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ON REFLECTIVE SUBCATEGORIES OF LOCALLY PRESENTABLE CATEGORIES

Dedicated to the memory of Horst Herrlich

J. ADÁMEK AND J. ROSICKÝ

ABSTRACT. Are all subcategories of locally finitely presentable categories that are closed under limits and λ-filtered colimits also locally presentable? For full subcategories the answer is affirmative. Makkai and Pitts proved that in the case $\lambda = \aleph_0$ the answer is affirmative also for all iso-full subcategories, *i. e.*, those containing with every pair of objects all isomorphisms between them. We discuss a possible generalization of this from $\aleph_0$ to an arbitrary λ.

1. Introduction

The lecture notes [8] of Horst Herrlich on reflections and coreflections in topology in 1968 with its introduction to categorical concepts in Part II was the first text on category theory we read, and it deeply influenced us. A quarter century later it was on Horst's impulse that we solved the problem of finding full reflective subcategories of **Top** whose intersection is not reflective [1]. That paper has started years of intense cooperation of the authors. It is therefore with deep gratitude that we dedicate our paper to the memory of Horst Herrlich.

The above mentioned lecture notes are also the first reference for the fact that a full, reflective subcategory of a complete category is complete and closed under limits (9.1.2 in [8]; P. J. Freyd [7] mentions this in Exercise 3F). In the present paper we study subcategories that are not necessarily full. They do not, in general, inherit completeness, see Example 2.1 below. However, if we restrict to complete subcategories, a necessary condition for reflectivity is closedness under limits. Now for full subcategories $\mathscr{K}$ of a locally presentable category an "almost" necessary condition is that $\mathscr{K}$ be closed under λ-filtered colimits for some λ, see Remark 2.3. Moreover, for full subcategories the Reflection Theorem of [2] (2.48) states the converse: a full subcategory is reflective if it is closed under limits and λ-filtered colimits. For non-full reflective subcategories the converse fails even in **Set**: a subcategory of **Set** closed under limits and filtered colimits need not be reflective, see Example 2.2 below. A beautiful result was proved by Makkai and Pitts [10]

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about *iso-full* subcategories of $\mathscr{L}$, i.e., those containing every isomorphism of $\mathscr{L}$ with domain and codomain in the subcategory:

THEOREM. [Makkai and Pitts] *Every iso-full subcategory of a locally finitely presentable category closed under limits and filtered colimits ($\lambda = \aleph_0$) is reflective.*

.

What about closedness under λ-filtered colimits for uncountable λ? As an example, take $\mathscr{L}$ to be the category of posets. Its subcategory of boolean σ-algebras and σ-homomorphisms is iso-full and closed under limits and $\aleph_1$-filtered colimits. Is it reflective? Yes, one can prove this using Freyd's Special Adjoint Functor Theorem. However, we have not (in spite of quite some effort) been able to answer the following general

OPEN PROBLEM. Is every iso-full subcategory of a locally λ-presentable category that is closed under limits and λ-filtered colimits reflective?

.

A substantial step in the proof of the above theorem due to Makkai and Pitts was proving that the given subcategory is closed under elementary subobjects. The main result of our paper is an "approximation" of the affirmative answer to the above open problem based on introducing the concept of a κ-elementary subobject (see Section 3). Every isomorphism is κ-elementary for all cardinals κ. And conversely, every monomorphism that is κ-elementary for all κ is an isomorphism. Our "approximate answer" to the above open problem substitutes closedness under κ-elementary subobjects for iso-fullness:

THEOREM. *Every subcategory $\mathscr{K}$ of a locally λ-presentable category closed under limits, λ-filtered colimits and λ-elementary subobjects is reflective. And it is itself locally λ-presentable.*

.

Unfortunately, there does not seem to exist a good categorical formulation of the concept of κ-elementary embedding (and we use therefore the model-theoretic formulation). Another approximation concerns abelian categories $\mathscr{L}$: the above open problem has an affirmative answer whenever the subcategory contains all zero morphisms. Indeed, in this case the subcategory will be proved to be full, thus our Reflection Theorem applies.

ACKNOWLEDGEMENT. We are grateful to Saharon Shelah for a consultation concerning a generalization of λ-good ultrafilters. We hoped to use such generalized ultrafilters for a direct extension of the proof due to Makkai and Pitts from $\aleph_0$ to λ. However, the result of the consultation is that the desired ultrafilters do not exist. Pity!

We are also grateful to the referee whose suggestions improved our presentation.

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2. Reflective Subcategories

Recall that a subcategory $\mathscr{K}$ of a category $\mathscr{L}$ is called *reflective* if the embedding $E : \mathscr{K} \hookrightarrow \mathscr{L}$ has a left adjoint. The left adjoint $F : \mathscr{L} \to \mathscr{K}$ is called a reflector, and the unit of adjunction has components

$$r_L : L \to FL \qquad \text{for } L \in \mathscr{L}$$

called the reflections. Recall further that a subcategory is called *replete* if it contains with every object all isomorphic ones.

The subcategory $\mathscr{K}$ is said to be *closed under limits* in $\mathscr{L}$ if it has limits and the embedding E preserves them. (There is a stronger concept demanding that every limit in $\mathscr{L}$ of a diagram in $\mathscr{K}$ lies in $\mathscr{K}$. For iso-full, replete subcategories these two concepts coincide.) Analogously for closure under finite limits, filtered colimits, *etc.*

2.1. Example. *An incomplete reflective subcategory of* **Pos**. Let $\mathscr{K}$ have as objects all posets with a least element 0 and a greatest element 1 such that $0 \neq 1$. Morphisms are monotone functions preserving 0 and 1. This category is

(i) iso-full,
(ii) reflective,

yet

(iii) incomplete: it does not have a terminal object.

Indeed, an isomorphism clearly preserves 0 and 1. A reflection of a poset is its embedding into the poset with a new 0 and a new 1. For every object K of $\mathscr{K}$ more than one morphism leads from the 3-chain to K, hence K is not terminal.

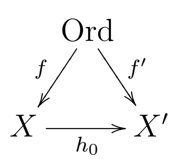
2.2. Examples. (1) *A subcategory* $\mathscr{K}$ *of* **Set** *that is*

(i) closed under limits and filtered colimits,

yet

(ii) not reflective.

Its objects are all sets $X \times \{f\}$, where X is a set and $f : \mathrm{Ord} \to X$ is a function that is, from some ordinal onwards, constant. Morphisms from $X \times \{f\}$ to $X' \times \{f'\}$ are all functions $h = h_0 \times h_1$ where h_0 makes the triangle



commutative.

The category $\mathscr{K}$ is equivalent to the category $\mathscr{K}'$ of algebras on nullary operations indexed by Ord that are from some ordinal onwards equal. The forgetful functor of $\mathscr{K}'$ preserves limits and filtered colimits, whence it easily follows that $\mathscr{K}$ is closed under limits and filtered colimits in **Set**.

However, $\mathscr{K}$ does not have an initial object (*i. e.*, the empty set has no reflection). Indeed, for every object K of $\mathscr{K}$ there exists an object K' of $\mathscr{K}$ with $\mathscr{K}(K, K') = \varnothing$: for $K = X \times \{f\}$ with f constant from i onwards choose any $K' = X' \times \{f'\}$ with $f'(i) \neq f'(i+1)$.

(2) *A subcategory* $\mathscr{K}$ *of* **Set** *that is*

(i) reflective and closed under filtered colimits,

yet

(ii) not iso-full.

Its objects are all sets $X \times \{f\}$, where X is a set and $f : X \to X$ is a function. Morphisms from $X \times \{f\}$ to $X' \times \{f'\}$ are all functions h such that $hf = f'h$.

The category $\mathscr{K}$ is equivalent to the category $\mathscr{K}'$ of algebras with one unary operation. The forgetful functor of $\mathscr{K}'$ is a right adjoint and preserves limits and filtered colimits, whence it easily follows that $\mathscr{K}$ is reflective and closed under filtered colimits in **Set**. But $\mathscr{K}$ is not iso-full.

2.3. Definition. A non-full subcategory $\mathscr{K}$ of $\mathscr{L}$ is said to be *closed under split subobjects* if for every object K of $\mathscr{K}$ and every split subobject of K in $\mathscr{L}$ there exists a split monomorphism of $\mathscr{K}$ representing the same subobject.

Analogously closure under other types of subobjects is defined.

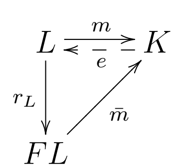
2.4. Proposition. *An iso-full reflective subcategory is closed under split subobjects iff it is full.*

Proof. Let $\mathscr{K}$ be reflective in $\mathscr{L}$.

(1) If $\mathscr{K}$ is full, then for every split subobject

$$m : L \to K, \quad e : K \to L, \quad em = \mathrm{id},$$

in $\mathscr{L}$ with $K \in \mathscr{K}$ there exists a unique morphism $\bar{m}$ of $\mathscr{K}$ such that the triangle



commutes. Then r_L is inverse to $e\bar{m}$. Indeed

$$(e\bar{m}) \cdot r_L = em = \mathrm{id}_L,$$

and for the other identity use the universal property: from the equality

$$r_L \cdot (e\bar{m}) \cdot r_L = r_L = \mathrm{id}_{FL} \cdot r_L$$

derive $r_L \cdot (e\bar{m}) = \mathrm{id}_L$.

Thus, $\bar{m}$ is the desired split monomorphism in $\mathscr{K}$: it represents the same subobject as does m.

(2) Let $\mathscr{K}$ be iso-full, reflective, and closed under split subobjects. For every object K of $\mathscr{K}$ the reflection of K splits: there exists a unique morphism $e_K : FK \to K$ of $\mathscr{K}$ with $e_K \cdot r_K = \mathrm{id}_K$. Therefore, r_K represents the same subobject as does some split monomorphism $\hat{r}_K : \hat{K} \to FK$ in $\mathscr{K}$; thus, we have an isomorphism $i_K : \hat{K} \to K$ (in $\mathscr{L}$, thus in $\mathscr{K}$) for which the triangle

$$\begin{array}{ccc} \hat{K} & \xrightarrow{i_K} & K \\ & {\scriptstyle \hat{r}_K}\searrow \quad \swarrow{\scriptstyle r_K} & \\ & FK & \end{array}$$

commutes.

We prove that every morphism $f : K_1 \to K_2$ of $\mathscr{L}$ between objects of $\mathscr{K}$ lies in $\mathscr{K}$. The naturality square

$$\begin{array}{ccccc} \hat{K}_1 & \overset{i_{K_1}}{\dashrightarrow} & K_1 & \xrightarrow{f} & K_2. \\ & {\scriptstyle \hat{r}_{K_1}}\searrow & \downarrow {\scriptstyle r_{K_1}} & & {\scriptstyle r_{K_2}}\downarrow \uparrow {\scriptstyle e} \\ & & FK_1 & \xrightarrow[Ff]{} & FK_2 \end{array}$$

yields

$$f = e \cdot Ff \cdot r_{K_1} = e \cdot Ff \cdot \hat{r}_{K_1} \cdot {i_{K_1}}^{-1}.$$

This is a composite of morphisms of $\mathscr{K}$: as for ${i_{K_1}}^{-1}$ recall the iso-fulness. ■

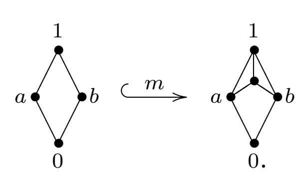
2.5. Example. *A non-full, iso-full reflective subcategory of* **Pos** *that with every poset K contains all split subposets K' as objects (but not necessarily the split monomorphism $m : K' \to K$).*

Let $\mathscr{K}$ be the subcategory of all join semilattices with 0 and all functions preserving finite joins. This is clearly a non-full but iso-full subcategory. A reflection of a poset L is its embedding $L \hookrightarrow \mathrm{Id}(L)$ into the poset of all ideals of L, *i. e.*, down-closed and up-directed subsets, ordered by inclusion.

Every split subobject (in **Pos**) of a semilattice K,

$$m : K' \to K, \quad e : K \to K', \quad em = \mathrm{id},$$

is itself a semilattice. Indeed, the join of a finite set $M \subseteq K$ is easily seen to be $e(\bigvee m[M])$. However, m itself need not preserve finite joins, as the following example demonstrates:



2.6. REMARK. The reflection of an initial object of $\mathscr{L}$ is clearly initial in $\mathscr{K}$. We say that $\mathscr{K}$ is *closed under initial objects* if $\mathscr{K}$ contains some initial object of $\mathscr{L}$ that is initial in $\mathscr{K}$.

2.7. PROPOSITION. *Let $\mathscr{L}$ be an abelian category. Every iso-full subcategory closed under finite limits and initial objects is full.*

PROOF. Let $\mathscr{K}$ be closed under finite limits and initial objects. By Proposition 2.4 we only need to prove that $\mathscr{K}$ is closed under split subobjects. First notice that for every object K of $\mathscr{K}$ the coproduct injections of $K \oplus K$ lie in $\mathscr{K}$. For example, the first coproduct injection $j : K \to K \oplus K$ is the equalizer of the second product projection $\pi_2 : K \oplus K \to K$ and the zero morphism 0. Now π_2 lies in $\mathscr{K}$ since $\mathscr{K}$ is closed under finite products, and 0 lies in $\mathscr{K}$ because $\mathscr{K}$ is also closed under initial objects. Consequently, j lies in $\mathscr{K}$.

For every split monomorphism $m : L \to K$, $K \in \mathscr{K}$, we prove that m lies in $\mathscr{K}$. There exists an object B of $\mathscr{L}$ such that m is the first coproduct injection of $K = L \oplus B$. For the first coproduct injection $j : K \to K \oplus K$ it follows that m is the equalizer

$$L \xrightarrow{m} K \underset{sj}{\overset{j}{\rightrightarrows}} K \oplus K,$$

where

$$s : L \oplus B \oplus L \oplus B \to L \oplus B \oplus L \oplus B$$

leaves the L-components unchanged and swaps the B-components; shortly:

$$s = \langle \pi_1, \pi_4, \pi_3, \pi_2 \rangle.$$

This follows easily from m being the first coproduct injection of $K = L \oplus B$. The morphism s lies in $\mathscr{K}$ since $\mathscr{K}$ is iso-full. Consequently, m lies in $\mathscr{K}$, as required. ■

2.8. EXAMPLE. *A non-full, reflective subcategory of* **Ab**. It has

objects: all powers of the group $\mathbb{Z}$,

morphisms: all $\mathbb{Z}^u : \mathbb{Z}^I \to \mathbb{Z}^J$, where $u : J \to I$ is a function and $\mathbb{Z}^u(h) = h \cdot u$.

This subcategory $\mathscr{K}$ is not full, since $\mathscr{K}(\mathbb{Z}^1, \mathbb{Z}^1) = \{\mathrm{id}_{\mathbb{Z}^1}\}$. $\mathscr{K}$ is reflective: the reflection of a group G is its canonical morphism $r : G \to \mathbb{Z}^{\mathbf{Ab}(G,\mathbb{Z})}$, $r(x)(h) = h(x)$. Indeed, for every group homomorphism $f : G \to \mathbb{Z}^I$ there exists a unique function $u : I \to \mathbf{Ab}(G, \mathbb{Z})$ with $f = \mathbb{Z}^u \cdot r$: put $u(i)(x) = f(x)(i)$.

.

We have asked, for a given locally λ-presentable category, whether the iso-full subcategories closed under limits and λ-filtered colimits are reflective in $\mathscr{L}$. Instead, we can ask whether those subcategories are themselves locally λ-presentable. This is an equivalent question:

2.9. Proposition. *Let $\mathscr{L}$ be a locally λ-presentable category. For iso-full and replete subcategories $\mathscr{K}$ closed under limits and λ-filtered colimits the following statements are equivalent:*

(i) $\mathscr{K}$ is reflective in $\mathscr{L}$

and

(ii) $\mathscr{K}$ is a locally λ-presentable category.

Proof. ii→i. We apply the Adjoint Functor Theorem of the following form proved in [2], Theorem 1.66: a functor between locally presentable categories is a right adjoint iff it preserves limits and λ-filtered colimits for some infinite cardinal λ. We conclude that the embedding $E : \mathscr{K} \to \mathscr{L}$ has a left adjoint.

i→ii. The left adjoint $F : \mathscr{L} \to \mathscr{K}$ of the inclusion $E : \mathscr{K} \to \mathscr{L}$ preserves λ-presentable objects. Indeed, given L λ-presentable in $\mathscr{L}$, we are to prove that FL is λ-presentable in $\mathscr{K}$. That is, $\mathscr{K}(FL, -)$ preserves colimits of λ-filtered diagrams $D : \mathscr{D} \to \mathscr{K}$. For a colimit of D, its image under E is a colimit of ED in $\mathscr{L}$. From the fact that $\mathscr{L}(L, -)$ preserves this colimit it easily follows that $\mathscr{K}(FL, -)$ preserves the colimit of D (using $F \dashv E$).

Let us next prove that $\mathscr{K}$ is cocomplete. Since the composite $EF : \mathscr{L} \to \mathscr{L}$ preserves λ-filtered colimits, there is a regular cardinal $\kappa \geq \lambda$ such that EF preserves κ-presentable objects (see [2], Uniformization Theorem 2.19). Let K be an object of $\mathscr{K}$ and consider the canonical diagram of EK consisting of all morphisms $l : L \to EK$ where L is a κ-presentable object in $\mathscr{L}$. This diagram is κ-filtered and, since $\mathscr{L}$ is locally κ-presentable, all the morphisms l form a colimit cocone. The reflections FL form a κ-filtered diagram in $\mathscr{K}$, and the unique morphisms $\bar{l} : FL \to K$ in $\mathscr{K}$ with $l = E\bar{l} \cdot r_L$ form a cocone. Since the objects EFL are κ-presentable in $\mathscr{L}$, the subdiagram consisting of all $E\bar{l} : EFL \to EK$ is cofinal in the canonical diagram of EK. Thus this diagram is κ-filtered has a colimit consisting of all $E\bar{l} : EFL \to EK$. Since $\mathscr{K}$ is iso-full and closed under lambda-filtered (thus under kappa-filtered) colimits, all the morphisms $\bar{l} : FL \to K$ form a colimit in $\mathscr{K}$. Thus $\mathscr{K}$ is κ-accessible and, since it is complete, it is cocomplete (see [2], 2.47).

We prove that the objects FL, where L ranges over λ-presentable objects of $\mathscr{L}$, form a strong generator of $\mathscr{K}$. Since these objects are λ-presentable in $\mathscr{K}$ (and form a set up to isomorphism), it follows that $\mathscr{K}$ is locally λ-presentable by Theorem 1.20 of [2]. Thus, our task is to prove that for every proper subobject $m : K \to K'$ in $\mathscr{K}$ there exists a morphism from some FL to K', where L is λ-presentable in $\mathscr{L}$, not factorizing through m. We know that m is not an isomorphism in $\mathscr{L}$ (since, $\mathscr{K}$ being iso-full, it would lie in $\mathscr{K}$). Since $\mathscr{L}$ is locally λ-presentable, there exists a morphism $p : L \to K'$, L λ-presentable in $\mathscr{L}$, that does not factorize through m. The unique morphism $\bar{p} : FL \to K'$ of $\mathscr{K}$ with $p = \bar{p}r$, where r is a reflection, does not factorize through m either: given u with $\bar{p} = mu$, we would have $p = mur$, a contradiction. ∎

3. Elementary Subobjects

We have mentioned the result of Makkai and Pitts that every iso-full subcategory $\mathscr{K}$ of a locally finitely presentable category $\mathscr{L}$ closed under limits and filtered colimits is reflective. A substantial step in the proof was to verify that in case $\mathscr{L}$ is the category $\mathbf{Str}\,\Sigma$ of structures of some finitary, many-sorted signature Σ, the given subcategory $\mathscr{K}$ is closed under elementary subobjects. Recall that a monomorphism $m : L \to K$ in $\mathbf{Str}\,\Sigma$ is called an *elementary embedding* provided that for every formula $\varphi(x_i)$ of first-order (finitary) logic with free variables x_i and every interpretation $p(x_i)$ of the variables in L the following holds.

$$L \text{ satisfies } \varphi(p(x_i)) \text{ iff } K \text{ satisfies } \varphi(m(p(x_i))). \tag{1}$$

(For many-sorted structures the variables also are assigned sorts and interpretations are required to preserve sorts.)

We now consider the infinitary first-order logic $\mathrm{L}_{\kappa\kappa}$, which allows conjunctions of fewer than κ formulas and quantification over fewer than κ variables. A monomorphism $m : L \to K$ is called a *κ-elementary embedding* if (1) holds for all formulas $\varphi(x_i)$ of $\mathrm{L}_{\kappa\kappa}$.

3.1. Example. (1) Every isomorphism is κ-elementary for all cardinals κ.

(2) The category of directed graphs is $\mathbf{Str}\,\Sigma$, where Σ consists of one binary relation R. If $m : L \to K$ is a κ-elementary embedding and L has fewer than κ vertices, then m is an isomorphism. Indeed, we can use the vertices of L as variables x_i ($i \in I$); let E be the set of all edges. The following formula describes L:

$$\bigvee_{(x_i,x_j)\in E} R(x_i,x_j) \;\wedge\; \bigvee_{(x_i,x_j)\notin E} \neg\, R(x_i,x_j) \;\wedge\; \bigvee_{\substack{i,j\in I\\ i\neq j}} \neg\, x_i = x_j \;\wedge\; (\forall x) \bigvee_{i\in I} x = x_i.$$

Since the formula holds in L for the identity interpretation, it holds in K for $x_i \mapsto m(x_i)$. This shows that m is invertible.

Consequently, the only monomorphisms that are κ-elementary embeddings for all κ are isomorphisms.

(3) More generally for every signature Σ: a morphism of $\mathbf{Str}\,\Sigma$ is a κ-elementary embedding for all κ iff it is an isomorphism.

3.2. Notation. Recall that every locally λ-presentable category $\mathscr{L}$ has a small full subcategory $\mathscr{L}_\lambda$ representing all λ-presentable objects up to isomorphism.

(a) We denote by

$$\Sigma_{\mathscr{L}}$$

the many-sorted signature of unary operation symbols with

sorts = objects of $\mathscr{L}_\lambda$,

and

operation symbols = morphisms of $\mathscr{L}_\lambda^{\mathrm{op}}$.

Thus a morphism $f : s \to t$ of $\mathscr{L}_\lambda^{\mathrm{op}}$ is a unary operation of input sort s and output sort t.

(b) Every object L of $\mathscr{L}$ defines a $\Sigma_{\mathscr{L}}$-algebra EL: The underlying many-sorted set has components

$$(EL)_s = \mathscr{L}(s, L) \qquad \text{for all } s \in \mathscr{L}_\lambda.$$

For an operation symbol $f : s \to t$ (morphism in $\mathscr{L}_\lambda^{\mathrm{op}}$) we define the operation of EL by precomposition with f:

$$(-) \cdot f : \mathscr{L}(s, L) \to \mathscr{L}(t, L).$$

(c) Every morphism $h : L \to L'$ of $\mathscr{L}$ defines a homomorphism $Eh : EL \to EL'$ of $\Sigma_{\mathscr{L}}$-algebras: its components $(Eh)_s : \mathscr{L}(s, L) \to \mathscr{L}(s, L')$ are given by postcomposition with h:

$$h \cdot (-) : \mathscr{L}(s, L) \to \mathscr{L}(s, L').$$

3.3. Lemma. [See [2], 1.26 and 1.27] *For every locally λ-presentable category $\mathscr{L}$ we have a full embedding*

$$E : \mathscr{L} \to \mathbf{Str}\, \Sigma_{\mathscr{L}}$$

defined as above. The full subcategory $E(\mathscr{L})$ is reflective and closed under λ-filtered colimits.

3.4. Definition. A subcategory $\mathscr{K}$ of a locally presentable category $\mathscr{L}$ is said to be *closed under κ-elementary embeddings* provided that there exists a signature Σ and a full, reflective embedding $E : \mathscr{L} \to \mathbf{Str}\, \Sigma$ preserving κ-filtered colimits such that for every morphism $m : L \to K$ of $\mathscr{L}$ with $K \in \mathscr{K}$ we have: if Em is a κ-elementary embedding, then L and m lie in $\mathscr{K}$.

3.5. Remark. Any such a subcategory is iso-full and replete (see 3.1(1)).

4. Iso-Full Reflective Subcategories

Recall from Makkai and Paré's [9] that a category $\mathscr{L}$ is called λ-accessible if it has λ-filtered colimits and a set of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathscr{L}$. An important result of [9] is that for a signature Σ and a cardinal λ the category $\mathbf{Elem}_\lambda\, \Sigma$ of Σ-structures and λ-elementary embeddings is κ-accessible for some $\kappa \geq \lambda$ (see also [2] 5.42). Moreover, following the proof of [2] 5.42, a Σ-structure is κ-presentable iff its underlying set has cardinality $< \kappa$.

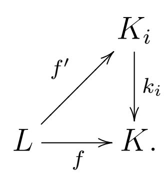
4.1. Theorem. *Let $\mathscr{L}$ be a locally presentable category. Every subcategory closed under limits, λ-filtered colimits and λ-elementary embeddings for some λ is reflective in $\mathscr{L}$.*

Proof. Since instead of the given λ every larger cardinal works as well, we can assume without loss of generality that $\mathscr{L}$ is locally λ-presentable and the embedding $E : \mathscr{L} \to \mathbf{Str}\,\Sigma$ of Definition 3.4 preserves λ-filtered colimits.

We are going to prove that the image $E(\mathscr{K})$ is a reflective subcategory of $\mathbf{Str}\,\Sigma$. This implies that it is reflective in $E(\mathscr{L})$, and since E defines an equivalence of the categories $\mathscr{L}$ and $E(\mathscr{L})$, it follows that $\mathscr{K}$ is reflective in $\mathscr{L}$.

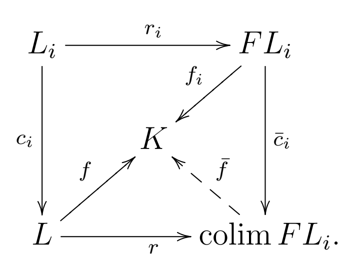
(1) We first prove that every λ-presentable Σ-structure $L \in \mathscr{L}$ has a reflection in $\mathscr{K}$. By the preceding remark $\mathbf{Elem}_\lambda\,\Sigma$ is κ-accessible for some $\kappa \geq \lambda$, and we take a set $\mathcal{A}$ of κ-presentable structures such that its closure under κ-filtered colimits is all of that category.

We are going to prove that the slice category $L \,/\, \mathscr{K}$ has an initial object (= reflection of L) by proving that the objects $f : L \to K$ with $E(K) \in \mathcal{A}$ form a solution set. Since $L/\mathscr{K}$ is complete, Freyd's Adjoint Functor Theorem then yields an initial object. Express $E(K)$ as a κ-filtered colimit $\bar{k}_i : \bar{K}_i \to E(K)$, $i \in I$, in $\mathbf{Elem}_\kappa\,\Sigma$ of objects $\bar{K}_i \in \mathcal{A}$. Since $\bar{k}_i$ is κ-elementary and $\mathscr{K}$ is closed under κ-elementary embeddings in $\mathscr{L}$, we obtain a κ-filtered diagram K_i in $\mathscr{K}$ whose image $E(K_i) = \bar{K}_i$ is the given diagram and whose colimit cocone $k_i : K_i \to K$ in $\mathscr{K}$ fulfils $Ek_i = \bar{k}_i$. Since this is a κ-filtered colimit in $\mathscr{L}$ and $\kappa \geq \lambda$, the morphism f factorizes through some k_i:

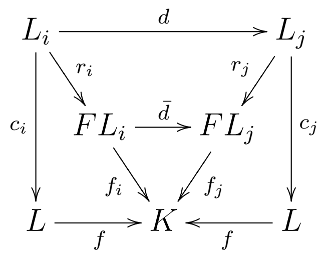


The object $f' : L \to K_i$ of $L \,/\, \mathscr{K}$ lies in the specified set.

(2) $\mathscr{L}$ is reflective in $\mathscr{K}$. Indeed, given an object L of $\mathscr{L}$, express it as a λ-filtered colimit $c_i : L_i \to L$ $(i \in I)$ of λ-presentable objects L_i of $\mathscr{L}$. By (1) we have reflections $r_i : L_i \to FL_i$ in $\mathscr{K}$, and it is easy to see that the objects FL_i form a λ-filtered diagram in $\mathscr{K}$ with a natural transformation having components r_i. Let $r : L \to \operatorname{colim}_{i\in I} FL_i$ be a colimit of that natural transformation. Then r is a reflection of L in $\mathscr{K}$. Indeed, use closedness of $\mathscr{K}$ under λ-filtered colimits: given a morphism $f : L \to K$ of $\mathscr{L}$ with $K \in \mathscr{K}$, for every i we get a unique morphism $f_i : FL_i \to K$ of $\mathscr{K}$ with $fc_i = f_i r_i$:



These morphisms form a cocone of the diagram of FL_i's: If $d : L_i \to L_j$ is a connecting morphism, the corresponding connecting morphism $\bar{d} : FL_i \to FL_j$ is the unique morphism of $\mathscr{K}$ with $\bar{d}r_i = r_j d$. Then in the following diagram



the inner triangle commutes, as desired, because it lies in $\mathscr{K}$ and commutes when precomposed with the universal arrow r_i (since $c_i = c_j d$). Consequently, there exists a unique morphism $\bar{f} : \operatorname{colim} FL_i \to K$ in $\mathscr{K}$ with $\bar{f} \cdot \bar{c}_i = f_i$ (where the $\bar{c}_i$ form the colimit cocone). The desired equality

$$f = \bar{f}r : L \to K$$

follows from the fact that c_i is a collectively epic cocone and the first diagram above commutes.

The uniqueness of $\bar{f}$ easily follows from the fact that the cocone $\bar{c}_i$ is collectively epic in $\mathscr{K}$. ■

Problem. Is any iso-full, replete, reflective subcategory of a locally presentable category which is closed under limits and λ-filtered colimits also closed under κ-elementary embeddings for some κ?

.

Makkai and Pitts proved that this is true for $\lambda = \aleph_0$ (with $\kappa = \aleph_0$).

4.2. Example. We apply Theorem 4.1 to Kan injectivity, a concept in order-enriched categories $\mathscr{L}$ introduced by Escardo [6] for objects and by Carvalho and Sousa [5] for morphisms.

An object K of $\mathscr{L}$ is said to be *Kan-injective* w. r. t. a morphism $h : X \to Y$ provided that every morphism $f : X \to K$ has a left Kan extension f/h along h with $f = (f/h) \cdot h$. That is, there is a morphism $f/h : Y \to K$ that satisfies

$$f = (f/h) \cdot h \tag{2}$$

and for every morphism $g : Y \to K$

$$f \leq g \cdot h \quad \text{implies} \quad f/h \leq g. \tag{3}$$

Carvalho and Sousa introduced in [5] the category $\mathscr{L}\,\mathrm{Inj}(\mathscr{H})$, for every class $\mathscr{H}$ of morphisms of $\mathscr{L}$: Objects are the those objects of $\mathscr{L}$ that are Kan-injective w. r. t. every $h \in \mathscr{H}$. Morphisms $u : K \to K'$ are those morphisms of $\mathscr{L}$ that *preserve Kan extensions*: given $h : X \to Y$ in $\mathscr{H}$, we have

$$u \cdot (f/h) = (uf)/h \qquad \text{for all } f : X \to K.$$

For example, in $\mathscr{L} = \mathbf{Pos}$ let $h : X \to Y$ be the embedding of a two-element discrete poset X into $Y = X \cup \{t\}$, t a top element. Then $\mathscr{L}\,\mathrm{Inj}\{h\}$ is the subcategory of join semilattices and their homomorphisms.

4.3. Remark. Recall that an *order-enriched locally λ-presentable* category is a category that is locally λ-presentable and enriched over $\mathbf{Pos}$ in such a way that a parallel pair $f_1, f_2 : L \to L'$ fulfils $f_1 \leq f_2$ whenever for every morphism $u : K \to L$, K λ-presentable, we have $f_1 \cdot u \leq f_2 \cdot u$.

.

The following result is proved in [3], but the present proof is simpler.

4.4. Proposition. *Let $\mathscr{H}$ be a set of morphisms of an order-enriched locally presentable category. Then $\mathscr{L}\,\mathrm{Inj}(\mathscr{H})$ is a reflective subcategory.*

Proof. Since $\mathscr{H}$ is a set, there exists a cardinal λ such that the given category $\mathscr{L}$ is locally λ-presentable and domains and codomains of members of $\mathscr{H}$ are all λ-presentable. As can be seen rather easily it follows that $\mathscr{L}\,\mathrm{Inj}(\mathscr{H})$ is closed under λ-filtered colimits in $\mathscr{L}$. A proof that $\mathscr{L}\,\mathrm{Inj}(\mathscr{H})$ is closed under limits can be found in [5]. Thus according to Theorem 4.1 it is sufficient to prove closedness under λ-elementary embeddings.

Let $\bar{\Sigma}_{\mathscr{L}}$ be the extension of the signature $\Sigma_{\mathscr{L}}$ of Notation 3.2 by a binary relation symbol $\leq_s$ for every sort $s \in \mathscr{L}_\lambda$. And let $\bar{E} : \mathscr{L} \to \mathbf{Str}\,\bar{\Sigma}_{\mathscr{L}}$ be the extension of the full embedding of Lemma 3.3 by interpreting, for every object L in $\mathscr{L}$, the symbol $\leq_s$ as the given partial order on $\mathscr{L}(s, L)$. For every member $h : s \to t$ of $\mathscr{H}$, since s and t lie in $\mathscr{L}_\lambda$, we have the unary operation symbol $h : t \to s$ in $\bar{\Sigma}_{\mathscr{L}}$ (interpreted as precomposition with h). The following formula, with variables x of sort s and y, z of sort t, expresses Kan-injectivity w. r. t. h:

$$\psi_h \;=\; (\forall x)\,(\exists y)\,([x = h(y)] \;\wedge\; (\forall z)\,([x \leq_s h(z)] \to [y \leq_t z]))\,.$$

Indeed, an object L of $\mathscr{L}$ has the property that ψ_h holds in $\bar{E}L$ iff for every element of sort s, *i. e.*, every morphism $f : s \to L$ in $\mathscr{L}$, there exists an element of sort t, *i. e.*, a morphism $\bar{f} : t \to L$, such that (i) $f = \bar{f} \cdot h$ and (ii) for every interpretation of z, *i. e.*, every morphism $g : t \to L$, if $f \leq g \cdot h$, then $\bar{f} \leq g$. This tells us precisely that $\bar{f} = f/h$.

Now let $m : L \to K$ be a morphism of $\mathscr{L}$ with K Kan-injective w. r. t. $\mathscr{H}$ and with $\bar{E}m$ a λ-elementary embedding. We prove that L is also Kan-injective, and that m preserves Kan extensions. Given $h \in \mathscr{H}$ we know that $\bar{E}K$ satisfies ψ_h, and this, since ψ_h has no free variables, implies that also $\bar{E}L$ satisfies ψ_h. That is, L is Kan-injective w. r. t. h (for all $h \in \mathscr{H}$). Next consider the formula $\psi'_h(x, y)$ with free variables x and y obtained by deleting the two quantifiers at the beginning of ψ_h. An interpretation of $\psi'_h(x, y)$ in $\bar{E}L$ is a pair of morphisms $f : s \to L$ and $\bar{f} : t \to L$ with $\bar{f} = f/h$. Analogously for interpretations in $\bar{E}K$. The assumption that $\bar{E}m$ is a λ-elementary embedding tells us: for an interpretation $f, f/h$ in $\bar{E}L$, it follows that $m \cdot f, m \cdot (f/h)$ is an interpretation in $\bar{E}K$, that is, $m \cdot (f/h) = (m \cdot f)/h$, as required. ∎

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