*Theory and Applications of Categories,* Vol. 33, No. 41, 2018, pp. 1318–1345.

# THE LOCALIC ISOTROPY GROUP OF A TOPOS

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ABSTRACT. It has been shown by J.Funk, P.Hofstra and B.Steinberg that any Grothendieck topos $\mathcal{T}$ is endowed with a canonical group object, called its isotropy group, which acts functorially on every object of the topos. We show that this group is in fact the group of points of a localic group object, called the localic isotropy group, which also acts on every object, and in fact also on every internal locale and on every $\mathcal{T}$-topos. This new localic isotropy group has better functoriality and stability property than the original version and sheds some light on the phenomenon of higher isotropy observed for the ordinary isotropy group. We prove in particular using a localic version of the isotropy quotient that any geometric morphism can be factored uniquely as a connected atomic geometric morphism followed by a so called "essentially anisotropic" geometric morphism, and that connected atomic morphisms are exactly the quotients by open isotropy actions, hence providing a form of Galois theory for general (unpointed) connected atomic geometric morphisms.

## 1. Introduction

In [4], J.Funk, P.Hofstra and B.Steinberg have introduced the idea of isotropy group of a topos. They have shown that any Grothendieck topos $\mathcal{T}$ has a canonical group object $Z_{\mathcal{T}}$ called the isotropy group of $\mathcal{T}$ which acts (also canonically) on every object of $\mathcal{T}$, and such that any morphism of $\mathcal{T}$ is compatible with this action. They have also been considering the "isotropy quotient" $\mathcal{T}_Z$ which is the full subcategory of $\mathcal{T}$ of objects on which the action of $Z_{\mathcal{T}}$ is trivial, it is a new Grothendieck topos (different from $\mathcal{T}$ if $Z_{\mathcal{T}}$ is non-trivial) endowed with a connected atomic geometric morphism $\mathcal{T} \to \mathcal{T}_Z$. It also happens that in some case this topos $\mathcal{T}_Z$ can have itself a non-trivial isotropy group and this construction can be iterated, which has been referred to as "higher isotropy" (see [3])

It has also been observed that this isotropy group is the internal automorphism group of the universal point of $\mathcal{T}$. For example if $\mathcal{T}$ is a classifying topos $\mathcal{S}[\mathbb{T}]$ for some geometric theory $\mathbb{T}$, then the isotropy group is the internal automorphism group of the universal model of $\mathbb{T}$ in $\mathcal{S}[\mathbb{T}]$. This was first conjectured by Steve Awodey and a result of this kind appears in the PhD thesis of his student Spencer Breiner ([1]). From there, following some classical ideas from topos theory (see for example [2]) it is natural to look at the

This work was partly supported by the Grant agency of the Czech republic under the grant P201/12/G028.

Received by the editors 2017-12-19 and, in final form, 2018-12-02.

Transmitted by Martin Hyland. Published on 2018-12-16.

2010 Mathematics Subject Classification: 18B25, 03G30.

Key words and phrases: Topos, Isotropy, localic groups.

automorphism group of a point (or of a model of a theory) not as a discrete group but as a topological or better a localic group. This suggests that the isotropy group should arise naturally as a localic group.

The goal of this paper is to develop this idea: We introduce in section 2 the isotropy group of an object in a general weak $(2,1)$-category with pseudo-limits. The localic isotropy group of a topos corresponds to a special case of this construction, in the category of Grothendieck toposes. The isotropy group of an object $X$ is an object that classifies pairs made of a generalized element of $X$ and an automorphisms of that element, it is a special case of the free loop space in homotopy theory. We show that every object of the slice category $\mathcal{C}_{/X}$ has an action of this isotropy group. In the case of topos theory this means that every topos over a fixed Grothendieck topos $\mathcal{T}$ has such an action by the localic isotropy group of $\mathcal{T}$. This includes in particular an action on every object of $\mathcal{T}$, via the action of the isotropy group on $\mathcal{T}_{/X} \to \mathcal{T}$. In section 3, we briefly review a well known result of A.Joyal and M.Tierney (in [8]) that every topos is a topos of equivariant sheaves on an open étale-complete localic groupoid, and we show how to compute explicitly the localic isotropy group and its isotropy action from such a description. This is our main source of example for this theory. In section 4 we introduce the notion of isotropy quotient adapted to the localic isotropy group, i.e. the fact that the subcategory of $\mathcal{T}$ of objects on which the isotropy action is trivial, is a topos $\mathcal{T}_G$ endowed with a hyperconnected geometric morphism $\mathcal{T} \to \mathcal{T}_G$. It is no longer the case in general that the quotient map is atomic. One can also consider isotropy quotients by arbitrary localic groups endowed with an "isotropy action" i.e. a morphism to the isotropy group.

Section 5 is the most important and technical. We focus on what happens when we take an isotropy quotient by a localic group which is locally positive (i.e. open, or overt), in this case one recovers that the map $\mathcal{T} \to \mathcal{T}_G$ to the isotropy quotient is atomic and connected, and contrary to the ordinary case one gets that the localic isotropy group of the isotropy quotient is nicely controlled by the isotropy group of the initial topos and the group which serves to construct the quotient, preventing in particular any kind of "higher isotropy" phenomenon in this case.

Conversely we also prove that any connected and atomic geometric morphism can be seen as an isotropy quotient by a locally positive isotropy group. Finally we see that any topos admits a "maximal positive isotropy quotient" which produces for any topos $\mathcal{T}$ a connected atomic geometric morphism $\mathcal{T} \to \mathcal{T}_{I^+}$ where $\mathcal{T}_{I^+}$ is "essentially anisotropic" i.e. the isotropy group of $\mathcal{T}_{I^+}$ has no locally positive sublocales other than 1. Applying this to an arbitrary basis gives a unique factorization of any geometric morphism into a connected atomic morphism followed by an essentially anisotropic morphism. However, this does not produces an orthogonal factorization system because the class of essentially anisotropic morphisms is not stable under composition.

Finally in section 6 we explain how the ordinary isotropy group mentioned in the beginning of the introduction (which we call the étale isotropy group) relates to our localic isotropy group and how the theory developed in [4] fits into ours.

All toposes considered in this paper are Grothendieck toposes over some base elementary topos $\mathcal{S}$ with a natural number object. By that we mean that they are (equivalent to) toposes of $\mathcal{S}$-valued sheaves over some $\mathcal{S}$-internal site, or equivalently that they are bounded $\mathcal{S}$-toposes. Morphisms of toposes are the geometric morphisms over $\mathcal{S}$. The 2-category of Grothendieck toposes and geometric morphisms over $\mathcal{S}$ is denoted Top (with the convention that 2-morphisms are the natural transformation between the inverse image functors).

In particular everything done in this paper can be done over an arbitrary base topos and we will use this to obtain relative versions of results proved over $\mathcal{S}$.

Also, even though Top actually is a strict to 2-category, we follow the $\infty$-categorical tradition to treat it as a weak 2-category (i.e. a bi-category). In particular all limits mentioned are by default pseudo-limits and when one says that two objects are isomorphic (or in an abuse of language “the same”) we actually just mean that they are equivalent (as actual isomorphism and strict limits are not the correct notion in a bicategory, for example, they are not preserved by equivalence of bicategories).

If $\mathcal{T}$ is a topos, we will tend to identify internal locales in $\mathcal{T}$ (i.e. $\mathcal{T}$-locales) with the corresponding localic $\mathcal{T}$-topos of $\mathcal{T}$-valued sheaves over them. In particular, the product in the category of $\mathcal{T}$-locale is denoted $\times_{\mathcal{T}}$ as it coincides with the fiber product over $\mathcal{T}$ in the category Top. If $\mathcal{L}$ is a locale in $\mathcal{T}$ (or a localic $\mathcal{T}$-topos) and $f$ a geometric morphism $f : \mathcal{E} \to \mathcal{T}$ we denote by $f^\sharp \mathcal{L}$ the pullback of the locale $\mathcal{L}$ to $\mathcal{E}$ along $f$, i.e. the $\mathcal{E}$-locale corresponding to the completion of the pullback of the $\mathcal{T}$-frame defining $\mathcal{L}$. It also corresponds to the pseudo-pullback in the 2-category of toposes of $\mathcal{L} \to \mathcal{T}$ along $f : \mathcal{E} \to \mathcal{T}$.

Finally the paper relies a lot on relatively advanced tools of topos theory and the reader will need some familiarity with those. Section 2 takes place in an arbitrary 2-category and uses almost no topos theory (one just needs to know that the 2-category of Grothendieck toposes over a base has finite pseudo-limits). Section 3 exploit the groupoid representation theorem of A.Joyal and M.Tierney ([8]), and some familiarity with it will clearly be of help. We refer to descent theory as well (also introduced, and very nicely presented in [8]). Section $C5$ of [6] also covers these topics. For Section 4 one needs to know what are hyperconnected geometric morphisms (see [6, A.4.6]) and one of the remarks relies on an understanding of how colimits of toposes are computed (or equivalently the existence of colimits and the existence of the object classifier topos $\mathcal{S}[\mathbb{O}]$, i.e. the classifying topos of the theory of objects). Some understanding of how the internal logic of toposes works will be useful at several places, but is only really necessary for section 5. Overall, section 5 is the one that really require a lot of topos theoretical material. Good knowledge of the theory of open geometric morphisms [6, C3.1] and of atomic geometric morphisms [6, C3.5], as well as basic descent theory (along open maps) as in [6, C5.1] (or in the nice exposition in [8] already mentioned above). The notion of fiberwise closedness for sublocales (from [6, C1.1], just above lemma C1.1.22) will be mentioned several time in section 5 and 6, but these can be omitted if the reader is unfamiliar with this notion, it is essentially just a slight modification of the notion of closed sublocale which in some situations is better behaved when one works over a non-boolean basis. Section 6 will require strictly less material, it will probably be more interesting to a reader that already knows about the classical isotropy theory of [4], but this is not completely necessary.

I also want to thank a few people: first this paper might have never existed if Pieter Hofstra and Jonathon Funk had not push me to write it after I mentioned these results to them. The second version paper has been notably influenced by a summary of this work written by Mike Shulman on nLab which was illuminating even to me. Finally, the anonymous referee made some very helpful remarks about the first version and pushed me to considerably improve several aspects of the paper, I'm also thankful for the number of typos he corrected.

## 2. Isotropy group and isotropy action in a 2-category

While we are mostly concerned in this paper with the isotropy group of toposes, it appears that the existence of such an isotropy group and isotropy action is a general fact about 2-categories. The anonymous referee suggested that we presented this theory in its general form. Though we haven't really been able to find references in the literature that present the following construction, it was clearly known before this work, and what follows is by no means new.

In this section we work with a general weak $(2,1)$-category $\mathcal{C}$, i.e. a bi-category in which every 2-cell is invertible. One writes $\circ_0$ for horizontal composition (i.e. composition of 1-arrows, horizontal composition of 2-arrows and whiskering of a 1-arrow and a 2-arrow) and $\circ_1$ for vertical composition of 2-arrows.

$\mathcal{C}$ is assumed to have all finite pseudo-limits. As we will only consider the appropriate 2-categorical notion, we will say limits instead of pseudo-limits, when we say that an object has a group structure, we mean that the operation $G \times G \to G$ is associative up to a 2-cell satisfying Mac Lane pentagon, and the "group-like" condition, which can be formulated for example as the fact that the map $G \times G \to G \times G$, given by multiplication on the first component and projection on the second component, is invertible, this can also be encoded as the existence of an "inverse" function $G \to G$. Similarly, an action of $G$ on an object $X$ is a map $G \times X$ with a compatibility 2-cell between the two natural maps $G \times G \times X \rightrightarrows X$ satisfying Mac Lane pentagon. By a presheaf on $\mathcal{C}$ we mean a pseudo-functor from $\mathcal{C}$ to the 2-category of groupoids, the Yoneda embedding from $\mathcal{C}$ to the bi-category of presheaves is denoted $\mathcal{Y}$. When the 2-cells necessary for any of these structures are clear we will not mention them to keep the exposition light.

Note that the only example we have in mind is the category $\mathrm{T{\scriptsize OP}}$ mentioned in the introduction, i.e. the 2-category of Grothendieck $\mathcal{S}$-toposes, geometric morphisms between them and (invertible) natural transformations between their inverse image functors. Here $\mathcal{S}$ is any elementary topos, with a natural number object. We will also look at the 2-category of groupoids to give a simple example.

2.1. Definition. *Fix $X \in \mathcal{C}$ one denotes by $\mathcal{I}_X$ the presheaf on $\mathcal{C}$:*

$$\mathcal{I}_X(T) = \{f : T \to X, \theta : f \Rightarrow f\}$$

*with the morphisms between* $(f, \theta)$ *and* $(f', \theta')$ *being the* $\eta : f \Rightarrow f'$ *such that* $\theta' \circ_1 \eta = \eta \circ_1 \theta$*, and functoriality along* $h : T' \to T$ *given by* $(f, \theta) \mapsto (f \circ_0 h, \theta \circ_0 h)$*.*

2.2. Proposition. *The presheaf* $\mathcal{I}_X$ *is representable by the pullback:*

$$\begin{tikzcd} I_X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

*Moreover, the two maps* $I_X \to X$ *are both equivalent to the map* $\mathcal{I}_X \to \mathcal{Y}(X)$ *given on presheaves by* $(f, \theta) \mapsto f$

Proof. Let $I_X$ be defined as the pullback in the proposition, $\mathcal{Y}(I_X)$ is equivalent to the presheaf:

$$\mathcal{Y}(I_X)(T) = \{f, g : T \to X; \theta_1, \theta_2 : f \Rightarrow g\}$$

with the natural choice of morphisms and functoriality. This presheaf is functorially equivalent to $\mathcal{I}_X$ via:

$$\begin{array}{rcl} (f, \theta) & \mapsto & (f, f, \theta, Id) \\ (f, \theta_2^{-1} \circ_1 \theta_1) & \leftarrow\!\shortmid & (f, g, \theta_1, \theta_2) \end{array}$$

■

The image by the Yoneda embedding of $I_X \times_X I_X$ is equivalent (over $X$) to the presheaf:

$$T \mapsto \{f : T \to X, \theta_1, \theta_2 : f \Rightarrow f\}$$

And more generally, the $n$-fold fiber product of $I_X$ over $X$ corresponds to the similar presheaf with $n$-automorphisms $\theta_1, \ldots, \theta_n : f \Rightarrow f$. One can hence define a map:

$$I_X \times_X I_X \to I_X$$

sending $(f, \theta_1, \theta_2)$ to $(f, \theta_1 \circ_1 \theta_2)$ (functoriality boils down to the exchange law).

2.3. Proposition. $I_X$ *is a group object in* $\mathcal{C}_{/X}$*, with the group structure given by composition of* 2*-cells.*

Indeed, one has an isomorphism between the two maps

$$I_X \times_X I_X \times_X I_X \rightrightarrows I_X$$

using the description of the $n$-fold product of $I_X$ over $X$ given above, the pentagon axiom can be checked by using the description of the 4-fold product and inverses are obtained using the same techniques.

2.4. Example. Let's give an elementary examples of this. Let $\mathcal{C}$ be the 2-category of groupoids. Take $X = BG$ to be the groupoid with one object $*$ and $G$ as its automorphism of $*$. As any groupoid is a coproduct of such groupoids, this is not far from being the general situation. The isotropy group of $BG$ is the groupoid that classifies a point of $BG$ together with an automorphisms of this point. This groupoids has one object $(*, g)$ for each element $g \in G$, and it has automorphisms given by $h : (*, g) \to (*, g')$ where $h$ is an element of $G$ such that $g'h = hg$, the map to $BG$ send $h : (*, g) \to (*, g')$ to $h : * \to *$. So it corresponds to the action groupoid of $G$ acting on itself by conjugation, with its canonical map to $BG$.

The 2-category $\mathcal{C}_{/BG}$ of groupoids with a map to $BG$ is equivalent to the 2-category of groupoids endowed with an action of $G$; given a groupoid $Y$ with a $G$ action one attaches to it a groupoid over $BG$ by forming the crossed product groupoid $Y \rtimes G$, with its canonical map to $* \rtimes G = BG$. Through this equivalence, the $I_X \in \mathcal{C}_{/X}$ we computed above can be identified with $G$ seen as a discrete groupoid (with no non-identity morphisms) and $G$ acting on it by conjugation. The multiplication of $G$: $G \times G \to G$ and the inverse map $G \to G$ are both equivariant for the conjugation action, so $G$ with the conjugation action is indeed a group object in the category of groupoids with an action of $G$.

2.5. Proposition. *Any object of the category $\mathcal{C}_{/X}$ has a canonical action, in $\mathcal{C}_{/X}$, by the group object $I_X$, called the isotropy action. Any morphism in $\mathcal{C}_{/X}$ is canonically equivariant for this action, more generally the functor from $\mathcal{C}_{/X}$ to the 2-category of objects of $\mathcal{C}_{/X}$ endowed with an action of $I_X$ endowing each object with its canonical action is fully faithful.*

The main point in the proof of this proposition is that a morphism in $\mathcal{C}_{/X}$ between two objects $f : A \to X$ and $g : B \to X$ is given by a pair of a 1-arrow $h : A \to B$ and a 2-arrow $\theta : f \to g \circ_0 h$, the action of $I_X$ on an object will be on the second component only. In particular, we emphasize that the action of $I_X$ on an object $Y$ of $\mathcal{C}_{/X}$, as a map $I_X \times_X Y \to Y$ can be non-trivial as a morphism in $\mathcal{C}_{/X}$ but its image by the "forgetful" functor $\mathcal{C}_{/X} \to \mathcal{C}$ will always be trivial (i.e. the projection map).

Proof. Let $f : Y \to X$ be an object of $\mathcal{C}_{/X}$. The product $I_X \times_X Y$ can be described as the presheaf on $\mathcal{C}$ :

$$Hom(Z, I_X \times_X Y) \simeq \{v : Z \to Y, \theta : f \circ v \Rightarrow f \circ v\}$$

with the obvious morphisms and functoriality. The map to $X$ is just $f \circ \pi_2 : (v, \theta) \mapsto f \circ v$. One can hence define the action $I_X \times_X Y \to Y$ as $\pi_2 : (v, \theta) \mapsto v$ but with the 2-cell $\theta$, instead of the identity, to fill the triangle:

$$\begin{tikzcd} I_x \times_X Y \arrow[r, "\pi_2"] \arrow[d, "f \circ \pi_2"] & Y \arrow[dl, "f"] \\ X & \end{tikzcd}$$

The fact that this is an action and all the other claims are easy verify from this description, for example, if $f : Y \to Z$ and $\alpha : g \to h \circ f$ is a morphism in $\mathcal{C}_{/X}$ from $g : Y \to X$ to $h : Z \to X$, then saying that $f$ is $I_X$-equivariant is the data of a 2-cell in $\mathcal{C}_{/X}$ making the square:

$$\begin{tikzcd} I_X \times_X Y \arrow[r] \arrow[d] & I_X \times_X Z \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

One can take the identity as this 2-cell, the non-trivial part is to check that this is indeed a morphism in $\mathcal{C}_{/X}$ i.e. that the two different 2-cells, obtained by both sides of the square which asserts that the map $I_X \times_X Y \to Z$ is a map in $\mathcal{C}_{/X}$, are equal. But on both sides they are just the composite of $\alpha$ with the canonical automorphism $\theta$ of the arrow $I_X \to X$. ■

2.6. Example. On the example of $BG$ in the 2-category of groupoids (see 2.4) the isotropy action corresponds to the fact that for any groupoid $Y$ with an action of $G$, the action map $G \times Y \to Y$ is $G$-equivariant when the $G$ component is endowed with the conjugation action. Hence any object of $\mathcal{C}_{/X}$ has a canonical (and functorial) action of $I_X = G$ with the conjugation action.

All the above can be applied to slices of the category $\mathcal{C}$ itself:

2.7. Definition. *If $f : X \to Y$ is a morphism in $\mathcal{C}$, one denotes by $I_{X/Y}$ the isotropy group of $X$ seen as an object of $\mathcal{C}_{/Y}$.*

One immediately obtains that:

- $I_{X/Y}$ can be defined as the pullback:

$$\begin{tikzcd} I_{X/Y} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X \times_Y X \end{tikzcd}$$

- As an object of $\mathcal{C}$, $I_{X/Y}$ represents the presheaf:

$$Z \mapsto \{v : Z \to X, \theta : v \Rightarrow v | f \circ_0 \theta = Id_{f \circ_0 v}\}.$$

  Indeed, the $\theta$ satisfying this last condition are exactly the 2-cell in $\mathcal{C}_{/Y}$.

- From the previous point it follows that there is a group morphism (over in $\mathcal{C}_{/X}$) $I_{X/Y} \to I_X$ and the isotropy action of $I_{X/Y}$ on an object of $\mathcal{C}_{/X}$ factors through the action of $I_X$ and this morphism.

  In terms of presheaves, this morphism just forget the condition $f \circ_0 \theta = Id_{f \circ_0 v}$.

2.8. Lemma. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Let $f^*I_Y$ be the pullback of the isotropy group of $Y$ to $X$, then there is a pullback square:*

$$\begin{tikzcd}
f^*I_Y \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \times_Y X \arrow[r] & X \times X
\end{tikzcd}$$

Proof. Indeed a morphism to this pullback is the data of a morphism $h$ to $X$, a pair of morphism $h_1$,$h_2$ to $X$, an isomorphism $\theta$ between $f \circ h_1$ and $f \circ h_2$ and a pair of isomorphism $h_1 \simeq h \simeq h_2$. But this boils down to the data of a morphism $h$ to $X$ together with an automorphism of $f \circ h$ as a morphism to $Y$, which is exactly a morphism to $f^*I_Y$. This can also be reformulated in terms of pullback squares:

The two squares below are pullback, so the outer rectangle also is:

$$\begin{tikzcd}
f^*I_Y \arrow[r] \arrow[d] & I_Y \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\
X \arrow[r, "f"] & Y \arrow[r, "\Delta"] & Y \times Y
\end{tikzcd}$$

and in

$$\begin{tikzcd}
f^*I_Y \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\
X \arrow[r, "\Delta"] & X \times X \arrow[r, "f \times f"] & Y \times Y
\end{tikzcd}$$

the rightmost square is a pullback for formal reasons and the outer rectangle is a pullback because of the observation above, so this proves that the leftmost square is also a pullback. ∎

2.9. Proposition. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$. There is a natural comparison map $I_X \to f^*I_Y$ which is a group morphism in $\mathcal{C}_{/X}$. Moreover this comparison map fits into a pullback square:*

$$\begin{tikzcd}
I_X \arrow[r] \arrow[d] & f^*I_Y \arrow[d] \\
X \arrow[r, "\Delta"] & X \times_Y X
\end{tikzcd}$$

Proof. The comparison map is easily defined in terms of maps of presheaves: A morphism to $f^*I_Y$ is the data of a morphism $v$ to $X$ together with an automorphism of $f \circ v$. A morphism to $I_X$ is the data of a morphism into $X$ together with an automorphism of this morphism. One easily associates to it a morphism to $f^*I_Y$ by simply applying $f \circ_0 \_$ to the automorphism.

This comparison map can be equivalently obtained from the pullback:

$$\begin{tikzcd}
I_X \arrow[r] \arrow[d] & f^*I_Y \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\
X \arrow[r, "\Delta"] & X \times_Y X \arrow[r, "Id\times Id"] & Y \times Y
\end{tikzcd}$$

Here the rightmost square is the pullback square of the lemma and the outer rectangle is the definition of $I_X$ as a pullback, which proves the existence of this comparison map, and moreover that this comparison map is a pullback of the diagonal map $X \to X \times_Y X$. ∎

2.10. Proposition. *Let $f : X \to Y$ be a geometric morphism, then $I_{X/Y}$ is the kernel of the comparison map above, i.e. the sequence:*

$$1 \to I_{X/Y} \to I_X \to f^*I_Y$$

*is exact.*

*In particular, if $I_Y = Y$ then $I_{X/Y} \simeq I_X$.*

Proof. This is clear on generalized points: A morphism to $I_{X/Y}$ is by definition a morphism $v$ to $X$ and an automorphism $\phi$ of $v$ such that $f \circ \phi$ is the identity. Such a couple $(v, \phi)$ without the last condition is the same as a morphism to $I_X$ and the last condition exactly says that the image into $f^*I_Y$ is the constant equal to the unit element.

This can also be obtained from the pullback square of proposition 2.9, indeed the kernel of $I_X \to f^*I_Y$ is the pullback of this map along the unit $X \to f^*I_Y$, but in:

$$\begin{tikzcd}
I_{X/Y} \arrow[r] \arrow[d] & X \arrow[d] \\
I_X \arrow[r] \arrow[d] & f^*I_Y \arrow[d] \\
X \arrow[r, "\Delta"] & X \times_Y X
\end{tikzcd}$$

the bottom square is the pullback square of proposition 2.9 and the outer rectangle is the definition of $I_{X/Y}$, so the top square is also a pullback. ∎

We conclude this section by some remarks more specific to the category of toposes. We now work in the category Top of Grothendieck $\mathcal{S}$-topos for $\mathcal{S}$ an elementary topos (with a natural number object), one call objects of Top just “toposes”. Equivalently, one works in the (equivalent) 2-category of bounded geometric morphism to $\mathcal{S}$. This category is seen as a $(2, 1)$-category by just dropping the non-invertible 2-arrows, it has all pseudo-limits (see [6, B4.1.1].

The category $\mathrm{TOP}_{/\mathcal{T}}$ is equivalent to the category of Grothendieck $\mathcal{T}$-topos, so the isotropy group of $\mathcal{T}$ is going to be a $\mathcal{T}$-topos and it will act on every other $\mathcal{T}$-topos. Among the $\mathcal{T}$-toposes one has in particular the "étale" $\mathcal{T}$-topos, which corresponds to the geometric morphisms of the form $\mathcal{T}_{/X} \to \mathcal{T}$ for $X$ a sheaf over $\mathcal{T}$ (i.e. an object of the topos $\mathcal{T}$). These form a full subcategory of the 2-category $\mathrm{TOP}_{/\mathcal{T}}$ which is equivalent to the 1-category of sheaves of sets over $\mathcal{T}$ (i.e. the category $\mathcal{T}$ itself), hence one recovers in this way an action of the isotropy group on objects of $\mathcal{T}$.

Another interesting subcategory of $\mathrm{TOP}_{\mathcal{T}}$ is the category of localic geometric morphisms to $\mathcal{T}$, which is known to be equivalent to the category of $\mathcal{T}$-locales, i.e. the opposite of the category of frame objects in $\mathcal{T}$. This subcategory has the advantage of being (equivalent to) a 1-category and hence avoids all the 2-categorical difficulties. Moreover the isotropy group of $\mathcal{T}$ itself belongs to this subcategory.

2.11. PROPOSITION. *For any topos $\mathcal{T}$, the morphism $I_{\mathcal{T}} \to \mathcal{T}$ is localic, i.e. it corresponds to an internal localic group in $\mathcal{T}$.*

PROOF. By lemma 1.2 of [7], the fact that there is an internal locale $I_{\mathcal{T}}$ in $\mathcal{T}$ with this property corresponds to the fact that the geometric morphism $\mathcal{I}_{\mathcal{T}} \to \mathcal{T}$ constructed above is localic, which is the case because the map $\Delta : \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is localic for any topos by [6, B3.3.8], and that the pullback of a localic morphism is again localic by [6, B3.3.6]. ∎

2.12. DEFINITION. *We define the* localic isotropy group $I_{\mathcal{T}}$ *of a topos $\mathcal{T}$ as this internal localic group object.*

One also similarly defines the localic relative isotropy group $I_{\mathcal{T}/\mathcal{E}}$ when $\mathcal{T} \to \mathcal{E}$ is in $\mathrm{TOP}_{/\mathcal{E}}$ as its isotropy group in the 2-category $\mathrm{TOP}_{/\mathcal{E}}$.

## 3. The case of localic groupoids

It is a remarkable result, due to A.Joyal and M.Tierney in [8], that every Grothendieck topos can be represented as the topos of equivariant sheaves on an open localic groupoid. If $\mathcal{G}$ is a localic groupoid with $G_0$ its space of objects and $G_1$ its space of arrows, an equivariant sheaf over $\mathcal{G}$ is a sheaf $\mathcal{F}$ over $G_0$ such that the étale space of $\mathcal{F}$, $Et\mathcal{F} \to G_0$ is endowed with an action of $G_1$:

$$G_1 \times_{G_0} Et\mathcal{F} \to Et\mathcal{F}$$

satisfying the usual axioms for a groupoid action (see for example [8], [11], or [6, B3.4.14(b)]) . The category of equivariant sheaves over a localic (or topological) groupoid always forms a Grothendieck topos.

The goal of this section is essentially to explain how to get a description of the localic isotropy group and its isotropy action in terms of such a groupoid (more precisely in terms of a representation by an étale-complete groupoid, see below). This is the main source of examples of easily computable isotropy groups. As any topos admits such a representation as a localic groupoid, it could be used to give an alternative definition of the localic isotropy group and of the isotropy action.

We start by reminding the reader of the key elements of the representation theorem of A.Joyal and M.Tierney ([8]). It relies on the following succession of key steps:

- For any topos $\mathcal{T}$, there exists an open surjection $G_0 \twoheadrightarrow \mathcal{T}$ with $G_0$ a locale.
- Given any open surjection as above, $G_1 = G_0 \times_{\mathcal{T}} G_0$ is a locale and the two projections $G_1 \rightrightarrows G_0$ are open surjections.
- $G_1 \rightrightarrows G_0$ has the structure of a localic groupoid. The source and target are the two projections, the identity is the diagonal map $G_0 \to G_0 \times_{\mathcal{T}} G_0$, the inverse operation is the exchange of components $G_0 \times_{\mathcal{T}} G_0 \to G_0 \times_{\mathcal{T}} G_0$, and composition is obtained as the projection forgetting the middle component in: $G_1 \times_{G_0} G_1 \simeq G_0 \times_{\mathcal{T}} G_0 \times_{\mathcal{T}} G_0 \to G_0 \times_{\mathcal{T}} G_0 = G_1$.
- The fact that open surjections are effective descent morphisms for sheaves exactly means that sheaves over $\mathcal{T}$ can be identified with sheaves over $G_0$ endowed with an action of $G_1$, i.e. $\mathcal{T}$ can be identified with the topos of equivariant sheaves on the groupoid $G_1 \rightrightarrows G_0$.

Now, open surjections are also of effective descent for locales, so one can also give a description of the category of $\mathcal{T}$-locales as $G_0$-locales, which are equivalent to locales with a map to $G_0$, together with an action of $G_1$ (in the sense of a map $G_1 \times_{G_0} \mathcal{L} \to \mathcal{L}$ satisfying the usual conditions). In fact, in [10], I.Moerdijk have shown that open surjections are of effective descent for toposes themselves (in an appropriate 2-categorical sense) so that one can even describe toposes over $\mathcal{T}$ as toposes over $G_0$ with an action of $G_1$ (again, in an appropriate 2-categorical sense).

Note that, as mentioned above, for any localic groupoid $G_1 \rightrightarrows G_0$, one can form a topos $\mathcal{T}$ of equivariant sheaves over it. If the groupoid is open (i.e. the two maps $G_1 \rightrightarrows G_0$ are open) the map $G_0 \to \mathcal{T}$ corresponding to the functor forgetting the action of $G_1$ is an open surjection, but $G_1$ does not necessarily coincide with $G_0 \times_{\mathcal{T}} G_0$. One hence needs the following definition, which as far as we know was first introduced by I.Moerdijk in [9]:

3.1. Definition. *One says that an open localic groupoid $G_1 \rightrightarrows G_0$ is* étale-complete *if it is equivalent to the groupoids $G_0 \times_{\mathcal{T}} G_0 \rightrightarrows G_0$ for some open surjection $G_0 \twoheadrightarrow \mathcal{T}$.*

Of course, in this definition, $\mathcal{T}$ is always the topos of equivariant sheaves on $G_1 \rightrightarrows G_0$. The results of this section will only apply to étale-complete groupoids, otherwise it is not possible to describe $\mathcal{T}$-locales in terms of the groupoids.

3.2. Proposition. *Given an open étale-complete localic groupoid $\mathcal{G} : (G_1 \rightrightarrows G_0)$ the isotropy group of the topos $\mathcal{T}$ of equivariant sheaves over $\mathcal{G}$, and its isotropy action can be described as follow:*

- *As a locale over $G_0$, it is the pullback of $G_1$ along the diagonal of $G_0$. I.e. in terms of generalized points (in the category of locales), it is the locale of pairs $(x, \theta)$ where $x \in G_0$ and $\theta$ is an automorphism of $x$ in $\mathcal{G}$.*

- *The group structure of $I_{\mathcal{T}}$ over $G_0$ is the restriction of the composition operation and inverse operation of groupoid structure on $G_1 \rightrightarrows G_0$. I.e. in terms of the generalized points description above, it is the composition and inverse of automorphisms in the $\theta$-component.*

- *The action of $G_1$ on $I_{\mathcal{T}}$ (making it a locale in $\mathcal{T}$) is the conjugation action. I.e. in terms of generalized points it attaches to a $(x, \theta) \in I_{\mathcal{T}}$ and a $g : x \to y$ in $G_1$, the element $(y, g\theta g^{-1}) \in I_{\mathcal{T}}$.*

- *Given any locale $\mathcal{L}$ over $G_0$ with an action of $G_1$, seen as a $\mathcal{T}$-locale, the isotropy action of $I_{\mathcal{T}}$ on $\mathcal{L}$ is simply the restriction of the action of $G_1$ on $\mathcal{L}$ to $I_{\mathcal{T}}$.*

Proof. We call $f$ the map $G_0 \to \mathcal{T}$. One will check all four points in terms of generalized points. The first point of the proposition is about describing the pullback of the isotropy group of $\mathcal{T}$ along $f$. i.e. the object which classifies the pairs $(x, \theta)$ where $x$ is a (generalized) point of $G_0$ and $\theta$ is an automorphism of $f(x)$ as a point of $\mathcal{T}$. On the other hand, by its definition as a pullback, $G_1$ classifies pairs of points $x, y$ in $G_0$ together with an isomorphism between their images in $\mathcal{T}$, so the pullback of $G_1$ along the diagonal map $G_0 \to G_0 \times G_0$ indeed classifies the data of a point $x \in G_0$ and $\theta : f(x) \to f(x)$ an automorphism of points of $\mathcal{T}$ and hence is isomorphic to the pullback of $I_{\mathcal{T}}$ to $G_0$. The second point is immediate from this description in terms of generalized points.

The third point is a little harder: One first needs to translate the action of $G_1$ on a pullback to $G_0$ of an object living over $\mathcal{T}$ in terms of generalized points. If $g : \mathcal{X} \to \mathcal{T}$ is a localic morphisms over $\mathcal{T}$, then the pullback $f^\sharp \mathcal{X}$ of $\mathcal{X}$ to $G_0$ classifies (up to unique equivalence) the triples of $p \in G_0, x \in \mathcal{X}$ and $\alpha : f(p) \overset{\sim}{\to} g(x)$ an isomorphism between their images in $\mathcal{T}$. The pullback $G_1 \times_{G_0} f^\sharp \mathcal{X}$ classifies (again up to equivalence) data of $q, p \in G_0$, $\theta : f(q) \overset{\sim}{\to} f(p)$, $x \in \mathcal{X}$ and $\alpha : f(p) \overset{\sim}{\to} g(x)$, such a generalized point can be sent (functorially) to $q$,$x$ and $\alpha \circ \theta$ in $f^\sharp \mathcal{X}$ defining a map $G_1 \times_{G_0} f^\sharp \mathcal{X} \to f^\sharp \mathcal{X}$ and this corresponds to this action.

Our description of the pullback $f^\sharp I_{\mathcal{T}}$ as classifying pairs of $p \in G_0$ and $\theta : f(p) \overset{\sim}{\to} f(p)$ is not in the form of a triple $(p, x, \alpha)$ as above. Such a description gives us instead the data of $p \in G_0$, $t \in \mathcal{T}$, $\theta' : t \overset{\sim}{\to} t$ and $\alpha : f(p) \overset{\sim}{\to} t$, the equivalence between the two description is given by $(p, t, \theta', \alpha) \mapsto (p, \alpha^{-1} t \alpha)$ in one direction and $(p, \theta) \mapsto (p, f(p), \theta, Id)$ in the other direction. Combining the description of the action the previous paragraph with these two equivalences immediately gives the conjugation action as claimed.

The fourth point is very similar. One takes $g : \mathcal{X} \to \mathcal{T}$ an arbitrary localic morphism. Its pullback $f^\sharp \mathcal{X}$ to $G_0$ can be represented as the object whose generalized points are triples $y \in G_0$, $x \in \mathcal{X}$ and $\theta : f(y) \overset{\sim}{\Rightarrow} g(x)$. If $(y, \alpha) \in f^\sharp I_{\mathcal{T}}$ is a generalized point, (i.e. $y$ is a generalized point of $G_0$ and $\theta$ is an automorphism of $f(y)$), then essentially by definition of the isotropy action it acts on the pullback $f^\sharp \mathcal{X}$ by composing the automorphisms $\theta$

by $\alpha$, but this indeed exactly corresponds to the restriction of the action of $G_1$ described above restricted to automorphisms. ■

This proposition allows to give a description of the localic isotropy group on a lot of examples. In fact as any topos admit such a description as a localic groupoid it can be virtually applied to any topos, though in some case the localic groupoid description may be very hard to find explicitly, for example for a presheaf topos. We mention the following very specific case:

3.3. REMARK. A localic group[1] is said to be étale-complete if it is étale-complete when seen as a localic groupoid whose $G_0$ is the terminal locale. This happens for example if $G$ is a discrete group, or if $G$ is a pro-discrete localic group, or $G$ is the localic automorphism group of some algebraic structure (for example if $G$ is the localic group $Aut(\mathbb{N})$, which happen to be a topological group assuming the axiom of choice). In this case, the associated topos $BG$ is the topos of sets endowed with a continuous action of $G$, Internal locales are locale with a $G$-action and the localic isotropy group is $G$ with its action on itself by conjugation. It acts on every $G$-Sets (or $G$-locales) by the canonical action of $G$. Exactly as in the case of the category of groupoids treated in 2.4 and 2.6, one has that given a group $G$ acting on some object $X$, the action map $G \times X \to X$ is $G$-equivariant when $G$ is endowed with its conjugation action.

## 4. Isotropy quotient

4.1. DEFINITION. *If $G$ is any localic group over $\mathcal{T}$ endowed with an isotropy action, i.e. a morphism to $I_{\mathcal{T}}$, we define $\mathcal{T}_G$ to be the full subcategory of $\mathcal{T}$ of objects on which the isotropy action of $G$ is trivial. $\mathcal{T}_G$ is called the* isotropy quotient *of $\mathcal{T}$ by $G$.*

4.2. PROPOSITION. *$\mathcal{T}_G$ is a topos, and the inclusion of $\mathcal{T}_G$ in $\mathcal{T}$ is the inverse image functor of a hyperconnected geometric morphism $p : \mathcal{T} \to \mathcal{T}_G$.*

See [6, A4.6] for the definition and basic properties of hyperconnected geometric morphisms.

PROOF. $\mathcal{T}_G$ is a full subcategory of $\mathcal{T}$ by definition, and because the action of $G$ is equivariant on all morphisms, it is stable under ($\mathcal{S}$-indexed) colimits, finite limits, subobjects and quotients. This is enough to imply the proposition. ■

Note that the isotropy quotient can be defined in a general 2-category as the following colimit (if it exists). If $G$ is a group object of $\mathcal{C}_X$ endowed with an isotropy action $G \to I_X$, the isotropy action of $G$ is determined by a certain 2-cell $\theta : f \Rightarrow f$ where $f$ is the map $G \to X$. One can then define the isotropy quotient of $X$ by $G$ as the object $X_G$ which is universal for having a morphism $v : X \to X_G$ such that $v \circ_0 \theta = Id$.

[1] If one does not assume the law of excluded middle, one might also want to assume that the map $G \to *$ is open. One can also work assuming that the group is compact instead, but we will not discuss this here.

Indeed this can be described as a type of pseudo-co-limits (it is a co-equifier), all such colimits exists in the category of toposes. Objects of the topos $X_G$ (defined as this colimit) can be described as morphism from $X_G$ to the topos $\mathcal{S}[\mathbb{O}]$ which classifies objects, but the universal property of $X_G$ shows that those are exactly the morphisms from $X$ to $\mathcal{S}[\mathbb{O}]$ on which the isotropy action is trivial, so this indeed corresponds to the topos $\mathcal{T}_G$ mentioned above. Now, very little can be said about this construction in a general 2-category and we will only use it in the case of toposes, where the first presentation given is considerably more useful.

4.3. Proposition. *Let $f : \mathcal{E} \to \mathcal{T}$ be a geometric morphism, let $G$ be a localic group over $\mathcal{E}$ endowed with a morphism $v : G \to I_{\mathcal{E}}$. The following conditions are equivalent:*

- *The composite $G \to I_{\mathcal{E}} \to f^{\sharp} I_{\mathcal{T}}$ is equal to 1.*
- *The morphism $v$ admits a (unique) factorization $G \to I_{\mathcal{E}/\mathcal{T}} \to I_{\mathcal{E}}$.*
- *The isotropy action of $G$ is trivial on every object of the form $f^*(X)$ for $X$ an object of $\mathcal{T}$.*
- *The geometric morphism $f$ factors as $\mathcal{E} \to \mathcal{E}_G \to \mathcal{T}$*

Proof. The equivalence of the first two points is exactly proposition 2.10. The equivalence of the second and the third points follows immediately from the universal property of $I_{\mathcal{E}/\mathcal{T}}$, and the equivalence between the last two points follows immediately from the definition of $\mathcal{E}_G$. ∎

The proposition above has an important corollary:

4.4. Corollary. *Let $\mathcal{E} \to \mathcal{T}$ be an isotropy quotient of a topos $\mathcal{E}$, I.e. $\mathcal{T} = \mathcal{E}_G$ for some localic group $G$ with an isotropy action $G \to I_{\mathcal{E}}$, then $\mathcal{T} = \mathcal{E}_{I_{\mathcal{E}/\mathcal{T}}}$.*

Proof. One has a factorization into $\mathcal{E} \to \mathcal{E}_{I_{\mathcal{E}/\mathcal{T}}} \to \mathcal{T}$ corresponding to the relative isotropy quotient of $\mathcal{E}$ over $\mathcal{T}$.

And as $\mathcal{E} \to \mathcal{E}_G$ factor through $\mathcal{T}$ one has that $G \to \mathcal{I}_{\mathcal{E}}$ factor through $I_{\mathcal{E}/\mathcal{T}}$ by proposition 4.3, and hence a second factorization $\mathcal{E} \to \mathcal{T} \to \mathcal{E}_{I_{\mathcal{E}/\mathcal{T}}}$, corresponding to the fact that $\mathcal{T}$ is an isotropy quotient by a "smaller" isotropy action.

In both cases the inverse image functors are inclusions of full subcategories so the existence of these two factorizations implies the result. ∎

The corollary above implies that one has a "Galois theory" classifying the isotropy quotient of a given topos $\mathcal{E}$ in terms of certain subgroups of its isotropy group: those that arise as $I_{\mathcal{E}/\mathcal{T}}$ for some isotropy quotient $f : \mathcal{E} \to \mathcal{T}$. It is also not hard to see that any subgroup that appears as $I_{\mathcal{E}/\mathcal{T}}$ for $f : \mathcal{E} \to \mathcal{T}$ a general geometric morphism also appears as $I_{\mathcal{E}/\mathcal{T}'}$ for $\mathcal{T}'$ the isotropy quotient $\mathcal{E}_{I_{\mathcal{E}/\mathcal{T}}}$. Unfortunately we are lacking of a good characterization of those.

4.5. Open problem. *What are the subgroups of $I_{\mathcal{T}}$ that appear as relative isotropy group $I_{\mathcal{T}/\mathcal{E}}$ of a geometric morphism $f : \mathcal{T} \to \mathcal{E}$ ?*

The description of the isotropy group of a topos represented by an étale-complete localic groupoid given in proposition 3.2 also produces a nice description of the isotropy quotient of such toposes:

4.6. Proposition. *If $\mathcal{T}$ is the topos of equivariant sheaves on an open étale-complete localic groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ then the maximal isotropy quotient of $\mathcal{T}$ is the topos of equivariant sheaves on $\mathcal{G}$ on which the action of $G_1$ restricted to the subspace of "automorphisms" $I \subset G_1$ is trivial.*

By the subspace of automorphisms we mean the pullback:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & G_1 \arrow[d] \\ G_0 \arrow[r, hook, "\Delta"] & G_0 \times G_0 \end{tikzcd}$$

The proposition follows immediately from the fact (proposition 3.2) that $I \to G_0$ above is (up to descent along $G_0 \to \mathcal{T}$) the isotropy group of the topos and that the isotropy action is obtained by restricting the action of $G_1$ to $I$ along the inclusion $I \hookrightarrow G_1$.

The more general isotropy quotient of such toposes also have a similar description: They are isotropy quotient by subgroups $N \subset I$ (over $G_0$) which are stable under the conjugation action of $G_1$ (which we would like to call "Normal subgroup") and the corresponding isotropy quotient are the sheaves on which the action of $N$ is trivial.

An easy special case of this is that if $G$ is an étale-complete open localic group, any normal subgroup $N \subset G$ define an isotropy action, and the corresponding isotropy quotient is the subcategory of objects of $BG$ on which the action of $N$ is trivial, i.e. it corresponds to the topos of $G/N$-sets. It is unclear to us though under what condition $G/N$ is again étale-complete, its étale-completion could in general be a quotient of $G$ by a larger normal subgroup $N \subset \overline{N} \subset G$. This being said we conjecture that in this specific case this does not happen (at least assuming excluded middle, otherwise some assumption of local positivity might be needed).

4.7. Example. Finally, it is important to note that without any assumptions on $G$ it is hard to say more about the map $\mathcal{T} \to \mathcal{T}_G$. Here is an interesting example where this map is relatively general:

Let $\mathcal{T}$ be the classifying topos of the theory of inhabited objects, i.e. the theory with one sort $\mathbb{O}$, with no terms and with only one axiom: $\exists x \in \mathbb{O}$.

Equivalently, $\mathcal{T}$ is the category of functors from the category of finite inhabited sets to the category of sets. One takes $G$ be the full isotropy group of $\mathcal{T}$, and we will see that the isotropy quotient $\mathcal{T}_G$ is just the terminal topos, i.e. the category of sets.

Indeed, for any finite inhabited set $X$, the functor $F \mapsto F(X)$ from $\mathcal{T}$ to Sets corresponds to a point of $\mathcal{T}$, and its automorphisms are exactly the automorphisms of $X$, in particular, this shows that for $F$ in $\mathcal{T}_G$ the action of the isomorphisms of $X$ on $F(X)$ should be trivial, as those can be factored into the isotropy action.

But one can easily see that a presheaf satisfying this condition is automatically constant, and hence that $\mathcal{T}_G$ is the category of sets.

Note that in this case, $\mathcal{T}$ does not "look like a category of group actions" at all and that the diagonal map $\mathcal{T} \to \mathcal{T} \times_{\mathcal{T}_G} \mathcal{T}$ is not a stable epimorphism (i.e. an epimorphism whose pullbacks are also epimorphisms) which is what we would need to apply the same kind of techniques as in the locally positive case treated in the next section.

## 5. Locally positive isotropy

In this section we will restrict our attention to the special case of an isotropy quotient $\mathcal{T}_G$ by a localic group $G \to I_{\mathcal{T}}$ such that $G$ is "locally positive", i.e. such that the map $G \to \mathcal{T}$ is open. It appears that in this case one has considerably more control on the isotropy quotient: one recovers that as in [4] the map $\mathcal{T} \to \mathcal{T}_G$ is connected atomic (proposition 5.4) and moreover one has a very precise control of the isotropy group of $\mathcal{T}_G$ (proposition 5.5). We will show that any connected atomic map can be obtained as such an isotropy quotient (proposition 5.9). We will show that any topos has a maximal locally positive isotropy group and that the corresponding isotropy quotient has no such locally positive isotropy action (proposition 5.8). In particular, this will produce (proposition 5.10) a unique factorization of any morphism into a connected atomic morphism followed by an "essentially anisotropic morphism", i.e. a morphism that has no relative locally positive isotropy action. This however does not exactly constitute an orthogonal factorization system (see remark 5.11).

We start by some recall on local positivity and open maps:

5.1. DEFINITION. *An open subspace $U$ of a locale $X$ is said to be* positive *if whenever $U$ is written as a union of open subspaces:*

$$U = \bigcup_{i \in I} U_i$$

*the indexing set is always inhabited: $\exists i \in I$.*

*A locale is said to be* locally positive *if every open subspace can be covered by positive open subspaces.*

If one uses classical logic, this notion is vacuous: "positive" is just equivalent to non-empty and every locale is locally positive, simply because any non-empty open subspace is the union of just itself and the empty open subspace is the union of the empty family. But within the internal logic of a topos it is a non-trivial notion:

5.2. Proposition. *A locale $\mathcal{L}$ internal to a topos $\mathcal{T}$ is internally locally positive, if and only if the geometric morphism:*

$$sh_{\mathcal{T}}(\mathcal{L}) \to \mathcal{T}$$

*is open. It is an open surjection if and only if $\mathcal{L}$ is internally positive and locally positive.*

Proof. This is [6, C3.1.17] ∎

So for a locale, locally positive is synonymous with "open" or "overt". We prefer the terminology "locally positive" to avoid the annoying double meaning of "open sublocales".

Note that in a locally positive locale $\mathcal{L}$ if $U = \bigcup U_i$ then one also has:

$$U = \bigcup_{i \text{ s.t. } U_i \text{ is positive.}} U_i.$$

Indeed, each $U_i$ is a union of positive open subspaces hence $U$ is the union of all the positive open subspaces which are included in one of the $U_i$, but each such open subspace is automatically included in a positive $U_i$, and hence $U$ is the union of the positive $U_i$.

5.3. Lemma. *Let $G$ be a localic group over $\mathcal{T}$ with a morphism $\theta : G \to I_{\mathcal{T}}$. Assume that the map from $G$ to $\mathcal{T}$ is an open geometric morphism, then the geometric morphism $p : \mathcal{T} \to \mathcal{T}_G$ is essential i.e. the inclusion functor $p^*$ has a left adjoint.*

Proof. Let $X$ be an object of $\mathcal{T}$, and let $\theta_X : G \times X \to X$ be the isotropy action of $G$ on $X$. We will define an equivalence relation on $X$ by the following internal formula:

$$x \sim y := \text{ the open subspace } \{g \in G | gx = y\} \text{ is positive}$$

One can see that (working internally in $\mathcal{T}$ and using that internally $G$ is locally positive) it is an equivalence relation. Let $X_G$ be the quotient of $X$ by this relation. Then:

- The action of $G$ on $X_G$ is trivial, i.e. $X_G \in \mathcal{T}_G$:

  Indeed, as any map in $\mathcal{T}$ the quotient surjection $X \to X_G$ is $G$-equivariant. Internally, let $x \in X$ and $G$ be the union for $y \in X$ of the open subspace $G_{x,y} = \{g | gx = y\}$. As $G$ is locally positive, one can also write $G$ as the union of those $G_{x,y}$ restricted to the $y$ such that $G_{x,y}$ is positive. In particular, all those $y$ are equivalent to $x$ and hence the action of $G$ on $x$ factors into the equivalence class of $x$ and hence is constant in $X_G$. This shows that $X_G$ is an object of $\mathcal{T}_G$.

- Every map from $X$ to an object of $\mathcal{T}_G$ factors through $X_G$:

  Let $f : X \to Y$ be any morphism, with $Y \in \mathcal{T}_G$. Internally in $\mathcal{T}$, let $x, y \in X$ such that $x \sim y$, we will show that $f(x) = f(y)$. If one works internally in (the topos of sheaves over) $G_{x,y}$ (see the remark just below the proof), one has a point $g$ of $G$ such that $y = gx$, hence as $Y$ has a trivial $G$ action and $f$ is $G$-invariant, one can prove internally in $G_{x,y}$ that $f(x) = f(y)$. But by assumption $G_{x,y}$ is positive in $\mathcal{T}$, hence this implies that $f(x) = f(y)$ internally in $\mathcal{T}$.

This shows that $X \mapsto X_G$ is a left adjoint to the inclusion functor $p^* : \mathcal{T}_G \to \mathcal{T}$ and hence concludes the proof. ■

It was pointed out to us that "working internally in the topos $G_{x,y}$", when $x$ and $y$ are already themselves variable of the internal language, is confusing. The following is intended to clarify what is meant by that:

- The "high level" way to make sense of it, is simply that proving an external result using some internal reasoning as done here is a constructively valid form a reasoning. So it can itself be used inside an internal proof, and hence have as input variable of the internal language. This is in this sense that I initially intended this proof to work.

- This being said, the proof can be rephrased as follow to avoid this. One consider a map $f : X \to Y$ where the isotropy action on $Y$ is trivial. In order to show that $f$ factor in $X_G$ one needs to show that the map $f$ equalize the two maps $R \rightrightarrows X$, where $R$ is the object representing the equivalence relation $\sim$. The sentence "let $x, y \in X$ such that $x \sim y$" can be replaced by "we work in $\mathcal{T}_{/R}$", indeed, internally in $\mathcal{T}_{/R}$ one has two canonical elements of $X$ which we call $x$ and $y$, which corresponds to the two maps $R \rightrightarrows X$, and these two elements satisfies $x \sim y$, moreover proving internally in $\mathcal{T}_{/R}$ that $f(x) = f(y)$ is exactly the claim that $f : X \to Y$ equalize $R \rightrightarrows X$. Now the locale "$G_{x,y}$" does makes external sense as a localic morphism $G_{\bullet,\bullet} \to \mathcal{T}_{/R}$: one first form the pullback (of locales over $\mathcal{T}$):

$$\begin{tikzcd} G_{\bullet,\bullet} \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \times G \times X \arrow[r, "\theta_X \times Id_X"] & X \times X \end{tikzcd}$$

  The map $G_{\bullet,\bullet} \to X \times G \times X$ is open (it is a pullback of $\Delta : X \to X \times X$ which is étale, hence open), and $G \to \mathcal{T}$ is open by assumption, so the composite $G_{\bullet,\bullet} \to X \times G \times X \overset{(\pi_1,\pi_3)}{\to} X \times X$ with the projection is also open. $R \to X \times X$ is by definition the image of this map (which is an open subspace of $X \times X$ as the map is open, hence a subobject of $X \times X$ in $\mathcal{T}$). So one obtains an open surjection $G_{\bullet,\bullet} \twoheadrightarrow R$. The end of the above proof (after the parenthesis) is then a proof, using the internal logic of $G_{\bullet,\bullet}$, that the composites $G_{\bullet,\bullet} \twoheadrightarrow R \rightrightarrows X \to Y$ are indeed equals, which proves that the composites $R \rightrightarrows X \to Y$ are equal as $G \twoheadrightarrow R$ is an open surjection.

  In fact this last part of the proof is simple enough so that it can be done without internal logic at all:

  One can form a commutative cube (where the front and back faces are pullbacks):

$$\begin{tikzcd}
G_{\bullet,\bullet} \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr, "f"] \arrow[dd, "\Delta"] & \\
& G'_{\bullet,\bullet} \arrow[rr] \arrow[dd] & & Y \arrow[dd, "\Delta"] \\
X \times G \times X \arrow[rr, "\theta_X \times Id_X"] \arrow[dr, "{(f,Id,f)}"'] & & X \times X \arrow[dr, "f \times f"] & \\
& Y \times G \times Y \arrow[rr, "\theta_Y \times Id_X"] & & Y \times Y
\end{tikzcd}$$

Now the action map $\theta_Y : Y \times G \to Y$ is be assumption just the projection to $Y$. So the total map of this cube is:

$$G_{\bullet,\bullet} \to X \times G \times X \to Y \times G \times Y \to Y \times Y$$

is given by the pair of maps:

$$G_{\bullet,\bullet} \to R \rightrightarrows X \to Y.$$

But on the other hand, by commutativity of the cube it can also be computed as:

$$G_{\bullet,\bullet} \to X \to Y \xrightarrow{\delta} Y \times Y$$

so the two map above are indeed equal as claimed.

The proof using internal logic is definitely shorter, and it is our opinion that it is also more intuitive and easier to understand than this second proof, but we have to admit that it also requires more trust in the correct use of a very subtle tool. We leave the question of which proof is better to the opinion of the reader.

5.4. Proposition. *If $G$ is a locally positive localic group in $\mathcal{T}$ endowed with an isotropy action (i.e. in particular $f : G \to \mathcal{T}$ is an open geometric morphism), then the natural map from $\mathcal{T} \to \mathcal{T}_G$ is atomic connected in the sense of [6, C3.5].*

Proof. We will prove that the inclusion functor[2] $f^* : sh(\mathcal{T}_G) \to sh(\mathcal{T})$ is a logical functor, i.e. that it preserves the power object. This will prove that the morphism is atomic, and it is connected because we already know that it is hyperconnected.

[2] “$Sh(\mathcal{T})$” is just our notation to distinguish a “topos” from its underlying category, in the same ways one usually distinguish between a locale $X$ from its underlying frame $\mathcal{O}(X)$ in order to avoid confusion on the direction of arrows.

Let $X \in \mathcal{T}_G$, let $\mathcal{P}(X)$ be its power object in $\mathcal{T}$, in order to see that $\mathcal{P}(X)$ is also a power object in $\mathcal{T}_G$ we just have to show that its isotropy $G$-action is trivial.

Let $Y$ be any object of $\mathcal{T}$, and let $V \subset X \times Y$ be any sub-object.

$V$ is in particular stable under the action of $G$. In particular if, internally, $x, y \in Y$ are equivalent under the equivalence relation constructed in the proof of lemma 5.3 and if $(v, x) \in V$ then $(v, y) \in V$ as well. Hence $V$ is the pullback of a subobject of $X \times Y_G$.

This proves that any morphism from $Y$ to $\mathcal{P}(X)$ can be factored into a morphism from $Y_G$ to $\mathcal{P}(X)$ and hence that $\mathcal{P}(X)$ is in $\mathcal{T}_G$ as claimed. ■

Note that the conclusion of proposition 5.4 can fail without the local positivity assumption, in fact proposition 5.9 below shows that an isotropy quotient $f : \mathcal{T} \to \mathcal{E}$ is atomic and connected only if it can be written as an isotropy quotient by a locally positive localic groups (although it may happen that a given isotropy quotient is both a quotient by a locally positive group and by a non locally positive group). The example given in 4.7 also provides an explicit example where the map to the isotropy quotient is not atomic.

5.5. Proposition. *Let $f : \mathcal{E} \to \mathcal{T}$ be a connected atomic geometric morphism. Then the comparison map:*

$$I_{\mathcal{E}} \to f^{\sharp} I_{\mathcal{T}}$$

*is an open surjection.*

So, in the case of a connected atomic morphism one has a “short exact” sequence of localic groups:

$$1 \to I_{\mathcal{E}/\mathcal{T}} \to I_{\mathcal{E}} \to f^{\sharp} I_{\mathcal{T}} \to 1,$$

where we just mean by that, that the map $I_{\mathcal{E}} \to f^{\sharp} I_{\mathcal{T}}$ is an open surjection with kernel $I_{\mathcal{E}/\mathcal{T}}$. But, as open surjections of locales are regular epimorphisms ([6, C.3.1.12]), this implies that $f^{\sharp} I_{\mathcal{T}}$ is the quotient of $I_{\mathcal{E}}$ by its localic normal subgroup $I_{\mathcal{E}/\mathcal{T}}$. Moreover as $f$ is an effective descent morphism of locales one can really think about $I_{\mathcal{T}}$ as being the quotient of $I_{\mathcal{E}}$ by $I_{\mathcal{E}/\mathcal{T}}$, in the sense that $I_{\mathcal{T}}$ can be recovered from $f^{\sharp} I_{\mathcal{T}}$ by descent. This is the proposition that allows us to have some control on the isotropy group of the isotropy quotient in the case where the isotropy quotient is by a locally positive localic group. See proposition 5.8 below for a typical examples of this sort of idea.

Proof. By proposition 2.9 the comparison map is a pullback of the diagonal map $\mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$. As open surjections are stable under pullback (see [6, C3.1.11]), it is enough to show that the diagonal of a connected atomic topos is an open surjection. It is open because of [6, C3.5.14], and it is a surjection by [6, C3.5.6] because it is a section of the morphism $\pi_1 : \mathcal{E} \times_{\mathcal{T}} \mathcal{E} \to \mathcal{E}$ which is (hyper)connected and atomic by [6, C3.5.12]. ■

We now want to show that among the locally positive localic groups with an isotropy action there is a terminal object $I^+_{\mathcal{T}}$ which defines a maximal connected atomic isotropy quotient. The idea is that thanks to the following proposition every locale (in particular $I_{\mathcal{T}}$) has a maximal locally positive sublocale. In particular, the following proposition is specifically meant to be interpreted internally in a topos.

5.6. Proposition. *Let $\mathcal{L}$ be any locale, then:*

- *There is a maximal locally positive sublocale $\mathcal{L}^+ \subset \mathcal{L}$.*
- *Any map from a locally positive locale to $\mathcal{L}$ factor through the inclusion $\mathcal{L}^+ \subset \mathcal{L}$.*
- *$\mathcal{L}^+ \subset \mathcal{L}$ is fiberwise closed (or weakly closed, see [6] just before C1.1.22) inside $\mathcal{L}$.*
- *If $G$ is a localic group then $G^+$ is a localic subgroup*

Proof. The existence of $\mathcal{L}^+$ follows from the fact that a co-product of a small family of locally positive locales is again locally positive and that if $X$ is locally positive and $f : X \to \mathcal{L}$ is a morphism then the (regular) image of $X$ in $\mathcal{L}$ is also locally positive (this follows from [6, C3.1.4(ii)] though the proof might actually be simpler than the translation of this lemma). This also implies the second point. The third point follows from the fact that the fiberwise closure of $\mathcal{L}^+$ in $\mathcal{L}$ is itself locally positive by [6, C3.1.14(ii)]. As for the last point: the terminal locale is locally positive, hence the unit of $G$ lies in $G^+$, and $G^+$ and $G^+ \times G^+$ are both locally positive, hence the inversion and the multiplication map restrict as maps $G^+ \to G^+$ and $G^+ \times G^+ \to G^+$, both because of the second point of the proposition. This shows that $G^+$ is a subgroup of $G$. ■

5.7. Definition. *One says that a geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is* completely anisotropic *if $I_{\mathcal{E}/\mathcal{T}} = \{1\}$ and* essentially anisotropic *if $I^+_{\mathcal{E}/\mathcal{T}} = \{1\}$.*

5.8. Proposition. *Let $f : \mathcal{E} \to \mathcal{T}$ be a geometric morphism, let $G$ be $(I_{\mathcal{E}/\mathcal{T}})^+$ endowed with its natural inclusion map to $I_{\mathcal{E}/\mathcal{T}}$, then the geometric morphism $\mathcal{E}_G \to \mathcal{T}$ is essentially anisotropic.*

Proof. To simplify notation, we work with $\mathcal{T}$ as our base topos, in particular all the isotropy groups mentioned below are considered over $\mathcal{T}$. Let $p$ be the map $\mathcal{E} \to \mathcal{E}_G$. It is connected and atomic by proposition 5.5 because $G$ is locally positive.

We want to prove that $I^+_{\mathcal{E}_G} = \{1\}$, i.e. that any morphism from a locally positive $\mathcal{E}_G$-locale $X$ to $I_{\mathcal{E}_G}$ is constant equal to the unit element. In the rest of the proof we will show instead that any morphism from a locally positive $\mathcal{E}$-locale to $p^\sharp I_{\mathcal{E}_G}$ is constant equal to the unit element. This is sufficient to conclude because if $f : X \to I_{\mathcal{E}_G}$ is as above then the pullback $p^\sharp X \to p^\sharp I_{\mathcal{E}_G}$ is a map from a locally positive $\mathcal{E}$-locale to $p^\sharp I_{\mathcal{E}_G}$ hence, by the claim above is constant, and as $p$ is of effective descent for locale, $p^\sharp$ is in particular faithful, so this implies that $f : X \to I_{\mathcal{E}_G}$ is constant equal to 1.

We fix such a map $X \to p^\sharp I_{\mathcal{E}_G}$ with $X$ a locally positive $\mathcal{E}$-locale. As the comparison map from $I_{\mathcal{E}}$ to $p^\sharp I_{\mathcal{E}_G}$ is an open surjection (by 5.5), if one forms the pullback $Y =$

$X \times_{p^\sharp I_{\mathcal{E}_G}} I_{\mathcal{E}}$ then the projection $Y \to X$ is also an open surjection, hence $Y$ is locally positive, and hence the second projection $Y \to I_{\mathcal{E}}$ factor into $I_{\mathcal{E}}^+ = G$.

But $G$ is in the kernel of the comparison map $I_{\mathcal{E}} \to p^\sharp I_{\mathcal{E}_G}$ hence, as $Y \to X$ is an open surjection, this implies that the map from $X$ to $I_{\mathcal{E}_G}$ is constant equal to 1 and hence proves the result. ■

We are now ready to prove that conversely any connected atomic map is canonically an isotropy quotient by a locally positive isotropy group:

5.9. Proposition. *Let $f : \mathcal{E} \to \mathcal{T}$ be a connected atomic morphism, then:*

- *The relative isotropy group $I_{\mathcal{E}/\mathcal{T}}$ is locally positive in $\mathcal{E}$.*
- *The topos $\mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ is equivalent to the topos of objects of $\mathcal{E}$ endowed with a $I_{\mathcal{E}/\mathcal{T}}$-action. Under this identification, $\Delta^*$ is the functor that forget the action, $\pi_1^*$ is the functor that endows an object with the trivial action and $\pi_2^*$ is the functor that endows an object with its canonical $I_{\mathcal{T}/\mathcal{E}}$-action.*
- *The natural map $\mathcal{E}_{I_{\mathcal{E}/\mathcal{T}}} \to \mathcal{T}$ is an equivalence of toposes.*

For the proof of this proposition we will need to use some results from descent theory. We refer the reader to [6, C5.1] for an introduction to descent theory which contains already a lot more than what we need.

Proof. One has a pullback square:

$$\begin{tikzcd}
I_{\mathcal{E}/\mathcal{T}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "\Delta"] \\
\mathcal{E} \arrow[r, "\Delta"] & \mathcal{E} \times_{\mathcal{T}} \mathcal{E}
\end{tikzcd}$$

but as $f$ is atomic, the arrow $\mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ is open (see [6, C3.5.14]), hence $I_{\mathcal{E}/\mathcal{T}} \to \mathcal{E}$ also is, which proves the first point.

The map $\pi_1 : \mathcal{E} \times_{\mathcal{T}} \mathcal{E} \to \mathcal{E}$ corresponds internally in $\mathcal{E}$ to a connected atomic topos which has a point given by $\Delta : \mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ hence by [6, C5.2.13] it can be identified with the topos of objects of $\mathcal{E}$ endowed with an action of the localic automorphism group of $\Delta$, but this is (by definition) the isotropy group $I_{\mathcal{E}/\mathcal{T}}$. Following the construction of the isotropy action shows that $\pi_2^*$ indeed corresponds to endowing any object with its isotropy action.

Moreover, $f$ (as any hyperconnected morphism) is an effective descent morphism for objects. Hence $\mathcal{T}$ is equivalent to the category of objects of $\mathcal{E}$ endowed with descent data relative to $p$. Once we replace $\mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ by the topos of objects of $\mathcal{E}$ endowed with an action of $I_{\mathcal{E}/\mathcal{T}}$, this descent data is described as an isomorphism between an object $X$ with the trivial isotropy action and $X$ with the canonical isotropy action which is the identity on $X$. Hence the category of such objects endowed with a descent data is just the category of objects whose isotropy action is trivial, which proves the third point. ■

The exact same proof, together with the fact that hyperconnected morphisms are also effective descent morphisms for locales, actually proves a stronger result: the category of locales over $\mathcal{T}$ is equivalent to the full subcategory of locales over $\mathcal{E}$ which have a trivial isotropy action.

In fact, using I.Moerdijk's result from [10] that open surjections (in particular hyperconnected morphisms) are effective descent morphisms in the 2-category of toposes, one can even deduce that the category $\mathrm{TOP}_{/\mathcal{T}}$ is equivalent to the category of toposes over $\mathcal{E}$ endowed with a trivialization over $\mathcal{E}$ of their isotropy action.

5.10. PROPOSITION. *Any geometric morphism $f : \mathcal{E} \to \mathcal{T}$ has a unique factorization (up to unique isomorphisms) as a connected atomic morphism followed by an essentially anisotropic morphism given by:*

$$\mathcal{E} \to \mathcal{E}_{I^+_{\mathcal{E}/\mathcal{T}}} \to \mathcal{T}$$

PROOF. The factorization given in the proposition is clearly a factorization as an atomic connected morphism (by proposition 5.8) followed by an essentially anisotropic morphism (by proposition 5.5). We will now prove the uniqueness of the factorization.

Let $\mathcal{E} \overset{p}{\to} \mathcal{F} \overset{a}{\to} \mathcal{T}$ be any such factorization. Let $G = I_{\mathcal{E}/\mathcal{F}}$. By proposition 5.9, $G$ is locally positive and $\mathcal{F}$ is canonically isomorphic to $\mathcal{E}_G$, and by propositions 2.10 and 5.5 applied with $\mathcal{T}$ as a base one has:

$$1 \to G \to I_{\mathcal{E}/\mathcal{T}} \twoheadrightarrow p^\sharp I_{\mathcal{F}/\mathcal{T}} \to 1$$

$G \to I_{\mathcal{E}/\mathcal{T}}$ factor through $I^+_{\mathcal{E}/\mathcal{T}}$ because $G$ is locally positive. Over $\mathcal{F}$, as $p$ is an open map, the topos $I^+_{\mathcal{E}/\mathcal{T}} \to \mathcal{E} \to \mathcal{F}$ is open, and hence its map to $I_{\mathcal{F}/\mathcal{T}}$ has to factor into $I^+_{\mathcal{F}/\mathcal{T}}$ which is $\{1\}$ because one has assumed that $\mathcal{F} \to \mathcal{T}$ is essentially anisotropic. Hence the map $I^+_{\mathcal{E}/\mathcal{T}} \to p^\sharp I_{\mathcal{F}/\mathcal{T}}$ is constant, and hence $G = I^+_{\mathcal{E}/\mathcal{T}}$ which concludes the proof. ■

One does not get an orthogonal factorization system or the unique lifting property because essentially anisotropic map are not stable under composition, as the following example will show:

5.11. EXAMPLE. Let $\mathcal{T}$ be the topos of sheaves over the real interval $[-1, 1]$ equivariant for the natural multiplication action of $\{1, -1\}$ on $[-1, 1]$ by multiplication, and let $\mathcal{E}$ be the topos of sets endowed with an action of $\{-1, 1\}$. There is a geometric morphism from $\mathcal{E}$ to $\mathcal{T}$ whose inverse image functor is the germ at 0 with the induced action of $\{-1, 1\}$.

The topos $\mathcal{T}$ is essentially anisotropic: $\mathcal{T}$ is attached to an étale localic groupoid (hence open and étale-complete), so its localic isotropy group can be computed using proposition 3.2. By construction, this isotropy is trivial over every point, except over 0 where it is $\{-1, 1\}$, with $-1$ being isolated. As the map $p : [-1, 1] \to \mathcal{T}$ is étale, one can compute the maximal positive subgroup of the isotropy group at the level of $[-1, 1]$, but any map $\mathcal{L} \to p^\sharp I \to [-1, 1]$ such that the composite is open have to avoid the element $-1$ in the fiber over $0 \in [-1, 1]$ as it is an isolated point. So the maximal positive subgroup of $I$ is reduced to 1, i.e. $\mathcal{T}$ is essentially anisotropic. The morphism from $\mathcal{E}$ to $\mathcal{T}$ is an inclusion:

its pullback along the map $p : [-1,1] \to \mathcal{T}$ is simply the inclusion $\{0\} \to [-1,1]$, hence it is completely anisotropic (in particular, essentially anisotropic) but the composite $\mathcal{E} \to *$ is not essentially anisotropic as $\mathcal{E}$ has $\{-1,1\}$ as isotropy group.

On the other hand, completely anisotropic maps are stable under composition because of proposition 2.10 applied relatively to the target of the composition, but it is not clear at all that this produces a factorization system as general isotropy quotient by non locally positive group can be relatively wild and we do not know if for example the isotropy quotient by the full isotropy group is always completely anisotropic or not.

## 6. Comparison to the "étale isotropy group" of J.Funk, P.Hofstra and B.Steinberg

In this section we relate the localic isotropy group $I_{\mathcal{T}}$ of a topos as introduced in the present paper to the isotropy group $Z_{\mathcal{T}}$ of $\mathcal{T}$ introduced by J.Funk, P.Hofstra and B.Steinberg in [4], and we explain how the two notions of isotropy quotient relate. The isotropy group of Funk, Hofstra and Steinberg will be called the "étale isotropy group" to distinguish it from the localic isotropy group. We refer to [4] for its definition. Though it is not necessary to have any knowledge of this previous work to follow this section: one can take the following proposition as the definition the étale isotropy group $Z_{\mathcal{T}}$ and then all the results of [4] that we might need can be seen as special case of the results of the present paper.

6.1. Proposition. *The étale isotropy group $Z_{\mathcal{T}}$ of $\mathcal{T}$, as defined in [4], is the group of points of $I_{\mathcal{T}}$, endowed with its natural isotropy action $Z_{\mathcal{T}} \to I_{\mathcal{T}}$.*

Proof. Let $Z_{\mathcal{T}}$ be the group of points of $I_{\mathcal{T}}$. For any object $X$ of $\mathcal{T}$, the morphisms from $X$ to $Z_{\mathcal{T}}$ are the same as the morphisms of toposes over $\mathcal{T}$ from $\mathcal{T}_{/X}$ to $I_{\mathcal{T}}$, hence they are the same as automorphisms of the morphism from $\mathcal{T}_{/X}$ to $\mathcal{T}$, which is exactly the universal property of the isotropy group defined in [4] (the compatibility with the group structure and the functoriality are immediately checked in same way). ■

Note that, as it is étale, the étale isotropy group is always locally positive, so the isotropy quotient constructed in [4] fits into the theory of quotients by locally positive isotropy groups of the previous section.

This localic picture can be used to understand the higher isotropy phenomenon of [3]:

We start with a topos $\mathcal{T}$, and $I$ its localic isotropy group. Taking the isotropy quotient in the sense of [4] amount to taking the isotropy quotient by $Z \to I$. And we want to understand the isotropy group of $\mathcal{T}_Z$ and whether it has a non-trivial étale isotropy group or not.

A quotient by an étale isotropy group give rise to a connected atomic morphisms $\mathcal{T} \to \mathcal{T}_Z$ (either by the results of [4] or by proposition 5.4), hence proposition 5.5 allows to have some control on the isotropy group of $\mathcal{T}_Z$: Let $\overline{Z}$ be the relative isotropy group

$I_{\mathcal{T}/\mathcal{T}_Z}$, It is a fiberwise closed subgroup of $I$ which contains $Z$ and such that the isotropy quotient by $Z$ and $\overline{Z}$ are the same, we conjecture that $\overline{Z}$ is always exactly the fiberwise closure of $Z$, but this is not known at this point. Then the quotient $I/\overline{Z}$ has unique descent data for the morphism $\mathcal{T} \to \mathcal{T}_Z$ (simply because $\overline{Z}$ acts trivially on it), and the isotropy group of $\mathcal{T}_Z$ is obtained from this descent data. Because of lemma [6, 3.5.5(ii)], points of the isotropy group of $\mathcal{T}_Z$ are also obtained by descent from the points of this quotient $I/\overline{Z}$. So the topos $\mathcal{T}_Z$ will have a non-trivial étale isotropy if and only if $I/\overline{Z}$ have points.

As $\overline{Z}$ contains all the points of $I$, one can indeed expect that it will often be the case that the quotient have no points, but it does not have to be true in general: a localic quotient of a localic group $I$ can have points that do not lift to points of $I$. Constructing localic groups that have no points, or not enough point in the topos of set is a hard task, but fortunately it is considerably easier to do so internally in a topos and one can easily find examples where the situation we just described happen, here is such an example:

6.2. Example. Consider the localic group $G = \mathbb{Q}^* \rtimes \mathbb{Q}_p$ where $\mathbb{Q}_p$ is the formal locale[3] of $p$-adic numbers, as an additive group, and $\mathbb{Q}^*$ is discrete and acts by multiplication on $\mathbb{Q}_p$.

Let $\mathcal{T}$ be the topos of sets endowed with a continuous (smooth) action of $G$. $G$ is locally profinite, so is indeed an étale-complete group, hence because of 3.3 the localic isotropy group of $\mathcal{T}$ is $G$ endowed with its conjugation action. A point of this isotropy group in $\mathcal{T}$ is a point of $G$ which is "smooth" for the conjugation action (i.e. has an open stabilizer for the conjugation action). These correspond exactly to the normal subgroups $\mathbb{Q}_p \subset G$: indeed elements of $\mathbb{Q}_p$ are stabilised by $\mathbb{Q}_p$ which is an open neighborhood of the unit, while no elements having a non-trivial component in the $\mathbb{Q}^*$ direction are stabilized by elements in $\mathbb{Q}_p$ and the neighbourhood of the unit all intersect with $\mathbb{Q}_p$. Hence the étale isotropy group of $\mathcal{T}$ is $\mathbb{Q}_p$ endowed with the discrete topology (a group for addition) and with the action of $G$ by multiplication by the $\mathbb{Q}^*$ component.

So $\mathcal{T}_Z$ is the subcategory of $\mathcal{T}$ of objects on which the $\mathbb{Q}_p$ component acts trivially, i.e. it identifies with the topos $B\mathbb{Q}^*$ of sets with an action $\mathbb{Q}^*$. The relative isotropy group $\overline{Z} = I_{\mathcal{T}/\mathcal{T}_Z}$ is $\mathbb{Q}_p$, this time as a localic group (still with the action by multiplication by the $\mathbb{Q}_p$ component), i.e. it is indeed (at least in this case) the closure of $Z$ in $I$. Finally, the quotient of $I/\overline{Z}$ is $\mathbb{Q}^*$ with trivial action, hence is already étale (and has a lot of points that don't lift to points of $I$).

It can of course happen that this second isotropy quotient again has non-trivial isotropy and so one, and this corresponds to the sequence of isotropy quotients studied in [3]. By the construction above, this sequence can be understood at the level of the localic isotropy group of the first topos: At each step one quotients the remaining group by the normal subgroup of all its points (the quotient having potentially new points so that the process

[3]In order to avoid to explain what that means, lets assume the axiom of choice and simply take $\mathbb{Q}_p$ to be the locally compact topological group of $p$-adic numbers, which is also a localic group because of local compactness. For an interested reader, the constructive definition of this locales have been sketched on mathoverflow at [5]

can keep going) and the successive higher isotropy quotients of the topos corresponds to the sequence of isotropy quotients by the increasing kernel of this sequence of quotients.

Unfortunately, reproducing the other examples of higher isotropy given in [3] seem difficult at this point: these mainly consist in presheaves toposes, and describing the localic isotropy group of a presheaf topos is difficult. I believe it might be possible to do it in full generality (more precisely, I believe it is possible to describe the positive part of the isotropy group of a general presheaf topos), but this would requires some work, in particular on the description of locales internal to a presheaf topos, and this is out of the scope of the present paper. We hope to come back to that question in a future work.

Finally, there is one case where the two theories agrees:

6.3. Proposition. *If the unit map $\mathcal{T} \to I^+_{\mathcal{T}}$ is open, then $I^+_{\mathcal{T}}$ is discrete and is the étale isotropy group $Z_{\mathcal{T}}$. This happens if $\mathcal{T}$ is locally essentially anisotropic, or for example if it is an étendu.*

By "locally essentially anisotropic" one just means that $\mathcal{T}$ admits an étale cover $\mathcal{T}_{/X} \twoheadrightarrow \mathcal{T}$ with $\mathcal{T}_{/X}$ essentially anisotropic. An étendu is a topos of equivariant sheaves over an étale localic groupoid. These are characterized by the fact that they are "locally localic" i.e. in particular locally anisotropic.

Proof. If the unit map of a localic group is open, then the diagonal map $G \to G \times G$ of the group is also open, because it is the pullback of the unit map along the map $(x, y) \to xy^{-1}$. The group of points always factors through $I^+_{\mathcal{T}}$, but under the assumption of the proposition $I^+_{\mathcal{T}}$ ends up being locally positive with an open diagonal. But [6, C3.1.15] exactly says that for any locale if $X \to \{*\}$ and $X \to X \times X$ are open then $X$ is discrete, hence $I^+_{\mathcal{T}}$ is discrete, and as it contains all points of $I_{\mathcal{T}}$ it is exactly the group of points, i.e. $Z_{\mathcal{T}}$.

We now show that when $\mathcal{T}$ is locally essentially anisotropic then this condition is satisfied:

For any slice $\mathcal{T}_{/X}$ of $\mathcal{T}$ the comparison map:

$$I_{\mathcal{T}_{/X}} \to X \times_{\mathcal{T}} I_{\mathcal{T}}$$

is an open inclusion because it is a pullback of the diagonal map $\mathcal{T}_{/X} \to \mathcal{T}_{/X} \times_{\mathcal{T}} \mathcal{T}_{/X}$, which is an open inclusion.

Note that if $\mathcal{L}$ is a locale over $\mathcal{T}_{/X}$ then $\mathcal{L}^+$ is the same whether one see $\mathcal{L}$ as a locale in $\mathcal{T}_{/X}$ or as a locale in $\mathcal{T}$ with a map to $X$, and it corresponds (internally in $\mathcal{T}$) to apply $^+$ to every fibers of this map to $X$. In particular one has that

$$(I_{\mathcal{T}} \times X)^+ = I^+_{\mathcal{T}} \times X,$$

and $I^+_{\mathcal{T}_{/X}} = (I_{\mathcal{T}} \times_{\mathcal{T}} X)^+ \cap I_{\mathcal{T}_{/X}}$.

Hence if $X$ is such that $I^+_{\mathcal{T}_{/X}} = \{1\}$, or more precisely $I^+_{\mathcal{T}_{/X}} = X$ as $X$ is the terminal object of $\mathcal{T}_{/X}$, as we mentioned above the map $I_{\mathcal{T}_{/X}} \to X \times_{\mathcal{T}} I_{\mathcal{T}}$ is open, hence $X \to X \times_{\mathcal{T}} \mathcal{I}^+_{\mathcal{T}}$ is open. Assuming $X$ is inhabited, this implies that the map $1 \to I^+_{\mathcal{T}}$ is open. ■

6.4. Remark. Even for an étendu, the localic group $I_{\mathcal{T}}$ does not have to be discrete. The example mentioned in 5.11 of the topos of equivariant sheaves over $[-1,1]$ with the action of $\{-1,+1\}$ by multiplication, is an étendu with non-trivial isotropy group (because the point corresponding to 0 has a non-trivial automorphism) but the étale isotropy group and the positive isotropy group (isomorphic because of the proposition above) are trivial as mentioned earlier.

Finally, while the localic theory explains and somehow solve[4] the higher isotropy phenomenon observed in [4] and [3], it is not clear that it does not produce a new sort of "higher isotropy". More precisely, we have very little control on the isotropy quotient by an isotropy group which is not locally positive[5], and we do not know the answer to the following question:

6.5. Open problem. *Given $\mathcal{T}$ a topos and $G$ its full localic isotropy group can the isotropy quotient $\mathcal{T}_G$ have a non-trivial localic isotropy group ?*

The description of the full isotropy quotient of a topos $\mathcal{T}$ in terms of presentation by an étale-complete localic groupoid $\mathcal{G}$ given in 4.6, gives a localic groupoid presentation of the isotropy action as the quotient of $\mathcal{G}$ by all its isomorphisms groups. We mean by that the localic groupoid $\mathcal{G}'$ which has the same space of objects as $\mathcal{G}$ and is universal for having a groupoid morphism from $\mathcal{G}$ sending all automorphisms to identities. It seems reasonable to expect (although not completely clear either) that this groupoid $\mathcal{G}'$ is always without automorphisms, which seems to suggest that the isotropy quotient should have no automorphisms. But the main problem is that this new groupoid $\mathcal{G}'$ is apparently in general not[6] étale-complete itself, and so this presentation of the isotropy quotient cannot be used to compute its isotropy group (with proposition 3.2). I don't see any reason either for the "étale-completion" of a localic groupoid with no automorphisms to also be without automorphisms.

Note that however these two types of "Higher isotropy" are very different from one other. The étale higher isotropy comes from the fact the étale isotropy group is "unnatural", in the technical sense that is not compatible to pullback, and also in the sense that it is sometimes regarded as unnatural from the point of view of topos theory to look at the set of points of locales, but there can of course be situations where this is a natural object to look at. On the other hand, the localic higher isotropy, if it exists, would be related to the fact that taking an isotropy quotient by a non locally positive isotropy group is a very poorly behaved operation, while in the étale case it fits into the case of isotropy quotient by locally positive groups treated in section 5 and so it is a very well behaved operation.

[4]In the sense that one has been able to obtain a factorization of geometric morphisms in connected atomic followed by essentially anisotropic by taking the isotropy quotient by the maximal positive isotropy group.

[5]One can probably also develop a similar theory to control isotropy quotient by *compact* localic groups, with no local positivity assumption, using descent along proper maps instead of open maps.

[6]It seems that this happen for example with the example 4.7.

## References

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