

# GENERATING THE ALGEBRAIC THEORY OF $C(X)$ : THE CASE OF PARTIALLY ORDERED COMPACT SPACES

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**ABSTRACT.** It is known since the late 1960’s that the dual of the category of compact Hausdorff spaces and continuous maps is a variety – not finitary, but bounded by  $\aleph_1$ . In this note we show that the dual of the category of partially ordered compact spaces and monotone continuous maps is an  $\aleph_1$ -ary quasivariety, and describe partially its algebraic theory. Based on this description, we extend these results to categories of Vietoris coalgebras and homomorphisms on ordered compact spaces. We also characterise the  $\aleph_1$ -copresentable partially ordered compact spaces.

## 1. Introduction

The motivation for this paper stems from two very different sources. Firstly, it is known since the end of the 1960’s that the dual of the category **CompHaus** of compact Hausdorff spaces and continuous maps is a variety – not finitary, but bounded by  $\aleph_1$ . In detail,

- in [Dus69] it is proved that the representable functor  $\text{hom}(-, [0, 1]): \mathbf{CompHaus}^{\text{op}} \rightarrow \mathbf{Set}$  is monadic,
- the unit interval  $[0, 1]$  is shown to be a  $\aleph_1$ -copresentable compact Hausdorff space in [GU71],
- a presentation of the algebra operations of  $\mathbf{CompHaus}^{\text{op}}$  is given in [Isb82], and
- a complete description of the algebraic theory of  $\mathbf{CompHaus}^{\text{op}}$  is obtained in [MR17].

It is also worth mentioning that, by the famous Gelfand duality theorem [Gel41], **CompHaus** is dually equivalent to the category of commutative  $C^*$ -algebras and homomorphisms; the algebraic theory of (commutative)  $C^*$ -algebras is extensively studied in [Neg71, PR89, PR93].

Our second source of inspiration is the theory of coalgebras. In [KKV04] the authors argue that the category **BooSp** of Boolean spaces and continuous maps “is an interesting base category for coalgebras”; among other reasons, due to the connection of the latter with finitary modal logic. A similar study based on the Vietoris functor on the category **Priest** of

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Priestley spaces and monotone continuous maps can be found in [CLP91, Pet96, BKR07]. Arguably, the categories **BooSp** and **Priest** are very suitable in this context because they are duals of finitary varieties (due to the famous Stone dualities [Sto36, Sto38a, Sto38b]), a property which extends to categories of coalgebras and therefore guarantees for instance good completeness properties.

In this note we go a step further and study the category **PosComp** of partially ordered compact spaces and monotone continuous maps, which was introduced in [Nac50] and forms a natural extension of the categories **CompHaus** and **Priest**. It remains open to us whether **PosComp**<sup>op</sup> is also a variety; however, based on the duality results of [HN18] and inspired by [Isb82], we prove that **PosComp**<sup>op</sup> is an  $\aleph_1$ -ary quasivariety and also give a partial description of its algebraic theory. Moreover, this description is sufficient to identify the dual of the category of coalgebras for the Vietoris functor  $V: \mathbf{PosComp} \rightarrow \mathbf{PosComp}$  as an  $\aleph_1$ -ary quasivariety. Finally, we characterise the  $\aleph_1$ -copresentable objects of **PosComp** as precisely the metrisable ones.

In this paper we assume the reader to be familiar with basic facts about varieties, quasivarieties, locally presentable categories (see, for instance, [AR94]) and quantale-enriched categories. For a gentle introduction to the latter subject we refer to [Law73, Stu14]. The general theory of categories enriched in a symmetric monoidal closed category can be found in [Kel82].

## 2. Preliminaries

In this section we recall the notion of partially ordered compact space introduced in [Nac50] together with some fundamental properties of these spaces.

2.1. DEFINITION. A **partially ordered compact space**  $(X, \leq, \tau)$  consists of a set  $X$ , a partial order  $\leq$  on  $X$  and a compact topology  $\tau$  on  $X$  so that the set

$$\{(x, y) \in X \times X \mid x \leq y\}$$

is closed in  $X \times X$  with respect to the product topology.

We will often write  $X$  instead of  $(X, \leq, \tau)$  to keep the notation simple. For every partially ordered compact space  $X$ , also the subset

$$\{(x, y) \in X \times X \mid x \geq y\}$$

is closed in  $X \times X$  since the mapping  $X \times X \rightarrow X \times X$ ,  $(x, y) \mapsto (y, x)$  is a homeomorphism. Therefore the diagonal

$$\Delta_X = \{(x, y) \in X \times X \mid x \leq y\} \cap \{(x, y) \in X \times X \mid x \geq y\}$$

is closed in  $X \times X$ , which tells us that the topology of a partially ordered compact space is Hausdorff. We denote the category of partially ordered compact spaces and monotone continuous maps by **PosComp**.

2.2. **EXAMPLE.** The unit interval  $[0, 1]$  with the usual Euclidean topology and the “greater or equal” relation  $\geq$  is a partially ordered compact space; via the mapping  $x \mapsto 1 - x$ , this space is isomorphic in **PosComp** to the space with the same topology and the “less or equal” relation  $\leq$ .

There is a canonical functor  $\mathbf{PosComp} \rightarrow \mathbf{Pos}$  from **PosComp** to the category **Pos** of partially ordered sets and monotone maps that forgets the topology. By the observation above, forgetting the order relation induces a functor  $\mathbf{PosComp} \rightarrow \mathbf{CompHaus}$  from **PosComp** to the category **CompHaus** of compact Hausdorff spaces and continuous maps. For more information regarding properties of **PosComp** we refer to [Nac65, GHK<sup>+</sup>80, Jun04, Tho09]; however, let us recall that:

2.3. **THEOREM.** *The category **PosComp** is complete and cocomplete, and both canonical forgetful functors  $\mathbf{PosComp} \rightarrow \mathbf{CompHaus}$  and  $\mathbf{PosComp} \rightarrow \mathbf{Pos}$  preserve limits.*

**PROOF.** This follows from the construction of limits and colimits in **PosComp** described in [Tho09]. ■

A cone  $(f_i: X \rightarrow Y_i)_{i \in I}$  in **PosComp** is initial with respect to the forgetful functor  $\mathbf{PosComp} \rightarrow \mathbf{CompHaus}$  if and only if

$$x_0 \leq x_1 \iff \forall i \in I. f_i(x_0) \leq f_i(x_1),$$

for all  $x_0, x_1 \in X$  (see [Tho09]). In particular, an embedding in **PosComp** is a morphism  $m: X \rightarrow Y$  that satisfies

$$x_0 \leq x_1 \iff m(x_0) \leq m(x_1)$$

for all  $x_0, x_1 \in X$ . Note that embeddings in **PosComp** are, up to isomorphism, closed subspace inclusions with the induced order.

The following result of Nachbin is crucial for our work.

2.4. **THEOREM.** *The unit interval  $[0, 1]$  is injective in **PosComp** with respect to embeddings.*

**PROOF.** See [Nac65, Theorem 6]. ■

The theorem above has the following important consequence.

2.5. **LEMMA.** *Let  $X$  be a partially ordered compact space,  $A, B \subseteq X$  closed subsets so that  $A \cap B = \emptyset$  and  $B = \downarrow B \cap \uparrow B$ . Then there is a family  $(f_u: X \rightarrow [0, 1])_{u \in [0, 1]}$  of monotone continuous maps that coincide on  $A$  and, moreover, satisfy  $f_u(y) = u$ , for all  $u \in [0, 1]$  and  $y \in B$ .*

**PROOF.** Put  $A_0 = A \cap \uparrow B$  and  $A_1 = A \cap \downarrow B$ . Then  $A_0$  and  $A_1$  are closed subsets of  $X$  and

$$A_0 \cap A_1 = A \cap \uparrow B \cap \downarrow B = A \cap B = \emptyset.$$

Moreover, for every  $x_0 \in A_0$  and  $x_1 \in A_1$ ,  $x_0 \not\leq x_1$ . In fact, if  $x_0 \geq y_0 \in B$  and  $x_1 \leq y_1 \in B$ , then  $x_0 \leq x_1$  implies

$$y_0 \leq x_0 \leq x_1 \leq y_1,$$

hence  $x_0 \in B$  which contradicts  $A \cap B = \emptyset$ . We define now the monotone continuous map

$$g: A_0 \cup A_1 \longrightarrow [0, 1]$$

$$x \longmapsto \begin{cases} 0 & \text{if } x \in A_0, \\ 1 & \text{if } x \in A_1. \end{cases}$$

By Theorem 2.4,  $g$  extends to a monotone continuous map  $g: A \rightarrow [0, 1]$ . Let now  $u \in [0, 1]$ . We define

$$f_u: A \cup B \longrightarrow [0, 1]$$

$$x \longmapsto \begin{cases} g(x) & \text{if } x \in A, \\ u & \text{if } x \in B. \end{cases}$$

Using again Theorem 2.4,  $f_u$  extends to a monotone continuous map  $f_u: X \rightarrow [0, 1]$ . ■

The following theorem implies in particular that  $\mathbf{PosComp}^{\text{op}}$  is a quasivariety (see [Adá04, Theorem 3.6]). In the next section we will give a more concrete description of the algebraic theory of  $\mathbf{PosComp}^{\text{op}}$ .

2.6. THEOREM. *The regular monomorphisms in  $\mathbf{PosComp}$  are, up to isomorphism, the closed subspaces with the induced order and the epimorphisms in  $\mathbf{PosComp}$  are precisely the surjections. Consequently,  $\mathbf{PosComp}$  has (Epi, Regular mono)-factorisations and the unit interval  $[0, 1]$  is a regular injective regular cogenerator of  $\mathbf{PosComp}$ .*

PROOF. An application of [AHS90, Proposition 8.7] tells that every regular monomorphism is an embedding, and the converse implication follows from Lemma 2.5.

Since faithful functors reflect epimorphisms, every surjective morphism of  $\mathbf{PosComp}$  is an epimorphism. Let  $f: X \rightarrow Y$  be an epimorphism in  $\mathbf{PosComp}$ , we consider its factorisation  $f = m \cdot e$  in  $\mathbf{PosComp}$  with  $e$  surjective and  $m$  an embedding. Hence, since  $m$  is a regular monomorphism and an epimorphism, we conclude that  $m$  is an isomorphism and therefore  $f$  is surjective.

Finally, the “order Urysohn property” for partially ordered compact spaces (see [Jun04, Lemma 2.2]) implies that  $[0, 1]$  is a regular cogenerator of  $\mathbf{PosComp}$ . ■

We close this section with the following characterisation of cofiltered limits in  $\mathbf{CompHaus}$  which goes back to [Bou66, Proposition 8, page 89] (see also [Hof02, Proposition 4.6] and [HNN16]).

2.7. THEOREM. Let  $D: I \rightarrow \mathbf{CompHaus}$  be a cofiltered diagram and  $(p_i: L \rightarrow D(i))_{i \in I}$  a cone for  $D$ . The following conditions are equivalent:

- (i) The cone  $(p_i: L \rightarrow D(i))_{i \in I}$  is a limit of  $D$ .
- (ii) The cone  $(p_i: L \rightarrow D(i))_{i \in I}$  is mono and, for every  $i \in I$ , the image of  $p_i$  is equal to the intersection of the images of all  $D(k: j \rightarrow i)$  with codomain  $i$ :

$$\mathrm{im} p_i = \bigcap_{j \rightarrow i} \mathrm{im} D(j \xrightarrow{k} i).$$

We emphasise that this intrinsic characterisation of cofiltered limits in  $\mathbf{CompHaus}$  is formally dual to the following well-known description of filtered colimits in  $\mathbf{Set}$  (see [AR94]).

2.8. THEOREM. Let  $D: I \rightarrow \mathbf{Set}$  be a filtered diagram and  $(c_i: D(i) \rightarrow C)_{i \in I}$  a compatible cocone  $(c_i: D(i) \rightarrow C)_{i \in I}$  for  $D$ . The following conditions are equivalent:

- (i) The cocone  $(c_i: D(i) \rightarrow C)_{i \in I}$  is a colimit of  $D$ .
- (ii) The cocone  $(c_i: D(i) \rightarrow C)_{i \in I}$  is epi and, for all  $i \in I$ , the coimage of  $c_i$  is equal to the cointersection of the coimages of all  $D(k: i \rightarrow j)$  with domain  $i$ :

$$c_i(x) = c_i(y) \iff \exists (i \xrightarrow{k} j) \in I. D(k)(x) = D(k)(y),$$

and  $x, y \in D(i)$ .

### 3. The quasivariety $\mathbf{PosComp}^{\mathrm{op}}$

In this section we show that  $\mathbf{PosComp}^{\mathrm{op}}$  is an  $\aleph_1$ -ary quasivariety and give a concrete presentation of the algebra structure of  $\mathbf{PosComp}^{\mathrm{op}}$ . To achieve this, we resort to [HN18] where  $\mathbf{PosComp}^{\mathrm{op}}$  is shown to be equivalent to the category of certain  $[0, 1]$ -enriched categories, for various quantale structures on the complete lattice  $[0, 1]$ . Arguably, the most convenient quantale structure is the **Łukasiewicz tensor** given by  $u \odot v = \max(0, u + v - 1)$ , for  $u, v \in [0, 1]$ . We recall that, for this quantale, a  $[0, 1]$ -category is a set  $X$  equipped with a mapping  $a: X \times X \rightarrow [0, 1]$  so that

$$1 \leq a(x, x) \quad \text{and} \quad a(x, y) \odot a(y, z) \leq a(x, z),$$

for all  $x, y \in X$ . Each  $[0, 1]$ -category  $(X, a)$  induces the order relation (that is, a reflexive and transitive relation)

$$x \leq y \text{ whenever } 1 \leq a(x, y) \quad (x, y \in X)$$

on  $X$ . A  $[0, 1]$ -category is called **separated** (also called skeletal) whenever this order relation is anti-symmetric. As we explain in Section 5, categories enriched in this quantale can be also thought of as metric spaces.

To state the duality result of [HN18], we need to impose certain (co)completeness conditions on  $[0, 1]$ -categories: we consider the category  $\mathbf{A}$  with objects all separated finitely cocomplete  $[0, 1]$ -categories with a monoid structure that, moreover, admit  $[0, 1]$ -powers (also called cotensors); the morphisms of  $\mathbf{A}$  are the finitely cocontinuous  $[0, 1]$ -functors preserving the monoid structure and the  $[0, 1]$ -powers. Alternatively, these structures can be described algebraically as sup-semilattices with actions of  $[0, 1]$ . Below we recall from [HN18] that the category  $\mathbf{A}$  together with its canonical forgetful functor  $\mathbf{A} \rightarrow \mathbf{Set}$  is an  $\aleph_1$ -ary quasivariety.

3.1. REMARK. The category  $\mathbf{A}$  is equivalent to the quasivariety defined by the following operations and implications (also known as quasi-equations). Firstly, the set of operation symbols consists of

- the nullary operation symbols  $\perp$  and  $\top$ ;
- the unary operation symbols  $- \odot u$  and  $- \pitchfork u$ , for each  $u \in [0, 1]$ ;
- the binary operation symbols  $\vee$  and  $\odot$ .

Secondly, the algebras for this theory should be sup-semilattices with a supremum-preserving action of  $[0, 1]$ ; writing  $x \leq y$  as an abbreviation for the equation  $y = x \vee y$ , this translates to the equations and implications

$$\begin{aligned} x \vee x &= x, & x \vee (y \vee z) &= (x \vee y) \vee z, & x \vee \perp &= x, & x \vee y &= y \vee x, \\ x \odot 1 &= x, & (x \odot u) \odot v &= x \odot (u \odot v), & \perp \odot u &= \perp, & (x \vee y) \odot u &= (x \odot u) \vee (y \odot u), \\ x \odot u &\leq x \odot v, & \bigwedge_{u \in S} (x \odot u \leq y) &\implies (x \odot v \leq y) & (S \subseteq [0, 1] \text{ countable, } v = \sup S). \end{aligned}$$

The algebras defined by the operations  $\perp$ ,  $\vee$  and  $- \odot u$  ( $u \in [0, 1]$ ) and the equations above are precisely the separated  $[0, 1]$ -categories with finite weighted colimits. Such a  $[0, 1]$ -category  $(X, a)$  has all powers  $x \pitchfork u$  ( $x \in X, u \in [0, 1]$ ) if and only if, for all  $u \in [0, 1]$ ,  $- \odot u$  has a right adjoint  $- \pitchfork u$  with respect to the underlying order. Therefore we add to our theory the implications

$$x \odot u \leq y \iff x \leq y \pitchfork u,$$

for all  $u \in [0, 1]$ . Finally, regarding  $\odot$ , we impose the commutative monoid axioms with neutral element the top-element:

$$x \odot y = y \odot x, \quad x \odot (y \odot z) = (x \odot y) \odot z, \quad x \odot \top = x, \quad \top \leq x.$$

Moreover, we require this multiplication to preserve suprema and the action  $- \odot u$  (for  $u \in [0, 1]$ ) in each variable:

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z), \quad x \odot \perp = \perp, \quad x \odot (y \odot u) = (x \odot y) \odot u.$$

3.2. REMARK. The unit interval  $[0, 1]$  becomes an algebra for the theory above with  $\odot = \ominus$  and  $v \pitchfork u = \min(1, 1 - u + v) = 1 - \max(0, u - v)$ , and the usual interpretation of all other symbols.

The following result is in [HN18].

3.3. THEOREM. *The functor*

$$C: \mathbf{PosComp}^{\text{op}} \longrightarrow \mathbf{A}$$

*sending  $f: X \rightarrow Y$  to  $Cf: CY \rightarrow CX$ ,  $\psi \mapsto \psi \cdot f$  is fully faithful, here the algebraic structure on*

$$CX = \{f: X \rightarrow [0, 1] \mid f \text{ is monotone and continuous}\}$$

*is defined pointwise.*

3.4. REMARK. We recall that, in order to define the functor of Theorem 3.3, it is important that all operation symbols are interpreted as monotone continuous functions  $[0, 1]^n \rightarrow [0, 1]$ . Furthermore, the theorem above remains valid if we augment the algebraic theory of  $\mathbf{A}$  with an operation symbol corresponding to a monotone continuous function  $[0, 1]^I \rightarrow [0, 1]$ . More precisely, let  $\aleph$  be a cardinal and  $h: [0, 1]^\aleph \rightarrow [0, 1]$  a monotone continuous map. If we add to the algebraic theory of  $\mathbf{A}$  an operation symbol of arity  $\aleph$ , then  $C: \mathbf{PosComp}^{\text{op}} \rightarrow \mathbf{A}$  lifts to a fully faithful functor from  $\mathbf{PosComp}^{\text{op}}$  to the category of algebras for this theory by interpreting the new operation symbol in  $CX$  by

$$(f_i)_{i \in I} \longmapsto (X \xrightarrow{\langle f_i \rangle_{i \in I}} [0, 1]^I \xrightarrow{h} [0, 1]).$$

By Theorem 3.3, all  $\mathbf{A}$ -morphisms of type  $CY \rightarrow CX$  preserves this new operation automatically.

3.5. REMARK. Note that  $1 - u = 0 \pitchfork u$ , for every  $u \in [0, 1]$ . Therefore we can express truncated minus  $v \ominus u = \max(0, v - u)$  in  $[0, 1]$  with the operations of  $\mathbf{A}$ :

$$v \ominus u = 0 \pitchfork (u \pitchfork v).$$

In particular, every subalgebra  $M \subseteq CX$  of  $CX$  is also closed under truncated minus.

In order to identify the image of  $C: \mathbf{PosComp}^{\text{op}} \longrightarrow \mathbf{A}$ , in [HN18] we use an adaptation of the classical ‘‘Stone–Weierstra theorem’’. This theorem affirms that certain subalgebras of  $CX$  are dense, which in turn depends on a notion of closure in quantale enriched categories. Such a closure operator is studied in [HT10]; however, as we observed in [HN18], for categories enriched in  $[0, 1]$  with the Łukasiewicz tensor, the topology defined by this closure operator coincides with the usual topology induced by the ‘‘sup-metric’’ on  $CX$ .

3.6. THEOREM. *Let  $X$  be a partially ordered compact space and  $m: A \hookrightarrow CX$  be a sub-object of  $CX$  in  $\mathbf{A}$  so that the cone  $(m(a): X \rightarrow [0, 1])_{a \in A}$  is point-separating and initial. Then  $m$  is dense in  $CX$ . In particular, if the  $[0, 1]$ -category  $A$  is Cauchy complete, then  $m$  is an isomorphism.*

One important consequence of Theorem 3.6 is the following proposition.

3.7. PROPOSITION. *The unit interval  $[0, 1]$  is  $\aleph_1$ -copresentable in  $\mathbf{PosComp}$ .*

PROOF. This can be shown with the same argument as in [GU71, 6.5.(c)]. Firstly, by Theorem 3.6,  $\text{hom}(-, [0, 1])$  sends every  $\aleph_1$ -codirected limit to a jointly surjective cocone. Secondly, using Theorem 2.8, this cocone is a colimit since  $[0, 1]$  is  $\aleph_1$ -copresentable in  $\mathbf{CompHaus}$ . ■

3.8. THEOREM. *The functor  $C: \mathbf{PosComp}^{\text{op}} \rightarrow \mathbf{A}$  corestricts to an equivalence between  $\mathbf{PosComp}^{\text{op}}$  and the full subcategory of  $\mathbf{A}$  defined by those objects  $A$  which are Cauchy complete and where the cone of all  $\mathbf{A}$ -morphisms from  $A$  to  $[0, 1]$  is point-separating.*

PROOF. See [HN18]. ■

Instead of working with Cauchy completeness, we wish to add an operation to the algebraic theory of  $\mathbf{A}$  so that, if  $M$  is closed under this operation in  $CX$ , then  $M$  is closed with respect to the topology of the  $[0, 1]$ -category  $CX$ . In the case of  $\mathbf{CompHaus}$ , this is achieved in [Isb82] using the operation

$$[0, 1]^{\mathbb{N}} \longrightarrow [0, 1], (u_n)_{n \in \mathbb{N}} \longmapsto \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u_n$$

on  $[0, 1]$ . For a compact Hausdorff space  $X$ , Isbell considers the set  $CX$  of all continuous functions  $X \rightarrow [-1, 1]$ . He observes that every subset  $M \subseteq CX$  closed under the operation above (defined now in  $[-1, 1]$ ), truncated addition and subtraction, is topologically closed. To see why, let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence in  $M$  with limit  $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ , we may assume that  $\|\varphi_{n+1} - \varphi_n\| \leq \frac{1}{2^{n+1}}$ , for all  $n \in \mathbb{N}$ . Then

$$\varphi = \varphi_0 + \frac{1}{2}(2(\varphi_1 - \varphi_0)) + \dots \in M.$$

However, this argument cannot be transported directly into the ordered setting since the difference  $\varphi_1 - \varphi_0$  of two monotone maps  $\varphi_0, \varphi_1: X \rightarrow [0, 1]$  is not necessarily monotone. To circumvent this problem, we look for a monotone and continuous function  $[0, 1]^{\mathbb{N}} \rightarrow [0, 1]$  which calculates the limit of “sufficiently many” sequences. We now make the meaning of “sufficiently many” more precise.

3.9. LEMMA. *Let  $M \subseteq CX$  be a subalgebra in  $\mathbf{A}$  and  $\psi \in CX$  with  $\psi \in \overline{M}$ . Then there exists a sequence  $(\psi_n)_{n \in \mathbb{N}}$  in  $M$  converging to  $\psi$  so that*

1.  $(\psi_n)_{n \in \mathbb{N}}$  is increasing, and
2. for all  $n \in \mathbb{N}$  and all  $x \in X: \psi_{n+1}(x) - \psi_n(x) \leq \frac{1}{2^n}$ .

PROOF. We can find  $(\psi_n)_{n \in \mathbb{N}}$  so that, for all  $n \in \mathbb{N}$ ,  $|\psi_n(x) - \psi(x)| \leq \frac{1}{n+1}$ . Then the sequence  $(\psi_n \ominus \frac{1}{n+1})_{n \in \mathbb{N}}$  converges to  $\psi$  too; moreover, since  $M \subseteq CX$  is a subalgebra, also  $\psi_n \ominus \frac{1}{n+1} \in M$ , for all  $n \in \mathbb{N}$ . Therefore we can assume that we have a sequence  $(\psi_n)_{n \in \mathbb{N}}$  in  $M$  with  $(\psi_n)_{n \in \mathbb{N}} \rightarrow \psi$  and  $\psi_n \leq \psi$ , for all  $n \in \mathbb{N}$ . Then the sequence  $(\psi_0 \vee \dots \vee \psi_n)_{n \in \mathbb{N}}$  has all its members in  $M$ , is increasing and converges to  $\psi$ . Finally, there is a subsequence of this sequence which satisfies the second condition above. ■



3.10. LEMMA. *Let*

$$\mathcal{C} = \{(u_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}} \mid (u_n)_{n \in \mathbb{N}} \text{ is increasing and } u_{n+1} - u_n \leq \frac{1}{2^n}, \text{ for all } n \in \mathbb{N}\}.$$

*Then every sequence in  $\mathcal{C}$  is Cauchy and  $\lim: \mathcal{C} \rightarrow [0, 1]$  is monotone and continuous.*

PROOF. Clearly, every element of  $\mathcal{C}$  is a Cauchy sequence and the function  $\lim: \mathcal{C} \rightarrow [0, 1]$  is monotone. To see that  $\lim$  is also continuous, let  $(u_n)_{n \in \mathbb{N}} \in \mathcal{C}$  with and  $\varepsilon > 0$ . Put  $u = \lim_{n \rightarrow \infty} u_n$ . Choose  $N \in \mathbb{N}$  so that  $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$  and  $u - u_N < \frac{\varepsilon}{2}$ . Then

$$U = \{(v_n)_{n \in \mathbb{N}} \in \mathcal{C} \mid |u - v_N| < \frac{\varepsilon}{2}\}$$

is an open neighbourhood of  $(u_n)_{n \in \mathbb{N}}$ . For every  $(v_n)_{n \in \mathbb{N}} \in U$  with  $v = \lim_{n \rightarrow \infty} v_n$ ,

$$|v - u| \leq |v - v_N| + |v_N - u| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;$$

which proves that  $\lim: \mathcal{C} \rightarrow [0, 1]$  is continuous. ■

Motivated by the two lemmas above, we are looking for a monotone continuous map  $[0, 1]^{\mathbb{N}} \rightarrow [0, 1]$  which sends every sequence in  $\mathcal{C}$  to its limit. Such a map can be obtained by combining  $\lim: \mathcal{C} \rightarrow [0, 1]$  with a monotone continuous retraction of the inclusion map  $\mathcal{C} \hookrightarrow [0, 1]^{\mathbb{N}}$ . The following result is straightforward to prove.

3.11. LEMMA. *The map  $\mu: [0, 1]^{\mathbb{N}} \rightarrow [0, 1]^{\mathbb{N}}$ ,  $(u_n)_{n \in \mathbb{N}} \mapsto (u_0 \vee \dots \vee u_n)_{n \in \mathbb{N}}$  is monotone and continuous.*

Clearly,  $\mu$  sends a sequence to an increasing sequence, and  $\mu((u_n)_{n \in \mathbb{N}}) = (u_n)_{n \in \mathbb{N}}$  for every increasing sequence  $(u_n)_{n \in \mathbb{N}}$ .

3.12. LEMMA. *The map  $\gamma: [0, 1]^{\mathbb{N}} \rightarrow [0, 1]^{\mathbb{N}}$  sending a sequence  $(u_n)_{n \in \mathbb{N}}$  to the sequence  $(v_n)_{n \in \mathbb{N}}$  defined recursively by*

$$v_0 = u_0 \quad \text{and} \quad v_{n+1} = \min \left( u_{n+1}, v_n + \frac{1}{2^n} \right)$$

*is monotone and continuous. Furthermore,  $\gamma$  sends an increasing sequence to an increasing sequence.*

PROOF. It is easy to see that  $\gamma$  is monotone. In order to verify continuity, we consider  $\mathbb{N}$  as a discrete topological space, this way  $[0, 1]^{\mathbb{N}}$  is an exponential in  $\mathbf{Top}$ . We show that  $\gamma$  corresponds via the exponential law to a (necessarily continuous) map  $f: \mathbb{N} \rightarrow [0, 1]^{([0, 1]^{\mathbb{N}})}$ . The recursion data above translate to the conditions

$$f(0) = \pi_0 \quad \text{and} \quad f(n+1)((u_m)_{m \in \mathbb{N}}) = \min \left( u_{n+1}, f(n)((u_m)_{m \in \mathbb{N}}) + \frac{1}{2^n} \right),$$

that is,  $f$  is defined by the recursion data  $\pi_0 \in [0, 1]^{([0,1]^{\mathbb{N}})}$  and

$$[0, 1]^{([0,1]^{\mathbb{N}})} \longrightarrow [0, 1]^{([0,1]^{\mathbb{N}})}, \varphi \longmapsto \min \left( \pi_{n+1}, \varphi + \frac{1}{2^n} \right).$$

Note that with  $\varphi: [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$  also  $\min(\pi_{n+1}, \varphi + \frac{1}{2^n}) : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$  is continuous. Finally, if  $(u_n)_{n \in \mathbb{N}}$  is increasing, then so is  $(v_n)_{n \in \mathbb{N}}$ . ■

We conclude that the map  $\gamma \cdot \mu: [0, 1]^{\mathbb{N}} \rightarrow \mathcal{C}$  is a retraction for the inclusion map  $\mathcal{C} \rightarrow [0, 1]^{\mathbb{N}}$  in **PosComp**. Therefore we define now:

3.13. DEFINITION. *Let  $\bar{\mathbf{A}}$  be the  $\aleph_1$ -ary quasivariety obtained by adding one  $\aleph_1$ -ary operation symbol to the theory of  $\mathbf{A}$  (see Remark 3.1). Then  $[0, 1]$  becomes an object of  $\bar{\mathbf{A}}$  by interpreting this operation symbol by*

$$\delta = \lim \cdot \gamma \cdot \mu: [0, 1]^{\mathbb{N}} \rightarrow [0, 1].$$

The (accordingly modified) functor  $C: \mathbf{PosComp} \rightarrow \bar{\mathbf{A}}$  is fully faithful (see Remark 3.4); moreover, by Proposition 3.7,  $C$  sends  $\aleph_1$ -codirected limits to  $\aleph_1$ -directed colimits in  $\bar{\mathbf{A}}$ .

3.14. DEFINITION. *Let  $\bar{\mathbf{A}}_0$  be the subcategory of  $\bar{\mathbf{A}}$  defined by those objects  $A$  where the cone of all morphisms from  $A$  to  $[0, 1]$  is point-separating.*

Hence,  $\bar{\mathbf{A}}_0$  is a regular epireflective full subcategory of  $\bar{\mathbf{A}}$  and therefore also a quasivariety. Moreover:

3.15. THEOREM. *The embedding  $C: \mathbf{PosComp}^{\text{op}} \rightarrow \bar{\mathbf{A}}$  corestricts to an equivalence functor  $C: \mathbf{PosComp}^{\text{op}} \rightarrow \bar{\mathbf{A}}_0$ . Hence,  $\bar{\mathbf{A}}_0$  is closed in  $\bar{\mathbf{A}}$  under  $\aleph_1$ -directed colimits and therefore also an  $\aleph_1$ -ary quasivariety (see [AR94, Remark 3.32]).*

#### 4. Vietoris coalgebras

Resorting to the results of Section 3, in this section we present immediate consequences for the category  $\text{CoAlg}(V)$  of coalgebras and homomorphisms, where  $V: \mathbf{PosComp} \rightarrow \mathbf{PosComp}$  is the Vietoris functor. In particular, we show that  $\text{CoAlg}(V)$ , as well as certain full subcategories, are also  $\aleph_1$ -ary quasivarieties.

Recall from [Sch93] (see also [HN18, Proposition 3.28]) that, for a partially ordered compact space  $X$ , the elements of  $VX$  are the closed upper subsets of  $X$ , the order on  $VX$  is containment  $\supseteq$ , and the sets

$$\{A \in VX \mid A \cap U \neq \emptyset\} \quad (U \subseteq X \text{ open lower})$$

and

$$\{A \in VX \mid A \cap K = \emptyset\} \quad (K \subseteq X \text{ closed lower})$$

generate the compact Hausdorff topology on  $VX$ . Furthermore, for  $f: X \rightarrow Y$  in  $\mathbf{PosComp}$ , the map  $Vf: VX \rightarrow VY$  sends  $A$  to the up-closure  $\uparrow f[A]$  of  $f[A]$ . A coalgebra  $(X, \alpha)$  for  $V$  consists of a partially ordered compact space  $X$  and a monotone continuous map  $\alpha: X \rightarrow VX$ . For coalgebras  $(X, \alpha)$  and  $(Y, \beta)$ , a homomorphism of coalgebras  $f: (X, \alpha) \rightarrow (Y, \beta)$  is a monotone continuous map  $f: X \rightarrow Y$  so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \downarrow & & \downarrow \beta \\ VX & \xrightarrow{Vf} & VY \end{array}$$

commutes. The coalgebras for the Vietoris functor and their homomorphisms form the category  $\mathbf{CoAlg}(V)$ ; moreover, there is a canonical forgetful functor  $\mathbf{CoAlg}(V) \rightarrow \mathbf{PosComp}$  which sends  $(X, \alpha)$  to  $X$  and keeps the maps unchanged. For the general theory of coalgebras we refer to [Adá05, Rut00].

As it is well-known, the functor  $V$  is part of a monad  $\mathbb{V} = (V, m, e)$  on  $\mathbf{PosComp}$  (see [Sch93]); here  $e_X: X \rightarrow VX$  sends  $x$  to  $\uparrow x$  and  $m_X: VVX \rightarrow VX$  is given by  $\mathcal{A} \mapsto \bigcup \mathcal{A}$ . In order to show that  $\mathbf{CoAlg}(V)^{\text{op}}$  is an  $\aleph_1$ -ary quasivariety, we will use this perspective together with the duality result for  $\mathbf{PosComp}_{\mathbb{V}}$  of [HN18] detailed below.

Let  $\bar{\mathbf{B}}$  denote the category with the same objects as  $\bar{\mathbf{A}}$  and morphisms those maps  $\varphi: A \rightarrow A'$  that preserve finite suprema and the action  $- \odot u$ , for all  $u \in [0, 1]$ , and satisfy

$$\varphi(x \odot y) \leq \varphi(x) \odot \varphi(y),$$

for all  $x, y \in A$ .

4.1. THEOREM. *The functor  $C: \mathbf{PosComp}^{\text{op}} \rightarrow \bar{\mathbf{A}}$  extends to a fully faithful functor  $C: \mathbf{PosComp}_{\mathbb{V}} \rightarrow \bar{\mathbf{B}}$  making the diagram*

$$\begin{array}{ccc} \mathbf{PosComp}_{\mathbb{V}}^{\text{op}} & \xrightarrow{C} & \bar{\mathbf{B}} \\ \uparrow & & \uparrow \\ \mathbf{PosComp}^{\text{op}} & \xrightarrow{C} & \bar{\mathbf{A}}_0 \end{array}$$

*commutative, where the vertical arrows denote the canonical inclusion functors.*

PROOF. See [HN18]. ■

Clearly, a coalgebra structure  $X \rightarrow VX$  for  $V$  can be also interpreted as an endomorphism  $X \rightleftarrows X$  in the Kleisli category  $\mathbf{PosComp}_{\mathbb{V}}$ . Therefore the category  $\mathbf{CoAlg}(V)$  is dually equivalent to the category with objects all pairs  $(A, a)$  consisting of an  $\bar{\mathbf{A}}_0$  object  $A$  and a  $\bar{\mathbf{B}}$ -morphism  $a: A \rightarrow A$ , and a morphism between such pairs  $(A, a)$  and  $(A', a')$  is an  $\bar{\mathbf{A}}_0$ -morphism  $A \rightarrow A'$  commuting in the obvious sense with  $a$  and  $a'$ .

4.2. THEOREM. *The category  $\mathbf{CoAlg}(V)$  of coalgebras and homomorphisms for the Vietoris functor  $V: \mathbf{PosComp} \rightarrow \mathbf{PosComp}$  is dually equivalent to an  $\aleph_1$ -ary quasivariety.*

PROOF. Just consider the algebraic theory of  $\bar{A}_0$  augmented by one unary operation symbol and by those equations which express that the corresponding operation is a  $\bar{B}$ -morphism. ■

In particular,  $\text{CoAlg}(V)$  is complete and the forgetful functor  $\text{CoAlg}(V) \rightarrow \text{PosComp}$  preserves  $\aleph_1$ -codirected limits. In fact, slightly more is shown in [HNN16]:

4.3. PROPOSITION. *The forgetful functor  $\text{CoAlg}(V) \rightarrow \text{PosComp}$  preserves codirected limits.*

We finish this section by pointing out some further consequences of our approach for certain full subcategories of  $\text{CoAlg}(V)$ . We will be guided by familiar concepts, namely reflexive and transitive relations, but note that our arguments apply to other concepts as well (for instance, idempotent relations).

Still thinking of a coalgebra structure  $\alpha: X \rightarrow VX$  as an endomorphism  $\alpha: X \rightrightarrows X$  in  $\text{PosComp}_{\mathbb{V}}$ , we say that  $\alpha$  is **reflexive** whenever  $1_X \leq \alpha$  in  $\text{PosComp}_{\mathbb{V}}$ , and  $\alpha$  is called **transitive** whenever  $\alpha \circ \alpha \leq \alpha$  in  $\text{PosComp}_{\mathbb{V}}$ ; with the local order in  $\text{PosComp}_{\mathbb{V}}$  being inclusion.

4.4. PROPOSITION. *The full subcategory of  $\text{CoAlg}(V)$  defined by all reflexive (or transitive or reflexive and transitive) coalgebras is dually equivalent to an  $\aleph_1$ -ary quasivariety. Moreover, this subcategory is coreflective in  $\text{CoAlg}(V)$  and closed under  $\aleph_1$ -directed limits.*

PROOF. First note that the functor  $C: \text{PosComp}_{\mathbb{V}} \rightarrow \bar{\mathbf{B}}$  preserves and reflects the local order of morphisms (defined pointwise, see [HN18]). Therefore, considering the corresponding  $\bar{\mathbf{B}}$ -morphism  $a: A \rightarrow A$ , the inequalities expressing reflexivity and transitivity can be formulated as equations in  $A$ . Then the assertion follows from [AR94, Theorem 1.66]. ■

For a class  $\mathcal{M}$  of monomorphisms in  $\text{CoAlg}(V)$ , a coalgebra  $X$  for  $V$  is called **coorthogonal** whenever, for all  $m: A \rightarrow B$  in  $\mathcal{M}$  and all homomorphisms  $f: X \rightarrow B$  there exists a (necessarily unique) homomorphism  $g: X \rightarrow A$  with  $m \cdot g = f$  (see [AR94, Definition 1.32] for the dual notion). We write  $\mathcal{M}^\top$  for the full subcategory of  $\text{CoAlg}(V)$  defined by those coalgebras which are coorthogonal to  $\mathcal{M}$ . From the dual of [AR94, Theorem 1.39] we obtain:

4.5. PROPOSITION. *For every set  $\mathcal{M}$  of monomorphisms in  $\text{CoAlg}(V)$ , the inclusion functor  $\mathcal{M}^\top \hookrightarrow \text{CoAlg}(V)$  has a right adjoint. Moreover, if  $\lambda$  denotes a regular cardinal larger or equal to  $\aleph_1$  so that, for every arrow  $m \in \mathcal{M}$ , the domain and codomain of  $m$  is  $\lambda$ -copresentable, then  $\mathcal{M}^\top \hookrightarrow \text{CoAlg}(V)$  is closed under  $\lambda$ -codirected limits.*

Another way of specifying full subcategories of  $\text{CoAlg}(V)$  uses coequations (see [Adá05, Definition 4.18]). For the Vietoris functor, the latter is a particular case of coorthogonality, and therefore we obtain the following result.

4.6. COROLLARY. *For every set of coequations in  $\text{CoAlg}(V)$ , the full subcategory of  $\text{CoAlg}(V)$  defined by these coequations is coreflective.*

## 5. $\aleph_1$ -copresentable spaces

It is shown in [GU71] that the  $\aleph_1$ -copresentable objects in **CompHaus** are precisely the metrisable compact Hausdorff spaces. We end this paper with a characterisation of the  $\aleph_1$ -copresentable objects in **PosComp** which resembles the one for compact Hausdorff spaces; to do so, we consider generalised metric spaces in the sense of Lawvere [Law73].

More precisely, we think of metric spaces as categories enriched in the quantale  $[0, 1]$ , ordered by the “greater or equal” relation  $\geq$ , with tensor product  $\oplus$  given by truncated addition:

$$u \oplus v = \min(1, u + v),$$

for all  $u, v \in [0, 1]$ . We note that the right adjoint  $\text{hom}(u, -)$  of  $u \oplus - : [0, 1] \rightarrow [0, 1]$  is defined by

$$\text{hom}(u, v) = v \ominus u = \max(0, v - u),$$

for all  $u, v \in [0, 1]$ .

5.1. **REMARK.** Via the isomorphism  $[0, 1] \rightarrow [0, 1]$ ,  $u \mapsto 1 - u$ , the quantale described above is isomorphic to the quantale  $[0, 1]$  equipped with the Łukasiewicz tensor used in Section 3. However, we decided to switch so that categories enriched in  $[0, 1]$  look more like metric spaces.

5.2. **DEFINITION.** A **metric space** is a pair  $(X, a)$  consisting of a set  $X$  and a map  $a: X \times X \rightarrow [0, 1]$  satisfying

$$0 \geq a(x, x) \quad \text{and} \quad a(x, y) \oplus a(y, z) \geq a(x, z),$$

for all  $x, y, z \in X$ . A map  $f: X \rightarrow Y$  between metric spaces  $(X, a)$ ,  $(Y, b)$  is called **non-expansive** whenever

$$a(x, x') \geq b(f(x), f(x')),$$

for all  $x, x' \in X$ . Metric spaces and non-expansive maps form the category **Met**.

5.3. **EXAMPLE.** The unit interval  $[0, 1]$  is a metric space with metric  $\text{hom}(u, v) = v \ominus u$ .

Our definition of metric space is not the classical one. Firstly, we consider only metrics bounded by 1; however, since we are interested in the induced topology and the induced order, “large” distances are irrelevant. Secondly, we allow distance zero for different points, which, besides topology, also allows us to obtain non-trivial orders. Every metric  $a$  on a set  $X$  defines the order relation

$$x \leq y \text{ whenever } 0 \geq a(x, y),$$

for all  $x, y \in X$ ; this construction defines a functor

$$O: \text{Met} \longrightarrow \text{Ord}$$

commuting with the canonical forgetful functors to **Set**. The following assertion is straightforward to prove.

5.4. LEMMA. *The functor  $O: \text{Met} \rightarrow \text{Ord}$  preserves limits and initial cones.*

A metric space is called **separated** whenever the underlying order is anti-symmetric, that is,

$$(0 \geq a(x, y) \ \& \ 0 \geq a(y, z)) \implies x = y,$$

for all  $x, y \in X$ .

Thirdly, we are not insisting on symmetry. However, every metric space  $(X, a)$  can be symmetrised by

$$a_s(x, y) = \max(a(x, y), a(y, x)).$$

For every metric space  $(X, a)$ , we consider the topology induced by the symmetrisation  $a_s$  of  $a$ . This construction defines the faithful functor

$$T: \text{Met} \longrightarrow \text{Top}.$$

We note that  $(X, a)$  is separated if and only if the underlying topology is Hausdorff. Let us recall:

5.5. LEMMA. *The functor  $T: \text{Met} \rightarrow \text{Top}$  preserves finite limits. In particular,  $T$  sends subspace embeddings to subspace embeddings.*

5.6. LEMMA. *Let  $(X, a)$  be a separated compact metric space. Then  $X$  equipped with the order and the topology induced by the metric  $a$  becomes a partially ordered compact space.*

PROOF. See [Nac65, Chapter II]. ■

5.7. EXAMPLE. The metric space  $[0, 1]$  of Example 5.3 induces the partially ordered compact Hausdorff space  $[0, 1]$  with the usual Euclidean topology and the “greater or equal” relation  $\geq$ .

5.8. DEFINITION. *A partially ordered compact space  $X$  is called **metrisable** whenever there is a metric on  $X$  which induces the order and the topology of  $X$ . We denote by  $\text{PosComp}_{\text{met}}$  the full subcategory of  $\text{PosComp}$  defined by all metrisable spaces.*

5.9. PROPOSITION.  $\text{PosComp}_{\text{met}}$  is closed under countable limits in  $\text{PosComp}$ .

PROOF. By Lemma 5.5,  $\text{PosComp}_{\text{met}}$  is closed under finite limits in  $\text{PosComp}$ . The argument for countable products is the same as in the classical case: for a family  $(X_n)_{n \in \mathbb{N}}$  of metrisable partially ordered compact Hausdorff spaces, with the metric  $a_n$  on  $X_n$  ( $n \in \mathbb{N}$ ), the structure of the product space  $X = \prod_{n \in \mathbb{N}} X_n$  is induced by the metric  $a$  defined by

$$a((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} a_n(x_n, y_n),$$

for  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in X$ . ■

For classical metric spaces it is known that the compact spaces are subspaces of countable powers of the unit interval; this fact carries over without any trouble to our case. Before we present the argument, let us recall that, for every metric space  $(X, a)$ , the cone  $(a(x, -): X \rightarrow [0, 1])_{x \in X}$  is initial with respect to the forgetful functor  $\mathbf{Met} \rightarrow \mathbf{Set}$ ; this is a consequence of the Yoneda Lemma for enriched categories (see [Law73]). Moreover,  $(X, a)$  is separated if and only if this cone is point-separating.

5.10. LEMMA. *Let  $(X, a)$  be a compact metric space. Then there exists a countable subset  $S \subseteq X$  so that the cone*

$$(a(z, -): X \rightarrow [0, 1])_{z \in S}$$

*is initial with respect to the forgetful functor  $\mathbf{Met} \rightarrow \mathbf{Set}$ .*

PROOF. Since  $X$  is compact, for every natural number  $n \geq 1$ , there is a finite set  $S_n$  so that the open balls

$$\{y \in X \mid a(x, y) < \frac{1}{n} \text{ and } a(y, x) < \frac{1}{n}\}$$

with  $x \in S_n$  cover  $X$ . Let  $S = \bigcup_{n \geq 1} S_n$ . We have to show that, for all  $x, y \in X$ ,

$$\bigvee_{z \in S} a(z, y) \ominus a(z, x) \geq a(x, y).$$

To see that, let  $\varepsilon = \frac{1}{n}$ , for some  $n \geq 1$ . By construction, there is some  $z \in S$  so that  $a(x, z) < \varepsilon$  and  $a(z, x) < \varepsilon$ . Hence,

$$(a(z, y) \ominus a(z, x)) + 2\varepsilon \geq a(z, y) + a(x, z) \geq a(x, y);$$

and the assertion follows. ■

5.11. PROPOSITION. *Every partially ordered compact space is a  $\aleph_1$ -cofiltered limit in  $\mathbf{PosComp}$  of metrisable spaces.*

PROOF. For a separated metric space  $X = (X, a)$ , the initial cone  $(a(x, -): X \rightarrow [0, 1])_{x \in S}$  of Lemma 5.10 is automatically point-separating, therefore there is an embedding  $X \hookrightarrow [0, 1]^{\mathbb{N}}$  in  $\mathbf{Met}$ . This proves that the full subcategory  $\mathbf{PosComp}_{\mathbf{met}}$  of  $\mathbf{PosComp}$  is small. Let  $X$  be a partially ordered compact space. By Proposition 5.9, the canonical diagram

$$D: X \downarrow \mathbf{PosComp}_{\mathbf{met}} \longrightarrow \mathbf{PosComp}$$

is  $\aleph_1$ -cofiltered. Moreover, the canonical cone

$$(f: X \rightarrow Y)_{f \in (X \downarrow \mathbf{PosComp}_{\mathbf{met}})} \tag{1}$$

is initial since (1) includes the cone  $(f: X \rightarrow [0, 1])_f$ . Finally, to see that (1) is a limit cone, we use Theorem 2.7: for every  $f: X \rightarrow Y$  with  $Y$  metrisable,  $\text{im}(f) \hookrightarrow Y$  actually belongs to  $\mathbf{PosComp}_{\mathbf{met}}$ , which proves

$$\text{im } f = \bigcap_{k: g \rightarrow f \in (X \downarrow \mathbf{PosComp}_{\mathbf{met}})} \text{im } D(k).$$

■

5.12. COROLLARY. *Every  $\aleph_1$ -copresentable object in  $\mathbf{PosComp}$  is metrisable.*

PROOF. Also here the argument is the same as for  $\mathbf{CompHaus}$ . Let  $X$  be a  $\aleph_1$ -copresentable object in  $\mathbf{PosComp}$ . By Proposition 5.11, we can present  $X$  as a limit  $(p_i: X \rightarrow X_i)_{i \in I}$  of a  $\aleph_1$ -cofiltered diagram  $D: I \rightarrow \mathbf{PosComp}$  where all  $D(i)$  are metrisable. Since  $X$  is  $\aleph_1$ -copresentable, the identity  $1_X: X \rightarrow X$  factorises as

$$X \xrightarrow{p_i} X_i \xrightarrow{h} X,$$

for some  $i \in I$ . Hence, being a subspace of a metrisable space,  $X$  is metrisable. ■

To prove that every metrisable partially ordered compact space  $X$  is  $\aleph_1$ -copresentable, we will show that every closed subspace  $A \hookrightarrow [0, 1]^I$  with countable  $I$  is an equaliser of a pair of arrows

$$[0, 1]^I \rightrightarrows [0, 1]^J$$

with countable  $J$ . For a symmetric metric on  $X$ , one can simply consider

$$[0, 1]^I \xrightarrow[0]{a(A, -)} [0, 1],$$

but in our non-symmetric setting this argument does not work since the map  $a(A, -)$  is in general not monotone.

5.13. LEMMA. *Let  $n \in \mathbb{N}$  and  $A \subseteq [0, 1]^n$  be a closed subset. Then there is a countable set  $J$  and monotone continuous maps*

$$[0, 1]^n \xrightarrow[k]{h} [0, 1]^J$$

*so that  $A \hookrightarrow [0, 1]^n$  is the equaliser of  $h$  and  $k$ . In particular,  $A$  is  $\aleph_1$ -copresentable.*

PROOF. We denote by  $d$  the usual Euclidean metric on  $[0, 1]^n$ . For every  $x \in [0, 1]^n$  with  $x \notin A$ , there is some  $\varepsilon > 0$  so that the closed ball

$$B(x, \varepsilon) = \{y \in [0, 1]^n \mid d(x, y) \leq \varepsilon\}$$

does not intersect  $A$ . Furthermore,  $B = \uparrow B \cap \downarrow B$ . Put

$$J = \{(k, x_1, \dots, x_n) \mid k \in \mathbb{N}, k \geq 1 \text{ and } x = (x_1, \dots, x_n) \in ([0, 1] \cap \mathbb{Q})^n \text{ and } B(x, \frac{1}{k}) \cap A = \emptyset\};$$

clearly,  $J$  is countable. For every  $j = (k, x_1, \dots, x_n) \in J$ , we consider the monotone continuous maps  $f_0, f_1: [0, 1]^n \rightarrow [0, 1]$  obtained in Lemma 2.5 and put  $h_j = f_0$  and  $k_j = f_1$ . Then  $A \hookrightarrow [0, 1]^n$  is the equaliser of

$$[0, 1]^n \xrightarrow[k=\langle k_j \rangle]{h=\langle h_j \rangle} [0, 1]^J$$

■



5.14. THEOREM. *Every metrisable partially ordered compact space is  $\aleph_1$ -copresentable in PosComp.*

PROOF. Let  $X$  be a metrisable partially ordered compact space. By Lemma 5.10, there is an embedding  $m: X \hookrightarrow [0, 1]^{\mathbb{N}}$  in PosComp. Moreover, with

$$J = \{F \subseteq \mathbb{N} \mid F \text{ is finite}\}$$

ordered by containment  $\supseteq$ ,  $(\pi_F: [0, 1]^{\mathbb{N}} \rightarrow [0, 1]^F)_{F \in J}$  is a limit cone of the codirected diagram

$$J^{\text{op}} \longrightarrow \text{PosComp}$$

sending  $F$  to  $[0, 1]^F$  and  $G \supseteq F$  to the canonical projection  $\pi: [0, 1]^G \rightarrow [0, 1]^F$ . For every  $F \in J$ , we consider the (Epi,Regular mono)-factorisation

$$X \xrightarrow{p_F} X_F \xrightarrow{m_F} [0, 1]^F$$

of  $\pi_F \cdot m: X \rightarrow [0, 1]^F$ . Then, using again Bourbaki's criterion (see Theorem 2.7),

$$(p_F: X \rightarrow A_F)_{F \in J}$$

is a limit cone of the codirect diagram

$$J^{\text{op}} \longrightarrow \text{PosComp}$$

sending  $F$  to  $X_F$  and  $G \supseteq F$  to the diagonal of the factorisation. By Lemma 5.13, each  $X_F$  is  $\aleph_1$ -copresentable, hence also  $X$  is  $\aleph_1$ -copresentable since  $X$  is a countable limit of  $\aleph_1$ -copresentable objects. ■

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