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POLYNOMIALS, FIBRATIONS AND DISTRIBUTIVE LAWS

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ABSTRACT. We study the structure of the category of polynomials in a locally cartesian closed category. Formalizing the conceptual view that polynomials are constructed from sums and products, we characterize this category in terms of the composite of the pseudomonads which freely add fibred sums and products to fibrations. The composite pseudomonad structure corresponds to a pseudo-distributive law between these two pseudomonads, which exists if and only if the base category is locally cartesian closed.

1. Introduction

The concept of a polynomial function on natural numbers, built out of sums and products, generalizes naturally to an abstract categorical setting. In a locally cartesian closed category $\mathbb{B}$, a polynomial is defined to be a diagram of shape

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J \tag{1}$$

in $\mathbb{B}$. This determines a functor between slice categories

$$\Sigma_t \Pi_f \Delta_s : \mathbb{B}/I \to \mathbb{B}/J$$

in terms of pullback functors $\Delta_{(-)}$ and their left and right adjoints $\Sigma_{(-)}$ and $\Pi_{(-)}$, and in the internal language of $\mathbb{B}$ represents an indexed family of polynomials

$$(X_i)_{i\in I} \mapsto \left(\sum_{a\in A_j} \prod_{b\in B_a} X_{s(b)} \right)_{j\in J.}$$

Notions of polynomial functors arise in a wide variety of fields (see [Gambino and Kock, 2013] for examples). The categories formed by their polynomial diagrams (also called containers in the computer science literature) provide a simplifying framework in which to work with such functors, and over the last decades the study of these categories has revealed a remarkably rich structure, for example [Abbott, Altenkirch and Ghani, 2003], [Altenkirch, Levy and Staton, 2010], [Gambino and Kock, 2013], [Weber, 2015a], [Walker, 2018].

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In most cases this structure can be constructed by hand, but conceptually it seems more natural to see the structure of polynomials as arising from general constructions of sums and products. In particular, the goal of this paper is to relate categories of polynomials to sums and products for fibrations. The category $\mathbf{Poly}_{\mathbb{B}}$ of polynomial diagrams of shape (1) in a category $\mathbb{B}$ is fibred over $\mathbb{B}$ by projecting to I or J. We show how these fibrations are constructed from the pseudomonad which freely adds sums to fibrations over $\mathbb{B}$ and its opposite which freely adds products.

In detail: we consider spans of fibrations and opfibrations as the 1-cells of a **2-Cat**-enriched bicategory in the sense of Carmody [1995], also called a 2-bicategory (Theorem 4.4). We construct a lax-idempotent pseudomonad $\Sigma_{\mathbb{B}}$ in this 2-bicategory whose pseudoalgebras are fibrations over $\mathbb{B}$ with sums and a colax-idempotent pseudomonad $\Pi_{\mathbb{B}}$ whose pseudoalgebras are fibrations with products. The category $\mathbf{Poly}_{\mathbb{B}}$ of polynomials with its projections to $\mathbb{B}$ is then constructed as the composite span $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ (Theorem 7.2). Moreover composition of polynomials is shown to correspond to a pseudo-distributive law of $\Pi_{\mathbb{B}}$ over $\Sigma_{\mathbb{B}}$ giving their composite $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ the structure of a pseudomonad. Since a span with the structure of a pseudomonad is equivalently a pseudo double category, we recover the pseudo double category of polynomials $\mathbb{P}\mathbf{oly}_{\mathbb{B}}$. The existence of such a pseudo-distributive law is shown to correspond exactly to the requirement that $\mathbb{B}$ be locally cartesian closed (Theorem 7.1).

While polynomial functors are usually defined as ordinary functors between categories, analogues of the pullbacks, sums and products used also make sense for internal categories in a context other than **Set**. We therefore prove our results as far as possible in the more general setting of an internal category $\mathbb{B}$ in some fixed ambient category $\mathcal{E}$.

The need for the weak 3-categorical structure of a 2-bicategory to organize fibrations comes from the fact that pullback and its right adjoint are only associative up to isomorphism. Accordingly there are various coherence conditions to be checked throughout, particularly in the construction of pseudo-distributive laws.

The outline of the paper is as follows. In Section 2 we recall some background on internal categories, spans and 2-bicategories. We then review the construction of internal fibrations and opfibrations as pseudoalgebras for 2-monads (Section 3), and organize these into a 2-bicategory (Section 4). In Section 5 we consider the interaction of pseudomonads via a pseudo-distributive law. Section 6 defines the pseudomonads for sums and products in the 2-bicategory of fibrations. Finally, in Section 7 we review the structure of the pseudo double category of polynomials, and recreate this structure from the composition of pseudomonads.

2. Preliminaries

We start by fixing some notation and reviewing the 2- and 3-categorical structures formed by internal categories and spans.

Let $\mathcal{E}$ be a category with pullbacks. We assume that an explicit choice of this structure is given, so each morphism $f : I \to J$ in $\mathcal{E}$ defines a pullback functor $\Delta_f : \mathcal{E}/J \to \mathcal{E}/I$.

We use $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ to denote internal categories in $\mathcal{E}$, so in other words $\mathbb{A}$ is a diagram

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, shift left=2, "s"] \arrow[r, "m" description] \arrow[r, shift right=2, "t"'] & A_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & A_0 \arrow[l, "i" description] \end{tikzcd}$$

in $\mathcal{E}$ satisfying the usual equations. Internal categories in $\mathcal{E}$ are the objects of a 2-category $\mathbf{Cat}(\mathcal{E})$, where a 1-cell $f : \mathbb{A} \to \mathbb{B}$ is an internal functor comprising morphisms $f_0 : A_0 \to B_0$ and $f_1 : A_1 \to B_1$ which preserve the internal category structure; and a 2-cell $\alpha : f \Rightarrow g$ between 1-cells $\mathbb{A} \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{B}$ is an internal natural transformation, that is a morphism $A_0 \longrightarrow B_1$ satisfying the required equations.

For a category $\mathcal{E}$ with pullbacks, we can also construct the bicategory $\mathbf{Span}(\mathcal{E})$ of spans in $\mathcal{E}$, where the objects are objects of $\mathcal{E}$ and a 1-cell from X to Y (written $X \nrightarrow Y$) is a span of morphisms over X and Y. The 2-cells are morphisms of spans, and composition of 1-cells is given by pullback:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"] & \\ X & & Y \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & & A \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & A \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

$\alpha : A \Rightarrow B$ $\qquad$ $C \circ A : X \nrightarrow Z$

To equip a 1-cell

$$\begin{tikzcd} & A_1 \arrow[dl, "c"'] \arrow[dr, "d"] & \\ A_0 & & A_0 \end{tikzcd}$$

in $\mathbf{Span}(\mathcal{E})$ with the structure of a monad is exactly to equip $A_1 \overset{d}{\underset{c}{\rightrightarrows}} A_0$ with the structure of identities $A_0 \xrightarrow{i} A_1$ and composition $A_1 \times_{A_0} A_1 \xrightarrow{m} A_1$ of an internal category $\mathbb{A}$ in $\mathcal{E}$.

If $\mathcal{K}$ is a 2-category with strict 2-pullbacks, then $\mathbf{Span}(\mathcal{K})$ inherits additional structure. It forms a *bicategory enriched in **2-Cat*** [Carmody, 1995], also called a *2-bicategory* [Weber, 2015a]. A 2-bicategory is a bicategory $\mathfrak{B}$ where each hom-category $\mathfrak{B}(X, Y)$ has the structure of a 2-category, horizontal composition $\circ : \mathfrak{B}(Y, Z) \times \mathfrak{B}(X, Y) \to \mathfrak{B}(X, Z)$ has the structure of a 2-functor, and the unit and associativity isomorphisms are 2-natural (see Definition A.1).

In the 2-bicategory $\mathbf{Span}(\mathcal{K})$, the 3-cells are 2-cells in $\mathcal{K}$

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] \arrow[dd, bend right] \arrow[dd, bend left] \arrow[dd, phantom, "\Rightarrow"] & \\
X & & Y \\
 & B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

which are vertical over X and Y.

A 2-bicategory is a stricter kind of tricategory, and every strict 3-category is a 2-bicategory. Composition makes each hom-2-category $\mathfrak{B}(X, X)$ into a monoidal 2-category. In what follows we will denote horizontal composition by juxtaposition, and suppress the unit and associativity 1-cells. In any 2-bicategory, the notions of pseudomonads, modules, bimodules and so on can be defined in a similar way to those in a bicategory (see Appendix A for details). There will be further examples of 2-bicategories in the rest of the paper.

For a category $\mathcal{E}$ with pullbacks, the 2-category $\mathbf{Cat}(\mathcal{E})$ has strict 2-pullbacks constructed pointwise, so we can form the 2-bicategory $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ of spans of internal functors. A 2-monad in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ is then an internal category $\mathbb{D}_1 \underset{c}{\overset{d}{\rightrightarrows}} \mathbb{D}_0$ in $\mathbf{Cat}(\mathcal{E})$, which is a strict double category in $\mathcal{E}$. This has the objects and morphisms of $\mathbb{D}_0$ as objects and vertical morphisms, objects of $\mathbb{D}_1$ as horizontal morphisms, and morphisms of $\mathbb{D}_1$ as 2-cells. A pseudomonad in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ is a pseudo double category in $\mathcal{E}$ (see [Grandis and Paré, 1999]), where vertical morphisms compose strictly but the composition of horizontal morphisms is only associative up to coherent isomorphism.

3. Fibrations of internal categories

The aim of this section is to set out some of the theory of internal fibrations, extending the view of cloven fibrations as algebras for a monad originally due to Street [1974]. We define what it means for an internal functor to be a fibration or opfibration, and define the opposite of a fibration.

In order to investigate what structure of **Set** is needed to formulate fibrations and later polynomials, we work instead in a general category $\mathcal{E}$, assumed fixed throughout. We start by merely assuming that $\mathcal{E}$ has pullbacks, and add other conditions as they are required. The case $\mathcal{E} = \mathbf{Set}$ will be a running example.

Let $\mathbb{B} \in \mathbf{Cat}(\mathcal{E})$ be an internal category in $\mathcal{E}$, that is, a 2-monad in $\mathbf{Span}(\mathcal{E})$. We consider a particular internal category in $\mathbf{Cat}(\mathcal{E})$, in other words a 2-monad on $\mathbb{B}$ in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ or a strict double category in $\mathcal{E}$.

The internal category of arrows $\mathbb{B}^{\mathbf{2}}$ is constructed using pullbacks in $\mathcal{E}$. This is the cotensor of $\mathbb{B}$ with the category $\mathbf{2} = \bullet \to \bullet$ in the 2-category $\mathbf{Cat}(\mathcal{E})$, so it is equipped with internal functors and an internal natural transformation

$$\begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathbb{B} \end{tikzcd}$$

and is universal with this data.

The internal category $\mathbb{B}^{\mathbf{2}}$ is equivalently described as the lax limit in $\mathbf{Cat}(\mathcal{E})$

$$\begin{tikzcd} & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[d, phantom, "\overset{\Leftarrow}{\varphi}"] & \\ \mathbb{B} & & \mathbb{B} \arrow[ll, "1_{\mathbb{B}}"] \end{tikzcd}$$

of the identity arrow on $\mathbb{B}$.

This universal property of $\mathbb{B}^{\mathbf{2}}$ applied to the internal natural transformations

$$\begin{tikzcd} & \mathbb{B} \arrow[dl, "1_{\mathbb{B}}"'] \arrow[dr, "1_{\mathbb{B}}"] \arrow[d, phantom, "="] & \\ \mathbb{B} & & \mathbb{B} \arrow[ll, "1_{\mathbb{B}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & & \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "="] & & \\ & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[d, phantom, "\overset{\Leftarrow}{\varphi}"] & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[d, phantom, "\overset{\Leftarrow}{\varphi}"] & \\ \mathbb{B} & & \mathbb{B} \arrow[ll, "1_{\mathbb{B}}"] & & \mathbb{B} \arrow[ll, "1_{\mathbb{B}}"] \end{tikzcd}$$

determines maps $\eta : \mathbb{B} \to \mathbb{B}^{\mathbf{2}}$ and $\mu : \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \to \mathbb{B}^{\mathbf{2}}$ giving the span

$$\begin{tikzcd} & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & \\ \mathbb{B} & & \mathbb{B} \end{tikzcd} \tag{2}$$

the structure of a 2-monad $\Phi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$.

3.1. EXAMPLE. [$\mathcal{E} = \mathbf{Set}$] If $\mathcal{E}$ is $\mathbf{Set}$, then $\mathbf{Cat}(\mathcal{E})$ is the category of small categories $\mathbf{Cat}$. The 2-monad $\Phi_{\mathbb{B}}$ is given by the usual category $\mathbb{B}^{\mathbf{2}}$ of arrows and commutative squares, with d and c the domain and codomain functors.

A 2-monad in a 2-bicategory acts by composition as a 2-monad on each of the hom-2-categories. Thus $\Phi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ defines by composition on one side a 2-monad on $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathbb{A}, \mathbb{B})$, and on the other a 2-monad on $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathbb{B}, \mathbb{C})$, for all internal categories $\mathbb{A}$, $\mathbb{C}$ in $\mathcal{E}$. Moreover, the definition of $\Phi_{\mathbb{B}}$ as a limit in a 2-category gives these 2-monads a form of uniqueness property which is characteristic of monads involving limits and colimits. Recall from [Kock, 1995] that a pseudomonad (T, η, μ) on a 2-category is *lax-idempotent* (also called Kock-Zöberlein) if the following equivalent conditions hold:

1. The multiplication μ is left adjoint to ηT with invertible counit.

2. The multiplication μ is right adjoint to $T\eta$ with invertible unit.

3. There is a modification $\theta : T\eta \to \eta T$ satisfying

$$\begin{tikzcd} 1 \arrow[r, "\eta"] & T \arrow[r, bend left, "T\eta"] \arrow[r, bend right, "\eta T"'] \arrow[r, phantom, "\Downarrow\theta"] & T^2 \end{tikzcd} = \begin{tikzcd} & T \arrow[dr, "T\eta"] & \\ 1 \arrow[ur, "\eta"] \arrow[dr, "\eta"'] \arrow[rr, phantom, "\cong"] & & T^2 \\ & T \arrow[ur, "\eta T"'] & \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, bend left, "T\eta"] \arrow[r, bend right, "\eta T"'] \arrow[r, phantom, "\Downarrow\theta"] & T^2 \arrow[r, "\mu"] & T \end{tikzcd} = \begin{tikzcd} & T^2 \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow l"] & \\ T \arrow[ur, "T\eta"] \arrow[dr, "\eta T"'] \arrow[rr, "1" description] & {} \arrow[d, phantom, "\Downarrow r"] & T. \\ & T^2 \arrow[ur, "\mu"'] & \end{tikzcd}$$

4. To give an object A a T-pseudoalgebra structure is exactly to give a left adjoint to the morphism $\eta_A : A \to TA$ with invertible counit.

Dually, a pseudomonad is *colax-idempotent* if the multiplication is right adjoint to ηT with invertible unit.

A pseudomonad in a 2-bicategory is called lax-idempotent if it acts as a lax-idempotent pseudomonad on the left, equivalently if it acts as a colax-idempotent pseudomonad on the right. Dually it is called colax-idempotent if it acts as a colax-idempotent pseudomonad on the left.

3.2. PROPOSITION. $\Phi_{\mathbb{B}} : \mathbb{B} \to\!\!\!\!\!\to \mathbb{B}$ *is a colax-idempotent 2-monad in* $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$.

PROOF. The 2-dimensional universal property of the arrow category $\mathbb{B}^{\mathbf{2}}$ determines two 2-cells

$$\begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[r, bend left, "1"] \arrow[r, bend right, "\eta c"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathbb{B}^{\mathbf{2}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[r, bend left, "\eta d"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathbb{B}^{\mathbf{2}}, \end{tikzcd}$$

which together define a 3-cell

$$\begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[r, bend left, "{(1,\eta d)}"] \arrow[r, bend right, "{(\eta c,1)}"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \end{tikzcd}$$

with the property that $\theta\eta = 1$ and $\mu\theta = 1$. If $\mathcal{E}$ is **Set**, then θ is the natural transformation which sends a morphism $f\colon A \to B$ in $\mathbb{B}$ to the diagram

$$\begin{tikzcd} B \arrow[d, "="'] & A \arrow[l, "f"'] \arrow[d, "f"] & A \arrow[l, "="'] \arrow[d, "="] \\ B & B \arrow[l, "="] & A. \arrow[l, "f"] \end{tikzcd}$$

Such a 3-cell θ corresponds to a modification as in the third condition of the above definition for $\Phi_{\mathbb{B}}$ acting on the right. Hence $\Phi_{\mathbb{B}}$ is colax-idempotent. ∎

We now take a closer look at the 2-monads that $\Phi_{\mathbb{B}}$ induces by composition.

Suppose now that $\mathcal{E}$ has a terminal object (so in fact it has all finite limits). Consider the slice $\mathbf{Cat}(\mathcal{E})/\mathbb{B}$ for an object $\mathbb{B}$. This can be identified with either of the hom-2-categories $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathbb{B}, 1)$ or $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(1, \mathbb{B})$. So composing in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ with the 2-monad $\Phi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ gives two 2-monads

$$\Phi_{\mathbb{B}} \circ (-) : \mathbf{Cat}(\mathcal{E})/\mathbb{B} \to \mathbf{Cat}(\mathcal{E})/\mathbb{B}$$
$$(-) \circ \Phi_{\mathbb{B}} : \mathbf{Cat}(\mathcal{E})/\mathbb{B} \to \mathbf{Cat}(\mathcal{E})/\mathbb{B}$$

which send an object $\mathbb{M} \to \mathbb{B}$ to the composites $\Phi_{\mathbb{B}} \circ \mathbb{M} \to \mathbb{B}^{\mathbf{2}} \xrightarrow{d} \mathbb{B}$ and $\mathbb{M} \circ \Phi_{\mathbb{B}} \to \mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ respectively, as in the diagrams

$$\begin{tikzcd}[column sep=small] & \Phi_{\mathbb{B}} \circ \mathbb{M} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\ \mathbb{M} \arrow[dr] & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & \\ & \mathbb{B} & & \mathbb{B} \end{tikzcd} \qquad (3)$$

$$\begin{tikzcd}[column sep=small] & & \mathbb{M} \circ \Phi_{\mathbb{B}} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\ & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \mathbb{M}. \arrow[dl] \\ \mathbb{B} & & \mathbb{B} & \end{tikzcd} \qquad (4)$$

To distinguish between the pseudoalgebras of the two 2-monads on $\mathbf{Cat}(\mathcal{E})/\mathbb{B}$, we call pseudoalgebras for $\Phi_{\mathbb{B}}$ with a left action as in (3) *left modules*, and pseudoalgebras for $\Phi_{\mathbb{B}}$ with a right action as in (4) *right modules*. Strict algebras are called strict left modules and strict right modules respectively.

3.3. Definition. *An internal functor $\mathbb{M} \to \mathbb{B}$ with the structure of a left module for $\Phi_{\mathbb{B}}$ is called a* fibration*, and with the structure of a right module an* opfibration*; strict left and right modules are* strict fibrations *and* strict opfibrations *respectively.*

Note that since $\Phi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ is colax-idempotent these are 'property-like' structures in the terminology of Kelly and Lack [1997]: a morphism can have at most one module structure up to isomorphism.

3.4. Example. [$\mathcal{E} = \mathbf{Set}$] To give a functor $\mathbb{M} \xrightarrow{p} \mathbb{B}$ in **Cat** the structure of a left module for $\Phi_{\mathbb{B}}$ is exactly to give p the structure of a cloven Grothendieck fibration, that is, to give a chosen cartesian lifting $f^*J \to J$ for each morphism $f : I \to pJ$ in $\mathbb{B}$.

Likewise to give $\mathbb{M} \xrightarrow{p} \mathbb{B}$ the structure of a right $\Phi_{\mathbb{B}}$-module is to give p the structure of a cloven Grothendieck opfibration, that is, to give a chosen opcartesian lifting $I \to f_! I$ for each morphism $f : pI \to J$ in $\mathbb{B}$. Strict fibrations and opfibrations correspond to split Grothendieck fibrations and opfibrations. A morphism of left $\Phi_{\mathbb{B}}$-modules is a functor over $\mathbb{B}$ preserving cartesian morphisms.

The morphism $\mathbb{B}^{\mathbf{2}} \xrightarrow{d} \mathbb{B}$ is naturally a fibration, and $\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ is an opfibration.

3.5. Example. [$\mathcal{E} = \mathbf{Set}$] A category $\mathbb{B}$ has (chosen) pullbacks if $\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ is also a (cloven) fibration: a functor

$$\begin{tikzcd}
\mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \arrow[dr, "\pi_2"] \arrow[ddd, "e"'] & & \\
& \mathbb{B}^{\mathbf{2}} \arrow[dr, "d"] & \\
& & \mathbb{B} \\
\mathbb{B}^{\mathbf{2}} \arrow[urr, "c"'] & &
\end{tikzcd}$$

in $\mathbf{Cat}$ gives c the structure of a left $\Phi_{\mathbb{B}}$-module exactly when e sends a cospan $I \xrightarrow{f} K \xleftarrow{g} J$ in $\mathbb{B}$ to a pullback of f along g.

For a general category $\mathcal{E}$ with finite limits, fibrations in $\mathbf{Cat}(\mathcal{E})$ can be defined representably, by the $\mathbf{Cat}$-enriched Yoneda embedding: all the constructions used to form $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, $\mathbb{B}^{\mathbf{2}}$ and $\Phi_{\mathbb{B}}$ are defined in terms of limits, and are preserved by each hom 2-functor $\mathbf{Cat}(\mathcal{E})(\mathbb{A}, -) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$. Thus an internal functor $\mathbb{M} \xrightarrow{p} \mathbb{B}$ is a fibration if and only if

$$\mathbf{Cat}(\mathcal{E})(\mathbb{C}, \mathbb{M}) \xrightarrow{p\,\circ -} \mathbf{Cat}(\mathcal{E})(\mathbb{C}, \mathbb{B})$$

is a fibration in $\mathbf{Cat}$ for each $\mathbb{C}$ in $\mathbf{Cat}(\mathcal{E})$, and for each $f : \mathbb{C} \to \mathbb{D}$ in $\mathbf{Cat}(\mathcal{E})$ the functor $\mathbf{Cat}(\mathcal{E})(\mathbb{D}, \mathbb{M}) \xrightarrow{-\circ f} \mathbf{Cat}(\mathcal{E})(\mathbb{C}, \mathbb{M})$ preserves cartesian and opcartesian morphisms.

We now show how fibrations can alternatively be viewed as internal categories in a certain category, using lax codescent objects. Taking opposite internal categories then gives a natural definition of the opposite of a fibration.

Recall from [Lack, 2002] that *coherence data* in a 2-category consists of a diagram

$$\begin{tikzcd}
X_3 \arrow[r, shift left=2, "p"] \arrow[r, "q" description] \arrow[r, shift right=2, "r"'] & X_2 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_1 \arrow[l, "e" description]
\end{tikzcd} \tag{5}$$

equipped with invertible 2-cells $\delta : de \to 1$, $\gamma : 1 \to ce$, $\kappa : dp \to dq$, $\lambda : cr \to cq$, and $\rho : cp \to dr$. A *codescent object* of this coherence data is a morphism $x : X_1 \to X$ together with an invertible 2-cell $\xi : xd \Rightarrow xc$ satisfying the coherence axioms

$$\begin{tikzcd}
& X_3 \arrow[rr, "p"] \arrow[dl, "r"'] \arrow[d, phantom, "\Swarrow\rho"] & & X_2 \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[d, phantom, "\Swarrow\xi"] & \\
X_2 \arrow[rr, "d"] \arrow[dr, "c"'] & \arrow[d, phantom, "\Swarrow\xi"] & X_1 \arrow[dr, "x"] & & X_1 \arrow[dl, "x"] \\
& X_1 \arrow[rr, "x"'] & & X &
\end{tikzcd}
=
\begin{tikzcd}
& X_3 \arrow[rr, "p"] \arrow[dl, "r"'] \arrow[dr, "q"] \arrow[d, phantom, "\Swarrow\lambda^{-1}"] & \arrow[d, phantom, "\Swarrow\kappa"] & X_2 \arrow[dr, "d"] & \\
X_2 \arrow[dr, "c"'] & & X_2 \arrow[rr, "d"] \arrow[dl, "c"] \arrow[d, phantom, "\Swarrow\xi"] & & X_1 \arrow[dl, "x"] \\
& X_1 \arrow[rr, "x"'] & & X &
\end{tikzcd}
\tag{6}$$

$$\begin{tikzcd}
& & X_1 \arrow[dr, "x"] & \\
X_1 \arrow[r, "e"] & X_2 \arrow[ur, "d"] \arrow[dr, "c"'] & \Downarrow\xi & X \\
& & X_1 \arrow[ur, "x"'] &
\end{tikzcd}
=
\begin{tikzcd}
& X_2 \arrow[dr, "d"] \arrow[d, phantom, "\Downarrow\delta"] & & \\
X_1 \arrow[ur, "e"] \arrow[rr, "1"] \arrow[dr, "e"'] & \Downarrow\gamma & X_1 \arrow[r, "x"] & X \\
& X_2 \arrow[ur, "c"'] & &
\end{tikzcd}$$

and such that the pair (x, ξ) is universal with this property. A *lax codescent object* is $x : X_1 \to X$ together with a not necessarily invertible $\xi : xd \Rightarrow xc$ satisfying the same coherence axioms and universal with this property.

Let $\mathbb{B}$ be an object of $\mathbf{Cat}(\mathcal{E})$ where $\mathcal{E}$ has finite limits, so we have objects B_0, B_1, and $B_2 = B_1 \times_{B_0} B_1$ in $\mathcal{E}$ and morphisms

$$\begin{tikzcd}
B_2 \arrow[r, shift left=2, "s"] \arrow[r, "m" description] \arrow[r, shift right=2, "t"'] & B_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B_0 \arrow[l, "i" description]
\end{tikzcd}$$

satisfying the required equations. Now B_0, B_1 and B_2 can also be considered as objects in $\mathbf{Cat}(\mathcal{E})$ by giving them the structure of discrete categories, making this diagram into a category object internal to $\mathbf{Cat}(\mathcal{E})$.

The diagram is an instance of the coherence data defined in (5). The inclusions of the discrete categories B_0 and B_1 into $\mathbb{B}$ and its arrow category commute with the corresponding internal domain and codomain functors

$$\begin{tikzcd}
B_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, hook, "x_1"'] & B_0 \arrow[d, hook, "x"] \\
\mathbb{B}^{\mathbf{2}} \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & \mathbb{B}
\end{tikzcd}$$

so we have an internal functor $x : B_0 \to \mathbb{B}$ and an internal natural transformation $\xi = \varphi x_1 : xd \Rightarrow xc$. These are the universal pair satisfying the coherence conditions in (6), making $\mathbb{B}$ into the lax codescent object of the coherence data.

Let $\mathbb{M} \xrightarrow{p} \mathbb{B}$ be a morphism in $\mathbf{Cat}(\mathcal{E})$. Forming the pullbacks of p along x, c, and t, we can construct (not necessarily discrete) categories $\mathbb{M}_0$, $\mathbb{M}_1$ and $\mathbb{M}_2$ and internal functors between them:

$$
\begin{tikzcd}
\mathbb{M}_2 \arrow[r, shift left=1, "m'"] \arrow[r, shift right=1, "t'"'] \arrow[d] & \mathbb{M}_1 \arrow[r, shift right=1, "c'"'] \arrow[d] & \mathbb{M}_0 \arrow[l, shift right=1, "i'"'] \arrow[r, hook, "x'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{M} \arrow[d, "p"] \\
B_2 \arrow[r, shift left=2, "s"] \arrow[r, "m" description] \arrow[r, shift right=2, "t"'] & B_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B_0 \arrow[l, "i" description] \arrow[r, hook, "x"] & \mathbb{B}.
\end{tikzcd}
\tag{7}
$$

The equations satisfied by m, t, i, and c force all corresponding squares to be pullbacks. Then unfolding the definition of the action of Φ_B on objects, morphisms and composable pairs in $\mathbb{M}$, it can be shown that:

3.6. Proposition. *An internal functor $\mathbb{M} \xrightarrow{p} \mathbb{B}$ is a strict fibration if and only if there are internal functors*

$$
\mathbb{M}_2 \xrightarrow{s'} \mathbb{M}_1 \xrightarrow{d'} \mathbb{M}_0
$$

such that the corresponding squares involving s and d commute and the top row of (7) is a category object in $\mathbf{Cat}(\mathcal{E})$.

The internal codomain functor $\mathbb{M}_1 \xrightarrow{c'} \mathbb{M}_0$ is a discrete opfibration since c is, and Weber [2015b] shows how to construct the lax codescent object for an internal category with this property in $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ has pullbacks. In this case the lax codescent object is exactly the internal category $\mathbb{M}$. More precisely, there exists an internal natural transformation $\xi' : x'd' \Rightarrow x'c'$ such that (x', ξ') is the lax codescent object of the top row and the corresponding square with ξ commutes.

3.7. Example. [$\mathcal{E} = \mathbf{Set}$] In $\mathbf{Cat}$, the category $\mathbb{M}_0$ consists of the objects of $\mathbb{M}$ with the morphisms of $\mathbb{M}$ that are p-vertical. The objects of $\mathbb{M}_1$ are pairs $(J \in \mathbb{M},\ f : I \to pJ \in \mathbb{B})$, which d' sends to the domain of the chosen cartesian lifting $f^*J \to J$. This lifting is the corresponding component of the natural transformation ξ'.

Projecting onto the object, morphism, and composable morphism parts of the categories in (7) gives *internal diagrams* in $\mathcal{E}$ over $\mathbb{B}$, as defined for example in [Johnstone, 1977]. Thus a strict fibration over $\mathbb{B}$ corresponds exactly to an internal category in the category $\mathcal{E}^{\mathbb{B}}$ of such diagrams.

Taking the opposite of this internal category corresponds to taking the opposites of all the internal categories and functors in the diagram (7). This will not affect the bottom row, since the categories are discrete, but the top row will have a new lax codescent object

$$
\begin{tikzcd}
\mathbb{M}_2^{op} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{M}_1^{op} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbb{M}_0^{op} \arrow[l] \arrow[r] & \mathbb{M}^{\circ}.
\end{tikzcd}
$$

The codomain $\mathbb{M}_1^{op} \xrightarrow{c'^{op}} \mathbb{M}_0^{op}$ is still a discrete opfibration so the lax codescent object exists in $\mathbf{Cat}(\mathcal{E})$. The universal property of the colimit then induces an internal functor $\mathbb{M}^{\circ} \xrightarrow{p^{op}} \mathbb{B}$, and by the previous proposition p^{op} has the structure of a fibration.

3.8. DEFINITION. *The fibration* $\mathbb{M}^\circ \xrightarrow{p^{op}} \mathbb{B}$ *is called the* opposite *of the fibration* p.

Using strict fibrations makes the correspondence between opposite fibrations and opposite internal categories clearer, but the above construction can be carried out in the same way for the non-strict fibrations of Definition 3.3. The analogue of Proposition 3.6 says that p is a fibration if there are internal functors s' and d' such that the corresponding squares commute and the top row of (7) is a weak form of category object: there are isomorphisms $d'i' \cong 1$ and $d'm' \cong d's'$ rather than equalities together with coherence conditions for these isomorphisms. The top row is however still an instance of coherence data, and taking the lax codescent object of the pointwise opposite of this row induces an internal functor $\mathbb{M}^\circ \xrightarrow{p^{op}} \mathbb{B}$ which has the structure of a fibration. We define p^{op} as the opposite of p.

In **Cat**, this construction gives the usual definition of the opposite of a cloven fibration:

3.9. EXAMPLE. [$\mathcal{E}$ = **Set**] The opposite of $\mathbb{M} \xrightarrow{p} \mathbb{B}$ in **Cat** is given by reversing the arrows of $\mathbb{M}$ which are vertical over $\mathbb{B}$. The category $\mathbb{M}^\circ$ has the same objects as $\mathbb{M}$, and as morphisms $A \to B$ over $pA \xrightarrow{u} pB$ the spans $A \xleftarrow{\alpha} M \xrightarrow{\beta} B$ in $\mathbb{M}$ where α is p-vertical and β is a chosen p-cartesian lifting of u.

4. Two-sided fibrations of internal categories

We now extend the definitions of fibration structure for a functor in the previous section to a span of functors. In this section we define what it means for a span of internal functors to be a two-sided fibration, define its opposite and organize two-sided fibrations into a 2-bicategory.

A span $\mathbb{A} \xleftarrow{q} \mathbb{M} \xrightarrow{p} \mathbb{B}$ can be acted on by both the 2-monads $\Phi_{\mathbb{A}} : \mathbb{A} \twoheadrightarrow \mathbb{A}$ and $\Phi_{\mathbb{B}} : \mathbb{B} \twoheadrightarrow \mathbb{B}$. It is a left $\Phi_{\mathbb{B}}$-module, in other words a pseudoalgebra for $\Phi_{\mathbb{B}}$, if p has the structure of a fibration and the structure map commutes with q (Definition A.3). Similarly it is a right $\Phi_{\mathbb{A}}$-module if q is an opfibration and the structure map commutes with p. We will construct a category of Φ-*bimodules*, that is, spans with the structure of a right $\Phi_{\mathbb{A}}$-module and left $\Phi_{\mathbb{B}}$-module in a compatible way. A bimodule for pseudomonads $S : Y \twoheadrightarrow Y$ and $T : X \twoheadrightarrow X$ is a 1-cell $M : Y \twoheadrightarrow X$ with the structure $(d, \delta, \bar{\delta})$ of a right S-module and the structure $(e, \varepsilon, \bar{\varepsilon})$ of a left T-module, together with an invertible 3-cell

$$\begin{tikzcd} TMS \arrow[r, "Td"] \arrow[d, "eS"'] \arrow[dr, phantom, "\swarrow\rho"] & TM \arrow[d, "e"] \\ MS \arrow[r, "d"'] & M \end{tikzcd}$$

which satisfies four coherence axioms showing compatibility with $\delta, \bar{\delta}, \varepsilon, \bar{\varepsilon}$. (Definition A.4)

4.1. Definition. *A span* $\mathbb{A} \xleftarrow{q} \mathbb{M} \xrightarrow{p} \mathbb{B}$ *is a* two-sided fibration *if it is a* $(\Phi_{\mathbb{A}}, \Phi_{\mathbb{B}})$-*bimodule.*

4.2. Example. [$\mathcal{E}$ = **Set**] In **Cat**, a span $\mathbb{A} \xleftarrow{q} \mathbb{M} \xrightarrow{p} \mathbb{B}$ is a two-sided fibration if and only if:

- p is a cloven fibration with q-vertical cartesian liftings $f^*J \to J$ for each morphism $I \xrightarrow{f} pJ$ in $\mathbb{B}$,
- q is a cloven opfibration with p-vertical opcartesian liftings $J \to g_!J$ for each $qJ \xrightarrow{g} K$ in $\mathbb{A}$,
- each canonical morphism $g_!f^*J \to f^*g_!J$ is an isomorphism.

In particular, every morphism $I \to J$ in the category $\mathbb{M}$ factors into three

$$I \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} J$$

where α is q-opcartesian p-vertical, β is p, q-vertical, and γ is p-cartesian q-vertical, and this factorization is unique up to unique vertical isomorphisms.

Two-sided fibrations were defined by Street [1974] under the name *bifibrations*. For each pair of objects $\mathbb{A}$ and $\mathbb{B}$, the two-sided fibrations from $\mathbb{A}$ to $\mathbb{B}$ assemble into a 2-category $\mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$. It has as objects bimodules, as 1-cells the maps of spans compatible with the fibration and opfibration structure, and as 2-cells the 2-cells of $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathbb{A}, \mathbb{B})$ (Definition A.5).

When $\mathcal{E}$ has sufficient structure, the 2-categories $\mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$ of two-sided fibrations in $\mathcal{E}$ additionally form the hom-2-categories of a 2-bicategory $\mathbf{Fib}(\mathcal{E})$. The composite

$$\begin{tikzcd} & \mathbb{N} \otimes \mathbb{M} \arrow[dl] \arrow[dr] & \\ \mathbb{A} & & \mathbb{C} \end{tikzcd}$$

(also written as $\mathbb{A} \leftarrow \mathbb{N}\mathbb{M} \to \mathbb{C}$) of bimodules $\mathbb{A} \leftarrow \mathbb{M} \to \mathbb{B}$ and $\mathbb{B} \leftarrow \mathbb{N} \to \mathbb{C}$ is given by composing as spans and then quotienting out by the action of $\Phi_{\mathbb{B}}$. More precisely, this quotient is constructed as a codescent object.

4.3. Proposition. *Let* $\mathfrak{B}$ *be a 2-bicategory such that each hom-2-category has codescent objects and composition preserves them. Then there is a 2-bicategory* $\mathbf{Bimod}(\mathfrak{B})$ *with pseudomonads in* $\mathfrak{B}$ *as objects,* (T, S)*-bimodules as 1-cells* $T \nrightarrow S$*, and morphisms and 2-cells of bimodules as 2-cells and 3-cells respectively.*

PROOF. Given two bimodules $M : T \to\!\!\!\!\!\to S$ and $N : S \to\!\!\!\!\!\to R$, the left S-module structure of M and right S-module structure of N define a diagram of coherence data in $\mathfrak{B}(T, R)$ as in (5):

$$NS^2M \;\overset{NSe}{\underset{dSM}{\substack{\longrightarrow\\ \xrightarrow{N\mu M} \\ \longrightarrow}}}\; NSM \;\overset{Ne}{\underset{dM}{\substack{\longrightarrow\\ \xleftarrow{N\eta M} \\ \longrightarrow}}}\; NM \tag{8}$$

Let $x\colon NM \to N \otimes M$ be the codescent object of this diagram. Since composition preserves codescent objects, $Rx\colon RNM \to R(N \otimes M)$ is the codescent object of the corresponding diagram composed with R. The right R-module structure of N and the universal property of the codescent object then give a morphism $R(N \otimes M) \to N \otimes M$, which makes $N \otimes M$ into a right R-module. Similarly, $N \otimes M$ can be given the structure of a left T-module, and a (T, R)-bimodule. The operation $\otimes$ extends to a 2-functor on the hom-2-categories of bimodules, and $\otimes$ is associative (up to coherent isomorphism) because composition in $\mathfrak{B}$ preserves the codescent objects in the construction. The identity for composition on S is S itself considered as an (S, S)-bimodule. ∎

Codescent objects in the hom-2-categories of the 2-bicategory $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ are just constructed as codescent objects in $\mathbf{Cat}(\mathcal{E})$. Unfortunately in general the 2-category $\mathbf{Cat}(\mathcal{E})$ does not have all finite colimits, even if $\mathcal{E}$ does. However a sufficient condition for all codescent objects to exist is that $\mathcal{E}$ have pullback-stable finite colimits and free cartesian monoids [Hermida, 2004].

So far $\mathcal{E}$ has been merely assumed to be a category with finite limits. In what follows we add the condition that $\mathcal{E}$ is locally cartesian closed, in other words that every slice category of $\mathcal{E}$ (including $\mathcal{E}/1 \cong \mathcal{E}$) is cartesian closed. We also assume that $\mathcal{E}$ either has countable colimits or is a topos with a natural numbers object. Then in either case $\mathcal{E}$ does have pullback-stable finite colimits and free cartesian monoids [Johnstone, 1977], so codescent objects exist in $\mathcal{E}$.

4.4. THEOREM. *There is a 2-bicategory* $\mathbf{Fib}(\mathcal{E})$ *with objects the same as* $\mathbf{Cat}(\mathcal{E})$*, two-sided fibrations as 1-cells, and morphisms and 2-cells of bimodules as 2-cells and 3-cells. The identity for composition on* $\mathbb{B}$ *is the span* $\Phi_{\mathbb{B}} : \mathbb{B} \to\!\!\!\!\!\to \mathbb{B}$.

PROOF. Given two internal categories $\mathbb{A}$ and $\mathbb{B}$, the hom-2-category is just the 2-category $\mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$ of two-sided fibrations, that is $(\Phi_{\mathbb{A}}, \Phi_{\mathbb{B}})$-bimodules.

The composite of two bimodules is defined as a codescent object as in (8), which exists by the assumptions on $\mathcal{E}$. As in the proof of Proposition 4.3, this composite has the structure of a bimodule if composition in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ preserves the codescent object. Composition in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ is given by pullback, which does not preserve all colimits in the hom-2-categories, even for $\mathcal{E} = \mathbf{Set}$. However for two-sided fibrations, the required pullbacks are taken along a fibration and an opfibration respectively. Since $\mathcal{E}$ is locally cartesian closed, fibrations and opfibrations are exponentiable in $\mathbf{Cat}(\mathcal{E})$ [Giraud, 1964] [Johnstone, 1993], and pullback along either morphism commutes with colimits. Thus $\otimes$ is a well-defined composition of bimodules giving $\mathbf{Fib}(\mathcal{E})$ the structure of a 2-bicategory. ∎

We now extend the definition of opposites to the case of two-sided fibrations $\mathbb{A} \xleftarrow{q} \mathbb{M} \xrightarrow{p} \mathbb{B}$. Consider the internal category $\mathbb{A} \times \mathbb{B}$ and the diagram of pullbacks in $\mathbf{Cat}(\mathcal{E})$

$$\begin{tikzcd}
\mathbb{M}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{M}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{M}_0 \arrow[r, hook, "x'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{M} \arrow[d, "{(q,p)}"] \\
A_2 \times B_2 \arrow[r, "m\times m"] & A_1 \times B_1 \arrow[r, "d\times c"] & A_0 \times B_0 \arrow[r, hook, "x"] & \mathbb{A} \times \mathbb{B}.
\end{tikzcd}$$

As in Proposition 3.6, the fibration structure of p and the opfibration structure of q induce morphisms $d',\ c' : \mathbb{M}_1 \to \mathbb{M}_0$ respectively, and the compatibility between the structures ensures that these can be extended to make the top row into a (weak) category object in $\mathbf{Cat}(\mathcal{E})$ with lax codescent object $\mathbb{M}$. Taking opposites of each internal category gives a new lax codescent object $\mathbb{M}^\circ$ and an internal functor $\mathbb{M}^\circ \xrightarrow{(q^{op},\, p^{op})} \mathbb{A} \times \mathbb{B}$, which defines a new two-sided fibration between $\mathbb{A}$ and $\mathbb{B}$.

4.5. Definition. *The span*

$$\begin{tikzcd}
 & \mathbb{M}^\circ \arrow[dl, "q^{op}"'] \arrow[dr, "p^{op}"] & \\
\mathbb{A} & & \mathbb{B}
\end{tikzcd}$$

is called the opposite two-sided fibration *of* $\mathbb{A} \xleftarrow{q} \mathbb{M} \xrightarrow{p} \mathbb{B}$.

4.6. Example. [$\mathcal{E} = \mathbf{Set}$] In $\mathbf{Cat}$ this corresponds to reversing the arrows of $\mathbb{M}$ which are vertical over both $\mathbb{A}$ and $\mathbb{B}$. The category $\mathbb{M}^\circ$ has the same objects as $\mathbb{M}$, and as morphisms $A \to B$ over $pA \xrightarrow{u} pB$ and $qA \xrightarrow{v} qB$ the diagrams $A \xrightarrow{\alpha} M \xleftarrow{\beta} N \xrightarrow{\gamma} B$ in $\mathbb{M}$ where α is a chosen q-opcartesian lifting of v which is p-vertical, γ is a chosen p-cartesian lifting of u which is q-vertical, and β is vertical for p and q.

Using the universal property of the codescent objects, a morphism of two-sided fibrations $(q,p) \to (r,s)$ over $\mathbb{A}$ and $\mathbb{B}$ will induce a morphism of two-sided fibrations $(q^{op}, p^{op}) \to (r^{op}, s^{op})$, and taking opposites extends to a pseudofunctor

$$(-)^{op} \colon \mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B}) \to \mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$$

for each $\mathbb{A}$ and $\mathbb{B}$. It is also clear that $((-)^{op})^{op} \cong 1$.

Since composition of two-sided fibrations is defined by a colimit of spans, which is stable under pullback along fibrations and opfibrations, composition commutes with opposites. That is, $\mathbb{N}^{op} \otimes \mathbb{M}^{op} \cong (\mathbb{N} \otimes \mathbb{M})^{op}$ naturally in $\mathbb{M}$ and $\mathbb{N}$.

5. Pseudo-distributivity

We now return to the 2-bicategory $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ and the 2-monad $\Phi_{\mathbb{B}} : \mathbb{B} \to\!\!\!\!\to \mathbb{B}$. In this section, we review the definition of a pseudo-distributive law between two pseudomonads in a 2-bicategory, and construct such a pseudo-distributive law between $\Phi_{\mathbb{B}}$ and its opposite span.

By the symmetry of $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, reversing $\Phi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ gives a span

$$\Psi_{\mathbb{B}} = \begin{tikzcd} & \mathbb{B}^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ \mathbb{B} & & \mathbb{B}, \end{tikzcd} \tag{9}$$

which is also a 2-monad on $\mathbb{B}$ in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$. Since $\Phi_{\mathbb{B}}$ is colax-idempotent, $\Psi_{\mathbb{B}}$ is lax-idempotent. Considered as an internal category in $\mathbf{Cat}(\mathcal{E})$, the span $\Psi_{\mathbb{B}}$ corresponds to the opposite internal category of $\Phi_{\mathbb{B}}$, with codomain and domain switched. A left module for $\Psi_{\mathbb{B}}$ is the reverse of a right $\Phi_{\mathbb{B}}$-module, or in other words a span $\mathbb{A} \xleftarrow{q} \mathbb{M} \xrightarrow{p} \mathbb{B}$ where the internal functor p has the structure of an opfibration and the structure map commutes with q.

Note that the span $\Psi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ is not a two-sided fibration, even when c is a fibration, as the compatibility condition between c and d does not hold. However although $\Psi_{\mathbb{B}}$ is not a $\Phi_{\mathbb{B}}$-module, we can still study the combination of module structures for $\Phi_{\mathbb{B}}$ and $\Psi_{\mathbb{B}}$ by considering pseudo-distributive laws between the two 2-monads.

A *pseudo-distributive law* of a pseudomonad $S : X \nrightarrow X$ over a pseudomonad $T : X \nrightarrow X$ in a 2-bicategory is defined by Marmolejo [1999] as a 2-cell $\lambda\colon ST \to TS$ together with invertible 3-cells

$$\begin{tikzcd} S^2T \arrow[r, "S\lambda"] \arrow[d, "\mu_S T"'] & STS \arrow[r, "\lambda S"] \arrow[d, phantom, "\swarrow\alpha"] & TS^2 \arrow[d, "T\mu_S"] \\ ST \arrow[rr, "\lambda"'] & & TS \end{tikzcd} \qquad \begin{tikzcd} ST^2 \arrow[r, "\lambda T"] \arrow[d, "S\mu_T"'] & TST \arrow[r, "T\lambda"] \arrow[d, phantom, "\nearrow\beta"] & T^2S \arrow[d, "\mu_T S"] \\ ST \arrow[rr, "\lambda"'] & & TS \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[d, "\eta_S T"'] \arrow[dr, "T\eta_S"] \arrow[dr, phantom, "\swarrow\gamma"', shift right=3] & \\ ST \arrow[r, "\lambda"'] & TS \end{tikzcd} \qquad \begin{tikzcd} S \arrow[d, "S\eta_T"'] \arrow[dr, "\eta_T S"] \arrow[dr, phantom, "\swarrow\delta"', shift right=3] & \\ ST \arrow[r, "\lambda"'] & TS \end{tikzcd}$$

satisfying nine coherence conditions. It is shown by Marmolejo and Wood [2008] that eight coherence conditions suffice. Here we have suppressed the associativity and unit constraints for S and T.

If such a pseudo-distributive law of S over T exists, then the composite TS has the structure of a pseudomonad, with unit $1 \xrightarrow{\eta_T} T \xrightarrow{T\eta_S} TS$ and multiplication $TSTS \xrightarrow{T\lambda S} TTSS \xrightarrow{\mu_T\mu_S} TS$. A left module for TS is a 1-cell E with the structure $(e, \varepsilon, \bar{\varepsilon})$ of a left S-module and the structure $(d, \delta, \bar{\delta})$ of a left T-module in a compatible way, in other words with an invertible 3-cell

$$\begin{tikzcd}
STE \arrow[r, "\lambda"] \arrow[d, "Sd"'] & TSE \arrow[r, "Te"] \arrow[d, phantom, "\swArrow\cong"] & TE \arrow[d, "d"] \\
SE \arrow[rr, "e"'] & & E
\end{tikzcd}$$

satisfying coherence axioms showing compatibility with $\delta, \bar{\delta}, \varepsilon, \bar{\varepsilon}$.

In the case when S is colax-idempotent and T is lax-idempotent, such as for $S = \Phi_{\mathbb{B}}$ and $T = \Psi_{\mathbb{B}}$ here, less data is required for a pseudo-distributive law, as shown by Marmolejo [1999]. Walker [2017] gives the following simplified form: if T is a lax-idempotent pseudomonad, then to give a pseudo-distributive law of a pseudomonad S over T it suffices to give the 2-cell $\lambda \colon ST \to TS$ and the invertible 3-cells α, γ and δ, satisfying the coherence conditions

1.

$$\begin{tikzcd}
ST \arrow[r, "\lambda"] \arrow[d, bend right, "S\eta T"'] \arrow[d, bend left, "ST\eta"] \arrow[d, phantom, "\overset{S\theta}{\Leftarrow}"] \arrow[dr, phantom, "\cong"] & TS \arrow[d, "TS\eta"] \arrow[dr, "T\eta S"] \arrow[dr, phantom, "\swArrow T\delta"'] & & \\
ST^2 \arrow[r, "\lambda T"'] & TST \arrow[r, "T\lambda"'] & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& ST \arrow[r, "\lambda"] \arrow[dl, "S\eta T"'] \arrow[d, "\eta ST"] \arrow[dl, phantom, "\swArrow\delta T"] \arrow[dr, phantom, "\cong"] & TS \arrow[d, bend right, "\eta TS"'] \arrow[d, bend left, "T\eta S"] \arrow[d, phantom, "\overset{\theta S}{\Leftarrow}"] & \\
ST^2 \arrow[r, "\lambda T"'] & TST \arrow[r, "T\lambda"'] & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}$$

2.

$$\begin{tikzcd}
& ST \arrow[dr, "\lambda"] & \\
T \arrow[ur, "\eta T"] \arrow[dr, phantom, "\cong"] & & TS \arrow[ul, phantom, "\nearrow\delta"] \\
1 \arrow[u, "\eta"] \arrow[r, "\eta"'] & S \arrow[uu, "S\eta"] \arrow[ur, "\eta S"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& ST \arrow[dr, "\lambda"] \arrow[d, phantom, "\nearrow\gamma"] & \\
T \arrow[ur, "\eta T"] \arrow[rr, "T\eta"'] \arrow[dr, phantom, "\cong"] & & TS \\
1 \arrow[u, "\eta"] \arrow[r, "\eta"'] & S \arrow[ur, "\eta S"'] &
\end{tikzcd}$$

3.

$$\begin{tikzcd}
& S^2 \arrow[dl, "S^2\eta"'] \arrow[d, "\mu"] \arrow[dr, "\eta S^2"] & \\
S^2T \arrow[r, phantom, "\cong"] \arrow[d, "\mu T"'] & S \arrow[r, phantom, "\cong"] \arrow[dl, "S\eta"] \arrow[dr, "\eta S"'] \arrow[d, phantom, "\swArrow\delta"] & TS^2 \arrow[d, "T\mu"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& S^2 \arrow[dl, "S^2\eta"'] \arrow[d, "S\eta S"] \arrow[dr, "\eta S^2"] \arrow[dl, phantom, "\swArrow S\delta"] \arrow[dr, phantom, "\swArrow\delta S"] & \\
S^2T \arrow[r, "S\lambda"'] \arrow[d, "\mu T"'] & STS \arrow[r, "\lambda S"'] \arrow[d, phantom, "\swArrow\alpha"] & TS^2 \arrow[d, "T\mu"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}$$

Here $\theta : T\eta \Rightarrow \eta T$ is the 3-cell defined by the lax-idempotence of T.

If a pseudo-distributive law of S over T exists, then it is unique up to isomorphism [Marmolejo and Wood, 2012].

To give such a pseudo-distributive law $\lambda : \Phi_{\mathbb{B}}\Psi_{\mathbb{B}} \to \Psi_{\mathbb{B}}\Phi_{\mathbb{B}}$ of $\Phi_{\mathbb{B}}$ over $\Psi_{\mathbb{B}}$ is equivalent to giving a lifting of $\Psi_{\mathbb{B}}$ to a pseudomonad on each 2-category $\Phi_{\mathbb{B}}$-$\mathbf{Mod}(\mathbb{A}, \mathbb{B})$ of left $\Phi_{\mathbb{B}}$-modules, pseudonaturally in $\mathbb{A}$, as shown in [Marmolejo, 2004] (see also [Cheng, Hyland and Power, 2004]). $\Psi_{\mathbb{B}}\Phi_{\mathbb{B}}$ then has the structure of a pseudomonad on $\mathbf{Span}(\mathcal{E})(\mathbb{A}, \mathbb{B})$, with $\Psi_{\mathbb{B}}\Phi_{\mathbb{B}}$-$\mathbf{Mod}(\mathbb{A}, \mathbb{B})$ biequivalent to the 2-category of left modules for this lifted pseudomonad. $\Psi_{\mathbb{B}}$ in fact lifts to a pseudomonad on each 2-category of two-sided fibrations $\mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$, since composition of spans with the pseudo-distributive law does not affect the right $\Phi_{\mathbb{A}}$-module structure.

5.1. Proposition. [$\mathcal{E} = \mathbf{Set}$] *When $\mathcal{E}$ is* **Set**, *there is a pseudo-distributive law of $\Phi_{\mathbb{B}}$ over $\Psi_{\mathbb{B}}$ in* $\mathbf{Span}(\mathbf{Cat})$ *if and only if the category $\mathbb{B}$ has pullbacks.*

Proof. If such a pseudo-distributive law from $\Phi_{\mathbb{B}}$ to $\Psi_{\mathbb{B}}$ exists, then since the identity morphism $\mathbb{B} \xrightarrow{=} \mathbb{B}$ is canonically a fibration,

$$\Psi_{\mathbb{B}}(\mathbb{B} \xrightarrow{=} \mathbb{B}) \quad = \quad \mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$$

will also be a fibration. In other words, $\mathbb{B}$ has pullbacks (Example 3.5).

Conversely, assume that $\mathbb{B}$ has pullbacks. The map sending a cospan in $\mathbb{B}$ to its (chosen) pullback extends to a functor $\lambda : \Phi_{\mathbb{B}}\Psi_{\mathbb{B}} \to \Psi_{\mathbb{B}}\Phi_{\mathbb{B}}$:

$$\begin{tikzcd}
 & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \cdot \arrow[dr] & & \cdot \arrow[dl] \\
\Phi_{\mathbb{B}}\Psi_{\mathbb{B}} \arrow[dddd] & \mathbb{B} & & \mathbb{B} & & \mathbb{B} & & \cdot \arrow[dd, mapsto] & \\
 & & & & & & & & \\
 & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & & & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
 & & & & & & \cdot \arrow[dr, dashed] & & \cdot \arrow[dl, dashed] \\
\Psi_{\mathbb{B}}\Phi_{\mathbb{B}} & \mathbb{B} & & \mathbb{B} & & \mathbb{B} & & \cdot &
\end{tikzcd}$$

The functor λ is clearly a 2-cell in $\mathbf{Span}(\mathbf{Cat})(\mathbb{B}, \mathbb{B})$. The required invertible 3-cell γ in

$$\begin{tikzcd}
\Psi_{\mathbb{B}} \arrow[d, "\eta\Psi_{\mathbb{B}}"'] \arrow[dr, "\Psi_{\mathbb{B}}\eta"] & \\
\Phi_{\mathbb{B}}\Psi_{\mathbb{B}} \arrow[r, "\lambda"'] \arrow[ur, phantom, "\Swarrow\gamma"] & \Psi_{\mathbb{B}}\Phi_{\mathbb{B}}
\end{tikzcd}$$

is defined for each object $A \xrightarrow{f} B$ of $\mathbb{B}^{\mathbf{2}}$ to be the unique isomorphism of spans

$$\begin{tikzcd}
& A \arrow[ddl, "1_A"'] \arrow[d, "\cong"'] \arrow[ddr, "f"] & \\
& A \times_B B \arrow[dl, "\Delta_f(1_B)"] \arrow[dr, "\Delta_{1_B}(f)"'] \arrow[d, phantom, "\lrcorner"] & \\
A \arrow[dr, dotted, "f"'] & & B \arrow[dl, dotted, "1_B"] \\
& B &
\end{tikzcd}$$

Similarly the invertible 3-cells δ and α are given by the natural isomorphisms of spans relating $\Delta_{1_B} g$ to g and $\Delta_h \Delta_g f$ to $\Delta_{gh} f$ for any morphisms $A \xrightarrow{f} B \xleftarrow{g} C \xleftarrow{h} D$ in $\mathbb{B}$.

The three coherence conditions required for a pseudo-distributive law all hold since there is a unique 3-cell fitting into each diagram. In particular, each side of the third condition is a natural isomorphism between the functors sending an object $A \xleftarrow{g} B \xleftarrow{h} C$ of $\Phi_{\mathbb{B}}\Phi_{\mathbb{B}}$ to the spans

$$A \xleftarrow{\quad gh \quad} C \xrightarrow{\quad 1_C \quad} C \quad \text{and}$$

$$A \xleftarrow{\Delta_{1_A}(gh)} A \times_A C \xrightarrow{\Delta_{gh}(1_A)} C$$

respectively, and there is a unique such natural isomorphism. Similarly the 3-cell in the first condition is given by the unique natural isomorphism relating $\Delta_f g$ to $\Delta_f \Delta_{1_B} g$ for any morphisms $A \xrightarrow{f} B \xleftarrow{g} C$ in $\mathbb{B}$, and the 3-cell in the third condition by the unique natural isomorphism relating $\Delta_{1_B} 1_B$ to 1_B for each object B of $\mathbb{B}$. ■

The structure of a category with chosen pullbacks is essentially algebraic [Freyd, 1972], and can be internalized (in at least one possible way) in a category $\mathcal{E}$ with finite limits by extending the definitions in Section 2. An object of $\mathbf{Cat}(\mathcal{E})$ then has the structure of pullbacks if it has this structure representably. This means that the argument of Example 3.5 relating pullbacks to a fibration structure on the codomain functor holds more generally for internal categories in $\mathcal{E}$. Focusing on fibrations as the key structures in this paper, we will take this correspondence as *defining* the particular choice of pullbacks for an internal category:

5.2. Definition. *An internal category* $\mathbb{B}$ has pullbacks *if c is a fibration.*

In the case of **Set**, since cloven fibrations are defined algebraically, the fibration structure of c determines the choice of pullback for each cospan in $\mathbb{B}$. Without the axiom of choice this is stronger than requiring the mere existence of pullbacks. In what follows an internal category having pullbacks, sums, products and so on will be defined as chosen structure rather than a property, generalizing the algebraic definition in **Set**.

With this definition, Proposition 5.1 holds more generally for an internal category:

5.3. Theorem. *There is a pseudo-distributive law of $\Phi_{\mathbb{B}}$ over $\Psi_{\mathbb{B}}$ in* $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ *if and only if $\mathbb{B}$ has pullbacks.*

Proof. As before, if such a pseudo-distributive law from $\Phi_{\mathbb{B}}$ to $\Psi_{\mathbb{B}}$ exists, then $\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ is a fibration so $\mathbb{B}$ has pullbacks.

For the converse, we reconstruct the above definition of λ and γ internally in $\mathbf{Cat}(\mathcal{E})$. Assuming $\mathbb{B}$ has pullbacks, there is a $\Phi_{\mathbb{B}}$-module structure map

$$\mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \xrightarrow{\quad e \quad} \mathbb{B}^{\mathbf{2}}$$

in $\mathbf{Cat}(\mathcal{E})$ as in Example 3.5. Since $\Phi_{\mathbb{B}}$ is colax-idempotent, e is right adjoint to the internal functor $(1, \eta c) : \mathbb{B}^{\mathbf{2}} \to \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}}$ with invertible unit. Composing the counit ε of this adjunction with the map $d\pi_1 : \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \to \mathbb{B}$ gives a 2-cell

$$\begin{tikzcd} \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \arrow[r, bend left, "de"] \arrow[r, bend right, "d\pi_1"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{B}, \end{tikzcd}$$

which by the universal property of the arrow category $\mathbb{B}^{\mathbf{2}}$ (Section 3) corresponds to a map $\tau : \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \to \mathbb{B}^{\mathbf{2}}$ satisfying $d\tau = de$, $c\tau = d\pi_1$, and $\varphi\tau = d\pi_1\varepsilon$.

The morphism

$$\mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \xrightarrow{(\tau,e)} \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}}$$

is then a map of spans $\Phi_{\mathbb{B}}\Psi_{\mathbb{B}} \to \Psi_{\mathbb{B}}\Phi_{\mathbb{B}}$, which we define to be λ.

To construct the 3-cell γ in

$$\begin{tikzcd} \mathbb{B}^{\mathbf{2}} \arrow[d, "(1,\eta c)"'] \arrow[dr, "(\eta d,1)"] & \\ \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \arrow[r, "(\tau,e)"'] \arrow[ur, phantom, "\Swarrow\gamma"] & \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}}, \end{tikzcd}$$

we require invertible 2-cells in $\mathbf{Cat}(\mathcal{E})$ of the form $\tau(1, \eta c) \Rightarrow \eta d$ and $e(1, \eta c) \Rightarrow 1_{\mathbb{B}^{\mathbf{2}}}$. The second of these is the invertible unit of the adjunction $(1, \eta c) \dashv e$, and the first is given by the 2-dimensional universal property of $\mathbb{B}^{\mathbf{2}}$ since $d\tau(1, \eta c) \cong d = d\eta d$ and $c\tau(1, \eta c) = d = c\eta d$. The 3-cell δ is defined similarly.

Constructing the 3-cell α requires an invertible 2-cell $e(1 \times \mu) \Rightarrow e(e \times 1)$, which is part of the $\Phi_{\mathbb{B}}$-module structure of e, and an invertible 2-cell $\tau(1 \times \mu) \Rightarrow \mu(1 \times \tau)((\tau, e) \times 1)$, which again is given by the 2-dimensional universal property of $\mathbb{B}^{\mathbf{2}}$.

The fact that these morphisms satisfy the coherence conditions required for a pseudo-distributive law now follows from the case $\mathcal{E} = \mathbf{Set}$: all the constructions used to form the objects $\mathbb{B}^{\mathbf{2}}$, $\Phi_{\mathbb{B}}$ and $\Psi_{\mathbb{B}}$ are defined in terms of limits, and are preserved by the jointly faithful hom 2-functors $\mathbf{Cat}(\mathcal{E})(\mathbb{A}, -) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$. ∎

6. Fibrations with sums and products

Using the pseudo-distributive law of the previous section, we now construct a pseudomonad and its opposite in $\mathbf{Fib}(\mathcal{E})$, and define what it means for a fibration to have sums and products.

From Theorem 5.3, $\Psi_{\mathbb{B}} : \mathbb{B} \twoheadrightarrow \mathbb{B}$ lifts to a pseudomonad $\Psi'_{\mathbb{B}}$ on each $\mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$ exactly when $\mathbb{B}$ has pullbacks. Since $\Phi_{\mathbb{B}}$ and $\Psi_{\mathbb{B}}$ are colax-idempotent and lax-idempotent respectively, if such a lifting exists then it is unique up to isomorphism. Suppose from now on that this lifting $\Psi'_{\mathbb{B}}$ exists. $\Psi'_{\mathbb{B}}$ inherits the structure of a lax-idempotent pseudomonad from $\Psi_{\mathbb{B}}$.

6.1. Definition. *A (two-sided) fibration* $\mathbb{A} \leftarrow \mathbb{M} \rightarrow \mathbb{B}$ has sums *if it has the structure of a left* $\Psi'_{\mathbb{B}}$*-module.*

Considered as a span $1 \twoheadrightarrow \mathbb{B}$, an internal functor $\mathbb{M} \rightarrow \mathbb{B}$ is a fibration with sums if it has the structure of both a fibration and an opfibration in a compatible way. In **Cat**, this definition of a fibration with sums reduces to the usual one [Jacobs, 1999]:

6.2. Example. [$\mathcal{E} = \mathbf{Set}$] To equip a cloven fibration $1 \leftarrow \mathbb{M} \xrightarrow{p} \mathbb{B}$ in **Cat** with sums is to give a left adjoint Σ_f for each reindexing functor $f^* : \mathbb{M}^J \rightarrow \mathbb{M}^I$, which satisfy the *Beck-Chevalley condition*: for every pullback square

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "k"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \tag{10}$$

in $\mathbb{B}$, the canonical map $\Sigma_g h^* \rightarrow f^* \Sigma_k$ is an isomorphism.

In particular, the codomain functor $\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ is always a fibration with sums when $\mathbb{B}$ has pullbacks, where the left adjoint $\Sigma_f : \mathbb{B}/I \rightarrow \mathbb{B}/J$ is given by composition with f.

Recall that composition in $\mathbf{Fib}(\mathcal{E})$ is given by bimodule tensor $\otimes$, in other words by a codescent object of composites of spans. Since $\Psi'_{\mathbb{B}} : \mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B}) \rightarrow \mathbf{Fib}(\mathcal{E})(\mathbb{A}, \mathbb{B})$ is given by composition with a span and pullback along d preserves colimits, $\Psi'_{\mathbb{B}}$ has a tensorial strength: that is a family of maps

$$\Psi'_{\mathbb{B}}(\mathbb{N}) \otimes \mathbb{M} \xrightarrow{\cong} \Psi'_{\mathbb{B}}(\mathbb{N} \otimes \mathbb{M})$$

natural in spans $\mathbb{M} \colon \mathbb{C} \twoheadrightarrow \mathbb{A}$ and $\mathbb{N} \colon \mathbb{A} \twoheadrightarrow \mathbb{B}$, which satisfy unit and associativity conditions [Kock, 1972]. Setting $\mathbb{N}$ to be the identity two-sided fibration $\Phi_{\mathbb{B}}$ shows that the pseudomonad $\Psi'_{\mathbb{B}}$ is given by composition in $\mathbf{Fib}(\mathcal{E})$ with the span $\Psi'_{\mathbb{B}}(\Phi_{\mathbb{B}}) : \mathbb{B} \twoheadrightarrow \mathbb{B}$.

6.3. Definition. *Let* $\Sigma_{\mathbb{B}} : \mathbb{B} \to\!\!\!\!\!\!+ \;\mathbb{B}$ *be the two-sided fibration* $\Psi'_{\mathbb{B}}(\Phi_{\mathbb{B}})$*, that is the span*

$$\begin{tikzcd}
 & & \mathbb{B}^{\mathbf{2}} \times_{\mathbb{B}} \mathbb{B}^{\mathbf{2}} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" very near start, rotate=-45] & & \\
 & \mathbb{B}^{\mathbf{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \mathbb{B}^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & \\
\mathbb{B} & & \mathbb{B} & & \mathbb{B}.
\end{tikzcd}$$

We then have:

6.4. Proposition. $\Sigma_{\mathbb{B}}$ *is a lax-idempotent pseudomonad in the 2-bicategory* $\mathbf{Fib}(\mathcal{E})$*, and composing with* $\Sigma_{\mathbb{B}}$ *on the right freely adds sums to fibrations.*

6.5. Example. $[\mathcal{E} = \mathbf{Set}]$ The pseudomonad $\Sigma_{\mathbb{B}}$ is the span

$$\begin{tikzcd}
 & \mathbb{B}^{\leftarrow\cdot\rightarrow} \arrow[dl, "l"'] \arrow[dr, "r"] & \\
\mathbb{B} & & \mathbb{B}
\end{tikzcd}$$

where the category $\mathbb{B}^{\leftarrow\cdot\rightarrow}$ has as objects the spans $I \leftarrow A \to J$ in $\mathbb{B}$ and as morphisms commuting diagrams

$$\begin{tikzcd}
I \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & J \arrow[d] \\
I' & A' \arrow[l] \arrow[r] & J'.
\end{tikzcd}$$

The functors l and r project such a morphism onto $I \to I'$ and $J \to J'$ respectively.

Having defined sums and opposites for fibrations, we can now consider their combination.

6.6. Definition. *A (two-sided) fibration* has products *if its opposite has sums.*

6.7. Example. $[\mathcal{E} = \mathbf{Set}]$ In $\mathbf{Cat}$, to equip a cloven fibration $1 \leftarrow \mathbb{M} \xrightarrow{p} \mathbb{B}$ with products is to give each reindexing functor $f^* : \mathbb{M}^J \to \mathbb{M}^I$ a right adjoint Π_f satisfying the Beck-Chevalley condition: for every pullback square as in (10), the canonical map $f^*\Pi_k \to \Pi_g h^*$ is an isomorphism.

In particular, this structure exists for the codomain functor $\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ if and only if $\mathbb{B}$ is locally cartesian closed [Freyd, 1972].

Similarly to how Definition 5.2 defines a choice of pullbacks for an internal category based on the example of $\mathbf{Cat}$, we use Example 6.7 to motivate a convenient choice of local cartesian closed structure for an internal category:

6.8. Definition. *An internal category* $\mathbb{B}$ *with pullbacks is* locally cartesian closed *if the codomain fibration* $\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$ *has products.*

Given a two-sided fibration $\mathbb{M}$, we can freely add products to $\mathbb{M}$ by taking the opposite fibration, adding sums, and then taking the opposite again. Since

$$(\Sigma_{\mathbb{B}} \otimes \mathbb{M}^{op})^{op} \cong (\Sigma_{\mathbb{B}})^{op} \otimes \mathbb{M}$$

it follows that:

6.9. Proposition. *The span* $\Pi_{\mathbb{B}} :\equiv (\Sigma_{\mathbb{B}})^{op} : \mathbb{B} \to\!\!\!\!\!\to \mathbb{B}$ *is a colax-idempotent pseudomonad in* $\mathbf{Fib}(\mathcal{E})$ *which freely adds products by composition on the right. A fibration has products if and only if it has the structure of a left* $\Pi_{\mathbb{B}}$*-module.*

6.10. Example. [$\mathcal{E} = \mathbf{Set}$] In $\mathbf{Cat}$, the pseudomonad $\Pi_{\mathbb{B}}$ is a span

$$\begin{tikzcd} & (\mathbb{B}^{\leftarrow\cdot\rightarrow})^{\circ} \arrow[dl, "l^{op}"'] \arrow[dr, "r^{op}"] & \\ \mathbb{B} & & \mathbb{B}, \end{tikzcd}$$

where the category $(\mathbb{B}^{\leftarrow\cdot\rightarrow})^{\circ}$ is given by reversing the arrows of $\mathbb{B}^{\leftarrow\cdot\rightarrow}$ that are vertical for both projections onto $\mathbb{B}$. So $(\mathbb{B}^{\leftarrow\cdot\rightarrow})^{\circ}$ has as objects the spans $I \leftarrow A \rightarrow J$ in $\mathbb{B}$ and as morphisms commuting diagrams

$$\begin{tikzcd} I \arrow[d] & A \arrow[l] \arrow[r] & J \arrow[d] \\ & E \arrow[u] \arrow[d] \arrow[ur] \arrow[dr, phantom, "\lrcorner" very near start] & \\ I' & A' \arrow[l] \arrow[r] & J' \end{tikzcd}$$

The functors l^{op} and r^{op} send such a morphism to $I \rightarrow I'$ and $J \rightarrow J'$ respectively.

7. Polynomials

This section presents the main results. After reviewing the structure of the pseudo double category of polynomials defined by Gambino and Kock [2013], we recover this structure from the composition of pseudomonads $\Sigma_{\mathbb{B}}$ and $\Pi_{\mathbb{B}}$ via a pseudo-distributive law.

Let $\mathbb{B}$ be a locally cartesian closed category (in $\mathbf{Set}$). A *polynomial* in $\mathbb{B}$ is a diagram

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

in $\mathbb{B}$. A functor $\mathbb{B}/I \rightarrow \mathbb{B}/J$ is called a polynomial functor if it is isomorphic to one of the form $\Sigma_t \Pi_f \Delta_s$ for some polynomial in $\mathbb{B}$. This name reflects the representation of such a functor as

$$(X_i)_{i \in I} \mapsto \left(\sum_{a \in A_j} \prod_{b \in B_a} X_{s(b)} \right)_{j \in J}$$

using the type families, sums and products of extensional dependent type theory [Seely, 1984] as the internal language of a locally cartesian closed category.

A *morphism of polynomials* is given by morphisms h, k, l, m in $\mathbb{B}$ making

$$\begin{tikzcd}
I \arrow[dd, "h"'] & B \arrow[l, "s"'] \arrow[r, "f"] & A \arrow[r, "t"] \arrow[dd, "k"] & J \arrow[dd, "l"] \\
& B' \times_{A'} A \arrow[u, "m"] \arrow[ur] \arrow[d] \arrow[d, phantom, "\lrcorner", very near start, shift left=2ex] & & \\
I' & B' \arrow[l, "s'"] \arrow[r, "f'"'] & A' \arrow[r, "t'"'] & J'
\end{tikzcd}$$

commute. Polynomials and morphisms of polynomials form a category $\mathbf{Poly}_{\mathbb{B}}$. Polynomials can also be composed: given two polynomials from I to J and J to K representing polynomial functors $P_F : \mathbb{B}/I \to \mathbb{B}/J$ and $P_G : \mathbb{B}/J \to \mathbb{B}/K$ respectively, the composite functor $P_G P_F : \mathbb{B}/I \to \mathbb{B}/K$ is also polynomial. This defines the horizontal composition of a pseudo double category

$$\mathbb{P}\mathbf{oly}_{\mathbb{B}} \quad = \quad \mathbf{Poly}_{\mathbb{B}} \rightrightarrows \mathbb{B}$$

which has $\mathbb{B}$ as its vertical category, polynomials in $\mathbb{B}$ as horizontal morphisms and morphisms of polynomials as 2-cells. It is equivalent as a pseudo double category to the pseudo double category $\mathbb{P}\mathbf{olyFun}_{\mathbb{B}}$ with slice categories as objects, polynomial functors as horizontal morphisms and $\mathbb{B}$-enriched natural transformations as 2-cells.

Additionally, Gambino and Kock [2013] show that the pseudo double category $\mathbb{P}\mathbf{oly}_{\mathbb{B}}$ has the structure of a *framed bicategory* in the sense of Shulman [2008] (equivalently a proarrow equipment as defined by Wood [1982]). This says in particular that the functor

$$\mathbf{Poly}_{\mathbb{B}} \to \mathbb{B} \times \mathbb{B}$$

projecting a polynomial onto its endpoints (I, J) is both a fibration and an opfibration.

We now see how this structure arises naturally as a two-sided fibration. From the previous section we have two pseudomonads on an internal category $\mathbb{B}$ with pullbacks in $\mathbf{Fib}(\mathcal{E})$: the pseudomonad $\Sigma_{\mathbb{B}}$ adding sums and its opposite $\Pi_{\mathbb{B}}$ adding products. In Theorem 5.3 we considered the interaction of two 2-monads on $\mathbb{B}$ in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, the 2-monad for fibrations $\Phi_{\mathbb{B}}$ of (2) and its opposite span $\Psi_{\mathbb{B}}$ of (9), by constructing a pseudo-distributive law. Mirroring that theorem, we consider the interaction between $\Sigma_{\mathbb{B}}$ and $\Pi_{\mathbb{B}}$.

In what follows we make a final restriction on the ambient category $\mathcal{E}$, and assume that $\mathcal{E}$ is a Grothendieck topos. In particular $\mathcal{E}$ has both countable colimits and a natural numbers object, which were used in Theorem 4.4 for the construction of $\mathbf{Fib}(\mathcal{E})$.

Recall from Definition 6.8 that an internal category is locally cartesian closed if its codomain fibration has products.

7.1. Theorem. *There exists a pseudo-distributive law of* $\Pi_{\mathbb{B}}$ *over* $\Sigma_{\mathbb{B}}$ *in* $\mathbf{Fib}(\mathcal{E})$ *if and only if* $\mathbb{B}$ *is locally cartesian closed.*

Proof. If such a pseudo-distributive law of $\Pi_{\mathbb{B}}$ over $\Sigma_{\mathbb{B}}$ exists, then as before $\Sigma_{\mathbb{B}}$ lifts to a pseudomonad on left $\Pi_{\mathbb{B}}$-modules. Since the identity $\mathbb{B} \xrightarrow{=} \mathbb{B}$ canonically has products,

$$\Sigma_{\mathbb{B}}(\mathbb{B} \xrightarrow{=} \mathbb{B}) \quad = \quad \mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B}$$

will also have products. In other words, $\mathbb{B}$ is locally cartesian closed.

Conversely, assume $\mathbb{B}$ is locally cartesian closed. We start with the case when $\mathcal{E} = \mathbf{Set}$. The composite fibration $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ is a span $\mathbb{B} \leftarrow \mathbb{M} \rightarrow \mathbb{B}$ where the category $\mathbb{M}$ has as objects diagrams $I \leftarrow B \rightarrow A \rightarrow J$ in $\mathbb{B}$, in other words polynomials, and as morphisms the morphisms of polynomials

$$\begin{tikzcd}
I \arrow[d] & B \arrow[l] \arrow[r] & A \arrow[r] \arrow[dd] & J \arrow[dd] \\
 & E \arrow[u] \arrow[ur] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & & \\
I' & B' \arrow[l] \arrow[r] & A' \arrow[r] & J'.
\end{tikzcd}$$

The composite $\Pi_B\Sigma_{\mathbb{B}}$ is a span $\mathbb{B} \leftarrow \mathbb{N} \rightarrow \mathbb{B}$ where $\mathbb{N}$ has the same objects as $\mathbb{M}$ and as morphisms the commuting diagrams

$$\begin{tikzcd}
I \arrow[dd] & B \arrow[l] \arrow[r] & A \arrow[r] & J \arrow[dd] \\
 & E \arrow[u] \arrow[r] \arrow[d] \arrow[ur, phantom, very near start, "\urcorner"] & F \arrow[u] \arrow[ur] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \\
I' & B' \arrow[l] \arrow[r] & A' \arrow[r] & J'.
\end{tikzcd}$$

If $\mathbb{B}$ is locally cartesian closed, there is a functor $\lambda : \mathbb{N} \rightarrow \mathbb{M}$ sending a diagram $I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ to the polynomial

$$I \leftarrow \Delta_t\Pi_t f \rightarrow \Pi_t f \rightarrow J$$

as in the diagram

$$\begin{tikzcd}
 & & \Delta_t\Pi_t f \arrow[r] \arrow[dl, "\varepsilon"'] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & \Pi_t f \arrow[dd] \\
 & B \arrow[dl, "s"'] \arrow[dr, "f"] & & \\
I & & A \arrow[r, "t"'] & J
\end{tikzcd}$$

where ε is the component at f of the counit of the adjunction $\Delta_t \dashv \Pi_t$. The Beck-Chevalley condition for Π ensures that λ preserves the cartesian and opcartesian morphisms in $\mathbb{N}$, so it defines a morphism $\Pi_{\mathbb{B}}\Sigma_{\mathbb{B}} \rightarrow \Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ in $\mathbf{Span}(\mathbf{Cat})(\mathbb{B}, \mathbb{B})$.

In the internal language of the locally cartesian closed category $\mathbb{B}$, the morphism λ represents a transformation

$$\left(\prod_{a \in A_j} \sum_{b \in B_a} X_{s(b)}\right)_{j \in J} \mapsto \left(\sum_{\varphi \in (\prod_{a \in A_j} B_a)} \prod_{a \in A_j} X_{s(\varphi a))}\right)_{j \in J.}$$

In other words, the distributive law corresponds to the distribution of products over sums.

The components of the 3-cell γ in the diagram

$$\begin{tikzcd} \mathbb{B}^{\leftarrow\cdot\rightarrow} \arrow[d, "\eta\Sigma_B"'] \arrow[dr, "\Sigma_{\mathbb{B}}\eta"] & \\ \mathbb{N} \arrow[r, "\lambda"'] \arrow[ur, phantom, "\Swarrow\gamma"] & \mathbb{M} \end{tikzcd}$$

are defined for each span $I \xleftarrow{s} A \xrightarrow{t} J$ as the unique isomorphism of polynomials

$$\begin{tikzcd} I \arrow[d, "="'] & A \arrow[l, "s"'] \arrow[r, "t"] & J \arrow[r, "1_J"] \arrow[d, "\cong"] & J \arrow[d, "="] \\ & E \arrow[u, "\cong"] \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \\ I & \Delta_t\Pi_t(1_A) \arrow[l] \arrow[r] & \Pi_t(1_A) \arrow[r] & J. \end{tikzcd}$$

Similarly the 3-cell δ is constructed from the canonical isomorphisms $\Pi_{1_J}t \cong t$ for each $A \xrightarrow{t} J$, and the 3-cell α from the isomorphisms $\Pi_{mt}f \cong \Pi_m\Pi_t f$ for morphisms $B \xleftarrow{f} A \xrightarrow{t} J \xrightarrow{m} K$.

As in the proof of Proposition 5.1, the coherence conditions follow by uniqueness, since by the universal properties of $\Pi_{(-)}$ and pullback there is a unique 3-cell fitting into each of the diagrams. For example, the source and target 2-cells of the second coherence condition in Section 5 are functors $\mathbb{B}^{\mathbf{2}} \to \mathbb{M}$ sending an object $A \xrightarrow{f} B$ of $\mathbb{B}^{\mathbf{2}}$ to the polynomials

$$B \longleftarrow \Delta_{1_A}\Pi_{1_A}1_A \rightarrow \Pi_{1_A}1_A \longrightarrow A \quad \text{and}$$

$$B \xleftarrow{\ f\ } A \xrightarrow{\ 1_A\ } A \xrightarrow{\ 1_A\ } A$$

respectively. There is a unique isomorphism of polynomials over A and B determining a natural transformation between these two functors.

Now consider the more general category $\mathcal{E}$. Unlike the case of the distributive law for $\Phi_{\mathbb{B}}$ and $\Psi_{\mathbb{B}}$ in Theorem 5.3, the above proof does not extend by representability to arbitrary $\mathbf{Fib}(\mathcal{E})$. As hom-functors do not preserve codescent objects in general, composition in the 2-bicategory $\mathbf{Fib}(\mathcal{E})$ is not representably defined. However, since $\mathcal{E}$ is a Grothendieck topos, $\mathbf{Cat}(\mathcal{E})$ is a 2-topos in the sense of Street [1982], that is a reflective sub-2-category of a 2-presheaf category $[C^{op}, \mathbf{Cat}]$ with left-exact reflector. In this case finite limits and colimits in $\mathbf{Cat}(\mathcal{E})$ interact as in $\mathbf{Cat}$ [Bourke and Garner, 2014], and the internal version of the above proof holds in $\mathbf{Fib}(\mathcal{E})$. ■

7.2. THEOREM. *For a locally cartesian closed category $\mathbb{B}$, the two-sided fibration $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}} : \mathbb{B} \nrightarrow \mathbb{B}$ is a pseudomonad which freely adds sums and products distributing over sums to fibrations over $\mathbb{B}$. As a pseudomonad object in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, in other words as a* *pseudo double category,* $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ *represents the pseudo double category of polynomials in* $\mathbb{B}$*. For* $\mathcal{E} = \mathbf{Set}$*, this is exactly the pseudo double category* $\mathbb{P}\mathbf{oly}_{\mathbb{B}}$ *of polynomials defined by Gambino and Kock [2013].*

Proof. Since $\Pi_{\mathbb{B}}$ is colax-idempotent and $\Sigma_{\mathbb{B}}$ is lax-idempotent, the pseudo-distributive law λ given by the previous theorem is the unique pseudo-distributive law between them up to isomorphism. λ gives the composite $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ the structure of a pseudomonad. As in Section 5, left modules for $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ in the 2-category $\mathbf{Fib}(\mathcal{E})(1, \mathbb{B})$ are those fibrations with the structure of a $\Sigma_{\mathbb{B}}$-module and a $\Pi_{\mathbb{B}}$-module in a compatible way, in other words fibrations with compatible sums and products.

Horizontal morphisms and 2-cells of $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ as a pseudo double category are exactly polynomials and morphisms of polynomials, and composition of horizontal morphisms is given by the multiplication for the pseudomonad $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$, which is

$$(\Sigma\Pi)(\Sigma\Pi) \cong \Sigma(\Pi\Sigma)\Pi \xrightarrow{\Sigma\lambda\Pi} \Sigma\Sigma\Pi\Pi \xrightarrow{\mu\mu} \Sigma\Pi.$$

In **Cat** this sends two polynomials $I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ and $J \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} K$ to

$$I \leftarrow M \rightarrow \Pi_g h \rightarrow K$$

as in the diagram

$$\begin{tikzcd}
 & & M \arrow[rr] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[ddl] & & \Delta_g\Pi_g h \arrow[r] \arrow[dl, "\varepsilon"'] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & \Pi_g h \arrow[dd] & \\
 & & & E \arrow[dl] \arrow[dr, "h"] \arrow[d, phantom, very near start, "\vee"] & & & \\
 & B \arrow[r, "f"'] \arrow[dl, "s"'] & A \arrow[dr, "t"'] & & D \arrow[r, "g"'] \arrow[dl, "u"] & C \arrow[dr, "v"] & \\
I & & & J & & & K,
\end{tikzcd}$$

which is shown to be their composite as polynomials by Gambino and Kock [2013]. ∎

8. Further remarks

We conclude with some comments on the proofs and possible extensions.

Firstly, it would be interesting to investigate a more conceptual proof of Theorem 7.1 by relating the pseudomonads $\Phi_{\mathbb{B}}$ and $\Psi_{\mathbb{B}}$ to *clubs* defined by Kelly [1992], which are monads interacting well with pullbacks. The two theorems 5.3 and 7.1 have a similar form, stating that to give a pseudo-distributive law $ST \to TS$ between two pseudomonads it suffices to give a S-module structure to T acting on a terminal object (in this case the terminal object $1 \leftarrow \mathbb{B} \xrightarrow{=} \mathbb{B}$ of $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(1, \mathbb{B})$ and $\mathbf{Fib}(\mathcal{E})(1, \mathbb{B})$), and a theorem of this form was proved by Garner [2008] using a generalization of clubs.

It is likely that the requirement that $\mathcal{E}$ be a Grothendieck topos in Theorem 7.1 could be weakened, as long as we impose sufficient conditions on $\mathcal{E}$ for the necessary codescent objects to behave well with respect to finite limits in $\mathbf{Cat}(\mathcal{E})$, that is, to satisfy a form of exactness. Bourke and Garner [2014] define exactness for 2-categories and show that for example $\mathbf{Cat}(\mathcal{E})$ is exact for a large class of colimits when $\mathcal{E}$ is exact in the 1-dimensional sense.

The pseudomonad $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ for a locally cartesian closed category $\mathbb{B}$ also has an alternative decomposition as two iterations of a more basic construction. Consider the pseudomonad acting on the slice category $\mathbf{Fib}/\mathbb{B} \cong \mathbf{Fib}(1, \mathbb{B})$ of fibrations over $\mathbb{B}$. Applying it to the domain fibration $\mathbb{B}^{\mathbf{2}} \xrightarrow{d} \mathbb{B}$ gives the category $\mathbf{Poly}_{\mathbb{B}}$ of polynomials and polynomial morphisms described above. Restricting to those polynomials where I is the terminal object of $\mathbb{B}$, we get the fibration

$$\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}(\mathbb{B} \xrightarrow{=} \mathbb{B}) = \Sigma_{\mathbb{B}}(\mathbb{B}^{\mathbf{2}} \xrightarrow{c} \mathbb{B})^{op},$$

that is, $\Sigma_{\mathbb{B}}(-)^{op}$ applied to the canonical fibration over $\mathbb{B}$. The fibre over the terminal object of $\mathbb{B}$ is then the category of non-indexed polynomials, as studied for example by Abbott [2003].

For a general fibration $\mathbb{M} \xrightarrow{p} \mathbb{B}$, we have

$$\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}p \cong \Sigma_{\mathbb{B}}(\Sigma_{\mathbb{B}}p^{op})^{op},$$

so the pseudomonad $\Sigma_{\mathbb{B}}\Pi_{\mathbb{B}}$ is given by iterating the construction $\Sigma_{\mathbb{B}}(-)^{op}$, as observed by Hyland [2007]. Thus $\mathbf{Poly}(-) :\equiv \Sigma_{\mathbb{B}}(-)^{op} = (\Pi_{\mathbb{B}}(-))^{op}$ can be considered as the basic construction of polynomials over a fibration.

When a category $\mathbb{B}$ (in $\mathbf{Set}$) is not locally cartesian closed, it can still make sense to consider polynomials in $\mathbb{B}$, as long as we restrict to those diagrams

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

for which Δ_s, Π_f and Σ_t are defined. For example, Weber [2015a] examines the case of a category with pullbacks, in which the polynomials are all the diagrams of this shape such that the middle morphism f is exponentiable. Alternatively, we don't have to require that the associated functor $\Sigma_t\Pi_f\Delta_s$ of a polynomial be defined on the full slice category $\mathbb{B}/I$, but only on a subcategory of it. Polynomial diagrams should then consist of morphisms for which pullback and its adjoints $\Sigma_{(-)}$ and $\Pi_{(-)}$ are defined on this subcategory.

In detail, we start with a class of morphisms in $\mathbb{B}$ which contains identities and is closed under composition. This means that these morphisms are the objects of a full subcategory $\mathbb{F}$ of $\mathbb{B}^{\mathbf{2}}$, such that the spans

$$\Phi_{\mathbb{F}} = \begin{tikzcd} & \mathbb{F} \arrow[dl, "c"'] \arrow[dr, "d"] & \\ \mathbb{B} & & \mathbb{B} \end{tikzcd} \quad \text{and} \quad \Psi_{\mathbb{F}} = \begin{tikzcd} & \mathbb{F} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ \mathbb{B} & & \mathbb{B} \end{tikzcd}$$

are submonads in $\mathbf{Span}(\mathbf{Cat})$ of $\Phi_{\mathbb{B}}$ and $\Psi_{\mathbb{B}}$ respectively.

As in Theorem 5.3, a pseudo-distributive law of $\Phi_{\mathbb{B}}$ over $\Psi_{\mathbb{F}}$ corresponds to a cloven fibration structure on $\mathbb{F} \xrightarrow{c} \mathbb{B}$, or equivalently to the existence of pullbacks of morphisms in $\mathbb{F}$ which are again in $\mathbb{F}$. The category $\mathbb{B}$ with such a class of morphisms $\mathbb{F}$ forms a *display map category* modelling dependent type theory [Hyland and Pitts, 1989], [Taylor, 1999].

The results about $\Sigma_{\mathbb{B}}$ and $\Pi_{\mathbb{B}}$ (Theorem 7.1) now generalize to the monads $\Sigma_{\mathbb{F}}$ and $\Pi_{\mathbb{F}}$: to give a pseudo-distributive law of $\Pi_{\mathbb{F}}$ over $\Sigma_{\mathbb{F}}$ corresponds to giving $\mathbb{F} \xrightarrow{c} \mathbb{B}$ the structure of products along $\mathbb{F}$-maps. Such a pseudo-distributive law is constructed by Hofstra [2011] for the case when $\mathbb{F}$ is the class of product projections in a cartesian closed category.

These generalizations of polynomials and their applications in dependent type theory are considered further in [von Glehn, 2015].

A. Some definitions

In this appendix, we spell out for completeness some of the categorical definitions used in the previous sections.

A bicategory $\mathfrak{B}$ is enriched in **2-Cat** (Section 2) when each hom-category $\mathfrak{B}(X, Y)$ has the structure of a 2-category, and this structure is preserved by horizontal composition. In detail:

A.1. Definition. [Carmody, 1995] *A* **2-Cat**-enriched bicategory $\mathfrak{B}$*, also called a* 2-bicategory*, consists of*

- *a collection of objects* $\mathrm{ob}\,\mathfrak{B}$*,*
- *a 2-category* $\mathfrak{B}(X, Y)$ *for each pair of objects* X, Y *in* $\mathfrak{B}$*, whose objects are called* 1-cells *and written* $f : X \to\!\!\!\!\to Y$*, whose morphisms are* 2-cells*, and whose 2-cells are* 3-cells *of* $\mathfrak{B}$*,*
- *a composition 2-functor* $\circ_{X,Y,Z} : \mathfrak{B}(Y, Z) \times \mathfrak{B}(X, Y) \to \mathfrak{B}(X, Z)$ *for each triple of objects* X, Y, Z*,*
- *an identity 2-functor* $1 \xrightarrow{1_X} \mathfrak{B}(X, X)$ *for each object* X*,*
- *a 2-natural isomorphism*

$$\begin{tikzcd} \mathfrak{B}(Z,W) \times \mathfrak{B}(Y,Z) \times \mathfrak{B}(X,Y) \arrow[r, "\circ\times 1"] \arrow[d, "1\times\circ"'] \arrow[dr, phantom, "\swarrow\alpha_{X,Y,Z,W}"] & \mathfrak{B}(Y,W) \times \mathfrak{B}(X,Y) \arrow[d, "\circ"] \\ \mathfrak{B}(Z,W) \times \mathfrak{B}(X,Z) \arrow[r, "\circ"'] & \mathfrak{B}(X,W) \end{tikzcd}$$

for each quadruple of objects X, Y, Z, W*,*

- *two 2-natural isomorphisms*

$$\begin{tikzcd}
 & 1\times\mathfrak{B}(X,Y) \arrow[dl, "\cong"'] \arrow[d, "1_Y\times 1"] \arrow[dl, phantom, shift left=3ex, "\Uparrow r_{X,Y}"] \\
\mathfrak{B}(X,Y) & \mathfrak{B}(Y,Y)\times\mathfrak{B}(X,Y) \arrow[l, "\circ"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{B}(X,Y)\times 1 \arrow[d, "1\times 1_X"'] \arrow[dr, "\cong"] \arrow[dr, phantom, shift right=3ex, "\Uparrow l_{X,Y}"] & \\
\mathfrak{B}(X,Y)\times\mathfrak{B}(X,X) \arrow[r, "\circ"'] & \mathfrak{B}(X,Y)
\end{tikzcd}$$

for each pair of objects X, Y,

such that the diagrams

$$\begin{tikzcd}[column sep=small]
((k\circ h)\circ g)\circ f \arrow[rr, "\alpha\circ 1"] \arrow[d, "\alpha"'] & & (k\circ(h\circ g))\circ f \arrow[d, "\alpha"] \\
(k\circ h)\circ(g\circ f) \arrow[dr, "\alpha"'] & & k\circ((h\circ g)\circ f) \arrow[dl, "1\circ\alpha"] \\
 & k\circ(h\circ(g\circ f)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(g\circ 1)\circ f \arrow[rr, "\alpha"] \arrow[dr, "l\circ 1"'] & & g\circ(1\circ f) \arrow[dl, "1\circ r"] \\
 & g\circ f &
\end{tikzcd}$$

commute for all 1-cells f, g, h, k for which the necessary composites are defined.

A.2. Definition. *A* pseudomonad *on an object X in a 2-bicategory $\mathfrak{B}$ is a pseudomonoid in the monoidal 2-category $\mathfrak{B}(X,X)$. Explicitly, a pseudomonad consists of a 1-cell $T\colon X \twoheadrightarrow X$, 2-cells $\eta\colon 1_X \to T$ and $\mu\colon T^2 \to T$ and invertible 3-cells*

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "\tau\Swarrow"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \arrow[dl, phantom, shift left=2ex, "l\Swarrow"] \arrow[dr, phantom, shift right=2ex, "\Searrow r"] & T \arrow[l, "\eta T"'] \arrow[dl, "1_T"] \\
 & T &
\end{tikzcd}$$

such that the following pasting diagrams of 3-cells are equal:

$$\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, "T\mu T"] & T^3 \arrow[dr, "T\mu"] \arrow[d, phantom, shift left=3ex, "\Swarrow T\tau"] & \\
T^3 \arrow[dr, "\mu T"'] \arrow[r, phantom, shift right=1ex, "\Swarrow \tau T"] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dr, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, phantom, "\cong"] & T^3 \arrow[d, "\mu T"] \arrow[dr, "T\mu"] \arrow[r, phantom, shift right=3ex, "\Swarrow\tau"] & \\
T^3 \arrow[r, "T\mu"] \arrow[dr, "\mu T"'] & T^2 \arrow[dr, "\mu"] \arrow[d, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

$$\begin{tikzcd}
T^2 \arrow[r, "T\eta T"] \arrow[dr, "="'] \arrow[dr, phantom, shift left=2ex, "\Swarrow lT"] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dr, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[r, "T\eta T"] \arrow[dr, "="'] \arrow[dr, phantom, shift left=2ex, "\Swarrow Tr"] & T^3 \arrow[r, "T\mu"] \arrow[d, "T\mu"] \arrow[dr, phantom, "="] & T^2 \arrow[d, "\mu"] \\
 & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

A 2-monad *is a pseudomonad for which the 3-cells are identities.*

A.3. DEFINITION. *A* left module *for a pseudomonad $T : X \nrightarrow X$ is a pseudoalgebra for T acting on the left hom-2-category, so in other words consists of a 1-cell $E : A \nrightarrow X$ with a 2-cell $e : TE \to E$ and invertible 3-cells*

$$\begin{tikzcd}
T^2E \arrow[r, "Te"] \arrow[d, "\mu E"'] \arrow[dr, phantom, "\varepsilon\Swarrow"] & TE \arrow[d, "e"] \\
TE \arrow[r, "e"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\eta E"] \arrow[dr, "1_E"'] & TE \arrow[d, "e"] \arrow[dl, phantom, "\bar\varepsilon\Swarrow"] \\
 & E
\end{tikzcd}$$

satisfying the coherence axioms:

$$\begin{tikzcd}
T^3E \arrow[r, "T^2e"] \arrow[d, "\mu TE"'] \arrow[dr, "T\mu E"] & T^2E \arrow[dr, "Te"] \arrow[d, phantom, "\Swarrow T\varepsilon"] & \\
T^2E \arrow[r, phantom, "\Swarrow\tau E"] \arrow[dr, "\mu E"'] & T^2E \arrow[r, "Te"] \arrow[d, "\mu E"] \arrow[dr, phantom, "\Swarrow\varepsilon"] & TE \arrow[d, "e"] \\
 & TE \arrow[r, "e"'] & E
\end{tikzcd}
=
\begin{tikzcd}
T^3E \arrow[r, "T^2e"] \arrow[d, "\mu TE"'] \arrow[dr, phantom, "\cong"] & T^2E \arrow[d, "\mu E"] \arrow[dr, "Te"] & \\
T^2E \arrow[r, "Te"] \arrow[dr, "\mu E"'] & TE \arrow[r, phantom, "\Swarrow\varepsilon"] \arrow[dr, "e"] \arrow[d, phantom, "\Swarrow\varepsilon"] & TE \arrow[d, "e"] \\
 & TE \arrow[r, "e"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
TE \arrow[r, "T\eta E"] \arrow[dr, "="'] & T^2E \arrow[r, "Te"] \arrow[d, "\mu E"] \arrow[dl, phantom, "\Swarrow lE"] \arrow[dr, phantom, "\Swarrow\varepsilon"] & TE \arrow[d, "e"] \\
 & TE \arrow[r, "e"'] & E
\end{tikzcd}
=
\begin{tikzcd}
TE \arrow[r, "T\eta E"] \arrow[dr, "="'] & T^2E \arrow[r, "Te"] \arrow[d, "Te"] \arrow[dl, phantom, "\Swarrow T\bar\varepsilon"] \arrow[dr, phantom, "="] & TE \arrow[d, "e"] \\
 & TE \arrow[r, "e"'] & E
\end{tikzcd}$$

It is a strict left T-module *if ε and $\bar\varepsilon$ are identities, in which case the coherence axioms are automatically satisfied.*

A morphism of left modules $(E, e, \varepsilon, \bar\varepsilon) \to (D, d, \delta, \bar\delta)$ *is given by a 2-cell $f : E \to D$ and an invertible 3-cell*

$$\begin{tikzcd}
TE \arrow[r, "Tf"] \arrow[d, "e"'] \arrow[dr, phantom, "\bar f\Swarrow"] & TD \arrow[d, "d"] \\
E \arrow[r, "f"'] & D
\end{tikzcd}$$

satisfying two more coherence axioms:

$$\begin{tikzcd}
T^2E \arrow[r, "T^2f"] \arrow[d, "\mu E"'] \arrow[dr, "Te"] & T^2D \arrow[dr, "Td"] \arrow[d, phantom, "\Swarrow T\bar f"] & \\
TE \arrow[r, phantom, "\Swarrow\varepsilon"] \arrow[dr, "e"'] & TE \arrow[r, "Tf"] \arrow[d, "e"] \arrow[dr, phantom, "\Swarrow\bar f"] & TD \arrow[d, "d"] \\
 & E \arrow[r, "f"'] & D
\end{tikzcd}
=
\begin{tikzcd}
T^2E \arrow[r, "T^2f"] \arrow[d, "\mu E"'] \arrow[dr, phantom, "\cong"] & T^2D \arrow[d, "\mu D"] \arrow[dr, "Td"] & \\
TE \arrow[r, "Tf"] \arrow[dr, "e"'] & TD \arrow[r, phantom, "\Swarrow\delta"] \arrow[dr, "d"] \arrow[d, phantom, "\Swarrow\bar f"] & TD \arrow[d, "d"] \\
 & E \arrow[r, "f"'] & D
\end{tikzcd}$$

$$
\begin{tikzcd}
& TE \arrow[r, "Tf"] \arrow[d, "e"] \arrow[dr, phantom, "\Swarrow \bar{f}"] & TD \arrow[d, "d"] \\
E \arrow[ur, "\eta E"] \arrow[r, "="'] \arrow[ur, phantom, shift right=3ex, "\Swarrow \bar{\varepsilon}"] & E \arrow[r, "f"'] & D
\end{tikzcd}
=
\begin{tikzcd}
& TE \arrow[r, "Tf"] \arrow[d, phantom, "\cong"] & TD \arrow[d, "d"] \\
E \arrow[ur, "\eta E"] \arrow[r, "f"'] & D \arrow[ur, "\eta D"] \arrow[r, "="'] \arrow[ur, phantom, shift right=3ex, "\Swarrow T\bar{\varepsilon}"] & D
\end{tikzcd}
$$

A 2-cell *between morphisms of left modules* $(f, \bar{f}) \to (g, \bar{g})$ *is a 3-cell* $\chi : f \Rightarrow g$ *such that* $\chi e \circ \bar{f} = \bar{g} \circ dT\chi$.

A right module *is a pseudoalgebra for* T *acting on the right hom-2-category.*

A.4. Definition. *A* bimodule *for pseudomonads* $S : Y \looparrowright Y$ *and* $T : X \looparrowright X$ *is a 1-cell* $M : Y \looparrowright X$ *with the structure* $(d, \delta, \bar{\delta})$ *of a right* S*-module and the structure* $(e, \varepsilon, \bar{\varepsilon})$ *of a left* T*-module, together with an invertible 3-cell*

$$
\begin{tikzcd}
TMS \arrow[r, "Td"] \arrow[d, "eS"'] \arrow[dr, phantom, "\gamma \Swarrow"] & TM \arrow[d, "e"] \\
MS \arrow[r, "d"'] & M
\end{tikzcd}
$$

which satisfies 4 coherence axioms showing compatibility with $\delta, \bar{\delta}, \varepsilon, \bar{\varepsilon}$.

A.5. Definition. *A* morphism of bimodules *is a 2-cell* $f : M \to N$ *together with invertible 3-cells* $\bar{f}$ *and* $\bar{g}$ *giving* $(f, \bar{f})$ *and* $(f, \bar{g})$ *the structure of a left module morphism and a right module morphism respectively, and satisfying the coherence axiom*

$$
\begin{tikzcd}
TMS \arrow[r, "TfS"] \arrow[d, "e_M S"'] \arrow[dr, "Td_M"] & TNS \arrow[dr, "Td_N"] \arrow[d, phantom, "\Swarrow T\bar{g}"] & \\
MS \arrow[dr, "d_M"'] \arrow[r, phantom, "\Swarrow \gamma_M"] & TM \arrow[r, "Tf"] \arrow[d, "e_M"] \arrow[dr, phantom, "\Swarrow \bar{f}"] & TN \arrow[d, "e_N"] \\
& M \arrow[r, "f"'] & N
\end{tikzcd}
=
\begin{tikzcd}
TMS \arrow[r, "TfS"] \arrow[d, "e_M S"'] \arrow[dr, phantom, "\bar{f}S \Swarrow"] & TNS \arrow[d, "e_N S"] \arrow[dr, "Td_N"] & \\
MS \arrow[r, "fS"] \arrow[dr, "d_M"'] \arrow[r, phantom, shift right=6ex, "\Swarrow \bar{g}"] & NS \arrow[dr, "d_N"] \arrow[r, phantom, "\Swarrow \gamma_N"] & TN \arrow[d, "e_N"] \\
& M \arrow[r, "f"'] & N
\end{tikzcd}
$$

A.6. Definition. *A* 2-cell *between morphisms of bimodules* $(f, \bar{f}, f') \to (g, \bar{g}, g')$ *is a 3-cell* $\chi : f \Rightarrow g$ *which is compatible with both the left module morphism and right module morphism structure.*

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