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NEW EXACTNESS CONDITIONS INVOLVING SPLIT CUBES IN PROTOMODULAR CATEGORIES

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ABSTRACT. We introduce and compare several new exactness conditions involving what we call split cubes. These conditions are motivated by their special cases, some which become familiar, in the pointed context, once we reformulate them with split cubes replaced with split extensions.

1. Introduction

The main purpose of the paper is to introduce and compare several new exactness conditions involving what we call split cubes. Some of these conditions can be thought of as *non-pointed analogues* of known categorical conditions. In particular our conditions include non-pointed analogues of: (i) the condition introduced and studied by F. Borceux, G. Janelidze and G. M. Kelly in [3] which they briefly called the axiom of normality of unions; and (ii) the condition requiring that an internal graph is multiplicative as soon as it is *star multiplicative*, introduced and studied by G. Janelidze in [10]. In addition to these conditions we introduce a condition which is new even in the semi-abelian context where it has the following consequences: (i) Huq commutativity is reflected by the change of base functors of the fibration of points, that is, $\mathbb{C}$ satisfies (SSH) in the sense of T. Van der Linden and the second author in [17] (Proposition 4.13); (ii) Huq commutators distribute over binary joins in each fiber of the fibration of points (Corollary 3.7).

Let us begin by recalling the necessary background and then formulating our conditions.

1.1. PROTOMODULARITY. In this subsection we recall the definition of a protomodular category as defined by D. Bourn [4]. In order to do so we will introduce some notation and terminology. For a category $\mathbb{C}$, using the notation of D. Bourn, we will write $\mathbf{Pt}(\mathbb{C})$ for the category with objects quadruples (A, B, α, β) where A and B are objects in $\mathbb{C}$, and $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{C}$ such that $\alpha\beta = 1_B$. A morphism from

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(A, B, α, β) to $(A', B', \alpha', \beta')$ in $\mathbf{Pt}(\mathbb{C})$ is a pair (f, g) where $f : A \to A'$ and $g : B \to B'$ are morphisms in $\mathbb{C}$ such that $g\alpha = \alpha' f$ and $f\beta = \beta' g$. We will also display such a morphism (f, g) as a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, shift left, "\alpha"] & A' \arrow[d, shift left, "\alpha'"] \\ B \arrow[r, "g"'] \arrow[u, shift left, "\beta"] & B' . \arrow[u, shift left, "\beta'"] \end{tikzcd} \tag{1}$$

The functor $P : \mathbf{Pt}(\mathbb{C}) \to \mathbb{C}$ defined on objects by $P(A, B, \alpha, \beta) = B$ and on morphisms by $P(f, g) = g$ was called the fibration of points by D. Bourn who first realized its importance in categorical algebra. When $\mathbb{C}$ has finite limits (or more generally, has pullbacks of split epimorphisms along arbitrary morphisms), this functor is a fibration with cartesian morphisms those morphisms (1) such that the diagram obtained by removing all upward directed arrows is a pullback. We call such a cartesian morphism a split pullback and call a morphism (1) a split pushout if the diagram obtained by removing all downward arrows is a pushout. Recall that for a split pullback the diagram obtained by removing all downward directed arrows is also necessarily a pullback, while (dually) for a split pushout the diagram obtained by removing all the upward directed arrows is a pushout. We write $\mathbf{Pt}(B)$ for the fiber of the functor P above an object B and denote its objects as triples (A, α, β) where A is an object in $\mathbb{C}$ and $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{C}$ such that $\alpha\beta = 1_B$. A morphism from (A, α, β) to (A', α', β') in $\mathbf{Pt}(B)$ is a morphism $f : A \to A'$ such that $\alpha = \alpha' f$ and $f\beta = \beta'$. For each morphism $p : E \to B$ in $\mathbb{C}$ we will denote by $p^* : \mathbf{Pt}(B) \to \mathbf{Pt}(E)$ the change of base functor along p which we will also call a pullback functor. A category $\mathbb{C}$ is protomodular [4] if it has pullbacks of split epimorphisms along arbitrary morphisms and each change of base functor of the fibration of points reflects isomorphisms. This can be reformulated in the following well-known way: a category $\mathbb{C}$ with pullbacks of split epimorphisms along arbitrary morphisms is protomodular if and only if for each split pullback (1) the morphisms f and β' are jointly extremal-epimorphic (in the sense we will recall in Subsection 1.3 below).

1.2. Semi-abelian categories. In [11] G. Janelidze, L. Márki and W. Tholen introduced the notion of a semi-abelian category, to play a similar role for the categories of groups, algebras, and other related structures as abelian categories play for abelian groups and modules. A category $\mathbb{C}$ is semi-abelian if it is pointed, has binary coproducts, is protomodular, and is exact in the sense of M. Barr [1]. The main result of [11] was to connect the older work (see [11] for references) beginning with S. Mac Lane's Duality for groups [14] to the newer work initiated by the new notion of protomodularity introduced by D. Bourn in [4].

1.3. Normality inside unions. In this paper we will say that a cospan

$$X \xrightarrow{f} Z \xleftarrow{g} Y, \tag{2}$$

in a category $\mathbb{C}$, forms a join if f and g are a pair of jointly strongly epimorphic monomorphisms. That is, for each diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[ddr, bend right, "f'"'] & Z \arrow[d, "\theta"] \arrow[dd, dashed, bend right, "\varphi"'] & Y, \arrow[l, "g"'] \arrow[ddl, bend left, "g'"] \\
 & Z' & \\
 & S \arrow[u, "m"'] &
\end{tikzcd} \tag{3}$$

where the solid arrows commute and m is a monomorphism, there exists (a necessarily unique) morphism $\varphi : Z \to S$ making the entire diagram commute. Recall that a cospan (2) is jointly extremal-epimorphic if it satisfies the same condition as for a jointly strongly epimorphic span, but restricted to diagrams of the form (3) where θ is an identity morphism. Recall also that the notions of jointly strongly epimorphic and jointly extremal-epimorphic coincide in any category with pullbacks. Note that if $\mathbb{C}$ has pullbacks, then the subobjects of each object in $\mathbb{C}$ form a meet-semilattice. Therefore, since in a meet-semilattice an element c is the join of a and b if and only if it is minimal amongst elements larger than a and b, it follows that a cospan (2) forms a join in the sense described here if and only if $(Z, 1_Z)$ is the join of (X, f) and (Y, f) as subobjects of Z.

In [3] F. Borceux, G. Janelidze and G. M. Kelly introduced a condition, for semi-abelian categories, that as mentioned above they briefly called the *axiom of normality of unions* which (in that context) is equivalent to:

1.4. Definition. *A pointed category* $\mathbb{C}$ *satisfies* the normality inside unions, *NU for short, if for each square of monomorphisms in* $\mathbb{C}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "i"'] & D
\end{tikzcd}$$

if f and g are normal monomorphisms and i and h form a join, then $hf = ig$ is a normal monomorphism.

1.5. Partially multiplicative graphs. Recall that a reflexive graph in a category $\mathbb{C}$ is a pentuple (C_1, C_0, d, c, e) where C_0 and C_1 are objects in $\mathbb{C}$, and $d, c : C_1 \to C_0$ and $e : C_0 \to C_1$ are morphisms in $\mathbb{C}$ such that $de = ce = 1_{C_0}$. Given a reflexive graph (C_1, C_0, d, c, e) in a category $\mathbb{C}$ with finite limits and an initial object 0 we may form the diagrams

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, shift left, "\tilde{d}"] & C_1 \arrow[d, shift left, "d"] \\ 0 \arrow[r, "!_{C_0}"'] \arrow[u, shift left, "\tilde{e}"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, dashed, "\langle k, \tilde{e}\tilde{d}\rangle"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "\tilde{e}\tilde{d}"'] & & \\ & C_1 \times_{\langle d, ck\rangle} X \arrow[r, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & C_1 \arrow[d, shift left, "d"] \\ & X \arrow[r, "ck"'] \arrow[u, shift left, "\langle eck, 1\rangle"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd}$$

where both squares are split pullbacks and $\langle k, \tilde{e}\tilde{d}\rangle$ is the unique morphism obtained from the universal property of the right hand pullback. Extending the definition of G. Janelidze in [10], to this slightly more general context, we will say that the reflexive graph (C_1, C_0, d, c, e) admits a star multiplication if there exists a morphism $p : C_1 \times_{\langle d,ck\rangle} X \to X$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\langle k, \tilde{e}\tilde{d}\rangle"] \arrow[dr, equal] & C_1 \times_{\langle d,ck\rangle} X \arrow[d, "p"] & X \arrow[l, "\langle eck, 1\rangle"'] \arrow[dl, equal] \\ & X & \end{tikzcd}$$

commute. Since both the right hand square and the whole rectangle in the diagram

$$\begin{tikzcd} X \arrow[r, "\langle k, \tilde{e}\tilde{d}\rangle"'] \arrow[rr, bend left, "k"] \arrow[d, shift left, "\tilde{d}"] & C_1 \times_{\langle d,ck\rangle} X \arrow[r, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & C_1 \arrow[d, shift left, "d"] \\ 0 \arrow[r, "\tilde{e}"] \arrow[rr, bend right, "!_{C_0}"'] \arrow[u, shift left, "\tilde{e}"] & X \arrow[r, "ck"] \arrow[u, shift left, "\langle eck, 1\rangle"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd}$$

are split pullbacks it follows that the left hand square is as well. This means that when, in addition, $\mathbb{C}$ is protomodular the morphisms $\langle k, \tilde{e}\tilde{d}\rangle$ and $\langle eck, 1\rangle$ are jointly strongly epimorphic and so such a p is unique (whenever it exists).

For a reflexive graph (C_1, C_0, d, c, e) and a morphism $j : I \to C_0$ we will say that (C_1, C_0, d, c, e) admits (I, j)-multiplicative structure if $((C_1, ej), (C_0, j), d, c, e)$ admits a star multiplication in $(I \downarrow \mathbb{C})$. Explicitly, this means forming the diagrams

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, shift left, "\tilde{d}"] & C_1 \arrow[d, shift left, "d"] \\ I \arrow[r, "j"'] \arrow[u, shift left, "\tilde{e}"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, dashed, "\langle k, \tilde{e}\tilde{d}\rangle"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "\tilde{e}\tilde{d}"'] & & \\ & C_1 \times_{\langle d, ck\rangle} X \arrow[r, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & C_1 \arrow[d, shift left, "d"] \\ & X \arrow[r, "ck"'] \arrow[u, shift left, "\langle eck, 1\rangle"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd}$$

where both squares are split pullbacks and $\langle k, \tilde{e}\tilde{d}\rangle$ is the unique morphism obtained from the universal property of the right hand pullback, there exists a morphism $p : C_1 \times_{\langle d,ck\rangle}$

$X \to X$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "{\langle k,\tilde{e}\tilde{d}\rangle}"] \arrow[dr, equal] & C_1 \times_{\langle d,ck\rangle} X \arrow[d, "p"] & X \arrow[l, "{\langle eck,1\rangle}"'] \arrow[dl, equal] \\
 & X &
\end{tikzcd}$$

commute. Recall that if $\mathbb{C}$ is protomodular with finite limits, then $(I \downarrow \mathbb{C})$ is protomodular with finite limits and an initial object. This means that in the protomodular context such a morphism p is unique (whenever it exists). We will call such a reflexive graph (I, j)-multiplicative.

For a protomodular category we introduce the following condition (generalizing the condition mentioned in Remark 4.7 of [10] and studied further by T. Van der Linden and the second author in [16]).

1.6. Condition. *For each reflexive graph (C_1, C_0, d, c, e) and each monomorphism $j : I \to C_0$, the reflexive graph (C_1, C_0, d, c, e) is multiplicative if and only if it is (I, j)-multiplicative.*

1.7. Conditions on split cubes. In this subsection we introduce several further new conditions. To do so it is convenient to introduce some terminology.

Throughout the rest of this paper by a split cube we will mean a diagram

$$\begin{tikzcd}
 & X \arrow[rr, "h_2"] \arrow[dl, "h_1"'] \arrow[dd, shift left, "\chi"] & & D_2 \arrow[dl, "i_2"'] \arrow[dd, shift left, "\delta_2"] \\
D_1 \arrow[rr, "i_1"'] \arrow[dd, shift left, "\delta_1"] & & A \arrow[dd, shift left, "\alpha"] & \\
 & I \arrow[uu, shift left, "\eta"] \arrow[rr, "f_2"'] \arrow[dl, "f_1"] & & E_2 \arrow[uu, shift left, "\epsilon_2"] \arrow[dl, "g_2"] \\
E_1 \arrow[uu, shift left, "\epsilon_1"] \arrow[rr, "g_1"'] & & B \arrow[uu, shift left, "\beta"] &
\end{tikzcd} \tag{4}$$

with $\alpha\beta = 1_B$, $\delta_1\epsilon_1 = 1_{E_1}$, $\delta_2\epsilon_2 = 1_{E_2}$, and $\chi\eta = 1_I$, such that the diagrams obtained by removing all upward and all downward directed arrows, respectively, commute. In this paper we will say that a split cube (4) is:

(i) *of type* LE1 if the two back faces are split pullbacks (i.e. (X, h_1, χ) and (X, h_2, χ) are pullbacks of δ_1 and f_1, and δ_2 and f_2, respectively);

(ii) *of type* LE2 if it is of type LE1 and the two front faces are split pullbacks (i.e. it is of type LE1, and (D_1, i_1, δ_1) and (D_2, i_2, δ_2) are pullbacks of α and g_1, and α and g_2, respectively);

(iii) *of type* RE if A together with the morphisms i_1, i_2 and β is the colimit of the diagram

$$\begin{tikzcd}
 & X \arrow[rr, "h_2"] \arrow[dl, "h_1"'] & & D_2 \\
D_1 & & & \\
 & I \arrow[uu, "\eta"] \arrow[rr, "f_2"'] \arrow[dl, "f_1"] & & E_2 \arrow[uu, "\epsilon_2"'] \arrow[dl, "g_2"] \\
E_1 \arrow[uu, "\epsilon_1"] \arrow[rr, "g_1"'] & & B. &
\end{tikzcd} \tag{5}$$

For a category $\mathbb{C}$ we compare the conditions:

1.8. Condition. *For each split cube* (4) *of type* LE1 *such that* g_1 *and* g_2 *form a join, if* i_1 *and* i_2 *form a join, then* (4) *is of type* LE2.

1.9. Condition. *For each split cube* (4) *of type* LE1 *such that* g_1 *and* g_2 *form a join, if* i_1 *and* i_2 *are monomorphisms and* (4) *is of type* RE, *then it is of type* LE2.

1.10. Condition. *For each split cube* (4) *such that* g_1 *and* g_2 *form a join, if* (4) *is of type* LE2, *then it is of type* RE.

1.11. Remark. *The Conditions 1.8, 1.9, and 1.10 might be thought of as* algebraic versions of Van Kampen type conditions *and could be phrased using an analogous functor to the functor* K_{g_1,g_2} *defined in Section 1 of [6]. We choose not to do so here since our more explicit conditions seem easier to work with when comparing these conditions to other categorical algebraic conditions.*

In the final section of the paper we give examples of categories satisfying Conditions 1.6, 1.8, 1.9 and 1.10. These categories include the category of groups, commutative rings, unital commutative rings, rings, unital rings, associative algebras, unital associative algebras and Lie algebras. Nevertheless, we end this section by directly showing that the category of commutative unital rings satisfies Condition 1.8.

1.12. Proposition. *The category of commutative unital rings satisfies Condition 1.8.*

Proof. Suppose that (4) is a split cube of type LE1 in the category of commutative unital rings such that g_1 and g_2, and i_1 and i_2 form joins. Without loss of generality we may assume that i_1, i_2, g_1, g_2, ϵ_1, ϵ_2 and β are inclusions. Let $\kappa : K \to X$ be the kernel of χ (in the category of commutative rings). Since (4) is of type LE1 it follows that $\lambda_1 = h_1\kappa$ and $\lambda_2 = h_2\kappa$ are kernels of δ_1 and δ_2, respectively. We will prove that $\theta = i_1h_1\kappa$ is the kernel of α and then use this to derive the desired conclusion. By adjusting K and κ if necessary we may again assume that λ_1, λ_2 and θ are inclusions. Suppose that a is an element of A such that $\alpha(a) = 0$. We will show that a is in K. Since A is the join of D_1

and D_2 it follows that for some positive integer n there exist $d_1, ..., d_n$ in D_1 and $d'_1, ..., d'_n$ in D_2 such that

$$a = \sum_{t=1}^{n} d_t d'_t.$$

Setting $e_t = \delta_1(d_t)$, $e'_t = \delta_2(d'_t)$, $k_t = d_t - e_t$ and $k'_t = d'_t - e'_t$ we have that $d_t = k_t + e_t$ and $d'_t = k'_t + e'_t$. Accordingly via distributivity we obtain

$$a = \sum_{t=1}^{n} [k_t k'_t + k_t e'_t + e_t k'_t + e_t e'_t]$$

Therefore, since K is an ideal of D_1 and D_2, and k_t and k'_t are in K it follows that $e_t k'_t$ and $k_t e'_t$ are in K. This means that

$$a = \sum_{t=1}^{n} [k_t k'_t + k_t e'_t + e_t k'_t] + \sum_{t=1}^{n} e_t e'_t = \sum_{t=1}^{n} [(k_t k'_t + k_t e'_t + e_t k'_t]$$

where the last equality follows from the fact that $\alpha(a) = 0$ and $\alpha(K) = \{0\}$ and hence $\sum_{t=1}^{n} e_t e'_t = \alpha(\sum_{t=1}^{n} e_t e'_t) = \alpha(a) = 0$. It follows that a is in K and hence K is the kernel of α as desired. Now suppose that $u : C \to E_1$ and $v : C \to A$ are morphisms such that $g_1 u = \alpha v$. Since $\alpha(v(c) - u(c)) = 0$ it follows that $v(c) - u(c)$ is in K and hence in D_1. Therefore since $u(c)$ is also in D_1 it follows that v factors through i_1 and so (D_1, δ_1, i_1) is a pullback of g_1 and α. Similarly (D_2, δ_2, i_2) is a pullback of g_2 and α and (4) is a split cube of type LE2 as required. ∎

2. Basic consequences and reformulations

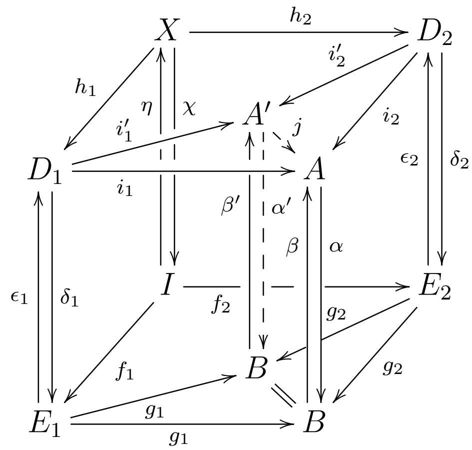
In this section we explain how the conditions on split cubes above are related to each other and what consequences they have for arbitrary protomodular categories (admitting certain colimits). In addition we give several reformulations of these conditions.

Let us begin by noting that since, for each category $\mathbb{C}$ and each object B in $\mathbb{C}$, connected limits and colimits in $(B \downarrow \mathbb{C})$ and $(\mathbb{C} \downarrow B)$ are calculated as in $\mathbb{C}$ and since, when $\mathbb{C}$ is finitely complete, a cospan in either $(B \downarrow \mathbb{C})$ or $(\mathbb{C} \downarrow B)$ is jointly strongly epimorphic if and only if the underlying cospan is jointly strongly epimorphic in $\mathbb{C}$ we obtain:

2.1. Proposition. *Let $\mathbb{C}$ be a finitely complete category. If the category $\mathbb{C}$ satisfies any of Conditions 1.6, 1.8, 1.9 or 1.10, then for each B in $\mathbb{C}$ the categories $(B \downarrow \mathbb{C})$, $(\mathbb{C} \downarrow B)$ and $\mathbf{Pt}(B)$ satisfy the same conditions.*

2.2. Proposition. *Let $\mathbb{C}$ be a protomodular category with finite colimits. If $\mathbb{C}$ satisfies Condition 1.9, then $\mathbb{C}$ satisfies Condition 1.10*

Proof. Given a split cube (4) of type LE2 such that g_1 and g_2 form a join, consider the diagram



(6)

where A' together with i'_1, i'_2 and β' is the colimit of (5), and α' and j are the unique morphisms such that $\alpha'\beta' = 1_B$, $\alpha' i'_1 = g_1\delta_1$ and $\alpha' i'_2 = g_2\delta_2$, and $j\beta' = \beta$, $ji'_1 = i_1$ and $ji'_2 = i_2$, respectively. Since i'_1 and i'_2 are monomorphisms (since i_1 and i_2 are) it follows from Condition 1.9 that the inner split "cube" is of type LE2. The claim is completed by noting that by protomodularity this means that j is an isomorphism and hence (4) is of type RE. ■

2.3. Proposition. *Let $\mathbb{C}$ be a regular category with finite colimits. The category $\mathbb{C}$ satisfies Condition 1.9 if and only if it satisfies Condition 1.8.*

Proof. The "only if" part follows from the observation that each split cube (4) of type RE such that g_1 and g_2 are jointly strongly epimorphic must have i_1 and i_2 jointly extremal-epimorphic and hence jointly strongly epimorphic. Indeed, for each subspan as displayed at the top of the commutative diagram

$$\begin{tikzcd}
& S & \\
D_1 \arrow[ur, "u"] \arrow[r, "i_1"'] & A \arrow[from=u, "m"] & D_2 \arrow[ul, "v"'] \arrow[l, "i_2"] \\
E_1 \arrow[u, "\delta_1"] \arrow[r, "g_1"'] & B \arrow[u, "\beta"] \arrow[uu, dashed, bend right, "\varphi"'] & E_2, \arrow[u, "\delta_2"'] \arrow[l, "g_2"]
\end{tikzcd}$$

since g_1 and g_2 are jointly strongly epimorphic there exists a morphism $\varphi : B \to S$ such that $m\varphi = \beta$. Since m is a monomorphism it follows that S together with the morphism u, v and φ is a subcone of a colimiting cone. This means that m is an isomorphism as required. Conversely given a split cube (4) of type LE1 such that g_1 and g_2 form a join, and i_1 and i_2 form a join, consider the diagram (6) constructed as in Proposition 2.2. Since i_1 and i_2 are jointly strongly epimorphic it follows that j is a regular epimorphism.

Now, consider the diagram

$$\begin{tikzcd}
& D_1 \arrow[rr, "i_1'"] \arrow[dl, "\tilde{j}_1"'] \arrow[dd, shift left, "\delta_1"] & & A' \arrow[dl, "j"'] \arrow[dd, shift left, "\alpha'"] \\
\tilde{D}_1 \arrow[rr, "\tilde{i}_1"'] \arrow[dd, shift left, "\tilde{\delta}_1"] & & A \arrow[dd, shift left, "\alpha"] & \\
& E_1 \arrow[uu, shift left, "\epsilon_1"] \arrow[rr, "g_1"'] \arrow[dl, equal] & & B \arrow[uu, shift left, "\beta'"] \arrow[dl, equal] \\
E_1 \arrow[uu, shift left, "\tilde{\epsilon}_1"] \arrow[rr, "g_1"'] & & B \arrow[uu, shift left, "\beta"] &
\end{tikzcd}$$

where $(\tilde{D}_1, \tilde{\delta}_1, \tilde{i}_1)$ is the pullback of g_1 and α, and $\tilde{j}_1$ and $\tilde{\epsilon}_1$ are the unique morphisms making the diagram into a split cube. Since $\tilde{i}_1\tilde{j}_1 = ji_1' = i_1$ and i_1 is a monomorphism it follows that $\tilde{j}_1$ is a monomorphism. On the other hand, since by Condition 1.9 (D_1, δ_1, i_1') is the pullback of g_1 and α' it follows that $\tilde{j}_1$ is the pullback of j and hence an isomorphism. From this it easily follows that (4) is of type LE2. ■

2.4. Remark. *The above proposition still holds if in its statement "regular category" is replaced by "finitely complete protomodular category". The only modification to the proof would be to note that j is no longer a regular epimorphism but is a strong epimorphism, and to replace "and hence an isomorphism. From this it easily follows that" beginning on the second last line by "and hence, by protomodularity, j is a monomorphism. This means that j is an isomorphism and hence".*

Recall that a category $\mathbb{C}$ is unital [2] if it is pointed, finitely complete and for each pair of objects X and Y in $\mathbb{C}$ the morphisms

$$X \xrightarrow{\langle 1,0\rangle} X \times Y \xleftarrow{\langle 0,1\rangle} Y$$

are jointly strongly epimorphic. Recall also that a pair of morphisms $f : A \to C$ and $g : B \to C$ in a unital category Huq-commute (going back to [9]) if there exists a morphism $\varphi : A \times B \to C$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\langle 1,0\rangle"] \arrow[dr, bend right, "f"'] & A \times B \arrow[d, "\varphi"] & B \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, bend left, "g"] \\
& C &
\end{tikzcd}$$

commutes. In [17] T. Van der Linden and the second author considered the condition on a category, which they denoted by (SSH), requiring that each change of base functor of the fibration of points reflects Huq commuting pairs of morphisms. It was shown that this condition implies the condition (SH) requiring the coincidence (in a certain sense) of the Smith and Huq commutators. In addition, in a previous paper [16] of the same authors, it was shown for a semi-abelian category, that (SH) is equivalent to the condition requiring that every star multiplicative graph is multiplicative. Here we show that Condition 1.10 implies that each change of base functor along a monomorphism reflects Huq commuting pairs, and also implies Condition 1.6. In Sections 3 and 4 we will show under additional conditions that Condition 1.10 implies (SSH) and that Condition 1.6 is equivalent to the condition requiring that every star multiplicative graph is multiplicative.

2.5. Proposition. *Let $\mathbb{C}$ be a protomodular category. If the category $\mathbb{C}$ satisfies Condition 1.10, then each change of base functor, along a monomorphism, of the fibration of points reflects Huq commuting pairs.*

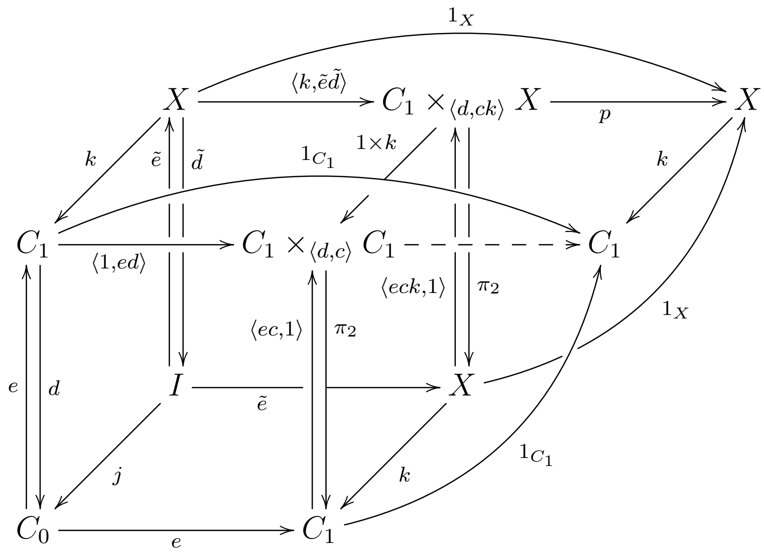
Proof. Let $p : E \to B$ be a monomorphism in $\mathbb{C}$ and let $f_1 : (A_1, \alpha_1, \beta_1) \to (A, \alpha, \beta)$ and $f_2 : (A_2, \alpha_2, \beta_2) \to (A, \alpha, \beta)$ be a pair of morphisms in $\mathbf{Pt}(B)$ such that $p^*(f_1) : (C_1, \gamma_1, \delta_1) \to (C, \gamma, \delta)$ and $p^*(f_2) : (C_2, \gamma_2, \delta_2) \to (C, \gamma, \delta)$ Huq-commute. Consider the diagram

$$
\begin{tikzcd}
& C_1 \arrow[rr, "\langle 1,\delta_2\gamma_1\rangle"] \arrow[dl, "q_1"'] \arrow[dd, shift left, "\gamma_1"] \arrow[rrrr, bend left, "p^*(f_1)"] & & C_1 \times_E C_2 \arrow[rr, "\varphi"'] \arrow[dl, "q_1\times q_2"'] \arrow[dd, shift left, "\pi_2"] & & C \arrow[dl, "q"] \\
A_1 \arrow[rr, "\langle 1,\beta_2\alpha_1\rangle"'] \arrow[dd, shift left, "\alpha_1"] \arrow[rrrr, bend left, "f_1"] & & A_1 \times_B A_2 \arrow[rr, dashed] \arrow[dd, shift left, "\pi_2"] & & A & \\
& E \arrow[rr, "\delta_2"'] \arrow[uu, shift left, "\delta_1"] \arrow[dl, "p"] & & C_2 \arrow[uu, shift left, "\langle\delta_1\gamma_2,1\rangle"] \arrow[dl, "q_2"] \arrow[uurr, bend right, "p^*(f_2)"'] & & \\
B \arrow[rr, "\beta_2"'] \arrow[uu, shift left, "\beta_1"] & & A_2 \arrow[uu, shift left, "\langle\beta_1\alpha_2,1\rangle"] \arrow[uurr, bend right, "f_2"'] & & &
\end{tikzcd}
$$

where q, q_1 and q_2 are pullback projections involved in the definition of p^*, and φ is the unique morphism exhibiting that $p^*(f_1)$ and $p^*(f_2)$ Huq-commute. The desired morphism $\psi : A_1 \times_B A_2 \to A$ is obtained using the fact that by Condition 1.10 the split cube is of type RE (i.e. ψ is the unique morphism such that $\psi\langle 1, \beta_2\alpha_1\rangle = f_1$, $\psi\langle \beta_1\alpha_2, 1\rangle = f_2$ and $\psi(q_1 \times q_2) = q\varphi$). ■

2.6. Proposition. *Let $\mathbb{C}$ be a protomodular category. If $\mathbb{C}$ satisfies Condition 1.10, then it satisfies Condition 1.6*

Proof. Let (C_1, C_0, d, c, e) be a reflexive graph which is (I, j)-multiplicative for some monomorphism $j : I \to C_0$. Consider the diagram



where all the objects and morphisms are defined as in the second half of Subsection 1.5. The existence of a multiplication $C_1 \times_{\langle d,c\rangle} C_1 \to C_1$ now follows from Condition 1.10. ■

The final part of this section is devoted to giving reformulations of Conditions 1.9 and 1.10 for regular protomodular categories with pushouts. To obtain these reformulations we will use several lemmas. Recall that a category $\mathbb{C}$ is Mal'tsev (see e.g. [2]) if it has finite limits and satisfies either (and hence both) of the equivalent conditions:

(i) every relation in $\mathbb{C}$ is difunctional;

(ii) for each object B in $\mathbb{C}$ the category $\mathbf{Pt}(B)$ is unital.

The following fact might be known but we couldn't find a reference.

2.7. Lemma. *Let $\mathbb{C}$ be a Mal'tsev category and*

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[d, shift left, "\gamma"] & A \arrow[d, shift left, "\alpha"] \\ D \arrow[r, "q"'] \arrow[u, shift left, "\delta"] & B \arrow[u, shift left, "\beta"] \end{tikzcd} \tag{7}$$

be a morphism of split epimorphisms. If p is a regular epimorphism and (7) is a split pullback, then (7) is a split pushout. This means that, when $\mathbb{C}$ is in addition regular with finite colimits, the change of base functor $q^* : \mathbf{Pt}(B) \to \mathbf{Pt}(D)$, along a regular epimorphism $q : D \to B$, has a left adjoint left inverse.

Proof. Consider the diagram

$$\begin{tikzcd}
C \times_A C \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] \arrow[d, shift left, "\gamma\times\gamma"] & C \arrow[l, "\langle 1,1\rangle" description] \arrow[r, "p"] \arrow[d, shift left, "\gamma"] & A \arrow[d, shift left, "\alpha"] \\
D \times_B D \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] \arrow[u, shift left, "\delta\times\delta"] & D \arrow[l, "\langle 1,1\rangle" description] \arrow[r, "q"'] \arrow[u, shift left, "\delta"] & B \arrow[u, shift left, "\beta"]
\end{tikzcd}$$

where $(C_1 \times_A C_1, \pi_1, \pi_2)$, $(D \times_B D, \pi_1, \pi_2)$ are the kernel pairs of p and q respectively, $\gamma \times \gamma$ and $\delta \times \delta$ are the canonical morphisms between these kernel pairs. It is easy to check that in such a situation the diagram

$$\begin{tikzcd}
C \times_A C \arrow[r, "\pi_1"] \arrow[d, "\gamma\times\gamma"'] & C \arrow[d, "\gamma"] \\
D \times_B D \arrow[r, "\pi_1"'] & D
\end{tikzcd}$$

is a pullback. Therefore, since $\mathbb{C}$ is Mal'tsev and hence $\mathbf{Pt}(D)$ is unital, the morphisms $\delta \times \delta : D \times_B D \to C \times_A C$ and $\langle 1, 1\rangle : C \to C \times_A C$ (being appropriate product inclusions in $\mathbf{Pt}(D)$) are jointly strongly epimorphic. Now suppose that $u : C \to W$ and $v : B \to W$ are morphisms such that $vq = u\delta$. Since $u\pi_1\langle 1, 1\rangle = u = u\pi_2\langle 1, 1\rangle$ and $u\pi_1(\delta \times \delta) = u\delta\pi_1 = vq\pi_1 = vq\pi_2 = u\delta\pi_2 = u\pi_2(\delta \times \delta)$ it follows that $u\pi_1 = u\pi_2$ and hence that there is a unique morphism $w : A \to W$ such that $wp = u$. The proof is completed by noting that q is necessarily an epimorphism and hence $w\beta = v$. ■

2.8. Remark. *The above lemma still holds if $\mathbb{C}$ is weakly Mal'tsev, [15], since in the proof we only use that the morphisms $\delta \times \delta$ and $\langle 1, 1\rangle$ are jointly epimorphic.*

2.9. Lemma. [4] *Let $\mathbb{C}$ be a protomodular category. For each diagram consisting of morphisms of split epimorphisms*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"'] \arrow[rr, bend left, "f"] \arrow[d, shift left, "\alpha_1"] & A_2 \arrow[r, "f_2"'] \arrow[d, shift left, "\alpha_2"] & A_3 \arrow[d, shift left, "\alpha_3"] \\
B_1 \arrow[r, "g_1"] \arrow[rr, bend right, "g"'] \arrow[u, shift left, "\beta_1"] & B_2 \arrow[r, "g_2"] \arrow[u, shift left, "\beta_2"] & B_3 \arrow[u, shift left, "\beta_3"]
\end{tikzcd} \tag{8}$$

if the left hand square as well as the outer rectangle are split pullbacks, then the right hand square is a split pullback.

2.10. Lemma. *Let $\mathbb{C}$ be a regular Mal'tsev category. For each diagram consisting of morphisms of split epimorphisms* (8) *such that the square on the left is a split pullback, if f is a monomorphism, and f_2 is a regular epimorphisms, then the outer rectangle is a split pullback. Furthermore if $\mathbb{C}$ is protomodular, then the right hand square is a split pullback.*

Proof. Consider the diagram

$$\begin{tikzcd}
A_1 \arrow[rr, "f_1"'] \arrow[rrrr, bend left, "f"] \arrow[dr, "u_1"] \arrow[dd, shift left, "\alpha_1"] & & A_2 \arrow[rr, "f_2"'] \arrow[dr, "u_2"] \arrow[dd, shift left, "\alpha_2"] & & A_3 \arrow[dd, shift left, "\alpha_3"] \\
& A_1' \arrow[rr, "p_1"] \arrow[dl, shift left, "\alpha_1'"] & & A_2' \arrow[ur, "p_2"'] \arrow[dl, shift left, "\alpha_2'"] & \\
B_1 \arrow[uu, shift left, "\beta_1"] \arrow[ur, shift left, "\beta_1'"] \arrow[rr, "g_1"] \arrow[rrrr, bend right, "g"'] & & B_2 \arrow[uu, shift left, "\beta_2"] \arrow[ur, shift left, "\beta_2'"] \arrow[rr, "g_2"] & & B_3 \arrow[uu, shift left, "\beta_3"]
\end{tikzcd}$$

where

- (A_2', α_2', p_2) is the pullback of g_2 and α_3;
- (A_1', α_1', p_1) is the pullback of g_1 and α_2';
- β_2' and β_1' are the canonical splittings of α_2' and α_1' respectively;
- u_1 and u_2 are the canonical morphisms into the two pullbacks.

Since $\mathbb{C}$ is a regular category and $\alpha_3 f_2 = g_2 \alpha_2$ is a regular epimorphism, it follows that g_2 is a regular epimorphism. Accordingly, since $\mathbb{C}$ is a regular Mal'tsev category, it follows that u_2 is a regular epimorphism (see e.g. Lemma 2.5.6 in [2]). Therefore, since u_1 is obtained by pullback from u_2 it is also a regular epimorphism. However since f is a monomorphism it follows that u_1 is as well, and hence must be an isomorphism. This means that the outer rectangle is the composite of two pullbacks and hence a pullback. The final conclusion follows from Lemma 2.9. ∎

2.11. Proposition. *Let $\mathbb{C}$ be a regular protomodular category with pushouts. The category $\mathbb{C}$ satisfies Condition 1.9 if and only if each split cube (4) of type* LE1 *where the bottom is a pushout, and g_1 and g_2 are monomorphisms, if i_1 and i_2 are monomorphisms and the top is a pushout, then (4) is of type* LE2.

Proof. The "only if" part follows from the fact that given a split cube (4) where the bottom is a pushout, one easily checks that it is of type RE if and only if the top is a pushout. Conversely given a split cube (4) of type RE and of type LE1 such that g_1 and

g_2 form a join, and i_1 and i_2 are monomorphisms, consider the diagram

$$\begin{tikzcd}
& X \arrow[rrr, "h_2"] \arrow[ddl, "h_1"'] \arrow[ddd, shift left, "\chi"] & & & D_2 \arrow[dll, "i_2'"'] \arrow[ddl, "i_2"] \arrow[ddd, shift left, "\delta_2"] \\
& & A' \arrow[dr, dashed, "p"] \arrow[ddd, dashed, shift left, "\alpha'"] & & \\
D_1 \arrow[urr, "i_1'"] \arrow[rrr, "i_1"'] \arrow[ddd, shift left, "\delta_1"] & & & A \arrow[ddd, shift left, "\alpha"] & \\
& I \arrow[uuu, shift left, "\eta"] \arrow[rrr, "f_2"'] \arrow[ddl, "f_1"'] & & & E_2 \arrow[uuu, shift left, "\epsilon_2"] \arrow[dll, "g_2'"'] \arrow[ddl, "g_2"] \\
& & B' \arrow[uuu, dashed, shift left, "\beta'"] \arrow[dr, dashed, "q"'] & & \\
E_1 \arrow[uuu, shift left, "\epsilon_1"] \arrow[urr, "g_1'"] \arrow[rrr, "g_1"'] & & & B \arrow[uuu, shift left, "\beta"] &
\end{tikzcd}$$

where (B', g_1', g_2') and (A, i_1', i_2') are pushouts of f_1 and f_2, and h_1 and h_2 respectively. The arrows α', β', p and q are obtained by the universal properties of the pushouts and make the inner part into a split "cube" of type LE1 and p and q together a morphism of split epimorphisms. As mentioned in the previous part of the proof the assumptions on the inner split cube make it of type RE which together with assumption that (4) is of type RE, makes the right hand square of the diagram

$$\begin{tikzcd}
D_1 \arrow[r, "i_1'"'] \arrow[rr, bend left, "i_1"] \arrow[d, shift left, "\epsilon_1"] & A' \arrow[r, "p"'] \arrow[d, shift left, "\alpha'"] & A \arrow[d, shift left, "\alpha"] \\
E_1 \arrow[u, shift left, "\delta_1"] \arrow[r, "g_1'"] \arrow[rr, bend right, "g_1"'] & B' \arrow[u, shift left, "\beta'"] \arrow[r, "q"] & B \arrow[u, shift left, "\beta"]
\end{tikzcd}$$

a split pushout. Indeed, just note that the universal property of A' implies that pairs of morphisms $u : A' \to W$ and $v : B \to W$ such that $u\beta' = vq$ are in bijection with cones over (5). Since q is necessarily a regular epimorphism (g_1 and g_2 are jointly strongly epimorphic) it follows that p is too being a pushout of q. The claim now follows from Lemma 2.10 applied to previous diagram (since the left hand square is by assumption a split pullback). ∎

2.12. Proposition. *Let $\mathbb{C}$ be a regular protomodular category with pushouts. The category $\mathbb{C}$ satisfies Condition 1.10 if and only if each split cube (4) of type* LE1 *where the bottom is a pushout and g_1 and g_2 are monomorphisms, if (4) is of type* LE2*, then the top is a pushout.*

Proof. As in the proof of Proposition 2.11 the "only if" part follows from the fact that given a split cube (4) where the bottom is a pushout, one easily checks that it is of type

RE if and only if the top is a pushout. Conversely given a split cube (4) of type LE2 where g_1 and g_2 form a join, consider the diagram

$$\begin{tikzcd}
 & X \arrow[rrr, "h_2"] \arrow[ddl, "h_1"'] \arrow[ddd, shift left, "\chi"] & & & D_2 \arrow[ddl, "i_2"] \arrow[dll, dashed, "i_2'"'] \arrow[ddd, shift left, "\delta_2"] \\
 & & A' \arrow[dr, "p"] \arrow[ddd, shift left, "\alpha'"] & & \\
D_1 \arrow[urr, dashed, "i_1'"] \arrow[rrr, "i_1"'] \arrow[ddd, shift left, "\delta_1"] & & & A \arrow[ddd, shift left, "\alpha"] & \\
 & I \arrow[uuu, shift left, "\eta"] \arrow[rrr, "f_2"] \arrow[ddl, "f_1"'] & & & E_2 \arrow[uuu, shift left, "\epsilon_2"] \arrow[dll, "g_2'"] \arrow[ddl, "g_2"] \\
 & & B' \arrow[uuu, dashed, shift left, "\beta'"] \arrow[dr, "q"] & & \\
E_1 \arrow[uuu, shift left, "\epsilon_1"] \arrow[urr, "g_1'"] \arrow[rrr, "g_1"'] & & & B \arrow[uuu, shift left, "\beta"] &
\end{tikzcd}$$

where (B', g_1', g_2') is a pushout of f_1 and f_2, and (A', α', p) is the pullback of q and α. The arrows β', i_1' and i_2' are obtained by the universal property of the pullback (A', α', p) and make the inner part into a split "cube" and p and q together a morphism of split epimorphisms. It follows by assumption and the remark at the beginning of the proof that the inner split cube is of type RE. Since g_1 and g_2 are jointly strongly epimorphic it follows that q is a regular epimorphism. This means that p being a pullback of q is also a regular epimorphism. Therefore, since by construction the diagram

$$\begin{tikzcd}
A' \arrow[r, "p"] \arrow[d, shift left, "\alpha'"] & A \arrow[d, shift left, "\alpha"] \\
B' \arrow[u, shift left, "\beta'"] \arrow[r, "q"'] & B \arrow[u, shift left, "\beta"]
\end{tikzcd}$$

is a split pullback it follows by Lemma 2.7 that it is a split pushout. One easily checks that this forces (4) to be of type RE. ∎

3. Pointed categories and categories with initial objects

The main purpose of this section is to explain what each of the Conditions 1.6, 1.8 and 1.10 mean when the underlying category is pointed, and finitely complete and cocomplete. We have:

3.1. PROPOSITION. *Let $\mathbb{C}$ be a protomodular category with initial object. The category $\mathbb{C}$ satisfies any one of Conditions 1.8, 1.9 or 1.10 if and only if it satisfies the restriction of the same condition to those split cubes where I is the initial object.*

Proof. For each split cube (4) and each morphism of split epimorphisms

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, shift left, "\chi'"] & X \arrow[d, shift left, "\chi"] \\ I' \arrow[r, "v"'] \arrow[u, shift left, "\eta'"] & I, \arrow[u, shift left, "\eta"] \end{tikzcd} \tag{9}$$

note that the split cube (4) is of type RE if the cube

$$\begin{tikzcd} & X' \arrow[rr, "h'_2"] \arrow[dl, "h'_1"'] \arrow[dd, shift left, "\chi'"] & & D_2 \arrow[dl, "i_2"'] \arrow[dd, shift left, "\delta_2"] \\ D_1 \arrow[rr, "i_1"] \arrow[dd, shift left, "\delta_1"] & & A \arrow[dd, shift left, "\alpha"] & \\ & I' \arrow[rr, "f'_2"] \arrow[dl, "f'_1"'] \arrow[uu, shift left, "\eta'"] & & E_2 \arrow[dl, "g_2"] \arrow[uu, shift left, "\epsilon_2"] \\ E_1 \arrow[rr, "g_1"'] \arrow[uu, shift left, "\epsilon_1"] & & B \arrow[uu, shift left, "\beta"] & \end{tikzcd} \tag{10}$$

with $h'_j = f_j u$ and $f'_j = f_j v$ is of type RE. Note also that the converse holds when u and η are jointly epimorphic. Now, since $\mathbb{C}$ is protomodular this means that if (9) is a split pullback, then (4) is of type RE, if and only if (10) is of type RE. The claim now follows, using this last observation where I' is an initial object and (9) is obtained by pulling back along the unique morphism $I' \to I$. To see why just note that the front faces of the split cubes (4) and (10) are the same and that, by Lemma 2.9, (10) is of type LE1 if and only if (4) is of type LE1. ■

3.2. Remark. *The previous proposition would remain true if "protomodular" was replaced by "finitely complete" and Condition 1.9 was dropped from the list of conditions.*

3.3. Proposition. *Let $\mathbb{C}$ be a pointed protomodular category. The category $\mathbb{C}$ satisfies Condition 1.6 if and only if reflexive graphs in $\mathbb{C}$ are multiplicative as soon as they are star multiplicative.*

Proof. It is sufficient to show that if a reflexive graph (C_1, C_0, d, c, e) admits an (I, j)-multiplicative structure for some $j : I \to C_0$, and $v : I' \to I$ is a morphism, then it admits an (I', jv)-multiplicative structure. Given a reflexive graph (C_1, C_0, d, c, d, e) and morphisms $j : I \to C_0$ and $v : I' \to I$ we can form the split pullbacks

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, shift left, "\tilde{d}"] & C_1 \arrow[d, shift left, "d"] \\ I \arrow[r, "j"'] \arrow[u, shift left, "\tilde{e}"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd} \quad \begin{tikzcd} C_1 \times_{\langle d, ck \rangle} X \arrow[r, shift left, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[l, shift left, "\langle eck, 1 \rangle"] \arrow[d, "ck"] \\ C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \end{tikzcd} \quad \begin{tikzcd} X' \arrow[r, "k'"] \arrow[d, shift left, "\tilde{d}'"] & C_1 \arrow[d, shift left, "d"] \\ I' \arrow[r, "jv"'] \arrow[u, shift left, "\tilde{e}'"] & C_0 \arrow[u, shift left, "e"] \end{tikzcd} \quad \begin{tikzcd} C_1 \times_{\langle d, ck' \rangle} X' \arrow[r, shift left, "\pi_2"] \arrow[d, "\pi_1"'] & X' \arrow[l, shift left, "\langle eck', 1 \rangle"] \arrow[d, "ck'"] \\ C_1 \arrow[r, shift left, "d"] & C_0. \arrow[l, shift left, "e"] \end{tikzcd}$$

Accordingly there are unique morphisms

$$u : X' \to X \text{ and } 1 \times u : C_1 \times_{\langle eck', 1 \rangle} X' \to C_1 \times_{\langle d, ck \rangle} X$$

such that $\tilde{d}u = v\tilde{d}'$, $ku = k'$ and $\pi_1(1 \times u) = \pi_1$ and $\pi_2(1 \times u) = u\pi_2$. Now suppose

$$p : C_1 \times_{\langle d,ck\rangle} X \to X$$

is an (I, j)-multiplicative structure for (C_1, C_0, d, c, e). Since one can show by protomodularity that $\tilde{d}p = \tilde{d}\pi_2$, it follows that the outer arrows of the diagram

$$\begin{tikzcd}
C_1 \times_{\langle d,ck'\rangle} X' \arrow[r, "1\times u"] \arrow[d, dashed, "p'"] \arrow[dd, bend right, "\tilde{d}'\pi_2"'] & C_1 \times_{\langle d,ck\rangle} X \arrow[d, "p"] \\
X' \arrow[r, "u"] \arrow[d, "\tilde{d}'"] & X \arrow[d, "\tilde{d}"] \\
I' \arrow[r, "v"'] & I
\end{tikzcd}$$

commute and hence since the lower rectangle is a pullback we obtain a unique morphism p' making the diagram commute. An easy calculation shows that p' is an (I', jv)-multiplicative structure for (C_1, C_0, d, c, e). ∎

It follows from Lemma 2.9 and Proposition 3.1 that, when $\mathbb{C}$ is a pointed protomodular category, Condition 1.8 is equivalent to the following condition. For each diagram

$$\begin{tikzcd}
 & D_1 \arrow[r, shift left, "\epsilon_1"] \arrow[d, "i_1"] & E_1 \arrow[l, shift left, "\delta_1"] \arrow[d, "g_1"] \\
X \arrow[ur, "h_1"] \arrow[dr, "h_2"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \\
 & D_2 \arrow[r, shift left, "\epsilon_2"] \arrow[u, "i_2"] & E_2 \arrow[l, shift left, "\delta_2"] \arrow[u, "g_2"']
\end{tikzcd} \tag{11}$$

where $\epsilon_1\delta_1 = 1_{E_1}$, $\epsilon_2\delta_2 = 1_{E_2}$, $\alpha\beta = 1_B$, h_1 and h_2 are the kernels of ϵ_1 and ϵ_2 respectively, and g_1 and g_2 form a join, if i_1 and i_2 form a join, then the composite i_1h_1 is the kernel of α. As we shall show, an equivalent statement can be formulated by replacing "i_1h_1 is the kernel of α" by "i_1h_1 is a normal monomorphism". To prove this it is sufficient to show that for a diagram (11) with $\epsilon_1\delta_1 = 1_{E_1}$, $\epsilon_2\delta_2 = 1_{E_2}$, $\alpha\beta = 1_B$, h_1 and h_2 being the kernels of ϵ_1 and ϵ_2, respectively, and the pairs g_1 and g_2, and i_1 and i_2 forming joins, the morphism α is necessarily the cokernel of i_1h_1. However, in such a situation, if $f : A \to C$ is a morphism such that $fi_1h_1 = 0$, then since $f\beta\alpha i_1h_1 = 0 = fi_1h_1$ and $f\beta\alpha i_1\delta_1 = f\beta\alpha\beta g_1 = f\beta g_1 = fi_1\delta_1$, and similarly $f\beta\alpha i_2h_2 = fi_2h_2$ and $f\beta\alpha i_2\delta_2 = fi_2\delta_2$ it follows by protomodularity that $f\beta\alpha i_1 = fi_1$ and $f\beta\alpha i_2 = fi_2$. This means, since i_1 and i_2 are jointly epimorphic, that $f = f\beta\alpha$ and hence, since $\alpha i_1h_1 = 0$ and α is an epimorphism, that α is the cokernel of i_1h_1.

3.4. Theorem. *Let $\mathbb{C}$ be a semi-abelian category (more generally an exact pointed protomodular category with binary joins of subobjects.) The category $\mathbb{C}$ satisfies Condition 1.8 if and only if $\mathbb{C}$ satisfies NU.*

PROOF. It easily follows from the discussion directly proceeding the statement of the theorem that NU implies Condition 1.8. Conversely, given a square

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "k'"'] & E \arrow[d, "g"] \\ E' \arrow[r, "g'"'] & B \end{tikzcd}$$

where k and k' are normal monomorphisms and g and g' form a join. We will show that gk is a normal monomorphism. Consider the diagram

$$\begin{tikzcd} & & & R' \arrow[rrr, shift left, "r'_1"] \arrow[dl, "i'"'] \arrow[dddl, "(g'\times g')r'"] & & & E' \arrow[lll, shift left, "s'"] \arrow[dl, "g'"] \arrow[dddl, bend left, "g'"] \\ X \arrow[urrr, "h'"] \arrow[dr, "h"'] \arrow[ddd, "gk"'] & & A \arrow[rrr, shift left, "\alpha"] \arrow[dd, "l"] & & & B \arrow[lll, shift left, "\beta"] \arrow[dd, equal] & \\ & R \arrow[ur, "i"] \arrow[rrr, shift left, "r_1"] \arrow[dr, bend right, "(g\times g)r"'] & & & E \arrow[lll, shift left, "s"] \arrow[ur, "g"'] \arrow[dr, bend right, "g"] & & \\ B \arrow[rr, "\langle 0,1\rangle"] & & B\times B \arrow[rrr, shift left, "\pi_1"] & & & B \arrow[lll, shift left, "\langle 1,1\rangle"] & \end{tikzcd}$$

where

- (R, r_1, r_2) and (R', r'_1, r'_2) are the associated equivalence relations of k and k' (i.e. the kernel pairs of the respective morphisms which k and k' are the kernel of);
- h and h' are kernels of r_1 and r'_1, respectively;
- $r = \langle r_1, r_2\rangle$, $r' = \langle r'_1, r'_2\rangle$;
- (A, l) is the join of $(R, (g \times g)r)$ and $(R', (g' \times g')r')$ in $B \times B$;
- i and i' are the respective inclusions into the join;
- $\alpha = \pi_1 l$;
- β is the (unique) morphism such that $\beta g = is$, $\beta g' = i's'$, and $l\beta = \langle 1, 1\rangle$ obtained using the fact that g' and g are jointly strongly epimorphic.

Note that $\alpha\beta = 1_B$, $gr_1 = \alpha i$ and $g'r'_1 = \alpha i'$. Now, since i and i' form a join it follows, from the reformulation of Condition 1.8 directly proceeding the proposition, that ih is the kernel of α and hence the diagram

$$\begin{tikzcd} X \arrow[r, "ih"] \arrow[d, "gk"'] & A \arrow[d, "l"] \\ B \arrow[r, "\langle 0,1\rangle"'] & B\times B \end{tikzcd}$$

is pullback. Therefore, since $l : A \to B \times B$ is a reflexive relation and hence an effective equivalence relation it follows that gk is a normal monomorphism as desired. ∎

3.5. Remark. *Note that according to Proposition 3.1 for a pointed protomodular category Condition 1.10 is equivalent to requiring for each diagram (11) where* $\epsilon_1\delta_1 = 1_{E_1}$, $\epsilon_2\delta_2 = 1_{E_2}$, $\alpha\beta = 1_B$, g_1 *and* g_2 *form a join, and* h_1, h_2 *and* i_1h_1 *are the kernels of* ϵ_1, ϵ_2 *and* α, *respectively, the object* A *together with the morphisms* i_1, i_2, β *is the colimit of the diagram*

$$\begin{tikzcd}
 & D_1 & E_1 \arrow[l, "\delta_1"'] \arrow[d, "g_1"] \\
X \arrow[ur, "h_1"] \arrow[dr, "h_2"'] & & B \\
 & D_2 & E_2. \arrow[l, "\delta_2"] \arrow[u, "g_2"']
\end{tikzcd}$$

3.6. Proposition. *Let* $\mathbb{C}$ *be a pointed protomodular category satisfying Condition 1.10. For each commutative diagram*

$$\begin{tikzcd}
 & & & T_1 \arrow[dll, bend right, "n_1"'] \arrow[dl, "i_1"] \\
S \arrow[r, "m"] & X & T \arrow[l, "n"'] & \\
 & & & T_2 \arrow[ull, bend left, "n_2"] \arrow[ul, "i_2"']
\end{tikzcd}$$

of monomorphisms, if i_1 *and* i_2 *form a join and* m *commutes with* n_1 *and* n_2, *then* m *commutes with* n.

Proof. Let $\varphi_1 : S \times T_1 \to X$ and $\varphi_2 : S \times T_2 \to X$ be the unique morphisms exhibiting that m and n_1, and m and n_2 commute. Since the split cube in the diagram

$$\begin{tikzcd}
 & S \arrow[rr, "\langle 1,0\rangle"] \arrow[dl, "\langle 1,0\rangle"'] \arrow[dd, shift left] & & S \times T_2 \arrow[dl, "1\times i_2"'] \arrow[dr, "\varphi_2"] \arrow[dd, shift left, "\pi_2"] & \\
S \times T_1 \arrow[rr, "1\times i_1"'] \arrow[rrrr, bend left, "\varphi_1"] \arrow[dd, shift left, "\pi_2"] & & S \times T \arrow[rr, dashed] \arrow[dd, shift left, "\pi_2"] & & X \\
 & 0 \arrow[rr] \arrow[dl] \arrow[uu, shift left] & & T_2 \arrow[dl, "i_2"] \arrow[uu, shift left, "\langle 0,1\rangle"] & \\
T_1 \arrow[rr, "i_1"'] \arrow[uu, shift left, "\langle 0,1\rangle"] & & T \arrow[uurr, bend right, "n"'] \arrow[uu, shift left, "\langle 0,1\rangle"] & &
\end{tikzcd}$$

satisfies the appropriate conditions it follows from Condition 1.10 that it is of type RE and hence there exists a unique morphism $\varphi : S \times T \to X$ such that $\varphi(1 \times i_1) = \varphi_1$, $\varphi(1 \times i_2) = \varphi_2$ and $\varphi\langle 0, 1\rangle = n$. It easily follows that m and n commute. ∎

Recall that a pointed category $\mathbb{C}$ is algebraically cartesian closed (as considered in [7] and named in [5]) when for each object B in $\mathbb{C}$ the pullback functor $(B \to 1)^* : \mathbb{C} \to \mathbf{Pt}(B)$ has a right adjoint. A category $\mathbb{C}$ is fiberwise algebraically cartesian closed when for each object B in $\mathbb{C}$ the category $\mathbf{Pt}(B)$ is algebraically cartesian closed. In [5] it was shown that a pointed regular unital category is algebraically cartesian closed if and only if it admits centralizers. In [8], for semi-abelian categories, it was shown that algebraic-cartesian-closedness implies the distributivity of Huq commutators, and furthermore the two conditions are equivalent under conditions that hold for every variety of universal algebras. Here we obtain:

3.7. Corollary. *Let $\mathbb{C}$ be an exact protomodular category with finite colimits. If $\mathbb{C}$ satisfies Condition 1.10, then for each object B in $\mathbb{C}$, Huq commutators in the category $\mathbf{Pt}(B)$ distribute over binary joins.*

Proof. Since for each B in $\mathbb{C}$ the category $\mathbf{Pt}(B)$ is semi-abelian and by Proposition 2.1 satisfies Condition 1.10, it is sufficient to prove that Huq commutators distribute over binary joins in a semi-abelian category satisfying Condition 1.10. According to the previous proposition we see for subobjects (S, m), (T_1, n_1) and (T_2, n_2) of X in $\mathbb{C}$ if m commutes with n_1 and n_2, then m commutes with the join of n_1 and n_2. The proof of Theorem 1.1 in [8] shows that this last condition implies distributivity of the Huq commutator in $\mathbb{C}$. ∎

4. Categories in which each object has global support

In this section we consider the Conditions 1.6, 1.8, 1.9 and 1.10 in (not necessarily pointed) contexts with finite limits where objects have global support. For a category $\mathbb{C}$ with finite limits recall that an object B in $\mathbb{C}$ has global support if the unique morphism into the terminal object 1 is an effective descent morphism (that is, the change of base functor $\mathbb{C} \to (\mathbb{C} \downarrow B)$ is monadic [12]). Examples of categories in which each object has global support include pointed categories with finite limits (since, for each B, the unique morphism $B \to 1$ being a split epimorphism is necessarily an effective descent morphism, see [13]) and exact categories with an initial object in which the unique morphism from the initial object to the terminal object is a regular epimorphism (since this forces each morphism with codomain 1 to be a regular epimorphism and exactness implies that regular epimorphisms are effective descent morphisms).

Recall, as explained above, if a finitely complete category $\mathbb{C}$ satisfies any of the conditions in Subsections 1.5 and 1.7 then for each B in $\mathbb{C}$ the category $\mathbf{Pt}(B)$ satisfies the same condition. For regular protomodular categories in which each object has global support and for Conditions 1.6, 1.8 and 1.10 we obtain converses. To obtain these converses we will use several results some of which seem to be interesting in their own right.

4.1. Proposition. *Let $\langle T, \eta, \mu \rangle$ be a monad on a category $\mathbb{C}$ with binary joins of subobjects. If T preserves binary joins (or more generally preserves those jointly strongly epimorphic cospans consisting of monomorphisms), then the forgetful functor U from the* *category $\mathbb{C}^T$ of algebras over the monad T to the category $\mathbb{C}$ preserves and reflects binary joins.*

PROOF. The fact that U reflects binary joins follows from the fact that it preserves monomorphisms and reflects isomorphisms. Now, suppose that

$$\begin{tikzcd} (A,\alpha) \arrow[r, "i"] & (C,\gamma) & (B,\beta) \arrow[l, "j"'] \end{tikzcd}$$

is a cospan forming a join in $\mathbb{C}^T$ and let (S,m) be the join of (A,i) and (B,j) as subobjects of C (in $\mathbb{C}$) as displayed in the diagram

$$\begin{tikzcd} & S \arrow[d, "m"] & \\ A \arrow[ur, "u"] \arrow[r, "i"'] & C & B. \arrow[ul, "v"'] \arrow[l, "j"] \end{tikzcd}$$

Since $T(u)$ and $T(v)$ are jointly strongly epimorphic (by assumption), the diagram

$$\begin{tikzcd} T(A) \arrow[r, "T(u)"] \arrow[ddr, bend right, "u\alpha"'] & T(S) \arrow[d, "\gamma T(m)"] \arrow[dd, dashed, bend right, "\sigma"'] & T(B) \arrow[l, "T(v)"'] \arrow[ddl, bend left, "v\beta"] \\ & C & \\ & S \arrow[u, "m"] & \end{tikzcd}$$

commutes, and m is a monomorphism, it follows that there is a unique morphism σ such that $m\sigma = \gamma T(m)$. Using again the fact that m is a monomorphism we see that (S,σ) is an algebra over the monad. Therefore, since the diagram

$$\begin{tikzcd} & (S,\sigma) \arrow[d, "m"] & \\ (A,\alpha) \arrow[ur, "u"] \arrow[r, "i"'] & (C,\gamma) & (B,\beta) \arrow[ul, "v"'] \arrow[l, "j"] \end{tikzcd}$$

commutes in $\mathbb{C}^T$, it follows that m is an isomorphism proving that i and j form a join in $\mathbb{C}$. ∎

4.2. PROPOSITION. *Let $U : \mathbb{X} \to \mathbb{C}$ be a functor preserving binary joins, and reflecting and preserving limits. If $\mathbb{C}$ satisfies Condition 1.8, then $\mathbb{X}$ satisfies the same condition.*

PROOF. Suppose that (4) is a split cube of type LE1, in $\mathbb{X}$, such that both g_1 and g_2, and i_1 and i_2 form joins. Since U preserves binary joins and limits it follows by Condition 1.8 that the image of (4) under U is of type LE2. However since U reflects limits this means that (4) is of type LE2 as required. ∎

4.3. Proposition. *Let $\langle T, \eta, \mu \rangle$ be a monad on a category $\mathbb{C}$ with finite limits and binary joins of subobjects. If T preserves binary joins (or more generally preserves those jointly strongly epimorphic cospans consisting of monomorphisms), and $\mathbb{C}$ satisfies Condition 1.10, then $\mathbb{C}^T$ satisfies the same condition.*

Proof. Suppose that (4) is a split cube of type LE2 in $\mathbb{C}^T$ such that g_1 and g_2 form a join. Since the forgetful functor U from $\mathbb{C}^T$ to $\mathbb{C}$ preserves limits and binary joins (by Proposition 4.1) it follows by Condition 1.10 that the image of (4) under U is of type RE. An easy calculation using the fact that $TU(i_1)$ and $TU(i_2)$ are jointly (strongly) epimorphic shows that under these conditions (4) is of type RE in $\mathbb{C}^T$. ∎

4.4. Proposition. *Let $\langle T, \eta, \mu \rangle$ be a monad on a protomodular category $\mathbb{C}$ with finite limits, let $((C_0, \gamma_0), (C_1, \gamma_1), d, c, e)$ be a reflexive graph in $\mathbb{C}^T$ and let $j : (I, \iota) \to (C_0, \gamma_0)$ be a morphism in $\mathbb{C}^T$. If T preserves binary joins (or more generally preserves those jointly strongly epimorphic cospans consisting of monomorphisms), then $((C_0, \gamma_0), (C_1, \gamma_1), d, c, e)$ is $((I, \iota), j)$-multiplicative if and only if (C_0, C_1, d, c, e) is (I, j)-multiplicative.*

Proof. The “only if” part follows from the fact that the forgetful functor U from the $\mathbb{C}^T$ to $\mathbb{C}$ preserves limits. Using the notation from Subsection 1.5 the “if” part follows easily from the fact that the morphisms

$$T(X) \xrightarrow{T(\langle k, \tilde{e}\tilde{d}\rangle)} T(C_1 \times_{\langle d, ck\rangle} X) \xleftarrow{T(\langle eck, 1\rangle)} T(X)$$

are jointly (strongly) epimorphic. ∎

4.5. Lemma. *Let $\mathbb{C}$ be a regular Mal'tsev category with terminal object 1. If*

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a cospan such that f and g forms a join and $X \to 1$ is a regular epimorphism, then for each W in $\mathbb{C}$ such that $W \to 1$ is a regular epimorphism the cospan

$$W \times X \xrightarrow{1 \times f} W \times Z \xleftarrow{1 \times g} W \times Y$$

forms a join.

Proof. Suppose that

$$\begin{tikzcd} & S \arrow[d, "\langle s_1, s_2\rangle"] & \\ W \times X \arrow[ur, "u"] \arrow[r, "1\times f"'] & W \times Z & W \times Y \arrow[ul, "v"'] \arrow[l, "1\times g"] \end{tikzcd}$$

is a monomorphism of cospans. We will begin by showing that s_2 is a regular epimorphism. Since $\pi_2 : W \times X \to X$ and $\pi_2 : W \times Y \to Y$ are regular epimorphisms (both being pullbacks of $W \to 1$) it follows that the cospan at the bottom of the diagram

$$\begin{tikzcd}
 & S \arrow[d, "s_2"] & \\
W \times X \arrow[ur, "u"] \arrow[r, "f\pi_2"'] & Z & W \times Y \arrow[ul, "v"'] \arrow[l, "g\pi_2"]
\end{tikzcd}$$

is jointly strongly epimorphic, and so s_2 is a regular epimorphism as desired. Now, consider the diagram

$$\begin{tikzcd}
(W \times X) \times S \arrow[r, "\langle s_1\pi_2,\pi_2\pi_1\rangle"] \arrow[d, dashed, "w"] \arrow[dd, bend right, "u\times 1"'] & W \times X \arrow[d, "u"] \\
D \arrow[r, "d_3"] \arrow[d, "\langle d_1,d_2\rangle"] & S \arrow[d, "\langle s_1,s_2\rangle"] \\
S \times S \arrow[r, "\langle s_1\pi_2,s_2\pi_1\rangle"'] & W \times Z
\end{tikzcd}$$

where the bottom rectangle is a pullback and w is the unique morphism making the diagram commute. Writing δ for the morphism exhibiting that S is difunctional, it follows that the diagram

$$\begin{tikzcd}
(W \times X) \times S \arrow[r, "w"] \arrow[d, "\pi_1\times 1"'] & D \arrow[r, "\delta"] \arrow[dr, "\langle s_1d_1,s_2d_2\rangle"'] & S \arrow[d, "\langle s_1,s_2\rangle"] \\
W \times S \arrow[rr, "1\times s_2"'] & & W \times Z
\end{tikzcd}$$

commutes. However, since both $\pi_1 \times 1$ and $1 \times s_2$ are regular epimorphisms (being pullbacks of the regular epimorphisms $X \to 1$ and $s_2 : S \to Z$ respectively), it follows that their composite is an extremal epimorphism and hence $\langle s_1, s_2\rangle$ is an isomorphism. ∎

As a corollary we obtain:

4.6. Lemma. *Let $\mathbb{C}$ be a regular Mal'tsev category such that each object has global support. For each B in $\mathbb{C}$ the functor T forming part of the monad $\langle T, \eta, \mu\rangle$ induced by the adjunction $f! \dashv f^* : \mathbb{C} \to (\mathbb{C} \downarrow B)$, where f is the unique morphism $B \to 1$, preserves binary joins.*

Proof. The claim follows from the previous lemma by noting that T is defined on an object (A, α) by $T(A, \alpha) = (B \times A, \pi_1)$ and on a morphism $f : (A, \alpha) \to (A', \alpha')$ by $T(f) = 1 \times f$. ∎

4.7. Lemma. *Let $\mathbb{C}$ be a regular Mal'tsev category such that each object has global support. For each B in $\mathbb{C}$ the functor P from $(B \downarrow \mathbb{C})$ to $\mathbf{Pt}(B)$ sending (C, δ) to $(B \times C, \pi_1, \langle 1, \delta\rangle)$ is monadic, and the functor T which is part of the induced monad $\langle T, \eta, \mu\rangle$ on $\mathbf{Pt}(B)$ preserves binary joins.*

PROOF. Let B be an object in $\mathbb{C}$ and let (C, γ) be an object in $(B \downarrow \mathbb{C})$. Since by assumption the unique morphism $f : C \to 1$ in $\mathbb{C}$ is an effective descent morphism it follows (directly or from Theorem 16 of [18] with Γ the constant functor onto B) that $f : (C, \gamma) \to (1, f\gamma)$ is an effective descent morphism in $(B \downarrow \mathbb{C})$. This means that $(B \downarrow \mathbb{C})$ is a regular Mal'tsev category such that each object has global support. The claim now follows from the previous lemma since the diagram

$$\begin{tikzcd}
(B \downarrow \mathbb{C}) \arrow[r, "P"] \arrow[d, "\cong"'] & \mathbf{Pt}(B) \\
((B \downarrow \mathbb{C}) \downarrow (1, f)) \arrow[r, "f^*"'] & ((B \downarrow \mathbb{C}) \downarrow (B, 1_B)), \arrow[u, "\cong"']
\end{tikzcd}$$

in which the vertical arrows are canonical isomorphisms, commutes. ∎

4.8. THEOREM. *Let $\mathbb{C}$ be a regular protomodular category with binary joins of subobjects such that each object has global support. For n equal to 1.6, 1.8 or 1.10, the following are equivalent:*

(a) the category $\mathbb{C}$ satisfies Condition n;

(b) for each B in $\mathbb{C}$ the category $(B \downarrow \mathbb{C})$ satisfies Condition n;

(c) for each B in $\mathbb{C}$ the category $\mathbf{Pt}(B)$ satisfies Condition n.

When in addition $\mathbb{C}$ has an initial object 0, these conditions are equivalent to

(d) the category $\mathbf{Pt}(0)$ satisfies Condition n.

PROOF. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) follow from Proposition 2.1. The implication (b) $\Rightarrow$ (a) follows easily from the discussion before Proposition 2.1, since a split cube (4) can be thought of as a split cube in $(I \downarrow \mathbb{C})$. The implications (c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a) (when $\mathbb{C}$ has an initial object) follow from Lemma 4.7 and Propositions 4.4, 4.2 and 4.3, when n is 1.6, 1.8 or 1.10, respectively. Finally when $\mathbb{C}$ has an initial object (c) trivially implies (d). ∎

4.9. REMARK. *Note that for n equal to 1.6, according to Proposition 3.3, "Condition n" in both (c) and (d) could be replaced by "the condition that star multiplicative graphs are multiplicative". When in addition $\mathbb{C}$ is exact, for n equal to 1.8, according to Theorem 3.4, "Condition n" in both (c) and (d) could be replaced by "NU".*

Note that the idea of studying non-pointed categories as categories of algebras over monads on pointed categories is not new. The first author learnt of this approach in private communication with G. Janelidze about his series of talks at the Australian Category Seminar on unpublished joint work with A. Carboni, G. M. Kelly and S. Lack.

Next we show that for a regular protomodular category, in which each object has global support, Condition 1.10 implies that change of base functors of the fibration of points reflect Huq commutativity. To do so we need some preliminary facts. Recall that for a descent morphism $p : E \to B$ (i.e. a pullback stable regular epimorphism) the comparison functor from $(\mathbb{C} \downarrow B)$ to the category of descent data associated to p is full and faithful [12]. Here we will use the following consequence:

4.10. Lemma. *Suppose $\mathbb{C}$ is a category with finite limits and $p : E \to B$ is a descent morphism with kernel pair $\pi_1, \pi_2 : E \times_B E \to E$. Let $p^* : \mathbf{Pt}(B) \to \mathbf{Pt}(E)$ and $\pi_1^*, \pi_2^* : \mathbf{Pt}(E) \to \mathbf{Pt}(E \times_B E)$ be the associated pullback functors, and $\phi : \pi_1^* p^* \to \pi_2^* p^*$ be the canonical isomorphism. For objects (A, α, β) and (A', α', β') in $\mathbf{Pt}(B)$, and for a morphism $h : p^*(A, \alpha, \beta) \to p^*(A', \alpha', \beta')$ in $\mathbf{Pt}(E)$, if the diagram*

$$\begin{tikzcd}
(\pi_1^* p^*)(A, \alpha, \beta) \arrow[r, "\pi_1^*(h)"] \arrow[d, "\phi_{(A,\alpha,\beta)}"'] & (\pi_1^* p^*)(A', \alpha', \beta') \arrow[d, "\phi_{(A',\alpha',\beta')}"] \\
(\pi_2^* p^*)(A, \alpha, \beta) \arrow[r, "\pi_2^*(h)"] & (\pi_2^* p^*)(A', \alpha', \beta')
\end{tikzcd}$$

commutes, then there exists a unique morphism $g : (A, \alpha, \beta) \to (A', \alpha', \beta')$ in $\mathbf{Pt}(B)$ such that $h = p^*(g)$.

4.11. Lemma. *Let $F : \mathbb{C} \to \mathbb{D}$ and $G_1, G_2 : \mathbb{D} \to \mathbb{E}$ be finite limit preserving functors between unital categories, and let $\phi : G_1 F \to G_2 F$ be a natural transformation, such that if $h : F(C) \to F(C')$ is a morphism with $\phi_{C'} G_1(h) = G_2(h) \phi_C$, then there exists $g : C \to C'$ such that $F(g) = h$. If F is faithful, then the functor F reflects Huq commuting pairs.*

Proof. Let $f : A \to C$ and $g : B \to C$ be morphisms in $\mathbb{C}$ such that $F(f)$ commutes with $F(g)$. Since F preserves limits this means that there exists a unique morphism $\theta : F(A \times B) \to F(C)$ such that the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(\langle 1,0 \rangle)"] \arrow[dr, bend right, "F(f)"'] & F(A \times B) \arrow[d, "\theta"] & F(B) \arrow[l, "F(\langle 0,1 \rangle)"'] \arrow[dl, bend left, "F(g)"] \\
& F(C) &
\end{tikzcd}$$

commutes. Since the morphisms $G_1F(\langle 1,0\rangle)$ and $G_1F(\langle 0,1\rangle)$ are jointly (strongly) epimorphic it follows that the diagram

$$\begin{tikzcd}
& G_1F(A) \arrow[rr, "G_1F(\langle 1,0\rangle)"] \arrow[dl, "\phi_A"'] \arrow[ddrr, bend right, "G_1F(f)"'] & & G_1F(A\times B) \arrow[dl, "\phi_{A\times B}"] \arrow[dd, "G_1(\theta)"] & & G_1F(B) \arrow[ll, "G_1F(\langle 0,1\rangle)"'] \arrow[dl, "\phi_B"] \arrow[ddll, bend left, "G_1F(g)"] \\
G_2F(A) \arrow[rr, "G_2F(\langle 1,0\rangle)"'] \arrow[ddrr, bend right, "G_2F(f)"'] & & G_2F(A\times B) \arrow[dd, "G_2(\theta)"] & & G_2F(B) \arrow[ll, "G_2F(\langle 0,1\rangle)"] \arrow[ddll, bend left, "G_2F(g)"] & \\
& & & G_1F(C) \arrow[dl, "\phi_C"] & & \\
& & G_2F(C) & & &
\end{tikzcd}$$

commutes, and hence there exists a morphism $\psi : A\times B \to C$ such that $F(\psi) = \theta$. Since F is a faithful, $F(f) = \theta F(\langle 1,0\rangle) = F(\psi)F(\langle 1,0\rangle) = F(\psi\langle 1,0\rangle)$ and similarly $F(g) = F(\psi\langle 0,1\rangle)$ it follows that f and g commute in $\mathbb{C}$. ∎

As a corollary of the last two lemmas we obtain:

4.12. PROPOSITION. *Let $\mathbb{C}$ be a Mal'tsev category with finite limits. If $p : E \to B$ is a descent morphism, then $p^* : \mathbf{Pt}(B) \to \mathbf{Pt}(E)$ reflects Huq commuting pairs.*

PROOF. Just apply Lemma 4.11 to the functors and natural transformations of Lemma 4.10. ∎

Combining this last proposition with Proposition 2.5 we obtain:

4.13. THEOREM. *Let $\mathbb{C}$ be a regular protomodular category such that each object has global support. If $\mathbb{C}$ satisfies Condition 1.10, then (SSH) holds in $\mathbb{C}$ (i.e. change of base functors of the fibration of points reflect Huq commutativity).*

PROOF. Let $p : E \to B$ be a morphism in $\mathbb{C}$. Since the diagram

$$\begin{tikzcd}
E \arrow[rr, "p"] \arrow[dr, "\langle 1,p\rangle"'] & & B \\
& E\times B \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

commutes, $\langle 1,p\rangle$ is a monomorphism and π_2 is regular epimorphism it follows by Propositions 2.5 and 4.12 that the functor $p^* \cong \langle 1,p\rangle^*\pi_2^*$ reflects Huq commutativity. ∎

5. Examples

In this section we give pointed and non-pointed examples satisfying the conditions above. Recall that every category of interest in the sense of [19] is a semi-abelian category satisfying NU (see [3]). Therefore, according to Theorem 3.4, every category of interest satisfies

Condition 1.8. In particular this means that the categories of groups, not necessarily unital rings and various types of algebras over a ring such as Lie, associative, commutative, and Leibniz all satisfy Condition 1.8. Furthermore, Proposition 4.2 implies that, any sub-category $\mathbb{X}$ of a category of interest $\mathbb{C}$ such that the forgetful functor $\mathbb{X} \to \mathbb{C}$ reflects limits and binary joins will also satisfy Condition 1.8. Amongst others this means that the categories of unital rings, of unital associative algebras over a fixed ring, and of unital Boolean rings (= Boolean algebras) also satisfy Condition 1.8. Further examples can be obtained, according to Proposition 2.1, by forming comma, (co)comma and categories of points in each of the previous examples. Finally according to Propositions 2.2, 2.3, 2.6 and Theorem 4.13 all of the previously mentioned categories also satisfy Conditions 1.6, 1.9, 1.10 and (SSH).

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