STABILITY FOR INNER FIBRATIONS REVISITED

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ABSTRACT. In this paper we prove a stability result for inner fibrations in terms of the wide, or fat join operation on simplicial sets. We also prove some additional results on inner anodyne morphisms that may be of independent interest.

1. Introduction

Recall (see [Ehlers and Porter (2000), Joyal (2002), Lurie (2009), Riehl and Verity (2015)]) that the *join* of simplicial sets A and B is the simplicial set $A \star B$ whose set of n-simplices for $n \geq 0$ is given by

$$(A \star B)_n = A_n \sqcup B_n \sqcup \bigsqcup_{i+j=n-1} A_i \times B_j.$$

For a fixed simplicial set A, the functor $A \star (-) \colon \mathcal{S}et_\Delta \to (\mathcal{S}et_\Delta)_{A/}$ commutes with colimits and hence admits a right adjoint. If $p \colon A \to X$ is an object in $(\mathcal{S}et_\Delta)_{A/}$ then its image under this right adjoint is the simplicial set $X_{p/}$ defined in Section 1.2.9 of [Lurie (2009)]. A fundamental result in the theory of ∞-categories is the following stability result due to Joyal (see Theorem 3.8 of [Joyal (2002)]) and Lurie (see Proposition 2.1.2.1 of [Lurie (2009)]).

1.1. THEOREM. [Joyal (2002), Lurie (2009)] *Suppose given a diagram*

$$L \xrightarrow{u} X \xrightarrow{p} S$$

of simplicial sets and maps between them. Suppose that $p \colon X \to S$ is an inner fibration between simplicial sets. If $K \subset L$ is a subcomplex and $v = u|_K$ then the induced map

$$X_{u/} \to X_{v/} \times_{S_{pv/}} S_{pu/}$$

is a left fibration which is a trivial Kan fibration if the inclusion $K \subset L$ is inner anodyne.

There is another join construction for simplicial sets, due originally to Joyal and developed further in [Lurie (2009)] (this notion of join also plays a prominent role in

[Riehl and Verity (2015)]). If A and B are simplicial sets then the *fat* or *wide join* is the simplicial set $A \diamond B$ defined by the pushout diagram

$$
\begin{CD}
\partial \Delta^1 \times A \times B @>>> \Delta^1 \times A \times B \\
@VVV @VVV \\
A \sqcup B @>>> A \diamond B
\end{CD}
$$

in $\mathcal{S}\mathrm{et}_\Delta$. It follows easily that the set $(A \diamond B)_n$ of n-simplices of $A \diamond B$ is given by

$$
(A \diamond B)_n = A_n \sqcup \coprod_{i=1}^{n-1} (A_n \times B_n) \sqcup B_n.
$$

The wide join has the advantage over the join construction described above in that it is simple to define, and is a fairly robust construction — it is a homotopy pushout in the Joyal model structure on simplicial sets. There is also a canonical comparison map $A \diamond B \to A \star B$ for any simplicial sets A and B, this turns out to be a categorical equivalence (see Proposition 4.2.1.2 of [Lurie (2009)]). In other respects, the wide join is not as convenient as the join construction; for instance, there is an isomorphism $\Delta^0 \diamond \Delta^0 \simeq \Delta^1$, but there is not an isomorphism $\Delta^0 \diamond \Delta^1 \simeq \Delta^2$ — one can check that $\Delta^0 \diamond \Delta^1$ has two non-degenerate 2-simplices.

For each fixed simplicial set K, the functor $K \diamond (-) \colon \mathcal{S}\mathrm{et}_\Delta \to (\mathcal{S}\mathrm{et}_\Delta)_{K/}$ preserves colimits (see Section 4.2.1 of [Lurie (2009)]) and hence admits a right adjoint whose value on a map $p \colon K \to X$ in $(\mathcal{S}\mathrm{et}_\Delta)_{K/}$ is the *wide slice* $X^{p/}$. We shall prove the following theorem in Section 3.

1.2. THEOREM. *Suppose given maps*

$$
A \xrightarrow{u} B \xrightarrow{f} X \xrightarrow{p} Y
$$

in $\mathcal{S}\mathrm{et}_\Delta$. If p is an inner fibration then the induced map

$$
X^{f/} \to X^{fu/} \times_{Y^{pfu/}} Y^{pf/}
$$

is a left fibration which is a trivial Kan fibration if u is inner anodyne.

This result is a strengthening of Proposition 4.2.1.6 from [Lurie (2009)] where the result above is proven under the assumption that $p \colon X \to Y$ is a categorical fibration. As far as we are aware Theorem 1.2 has not appeared in the literature before. The proof of Theorem 1.2 will be easy, once we have

1.3. THEOREM. *Suppose $u \colon A \to B$ and $v \colon C \to D$ are monomorphisms of simplicial sets. If v is left anodyne then the canonical map*

$$
u \diamond' v \colon B \diamond C \cup A \diamond D \to B \diamond D
$$

is inner anodyne.

In Section 3 we shall also prove the following result, which gives an alternative characterization of cocartesian morphisms.

1.4. THEOREM. *Let $p\colon X \to S$ be an inner fibration of simplicial sets. Suppose that $u\colon x \to y$ is an edge of X. Then the following statements are equivalent:*

1. *u is a p-cocartesian morphism of X;*

2. *the induced map $X^{u/} \to X^{x/} \times_{S^{p(x)/}} S^{p(u)/}$ is a trivial Kan fibration.*

Our main technical tool for the proof of Theorem 1.3 will be the following theorem which is of interest in its own right.

1.5. THEOREM. *The class of inner anodyne maps has the right cancellation property.*

Recall (see [Joyal and Tierney (2007)]) that a class $\mathcal{A}$ of monomorphisms in Set_Δ is said to satisfy the *right cancellation* property if the following is true: for all monomorphisms of simplicial sets $i\colon A \to B$ and $j\colon B \to C$, if $i, ji \in \mathcal{A}$ then $j \in \mathcal{A}$.

In summary then this paper is as follows. In Section 2 we prove Theorem 1.5 and establish some corollaries of it. In Section 3 we prove Theorem 1.3, the stability theorem (Theorem 1.2) and we prove Theorem 1.4 giving an alternate description of cocartesian morphisms in terms of the wide slice. For the most part we will use the notation from [Lurie (2009)]; thus Set_Δ will denote the category of simplicial sets, for instance.

2. Inner anodyne maps

The aim of this section is to prove Theorem 1.5 and explore some corollaries of it. To begin with, we give some examples of classes of monomorphisms with the right cancellation property. The class of anodyne maps in Set_Δ has the right cancellation property (this is a simple consequence of the 2-out-of-3 property for weak homotopy equivalences). Less obvious is the following result due to Joyal (see Corollary 8.15 of [Joyal (2008)]) and Lurie (see Corollary 4.1.2.2 of [Lurie (2009)]).

2.1. PROPOSITION. [Joyal (2008), Lurie (2009)] *The class of left anodyne maps in Set_Δ satisfies the right cancellation property.*

The right cancellation property forms part of several criteria for showing that a class of monomorphisms contains a given class as a subclass. For example we have the following result due to Joyal and Tierney (see Lemma 3.5 of [Joyal and Tierney (2007)])

2.2. PROPOSITION. [Joyal and Tierney (2007)] *Let $\mathcal{A}$ be a saturated class of monomorphisms in Set_Δ satisfying the right cancellation property. If $\mathcal{A}$ contains the inclusions $I_n \subset \Delta^n$ for all $n \geq 2$ then $\mathcal{A}$ contains every inner anodyne map.*

Here I_n denotes the *n-chain*, i.e.

$$I_n = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \cdots \cup \Delta^{\{n-1,n\}}.$$

From [Stevenson (2017)] we have an analogous criteria for left anodyne maps which we shall make use of in the following section.

2.3. PROPOSITION. [Stevenson (2017)] *Let $\mathcal{A}$ be a saturated class of monomorphisms in* Set_Δ *which satisfies the right cancellation property. Then the following statements are equivalent:*

1. *$\mathcal{A}$ contains the class of left anodyne morphisms;*

2. *$\mathcal{A}$ contains the initial vertex maps $\Delta^{\{0\}} \to \Delta^n$ for all $n \geq 1$;*

3. *$\mathcal{A}$ contains the horn inclusions $h_n^0 \colon \Lambda_0^n \subset \Delta^n$ for all $n \geq 1$.*

We now turn our attention to the proof of Theorem 1.5. We will need the following lemma.

2.4. LEMMA. *Let $p \colon X \to S$ be an inner fibration between simplicial sets and suppose given a commutative diagram*

$$
\begin{array}{ccc}
(\Delta^1 \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) & \xrightarrow{\quad u \quad} & X \\
\downarrow & & \downarrow p \\
\Delta^1 \times \Delta^n & \xrightarrow[pr_1]{} \Delta^n \xrightarrow[v]{} & S
\end{array}
$$

where $n \geq 1$. If the edge $u|\Delta^1 \times \{0\}$ is an equivalence in the ∞-category $X_{v(0)}$ then the square has a diagonal filler.

Here $X_{v(0)}$ denotes the ∞-category which is the fiber of $p \colon X \to S$ over the vertex $v(0)$ of S.

PROOF. By pulling back $p \colon X \to S$ to Δ^n via v we may suppose without loss of generality that S and X are ∞-categories. We recall some facts about the structure of the simplicial set $\Delta^1 \times \Delta^n$. This simplicial set can be regarded as the nerve of the category $[1] \times [n]$, which may be pictured as

$$
\begin{array}{ccccc}
0 & \longrightarrow & 1 \longrightarrow \cdots \longrightarrow & n \\
\downarrow & & \downarrow & & \downarrow \\
0' & \longrightarrow & 1' \longrightarrow \cdots \longrightarrow & n'.
\end{array}
$$

The simplicial set $\Delta^1 \times \Delta^n$ is a union of $n+1$ non-degenerate $(n+1)$-simplices σ_0, σ_1, ..., σ_n where

$$
\sigma_0 = \Delta^{\{0,0',\ldots,n'\}}, \quad \sigma_1 = \Delta^{\{0,1,1',\ldots,n'\}}, \ldots, \sigma_n = \Delta^{\{0,1,\ldots,n,n'\}}.
$$

Following the proof of Proposition 2.1.2.6 from [Lurie (2009)], we define a chain of inclusions

$$
A(n+1) \subset A(n) \subset \cdots \subset A(1) \subset A(0)
$$

of $\Delta^1 \times \Delta^n$ where

$$
A(n+1) = \{0\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n
$$

and where $A(k)$ denotes the union of $A(k+1)$ and the simplex σ_k (together with all of its faces). Then $A(k)$ is the pushout

$$A(k+1) \cup_{\Lambda_k^{n+1}} \Delta^{n+1}$$

and $A(0) = \Delta^1 \times \Delta^n$. Since the inclusions $A(k+1) \subset A(k)$ are inner anodyne for $1 \leq k \leq n$ we are reduced to the problem of showing that we may extend the map $\Lambda_0^{n+1} \to X_{\sigma_0}$ induced by u, over the simplex σ_0 (here X_{σ_0} denotes the pullback of X to σ_0). By hypothesis the edge $u|\Delta^1 \times \{0\}$ is an equivalence in $X_{v(0)}$ and hence in X_{σ_0}; the result then follows from Proposition 1.2.4.3 of [Lurie (2009)]. ∎

We now give the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. Suppose that $i \colon A \to B$ and $j \colon B \to C$ are monomorphisms of simplicial sets such that i and ji are inner anodyne. We will prove that j is inner anodyne. It suffices to show that j has the left lifting property against all inner fibrations.

Let $p \colon X \to Y$ be an inner fibration between simplicial sets X and Y and suppose given a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{u} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{v} & Y. \end{array}$$

We will show that there exists a diagonal filler for this diagram. Since ji is inner anodyne we may choose a map $\phi \colon C \to X$ such that $p\phi = v$ and $\phi ji = ui$. Observe that ϕj and u lie in the same fiber of the canonical map $X^B \to Y^B \times_{Y^A} X^A$. Since this map is a trivial Kan fibration (on account of the hypothesis on i — see Corollary 2.3.2.5 of [Lurie (2009)]) it follows that there exists a homotopy $h \colon \phi j \to u$ relative to A over Y. Thus h determines a map $h \colon \Delta^1 \times B \to X$ such that $h|\{0\} \times B = \phi j$, $h|\{1\} \times B = u$, $ph = vj\pi_B$, $h(1 \times i) = ui\pi_A$, and $h|\Delta^1 \times \{b\}$ is an equivalence in $X_{v(b)}$ for all vertices b of B.

Since $ph = vj\pi_B = v\pi_C(1 \times j)$, the maps h and ϕ determine a map $w \colon \{0\} \times C \cup \Delta^1 \times B \to X$ forming part of a commutative diagram

$$\begin{array}{ccc} \{0\} \times C \cup \Delta^1 \times B & \xrightarrow{\quad w \quad} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^1 \times C & \xrightarrow[\pi_C]{} C \xrightarrow{v} & Y. \end{array}$$

Observe that $w|\Delta^1 \times \{b\}$ is an equivalence in the ∞-category $X_{v(b)}$ for every vertex b of B. To finish the proof it suffices to find a diagonal filler $d \colon \Delta^1 \times C \to X$ for this diagram, for then $d|\{1\} \times C \to X$ is a diagonal filler for the original diagram above.

Writing $C = B \cup \bigcup_{n \geq 1} \mathrm{sk}_n C$, where $\mathrm{sk}_n C$ denotes the n-skeleton of C and noting that $A_0 = B_0 = C_0$, we see that it suffices to prove by induction on $n \geq 0$ that there is a

diagonal filler for the induced diagrams

$$\{\,0\,\} \times (B \cup \mathrm{sk}_{n+1}C) \cup \Delta^1 \times (B \cup \mathrm{sk}_n C) \longrightarrow X$$

$$\Delta^1 \times (B \cup \mathrm{sk}_{n+1}C) \longrightarrow B \cup \mathrm{sk}_{n+1}C \xrightarrow[v]{} Y.$$

But the map $B \cup \mathrm{sk}_n C \to B \cup \mathrm{sk}_{n+1}C$ forms part of a pushout diagram

$$\bigsqcup \partial \Delta^{n+1} \longrightarrow B \cup \mathrm{sk}_n C$$

$$\bigsqcup \Delta^{n+1} \longrightarrow B \cup \mathrm{sk}_{n+1}C$$

where the coproduct is over the set of non-degenerate $(n+1)$-simplices of C which do not belong to B. Hence we may apply Lemma 2.4 to conclude that the required diagonal fillers exist. ∎

We give some corollaries. First we have

2.5. LEMMA. *Suppose given a commutative diagram*

$$
\begin{array}{ccccc}
A_1 & \xleftarrow{\ u\ } & A_2 & \longrightarrow & A_3 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\
B_1 & \xleftarrow{\ v\ } & B_2 & \longrightarrow & B_3
\end{array}
$$

of simplicial sets where f_1, f_2, f_3 are inner anodyne and u, v are monomorphisms. If the left hand square is a pullback, then the induced map

$$A_1 \cup_{A_2} A_3 \to B_1 \cup_{B_2} B_3$$

is inner anodyne.

PROOF. The induced map factors as

$$A_1 \cup_{A_2} A_3 \to A_1 \cup_{A_2} B_3 \simeq A_1 \cup_{A_2} B_2 \cup_{B_2} B_3 \to B_1 \cup_{B_2} B_3.$$

The first map in this composite is inner anodyne since it is the pushout of an inner anodyne map. The induced map $A_1 \cup B_2 = A_1 \cup_{A_2} B_2 \to B_1$ is inner anodyne by Theorem 1.5. It follows that the second map in the composite above is also the pushout of an inner anodyne map. ∎

2.6. COROLLARY. *Suppose that $u\colon A \to B$ is an inner anodyne map in Set_Δ. Then the induced map $u \diamond C\colon A \diamond C \to B \diamond C$ is inner anodyne for any simplicial set C.*

PROOF. This follows from two applications of Lemma 2.5 and the definition of the wide join. ∎

3. Stability of inner fibrations

In this section we prove Theorems 1.2, 1.3 and 1.4. We begin with the following observation: if $A \subset C$ and $B \subset D$ are subcomplexes, then $A \diamond B$ is naturally a subcomplex of $C \diamond D$ — this follows from the description of the set of n-simplices of the wide join given in the Introduction.

Next, we prove the following proposition.

3.1. PROPOSITION. *Suppose that $u\colon A \to B$ and $v\colon C \to D$ are monomorphisms of simplicial sets. If u is inner anodyne then so is the canonical map*

$$u \diamond' v\colon A \diamond D \cup B \diamond C \to B \diamond D.$$

PROOF. We have a commutative diagram

$$
\begin{array}{ccc}
A \diamond C & \longrightarrow & A \diamond D \\
\downarrow & & \downarrow \\
B \diamond C & \longrightarrow & B \diamond D
\end{array}
$$

in which the vertical maps are inner anodyne by Corollary 2.6. It follows that the canonical map

$$A \diamond D \to A \diamond D \cup B \diamond C$$

is inner anodyne, since it is the pushout of an inner anodyne map. Now the composite map

$$A \diamond D \to A \diamond D \cup B \diamond C \to B \diamond D$$

is inner anodyne, and so is the left hand map. Therefore the right hand map, i.e. the map $u \diamond' v$, is inner anodyne by Theorem 1.5. ∎

Our aim now is to prove Theorem 1.3.

PROOF OF THEOREM 1.3. We make a series of reductions to progressively simpler cases. We show first it suffices to prove the proposition in the special case when $A = \emptyset$. Assuming the truth of the proposition in this case for the moment, we see that in the commutative diagram

$$
\begin{array}{ccc}
A \diamond C \cup_C D & \longrightarrow & B \diamond C \cup_C D \\
\downarrow & & \downarrow \\
A \diamond D & \longrightarrow & B \diamond D.
\end{array}
$$

the vertical maps are inner anodyne. The composite map

$$B \diamond C \cup_C D \to B \diamond C \cup A \diamond D \to B \diamond D$$

is inner anodyne, and the left hand map in this composite is inner anodyne since it is a pushout of an inner anodyne map. Therefore, by the right cancellation property (Theorem 1.5) of inner anodyne maps it follows that $u \diamond' v$ is also inner anodyne.

Assume now that $A = \emptyset$. Since the class of inner anodyne maps is weakly saturated, and the functors $(-) \diamond C \colon \mathcal{S}\mathrm{et}_\Delta \to (\mathcal{S}\mathrm{et}_\Delta)_{C/}$ and $(-) \diamond D \colon \mathcal{S}\mathrm{et}_\Delta \to (\mathcal{S}\mathrm{et}_\Delta)_{D/}$ are compatible with colimits (see Section 1), a standard argument using the skeletal filtration of B and Lemma 2.5 shows that we may reduce to the special case in which u is the inclusion $\emptyset \subset \Delta^n$ for some $n \geq 0$. If $n \geq 2$ let $I_n \subset \Delta^n$ denote the inclusion of the n-chain. In the commutative diagram of monomorphisms

$$
\begin{array}{ccc}
I_n \diamond C \cup_C D & \longrightarrow & \Delta^n \diamond C \cup_C D \\
\downarrow & & \downarrow \\
I_n \diamond D & \longrightarrow & \Delta^n \diamond D
\end{array}
$$

the horizontal maps are inner anodyne by Corollary 2.6. Therefore, by the right cancellation property of inner anodyne maps (Theorem 1.5) it suffices to prove that the left hand vertical map is inner anodyne. Therefore we are reduced (by Lemma 2.5) to the special case in which u is the inclusion $\emptyset \subset \Delta^0$ or $\emptyset \subset \Delta^1$. In the case where u is the inclusion $\emptyset \subset \Delta^1$, the map

$$
\Delta^1 \diamond C \cup_C D \to \Delta^1 \diamond D
$$

is isomorphic to the map

$$
D \cup_{D \times \Delta^1} \left((\Delta^0 \diamond C \cup_C D) \times \Delta^1 \right) \to D \cup_{D \times \Delta^1} \left((\Delta^0 \diamond D) \times \Delta^1 \right).
$$

Since the pushout of an inner anodyne map is inner anodyne, we are finally reduced to the special case in which u is the inclusion $\emptyset \subset \Delta^0$.

Let $\mathcal{A}$ be the class of monomorphisms $v \colon C \to D$ in $\mathcal{S}\mathrm{et}_\Delta$ for which $u \diamond' v$ is inner anodyne. To complete the proof of the proposition it suffices to show that $\mathcal{A}$ contains every left anodyne map. Clearly $\mathcal{A}$ is saturated, and satisfies the right cancellation property by Theorem 1.5. Therefore, by Proposition 2.3 it suffices to check that $\mathcal{A}$ contains the initial vertex maps $\Delta^{\{0\}} \to \Delta^n$ for all $n \geq 0$. If v is the initial vertex map $\Delta^{\{0\}} \to \Delta^n$, then $u \diamond' v$ is isomorphic to the canonical inclusion

$$
i \colon (\Delta^1 \times \{0\}) \cup (\{1\} \times \Delta^n) \hookrightarrow \Delta^0 \diamond \Delta^n.
$$

The simplicial set $\Delta^0 \diamond \Delta^n = (\Delta^1 \times \Delta^n) \cup_{\{0\} \times \Delta^n} \Delta^0$ is the union of the simplicial subsets $S_0 = \sigma_0, S_1, \ldots, S_n$ where

$$
S_1 = \sigma_1 \cup_{\Delta^{\{0,1\}}} \Delta^0, \ S_2 = \sigma_2 \cup_{\Delta^{\{0,1,2\}}} \Delta^0, \ldots, S_n = \sigma_n \cup_{\Delta^{\{0,\ldots,n\}}} \Delta^0,
$$

and where the σ_i are the non-degenerate $(n+1)$-simplices of $\Delta^1 \times \Delta^n$ described in Lemma 2.4 above. Let us write $T_0 = S_0$ and $T_i = T_{i-1} \cup S_i$ for $i \geq 1$. Then we have a sequence of inclusions

$$
T_0 \subset T_1 \subset \cdots \subset T_n = \Delta^0 \diamond \Delta^n.
$$

The map i above factors as

$$
(\Delta^1 \times \{0\}) \cup (\{1\} \times \Delta^n) \hookrightarrow T_0 \hookrightarrow T_n = \Delta^0 \diamond \Delta^n
$$

and the inclusion $(\Delta^1 \times \{0\}) \cup (\{1\} \times \Delta^n) \hookrightarrow T_0$ is clearly inner anodyne. To complete the proof we need to show that $T_0 \subset T_n$ is inner anodyne. Clearly it suffices to prove that $T_{i-1} \subset T_i$ is inner anodyne for $i = 1, \ldots, n$. For each $i \geq 1$ we have a pushout diagram

$$
\begin{array}{ccc}
S_i \cap T_{i-1} & \longrightarrow & T_{i-1} \\
\downarrow & & \downarrow \\
S_i & \longrightarrow & T_i
\end{array}
$$

with

$$
S_i \cap T_{i-1} = \Delta^{\{0,\ldots,i-1,i',\ldots,n'\}} \cup_{\Delta^{\{0,\ldots,i-1\}}} \Delta^0.
$$

It suffices to prove that the map $S_i \cap T_{i-1} \to S_i$ is inner anodyne for each $i \geq 1$. This can be done using Theorem 1.5 and the fact that the maps

$$
\Delta^{\{0,\ldots,i-1\}} \cup \Delta^{\{i-1,i',\ldots,n'\}} \to \Delta^{\{0,\ldots,i-1,i',\ldots,n'\}}
$$

and

$$
\Delta^{\{0,\ldots,i\}} \cup \Delta^{\{i,i',\ldots,n'\}} \to \Delta^{\{0,\ldots,i,i',\ldots,n'\}}
$$

are inner anodyne.   ∎

We now prove Theorem 1.2.

PROOF OF THEOREM 1.2. By adjointness, it suffices to show that p has the right lifting property against every map of the form

$$
u \diamond' v \colon B \diamond C \cup A \diamond D \to B \diamond D
$$

where $v \colon C \to D$ is left anodyne. But $u \diamond' v$ is inner anodyne by Theorem 1.3. The case where u is inner anodyne follows from Proposition 3.1.   ∎

The remainder of the paper is devoted to the proof of Theorem 1.4. Let us say that an edge $u \colon \Delta^1 \to X$ is *wide p-cocartesian* if the induced map $X^{u/} \to X^{x/} \times_{S^{p(x)/}} S^{p(u)/}$ is a trivial Kan fibration. Thus to prove Theorem 1.4 we must prove that an edge of X is p-cocartesian if and only if it is wide p-cocartesian.

Suppose first that $u \colon \Delta^1 \to X$ is a wide p-cocartesian edge. We need to show that u is p-cocartesian. Therefore we need to show that in any commutative diagram of the form

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} & & \\
\downarrow \quad \searrow^{u} & & \\
\Lambda_0^{n+2} & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow^{p} \\
\Delta^{n+2} & \longrightarrow & S
\end{array}
$$

where $n \geq 0$ the indicated map exists making the diagram commute. Since u is wide p-cocartesian we may find the indicated diagonal filler in the diagram

$$
\begin{array}{ccccc}
\Delta^1 \diamond \partial\Delta^n \cup \{\,0\,\} \diamond \Delta^n & \longrightarrow & \Lambda_0^{n+2} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
\Delta^1 \diamond \Delta^n & \longrightarrow & \Delta^{n+2} & \longrightarrow & S.
\end{array}
$$

We claim that the map $j \colon \Delta^1 \diamond \partial\Delta^n \cup \{\,0\,\} \diamond \Delta^n \to \Lambda_0^{n+2}$ is a categorical equivalence. Proposition 4.2.1.2 of [Lurie (2009)] implies that the maps $\Delta^1 \diamond \partial\Delta^n \to \Delta^1 \star \partial\Delta^n$ and $\{\,0\,\} \diamond \Delta^n \to \{\,0\,\} \star \Delta^n$ are categorical equivalences. Therefore, since the Joyal model structure is left proper, it suffices to prove that the canonical map

$$
\Delta^1 \diamond \partial\Delta^n \cup_{\{\,0\,\}\diamond\partial\Delta^n} \{\,0\,\} \star \partial\Delta^n \to \Delta^1 \star \partial\Delta^n
$$

is a categorical equivalence. This map forms part of the composite map

$$
\Delta^1 \diamond \partial\Delta^n \to \Delta^1 \diamond \partial\Delta^n \cup_{\{\,0\,\}\diamond\partial\Delta^n} \{\,0\,\} \star \partial\Delta^n \to \Delta^1 \star \partial\Delta^n
$$

and hence it suffices to prove that the first map in this composite is a categorical equivalence. But this map is a pushout of the categorical equivalence $\{\,0\,\} \diamond \partial\Delta^n \to \{\,0\,\} \star \partial\Delta^n$ along the inclusion $\{\,0\,\} \diamond \partial\Delta^n \to \Delta^1 \diamond \partial\Delta^n$ and hence is a categorical equivalence since the Joyal model structure is left proper.

It follows that we may factor the map j as $j = qi$ where $i \colon \Delta^1 \diamond \partial\Delta^n \cup \{\,0\,\} \diamond \Delta^n \to A$ is an acyclic cofibration in the Joyal model structure and $q \colon A \to \Lambda_0^{n+2}$ is a trivial Kan fibration. Let

$$
\begin{array}{ccc}
\Delta^1 \diamond \partial\Delta^n \cup \{\,0\,\} \diamond \Delta^n & \xrightarrow{\;i\;} & A \\
\downarrow & & \downarrow \\
\Delta^1 \diamond \Delta^n & \longrightarrow & B
\end{array}
$$

denote the pushout diagram and note that in the induced diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;q\;} & \Lambda_0^{n+2} \\
\downarrow & & \downarrow \\
B & \longrightarrow & \Delta^{n+2}
\end{array}
$$

the map $B \to \Delta^{n+2}$ is a categorical equivalence by Proposition 4.2.1.2 of [Lurie (2009)] and the fact that the canonical map $\Delta^1 \diamond \Delta^n \to B$ in the diagram above is an acyclic cofibration. We have another commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & \Lambda_0^{n+2} & \longrightarrow & X \\
\downarrow & & & \nearrow & \downarrow{\scriptstyle p} \\
B & \longrightarrow & \Delta^{n+2} & \longrightarrow & S
\end{array}
$$

where the map $B \to X$ is the canonical map from the pushout. Factor the map $B \to \Delta^{n+2}$ as $B \to B' \to \Delta^{n+2}$, where $B \to B'$ is inner anodyne and $B' \to \Delta^{n+2}$ is an inner fibration (and hence a categorical fibration). It follows that $B' \to \Delta^{n+2}$ is a trivial Kan fibration. Since $B \to B'$ is inner anodyne and $p \colon X \to S$ is an inner fibration, we may find a map $B' \to X$ forming part of a commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & \Lambda_0^{n+2} & \longrightarrow & X \\
\downarrow & & & \nearrow & \downarrow{\scriptstyle p} \\
B' & \longrightarrow & \Delta^{n+2} & \longrightarrow & S
\end{array}
$$

To complete the proof that the edge u is p-cocartesian it suffices to prove that $\Lambda_0^{n+2} \to \Delta^{n+2}$ is a retract of $A \to B'$. Since $A \to \Lambda_0^{n+2}$ is a trivial Kan fibration we may choose a section $\Lambda_0^{n+2} \to A$. Similarly, since $B' \to \Delta^{n+2}$ is a trivial Kan fibration, we may extend the induced map $\Lambda_0^{n+2} \to B'$ along the inclusion $\Lambda_0^{n+2} \to \Delta^{n+2}$ to obtain a section of $B' \to \Delta^{n+2}$, which exhibits the desired retraction.

Suppose now that $u \colon \Delta^1 \to X$ is p-cocartesian. We need to show that u is wide p-cocartesian. Therefore we need to show that the indicated diagonal filler exists in every commutative diagram of the form

$$
\begin{array}{ccc}
\Delta^1 & & \\
\downarrow & \searrow^{u} & \\
\Delta^1 \diamond \partial\Delta^n \cup \Delta^{\{0\}} \diamond \Delta^n & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow{\scriptstyle p} \\
\Delta^1 \diamond \Delta^n & \longrightarrow & S.
\end{array}
$$

A short calculation shows that $\Delta^1 \diamond \partial\Delta^n \cup \Delta^{\{0\}} \diamond \Delta^n$ is isomorphic to

$$
\Delta^1 \times A(n+1) \cup_{\partial\Delta^1 \times A(n+1)} \Delta^1 \sqcup \Delta^n
$$

using the notation from the proof of Lemma 2.4. So the lifting problem above may be re-phrased as the problem of finding the indicated diagonal filler in the diagram

$$
\begin{array}{ccc}
\Delta^1 & & \\
\downarrow & \searrow^{u} & \\
\Delta^1 \times A(n+1) \cup_{\partial\Delta^1 \times A(n+1)} \Delta^1 \sqcup \Delta^n & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow{\scriptstyle p} \\
\Delta^1 \times A(0) \cup_{\partial\Delta^1 \times A(0)} \Delta^1 \sqcup \Delta^n & \longrightarrow & S.
\end{array}
$$

Since the maps $A(n+1-i) \to A(n-i)$ are inner anodyne for all $i = 0, 1, \ldots, n-1$ it follows from Lemma 2.5 that we are reduced to proving that the indicated diagonal filler

exists in the diagram

$$
\begin{array}{c}
\Delta^1 \\
\downarrow \qquad \searrow{}^{u} \\
\Delta^1 \times A(1) \cup_{\partial\Delta^1 \times A(1)} \Delta^1 \sqcup \Delta^n \longrightarrow X \\
\downarrow \qquad\qquad\qquad \nearrow \qquad \downarrow{}^{p} \\
\Delta^1 \times A(0) \cup_{\partial\Delta^1 \times A(0)} \Delta^1 \sqcup \Delta^n \longrightarrow S.
\end{array}
$$

Observe that the canonical map $\partial\Delta^1 \times A(0) \to \Delta^1 \sqcup \Delta^n$ induces a pushout diagram

$$
\begin{array}{ccc}
\Delta^1 \times A(1) \cup \partial\Delta^1 \times A(0) & \longrightarrow & \Delta^1 \times A(1) \cup_{\partial\Delta^1 \times A(1)} \Delta^1 \sqcup \Delta^n \\
\downarrow & & \downarrow \\
\Delta^1 \times A(0) & \longrightarrow & \Delta^1 \times A(0) \cup_{\partial\Delta^1 \times A(0)} \Delta^1 \sqcup \Delta^n
\end{array}
$$

and that the induced map $\{\,0\,\} \times \Delta^1 \times \{\,0\,\} \to X$ is isomorphic to the map $u\colon \Delta^1 \to X$ (here $\{\,0\,\} \times \Delta^1 \times \{\,0\,\}$ is thought of as a subcomplex of $\partial\Delta^1 \times A(0)$ in the obvious way). Therefore, it suffices to show that the indicated diagonal filler exists in every commutative diagram of the form

$$
\begin{array}{ccc}
\Delta^1 \times A(1) \cup \partial\Delta^1 \times A(0) & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow{}^{p} \\
\Delta^1 \times A(0) & \longrightarrow & S.
\end{array}
$$

in which the induced map $\{\,0\,\} \times \Delta^1 \times \{\,0\,\} \to X$ represents a p-cocartesian edge of X. Therefore, since the map $A(1) \to A(0)$ is obtained as a pushout of $\Lambda_0^{n+1} \to \Delta^{n+1}$, we are finally reduced to proving the following lemma.

3.2. LEMMA. *Let $n \geq 1$ and suppose given a commutative diagram*

$$
\begin{array}{ccc}
\Delta^1 \times \Lambda_0^n \cup \partial\Delta^1 \times \Delta^n & \xrightarrow{u} & X \\
\downarrow & \overset{f}{\nearrow} & \downarrow{}^{p} \\
\Delta^1 \times \Delta^n & \longrightarrow & S.
\end{array}
$$

of simplicial sets where $p\colon X \to S$ is an inner fibration. If $u|\{\,0\,\}\times\Delta^{\{0,1\}}$ is a p-cocartesian edge of X then the indicated diagonal filler exists.

PROOF. We remark first that

$$
\Delta^1 \times \partial_i\Delta^n = \partial_{i+1}\sigma_0 \cup \cdots \cup \partial_{i+1}\sigma_{i-1} \cup \partial_i\sigma_{i+1} \cup \cdots \cup \partial_i\sigma_n.
$$

where $\partial_i\Delta^n$ denotes the i-th face of Δ^n (i.e. $\partial_i\Delta^n = \Delta^{\{0,\ldots,\hat{i},\ldots,n\}}$), and where $\partial_i\sigma_j$ denotes the i-th face of the $(n+1)$- simplex σ_j.

Write $B(n+1) = \Delta^1 \times \Lambda_0^n \cup \partial\Delta^1 \times \Delta^n$, and assuming that $B(i+1)$ has been defined, let $B(i)$ equal the union of $B(i+1)$ and the $(n+1)$-simplex σ_{n-i}, together with all of its faces. Thus $B(0) = \Delta^1 \times \Delta^n$ and for each $i = 0, 1, \ldots, n$ we have a pushout diagram

$$
\begin{array}{ccc}
B(i+1) \cap \sigma_{n-i} & \longrightarrow & B(i+1) \\
\downarrow & & \downarrow \\
\sigma_{n-i} & \longrightarrow & B(i).
\end{array}
$$

We construct the diagonal filler f from the statement of the lemma by descending induction on n. By the remark above we have an identification

$$
B(n+1) \cap \sigma_0 = \bigcup_{i \neq 1} \partial_i \sigma_0
$$

so that the inclusion $B(n+1) \cap \sigma_0 \subset \sigma_0$ is isomorphic to the inclusion $\Lambda_1^{n+1} \subset \Delta^{n+1}$. It follows that the extension $f|B(n)$ exists since $p \colon X \to S$ is an inner fibration.

Assume that $f|B(i+1)$ has been constructed for $0 < i < n$. We have

$$
B(i+1) \cap \sigma_{n-i} = \bigcup_{j \neq 0, n-i+1} \partial_j \sigma_{n-i}
$$

and

$$
B(i+1) \cap \partial_0 \sigma_{n-i} = \bigcup_{j \neq n-i} \partial_j \partial_0 \sigma_{n-i}.
$$

Thus the inclusion $B(i+1) \cap \partial_0 \sigma_{n-i} \subset \partial_0 \sigma_{n-i}$ is isomorphic to the inner anodyne inclusion $\Lambda_{n-i}^n \subset \Delta^n$. Since p is an inner fibration, it follows that $f|B(i+1)$ extends to a map $f|B(i+1) \cup \partial_0 \sigma_{n-i}$. We now have

$$
(B(i+1) \cup \partial_0 \sigma_{n-i}) \cap \sigma_{n-i} = \bigcup_{j \neq n-i+1} \partial_j \sigma_{n-i}
$$

and it follows that the desired extension $f|B(i)$ exists, using the fact that p is an inner fibration again. Finally, we have

$$
B(1) \cap \sigma_n = \bigcup_{j \neq 0} \partial_j \sigma_n
$$

and hence the extension f exists because $u|\Delta^{\{0,1\}}$ is p-cocartesian. $\blacksquare$

References

P.J. Ehlers and T. Porter, Joins for (augmented) simplicial sets, *J. Pure Appl. Algebra* **145** (2000) no. 1 37–44.

A. Joyal, Quasi-categories and Kan complexes, *J. Pure Appl. Algebra* **175** (2002) 207–222.

A. Joyal, *The theory of quasicategories and its applications*, preprint (2008).

A. Joyal and M. Tierney, Quasi-categories vs Segal spaces, in *Categories in algebra, geometry and mathematical physics*, 277–326, Contemp. Math. **431** Amer. Math. Soc., Providence, RI, 2007.

J. Lurie, *Higher topos theory*, Annals of Mathematics Studies **170**, Princeton University Press, Princeton, NJ, 2009.

E. Riehl and D. Verity, The 2-category theory of quasi-categories, *Adv. Math.*, **280** (2015) 549–642.

D. Stevenson, Covariant model structures and simplicial localization, *North-West. Eur. J. Math.*, **3** (2017), 141–203.

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