom}(X, X)$	$\prod_{m \in M} X$ with the usual action
Any monoidal category **C**	**C** trivially acting (i.e. $A \bullet X = X$) on any category $\mathbb{X}$	A monoid M in $\mathbf{Mon}(\mathbf{C})$	Every object has a unique action of M	A terminal object in $\mathbf{Mon}(\mathbf{C})$, provided it exists	X
Set regarded as a monoidal category with $\otimes = \times$	$\mathbb{X}$ a category with coproducts, and $A \bullet X = \coprod_{a \in A} X$ with evident remaining structure	An ordinary monoid M	An ("external") M-action on an object in $\mathbb{X}$ in the usual sense	$\mathrm{End}(X)$, the monoid of endomorphisms of X	$\mathrm{Ran}_M(X)$ the right Kan extension of X (considered as a functor from the category $\mathbf{1}$ to $\mathbb{X}$) along $\mathbf{1} \to M$ provided that Kan extension does exist

Recalling that for each monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$ the functor $Q_{(M,e,m)} : P^{-1}(M, e, m) \to \mathbb{X}$ is essentially the forgetful functor from the category of $(M \bullet -)$-algebras to $\mathbb{X}$ it follows that:

5.10. Proposition. *The span* $\underline{\mathbf{Act}(\mathcal{A})}$ *has* LFLA.

Recall that for the monoidal category $\mathbf{C}$ and for a category $\mathbb{I}$ we can construct a monoidal category $\mathbf{C}^{\mathbb{I}} = (\mathbb{C}^{\mathbb{I}}, I^{\mathbb{I}}, \otimes^{\mathbb{I}}, \alpha^{\mathbb{I}}, \lambda^{\mathbb{I}}, \rho^{\mathbb{I}})$ where $\mathbb{C}^{\mathbb{I}}$ is the usual functor category, $I^{\mathbb{I}}$ is the constant functor onto I, $\otimes^{\mathbb{I}}$ is defined point-wise, and $\alpha^{\mathbb{I}}$, $\lambda^{\mathbb{I}}$ and $\rho^{\mathbb{I}}$ are natural transformations whose components are natural transformations defined point-wise. Furthermore the monoidal action $\mathcal{A}$ induces a monoidal action $\mathcal{A}^{\mathbb{I}} = (\mathbf{C}^{\mathbb{I}}, \mathbb{X}^{\mathbb{I}}, \bullet^{\mathbb{I}}, \theta^{\mathbb{I}}, \gamma^{\mathbb{I}})$ where $\bullet^{\mathbb{I}}$ is defined point-wise and $\theta^{\mathbb{I}}$ and $\gamma^{\mathbb{I}}$ are natural transformations whose components are natural transformations defined point-wise.

Note that the canonical isomorphisms (the vertical arrows) make the diagram

$$
\begin{tikzcd}
\mathbf{Mon}(\mathbf{C})^{\mathbb{I}} \arrow[d] & \mathbf{Act}(\mathcal{A})^{\mathbb{I}} \arrow[l, "P^{\mathbb{I}}"'] \arrow[r, "Q^{\mathbb{I}}"] \arrow[d] & \mathbb{X}^{\mathbb{I}} \arrow[d, equal] \\
\mathbf{Mon}(\mathbf{C}^{\mathbb{I}}) & \mathbf{Act}(\mathcal{A}^{\mathbb{I}}) \arrow[l, "P"'] \arrow[r, "Q"] & \mathbb{X}^{\mathbb{I}}.
\end{tikzcd}
\tag{5.9}
$$

commute. Therefore as a corollary of Theorem 3.5 via Theorem 5.2 we obtain:

5.11. THEOREM. *For the monoidal action $\mathcal{A}$. If $\mathbb{X}$ and $\mathbb{C}$ have finite limits, then the following conditions are equivalent:*

(a) the induced action $\mathcal{A}^{\mathbf{2}}$ has representable monoid actions;

(b) for each finite category $\mathbb{I}$ the induced action of $\mathcal{A}^{\mathbb{I}}$ has representable monoid actions.

and imply that the action of $\mathcal{A}$ has representable monoid actions.

We could write down a similar (essentially dual) theorem about exponentiable monoid actions. However, using Theorems 3.5, 5.6 and 5.8 we will prove a stronger result (see Theorem 5.14 (ii) below.) To do so we use the following proposition which is closely related to Theorem 2.12 of [21]. The proof given in [21] can easily be adapted as follows:

5.12. PROPOSITION. *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{I}$ be categories such that $\mathbb{B}$ and $\mathbb{C}$ have finite limits and products indexed by $\mathbb{I}_1$, let $H : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ be a functor, and let $H^{\mathbb{I}} : \mathbb{A}^{\mathbb{I}} \times \mathbb{B}^{\mathbb{I}} \to \mathbb{C}^{\mathbb{I}}$ be the induced functor. For A in $\mathbb{A}^{\mathbb{I}}$, the functor $H^{\mathbb{I}}(A, -) : \mathbb{B}^{\mathbb{I}} \to \mathbb{C}^{\mathbb{I}}$ has a right adjoint if for each object I in $\mathbb{I}$ the functors $H(A(I), -) : \mathbb{B} \to \mathbb{C}$ have right adjoints.*

PROOF. Let $J : \mathbb{I}_0 \to \mathbb{I}$ be the inclusion of the objects of $\mathbb{I}$ as a discrete category. The product assumptions on $\mathbb{B}$ and $\mathbb{C}$ allow us to conclude the functors $\mathbb{B}^J : \mathbb{B}^{\mathbb{I}} \to \mathbb{B}^{\mathbb{I}_0}$ and $\mathbb{C}^J : \mathbb{C}^{\mathbb{I}} \to \mathbb{C}^{\mathbb{I}_0}$ have right adjoints given by right Kan extensions along J. Note that the diagram

$$\begin{tikzcd} \mathbb{B}^{\mathbb{I}} \arrow[r, "H^{\mathbb{I}}(A,-)"] \arrow[d, "\mathbb{B}^J"'] & \mathbb{C}^{\mathbb{I}} \arrow[d, "\mathbb{C}^J"] \\ \mathbb{B}^{\mathbb{I}_0} \arrow[r, "H^{\mathbb{I}_0}(AJ,-)"'] & \mathbb{C}^{\mathbb{I}_0} \end{tikzcd} \tag{5.10}$$

commutes and the vertical functors are comonadic. Therefore, since the bottom functor has right adjoint as soon as for each I in $\mathbb{I}$ the functors $H(A(I), -)$ have a right adjoint, we see that the claim follows from a standard adjoint lifting theorem (see e.g. Lemma 2.1 of [21]). ∎

5.13. REMARK. *One can give a longer alternative proof of the previous proposition dropping the limit requirements on $\mathbb{C}$.*

5.14. THEOREM. *For the monoidal action $\mathcal{A}$, suppose that $\mathbb{X}$ and $\mathbb{C}$ have finite limits.*

(i) If $\mathcal{A}$ is a strong action, and for each object X in $\mathbb{X}$ the functor $- \bullet X$ has a right adjoint, then for each finite category $\mathbb{I}$ the induced action of $\mathbf{C}^{\mathbb{I}}$ on $\mathbb{X}^{\mathbb{I}}$ has representable monoid actions.

(ii) For each monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$, the functor $M \bullet -$ has a right adjoint if and only if for each finite category $\mathbb{I}$ the induced action of $\mathbf{C}^{\mathbb{I}}$ on $\mathbb{X}^{\mathbb{I}}$ has exponentiable monoid actions.

PROOF.

(i) It is immediate that if $\mathcal{A}$ is a strong action so is $\mathcal{A}^{\mathbf{2}}$. Therefore, since for each (X_1, X_2, χ) in $\mathbb{X}^{\mathbf{2}}$ by Proposition 5.12 the functor $- \bullet (X_1, X_2, \chi)$ has a right adjoint it follows by Corollary 5.3 that $\mathcal{A}^{\mathbf{2}}$ has representable monoid actions, and hence by Theorem 5.11 so does $\mathcal{A}^{\mathbb{I}}$.

(ii) The "if" part follows immediately from Theorems 5.6 and 5.8. For the "only if" part, suppose that for each monoid (M, e, m) in $\mathbf{Mon}(\mathbf{C})$ the functor $M \bullet -$ has a right adjoint. It follows from Proposition 5.12 that for each $((M, e, m), (M', e', m'), f)$ in $\mathbf{Mon}(\mathbf{C})^{\mathbf{2}}$ the functor

$$(M, M', f) \bullet - : \mathbb{X}^{\mathbf{2}} \to \mathbb{X}^{\mathbf{2}} \tag{5.11}$$

has a right adjoint, and hence by Theorem 5.8 the span $\underline{\mathbf{Act}(\mathcal{A}^{\mathbf{2}})}$ has LFRA. Theorem 3.5 now tells us that $\underline{\mathbf{Act}(\mathcal{A}^{\mathbb{I}})}$ has LFRA and hence by Theorem 5.6 the action $\mathcal{A}^{\mathbb{I}}$ has exponentiable monoid actions as required.

∎

5.15. REMARK. *Using the fact that the span $\underline{\mathbf{Act}(\mathcal{A})}$ is right regular it is possible to construct a second proof of (ii) above using Proposition 4.8*

6. Split extensions

Recall that for a pointed category $\mathbb{C}$, a split extension (of B with kernel X) is a diagram in $\mathbb{C}$

$$X \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{6.1}$$

where κ is the kernel of α, and $\alpha\beta = 1_B$. A morphism of split extensions is a diagram in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "\kappa"] \arrow[d, "u"'] & A \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & B \arrow[l, shift left, "\beta"] \arrow[d, "w"] \\ X' \arrow[r, "\kappa'"] & A' \arrow[r, shift left, "\alpha'"] & B' \arrow[l, shift left, "\beta'"] \end{tikzcd} \tag{6.2}$$

where the top and bottom rows are split extensions (the domain and codomain respectively), and $v\kappa = \kappa' u$, $v\beta = \beta' w$ and $w\alpha = \alpha' v$. In this section we consider the span $\underline{\mathbf{SE}(\mathbb{C})} =$

$$\mathbb{C} \xleftarrow{P} \mathbf{SplExt}(\mathbb{C}) \xrightarrow{K} \mathbb{C} \tag{6.3}$$

where $\mathbf{SplExt}(\mathbb{C})$ is the category of split extensions and P and K are the functors sending the split extension (6.1) to B and X respectively. When $\mathbb{C}$ is semi-abelian, in the sense of G. Janelidze, L. Màrki and W. Tholen in [20], it was shown by F. Borceux, G. Janelidze and G. M. Kelly in [5] that the category $\mathbf{SplExt}(\mathbb{C})$ is equivalent to a category $\mathbf{Act}(\mathbf{C}, \mathbb{C}, \bullet, \theta, \gamma)$ where $\mathbf{C}$ is the monoidal category with tensor defined by binary coproduct in $\mathbb{C}$. Furthermore one easily checks that this equivalence is such that it makes the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[d] & \mathbf{SplExt}(\mathbb{C}) \arrow[l, "P"'] \arrow[r, "K"] \arrow[d] & \mathbb{C} \arrow[d, equal] \\ \mathbf{Mon}(\mathbf{C}) & \mathbf{Act}(\mathbf{C}, \mathbb{C}, \bullet, \theta, \gamma) \arrow[l, "P"'] \arrow[r, "Q"] & \mathbb{C}, \end{tikzcd} \tag{6.4}$$

where the vertical functor on the left is the isomorphism which sends each object C in $\mathbb{C}$ to C equipped with the unique monoid structure on it, commute. We will see that this means that for a semi-abelian category some of the results in this section could be obtained from the results in the previous section. However, there are examples for which this equivalence no longer exists, to which results in this section will be applicable. In particular some of the results below will be applicable to the category of finite groups or more generally to a category of internal groups in a finitely complete cartesian closed category.

For a pointed category $\mathbb{C}$ and for an object X in $\mathbb{C}$, a terminal object (when it exists) in the fibre $K^{-1}(X)$ of the functor $K : \mathbf{SplExt}(\mathbb{C}) \to \mathbb{C}$ is essentially by definition a *generic split extension with kernel* X in the sense of [5]. Accordingly we will say that a pointed category $\mathbb{C}$ has generic split extensions when the fibres of K have terminal objects. Therefore, since the span $\underline{\mathbf{SE}(\mathbb{C})}$ is right regular, as soon as $\mathbb{C}$ has finite limits, applying Proposition 4.2 to the span $\underline{\mathbf{SE}(\mathbb{C})}$ we obtain:

6.1. Proposition. *If $\mathbb{C}$ is pointed and finitely complete, then the following conditions are equivalent:*

(a) $\mathbb{C}$ has generic split extensions;

(b) the span $\underline{\mathbf{SE}(\mathbb{C})}$ has RFRA.

As a corollary we obtain:

6.2. Theorem. *If $\mathbb{C}$ is semi-abelian, then the following conditions are equivalent:*

(a) $\mathbb{C}$ is action representable;

(b) the span $\underline{\mathbf{SE}(\mathbb{C})}$ has RFRA.

Proof. The equivalence of (a) and (b) follows from the previous proposition and from [5] where it was proved that a semi-abelian category is action representable if and only if it has generic split extensions. ∎

For $\mathbb{C}$ pointed with finite limits, if in our description of the category $\mathbf{SplExt}(\mathbb{C})$ we omitted all kernels from the diagrams involved we would obtain an equivalent category which is called by D. Bourn the category of points and denoted $\mathbf{Pt}(\mathbb{C})$. Recall that the functor $\pi : \mathbf{Pt}(\mathbb{C}) \to \mathbb{C}$ sending a split epimorphism to its codomain is called the fibration of points. Since the composite of the forgetful functor $\mathbf{SplExt}(\mathbb{C}) \to \mathbf{Pt}(\mathbb{C})$ and π gives the functor P, it follows that for each B in $\mathbb{C}$ the functor K_B is "up to equivalence" what is known as the kernel functor $\mathrm{Ker}_B : \mathbf{Pt}_B(\mathbb{C}) \to \mathbb{C}$ (i.e. the functor sending a split epimorphism in the fibre of $\pi^{-1}(B) = \mathbf{Pt}_B(\mathbb{C})$ to a chosen kernel). This means that:

6.3. Proposition. *If $\mathbb{C}$ is pointed and finitely complete, then the following conditions are equivalent:*

(a) for each B the functor Ker_B has a right adjoint;

(b) the span $\underline{\mathbf{SE}(\mathbb{C})}$ has LFRA.

For a category $\mathbb{C}$ with finite limits, recall that $\mathbb{C}$ is said to be locally algebraically cartesian closed when each change of base functor between fibres of π has a right adjoint. According to Theorem 5.1 of [15], for a pointed (Bourn-)protomodular category [7], this is the case whenever each kernel functor has a right adjoint and hence we obtain:

6.4. Theorem. *If $\mathbb{C}$ is pointed protomodular and finitely complete, then the following conditions are equivalent:*

(a) the category $\mathbb{C}$ is locally algebraically cartesian closed;

(b) the span $\underline{\mathbf{SE}(\mathbb{C})}$ has LFRA.

Examples of locally algebraically cartesian closed categories include the categories of groups (internal to any finitely complete cartesian closed category) and Lie algebras over any commutative ring (for details see [14], [15] and [8]). Note that:

6.5. Remark. *X. G. Martinez and T. Van der Linden noticed that Proposition 6.9 of [15] is incorrect, so that the category of commutative rings satisfying $xyz = 0$ is not a locally algebraically cartesian closed category (see [12]). Using the notation from Proposition 6.9 and its proof, the error in the proof is that R is defined on objects and claimed to be a functor, but in fact can't be made into a functor.*

Using the well-known fact that for a pointed finitely complete category the existence of a left adjoint to each kernel functor is equivalent to the existence of binary coproducts we obtain:

6.6. Theorem. *If $\mathbb{C}$ is pointed and finitely complete, then the following conditions are equivalent:*

(a) $\mathbb{C}$ has binary coproducts;

(b) the span $\underline{\mathbf{SE}(\mathbb{C})}$ has LFLA.

Next we investigate, in the pointed protomodular context, what condition RFLA means for the span $\underline{\mathbf{SE}(\mathbb{C})}$. Recall that a pointed finitely complete protomodular category is additive if and only if for each object X the morphisms 1_X and 1_X commute in the sense of S. A. Huq [18] i.e. there is a (unique) morphism $\varphi : X \times X \to X$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "1_X"'] & X \times X \arrow[d, "\varphi"] & X \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "1_X"] \\ & X & \end{tikzcd} \tag{6.5}$$

commutes. Recall also that the existence of φ in (6.5) makes $(X, \varphi, 0)$ an internal abelian group. For reference, the reader may consult [2] or more explicitly, in semi-abelian case Theorem 6.9 and Corollary 7.3 of [1].

6.7. Theorem. *If $\mathbb{C}$ is pointed protomodular and finitely complete, then the following conditions are equivalent:*

(a) $\mathbb{C}$ is additive;

(b) the span $\underline{\mathbf{SE}(\mathbb{C})}$ has RFLA.

(c) the span $\underline{\mathbf{SE}(\mathbb{C})}$ and all its duals have RFRA.

Proof. (b)⇒(a): Let X and B be objects in $\mathbb{C}$ and let the upper row of the diagram

$$\begin{tikzcd} X \arrow[r, "\bar{\kappa}"] \arrow[d, equal] & \bar{A} \arrow[r, shift left, "\bar{\alpha}"] \arrow[d, dashed, "v"] & \bar{B} \arrow[l, shift left, "\bar{\beta}"] \arrow[d, dashed] & B \arrow[l, "\eta_B"'] \arrow[d, equal] \\ X \arrow[r, "1_X"] & X \arrow[r, shift left] & 0 \arrow[l, shift left] & B \arrow[l] \end{tikzcd} \tag{6.6}$$

be an initial object in $(B \downarrow P_X)$. It follows that there is a unique morphism as shown above and hence by protomodularity that

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"] & X \times \bar{B} \arrow[r, shift left, "\pi_2"] & \bar{B} \arrow[l, shift left, "\langle 0,1\rangle"] & B \arrow[l, "\eta_B"'] \end{tikzcd} \tag{6.7}$$

is an initial object in $(B \downarrow P_X)$. Accordingly, we obtain the unique morphism as displayed in the upper part of the diagram

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times \bar{B} \arrow[r, shift left, "\pi_2"] \arrow[d, "f"] & \bar{B} \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[d, "g"] & B \arrow[l, "\eta_B"'] \arrow[d, equal] \\ X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times B \arrow[r, shift left, "\pi_2"] \arrow[d, "1\times\eta_B"] & B \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[d, "\eta_b"] & B \arrow[l, "1_B"'] \arrow[d, equal] \\ X \arrow[r, "\langle 1,0\rangle"] & X \times \bar{B} \arrow[r, shift left, "\pi_2"] & \bar{B} \arrow[l, shift left, "\langle 0,1\rangle"] & B \arrow[l, "\eta_B"'] \end{tikzcd} \tag{6.8}$$

whose composite with the morphism displayed in the lower part of the diagram must be the identity morphism. This means that g is an isomorphism and hence by protomodularity that the middle row of the above diagram is an initial object in $(B \downarrow P_X)$. In particular when $B = X$ we obtain the unique morphism

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times X \arrow[r, shift left, "\pi_2"] \arrow[d, "\varphi"] & X \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[d, "1_X"] & X \arrow[l, "1_X"'] \arrow[d, equal] \\
X \arrow[r, "\langle 1,0\rangle"] & X \times X \arrow[r, shift left, "\pi_2"] & X \arrow[l, shift left, "\langle 1,1\rangle"] & X \arrow[l, "1_X"']
\end{tikzcd} \tag{6.9}$$

meaning that 1_X commutes with itself and hence $\mathbb{C}$ is additive. Since trivially (c)$\Rightarrow$(b), it remains to prove that (a)$\Rightarrow$(c). However this easily follows from the fact that if $\mathbb{C}$ is additive, then for each object C in $\mathbb{C}$ the functors P_C and K_C are both parts of equivalences of categories. ■

Since for each category $\mathbb{I}$ there is an isomorphism (the vertical functor) making the diagram

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{I}} \arrow[d, equal] & \mathbf{SplExt}(\mathbb{C})^{\mathbb{I}} \arrow[l, "P^{\mathbb{I}}"'] \arrow[r, "K^{\mathbb{I}}"] \arrow[d] & \mathbb{C}^{\mathbb{I}} \arrow[d, equal] \\
\mathbb{C}^{\mathbb{I}} & \mathbf{SplExt}(\mathbb{C}^{\mathbb{I}}) \arrow[l, "P"'] \arrow[r, "K"] & \mathbb{C}^{\mathbb{I}}
\end{tikzcd} \tag{6.10}$$

commute, according to Proposition 6.1 and Theorem 3.5 as well as Theorem 4.8 of [16], we obtain:

6.8. Corollary. *If $\mathbb{C}$ is pointed and finitely complete, then Conditions 3.5 (a) - (h) and 3.6, for the span $\underline{\mathbf{SE}(\mathbb{C})}$, are all equivalent to $\mathbb{C}^{\mathbf{2}}$ having generic split extensions. Furthermore, if $\mathbb{C}$ is semi-abelian, then these conditions are all further equivalent to $\mathbb{C}$ being action representable and having normalizers.*

While, according to Proposition 4.8 and Theorems 6.4 and 3.5 we obtain:

6.9. Corollary. *If $\mathbb{C}$ is pointed protomodular and finitely complete, then Conditions 3.5 (a) - (h) and 3.6, for the span $\underline{\mathbf{SE}(\mathbb{C})}_\circ$, are all equivalent to $\mathbb{C}$ being locally algebraically cartesian closed.*

In particular, by considering Condition 3.6, for $\underline{\mathbf{SE}(\mathbb{C})}$ and $\underline{\mathbf{SE}(\mathbb{C})}_\circ$, when $\mathbb{C}$ is semi-abelian, we obtain II(e) and II(f) of the introduction, respectively.

Note that both II(e) and II(f) are equivalent to the same condition restricted to where ϕ is required to be an identity morphism. These restricted conditions although simpler are no longer symmetric.

One can show that requiring the existence of universal liftings, as in II(e), where θ and ϕ are monomorphisms is equivalent (via Theorem 2.8 of [9]) to the existence of normalizers. This leads to the question of whether it might be interesting to study the symmetric condition requiring universal liftings as in II(f) but restricted to monomorphisms. This condition turns out to be closely related to algebraic coherence [10] and is part of the subject of a forthcoming joint paper with D. Bourn, A. S. Cigoli and T. Van der Linden.

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