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A NOTE ON INJECTIVE HULLS OF POSEMIGROUPS

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ABSTRACT. In this note, we prove the existence of $\mathscr{E}_{\leq}$-injective hulls in the category $\mathbf{PoSgr}_{\leq}$ of posemigroups and their submultiplicative order-preserving maps; here $\mathscr{E}_{\leq}$ denotes the class of those morphisms $h\colon A \to B$ for which $h(a_1)\cdots h(a_n) \leq h(a)$ always implies $a_1\cdots a_n \leq a$. The result of this note subsumes the results given by Lambek et al. (2012) and by Zhang and Laan (2014).

1. Introduction

The injective hull, though not under that name, was first obtained in [Baer (1940)]. The best-known early discussion of injective hulls is in [Eckmann and Schopf (1953)], who introduced them as maximal essential extensions. Banaschewski and Bruns [1967] characterized the injective hulls in the category of posets. In 1970, Bruns and Lakser characterized the injective hulls in the category of semilattices. As we all know, the injective objects in the category of pomonoids with pomonoid homomorphisms are trivial [Lambek et al. (2012)]. Hence, Lambek et al. changed a few of the definitions of pomonoid homomorphisms, chose submultiplicative order-preserving maps as morphisms of the category $\mathbf{PoMon}_{\leq}$ of pomonoids to investigate the injectivity in $\mathbf{PoMon}_{\leq}$, where a *submultiplicative map* between pomonoids is a map $f\colon (A,\cdot,\leq) \to (B,*,\leq)$ such that $f(a)*f(a') \leq f(a\cdot a')$ for all $a,a' \in A$. Also, they proved that every pomonoid has an injective hull and gave the concrete form of the injective hulls. Based on the work of Lambek et al., dropping out the unit, Zhang and Laan intended to consider the injective hulls in the category $\mathbf{PoSgr}_{\leq}$ of posemigroups, and constructed the injective hulls for a certain class of posemigroups with respect to a class of order embedding. However, for the question whether every posemigroup has an $\mathscr{E}_{\leq}$-injective hull in $\mathbf{PoSgr}_{\leq}$, Zhang and Laan did not give an answer. In this note, we shall give an affirmative answer and obtain the concrete form of injective hulls for general posemigroups. Particularly, when a posemigroup S is a pomonoid, the injective hull of S in $\mathbf{PoSgr}_{\leq}$ we obtain is exactly the injective hull of S in $\mathbf{PoMon}_{\leq}$ Lambek et al. gave.

A *partially ordered semigroup* (*posemigroup* for short) is a semigroup $(S,\cdot)$ with a partial order $\leq$ on S which is compatible with the partial order $\leq$, that is, $x \leq y \Rightarrow x\cdot z \leq y\cdot z$ and $z\cdot x \leq z\cdot y$ for any z in S. A *quantale* is a complete lattice Q with an associative binary operation &

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satisfying $a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i)$ and $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$ for any $a \in Q$ and $\{b_i\}_{i\in I} \subseteq Q$. A *quantic nucleus* j on a quantale Q is a submultiplicative closure operator. It is easy to show that $j(a\&b) = j(a\&j(b)) = j(j(a)\&b) = j(j(a)\&j(b))$ for all $a,b \in Q$. Furthermore, we can also prove that the set Q_j of fixed points of j is a quantale with $a \otimes b = j(a\&b)$ and $\bigvee\limits_{i\in I}^{Q_j} a_i = j(\bigvee\limits_{i\in I}^{Q} a_i)$. Clearly, every quantale is necessarily a posemigroup. For a given semigroup $(S,\cdot)$, we denote by $\mathscr{P}(S)$ the set of all subsets of S. Clearly, $(\mathscr{P}(S),\bullet)$ forms a quantale under the inclusive order, where $A \bullet B = \{a\cdot b : a \in A, b \in B\}$. When $A = \{a\}$, we write $A \bullet B$ and $B \bullet A$ simply as $a \cdot B$ and $B \cdot a$.

Let $\mathscr{C}$ be a category and let $\mathscr{M}$ be a class of morphisms in $\mathscr{C}$. Then an object S in $\mathscr{C}$ is $\mathscr{M}$-*injective* in $\mathscr{C}$ provided that for any morphism $h\colon A \to B$ in $\mathscr{M}$ and any morphism $f\colon A \to S$ in $\mathscr{C}$ there exists a morphism $g\colon B \to S$ such that $g \circ h = f$. A morphism $\eta : A \to B$ in $\mathscr{M}$ is called $\mathscr{M}$-*essential* provided that every morphism $\psi\colon B \to C$ in $\mathscr{C}$ for which the composite $\psi \circ \eta$ is in $\mathscr{M}$ is itself in $\mathscr{M}$. An object $H \in \mathscr{C}$ is said to be an $\mathscr{M}$-injective hull of an object of S provided that H is $\mathscr{M}$-injective and there exists an $\mathscr{M}$-essential morphism $S \to H$.

For notions and concepts concerned, but not explained, please refer to [Lambek et al. (2012)], [Zhang and Laan (2014)] and [Adámek et al. (2004)].

2. Main results

In this section, we shall show that every posemigroup has an $\mathscr{E}_{\leq}$-injective hull, and give its concrete form. Firstly, we review some main results on injective hulls in the category $\mathbf{PoMon}_{\leq}$ and $\mathbf{PoSgr}_{\leq}$.

2.1. THEOREM. [Lambek et al. (2012)] *A pomonoid is injective (with respect to embeddings) if and only if it is a quantale.*

2.2. THEOREM. [Lambek et al. (2012)] *The map $\eta(x \mapsto x{\downarrow})\colon A \to \mathscr{Q}(A)$ is an essential embedding and thus is an embedding into an injective hull.*

2.3. THEOREM. [Zhang and Laan (2014)] *Let $(S,\cdot,\leq)$ be a posemigroup. Then S is $\mathscr{E}_{\leq}$-injective in $\mathbf{PoSgr}_{\leq}$ if and only if S is a quantale.*

2.4. THEOREM. [Zhang and Laan (2014)] *Let S be a posemigroup such that $cl(s{\downarrow}) = s{\downarrow}$ for every $s \in S$. Then $\mathscr{D}(S)$ is an $\mathscr{E}_{\leq}$-injective hull in $\mathbf{PoSgr}_{\leq}$.*

In what follows we shall prove our main result.

Let $(S,\cdot,\leq)$ be a posemigroup and $a \in S$. Then we denote by $a{\downarrow}$ the set $\{b \in S : b \leq a\}$. For any subset X of S, we define its closure by $X^\star$

$$X^\star \triangleq X^{\mathrm{ul}} \cap X^L \cap X^R \cap X^T$$

where

$$X^{\mathrm{ul}} \triangleq \{s \in S : \forall\, b \in S, X \subseteq b{\downarrow} \Rightarrow s \leq b\}$$

$$X^L \triangleq \{s \in S : \forall\, a,b \in S, X \cdot a \subseteq b{\downarrow} \Rightarrow s \cdot a \leq b\}$$
$$X^R \triangleq \{s \in S : \forall\, a,b \in S, a \cdot X \subseteq b{\downarrow} \Rightarrow a \cdot s \leq b\}$$
$$X^T \triangleq \{s \in S : \forall\, a,b,c \in S, a \cdot X \cdot c \subseteq b{\downarrow} \Rightarrow a \cdot s \cdot c \leq b\}$$

It is easy to verify that $X^\star = X^T$ when S is a pomonoid.

2.5. LEMMA. [Xia et al. (2016)] *Let $(S,\cdot,\leq)$ be a posemigroup. Then $(\cdot)^\star$ is a quantic nucleus on the power-set quantale $(\mathscr{P}(S),\bullet)$, that is, $(\mathbf{S}^\star,\otimes)$ is a quantic quotient of $(\mathscr{P}(S),\bullet)$, where $\mathbf{S}^\star = \{A \in \mathscr{P}(S) : A^\star = A\}$, $X \otimes Y = (X \bullet Y)^\star$.*

In [Xia et al. (2016)], we prove that $(\cdot)^\star$ is also topological, that is, $\{x\}^\star = x{\downarrow}$ for all $x \in S$. We define now a map $\eta_S \colon S \to \mathbf{S}^\star$ as follows $\eta_S(x) = \{x\}^\star$ for all $x \in S$. In [Xia et al. (2016)], we prove that $(\eta_S, S^\star)$ is the least quantale completion of S. In the following we shall see that η_S is in fact $\mathscr{E}_\leq$-essential in $\mathbf{PoSgr}_\leq$, that is, $\mathbf{S}^\star$is an $\mathscr{E}_\leq$-injective hull of S in $\mathbf{PoSgr}_\leq$.

2.6. THEOREM. *Let $(S,\cdot,\leq)$ be a posemigroup. Then $\mathbf{S}^\star$is an $\mathscr{E}_\leq$-injective hull of S in $\mathbf{PoSgr}_\leq$.*

PROOF. One can easily see that $(x{\downarrow})^\star = x{\downarrow}$ for any $x \in S$. In the following we shall prove that η_S is an $\mathscr{E}_\leq$-essential morphism in $\mathbf{PoSgr}_\leq$. Clearly, η_S belongs to $\mathscr{E}_\leq$. By Theorem 2.3, $\mathbf{S}^\star$ is $\mathscr{E}_\leq$-injective. It remains to show that η_S is $\mathscr{E}_\leq$-essential. Let $\psi \colon \mathbf{S}^\star \to (B,\leq,*)$ be a morphism in $\mathbf{PoSgr}_\leq$ such that $\psi \circ \eta_S \in \mathscr{E}_\leq$. We shall verify that $\psi \in \mathscr{E}_\leq$. Suppose that $\psi(X_1) * \psi(X_2) * \cdots * \psi(X_n) \leq \psi(Y)$ in B, where $X_1, X_2, \cdots, X_n, Y \in \mathbf{S}^\star$. We need to show that $X_1 \otimes X_2 \otimes \cdots \otimes X_n \subseteq Y$. In fact, we only need to prove that $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^{\mathrm{ul}} \cap Y^L \cap Y^R \cap Y^T$.

(1) $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^{\mathrm{ul}}$.

Assume that $x_1 \in X_1, \cdots, x_n \in X_n$. We let $b \in S$ and $Y \subseteq b{\downarrow}$. Then we have

$$\begin{aligned}
(\psi \circ \eta_S)(x_1) * \cdots * (\psi \circ \eta_S)(x_n) &= \psi(x_1{\downarrow}) * \cdots * \psi(x_n{\downarrow}) \\
&\leq \psi(X_1) * \cdots * \psi(X_n) \\
&\leq \psi(Y) \\
&\leq \psi(b{\downarrow}) \\
&= (\psi \circ \eta_S)(b).
\end{aligned}$$

It follows from the case that $\psi \circ \eta_S \in \mathscr{E}_\leq$ that $x_1 \cdot x_2 \cdot \;\cdots\; \cdot x_n \leq b$. Hence, $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq b{\downarrow}$, which implies $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^{\mathrm{ul}}$.

(2) $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^L$.

Suppose that $x_1 \in X_1, \cdots, x_n \in X_n$. We let $a, b \in S$ and $Y \cdot a \subseteq b{\downarrow}$. Then we have

$$\begin{aligned}
(\psi \circ \eta_S)(x_1) * \cdots * (\psi \circ \eta_S)(x_n) &= \psi(x_1{\downarrow}) * \cdots * \psi(x_n{\downarrow}) * \psi(a{\downarrow}) \\
&\leq \psi(X_1) * \cdots * \psi(X_n) * \psi(a{\downarrow}) \\
&\leq \psi(Y) * \psi(a{\downarrow}) \\
&\leq \psi(Y \otimes a{\downarrow}) \\
&\leq \psi((Y \cdot a)^\star) \\
&\leq \psi(b{\downarrow}) \\
&= (\psi \circ \eta_S)(b),
\end{aligned}$$

It follows from the case that $\psi \circ \eta_s \in \mathscr{E}_{\leq}$ that $x_1 \cdot x_2 \cdot \cdots \cdot x_n \cdot a \leq b$. Hence, $(X_1 \bullet X_2 \bullet \cdots \bullet X_n) \cdot a \subseteq b\downarrow$, which implies $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^L$.

(3) Similar to (2), we can prove that $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^R$ and $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^T$.

By (1)-(3), we conclude that $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y^{\mathrm{ul}} \cap Y^L \cap Y^R \cap Y^T$, that is, $X_1 \bullet X_2 \bullet \cdots \bullet X_n \subseteq Y$. Therefore, $\mathbf{S}^{\star}$ is an $\mathscr{E}_{\leq}$-injective hull of S in $\mathbf{PoSgr}_{\leq}$. ∎

2.7. REMARK. $\mathbf{S}^{\star}$ is the injective hull of S in $\mathbf{PoMon}_{\leq}$ whenever S is a pomonoid.

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