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DELIGNE GROUPOID REVISITED

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ABSTRACT. We show that for a differential graded Lie algebra $\mathfrak{g}$ whose components vanish in degrees below -1 the nerve of the Deligne 2-groupoid is homotopy equivalent to the simplicial set of $\mathfrak{g}$-valued differential forms introduced by V. Hinich [Hinich, 1997].

1. Introduction

The principal result of the present note compares two spaces (simplicial sets) naturally associated with a nilpotent differential graded Lie algebra (DGLA) subject to certain restrictions. Our interest in this problem has its origins in formal deformation theory of associative algebras and, more generally, algebroid stacks ([Bressler, Gorokhovsky, Nest & Tsygan, 2007]). The results of the present note are used in [Bressler, Gorokhovsky, Nest & Tsygan, 2015] to deduce a quasi-classical description of the deformation theory of a gerbe from the formality theorem of M. Kontsevich ([Kontsevich, 2003]).

To a nilpotent DGLA $\mathfrak{h}$ which satisfies the additional condition

$$\mathfrak{h}^i = 0 \text{ for } i < -1 \tag{1}$$

P. Deligne [Deligne, 1994] and, independently, E. Getzler [Getzler, 2009] associated a (strict) 2-groupoid which we denote $\mathrm{MC}^2(\mathfrak{h})$ and refer to as the Deligne 2-groupoid.

Our principal result (Theorem 4.2) compares the simplicial nerve $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h})$ of the 2-groupoid $\mathrm{MC}^2(\mathfrak{h})$, $\mathfrak{h}$ a nilpotent DGLA satisfying (1), to another simplicial set, denoted $\Sigma(\mathfrak{h})$, introduced by V. Hinich [Hinich, 1997]:

1.1. THEOREM. *(Main theorem) Suppose that $\mathfrak{h}$ is a nilpotent DGLA such that $\mathfrak{h}^i = 0$ for $i < -1$. Then, the simplicial sets $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h})$ and $\Sigma(\mathfrak{h})$ are weakly homotopy equivalent.*

In the case when the nilpotent DGLA $\mathfrak{h}$ satisfies $\mathfrak{h}^i = 0$ for $i < 0$ and, consequently, $\mathrm{MC}^2(\mathfrak{h})$ is an ordinary groupoid a homotopy equivalence between $\Sigma(\mathfrak{h})$ and the nerve of $\mathrm{MC}^2(\mathfrak{h})$ was constructed by V. Hinich in [Hinich, 1997].

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Differential graded Lie algebras satisfying (1) arise in formal deformation theory of algebraic structures such as Lie algebras, commutative algebras, associative algebras to name a few. In what follows we shall concentrate on the latter example. Let k denote an algebraically closed field of characteristic zero. For an associative algebra A over k the shifted Hochschild cochain complex $C^\bullet(A)[1]$ has a canonical structure of a DGLA under the Gerstenhaber bracket; we denote this DGLA by $\mathfrak{g}(A)$ for short. Suppose that $\mathfrak{m}$ is a nilpotent commutative k-algebra (without unit). Then, $\mathfrak{g}(A) \otimes_k \mathfrak{m}$ is a nilpotent DGLA which satisfies (1). Thus, the Deligne 2-groupoid $\mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m})$ is defined. For an Artin k-algebra R with maximal ideal $\mathfrak{m}_R$ the 2-groupoid $\mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$ is naturally equivalent to the 2-groupoid of R-deformations of the algebra A. In this sense the DGLA $\mathfrak{g}(A)$ controls the formal deformation theory of A.

The reason for considering the space $\Sigma(\mathfrak{h})$ is that it is defined not just for a DGLA (V. Hinich, [Hinich, 1997]), but, more generally, for any nilpotent L_∞ algebra (E. Getzler, [Getzler, 2009]). Homotopy invariance properties of the functor Σ (Proposition 3.9), the theory of J.W. Duskin ([Duskin, 2001/02]) and the theorem above yield the following result. If $\mathfrak{h}$ is a DGLA satisfying (1), $\mathfrak{g}$ is a L_∞ algebra L_∞-quasi-isomorphic to $\mathfrak{h}$ and $\mathfrak{m}$ is a nilpotent commutative k-algebra, then $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h} \otimes_k \mathfrak{m})$ is homotopy equivalent to $\Sigma(\mathfrak{g} \otimes_k \mathfrak{m})$. Thus, the 2-groupoid $\mathrm{MC}^2(\mathfrak{h} \otimes_k \mathfrak{m})$ can be reconstructed, up to equivalence, from the space $\Sigma(\mathfrak{g} \otimes_k \mathfrak{m})$. The situation envisaged above arises naturally. Any DGLA $\mathfrak{h}$ is L_∞-quasi-isomorphic to an L_∞ algebra with trivial univalent operation (the differential).

The paper is organized as follows. In Section 2 we review various constructions of nerves of 2-groupoids and their properties. In section 3 we recall the definitions of the functor Σ (3.4) and of the Deligne 2-groupoid (3.10) and prove basic properties thereof. The proof of the main theorem (Theorem 4.2) given in Section 4 proceeds by exhibiting canonical weak homotopy equivalences from $\Sigma(\mathfrak{h})$ and $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h})$ to a third naturally defined simplicial set.

2. The homotopy type of a strict 2-groupoid

2.1. Nerves of simplicial groupoids.

2.1.1. Simplicial groupoids. In what follows a *simplicial category* is a category enriched over the category of simplicial sets. A small simplicial category consists of a set of objects and a simplicial set of morphisms for each pair of objects.

A simplicial category $\mathtt{G}$ is a particular case of a simplicial object $[p] \mapsto \mathtt{G}_p$ in Cat whose simplicial set of objects $[p] \mapsto N_0\mathtt{G}_p$ is constant.

A simplicial category is a simplicial groupoid if it is a groupoid in each (simplicial) degree.

2.1.2. The naïve nerve. Suppose that $\mathtt{G}$ is a simplicial category. Applying the nerve functor degree-wise we obtain the bi-simplicial set $N\mathtt{G}\colon ([p],[q]) \mapsto N_q\mathtt{G}_p$ whose diagonal we denote by $\mathcal{N}\mathtt{G}$ and refer to as the *naïve nerve* of $\mathtt{G}$.

2.1.3. THE SIMPLICIAL NERVE. For a simplicial category $\mathtt{G}$ the *simplicial nerve*, also known as the homotopy coherent nerve, $\mathfrak{N}\mathtt{G}$ is represented by the cosimplicial object in $[p] \mapsto \Delta^p_{\mathfrak{N}} \in Cat_\Delta$, i.e

$$\mathfrak{N}_p\mathtt{G} = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\Delta^p_{\mathfrak{N}}, \mathtt{G}).$$

Here, $\Delta^p_{\mathfrak{N}}$ is the canonical free simplicial resolution of $[p]$ which admits the following explicit description ([Cordier, 1982]).

The set of objects of $\Delta^p_{\mathfrak{N}}$ is $\{0, 1, \ldots, p\}$. For $0 \le i \le j \le p$ the simplicial set of morphisms is given by $\mathrm{Hom}_{\Delta^p_{\mathfrak{N}}}(i, j) = N\mathcal{P}(i, j)$. The category $\mathcal{P}(i, j)$ is a sub-poset of $2^{\{0,\ldots,p\}}$ (with the induced partial ordering whereby viewed as a category) given by

$$\mathcal{P}(i, j) = \{I \subset \mathbb{Z} \mid (i, j \in I) \,\&\, (k \in I \implies i \leqslant k \leqslant j)\}.$$

The composition in $\Delta^p_{\mathfrak{N}}$ is induced by functors

$$\mathcal{P}(i, j) \times \mathcal{P}(j, k) \to \mathcal{P}(i, k) \colon (I, J) \mapsto I \cup J.$$

In particular, $\Delta^0_{\mathfrak{N}} = [0]$ and $\Delta^1_{\mathfrak{N}} = [1]$

We refer the reader to [Hinich, 2007] for applications to deformation theory and to [Lurie, 2009] for the connection with higher category theory. The simplicial nerve of a simplicial groupoid is a Kan complex which reduces to the usual nerve for ordinary groupoids.

Since $\Delta^0_{\mathfrak{N}} = [0]$ (respectively, $\Delta^1_{\mathfrak{N}} = [1]$) it follows that $\mathfrak{N}_0\mathtt{G}$ (respectively, $\mathfrak{N}_1\mathtt{G}$) is the set of objects (respectively, the set of morphisms) of $\mathtt{G}_0$.

2.1.4. COMPARISON OF NERVES. We refer the reader to [Hinich, 2007] for the definition of the canonical map of simplicial sets $\mathcal{N}\mathtt{G} \to \mathfrak{N}\mathtt{G}$. In what follows we will make use of the following result of loc. cit.

2.2. THEOREM. ([Hinich, 2007], Corollary 2.6.3) *For any simplicial groupoid* $\mathtt{G}$ *the canonical map* $\mathcal{N}\mathtt{G} \to \mathfrak{N}\mathtt{G}$ *is a weak homotopy equivalence.*

2.3. STRICT 2-GROUPOIDS.

2.3.1. FROM STRICT 2-GROUPOIDS TO SIMPLICIAL GROUPOIDS. Suppose that $\mathtt{G}$ is a strict 2-groupoid, i.e. a groupoid enriched over the category of groupoids. Thus, for every $g, g' \in \mathtt{G}$, we have the groupoid $\mathrm{Hom}_{\mathtt{G}}(g, g')$ and the composition is strictly associative.

The nerve functor $[p] \mapsto N_p(\cdot) := \mathrm{Hom}_{\mathrm{Cat}}([p], \cdot)$ commutes with products. Let $\mathtt{G}_p$ denote the category with the same objects as $\mathtt{G}$ and with morphisms defined by $\mathrm{Hom}_{\mathtt{G}_p}(g, g') = N_p \mathrm{Hom}_{\mathtt{G}}(g, g')$; the composition of morphisms is induced by the composition in $\mathtt{G}$. Note that the groupoid $\mathtt{G}_0$ is obtained from $\mathtt{G}$ by forgetting the 2-morphisms.

The assignment $[p] \mapsto \mathtt{G}_p$ defines a simplicial object in groupoids with the constant simplicial set of objects, i.e. a simplicial groupoid which we denote by $\widetilde{\mathtt{G}}$.

2.4. Lemma. *The simplicial nerve $\mathfrak{N}\widetilde{\mathtt{G}}$ admits the following explicit description:*

1. *There is a canonical bijection between $\mathfrak{N}_0\widetilde{\mathtt{G}}$ and the set of objects of $\mathtt{G}$.*
2. *For $n \geq 1$ there is a canonical bijection between $\mathfrak{N}_n\widetilde{\mathtt{G}}$ and the set of data of the form $((\mu_i)_{0\leq i\leq n}, (g_{ij})_{0\leq i<j\leq n}, (c_{ijk})_{0\leq i<j<k\leq n})$, where (μ_i) is an $(n+1)$-tuple of objects of $\mathtt{G}$, (g_{ij}) is a collection of 1-morphisms $g_{ij}\colon \mu_j \to \mu_i$ and (c_{ijk}) is a collection of 2-morphisms $c_{ijk} : g_{ij}g_{jk} \to g_{ik}$ which satisfies*

$$c_{ijl}c_{jkl} = c_{ikl}c_{ijk} \tag{2}$$

   (in the set of 2-morphisms $g_{ij}g_{jk}g_{kl} \to g_{il}$).

For a morphism $f\colon [m] \to [n]$ in Δ the induced structure map $f^*\colon \mathfrak{N}_n\widetilde{\mathtt{G}} \to \mathfrak{N}_m\widetilde{\mathtt{G}}$ is given (under the above bijection) by $f^*((\mu_i), (g_{ij}), (c_{ijk})) = ((\nu_i), (h_{ij}), (d_{ijk}))$, where $\nu_i = \mu_{f(i)}$, $h_{ij} = g_{f(i),f(j)}$, $d_{ijk} = c_{f(i),f(j),f(k)}$ (cf. [Duskin, 2001/02]).

Proof. An n-simplex of $\mathfrak{N}\widetilde{\mathtt{G}}$ is the following collection of data:

1. objects $\mu_0, \ldots, \mu_n$ of $\mathtt{G}$;
2. morphisms of simplicial sets $N\mathcal{P}(i,j)) \to N\operatorname{Hom}_{\mathtt{G}}(\mu_i, \mu_j))$ intertwining the maps induced on the nerves by composition functors $\mathcal{P}(i,j) \times \mathcal{P}(j,k) \to \mathcal{P}(i,k)$ and $\operatorname{Hom}_{\mathtt{G}}(\mu_i, \mu_j) \times \operatorname{Hom}_{\mathtt{G}}(\mu_j, \mu_k) \to \operatorname{Hom}_{\mathtt{G}}(\mu_i, \mu_k)$.

Since the nerve functor is fully faithful, the above data are equivalent to the following:

1. objects $\mu_0, \ldots, \mu_n$ of $\mathtt{G}$;
2. for any $I \in N_0\mathcal{P}(i,j)$, a 1-morphism $g_I : \mu_j \to \mu_i$ in $\mathtt{G}$;
3. for any morphism $J \to I$ in $\mathcal{P}(i,j)$, a 2-morphism $c_{IJ}\colon g_J \to g_I$, such that

$$c_{IJ}c_{JK} = c_{IK} \tag{3}$$

   These data have to be compatible with the composition pairings $\mathcal{P}(i,j)\times\mathcal{P}(j,k) \to \mathcal{P}(i,k)$ and $\operatorname{Hom}_{\mathtt{G}}(\mu_i, \mu_j) \times \operatorname{Hom}_{\mathtt{G}}(\mu_j, \mu_k) \to \operatorname{Hom}_{\mathtt{G}}(\mu_i, \mu_k)$.

Let $g_{ij}\colon \mu_j \to \mu_i$ denote the morphism $g_{\{i,j\}}$. By compatibility with compositions, if $I = \{i, i_1, \ldots, i_k, j\}$ then $g_I = g_{ii_1} \ldots g_{i_kj}$. Let c_{ijk} denote the two-morphism $c_{\{i,j,k\},\{i,k\}}\colon g_{ik} \to g_{ij}g_{jk}$. Now, by virtue of (3) and of compatibility with compositions, c_{ijk} satisfy the two-cocycle identity (3) and determine c_{IJ} for any I, J. ∎

In what follows, for a strict 2-groupoid $\mathtt{G}$, we will denote by $\mathcal{N}\mathtt{G}$ (respectively $\mathfrak{N}\mathtt{G}$) the naïve (respectively simplicial) nerve of the associated simplicial groupoid $\widetilde{\mathtt{G}}$.

3. Homotopy types associated with L_∞-algebras

3.1. L_∞-ALGEBRAS. We follow the notation of [Getzler, 2009] and refer the reader to loc. cit. for details.

Recall that an L_∞-algebra is a graded vector space $\mathfrak{g}$ equipped with operations

$$\bigwedge^k \mathfrak{g} \to \mathfrak{g}[2-k] \colon x_1 \wedge \ldots \wedge x_k \mapsto [x_1, \ldots, x_k]$$

defined for $k = 1, 2, \ldots$. which satisfy a sequence of Jacobi identities.

It follows from the Jacobi identities that the unary operation $[.]\colon \mathfrak{g} \to \mathfrak{g}[1]$ is a differential, which we will denote by δ.

An L_∞-algebra is *abelian* if all operations with valency two and higher (i.e. all operations except for δ) vanish. In other words, an abelian L_∞-algebra is a complex. An L_∞-algebra structure with vanishing operations of valency three and higher reduces to a structure of a DGLA.

The *lower central series* of an L_∞-algebra $\mathfrak{g}$ is the canonical decreasing filtration $F^\bullet\mathfrak{g}$ with $F^i\mathfrak{g} = \mathfrak{g}$ for $i \leq 1$ and defined recursively for $i \geq 1$ by

$$F^{i+1}\mathfrak{g} = \sum_{k=2}^{\infty} \sum_{\substack{i=i_1+\cdots+i_k \\ i_k \leqslant i}} [F^{i_1}\mathfrak{g}, \ldots, F^{i_k}\mathfrak{g}].$$

An L_∞-algebra is *nilpotent* if there exists an i such that $F^i\mathfrak{g} = 0$.

3.1.1. MAURER-CARTAN ELEMENTS. Suppose that $\mathfrak{g}$ is a nilpotent L_∞-algebra. For $\mu \in \mathfrak{g}^1$ let

$$\mathcal{F}(\mu) = \delta\mu + \sum_{k=2}^{\infty} \frac{1}{k!}[\mu^{\wedge k}]. \tag{4}$$

The element $\mathcal{F}(\mu)$ of $\mathfrak{g}^2$ is called the *curvature* of μ. For any $\mu \in \mathfrak{g}^1$ the curvature $\mathcal{F}(\mu)$ satisfies the Bianchi identity ([Getzler, 2009], Lemma 4.5)

$$\delta\mathcal{F}(\mu) + \sum_{k=1}^{\infty} \frac{1}{k!}[\mu^{\wedge k}, \mathcal{F}(\mu)] = 0. \tag{5}$$

An element $\mu \in \mathfrak{g}^1$ is called a *Maurer-Cartan element* (of $\mathfrak{g}$) if it satisfies the condition $\mathcal{F}(\mu) = 0$. The set of Maurer-Cartan elements of $\mathfrak{g}$ will be denoted $\mathrm{MC}(\mathfrak{g})$:

$$\mathrm{MC}(\mathfrak{g}) := \{\mu \in \mathfrak{g}^1 \mid \mathcal{F}(\mu) = 0\}.$$

The set $\mathrm{MC}(\mathfrak{g})$ is pointed by the distinguished element $0 \in \mathfrak{g}^1$.

Suppose that $\mathfrak{a}$ is an abelian L_∞-algebra. Then,

$$\mathrm{MC}(\mathfrak{a}) = Z^1(\mathfrak{a}) := \ker(\delta\colon \mathfrak{a}^1 \to \mathfrak{a}^2),$$

hence is equipped with a canonical structure of an abelian group.

3.1.2. Central extensions. Suppose that $\mathfrak{g}$ is a L_∞-algebra and $\mathfrak{a}$ is a subcomplex of $(\mathfrak{g}, \delta)$ such that $[\mathfrak{a} \wedge \mathfrak{g}^{\wedge k}] = 0$ for all $k \geq 1$. In this case we will say that $\mathfrak{a}$ *is central in* $\mathfrak{g}$.

If $\mathfrak{a}$ is central in $\mathfrak{g}$, then there is a unique structure of an L_∞-algebra on $\mathfrak{g}/\mathfrak{a}$ such that the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ is a map of L_∞-algebras. If $\mathfrak{g}$ is nilpotent, then so is $\mathfrak{g}/\mathfrak{a}$.

In what follows we assume that $\mathfrak{g}$ is a nilpotent L_∞-algebra and $\mathfrak{a}$ is central in $\mathfrak{g}$.

3.2. Lemma.

1. *The addition operation on* $\mathfrak{g}^1$ *restricts to a free action of the abelian group* $\mathrm{MC}(\mathfrak{a})$ *on the set* $\mathrm{MC}(\mathfrak{g})$.
2. *The map* $\mathrm{MC}(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$ *is constant on the orbits of the action.*
3. *The induced map* $\mathrm{MC}(\mathfrak{g})/\,\mathrm{MC}(\mathfrak{a}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$ *is injective.*

Proof. Suppose that $\alpha \in \mathfrak{a}^1$ and $\mu \in \mathfrak{g}^1$. Since $\mathfrak{a}$ is central in $\mathfrak{g}$, $[(\alpha+\mu)^{\wedge k}] = [\mu^{\wedge k}]$ for $k \geq 2$ and $\mathcal{F}(\alpha + \mu) = \delta\alpha + \mathcal{F}(\mu)$ (in the notation of (4)). Therefore, $\mathrm{MC}(\mathfrak{a}) + \mathrm{MC}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g})$. In other words, the addition operation in $\mathfrak{g}^1$ restricts to an action of the abelian group $\mathrm{MC}(\mathfrak{a})$ on the set $\mathrm{MC}(\mathfrak{g})$ which is obviously free. Since the map $\mathrm{MC}(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$ is the restriction of the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ constant on the orbits of the action, i.e. factors through $\mathrm{MC}(\mathfrak{g})/\,\mathrm{MC}(\mathfrak{a})$, and the induced map $\mathrm{MC}(\mathfrak{g})/\,\mathrm{MC}(\mathfrak{a}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$ is injective. ■

3.2.1. The obstruction map. The image of the map $\mathrm{MC}(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$ may be described in terms of the obstruction map (6) which we construct presently.

If $\mu \in \mathfrak{g}^1$ and $\mu + \mathfrak{a}^1 \in \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$, then $\mathcal{F}(\mu + \mathfrak{a}^1) = \mathcal{F}(\mu) + \delta\mathfrak{a}^1 \subset \mathfrak{a}^2$ and the Bianchi identity (5) reduces to $\delta\mathcal{F}(\mu + \mathfrak{a}^1) = 0$, i.e. the assignment $\mu + \mathfrak{a}^1 \mapsto \mathcal{F}(\mu + \mathfrak{a}^1)$ gives rise to a well-defined map

$$o_2\colon\ \mathrm{MC}(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}) \tag{6}$$

(notation borrowed from [Goldman, Millson, 1988], 2.6).

3.3. Lemma. *The sequence of pointed sets*

$$0 \to \mathrm{MC}(\mathfrak{g})/\,\mathrm{MC}(\mathfrak{a}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a}) \xrightarrow{o_2} H^2(\mathfrak{a}) \tag{7}$$

is exact.

Proof. If $\mathcal{F}(\mu + \mathfrak{a}^1) \subset \delta\mathfrak{a}^1$, then there exists $\alpha \in \mathfrak{a}^1$ such that $\mathcal{F}(\mu + \alpha) = 0$, i.e. $\mu + \mathfrak{a}^1$ is in the image of $\mathrm{MC}(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$. ■

3.4. The functor Σ. In what follows we denote by Ω_n, $n = 0, 1, 2, \ldots$ the commutative differential graded algebra over $\mathbb{Q}$ with generators $t_0, \ldots, t_n$ of degree zero and $dt_0, \ldots, dt_n$ of degree one subject to the relations $t_0 + \cdots + t_n = 1$ and $dt_0 + \cdots + dt_n = 0$. The differential $d\colon \Omega_n \to \Omega_n[1]$ is defined by $t_i \mapsto dt_i$ and $dt_i \mapsto 0$. The assignment $[n] \mapsto \Omega_n$ extends in a natural way to a simplicial commutative differential graded algebra.

3.4.1. The simplicial set $\Sigma(\mathfrak{g})$. For a nilpotent L_∞-algebra $\mathfrak{g}$ and a non-negative integer n let

$$\Sigma_n(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g} \otimes \Omega_n).$$

Equipped with structure maps induced by those of $\Omega_\bullet$ the assignment $n \mapsto \Sigma_n(\mathfrak{g})$ defines a simplicial set denoted $\Sigma(\mathfrak{g})$.

The simplicial set $\Sigma(\mathfrak{g})$ was introduced by V. Hinich in [Hinich, 1997] for DGLA and used by E. Getzler in [Getzler, 2009] (where it is denoted $\mathrm{MC}_\bullet(\mathfrak{g})$) for general nilpotent L_∞-algebras.

3.4.2. Abelian DGLA. If $\mathfrak{a}$ is an abelian L_∞-algebra, then $\Sigma(\mathfrak{a})$ is given by $\Sigma_n(\mathfrak{a}) = Z^1(\Omega_n \otimes \mathfrak{a}) = Z^0(\Omega_n \otimes \mathfrak{a}[1])$ and has a canonical structure of a simplicial abelian group. In particular, it is a Kan simplicial set.

Recall that the Dold-Kan correspondence associates to a complex of abelian groups A a simplicial abelian group $K(A)$ defined by $K(A)_n = Z^0(C^\bullet([n]; A))$, the group of cocycles of (total) degree zero in the complex of simplicial cochains on the n-simplex with coefficients in A.

The integration map $\int : \Omega_n \otimes \mathfrak{a} \to C^\bullet([n]; \mathfrak{a})$ induces a homotopy equivalence

$$\int : \Sigma(\mathfrak{a}) \to K(\mathfrak{a}[1]); \tag{8}$$

see [Getzler, 2009], Section 3. Thus, $\pi_i\Sigma(\mathfrak{a}) \cong H^{1-i}(\mathfrak{a})$.

3.4.3. Central extensions. Suppose that $\mathfrak{g}$ is a nilpotent L_∞-algebra and $\mathfrak{a}$ is a central subalgebra in $\mathfrak{g}$. Then, for $n = 0, 1, \ldots$, $\Omega_n \otimes \mathfrak{a}$ is central in $\Omega_n \otimes \mathfrak{g}$.

3.5. Lemma.

1. *The addition operation on* $(\Omega_n \otimes \mathfrak{g})^1$ *induces a principal action of the simplicial abelian group* $\Sigma(\mathfrak{a})$ *on the simplicial set* $\Sigma(\mathfrak{g})$.
2. *The map* $\Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{g}/\mathfrak{a})$ *factors through* $\Sigma(\mathfrak{g})/\Sigma(\mathfrak{a})$.
3. *The induced map* $\Sigma(\mathfrak{g})/\Sigma(\mathfrak{a}) \to \Sigma(\mathfrak{g}/\mathfrak{a})$ *is injective.*

Proof. Follows from Lemma 3.2 and the naturality properties of the constructions in 3.1.2. ∎

For $n = 0, 1, \ldots$ the map $([n] \to [0])^* \colon \mathbb{Q} \to \Omega_n$ is a quasi-isomorphism, with the quasi-inverse provided by the map induced by any morphism $[0] \to [n]$. Therefore, the map $\mathfrak{a} \to \Omega_n \otimes \mathfrak{a}$ is a quasi-isomorphism as well. The induced isomorphisms $H^2(\mathfrak{a}) \cong H^2(\Omega_n \otimes \mathfrak{a})$ give rise to the isomorphism of the constant simplicial set $H^2(\mathfrak{a})$ and $n \mapsto H^2(\Omega_n \otimes \mathfrak{a})$.

The maps

$$o_{2,n} \colon \Sigma_n(\mathfrak{g}/\mathfrak{a}) = \mathrm{MC}(\Omega_n \otimes \mathfrak{g}/\mathfrak{a}) \to H^2(\Omega_n \otimes \mathfrak{a}) \cong H^2(\mathfrak{a})$$

assemble into the map of simplicial sets

$$o_2\colon \Sigma(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}). \tag{9}$$

which factors as $\Sigma(\mathfrak{g}/\mathfrak{a}) \to \pi_0\Sigma(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a})$.

Let $\Sigma(\mathfrak{g}/\mathfrak{a})_0 = o_2^{-1}(0)$. Thus, by (7), $\Sigma(\mathfrak{g}/\mathfrak{a})_0$ is a union of connected components of $\Sigma(\mathfrak{g}/\mathfrak{a})$ equal to the range of the map $\Sigma(\mathfrak{g})/\Sigma(\mathfrak{a}) \to \Sigma(\mathfrak{g}/\mathfrak{a})$.

It follows that the map $\Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{g}/\mathfrak{a})_0$ is a principal fibration with group $\Sigma(\mathfrak{a})$, in particular, a Kan fibration ([May, 1967], Lemma 18.2).

3.6. Lemma. *Suppose that* $\mathfrak{g}$ *is a nilpotent* L_∞*-algebra. Then,* $\Sigma(\mathfrak{g})$ *is a Kan simplicial set.*

Proof. If $\mathfrak{g}$ is an abelian L_∞-algebra then $\Sigma(\mathfrak{g})$ is a simplicial group and therefore a Kan simplicial set.

Let $F^\bullet\mathfrak{g}$ denote the lower central series. Assume that $Gr^i_F\mathfrak{g} \neq 0$ if and only if $0 \leq i \leq n$; that is, $\mathfrak{g}$ is *nilpotent of length* n. By induction assume that $\Sigma(\mathfrak{h})$ is a Kan simplicial set for any nilpotent L_∞-algebra $\mathfrak{h}$ of length at most $n-1$.

Since $\mathfrak{g}$ is nilpotent of length n, it follows that $F^n\mathfrak{g} = Gr^n\mathfrak{g}$ is central in $\mathfrak{g}$ and $\mathfrak{g}/F^n\mathfrak{g}$ is nilpotent of length $n-1$. Therefore, $\Sigma(\mathfrak{g}/F^n\mathfrak{g})$ is a Kan simplicial set and so is $\Sigma(\mathfrak{g}/F^n\mathfrak{g})_0$. Since $\Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{g}/F^n\mathfrak{g})_0$ is a Kan fibration it follows that $\Sigma(\mathfrak{g})$ is a Kan simplicial set as well. ∎

3.7. Lemma. *Suppose that* $\mathfrak{g}$ *is a nilpotent* L_∞*-algebra such that* $\mathfrak{g}^q = 0$ *for* $q \leq -k$*,* k *a positive integer. Then, for any connected component* X *of* $\Sigma(\mathfrak{g})$*,* $\pi_i(X) = 0$ *for* $i > k$*.*

Proof. Suppose that $\mathfrak{g}$ is an abelian L_∞-algebra. Then, $\pi_i\Sigma(\mathfrak{g}) \cong H^{1-i}(\mathfrak{g})$. For an L_∞-algebra $\mathfrak{g}$ which is not necessarily abelian the statement follows by induction on the nilpotency length, the abelian case establishing the base of the induction.

Let $F^\bullet\mathfrak{g}$ denote the lower central series. Assume that $Gr^i_F\mathfrak{g} \neq 0$ if and only if $0 \leq i \leq n$; that is, $\mathfrak{g}$ is *nilpotent of length* n. By induction assume that the conclusion holds for all nilpotent L_∞-algebras of length at most $n-1$.

Since $\mathfrak{g}$ is nilpotent of length n, it follows that $F^n\mathfrak{g} = Gr^n\mathfrak{g}$ is central in $\mathfrak{g}$ and $\mathfrak{g}/F^n\mathfrak{g}$ is nilpotent of length $n-1$. Let $X \subseteq \Sigma(\mathfrak{g})$ be a connected component of $\Sigma(\mathfrak{g})$ and let $Y \subseteq \Sigma(\mathfrak{g}/F^n\mathfrak{g})$ be the image of X under the map induced by the quotient map $\mathfrak{g} \to \mathfrak{g}/F^n\mathfrak{g}$. Then, $X \to Y$ is a principal fibration with group the connected component of the identity in $\Sigma(F^n\mathfrak{g})$. The desired vanishing of higher homotopy groups of X follows from the induction hypotheses using the long exact sequence of homotopy groups. ∎

3.7.1. Homotopy invariance.

3.8. Lemma. *Suppose that* $f\colon \mathfrak{a} \to \mathfrak{b}$ *is a quasi-isomorphism of abelian* L_∞*-algebras. Then, the induced map* $\Sigma(f)\colon \Sigma(\mathfrak{a}) \to \Sigma(\mathfrak{b})$ *is a weak homotopy equivalence.*

PROOF. Note that $\Sigma(f)$ is a morphism of simplicial abelian groups. It is sufficient to show that the maps $\pi_n\Sigma(f)\colon \pi_n\Sigma(\mathfrak{a}) \to \pi_n\Sigma(\mathfrak{b})$ are isomorphisms for $n \geqslant 0$. To this end note that $\pi_n\Sigma(f)$ factors as the composition of isomorphisms

$$\pi_n\Sigma(\mathfrak{a}) \cong H^{1-n}(\mathfrak{a}) \xrightarrow{H^{1-n}(\Sigma(f))} H^{1-n}(\mathfrak{b}) \cong \pi_n\Sigma(\mathfrak{b}).$$

∎

3.9. PROPOSITION. ([Getzler, 2009], Proposition 4.9) *Suppose that $f\colon \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of L_∞-algebras and R is an Artin algebra with maximal ideal $\mathfrak{m}_R$. Then, the map $\Sigma(f \otimes \mathtt{Id})\colon \Sigma(\mathfrak{g} \otimes \mathfrak{m}_R) \to \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R)$ is a weak homotopy equivalence.*

PROOF. We use induction on the nilpotency length of $\mathfrak{m}_R$, which is to say the largest integer l such that $\mathfrak{m}_R^l \neq 0$.

If $\mathfrak{m}_R^2 = 0$, then $f \otimes \mathtt{Id}\colon \mathfrak{g} \otimes \mathfrak{m}_R \to \mathfrak{h} \otimes \mathfrak{m}_R$ is a quasi-isomorphism of abelian L_∞-algebras and the claim follows from Lemma 3.8.

Suppose that $\mathfrak{m}_R^{l+1} = 0$. By the induction hypothesis

- the map $\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l) \to \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)$ is a weak homotopy equivalence and
- the map $\pi_0\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l) \to \pi_0\Sigma(\mathfrak{h} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)$ is a bijection.

The map $f \otimes \mathtt{Id}_{\mathfrak{m}_R^l}$ is a quasi-isomorphism of abelian L_∞-algebras, therefore the map $H^2(\mathfrak{g} \otimes \mathfrak{m}_R^l) \to H^2(\mathfrak{h} \otimes \mathfrak{m}_R^l)$ is an isomorphism. The commutativity of

$$\begin{array}{ccc} \pi_0\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l) & \longrightarrow & \pi_0\Sigma(\mathfrak{h} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l) \\ \downarrow & & \downarrow \\ H^2(\mathfrak{g} \otimes \mathfrak{m}_R^l) & \longrightarrow & H^2(\mathfrak{h} \otimes \mathfrak{m}_R^l) \end{array}$$

implies that the map

$$\pi_0\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)_0 \to \pi_0\Sigma(\mathfrak{h} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)_0$$

is a bijection. Therefore, the map

$$\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)_0 \to \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)_0$$

is a weak homotopy equivalence. The map $\Sigma(f)$ restricts to a map of principal fibrations

$$\begin{array}{ccc} \Sigma(\mathfrak{g} \otimes \mathfrak{m}_R) & \longrightarrow & \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R) \\ \downarrow & & \downarrow \\ \Sigma(\mathfrak{g} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)_0 & \longrightarrow & \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R/\mathfrak{m}_R^l)_0 \end{array}$$

relative to the map of simplicial groups $\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R^l) \to \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R^l)$. The latter is a weak homotopy equivalence by Lemma 3.8. Therefore, so is the map $\Sigma(\mathfrak{g} \otimes \mathfrak{m}_R) \to \Sigma(\mathfrak{h} \otimes \mathfrak{m}_R)$. ∎

3.10. Deligne groupoids.

3.10.1. Gauge transformations. Suppose that $\mathfrak{h}$ is a nilpotent DGLA. Then, $\mathfrak{h}^0$ is a nilpotent Lie algebra. The unipotent group $\exp \mathfrak{h}^0$ acts on the space $\mathfrak{h}^1$ by affine transformations. The action of $\exp X$, $X \in \mathfrak{h}^0$, on $\gamma \in \mathfrak{h}^1$ is given by the formula

$$(\exp X) \cdot \gamma = \gamma - \sum_{i=0}^{\infty} \frac{(\mathtt{ad}\, X)^i}{(i+1)!}(\delta X + [\gamma, X]). \tag{10}$$

The effect of the above action on the curvature $\mathcal{F}(\gamma) = \delta\gamma + \frac{1}{2}[\gamma, \gamma]$ is given by

$$\mathcal{F}((\exp X) \cdot \gamma) = \exp(\mathtt{ad}\, X)(\mathcal{F}(\gamma)). \tag{11}$$

3.10.2. The functor MC^1. Suppose that $\mathfrak{h}$ is a nilpotent DGLA. It follows from (11) that gauge transformations (10) preserve the subset of Maurer-Cartan elements $\mathrm{MC}(\mathfrak{h}) \subset \mathfrak{h}^1$.

We denote by $\mathrm{MC}^1(\mathfrak{h})$ the Deligne groupoid (denoted $\mathcal{C}(\mathfrak{h})$ in [Hinich, 1997]) defined as the groupoid associated with the action of the group $\exp \mathfrak{h}^0$ by gauge transformations on the set $\mathrm{MC}(\mathfrak{h})$.

Thus, $\mathrm{MC}^1(\mathfrak{h})$ is the category with the set of objects $\mathrm{MC}(\mathfrak{h})$. For $\gamma_1, \gamma_2 \in \mathrm{MC}(\mathfrak{h})$, $\mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2)$ is the set of gauge transformations between γ_1, γ_2. The composition

$$\mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_2, \gamma_3) \times \mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2) \to \mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_3)$$

is given by the product in the group $\exp(\mathfrak{h}^0)$.

3.10.3. The functor MC^2. For $\mathfrak{h}$ as above satisfying the additional vanishing condition $\mathfrak{h}^i = 0$ for $i < -1$ we denote by $\mathrm{MC}^2(\mathfrak{h})$ the Deligne 2-groupoid as defined by P. Deligne [Deligne, 1994] and independently by E. Getzler, [Getzler, 2009]. Below we review the construction of Deligne 2-groupoid of a nilpotent DGLA following [Getzler, 2009, Getzler, 2002] and references therein.

The objects and the 1-morphisms of $\mathrm{MC}^2(\mathfrak{h})$ are those of $\mathrm{MC}^1(\mathfrak{h})$. That is, for $\gamma_1, \gamma_2 \in \mathrm{MC}(\mathfrak{h})$ the set $\mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2)$ is the set of objects of the groupoid $\mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2)$. The morphisms in $\mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2)$ (i.e. the 2-morphisms of $\mathrm{MC}^2(\mathfrak{h})$) are defined as follows.

For $\gamma \in \mathrm{MC}(\mathfrak{h})$ let $[\cdot, \cdot]_\gamma$ denote the Lie bracket on $\mathfrak{h}^{-1}$ defined by

$$[a,\, b]_\gamma = [a,\, \delta b + [\gamma,\, b]]. \tag{12}$$

Equipped with this bracket, $\mathfrak{h}^{-1}$ becomes a nilpotent Lie algebra. We denote by $\exp_\gamma \mathfrak{h}^{-1}$ the corresponding unipotent group, and by

$$\exp_\gamma \colon \mathfrak{h}^{-1} \to \exp_\gamma \mathfrak{h}^{-1}$$

the corresponding exponential map. If γ_1, γ_2 are two Maurer-Cartan elements, then the group $\exp_{\gamma_2} \mathfrak{h}^{-1}$ acts on $\mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2)$. For $\exp_{\gamma_2} t \in \exp_{\gamma_2} \mathfrak{h}^{-1}$ and $\mathrm{Hom}_{\mathrm{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2)$ the action is given by

$$(\exp_{\gamma_2} t) \cdot (\exp X) = \exp(\delta t + [\gamma_2, t]) \exp X \in \exp \mathfrak{h}^0.$$

By definition, $\mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2)$ is the groupoid associated with the above action.

The horizontal composition in $\mathrm{MC}^2(\mathfrak{h})$, i.e. the map of groupoids

$$\otimes : \mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\exp X_{23}, \exp Y_{23}) \times \mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\exp X_{12}, \exp Y_{12}) \to \mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\exp X_{23} \exp X_{12}, \exp X_{23} \exp Y_{12}),$$

where $\gamma_i \in \mathrm{MC}(\mathfrak{h})$, $\exp X_{ij}, \exp Y_{ij}$, $1 \le i, j \le 3$ is defined by

$$\exp_{\gamma_3} t_{23} \otimes \exp_{\gamma_2} t_{12} = \exp_{\gamma_3} t_{23} \exp_{\gamma_3}(\exp(\mathtt{ad}\, X_{23})(t_{12})),$$

where $\exp_{\gamma_j} t_{ij} \in \mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\exp X_{ij}, \exp Y_{ij})$.

3.11. Remark. There is a canonical map of 2-groupoids $\mathrm{MC}^1(\mathfrak{h}) \to \mathrm{MC}^2(\mathfrak{h})$ which induces a bijection $\pi_0(\mathrm{MC}^1(\mathfrak{h})) \to \pi_0(\mathrm{MC}^2(\mathfrak{h}))$ on sets of isomorphism classes of objects.

3.12. Properties of $\mathfrak{N}\,\mathrm{MC}^2$.

3.12.1. Abelian DGLA.

3.13. Lemma. *Suppose that $\mathfrak{a}$ is an abelian DGLA satisfying $\mathfrak{a}^i = 0$ for $i < -1$. Then, the simplicial sets $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a})$ and $K(\mathfrak{a}[1])$ are isomorphic naturally in $\mathfrak{a}$.*

Proof. The claim is an immediate consequence of the definitions and the explicit description of the nerve of $\mathrm{MC}^2(\mathfrak{a})$ given in Lemma 2.4. ∎

Combining Lemma 3.13 with the integration map (8) we obtain the map of simplicial abelian groups

$$\int : \Sigma(\mathfrak{a}) \to \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a}) \tag{13}$$

which is a weak homotopy equivalence.

3.13.1. Central extensions. Suppose that $\mathfrak{g}$ is a nilpotent DGLA satisfying $\mathfrak{g}^i = 0$ for $i < -1$ and $\mathfrak{a}$ is a central subalgebra in $\mathfrak{g}$. Note that MC^2 commutes with products, $\mathfrak{N}$ commutes with products and the addition map $+\colon \mathfrak{a} \times \mathfrak{g} \to \mathfrak{g}$ is a morphism of DGLAs. Thus, we obtain an action of the simplicial abelian group $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a})$ on the simplicial set $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g})$

$$\mathfrak{N}\,\mathrm{MC}^2(+)\colon \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a}) \times \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}) \to \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}).$$

Note that the group structure on $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a})$ is obtained from the case $\mathfrak{a} = \mathfrak{g}$. Clearly, the action is free and the map $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}) \to \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})$ factors through $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g})/\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a})$.

3.13.2. The obstruction map.

3.14. Lemma. *The obstruction map* (6) *factors as*

$$\mathrm{MC}(\mathfrak{g}/\mathfrak{a}) \to \pi_0 \mathrm{MC}^2(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a})$$

Proof. Suppose $\mu + \mathfrak{a}^1 \in \mathrm{MC}(\mathfrak{g}/\mathfrak{a})$. It follows from the formula (10) that

$$\exp(X + \mathfrak{a}^0) \cdot (\mu + \mathfrak{a}^1) = (\exp X) \cdot \mu + \mathfrak{a}^1.$$

The formula (11) implies that

$$\mathcal{F}(\exp(X + \mathfrak{a}^0) \cdot (\mu + \mathfrak{a}^1)) = \mathcal{F}((\exp X) \cdot \mu) + \delta\mathfrak{a}^1 = \exp(\mathtt{ad}\, X)(\mathcal{F}(\mu) + \delta\mathfrak{a}^1).$$

Since $\mathcal{F}(\mu)+\delta\mathfrak{a}^1 \subset \mathfrak{a}^2$, it follows that $\exp(\mathtt{ad}\, X)(\mathcal{F}(\mu)+\delta\mathfrak{a}^1) = \mathcal{F}(\mu)+\delta\mathfrak{a}^1$ or, equivalently, $o_2(\exp(X + \mathfrak{a}^0) \cdot (\mu + \mathfrak{a}^1)) = o_2(\mu + \mathfrak{a}^1)$. ■

Recall (Lemma 2.4) that an n-simplex of $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})$, i.e. an element of $\mathfrak{N}_n\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})$ includes, among other things, a collection of $n+1$ gauge-equivalent Maurer-Cartan elements of $\mathfrak{g}/\mathfrak{a}$. By Lemma 3.14 all of these Maurer-Cartan elements give rise to the same element of $H^2(\mathfrak{a})$ under the map (6). Therefore, the assignment of this common value to an element of $\mathfrak{N}_n\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})$ give rise to a well-defined map

$$o_{2,n} \colon \mathfrak{N}_n\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}) \tag{14}$$

for each $n = 0, 1, 2, \ldots$ such that the sequence of pointed sets

$$0 \to \mathfrak{N}_n\,\mathrm{MC}^2(\mathfrak{g})/\mathfrak{N}_n\,\mathrm{MC}^2(\mathfrak{a}) \to \mathfrak{N}_n\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a}) \xrightarrow{o_{2,n}} H^2(\mathfrak{a})$$

is exact. The maps (14) assemble into a map of simplicial sets

$$o_2 \colon \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a}) \xrightarrow{o_2} H^2(\mathfrak{a}),$$

where $H^2(\mathfrak{a})$ is constant. Let $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})_0 = o_2^{-1}(0)$. The simplicial subset $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})_0$ is a union of connected components of $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})$ equal to the range of the map $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g})/\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a}) \to \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})$.

It follows that $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}) \to \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{g}/\mathfrak{a})_0$ is a principal fibration with the group $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{a})$.

4. $\mathfrak{N}\,\mathrm{MC}^2$ vs. Σ

In this section we show that for a DGLA $\mathfrak{h}$ satisfying $\mathfrak{h}^i = 0$ for $i < -1$ the simplicial sets $\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h})$ and $\Sigma(\mathfrak{h})$ are isomorphic in the homotopy category of simplicial sets.

4.1. The main theorem. Let $\boldsymbol{\Sigma}^2_n(\mathfrak{h}) = \widetilde{\mathrm{MC}^2(\Omega_n \otimes \mathfrak{h})}$, where the latter is the simplicial groupoid associated with the strict 2-groupoid $\mathrm{MC}^2(\Omega_n \otimes \mathfrak{h})$ (see 2.3.1). Let $\boldsymbol{\Sigma}^2(\mathfrak{h})\colon [n] \mapsto \boldsymbol{\Sigma}^2_n(\mathfrak{h})$ denote the corresponding simplicial object in simplicial groupoids. Note that $\Sigma(\mathfrak{h})$ is the simplicial set of objects of $\boldsymbol{\Sigma}^2(\mathfrak{h})$, hence there is a canonical map

$$\Sigma(\mathfrak{h}) \to \mathfrak{N}\boldsymbol{\Sigma}^2(\mathfrak{h}). \tag{15}$$

The map $\mathbb{Q} \to \Omega_\bullet$ of simplicial DGA induces the map of simplicial objects in simplicial groupoids

$$\mathrm{MC}^2(\mathfrak{h}) \to \boldsymbol{\Sigma}^2(\mathfrak{h}). \tag{16}$$

Consider the diagram

$$\Sigma(\mathfrak{h}) \xrightarrow{(15)} \mathfrak{N}\boldsymbol{\Sigma}^2(\mathfrak{h}) \xleftarrow{\mathfrak{N}((16))} \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h}). \tag{17}$$

4.2. Theorem. *Suppose that $\mathfrak{h}$ is a nilpotent DGLA satisfying $\mathfrak{h}^i = 0$ for $i < -1$. Then, the morphisms* (15) *and* $\mathfrak{N}((16))$ *are weak homotopy equivalences so that the diagram* (17) *represents an isomorphism $\Sigma(\mathfrak{h}) \cong \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h})$ in the homotopy category of simplicial sets.*

The rest of Section 4 is devoted to a proof of Theorem 4.2 which borrows techniques from the proof of Proposition 3.2.1 of [Hinich, 2004].

4.3. The map (15) is a weak homotopy equivalence. Let $\boldsymbol{\Sigma}^1(\mathfrak{h})$ denote the simplicial object in groupoids defined by $\boldsymbol{\Sigma}^1_n(\mathfrak{h}) = \mathrm{MC}^1(\Omega_n \otimes \mathfrak{h})$. Note that $\Sigma(\mathfrak{h})$ is the simplicial set of objects of $\boldsymbol{\Sigma}^1(\mathfrak{h})$ and hence there is a canonical map

$$\Sigma(\mathfrak{h}) \to \mathcal{N}\boldsymbol{\Sigma}^1(\mathfrak{h}); \tag{18}$$

by Remark 3.11 there is a canonical map of simplicial objects in simplicial groupoids

$$\boldsymbol{\Sigma}^1(\mathfrak{h}) \to \boldsymbol{\Sigma}^2(\mathfrak{h}). \tag{19}$$

The map (15) is equal to the composition

$$\Sigma(\mathfrak{h}) \xrightarrow{(18)} \mathcal{N}\boldsymbol{\Sigma}^1(\mathfrak{h}) \xrightarrow{\mathcal{N}((19))} \mathcal{N}\boldsymbol{\Sigma}^2(\mathfrak{h}) \to \mathfrak{N}\boldsymbol{\Sigma}^2(\mathfrak{h}),$$

where the last map is the weak homotopy equivalence of Theorem 2.2.

4.4. Lemma. ([Hinich, 2004], Proposition 3.2.1) *The map* (18) *is a weak homotopy equivalence.*

Proof. Let $G_n(\mathfrak{h}) := \exp((\Omega_n \otimes \mathfrak{h})^0)$. Then, $G(\mathfrak{h})\colon [n] \mapsto G_n(\mathfrak{h})$ is a simplicial group acting on $\Sigma(\mathfrak{h})$, and $\boldsymbol{\Sigma}(\mathfrak{h})$ is the associated groupoid. Therefore,

$$N_q\boldsymbol{\Sigma}(\mathfrak{h}) = \Sigma(\mathfrak{h}) \times G(\mathfrak{h})^{\times q}$$

and the map

$$\Sigma(\mathfrak{h}) \to N_q\boldsymbol{\Sigma}(\mathfrak{h})$$

is a weak homotopy equivalence because $G(\mathfrak{h})$ is contractible. ∎

4.5. Proposition. *The map $\mathcal{N}((19))$ is a weak homotopy equivalence.*

Proof. Let $\mathbf{\Gamma}^1(\mathfrak{h})$ (respectively, $\mathbf{\Gamma}^2(\mathfrak{h})$) denote the full subcategory of $\mathbf{\Sigma}^1(\mathfrak{h})$ (respectively, of $\mathbf{\Sigma}^2(\mathfrak{h})$) whose set of objects is $\mathrm{MC}(\mathfrak{h})$ (a constant simplicial set). There is a commutative diagram

$$\begin{array}{ccc} \mathbf{\Gamma}^1(\mathfrak{h}) & \longrightarrow & \mathbf{\Gamma}^2(\mathfrak{h}) \\ \downarrow & & \downarrow \\ \mathbf{\Sigma}^1(\mathfrak{h}) & \xrightarrow{(19)} & \mathbf{\Sigma}^2(\mathfrak{h}) \end{array}$$

The vertical arrows induce weak homotopy equivalences on respective nerves since, for each n the functors $\mathbf{\Gamma}^1(\mathfrak{h})_n \to \mathbf{\Sigma}^1(\mathfrak{h})_n = \mathrm{MC}^1(\Omega_n \otimes \mathfrak{h})$ and $\mathbf{\Gamma}^2(\mathfrak{h})_n \to \mathbf{\Sigma}^2(\mathfrak{h})_n = \mathrm{MC}^2(\Omega_n \otimes \mathfrak{h})$ are equivalences by [Hinich, 2001], Proposition 8.2.5.

The map $\mathbf{\Gamma}^1(\mathfrak{h}) \to \mathbf{\Gamma}^2(\mathfrak{h})$ induces a bijection between sets of isomorphism classes of objects. For $\mu \in \mathrm{MC}(\mathfrak{h})$, $\mathrm{Hom}_{\mathbf{\Gamma}^2(\mathfrak{h})}(\mu, \mu)$ is naturally identified with the nerve of the groupoid associated to the action of the simplicial group $H(\mathfrak{h}, \mu)\colon [n] \mapsto \exp((\Omega_n \otimes \mathfrak{h})_\mu)$ on the simplicial set $\mathrm{Hom}_{\mathbf{\Gamma}^1(\mathfrak{h})}(\mu, \mu)$. Since the group $H(\mathfrak{h}, \mu)$ is contractible (it is isomorphic as a simplicial set to $[n] \mapsto \Omega^0_n \otimes \mathfrak{h}^{-1}$) the induced map $\mathrm{Hom}_{\mathbf{\Gamma}^1(\mathfrak{h})}(\mu, \mu) \to \mathrm{Hom}_{\mathbf{\Gamma}^2(\mathfrak{h})}(\mu, \mu)$ is an equivalence. ∎

4.6. The map $\mathfrak{N}((16))\colon \mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h}) \to \mathfrak{N}\mathbf{\Sigma}^2(\mathfrak{h})$ is a weak homotopy equivalence. It suffices to show that the map

$$\mathfrak{N}\,\mathrm{MC}^2(\mathfrak{h}) \to \mathfrak{N}\,\mathrm{MC}^2(\Omega_n \otimes \mathfrak{h})$$

is a weak homotopy equivalence for all n. This follows from Proposition 4.7.

4.7. Proposition. *Suppose that $\mathfrak{h}$ is a nilpotent DGLA concentrated in degrees greater than or equal to -1. The functor*

$$\mathrm{MC}^2(\mathfrak{h}) \to \mathrm{MC}^2(\Omega_n \otimes \mathfrak{h}) \tag{20}$$

is an equivalence.

Proof. The induced map $\pi_0((20))$ is a bijection by Remark 3.11 and (the proof of) [Hinich, 1997], Lemma 2.2.1. The result now follows from Lemma 4.8 below. ∎

4.8. Lemma. *Suppose $\mu \in \mathrm{MC}(\mathfrak{h})$. The functor*

$$\mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{h})}(\mu, \mu) \to \mathrm{Hom}_{\mathrm{MC}^2(\Omega_n \otimes \mathfrak{h})}(\mu, \mu) \tag{21}$$

is an equivalence.

Proof. According to the description given in 3.10.3, for any nilpotent DGLA $(\mathfrak{g}, \delta)$ with $\mathfrak{g}^i = 0$ for $i < -1$ and $\mu \in \mathrm{MC}(\mathfrak{g})$ the groupoid $\mathrm{Hom}_{\mathrm{MC}^2(\mathfrak{g})}(\mu, \mu)$ is isomorphic to the groupoid associated with the action of the group $\exp_\mu \mathfrak{g}^{-1}$ on the set $\exp(\ker(\delta_\mu^{-1})) \subset \exp(\mathfrak{g}^0)$ where $\delta_\mu = \delta + [\mu, .]$.

Note that, for any $X \in \ker(\delta_\mu^{-1})$, the automorphism group $\mathrm{Aut}(\exp(X))$ is isomorphic to (the additive group) $\ker(\delta_\mu^{-1})$.

The map

$$([n] \to [0])^* \otimes \mathtt{Id}\colon (\mathfrak{h}, \delta) \to (\Omega_n \otimes \mathfrak{h}, d + \delta) \tag{22}$$

is a quasi-isomorphism of DGLA with the quasi-inverse given by the evaluation map $\mathrm{ev}_0 := ([0] \to [n])^* \otimes \mathtt{Id}\colon \Omega_n \otimes \mathfrak{h} \to \mathfrak{h}$ (for any choice of a morphism $[0] \to [n]$) which is a morphism of DGLA as well. The same maps are mutually quasi-inverse quasi-isomorphisms of DGLA

$$(\mathfrak{h}, \delta_\mu) \rightleftarrows (\Omega_n \otimes \mathfrak{h}, d + \delta_\mu).$$

Since (22) is a quasi-isomorphism and both DGLA are concentrated in degrees greater than or equal to -1, the induced map $\ker(\delta_\mu^{-1}) \to \ker((d + \delta_\mu)^{-1})$ an isomorphism, hence so are the maps of automorphism groups.

Since the map (21) admits a left inverse (namely, ev_0) it remains to show that the induced map on sets of isomorphism classes is surjective. Note that, since ev_0 is a surjective quasi-isomorphism, the map $d + \delta_\mu\colon \ker(\mathrm{ev}_0)^{-1} \to \ker(\mathrm{ev}_0)^0 \bigcap \ker((d + \delta_\mu)^0)$ is an isomorphism.

Consider $X \in (\Omega_n \otimes \mathfrak{g})^0$. Then, $X = \mathrm{ev}_0(X) + Y$ with $Y \in \ker(\mathrm{ev}_0)$, and $(d + \delta_\mu)X = 0$ if and only if $\delta_\mu \mathrm{ev}_0(X) = 0$ and $(d + \delta_\mu)Y = 0$.

Suppose $X \in \ker((d + \delta_\mu)^0)$. Then, $\exp(X) = \exp(\mathrm{ev}_0(X)) \cdot \exp(Z)$ where $Z \in \ker(\mathrm{ev}_0)^0 \bigcap \ker((d + \delta_\mu)^0)$, and, therefore, $Z = (d + \delta_\mu)U$ for a *uniquely determined* U. ■

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